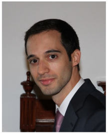
Alternating Direction Method of Multipliers in Imaging: Overview of a Line of Work

Mário A. T. Figueiredo

Joint work with **José Bioucas-Dias**, Manya Afonso, Mariana Almeida, Afonso Teodoro, Marina Ljubenovic, ...

1960-2020



Instituto de Telecomunicações, Instituto Superior Técnico, Universidade de Lisboa, Portugal

ADMM, Edinburgh, 2022

Outline

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Alternating Direction Methof of Multipliers (ADMM)

- Canonical problem:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n,\, \mathbf{z}\in\mathbb{R}^m} \quad & f(\mathbf{x}) + g(\mathbf{z}) \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} = \mathbf{b} \end{aligned}$$

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- Often used to re-write problems of the form

$$\min_{\mathbf{x}} \; f(\mathbf{x}) + g(\mathbf{H}\mathbf{x})$$

as

$$\min_{\mathbf{x},\mathbf{z}} \; f(\mathbf{x}) + g(\mathbf{z}) \quad \text{subject to} \quad \mathbf{H}\mathbf{x} = \mathbf{z}$$

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- Canonical ADMM (in scaled form)

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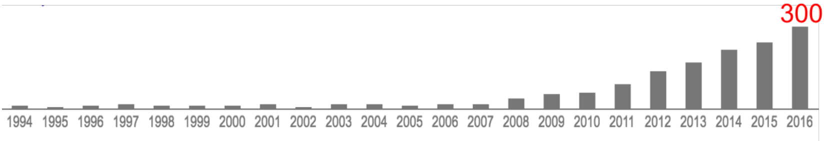
- *Cornerstone* work in the 1990s by Eckstein and Bertsekas [1992]

Explosion of Interest in ADMM

- Citations to paper by Eckstein and Bertsekas [1992]:



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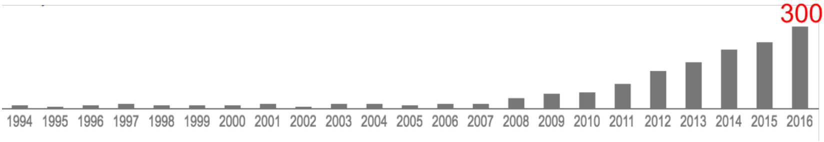

- Citations to review paper by Boyd et al. [2011]:

Classical Convergence Result

- Problem: $\min_{\mathbf{x}} \; f(\mathbf{x}) + g(\mathbf{H}\mathbf{x})$

- ADMM:

$$\mathbf{x}^{(k+1)} = \arg\min_{\mathbf{x}} f(\mathbf{x}) + \frac{\rho}{2}\big\|\mathbf{H}\mathbf{x} - \mathbf{v}^{(k)} - \mathbf{u}^{(k)}\big\|_2^2$$
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Theorem (Eckstein and Bertsekas [1992] (simplified version))

Let $\mathbf{H}$ have full column rank, and $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $g : \mathbb{R}^m \to \bar{\mathbb{R}}$ be closed, proper, convex functions; let $\mathbf{v}_0, \mathbf{u}_0 \in \mathbb{R}^m$, and $\rho > 0$ be given. Then $(\mathbf{x}^{(k)})_{k=1,2,\ldots}$ converges to a solution, if one exists. If no solution exists, then at least one of the sequences $(\mathbf{v}^{(k)})_{k=1,2,\ldots}$ or $(\mathbf{u}^{(k)})_{k=1,2,\ldots}$ diverges.

Classical Convergence Result

- Problem: $\min_{\mathbf{x}} \; f(\mathbf{x}) + g(\mathbf{Hx})$

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$$\mathbf{x}^{(k+1)} = \arg\min_{\mathbf{x}} f(\mathbf{x}) + \frac{\rho}{2}\big\|\mathbf{Hx} - \mathbf{v}^{(k)} - \mathbf{u}^{(k)}\big\|_2^2 \qquad \text{(not a prox)}$$

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- **Prox**imity operator: $\text{prox}_\phi(\mathbf{u}) := \arg\min_{\mathbf{x}} \; \phi(\mathbf{x}) + \frac{1}{2}\|\mathbf{u} - \mathbf{x}\|_2^2$

  Moreau [1965]

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- SALSA, PIDAL, PIDSplit, SDMM
  [Figueiredo and Bioucas-Dias, 2010], [Setzer et al., 2010], [Combettes and Pesquet, 2011]

