Implosion, Contraction and Moore-Tachikawa.

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We give a survey of the implosion construction, extending some of its aspects relating to hypertoric geometry from type A to a general reductive group, and interpret it in the context of the Moore-Tachikawa category. We use these ideas to discuss how the contraction construction in symplectic geometry can be generalised to the hyperkähler or complex symplectic situation.

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Dedication.

Dedicated to Oscar Garcia-Prada on the occasion of his 60th birthday.

1. Introduction

The symplectic implosion construction of Guillemin, Jeffrey and Sjamaar [18] is an Abelianisation construction for Hamiltonian group actions. Given a symplectic space M with such an action of a compact Lie group K, the implosion M_{impl} carries an action of the maximal torus T of K, such that the symplectic reduction of the implosion M_{impl} by T at a level in the closed positive Weyl chamber is the same as the reduction of M by K.

If M actually arises as a Kähler variety, projective over an affine, equipped with a linearised action of $K_{\mathbb{C}}$ and compatible Kähler structure, M_{impl} can also be understood as the non-reductive geometric invariant theory quotient of M by U, a maximal unipotent subgroup of $G = K_{\mathbb{C}}$ [24].

Implosion was generalised to the hyperkähler situation in [9] in the $SU(n)$ case. In the general case, starting from an arbitrary compact connected Lie group, work of Ginzburg and Riche [17] shows we have at least a complex symplectic version of the implosion (though the hyperkähler structure has not been shown yet, it is expected to exist, and we will refer to it as such). The hyperkähler version takes inspiration from both the (real) symplectic and algebro-geometric version, but is overall a step up in complexity. This is parallel to other operations in equivariant geometry that come in three flavours, such as symplectic reduction [27], geometric invariant theory [29], and hyperkähler reduction [21], or (for abelian actions) symplectic and algebraic cutting [26, 13] and hyperkähler modifications [11].

In this note we expand this viewpoint by introducing the complex symplectic version of contraction, which so far was understood only in an algebraic [30] or real symplectic [20] context (see §5 for a survey of these constructions).Unlike reduction or implosion, contraction does not change the dimension of the space one starts from, but gives rise to an additional action of the maximal torus, commuting with the actions of K or G.

It is common to all these operations that they can be reduced to a universal version, which is the result of applying them to the cotangent bundle T^*K (in the real symplectic case), the complex group G (in the algebraic case), or the complex cotangent bundle T^*G (in the complex symplectic case). The resulting spaces are often familiar – the symplectic reductions of T^*K are coadjoint K orbits (or flag varieties for G), for example. For abelian K the symplectic cuts of T^*K, or algebraic cuts of G, give toric varieties, and the hyperkähler modifications of T^*G are the hypertoric varieties.

This in turn means that they naturally fit in with the concept of Moore-Tachikawa categories, whose objects are groups, morphisms are spaces with group actions, and composition is done by reduction of products. Indeed, the universal implosions are morphisms in this category between a group and its maximal torus, and the universal contraction is the composition of this morphism with itself, suitably interpreted.

2. Hyperkähler implosion

In this section we survey symplectic and hyperkähler implosion.

Let T be a maximal torus for a compact Lie group K, with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$. By using the weight decomposition for the adjoint representation, $\mathfrak{t}$ has a canonical complement, and hence we have $\mathfrak{t}^* \subset \mathfrak{k}^*$, and also $\mathfrak{t}^*_+ \subset \mathfrak{k}^*$ for the choice of a positive Weyl chamber $\mathfrak{t}^*_+$. If M is a (real) Hamiltonian K-manifold, with moment map $\mu : M \to \mathfrak{k}^*$, the symplectic implosion M_{impl} is (as a set) $\mu^{-1}(\mathfrak{t}^*_+)/\sim$, where

$$x \sim y \Leftrightarrow \mu(x) = \mu(y) \text{ and } y \in [K_{\mu(x)}, K_{\mu(x)}].x.$$

It is typically not a smooth manifold, but a stratified Hamiltonian space – the residual map to $\mathfrak{t}_+^* \subset \mathfrak{t}^*$ is the moment map for a T-action. Applied to T^*K (as a K-space with the action induced from K acting on itself as $k.h = hk^{-1}$), this operation gives the universal implosion

$$(T^*K)_{\mathrm{impl}} = K \times \mathfrak{t}_+^* \,/\sim . \tag{2.1}$$

Besides the T-action, $(T^*K)_{\mathrm{impl}}$ still has the residual K-action that comes from multiplication on the left, since if a space has two commuting actions by compact groups K and L, the implosion by K still has an action of L, and moreover implosion and symplectic reduction commute:

$$M_{K-\mathrm{impl}} /\!\!/ L \cong (M /\!\!/ L)_{K-\mathrm{impl}}. \tag{2.2}$$

This last property also explains the relevance of the universal implosion: for any symplectic K-manifold, one has a canonical equivariant isomorphism

$$M \cong (M \times T^*K) /\!\!/ \Delta K, \tag{2.3}$$

where the quotient in the right-hand side uses the diagonal action of K acting on M and on T^*K by multiplication from the right – this quotient space still inherits a K-action coming from the left action of K on T^*K.

Also for GIT quotients and complex symplectic reduction, one has an analogue of (2.3), with the role of T^*K now played by respectively $G = K_{\mathbb{C}}$ or T^*G (for simplicity of notation we shall indicate both symplectic reduction and GIT quotients by $/\!\!/$, and hyperkähler and complex symplectic reduction by $/\!\!/\!\!/$). Remark that this is not quite true for the metric aspects in Kähler or hyperkähler quotients: there is no known metric on G or T^*G that would make the analogues of (2.3) an isometry.

Because of the commutativity of implosion and reduction, as in (2.2), when applying the former to both sides of (2.3), we obtain

$$M_{\mathrm{impl}} \cong (M \times (T^*K)_{\mathrm{impl}}) /\!\!/ \Delta K,$$

which shows how to recover any implosion from the universal implosion.

The universal symplectic implosion $(T^*K)_{\mathrm{impl}}$ can be identified (as a stratified $K \times T$ space) with $G /\!\!/ U = \overline{G/U}^{\mathrm{Aff}} = \mathrm{Spec}\left(k[G]^U\right)$, for U the unipotent radical of the Borel subgroup of G determined by the maximal torus $T_{\mathbb{C}}$ and the positive Weyl chamber $\mathfrak{t}_+^*$. Even though U is not a reductive group, the invariant ring $k[G]^U$ is still finitely generated, which makes this definition sensible.

This is done as follows (see Section 6 in [18] and Appendix in [20]): given the choice of a finite set of generators Π for the semigroup of dominant weights of G, one can embed

$$G /\!\!/ U \hookrightarrow E = \oplus_{\varpi \in \Pi} V_{\varpi}. \tag{2.4}$$

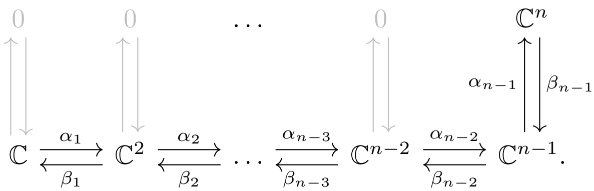
Here V_ϖ is the irreducible representation of G with highest weight ϖ, and we moreover choose highest-weight vectors $v_\varpi \in V_\varpi$ and K-invariant Hermitian inner products on the V_ϖ such that $\|v_\varpi\| = 1$. The image of $G/\!\!/U$ is now the closure of the G-orbit of $\sum_{\varpi\in\Pi} v_\varpi$, and $G/\!\!/U$ inherits a Kähler structure from the embedding in E.

