Category Theory 2

Natural transformations and equivalence

This is to accompany the reading of 11–17 October. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

Some questions on these sheets require knowledge of other areas of mathematics; skip over any that you haven't the background for. That aside, I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Write down three examples of natural transformations that aren't in the notes.

2. Prove that a natural transformation is a natural isomorphism if and only if each of its components is an isomorphism (Lemma 1.3.6).

3. *Linear algebra can be done equivalently with matrices or with linear maps...*

   Fix a field k. Let **Mat** be the category whose objects are the natural numbers and with

   $$\mathbf{Mat}(m, n) = \{n \times m \text{ matrices over } k\}.$$

   Prove that **Mat** is equivalent to **FDVect**, the category of finite-dimensional vector spaces over k. Does your equivalence involve a *canonical* functor from **Mat** to **FDVect**, or from **FDVect** to **Mat**?

   (Hints: (i) Part of the exercise is to work out what composition in the category **Mat** *is supposed to be; there's only one sensible possibility. (ii) It's easier if you use 1.3.12. (iii) The word 'canonical' means something like 'God-given' or 'definable without making any arbitrary choices'.)*

4. Let G be a group. For any $g \in G$ there is a unique homomorphism $\phi : \mathbb{Z} \longrightarrow G$ satisfying $\phi(1) = g$, so elements of G are essentially the same as homomorphisms $\mathbb{Z} \longrightarrow G$. These in turn are the same as functors $\mathbb{Z} \longrightarrow G$, where groups are regarded as one-object categories. Natural isomorphism therefore defines an equivalence relation on the elements of G. What is this equivalence relation, in group-theoretic terms?

   (First have a guess. For a general group G, what equivalence relations on G can you think of?)

5. A **permutation** on a set X is a bijection $X \longrightarrow X$. Let $\mathbf{Sym}(X)$ be the set of permutations on X. A **total order** on a set X is an order $\leq$ such that for all $x, y \in X$, either $x \leq y$ or $y \leq x$; so a total order on a finite set amounts to a way of placing its elements in sequence. Let $\mathbf{Ord}(X)$ be the set of total orders on X.

   Let $\mathcal{B}$ be the category of finite sets and bijections.

   (a) Give a definition of **Sym** on morphisms of $\mathcal{B}$ so that **Sym** becomes a functor $\mathcal{B} \longrightarrow \mathbf{Set}$. Do the same for **Ord**. Both your definitions should be canonical (no arbitrary choices).

   (b) Show that there is no natural transformation $\mathbf{Sym} \longrightarrow \mathbf{Ord}$. *(Hint: consider the identity permutation.)*

   (c) If X is an n-element set, how many elements do the sets $\mathbf{Sym}(X)$ and $\mathbf{Ord}(X)$ have?

   Conclude that $\mathbf{Sym}(X) \cong \mathbf{Ord}(X)$ for all $X \in \mathcal{B}$, but not naturally in X.