A Closer Look

- A closer look at the algorithm

$$\mathbf{x}_{k+1} = \left(\sum_{j=1}^{J} \mathbf{H}_j^T \mathbf{H}_j\right)^{-1} \sum_{j=1}^{J} \mathbf{H}_j \left(\mathbf{z}_k^{(j)} - \mathbf{u}_k^{(j)}\right)$$

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- Hinges on: fast matrix inversion; simple proximity operators

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- Hinges on: fast matrix inversion; simple proximity operators
- Matrix inverse independent of ρ_k (good, if not kept constant)

Outline

Image Restoration/Reconstruction

- General formulation: $$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \Psi(\mathbf{A}\mathbf{x},\mathbf{y}) + \Phi(\mathbf{P}\mathbf{x}) + \iota_C(\mathbf{x})$$

where $\mathbf{y}$ are observations and $\iota_C(\mathbf{x}) = \begin{cases} 0 & \Leftarrow \quad \mathbf{x} \in C \\ +\infty & \Leftarrow \quad \mathbf{x} \notin C \end{cases}$

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- $\mathbf{A}$: linear (observation) operator; *e.g.*, blur, tomographic projections, partial Fourier observations (MRI),...

Image Restoration: Observation Models Ψ

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$$\text{prox}_{\iota_C}(\mathbf{u}) = \text{proj}_C(\mathbf{u});$$

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- The proximity operator of ι_C is simply an Euclidean projection:

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e.g., if $C = \mathbb{R}^n_+$, then $\big(\text{proj}_C(\mathbf{u})\big)_i = \max\{0, u_i\}$

Image Restoration: Regularizers Φ

- General formulation: $$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \Psi(\mathbf{A}\mathbf{x},\mathbf{y}) + \Phi(\mathbf{P}\mathbf{x}) + \iota_C(\mathbf{x})$$

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- Classical regularizers with simple proximity operators

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- Classical regularizers with simple proximity operators

  ✓ ℓ_1 norm: $\big(\mathsf{prox}_{\tau\|\cdot\|_1}(\mathbf{u})\big)_i = \mathsf{sign}(u_i)\max\{0, u_i - \tau\}$

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- Total variation can be written as $\Phi \circ \mathbf{P}$, where

$$\mathbf{P} : \mathbb{R}^n \rightarrow (\mathbb{R}^2)^n, \text{ with } (\mathbf{P}\mathbf{x})_i = \begin{bmatrix} x_i - x_{h(i)} \\ x_i - x_{v(i)} \end{bmatrix}, \text{ and } \Phi(\mathbf{v}) = \sum_i \|\mathbf{v}_i\|_2$$

with $h(i)$ and $v(i)$ the horizontal and vertical neighbours of pixel i

Image Restoration: Synthesis Formulation

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- Via the Sherman-Morrison-Woodbury formula

$$(\mathbf{A}^T\mathbf{A}+\mathbf{P}^T\mathbf{P})^{-1} = \left(\mathbf{W}^T\mathbf{B}^T\mathbf{B}\mathbf{W}+\mathbf{I}\right)^{-1} = \mathbf{I}-\mathbf{W}^T\mathbf{B}^T\left(\mathbf{B}^T\mathbf{B}+\mathbf{I}\right)^{-1}\mathbf{B}\mathbf{W}$$

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- Can $\mathbf{B}^T\mathbf{B}+\mathbf{I}$ be inverted efficiently?

Image Restoration: Analysis Formulation

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- Matrix inversion:

$$\left(\mathbf{B}^T\mathbf{B} + \mathbf{P}^T\mathbf{P}\right)^{-1} = \left(\mathbf{B}^T\mathbf{B} + \mathbf{I}\right)^{-1}$$

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Image Restoration: Constrained (Morozov) Formulations

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$$\hat{\mathbf{x}} \in \min_{\mathbf{x}\in\mathbb{R}^n} \Phi(\mathbf{P}\mathbf{x}) \;\text{ subject to }\; \Lambda(\mathbf{A}\mathbf{x}, \mathbf{y}) \leq 1$$

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- Can be written in the general formulation, with

$$\Psi(\mathbf{A}\mathbf{x},\mathbf{y}) = \iota_{D(\mathbf{y})}(\mathbf{A}\mathbf{x}), \;\text{ with }\; D(\mathbf{y}) = \{\mathbf{x} : \Lambda(\mathbf{x},\mathbf{y}) \leq 1\}$$

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- Classical example: $\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \Phi(\mathbf{P}\mathbf{x})$ s.t. $\xi\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2 \leq 1$

  Thus, $D(\mathbf{y})$ is a unit Euclidean ball around $\mathbf{y}$; projection is trivial.

