**Category Theory 1**

Categories and functors

This is to accompany the reading of 1–7 October and the lecture of 8 October. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

Some questions on these sheets require knowledge of other areas of mathematics; skip over any that you haven't the background for. That aside, I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Write down three examples of (a) categories, and (b) functors, that weren't given in lectures.

2. Show that functors preserve isomorphism. That is, prove that if $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor and $A, A' \in \mathcal{A}$ with $A \cong A'$, then $F(A) \cong F(A')$.

3. Two categories $\mathcal{A}$ and $\mathcal{B}$ are **isomorphic**, written $\mathcal{A} \cong \mathcal{B}$, if they are isomorphic as objects of **CAT**.

   (a) Let $G$ be a group, regarded as a one-object category. What is the opposite of $G$? Prove that $G$ is isomorphic to $G^{\mathrm{op}}$.

   (b) Find a monoid not isomorphic to its opposite.

4. Is there a functor $Z : \mathbf{Gp} \longrightarrow \mathbf{Gp}$ with the property that $Z(G)$ is the centre of $G$ for all groups $G$?

5. *Sometimes we meet functors whose domain is a product $\mathcal{A} \times \mathcal{B}$ of categories. In this question we'll show that such a functor can be regarded as an interlocking pair of families of functors, one defined on $\mathcal{A}$ and one defined on $\mathcal{B}$. This is very like the situation with bilinear and linear maps.*

   Let $F : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ be a functor. For each $A \in \mathcal{A}$, there is an induced functor $F^A : \mathcal{B} \longrightarrow \mathcal{C}$ defined on objects $B \in \mathcal{B}$ by $F^A(B) = F(A, B)$ and on arrows $g$ of $\mathcal{B}$ by $F^A(g) = F(1_A, g)$. Similarly, for each $B \in \mathcal{B}$ there is an induced functor $F_B : \mathcal{A} \longrightarrow \mathcal{C}$. Show that the families of functors $(F^A)_{A \in \mathcal{A}}$ and $(F_B)_{B \in \mathcal{B}}$ satisfy the following conditions:

   - if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $F^A(B) = F_B(A)$
   - if $f : A \longrightarrow A'$ in $\mathcal{A}$ and $g : B \longrightarrow B'$ in $\mathcal{B}$ then $F^{A'}(g) \circ F_B(f) = F_{B'}(f) \circ F^A(g)$.

   Then show that given families $(F^A)_{A \in \mathcal{A}}$ and $(F_B)_{B \in \mathcal{B}}$ of functors satisfying these two conditions, there is a unique functor $F : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ inducing them.