We can also consider an extra action of $T_{\mathbb{C}}$ on E (commuting with the G action), acting diagonally on V_ϖ with weight $-\varpi$, and look at the affine toric variety inside $G/\!\!/U$ that is the closure of the $T_{\mathbb{C}}$-orbit of $\sum_{\varpi\in\Pi} v_\varpi$. The image of the T-moment map of this toric variety is $-\,\mathfrak{t}^*_+$, and the non-negative part of the toric variety provides a section of this moment map. We can now use minus this section to create a subjective map from $K \times \mathfrak{t}^*_+$ into $G/\!\!/U \subset E$, which (using (2.1)) induces an isomorphism of symplectic stratified spaces $(T^*K)_{\mathrm{impl}}$.

This identification is the basis for the correspondence between symplectic reduction and quotients in non-reductive GIT, though this only concerns those quotients on the algebraic side where the action of the group U extends to one of G, in which case the invariant ring will be finitely generated.

When it comes to a hyperkähler or complex symplectic analogue of implosion, as it too commutes with complex symplectic reduction, the key case is again the implosion of T^*G. We recall that this space is complex symplectic with $G\times G$-action, and in fact, by the work of Kronheimer [25], is even known to be hyperkähler. One way to view its implosion is as the complex symplectic reduction of T^*G by U, acting from the right. If we identify $T^*G \cong G \times \mathfrak{g}^*$, then the complex moment map for the U-action is given by the projection of the second factor on $\mathfrak{u}^*$, where $\mathfrak{u}$ is the Lie algebra of U. The level set is the product affine variety $G \times \mathfrak{u}^\circ$ (where $\mathfrak{u}^\circ \subset \mathfrak{g}^*$ is the annihilator of $\mathfrak{u}$), which has a $G \times U$-action, but not a $G \times G$-action, and as a result it is not directly clear that the invariant ring $k[G \times \mathfrak{u}^\circ]^U$ is finitely generated. Alternatively, remark that if $(G \times \mathfrak{u}^\circ)/\!\!/U = \mathrm{Spec}\left(k[G \times \mathfrak{u}^\circ]^U\right)$ were well-defined as an affine variety, it would contain the cotangent space $T^*(G/U)$, and hence the question is whether $T^*(G/U)$ is a quasi-affine variety.

For $K = SU(n)$, the question of finite generation was answered affirmatively by Dancer, Kirwan and Swann in [9], where it was shown that the resulting variety could also be constructed through quivers as a hyperkähler quotient of a vector space by a compact group (which in particular showed that the space is not just complex symplectic, but in fact hyperkähler on its smooth locus). In particular, the following quiver diagram is considered:



(We follow the presentation of the quiver given in [39] here, including the vacuous light-coloured part, to indicate this can be understood as a special case of a framed and doubled quiver, as in the context of Nakajima quiver varieties.) One can take the affine hyperkähler quotient (at zero for all moment maps) of

$$M = \{(\alpha_i, \beta_i)\} = \bigoplus_{i=1}^{r-1} \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1}) \oplus \mathrm{Hom}(\mathbb{C}^{i+1}, \mathbb{C}^i)$$

by the natural action of $\widetilde{H} = \prod_{i=1}^{n-1} U(i)$, to obtain the nilpotent cone inside $\mathfrak{sl}(n) \cong \mathfrak{sl}(n)^*$. When one takes the hyperkähler quotient by $H = \prod_{i=1}^{r-1} SU(i)$, however, one obtains an affine variety containing $T^*(SL(n)/U)$ as a dense open subvariety of codimension at least two (Theorem 7.18 in [9]). As a result, $k[SL(n) \times \mathfrak{u}^\circ]^U$ is finitely generated, and

$$M /\!\!/\!\!/ H = \mathrm{Spec}\left(k[SL(n) \times \mathfrak{u}^\circ]^U\right) = (SL(n) \times \mathfrak{u}^\circ) /\!\!/ U,$$

where the quotient is taken in the sense of non-reductive GIT.

The finite generation of $k[G \times \mathfrak{u}^\circ]^U$ for general reductive G was established by Ginzburg and Riche (Lemma 3.6.2 in [17]), and further studied by Ginzburg and Kazhdan in [15, 16]. We shall refer to the resulting affine space, as a singular complex symplectic variety, as the (right) universal implosion $Q_G = (G \times \mathfrak{u}^\circ) /\!\!/ U$. Remark that Q_G is always normal, following Proposition 1 in [36].

A key ingredient in the work of Ginzburg-Riche and Ginzburg-Kazhdan is an action of the Weyl group W_G of G on Q_G. In the real symplectic or algebraic case, we have an action of $K \times T$ (or its complexification) on $T^* K_{\mathrm{impl}} \cong G /\!\!/ U$, but in the complex symplectic case Q_G has an action of $G \times (W_G \ltimes T_{\mathbb{C}})$. From the point of view of the quiver-construction in type A_n of [9], this action of W_G was constructed recently by Wang in [39]. Ginzburg-Kazhdan also conjectured that Q_G has symplectic singularities (in the sense of [1]). This was proved for type A_n using the quiver construction by Jia [22, 23], and in the general case by Gannon [14].

Finally, Ginzburg and Kazhdan establish an affine embedding of Q_G (Corollary 7.2.3 in [15], which follows from the proof of Lemma 3.6.2 in [17]):

Proposition 2.1. *There is a W_G-equivariant closed embedding of affine varieties*

$$Q_G \to (\mathfrak{g}^* \oplus \mathfrak{t}^*_{\mathbb{C}}) \times \prod_{w \in W_G} G /\!\!/ U. \tag{2.5}$$

Here the component going to $\mathfrak{g}^* \oplus \mathfrak{t}_{\mathbb{C}}^*$ is the moment map for the $G \times T_{\mathbb{C}}$-action, and the component going to each copy of $G/\!\!/U$ is given by the composition of the action of $w \in W_G$ on Q_G, and the map $Q_G \to G/\!\!/U$ that is the affinization of the natural projection $T^*(G/U) \to G/U$. The W_G-action on the target is the coadjoint action on $\mathfrak{t}_{\mathbb{C}}$, and the permutation of the factors of $\prod_{w \in W_G} G/\!\!/U$, while $\mathfrak{g}$ is left invariant. One can combine (2.5) with (2.4) for each $G/\!\!/U$ factor to obtain an embedding into affine space that provides the counterpart to (2.4)

3. A hypertoric variety related to Q_G

Inside each universal symplectic implosion $(T^*K)_{\mathrm{impl}} \cong G/\!\!/U$ one can embed an affine toric variety X_G, preserved by the action of $T_{\mathbb{C}}$, whose image under the T-moment map is $\mathfrak{t}_+^*$, and whose K-sweep is all of $(T^*K)_{\mathrm{impl}}$, see [§7] of [18] and the appendix to [20]. In [10], an analogue of this was shown for the universal hyperkähler implosion for $SL(n)$: there exists a morphism of the affine hypertoric variety Y_G, associated with the hyperplane arrangement given by the Weyl chambers for G, to Q_G, which is generically an embedding. We generalise this here to the case of arbitrary reductive groups.

We briefly recall the construction of hypertoric varieties, following [5, 32]. For our purpose, it suffices to restrict to affine ones. These are determined by an arrangement $\mathcal{A}$ of N distinct hyperplanes $H_1, \dots, H_N$ in $\mathfrak{t}^*$, all containing the origin. For each hyperplane, make a choice of a minimal normal vector $\alpha_i \in \mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{C}}) \subset \mathfrak{t}$. These choices determine a group homomorphism $\mathbb{G}_m^N \to T_{\mathbb{C}}$, which we require to be surjective. It has kernel $L \lhd \mathbb{G}_m^N$, and if we let $\mathbb{G}_m^N$ act diagonally on $\mathbb{C}^N$, and via co-tangent lift on $T^*\mathbb{C}^N \cong \mathbb{C}^{2N}$, then the hypertoric variety is constructed as the complex symplectic reduction of $\mathbb{C}^{2N}$ by L:

$$\mathfrak{M}(\mathcal{A}) = \mathbb{C}^{2N}/\!\!/\!\!/L.$$

Up to isomorphism this is independent of the choices of the α_i. We will denote elements of $\mathbb{C}^{2N}$ as N-tuples of pairs (a_i, b_i), and will denote the equivalence class of such tuples representing elements in $\mathfrak{M}(\mathcal{A})$ as $[\underline{a}, \underline{b}]$.