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- Applies both to synthesis and analysis formulations

Image Restoration: Matrix Inversions

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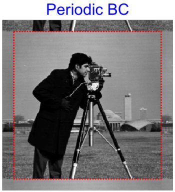
- Cost is at most $O(n \log n)$

Non-periodic Deconvolution



- Periodic boundary conditions are usually unnatural

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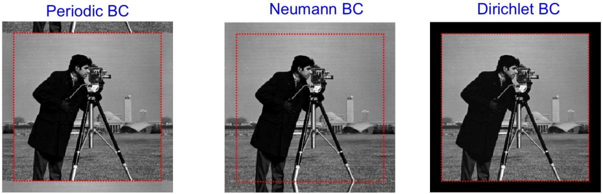

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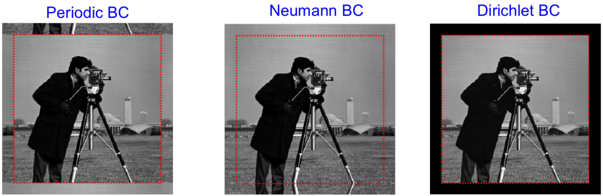
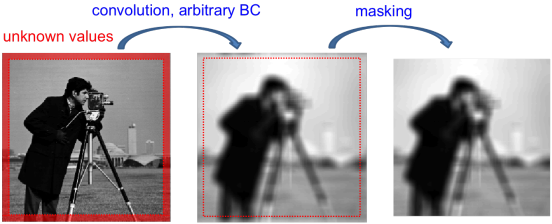


- Periodic boundary conditions are usually unnatural
- ...as are other standard BC: Neumann, Dirichlet.
- A more natural choice: unknown boundaries [Reeves, 2005], [Chan et al., 2005], [Almeida and Figueiredo, 2013a], [Ramani and Fessler, 2013]



Non-periodic Deconvolution (2)

- Gaussian noise model: $\Psi(\mathbf{B}\mathbf{x}, \mathbf{y}) = \frac{1}{2\sigma^2} \| \overbrace{\mathbf{M}}^{\text{mask}} \underbrace{\mathbf{U}^H \mathbf{F} \mathbf{U}}_{\text{period. conv.}} \mathbf{x} - \mathbf{y} \|_2^2$

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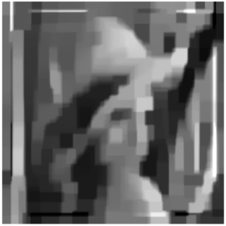
- Similar formulations:
  - ✓ deconvolution + inpainting ($\mathbf{M}$ masks the boundary and missing pixels)
  - ✓ super-resolution (filtering + downsampling mask)

Deconvolution with Unknown Boundaries: Example

original (256 × 256)

observed (238 × 238)

Assuming periodic BC



FA-BC (ISNR = -2.52dB)

Edge tapering

FA-ET (ISNR = 3.06dB)

Unknown BC by ADMM

FA-MD (ISRN = 10.63dB)

Deconvolution + Inpainting with Unknown BC: Example

original (256×256)

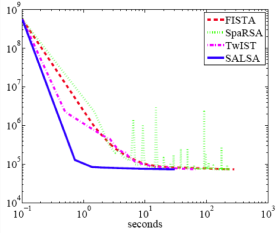
observed (238×238)

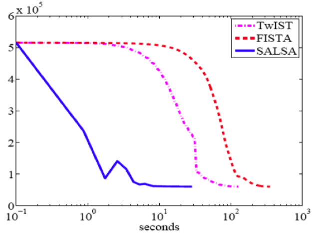
FA-CG (SNR = 20.58dB)

FA-MD (SNR = 20.57dB)

Speed

- Benchmark deblurring problem (9×9 blur, 40dB SNR, Haar frame, ℓ_1) and inpainting problem (50% missing data) [Afonso et al., 2011]





