Category Theory 10

Interaction of (co)limits with adjunctions

This is to accompany the final batch of reading, beginning on 5 December. In the week of 7–11 January, there will be a lecture on this material and a tutorial on this sheet. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Consider the following three conditions on a functor U from a locally small category $\mathcal{A}$ to **Set**:

   A. U has a left adjoint **R**. U is representable **L**. U preserves limits.

   (a) Show that $\mathbf{A} \Rightarrow \mathbf{R} \Rightarrow \mathbf{L}$.
   (b) Show that if $\mathcal{A}$ has sums then $\mathbf{R} \Rightarrow \mathbf{A}$.

   (If $\mathcal{A}$ satisfies the hypotheses of the Special Adjoint Functor Theorem then $\mathbf{L} \Rightarrow \mathbf{A}$ *and the three conditions are equivalent.)*

2. (a) Prove that left adjoints preserve colimits and right adjoints preserve limits.
   (b) Prove that the forgetful functor $U : \mathbf{Gp} \longrightarrow \mathbf{Set}$ has no right adjoint.
   (c) Prove that the chain of adjunctions $C \dashv D \dashv O \dashv I$ in Sheet 4, q.4 extends no further in either direction.

3. (a) Show that every presheaf on a small category is a colimit of representable presheaves.
   (b) What does it mean for a category to be **cartesian closed**? Show that for any small category $\mathbb{A}$, the presheaf category $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed. (You may assume that limits and colimits in presheaf categories exist and are computed pointwise.)

4. (a) Let

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ A' & \xrightarrow[f]{} & A \end{array}$$

   be a pullback square in some category. Show that if m is monic then so is m'. *(We already know that this holds in the category of sets: Example 4.1.16.)*

   A category $\mathcal{A}$ is **well-powered** if for each $A \in \mathcal{A}$, the class of subobjects of A is small—that is, a set. All of our usual examples of categories are well-powered. Let $\mathcal{A}$ be a well-powered category with pullbacks, and write $\mathrm{Sub}(A)$ for the set of subobjects of an object $A \in \mathcal{A}$.

   (b) Deduce from (a) that any map $A' \xrightarrow{f} A$ in $\mathcal{A}$ induces a map $\mathrm{Sub}(A) \xrightarrow{\mathrm{Sub}(f)} \mathrm{Sub}(A')$.
   (c) Show that this determines a functor $\mathrm{Sub} : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. *(Hint: Sheet 7, q.3.)*
   (d) For some categories $\mathcal{A}$, Sub is representable. A **subobject classifier** for $\mathcal{A}$ is an object $\Omega \in \mathcal{A}$ such that $\mathrm{Sub} \cong H_\Omega$. Prove that 2 is a subobject classifier for **Set**.