In the case of $\mathfrak{M}(\mathcal{A}) = Y_G$, N is half the number of roots of the associated root system of G. It will be most convenient in this case if the choice of the normal vectors α_i is induced by the choice of a set of simple positive roots, such that the positive Weyl chamber $\mathfrak{t}_+^*$ is cut out by the equations $\alpha_i(x) \geq 0, \forall i$. As could be expected by the symmetry of the hyperplane arrangement for Y_G, this hypertoric variety also has an action of $W_G \ltimes T_{\mathbb{C}}$ (which the toric variety X_G does not have). This action plays a key role in the construction of the desired map $Y_G \to Q_G$, even though, surprisingly, this map will not be W_G-equivariant.

Lemma 3.1. *The $T_{\mathbb{C}}$-action on the affine hypertoric variety Y_G naturally extends to a $W_G \ltimes T_{\mathbb{C}}$-action (where W_G is the Weyl group of G) such that all moment maps for the action of T and $T_{\mathbb{C}}$ on Y_G are W_G-equivariant.*

This fact, and the proof below, are valid for any affine hypertoric variety determined by a hyperplane arrangement acted on by a finite group.

Proof. The action of the Weyl group on $\mathfrak{t}^*$ permutes the hyperplanes in the arrangement, so we get a homomorphism $\sigma : W_G \to S_N$. Moreover, for each $w \in W_G$ and $i \in \{1, \ldots, N\}$ we get a sign $s_{w,i} \in \{\pm 1\}$ (a cocycle for the permutation action), determined by $w(\alpha_i) = s_{w,i}\alpha_{\sigma(w)(i)}$. We obtain an action ϕ of W_G on $\mathbb{G}_m^N$, given by

$$\phi(w)(t_1, \ldots, t_N) = (t_{\sigma(w)(1)}^{s_{w,1}}, \ldots, t_{\sigma(w)(N)}^{s_{w,N}}),$$

which makes the morphism $\mathbb{G}_m^N \to T_{\mathbb{C}}$ (determined by the α_i) W_G-equivariant. As a result, ϕ preserves L.

Finally, we can extend the action of $\mathbb{G}_m^N$ on $\mathbb{C}^{2N}$ to an action of $W_G \ltimes \mathbb{G}_m^N$ (note that this does not work for the action of $\mathbb{G}_m^N$ on $\mathbb{C}^N$). Denoting elements of $\mathbb{C}^{2N}$ as N-tuples of pairs $(a_i, b_i) \in \mathbb{C}^2$, then each $w \in W_G$ permutes the pairs via σ, and moreover switches $(a_{\sigma(w)(i)}, b_{\sigma(w)(i)})$ to $(b_{\sigma(w)(i)}, a_{\sigma(w)(i)}))$ if $s_{w,i} = -1$.

We immediately get that, after taking the complex symplectic reduction of $\mathbb{C}^{2N}$ by L (at zero-level of the moment maps), the action of $W_G \ltimes \mathbb{G}_m^N$ descends to an action of $\left(W_G \ltimes \mathbb{G}_m^N\right) \big/ L \cong W_G \ltimes T_{\mathbb{C}}$ on the hypertoric variety. Moreover, all the moment maps for the action of T on Y_G are W_G-equivariant. □

We now want to define a morphism from Y_G into the target of (2.5), whose image is contained in the image of Q_G. The components of this map into both $\mathfrak{g}^*$ and $\mathfrak{t}_{\mathbb{C}}^*$ are just given by the moment map for the action of $T_{\mathbb{C}}$ on Y_G – remark that such a map from Y_G into $\mathfrak{g}^*$ is not W_G-invariant, whereas the component into $\mathfrak{g}^*$ of (2.5) is W_G-invariant (this at the end will result in the map $Y_G \to Q_G$ not being W_G-equivariant).

The components of the map into the various copies of $G/\!\!/U$ will land in $X_G \subset G/\!\!/U$ for each w. If Y_G were to be a smooth hypertoric variety, it would be covered by cotangent bundles to the various components of its extended core (see [32], Remark 2.1.6). These components are Lagrangian subvarieties, each isomorphic to X_G, and permuted by the action of W_G on Y_G. Under the real moment map their images are given by the (closed) Weyl chambers in $\mathfrak{t}^*$. The various projections $T^*X_G \to X_G$ would extend to give maps $Y_G \to X_G \subset G/\!\!/U$, and these would give the remaining components of the morphism from Y_G into the target of (2.5).

However, the variety Y_G will be singular in general, and therefore this argument needs a bit more care. We can use a smooth stratification of affine hypertoric varieties introduced in [31], §2. Given a hyperplane arrangement $\mathcal{A}$, and a subset $F \subset \{1, \ldots, N\}$, the affine subspace $\cap_{i \in F} H_i$ is denoted H_F. If $F = \{i | H_i \subset H_F\}$, F is called a *flat*. For every flat F, $\mathcal{A}^F$ is defined to be the hyperplane arrangement $\{H_i \cap H_F | i \notin F\}$ in the affine space H_F. Similarly, $\mathcal{A}_F$ is defined to be the hyperplane arrangement $\{H_i/H_F | i \in F\}$ in $\mathfrak{t}^* / H_F$. We have that the $\mathfrak{M}(\mathcal{A}^F)$ are all symplectic subvarieties of $\mathfrak{M}(\mathcal{A}^{\emptyset}) = \mathfrak{M}(\mathcal{A})$. Finally, Proudfoot and Webster define $\mathring{\mathfrak{M}}(\mathcal{A}^F)$ to be

$$\mathring{\mathfrak{M}}(\mathcal{A}^F) = \left\{ [\underline{a}, \underline{b}] \in \mathfrak{M}(\mathcal{A}) \;\middle|\; a_i = b_i = 0 \;\Leftrightarrow\; i \in F \right\}$$

(this is always a smooth subvariety). It is shown in Lemma 2.4 of [31] that the decomposition

$$\mathfrak{M}(\mathcal{A}) = \bigsqcup_{F \text{ flat}} \mathring{\mathfrak{M}}(\mathcal{A}^F)$$

is a stratification, with a normal slice to each $\mathfrak{M}(\mathcal{A}^F)$ provided by $\mathfrak{M}(\mathcal{A}_F)$. Since all of these strata are even-dimensional, and two-dimensional hypertoric varieties (when hyperplanes are not repeated) are always smooth, this immediately gives us

Lemma 3.2. *The singular locus of an affine hypertoric variety will have (complex) codimension at least four.*

In particular we can focus on $\mathfrak{M}(\mathcal{A})^g = \mathring{\mathfrak{M}}(\mathcal{A}) \cup \bigcup_{i \in \{1,\ldots,N\}} \mathring{\mathfrak{M}}(\mathcal{A}^{\{H_i\}})$.

Furthermore, for each $V \subset \{1, \ldots, N\}$, we can look at the cone $\mathcal{C}_V$ cut out by $\alpha_i(x) \geq 0$ if $i \in V$ and $\alpha_i(x) \leq 0$ if $i \notin V$. If this cone is top-dimensional, we say V is *broad*, and we define

$$\mathcal{X}(V) = \left\{ \; [\underline{a}, \underline{b}] \;\middle|\; b_i = 0 \text{ if } i \in V \text{ and } a_j = 0 \text{ if } j \notin V \right\}.$$

The $\mathcal{X}(V)$, for all broad V, are the components of the *extended core*. They are Lagrangian subvarieties of Y_G, that are toric for the action of $T_{\mathbb{C}}$, with at worst finite quotient singularities, and whose image under the real moment map is given by $\mathcal{C}_V$. They will intersect the singular locus of Y_G, but we can restrict our attention to what happens in $\mathfrak{M}(\mathcal{A})^g$. We put $\mathcal{X}(V)^g = \mathcal{X}(V) \cap \mathfrak{M}(\mathcal{A})^g$, which is the subvariety of the toric variety $\mathcal{X}(V)$ consisting of the open orbit and the codimension-one orbits.