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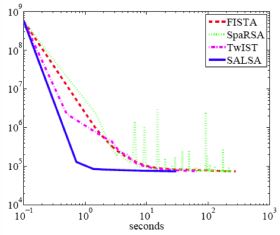
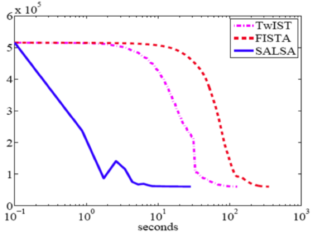
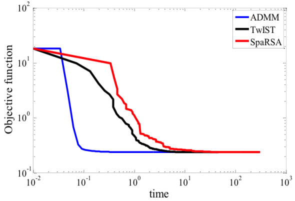
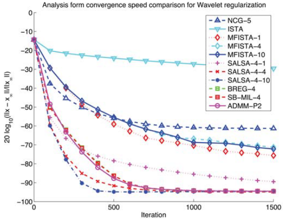




- Deconvolution with unknown BC [Almeida and Figueiredo, 2013a], [Ramani and Fessler, 2013]





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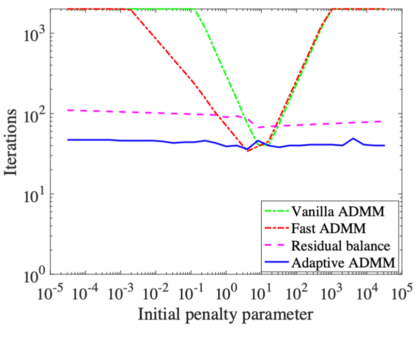
- Key issue: choosing parameter ρ_k
- Barzilai-Borwein-type method on the dual [Xu et al., 2016]



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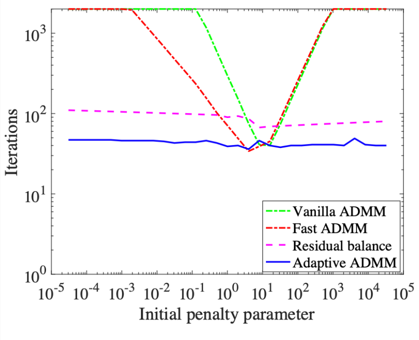

- Extension to over-relaxed and distributed ADMM [Xu et al., 2017a,b]

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  - ✓ Periodic or unknown boundaries
  - ✓ Blind deconvolution (later)
- Convergence guaranteed by classical results [Eckstein and Bertsekas, 1992] ...functions are closed, proper, convex; matrices have full column rank (except blind deconvolution)

Outline

1. Introduction: ADMM et al. (2007-2011)
2. Image Restoration/Reconstruction (2011-2014)
3. Plug-and-Play and Class-Adaptation (2015-2020)
4. Blind Restoration: Non-Convex Optimization (2013-2019)
5. Hyperspectral Imaging (2017-2020)
6. Final Remarks

Denoising Step in ADMM

- Restoration (w/ Gauss noise): $\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2 + \Phi(\mathbf{x})$

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- Can we use one of these denoisers instead of a proximity operator?

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- Plug a black-box denoiser into ADMM [Venkatakrishnan et al., 2013]

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- ...what about convergence of PnP-ADMM? [Sreehari et al., 2016, Teodoro et al., 2017b, 2019, Chan et al., 2017, Xu et al., 2020, ..., ...]

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$$p(\mathbf{x}_i) = \sum_{j=1}^{K} \alpha_j \, \mathcal{N}(\mathbf{x}_i; \boldsymbol{\mu}_j, \mathbf{C}_j)$$

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Plug-and-Play ADMM: Deblurring of Generic Images

- Generic GMM prior

Image:	Cameraman						House					
Experiment:	1	2	3	4	5	6	1	2	3	4	5	6
IDD-BM3D [Danielyan et al., 2012]	**8.85**	**7.12**	**10.45**	**3.98**	**4.31**	**4.89**	**9.95**	**8.55**	**12.89**	**5.79**	**5.74**	**7.13**
ADMM-GMM [Teodoro et al., 2016]	8.39	6.36	9.80	3.47	4.16	4.88	9.66	8.22	12.43	5.50	5.42	6.82

(a) Original (b) Blurred (c) IDD-BM3D (d) ADMM-GMM

- For generic natural images: competitive, but does not beat state-of-the-art

Class-Adapted GMM-Based Restoration

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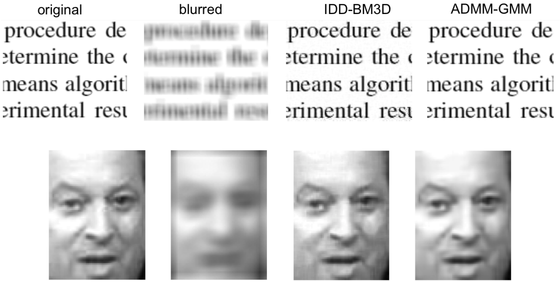