Lemma 3.3. *The cotangent bundles $T^*\mathcal{X}(V)^g$, for all broad V, are symplectic subvarieties that cover $\mathfrak{M}(\mathcal{A})^g$. Each of these $T^*\mathcal{X}(V)^g$ has (complex) codimension at least two in $\mathfrak{M}(\mathcal{A})$.*

Proof. It suffices to remark that

$$T^*\mathcal{X}(V)^g = \left\{ \; [\underline{a}, \underline{b}] \;\middle|\; \text{at most one of the } a_i, i \in V, \text{ and } b_j, j \notin V, \text{ is zero} \; \right\}. \qquad \square$$

We now return our focus to the particular hypertoric variety $\mathfrak{M}(\mathcal{A}) = Y_G$. Here the cotangent bundles $T^*\mathcal{X}(V)^g \subset Y_G$, for broad V, are permuted by the W_G-action. Moreover, for $\Omega = \{1, \ldots, N\}$, we have $\mathcal{X}(\Omega) = X_G \subset Y_G$.

The natural projection $T^*\mathcal{X}(\Omega)^g \to X_G$ extends to a morphism $Y_G \to X_G$ by the algebraic version of Hartogs' theorem, since affine hypertoric varieties are always normal (see see [2], Proof of 4.11, or [38], §4) and Lemma 3.3. Precomposing this map with the action of $w \in W_G$ on Y_G gives us the desired components into the other copies of X_G, to finally obtain a morphism of affine varieties

$$Y_G \to (\mathfrak{g}^* \oplus \mathfrak{t}_{\mathbb{C}}^*) \times \prod_{w \in W_G} X_G. \tag{3.1}$$

Lemma 3.4. *The map (3.1) restricted to $\overset{\circ}{Y}_G \subset Y_G$ is an embedding.*

Remark that the locus on which the map $Y_G \to Q_G$ is an embedding is the same as the one described in [10] using the quiver construction for $Q_{SL(n,\mathbb{C})}$.

Proof. Similar to Lemma 3.3, we have that $\overset{\circ}{Y}_G$ is covered by $T^*\left(\mathcal{X}(V) \cap \overset{\circ}{Y}_G\right)$, for broad V. Each of the latter, in turn, is

$$T^*\left(\mathcal{X}(V) \cap \overset{\circ}{Y}_G\right) = \left\{ [\underline{a}, \underline{b}] \;\middle|\; a_i \neq 0 \text{ if } i \in V \text{ and } b_j \neq 0 \text{ if } j \notin V \right\},$$

which is isomorphic to $T^*T_{\mathbb{C}}$. It therefore suffices to remark that, for any Lie group H, the map $T^*H \to H \times \mathfrak{h}^*$, given by the natural projection onto H, and the moment map for the cotangent lift of the action $H \circlearrowright H$ by multiplication by inverses on the right, is an isomorphism. Hence the map from $T^*\left(\mathcal{X}(\Omega) \cap \overset{\circ}{Y}_G\right)$ to $(T_{\mathbb{C}} \subset X_G \subset G/\!\!/U) \times \mathfrak{t}_{\mathbb{C}}^*$ is an embedding. As W_G permutes the broad V, the full map (3.1) is an embedding on $\overset{\circ}{Y}_G$. □

Proposition 3.1. *The image of (3.1) is contained in the image of (2.5), hence we have a morphism $Y_G \to Q_G$ that is an embedding on $\overset{\circ}{Y}_G$.*

Proof. As (2.5) is a closed embedding, it suffices to show this for the dense open subvariety $T^*\left(\mathcal{X}(\Omega) \cap \overset{\circ}{Y}_G\right) \cong T^*T_{\mathbb{C}}$. We can embed $T^*T_{\mathbb{C}}$ naturally into $T^*(G/U) \subset Q_G$, by

$$(t, \alpha) \in T_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}^* \mapsto \overline{(t, \alpha)} \in T^*(G/U) \cong (G \times \mathfrak{u}^\circ)/U.$$

(We use here that $\mathfrak{t}_C^*$ naturally is a subspace of $\mathfrak{u}^\circ$.) As it is straightforward to verify that the images of $T^*T_{\mathbb{C}}$ under (2.5) and (3.1) are identical, we are done. □

4. Implosion and the Moore-Tachikawa category

In this section, we discuss the Moore-Tachikawa category HS and explain how it gives a useful organising principle for implosion and contraction.

The objects of HS, as introduced in [28], are complex semisimple groups and the morphisms from G_1 to G_2 are complex symplectic varieties with linearised, Hamiltonian, $G_1 \times G_2$-actions. In the setup of [28] one also assumes a commuting circle action acting on the complex symplectic form with weight 2, but we do not need that for our discussion, and we will also allow for objects to be complex reductive groups.

The composition $Y \circ X$ of morphisms $X \in \mathrm{Hom}_{HS}(G_1, G_2)$ and $Y \in \mathrm{Hom}_{HS}(G_2, G_3)$ is given by the complex symplectic quotient

$$Y \circ X = (X \times Y) /\!/\!/ \Delta G_2 \tag{4.1}$$

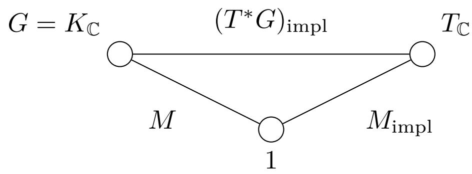
where ΔG_2 denotes the diagonally embedded G_2 in the symmetry group $G_1 \times (G_2 \times G_2) \times G_3$ of $X \times Y$. This quotient is complex symplectic with residual $G_1 \times G_3$-action so lies in $\mathrm{Hom}_{HS}(G_1, G_3)$ as required. If K is a compact Lie group with $G = K_{\mathbb{C}}$, then the Kronheimer space T^*G [25] is complex symplectic with $G \times G$-action and defines the identity element in $\mathrm{Hom}_{HS}(G, G)$, due to the complex analogue of (2.3).

More generally, we could consider a Moore-Tachikawa category in any setting where one has a reduction theory for group actions on spaces of some kind, provided that reduction by commuting actions commutes, and there exists an identity. This is the case for Hamiltonian actions of compact groups on symplectic manifolds and stratified spaces, and linearised actions of reductive groups on varieties projective over an affine. As mentioned above, an identity is missing in the case of Kähler reduction of Kähler manifolds by compact groups, or hyper-Kähler reduction – in those cases one only has a semi-category.

In [28] it was conjectured that a choice of group G would give a functor $\eta_G : Bo_2 \to H$, where Bo_2 is the category of 2-bordism, mapping the cylinder to T^*G. The Riemann sphere $\mathbb{CP}^1$ maps to the BFM space (or universal centraliser) [3], while the once-punctured sphere maps to the product of G and the Slodowy slice. This is actually a symmetric monoidal functor, where the monoidal structure on HS is given by the product of groups and of spaces, and the dual of a group is just itself. The functor thus defines a 2-dimensional TQFT. Recently Ginzburg and Kazhdan [15, 16] and Bielawski [4] have defined the image under η_G of the three-times punctured sphere — together with the above data this determines the functor. Physically, $\eta_G(C)$ represents the Higgs branch of a $N = 2$ supersymmetric theory associated to a punctured Riemann surface C with defects located at the punctures, associated to homomorphisms $\mathfrak{su}(2) \to \mathfrak{g}$.

As mentioned, in the original Moore-Tachikawa discussion the groups are taken to be semisimple. In what follows we shall find it useful to extend the formalism to more general reductive groups.

As discussed in [12] and [7], the universal hyperkähler implosion for a compact K with maximal torus T may be viewed as an element of $\mathrm{Hom}_{HS}(G, T_{\mathbb{C}})$. The process of imploding a complex symplectic manifold M with G-action, viewed as an element of $\mathrm{Hom}_{HS}(1, G)$, to obtain a manifold M_{impl} with $T_{\mathbb{C}}$-action, is now exactly that of composition of morphisms with the universal implosion.

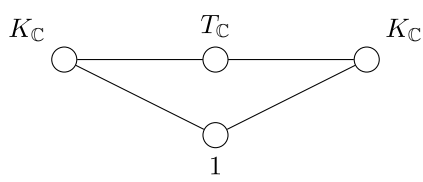


Of course, one could also define a similar picture with compact groups replacing complex reductive groups, and with (real) symplectic manifolds playing the role of morphisms. Again, the universal symplectic implosion $(T^*K)_{\mathrm{impl}}$ for a compact K is an element of $\mathrm{Hom}_{HS}(K, T)$, and implosion of a symplectic K-manifold as defined in [18] is just composition with the universal implosion. Explicitly, we have

$$M_{\mathrm{impl}} = (M \times (T^*K)_{\mathrm{impl}}) /\!\!/ \Delta K.$$

5. Symplectic and algebraic contraction

Returning to our definition of composition in HS, we see that if the middle group G_2 is *Abelian*, then in fact $Y \circ X$ has an action not just of $G_1 \times G_3$, but of $G_1 \times G_2 \times G_3$, because the *anti*-diagonally embedded G_2 commutes with the diagonal G_2 and hence descends to the quotient. In particular, if we have a space with G-action, we may compose it with the (right) universal implosion (viewed as a morphism from G to $T_{\mathbb{C}}$) to obtain a space with $T_{\mathbb{C}}$-action, i.e. we form the implosion. Then we may compose again with the (left) universal implosion considered as a morphism in the reverse direction, and get a space with $G \times T_{\mathbb{C}}$-action, where the $T_{\mathbb{C}}$ factor arises because the middle group $T_{\mathbb{C}}$ in the composition is Abelian.



In the setting of real symplectic manifolds with compact group actions, the resulting space is the symplectic contraction of [20]. The dimension calculation in the real case is as follows, denoting the contraction of X by X_{sc}:

$$\dim X_{\mathrm{impl}} = \dim X + \dim (T^*K)_{\mathrm{impl}} - 2\dim K = \dim X + \operatorname{rank} K - \dim K$$

$$\begin{aligned} \dim X_{\mathrm{sc}} &= \dim X_{\mathrm{impl}} + \dim (T^*K)_{\mathrm{impl}} - 2\dim T && (5.1)\\ &= \dim X + \operatorname{rank} K - \dim K + (\dim K + \operatorname{rank} K) - 2\operatorname{rank} K && (5.2)\\ &= \dim X. && (5.3) \end{aligned}$$

So the contraction keeps the dimension the same but enhances the K-action to a $K \times T$-action.

As with implosion, the key example is the universal contraction, i.e. the contraction of T^*K with $K \times K \times T$-action. We recover the contraction of X by taking the product with the universal contraction and reducing by K. In the symplectic case

$$(T^*K)_{\text{sc}} = ((T^*K)_{\text{right impl}} \times (T^*K)_{\text{left impl}}) /\!\!/ T.$$

Recall that the left K-action on T^*K is

$$h.(k,v) \mapsto (hk, v)$$

with moment map

$$\mu_L : (k,v) \mapsto -k.v$$

where $k.v$ denotes the coadjoint action of k on v. The right action is

$$h.(k,v) \mapsto (kh^{-1}, h.v)$$

with the moment map

$$\mu_R : (k,v) \mapsto v$$

so that points of $(T^*K)_{\text{right impl}}$ are represented by pairs (k,v) with $k \in K$ and v in the closed positive Weyl chamber $\bar{C}$. Points of $(T^*K)_{\text{left impl}}$ are represented by pairs (k,v) with $k \in K$ and $-k.v \in \bar{C}$. The involution $(k,v) \mapsto (k^{-1}, -k.v)$ intertwines the left and right actions and moment maps, and induces an identification of the corresponding implosions.

In [20], a map $\Phi : T^*K \to (T^*K)_{\text{sc}}$ is defined by

$$(k,v) \mapsto [(kh^{-1}, h.v), (h,v)] \tag{5.4}$$

where h is such that $h.v$ lies in the chosen closed positive Weyl chamber. We use square brackets to denote the fact we are passing to the quotient by T. Note that h is uniquely defined up an element of K_σ, where K_σ is the stabiliser corresponding to the fact σ of $\bar{C}$ containing $h.v$. But we collapse by $[K_\sigma, K_\sigma]$ in forming the implosion, and by T in taking the symplectic quotient to form $(T^*K)_{\text{sc}}$, so the map (5.4) is well-defined.

This map is surjective, as proven in [20], because given any $[(k,w),(h,v)] \in (T^*K)_{\text{sc}}$, we have $w = h.v \in \bar{C}$ by the 0-level set condition for the T-moment map. Now we see that

$$\Phi : (kh, v) \mapsto [(k, h.v), (h,v)] = [(k,w),(h,v)].$$

As a result, we can understand $(T^*K)_{\text{sc}}$ or, in fact, any M_{sc} set-theoretically as $M_{\text{sc}} = M/\sim$, where

$$x \sim y \Leftrightarrow \mu(x) = \mu(y) \text{ and } y \in [K_{\mu(x)}, K_{\mu(x)}].x,$$

where $K_{\mu(x)}$ is the stabiliser in K of $\mu(x)$ for the coadjoint action.

The algebraic counterpart of symplectic contraction is given by Popov's notion of horospherical contraction [30] for varieties with an action of a reductive group. This was initially defined for an affine G-variety X as follows: the ring $k[X]$ is equipped with a poly-filtration obtained from the decomposition into isotypical components $k[X]_{(\lambda)}$ for G (where λ runs over the dominant weights). By choosing a regular dominant co-weight $\rho \in \mathrm{Hom}(U(1), T) \cap \mathfrak{t}_+$, we can make this into an $\mathbb{N}$-filtered ring by putting

$$k[X]_{\leq n} = \bigoplus_{\rho(\lambda) \leq n} k[X]_{(\lambda)}.$$

The horospherical contraction of X is now determined by the associated graded of this algebra:

$$X_{\mathrm{hc}} = \mathrm{Spec}(\mathrm{gr}(k[X])),$$

which is independent of the choice of ρ. (This definition naturally extends to a broader setting of varieties that are projective over an affine.) The Rees construction moreover naturally creates a $\mathbb{G}_m$-equivariant flat family $\pi : \chi \to \mathbb{A}^1$, the generic fiber of which is isomorphic to X, and the fiber over 0 is given by X_{hc}.

The universal horospherical contraction is the asymptotic semigroup $\mathrm{As}(G)$, as defined by Vinberg [37]. This is the central fiber in a degeneration of G known as the Vinberg monoid S_G [35]. This is obtained via the Rees construction as above, but without switching to an $\mathbb{N}$-filtration by a choice of ρ as above – this results in a flat fibration $\pi_G : S_G \to \mathbb{A}_G$ over a higher dimensional base, which is the (smooth) affine toric variety for the torus T/Z_K (where Z_K is the center of K) given by the cone spanned by the positive simple roots. Vinberg shows that S_G is a reductive monoid, with group of units $S_G^\times \cong (G \times T_\mathbb{C})/Z_G$. A choice of a regular dominant ρ as before gives rise to a morphism of monoids $\mathbb{A}^1 \to \mathbb{A}_G$, and the flat family over $\mathbb{A}^1$ induced by this choice is the base-change of $\pi : S_G \to \mathbb{A}_G$ by this morphism.