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Image class:	Text						Face					
Experiment:	1	2	3	4	5	6	1	2	3	4	5	6
BSNR	26.07	20.05	40.00	15.95	24.78	18.11	28.28	22.26	40.00	15.89	26.22	15.37
Input PSNR	14.14	14.13	12.13	16.83	14.48	28.73	25.61	22.54	20.71	26.49	24.79	30.03
IDD-BM3D	11.97	8.91	16.29	5.88	6.81	4.87	13.66	11.16	14.96	7.31	10.33	6.18
ADMM-GMM	**16.24**	**11.55**	**23.11**	**8.88**	**10.77**	**8.34**	**15.05**	**12.59**	**17.28**	**8.84**	**11.69**	**7.32**

Convergence

- PnP-ADMM with a patch-based GMM-MMSE denoiser

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \left(\mathbf{A}^T\mathbf{A} + \rho\mathbf{I}\right)^{-1}\left(\mathbf{A}^T\mathbf{y} + \rho(\mathbf{z}_k + \mathbf{u}_k)\right)\\
\mathbf{z}_{k+1} &= \text{denoiser}\left(\mathbf{x}_{k+1} - \mathbf{u}_k, 1/\rho\right)\\
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- Does the patch-based GMM-MMSE denoiser satisfy these conditions?

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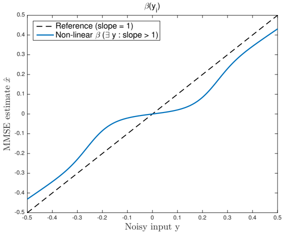
$$\hat{x} = \mathbb{E}[X|y] = \frac{\frac{\tau_1\, y}{\tau_1+1}\, \beta_1(y) + \frac{\tau_2\, y}{\tau_2+1}\, \beta_2(y)}{\beta_1(y) + \beta_2(y)}, \qquad \text{where } \beta_i(y) = \mathcal{N}(y; 0, \tau_i+1)$$



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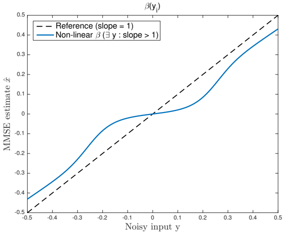
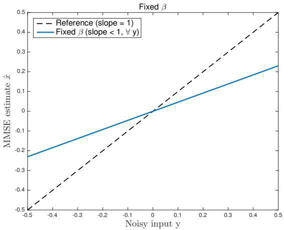
- With β_i fixed: $\hat{x} = y\left(\beta_1\,\frac{\tau_1}{\tau_1+1} + \beta_2\,\frac{\tau_2}{\tau_2+1}\right)/(\beta_1+\beta_2)$





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- Key properties of $\mathbf{W}$ [Teodoro et al., 2019]: for any $\sigma^2 > 0$,

$$\mathbf{W}(\sigma^2) = \mathbf{W}(\sigma^2)^T, \qquad \mathbf{W}(\sigma^2) \succeq 0, \qquad \lambda_{\text{max}}\big(\mathbf{W}(\sigma^2)\big) < 1$$

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- Conclusion: the problem has a solution and PnP-ADMM converges

Outline

Blind Deblurring

- Blind image deblurring/deconvolution

$$\mathbf{y} = \mathbf{h} * \mathbf{x} + \mathbf{n}$$

where both $\mathbf{x}$ and $\mathbf{h}$ are unknown

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- Joint criterion (under Gaussian noise) [Almeida and Figueiredo, 2013b]

$$(\hat{\mathbf{x}}, \hat{\mathbf{h}}) \in \arg\min_{\mathbf{x},\mathbf{h}} \underbrace{\frac{1}{2}\|\mathbf{h} * \mathbf{x} - \mathbf{y}\|_2^2 + \Phi(\mathbf{x}) + \Psi(\mathbf{h})}_{O(\mathbf{x},\mathbf{h})}$$

  where Φ and Ψ are regularizers

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- Blind image deblurring/deconvolution

$$\mathbf{y} = \mathbf{h} * \mathbf{x} + \mathbf{n} = \mathbf{H}(\mathbf{h})\,\mathbf{x} + \mathbf{n} = \mathbf{X}(\mathbf{x})\,\mathbf{h} + \mathbf{n}$$

  where both $\mathbf{x}$ and $\mathbf{h}$ are unknown

- Joint criterion (under Gaussian noise) [Almeida and Figueiredo, 2013b]

$$(\hat{\mathbf{x}}, \hat{\mathbf{h}}) \in \arg\min_{\mathbf{x},\mathbf{h}} \underbrace{\frac{1}{2}\|\mathbf{h} * \mathbf{x} - \mathbf{y}\|_2^2 + \Phi(\mathbf{x}) + \Psi(\mathbf{h})}_{O(\mathbf{x},\mathbf{h})}$$

  where Φ and Ψ are regularizers

- Even if Φ and Ψ are convex, this is a non-convex problem

Algorithm

- Proximal alternating minimization [Attouch et al., 2007]