With a suitable (but not unique) choice of Kähler structure, we can identify X_{sc} and X_{hc}, see §5 and 6 in [20]. This is done, following Harada-Kaveh [19], by means of the gradient-Hamiltonian vector field

$$V_\pi = -\frac{\nabla \Re(\pi)}{\|\nabla \Re(\pi)\|^2} \tag{5.5}$$

on the smooth locus of the total space of the degeneration χ, whose flow induces a continuous surjective map $X \to X_{\mathrm{hc}}$ that generically is a symplectomorphism. In general the results of [19] just guarantee existence of this map, but in the case of a degeneration corresponding to a horospherical contraction, as we are considering, it can be made explicit. What enables this is a reduction to the universal case $\pi_G : S_G \to \mathbb{A}_G$, where the monoid structure of S_G, and Lie group decompositions for $S_G^\times$, make the flow for V_π tractable. This allows the identification of $\mathrm{As}(G)$ and $(T^*K)_{\mathrm{sc}}$, which in turn implies the same for X_{hc} and X_{sc}.

6. Complex symplectic contraction

Given the interpretation for the contraction in the real symplectic and algebraic situations in terms of implosion, it is now natural to take the product of the complex symplectic implosion of a space M and the left implosion of T^*G, and reduce this by the diagonal complex torus action. We shall refer to this as the complex symplectic contraction, and denote it as M_{csc}.

The right implosion is the complex symplectic quotient, in the GIT sense, of $T^*G = G \times \mathfrak{g}^*$ by the maximal unipotent subgroup U of G, that is, the GIT quotient $Q^R = (G \times \mathfrak{u}^\circ)/\!/U$. This can also be viewed as $(G \times \mathfrak{b})/\!/U$ where $\mathfrak{b}$ denotes the Borel algebra. We could also form the left implosion

$$Q^L = \left\{(g, v) \in G \times \mathfrak{g} \;\middle|\; g.v \in \mathfrak{b}\right\} /\!/ U$$

where U now acts on the left, i.e. by translation in the G-factor. As before, the involution $(g, v) \mapsto (g^{-1}, -g.v)$ gives an identification between left and right implosions.

Definition 6.1. We now define the universal complex symplectic contraction to be the complex-symplectic quotient by the diagonal complex torus $T_{\mathbb{C}}$ of the product of right and left implosions:

$$(T^*G)_{\mathrm{csc}} = (Q^R \times Q^L)/\!/\!/_0 T_{\mathbb{C}}.$$

Now M_{csc} can be obtained by taking the product of M with the universal example $(T^*G)_{\mathrm{csc}}$ and reducing by G.

The above discussion shows the following.

Proposition 6.1. *$(T^*G)_{\mathrm{csc}}$ has the same dimension as T^*G but the complex-symplectic $G \times G$-action is enhanced to a $G \times T_{\mathbb{C}} \times G$-action.*

Similarly M_{csc} has the same dimension as M but its G-action is enhanced to a $G \times T_{\mathbb{C}}$-action.

Remark 6.1. The complex-symplectic reduction of $(T^*G)_{\mathrm{csc}}$ by the $T_{\mathbb{C}}$-action is the same as the product of two copies of the reduction of the implosion by $T_{\mathbb{C}}$ – in particular if we reduce at level zero we obtain the product of two copies of the nilpotent cone.

Let us discuss some properties of the universal example. How could we mimic the map (5.4) from the symplectic case? We want a map $T^*G \to (T^*G)_{\mathrm{csc}} = (Q^R \times Q^L)/\!/\!/_0 T_{\mathbb{C}}$. Given $(g, v) \in T^*G$, we can choose $h \in G$ with $h.v \in \mathfrak{b}$ as every element of $\mathfrak{g}$ is conjugate via the adjoint action to an element of the fixed Borel algebra $\mathfrak{b}$. So we could try to define

$$\Psi : (g, v) \mapsto [(gh^{-1}, h.v), (h, v)] \tag{6.1}$$

and the right-hand side then lies in the 0-level set of the $T_{\mathbb{C}}$-action as required.

However, the problem here is that the condition $h.v \in \mathfrak{b}$ does not specify h up to an element of the Borel subgroup B in general. It is therefore natural to replace the domain T^*G by $G \times \tilde{\mathfrak{g}}$ where $\tilde{\mathfrak{g}}$ is the Grothendieck simultaneous resolution. That is, we consider

$$G \times \tilde{\mathfrak{g}} = \left\{ (g, v, \mathfrak{b}_1) \;\middle|\; g \in G, v \in \mathfrak{b}_1 \,:\, \mathfrak{b}_1 \in \mathcal{B} \right\}$$

where $\mathcal{B} = G/B$ is the variety of Borel algebras in $\mathfrak{g}$.

Proposition 6.2. *The formula (6.1) above defines a map from $G \times \tilde{\mathfrak{g}}$ to the complex-symplectic contraction $(T^*G)_{\mathrm{csc}}$.*

Proof. Given $(g, v\,\mathfrak{b}_1) \in G \times \tilde{\mathfrak{g}}$, there is now $h \in G$ with $h.\,\mathfrak{b}_1 = \mathfrak{b}$ (so $h.v \in \mathfrak{b}$), and h *is* unique up to an element of B. Therefore the right-hand side of (6.1) is well-defined, as the terms in round brackets are elements of the quotients by the maximal unipotent U, and the square bracket denotes the equivalence class under the $T_{\mathbb{C}}$-action. □

Remark 6.2. We could generalise this construction to any X with Hamiltonian $G_{\mathbb{C}}$-action, and form

$$\left\{ (x, \mathfrak{b}_1) \;\middle|\; \mu(x) \in \mathfrak{b}_1 \right\} \subset X \times \mathcal{B}.$$

If $X = T^*G$ then the moment map is projection to the second factor, and identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ we obtain $G \times \tilde{\mathfrak{g}}$ as above.

Recall from [9] that every orbit of U in the open set

$$\left\{ (g, v) \in G \times \mathfrak{b} \;\middle|\; \mathfrak{t}-\text{component of } v \text{ is } \in \mathfrak{t}_{\mathrm{reg}} \right\}$$

(where $\mathfrak{t}_{\mathrm{reg}}$ consists of the regular elements in $\mathfrak{t}$) has a unique representative with $v \in \mathfrak{t}_{\mathrm{reg}}$. We therefore have a set $Q^R_\circ = G \times \mathfrak{t}_{\mathrm{reg}}$ contained in Q^R, and of the same dimension as Q^R, as well as an analogous subset $Q^L_\circ \subset Q^L$. We may consider the locus $Q^R_\circ \times Q^L_\circ$ and its projection $(T^*G)_{\mathrm{csc},\circ}$ to the symplectic quotient by $T_{\mathbb{C}}$. Points in this locus are represented by pairs

$$((g_1, v), (g_2, w)) \;:\; g_1, g_2 \in G \;:\; v,\; g_2.w \in \mathfrak{t}_{\mathrm{reg}}$$

subject to the zero-level set condition for the $T_{\mathbb{C}}$-moment map

$$v - g_2.w = 0.$$

Proposition 6.3. *The locus $(T^*G)_{\mathrm{csc},\circ}$ is contained in the image of the map Ψ.*

Proof. For such points, we can proceed as in the real symplectic case, for now $v = g_2.w \in \mathfrak{t} \subset \mathfrak{b}$ and $(g_1 g_2, w, \mathfrak{b}_1) \in G \times \tilde{\mathfrak{g}}$ maps to

$$[(g_1, g_2.w), (g_2, w)] = [(g_1, v), (g_2, w)],$$

where we choose $\mathfrak{b}_1$ to be the Borel algebra containing w such that $g_2.\,\mathfrak{b}_1 = \mathfrak{b}$. So we have surjectivity onto the locus represented by $Q^R_\circ \times Q^L_\circ$. □

Remark 6.3. We could also see this by recalling that we have an identification $G \times_B \mathfrak{b} \cong \tilde{\mathfrak{g}}$ given by

$$(g', x) \mapsto (g'.x, g'B/B).$$

So Ψ can be viewed as a map from $G \times G \times_B \mathfrak{b}$ to the contraction given by the composition

$$(g, g', x) \mapsto (g, g'.x, g'B/B) \mapsto [(gg', x), ((g')^{-1}, g'.x)]$$

as we can take $h = (g')^{-1}$ in the formula (6.1).