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Initialization: $\hat{\mathbf{x}} = \mathbf{y}$, $\hat{\mathbf{h}}$ - identity filter
while stopping criterion is not satisfied **do**

$$\hat{\mathbf{x}} \leftarrow \arg\min_{\mathbf{x}} \; O(\mathbf{x}, \hat{\mathbf{h}}) + \frac{\rho_x}{2}\|\mathbf{x} - \hat{\mathbf{x}}^{\text{previous}}\|^2$$

$$\hat{\mathbf{h}} \leftarrow \arg\min_{\mathbf{h}} \; O(\hat{\mathbf{x}}, \mathbf{h}) + \frac{\rho_h}{2}\|\mathbf{h} - \hat{\mathbf{h}}^{\text{previous}}\|^2$$

end while

Algorithm

- Proximal alternating minimization [Attouch et al., 2007]
- Solver for each minimization: ADMM/SALSA

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- Image regularizer: class-adapted plug-and-play priors

Algorithm

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end while

- Image regularizer: class-adapted plug-and-play priors
- Filter regularizer: positivity and support, or sparsity

Priors

- Plug-and-play image priors:

Priors

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  - ✓ **GMM**-based patch denoiser, trained on a dataset of clean images (text, faces, fingerprint)

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  - ✓ Constraint: positivity and maximum support

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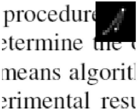
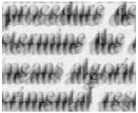
  - ✓ **Dictionary**-based patch denoiser, learned from clean images (same classes)
  - ✓ General-purpose **BM3D** denoiser.
- Blur filter priors
  - ✓ Constraint: positivity and maximum support
  - ✓ Sparsity (adequate for motion blur)

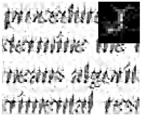
Results: GMM-based prior for text images



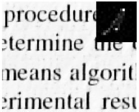
original



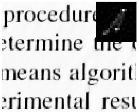
blurred



[Pan et al., 2014]



BM3D: 9.97 dB



GMM: 11.16 dB

Experiments

original

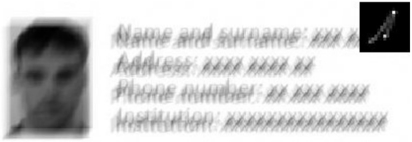
blurred

[Almeida and Figueiredo, 2013b]
0.36 dB

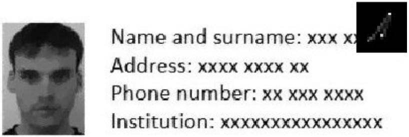
BM3D: 0.66 dB

GMM: 1.19 dB

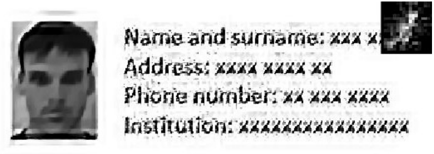
Experiments



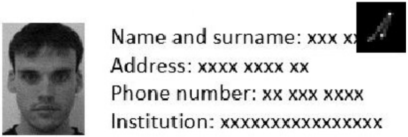
(a) Blurred image



(b) [Almeida and Figueiredo, 2013b]

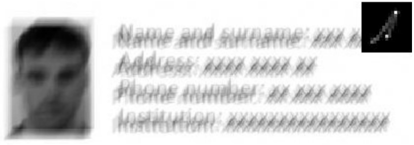


(c) [Pan et al., 2014]

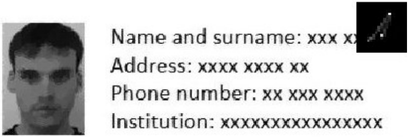


(d) Proposed

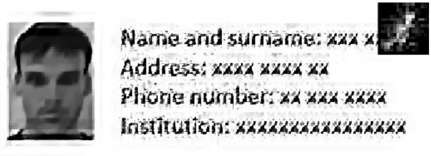
Experiments



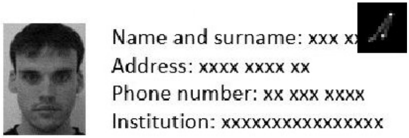
(a) Blurred image



(b) [Almeida and Figueiredo, 2013b]



(c) [Pan et al., 2014]

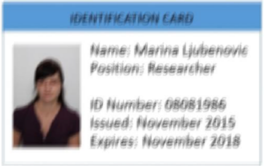


(d) Proposed

- Uses a concatenation of two dictionaries: face and text

Experiments

original



blurred

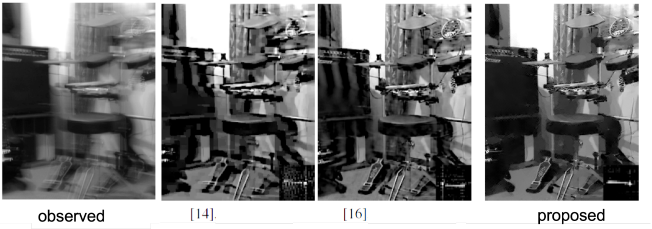
[Krishnan et al, 2011]

[Xu and Jia, 2011]

[Pan et al, 2014]

proposed

Blind Deconvolution: Real Examples

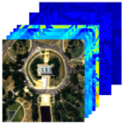


Results from [Almeida and Figueiredo, 2013b]

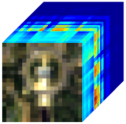
Outline

An Extreme Case of Adaptation: Hyperspectral Fusion

- Spectral-spatial resolution trade-off:



Multi-spectral:
high spatial resolution
low spectral resolution



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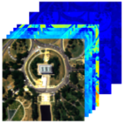
An Extreme Case of Adaptation: Hyperspectral Fusion

- Spectral-spatial resolution trade-off:



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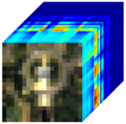
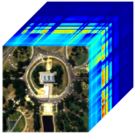


Hyper-spectral:
low spatial resolution
high spectral resolution

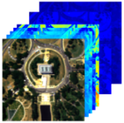
- Fuse MS and HS data:

  high spatial & spectral resolutions



An Extreme Case of Adaptation: Hyperspectral Fusion

- Spectral-spatial resolution trade-off:



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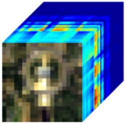

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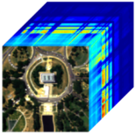

- Extreme case: pansharpening (panchromatic rather than MS image).

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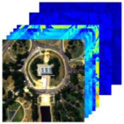

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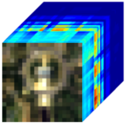

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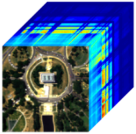

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Hyperspectral Fusion: Formulation

- Observation model [Simões et al., 2015]

$$
\begin{aligned}
\mathbf{Y}_h &= \overbrace{\mathbf{E}\mathbf{X}}^{\mathbf{Z}}\mathbf{B}\mathbf{M} + \mathbf{N}_h \\
\mathbf{Y}_m &= \mathbf{R}\underbrace{\mathbf{E}\mathbf{X}}_{\mathbf{Z}} + \mathbf{N}_m
\end{aligned}
$$

hyperspectral data $\in \mathbb{R}^{L_h \times n_h}$

multispectral data $\in \mathbb{R}^{L_m \times n_m}$

$L_h > L_m$ and $n_h < n_m$

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$$L_h > L_m \text{ and } n_h < n_m$$

✓ $\mathbf{E} \in \mathbb{R}^{L_h \times p}$: the p-dimensional subspace containing the fused image $\mathbf{Z}$

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✓ $\mathbf{X} \in \mathbb{R}^{p \times n_h}$: the corresponding coefficients ($p \ll L_h$)

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✓ $(\mathbf{B}\,\mathbf{M}) \in \mathbb{R}^{n_m \times n_h}$ models spatial convolution & subsampling

✓ $\mathbf{R} \in \mathbb{R}^{L_m \times L_h}$ models the spectral responses of the MS sensors

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✓ $\mathbf{R} \in \mathbb{R}^{L_m \times L_h}$ models the spectral responses of the MS sensors

✓ $\mathbf{N}_h$ and $\mathbf{N}_m$ model noise

Hyperspectral Fusion via PnP-ADMM

- Assuming Gaussian noise:

$$\widehat{\mathbf{X}} \in \arg \min_{\mathbf{X} \in \mathbb{R}^{p \times n_h}} \frac{1}{2} \|\mathbf{EXBM} - \mathbf{Y}_h\|_F^2 + \frac{\lambda_m}{2} \|\mathbf{REX} - \mathbf{Y}_m\|_F^2 + \text{``}\phi(\mathbf{X})\text{''}$$