Remark 6.4. As the complex symplectic implosion has an action of $W_G \ltimes T_{\mathbb{C}}$, and

$$\Delta T_{\mathbb{C}} \trianglelefteq \Delta W_G \ltimes (T_{\mathbb{C}} \times T_{\mathbb{C}}) \subset (W_G \ltimes T_{\mathbb{C}}) \times (W_G \ltimes T_{\mathbb{C}}),$$

we see that the contraction inherits an action of the diagonally embedded Weyl group also. In particular the universal complex symplectic contraction T^*G_{csc} has an action of $G \times G \times (W_G \ltimes T_{\mathbb{C}})$.

Proposition 6.4. *The real symplectic contraction sits naturally inside the complex-symplectic contraction.*

Proof. Recall from §4 of [6] that the real symplectic implosion $G /\!\!/ U$ arises as the fixed point locus of the $\mathbb{C}^*$-action on the hyperkähler implosion induced from scaling in the $\mathfrak{u}^\circ$ factor. The $T_{\mathbb{C}}$-moment map is equivariant for this action, so the product of left and right real symplectic implosions lies in the zero locus of the $T_{\mathbb{C}}$-moment map in $Q^R \times Q^L$, and taking $T_{\mathbb{C}}$-quotients the statement follows. □

7. Examples

Let us consider the simplest case, where $K = SU(2)$.

We first look at the universal symplectic contraction. The universal symplectic implosion is $\mathbb{C}^2$ with the flat Kähler structure and $U(2)$-action, so the contraction is the symplectic quotient

$$(\mathbb{C}^2 \times \mathbb{C}^2) /\!\!/ U(1),$$

where $U(1)$ acts on the copies of $\mathbb{C}^2$ by scalar multiplication by $e^{i\theta}$ and $e^{-i\theta}$ respectively. Equivalently, this is the GIT quotient by $\mathbb{C}^*$ with action

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} tv_1 \\ tv_2 \end{pmatrix}, \quad : \quad (w_1 \ \ w_2) \mapsto (t^{-1}w_1 \ \ t^{-1}w_2).$$

The invariants are given by the entries of

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = v \otimes w = (v_i w_j)_{i,j=1}^2.$$

This is the affine variety in $\mathbb{C}^4$ defined by $XW - YZ = 0$, which is the symplectic contraction for $SU(2)$. Indeed, in this case the Vinberg monoid $S_{SL(2)}$ is just the space of 2×2 matrices, fibering over $\mathbb{C}$ by the determinant, so that the central fiber, the asymptotic semigroup, consists of the singular matrices.

To understand the flow for the vectorfield (5.5), where $\pi = \det$, we can argue as follows (cf.[20]). Since both π and the Kähler metric on $S_{SL(2)}$ are invariant under left and right multiplication by matrices in $SU(2)$, the flow for V_π is equivariant for these actions as well. Now consider the KAK-type Cartan decomposition for $S^\times_{SL(2)} = GL(2,\mathbb{C})$ (A here consists of the diagonal matrices in $GL(2,\mathbb{R})$ with positive entries). If $x \in SL(2,\mathbb{C})$ is contained in $k_1 A k_2$ for some $k_1, k_2 \in SU(2)$, by the equivariance, the entire flow of x under V_π will be as well. So it suffices to just look at elements $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in (A \cap SL(2,\mathbb{C}))$. Level sets of π restricted to A are just given by components of hyperbola

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \;\middle|\; x, y \in \mathbb{R}_+ \text{ with } xy = \lambda \right\},$$

and hence the flow lines for V_π, which are everywhere orthogonal to them, lie on hyperbola given by $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ with $x^2 - y^2 = c$. This in turn allows us to determine exactly what the flow for time $t = 1$ does to elements in $A \cap SL(2,\mathbb{C})$:

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mapsto \begin{cases} \begin{pmatrix} \sqrt{x^2 - x^{-2}} & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \; x \geq 1, \\ \\ \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{x^{-2} - x^2} \end{pmatrix} & \text{if } \; x \leq 1. \end{cases}$$

We can translate this cleanly to all of $SL(2,\mathbb{C})$ using the polar decomposition as follows:

$$SL(2,\mathbb{C}) \ni B = UP = U\sqrt{B^*B} \mapsto U\sqrt{B^*B - \gamma\,\mathrm{Id}}, \tag{7.1}$$

where γ is the smallest eigenvalue of B^*B. From this one immediately sees that the moment maps for the actions of $SU(2)$ given by multiplication on the left and the right are invariant under this map. Moreover, the stabilisers for these actions remain the same, except when $B^*B = \mathrm{Id}$, i.e. when $B \in SU(2)$ (all of which get sent to the zero matrix). So (7.1) just collapses $SU(2) \subset SL(2.\mathbb{C})$, which, using the Cartan decomposition, we can think of as the zero section of $T^*SU(2) \cong SL(2,\mathbb{C})$. This is of course the exactly the same as what happens for $T^*SU(2)_{\mathrm{sc}}$.

In the hyperkähler case, the universal implosion is $Q_{SL(2,\mathbb{C})} = \mathbb{H}^2 = \mathrm{Hom}(\mathbb{C},\mathbb{C}^2) \oplus \mathrm{Hom}(\mathbb{C}^2,\mathbb{C})$ with $SU(2)$-action $(\alpha,\beta) \mapsto (g\alpha, \beta g^{-1})$ and $U(1)$-action $(\alpha,\beta) \mapsto (e^{-i\theta}\alpha, e^{i\theta}\beta)$.

To see the action of the Weyl group $W_{SL(2,\mathbb{C})} = \{1, \gamma\}$, it is easiest to use the identification $SL(2,\mathbb{C}) = Sp(2,\mathbb{C})$. If we take $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then for any $g \in SL(2,\mathbb{C})$ we have $g^T J = J g^{-1}$. We can now put

$$\gamma.(\alpha,\beta) = \left((\beta J)^T, (J\alpha)^T\right), \text{ i.e. } \gamma.\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, (\beta_1\ \beta_2)\right) = \left(\begin{pmatrix} -\beta_2 \\ \beta_1 \end{pmatrix}, (\alpha_2\ -\alpha_1)\right).$$

This all indeed combines to given an action of $SL(2,\mathbb{C}) \times \left(W_{SL(2,\mathbb{C})} \ltimes \mathbb{C}^*\right)$ on $\mathbb{H}^2$ (where $\gamma.\lambda = \lambda^{-1}$ for $\lambda \in \mathbb{C}^*$). Note that the $SL(2,\mathbb{C})$-moment map $(\alpha\beta)_0$ (ie. the tracefree part of $\alpha\beta$) is Weyl-invariant, while the $\mathbb{C}^*$-moment map $\beta\alpha$ changes sign under the Weyl action. The hypertoric variety $Y_{SL(2,\mathbb{C}}$ is now just $\mathbb{H} \cong \mathbb{C}^2$, which embeds into $Q_{SL(2,\mathbb{C})}$ by $(\alpha,\beta) \mapsto \left(\begin{pmatrix} \alpha \\ 0 \end{pmatrix}, (\beta\ 0)\right)$ – remark that this is not $W_{SL(2,\mathbb{C})}$-equivariant.

For the universal complex symplectic contraction, we need to consider the hyperkähler quotient of two copies of the implosion by the diagonal $U(1)$-action. We can view this as $\mathbb{C}^4 \times (\mathbb{C}^4)^*$ with action $(v,w) \mapsto (t^{-1}v, tw)$. So we must impose the complex moment map condition $\mu_{\mathbb{C}} := \langle v, w\rangle = 0$ and factor out the complexified action.

Again we can form the invariants $v \otimes w = (v_i w_j)_{i,j=1}^4$. We obtain a hyperplane in the cone over the Segre variety defined by the vanishing of all 2×2 minors. If, for example, we set $v_1 = v_2 = w_3 = w_4 = 0$ then the condition $\mu_{\mathbb{C}} = 0$ is automatically satisfied and we recover the (real) symplectic contraction given by coordinates $v_3w_1, v_3w_2, v_4w_1, v_4w_2$ with one quadratic relation.

More concretely, it is well-known that our space can be identified with the Swann bundle $\mathcal{U}(Gr_2\mathbb{C}^4)$ of the Grassmannian of complex 2-planes in $\mathbb{C}^4$. Explicitly, $v \otimes w$ defines an element of the closure of the highest root (or minimal) nilpotent orbit in $\mathfrak{sl}(4,\mathbb{C})$, and this is the total space of the Swann bundle, see page 432 in [34]. If we reduce at a generic nonzero level we instead obtain the Calabi space $T^*\mathbb{CP}^3$.

In each case the hyperkähler space has an $SU(4)$-action preserving the hyperkähler structure – in the case of $\mathcal{U}(Gr_2\mathbb{C}^4)$ this is induced from the $SU(4)$-action on the quaternionic Kähler Wolf space $Gr_2(\mathbb{C}^4) = SU(4)/S(U(2)\times U(2))$. This $SU(4)$-action includes two commuting $SU(2)$'s, as well as a circle action commuting with each $SU(2)$ factor, in accordance with our interpretation of the space as a contraction for $SU(2)$.

In this case the Weyl group action on the complex symplectic contraction had already been observed by Swann, see Proposition 4.4.1 in [33], as the Galois action for the covering of a nilpotent orbit closure in $\mathfrak{sp}(2,\mathbb{C})$ by the highest root nilpotent orbit closure in $\mathfrak{sl}(4,\mathbb{C})$. Indeed, if we identify $\mathbb{C}^4 \otimes \mathbb{C}^4$ with $\mathfrak{gl}(4) \cong \mathbb{C}^4 \otimes (\mathbb{C}^4)^*$ by means of $v \otimes w \mapsto v \otimes (\widetilde{J}w)^T$ (where $\widetilde{J} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ and J is as above), then $\mathfrak{sp}(2)$

corresponds to $\mathrm{Sym}^2 \mathbb{C}^4 \subset \mathbb{C}^4 \otimes \mathbb{C}^4$. The action of γ is then just the restriction of the transposition of the factors of $\mathbb{C}^4 \otimes \mathbb{C}^4$, which on $\mathfrak{gl}(4)$ translates to $A \mapsto \widetilde{J} A^T \widetilde{J}$.

The locus $Q_\circ^R$ of §6 corresponds to the locus where the $T_\mathbb{C} = \mathbb{C}^*$-moment map $\beta\alpha = \alpha_1\beta_1 + \alpha_2\beta_2$ is nonzero. The associated locus in the contraction represented by points of $Q_\circ^R \times Q_\circ^L$ is therefore the set of points where the sum of the moment maps from the right and left implosions is zero, but the values of the individual maps are nonzero.

Remark 7.1. As described in Section 2, for $K = SU(n)$ we have a quiver description of the implosion, as the space of full flag quivers

$$0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} \mathbb{C} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \mathbb{C}^2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} \mathbb{C}^{n-1} \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} \mathbb{C}^n,$$

with $\alpha_0 = \beta_0 = 0$, satisfying the equations

$$\alpha_{i-1}\beta_{i-1} - \beta_i\alpha_i = \lambda_i^\mathbb{C} I \qquad (i = 1, \dots, n-1), \tag{7.2}$$

for free complex scalars $\lambda_1^\mathbb{C}, \dots, \lambda_{n-1}^\mathbb{C}$, modulo the natural action of $\prod_{i=1}^{n-1} SL(i, \mathbb{C})$

$$\alpha_i \mapsto g_{i+1}\alpha_i g_i^{-1}, \quad \beta_i \mapsto g_i\beta_i g_{i+1}^{-1} \quad (i = 1, \dots, n-2) \tag{7.3}$$

$$\alpha_{n-1} \mapsto \alpha_{n-1} g_{n-1}^{-1}, \quad \beta_{n-1} \mapsto g_{n-1}\beta_{n-1}.$$

This can be identified with the hyperkähler quotient of the space of full flag quivers by the action of $\prod_{i=1}^{n-1} SU(i)$. Note that if $n = 2$ this just reduces to $\mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \oplus \mathrm{Hom}(\mathbb{C}^2, \mathbb{C})$ as in the example above since the remaining equation 7.2) is automatic. For higher n we do not in general have a simple description of the space, although for $n = 3$ it is the Swann bundle of the Grassmannian $SO(8)/(SO(4) \times SO(4))$, that is the closure of the minimal nilpotent orbit in $\mathfrak{so}(8, \mathbb{C})$.

We have a residual action of $SL(n)$ by extending (7.3) to $i = n$, as well as a complex torus action complexifying the hyperkähler action of of $T = \prod_{i=1}^{n-1} U(i)/SU(i)$, whose moment map gives the λ_i. In general, elements of the contraction may therefore be represented by T-equivalence classes of pairs of quivers satisfying the equations, but with opposite λ_i.

Remark 7.2. In [8] an alternative approach to implosion was introduced using Nahm's equations

$$\frac{dT_i}{dt} + [T_0, T_i] = [T_j, T_k] \quad : \quad (ijk) \text{ cyclic permutation of } (123).$$

The idea here is to consider Nahm data on the half line $[0, \infty)$ asymptotic to a commuting triple (τ_1, τ_2, τ_3) of elements of a fixed Cartan algebra. We further collapse by gauge transformations asymptotic at infinity to the commutator of the common centraliser C of the triple (so no collapsing occurs on the open dense set where C is the maximal torus). We obtain a stratified pseudo-hyperkähler space with an action of T, represented by gauge transformations asymptotic to values in T at infinity.

Moreover, the moment map for the T-action is just evaluation of the Nahm data at infinity, i.e. the triple (τ_1, τ_2, τ_3). The metric becomes a genuine positive definite hyperkähler metric on the level sets of the moment map, and the quotient gives the Kostant variety as required.

This construction will in general give a rather different space from the implosion that we have discussed – in particular the structure as an algebraic variety is still unclear. However, we may mimic the contraction construction and take the hyperkähler quotient by T of two copies of this space, and obtain a space with the correct dimension and action. Being in the zero-level set of the T-action means the two sets of Nahm data are equal and opposite at infinity.

Using the symmetry $T_i(t) \mapsto -T_i(-t)$ of the Nahm equations, we may regard such configurations as giving Nahm data on two half lines that match at the respective points at infinity. Note that in general, we have a scaling symmetry $T_i(t) \mapsto cT_i(ct)$ of the Nahm equations that expands the interval of definition by a factor of c^{-1} if $0 < c < 1$, and under this symmetry the Nahm matrices scale by c but their derivatives scale by c^2. Recall that $T^*K_{\mathbb{C}}$ may be identified with the space of $\mathfrak{k}$-valued solutions to Nahm's equations smooth on a finite interval. Intuitively, we may view the above construction as a limiting case of stretching out the interval so that the Nahm matrices are almost commuting in the interior.

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