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- ...which fits nicely the SALSA template ($J = 3$): $\quad \min_{\mathbf{x}} \sum_{j=1}^{J} g_j\big(\mathbf{H}_j\, \mathbf{x}\big)$

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- Matrix inversion computable via FFT (with periodic or unknown BC)

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- Proximity operators:
  - ✓ The one involving $\mathbf{RE}$: a single $p \times p$ inversion; decoupled across pixels

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  - ✓ The one involving $\mathbf{RE}$: a single $p \times p$ inversion; decoupled across pixels
  - ✓ The one involving $\mathbf{BM}$: solved by FFT, decoupled across bands

Hyperspectral Fusion via PnP-ADMM

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  - ✓ The prox of ϕ is replaced by an adapted GMM-based denoiser

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- ...which fits nicely the SALSA template ($J = 3$): $\min_{\mathbf{x}} \sum_{j=1}^{J} g_j\big(\mathbf{H}_j\, \mathbf{x}\big)$
- Matrix inversion computable via FFT (with periodic or unknown BC)
- Proximity operators:
  - ✓ The one involving $\mathbf{RE}$: a single $p \times p$ inversion; decoupled across pixels
  - ✓ The one involving $\mathbf{BM}$: solved by FFT, decoupled across bands
  - ✓ The prox of ϕ is replaced by an adapted GMM-based denoiser
- The GMM is learned from patches of $\mathbf{Y}_m$ (high spatial resolution) [Teodoro et al., 2019]

Hyperspectral Fusion: Synthetic Example

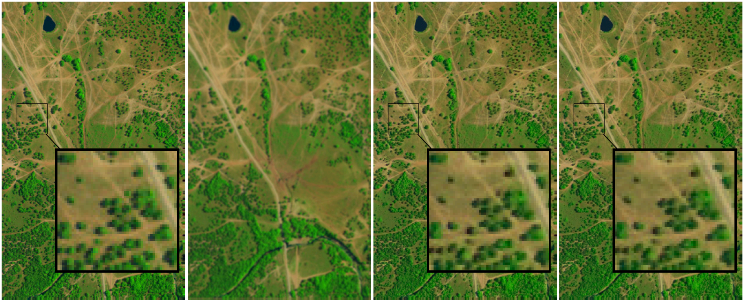
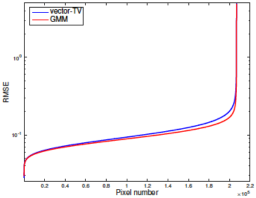


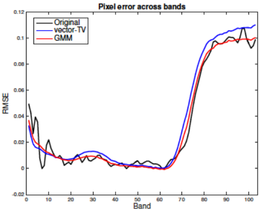
Table 3: HS and MS fusion on RTerrain dataset.

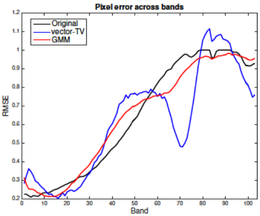
	Exp. 1 (PAN)			Exp. 2 (PAN)			Exp. 3 (R,G,B,N-IR)			Exp. 4 (R,G,B,N-IR)		
SNR ($\mathbf{Y}_m$)	50dB			30dB			50dB			30dB		
SNR ($\mathbf{Y}_h$)	50dB			20dB			50dB			20dB		
Metric	ERGAS	SAM	SRE	ERGAS	SAM	SRE	ERGAS	SAM	SRE	ERGAS	SAM	SRE
HySure	2.62	5.34	21.46	2.77	5.35	20.86	1.08	2.68	28.71	1.53	3.42	26.07
Proposed	2.58	**5.15**	**21.69**	**2.75**	**5.33**	**21.12**	**0.91**	**2.20**	**30.86**	**1.29**	**2.85**	**27.85**
ADMM-BM3D	**2.57**	5.17	21.65	2.76	5.36	21.08	0.93	2.22	30.80	1.31	2.91	27.72

[Teodoro et al., 2017a]

Hyperspectral Fusion: Synthetic Example







Outline

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- Plug-and-play (PnP) denoisers "can" be used with ADMM
- Convergence properties of PnP-ADMM with fixed linear denoiser
- Extension to blind deblurring (non-convex)
- Ideally suited for hyperspectral imaging

Thank you.

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