SALEM NUMBERS, PISOT NUMBERS, MAHLER MEASURE AND GRAPHS

JAMES MCKEE AND CHRIS SMYTH

ABSTRACT. We use graphs to define sets of Salem and Pisot numbers, and prove that the union of these sets is closed, supporting a conjecture of Boyd that the set of all Salem and Pisot numbers is closed. We find all trees that define Salem numbers. We show that for all integers n the smallest known element of the n-th derived set of the set of Pisot numbers comes from a graph. We define the Mahler measure of a graph, and find all graphs of Mahler measure less than $\frac{1}{2}(1+\sqrt{5})$. Finally, we list all small Salem numbers known to be definable using a graph.

1. INTRODUCTION

The work described in this paper arose from the following idea: that one way of studying algebraic integers might be by associating combinatorial objects with them. Here, we try to do this for two particular classes of algebraic integers, Salem numbers and Pisot numbers, the associated combinatorial objects being graphs. We also find all graphs of small Mahler measure. All but one of these measures turns out to be a Salem number.

A *Pisot number* is a real algebraic integer $\theta > 1$, all of whose other Galois conjugates have modulus strictly less than 1. A *Salem number* is a real algebraic integer $\tau > 1$, whose other conjugates all have modulus at most 1, with at least one having modulus exactly 1. It follows that the minimal polynomial $P(z)$ of τ is *reciprocal* (that is, $z^{\deg P}P(1/z) = P(z)$), that τ^{-1} is a conjugate of τ, that all conjugates of τ other than τ and τ^{-1} have modulus exactly 1, and that $P(z)$ has even degree. The set of all Pisot numbers is traditionally (if a little unfortunately) denoted S, with T being used for the set of all Salem numbers.

We call a graph G a *Salem graph* if either

- it is nonbipartite, has only one eigenvalue $\lambda > 2$ and no eigenvalues in $(-\infty,-2)$;
  or
- it is bipartite, has only one eigenvalue $\lambda > 2$ and only the eigenvalue $-\lambda$ in $(-\infty,-2)$.

We call a Salem graph *trivial* if it is nonbipartite and $\lambda \in \mathbb{Z}$, or it is bipartite and $\lambda^2 \in \mathbb{Z}$. For a nontrivial Salem graph, its associated Salem number $\tau(G)$ is then the larger root of $z + 1/z = \lambda$ in the nonbipartite case, and of $\sqrt{z} + 1/\sqrt{z} = \lambda$ in the bipartite case. (Proposition 6 shows that $\tau(G)$ is indeed a Salem number.) We call $\tau(G)$ a *graph Salem number*, and denote by T_{graph} the set of all graph Salem numbers. (For a trivial Salem graph G, $\tau(G)$ is a reciprocal quadratic Pisot number.)

Our first result is the following.

Date: March 22, 2005.

2000 *Mathematics Subject Classification.* 11R06, 05C50.

Theorem 1. *The set of limit points of T_{graph} is some set S_{graph} of Pisot numbers. Furthermore, $T_{\text{graph}} \cup S_{\text{graph}}$ is closed.*

In [MRS, Corollary 9], a construction was given for certain subsets S^* of S and T^* of T, using a restricted class of graphs (star-like trees). We showed that T^* had its limit points in S^*, and that (like S) S^* was closed in $\mathbb{R}$. The main aim of this paper is to push these ideas as far as we can.

We call elements of S_{graph} *graph Pisot numbers*. The proof of Theorem 1 reveals a way to represent graph Pisot numbers by bi-vertex-coloured graphs, which we call *Pisot graphs*.

Since Boyd has long conjectured that S is the set of limit points of T, and that therefore $S \cup T$ is closed ([Bo]), our result is a step in the direction of a proof of his conjecture. However, we can find elements in $T - T_{\text{graph}}$ (see Section 11) and elements in $S - S_{\text{graph}}$ (see Corollary 20), so that graphs do not tell the whole story.

It is clearly desirable to describe all Salem graphs. While we have not been able to do this completely, we are able in Proposition 7 to restrict the class of graphs that can be Salem graphs. Naturally enough, we call a Salem graph that happens to be a tree a *Salem tree*. In Section 7 we completely describe all Salem trees.

In Section 9 we show that the smallest known elements of the k-th derived set of S belong to the k-th derived set of S_{graph}. In Section 10, we find all graphs having Mahler measure at most $\frac{1}{2}(1+\sqrt{5})$. Finally, in Section 11 we list some small Salem numbers coming from graphs.

Acknowledgments. We are very grateful to Peter Rowlinson for providing us with many references on graph eigenvalues. We also thank the referees for helpful comments.

2. Lemmas on graph eigenvalues

For a graph G, recall that its eigenvalues are defined to be those of its adjacency matrix $A = (a_{ij})$, where $a_{ij} = 1$ if the ith and jth vertices are joined by an edge ('adjacent'), and 0 otherwise. Because A is symmetric, all eigenvalues of G are real.

The following facts are essential ingredients in our proofs.

Lemma 2 (Interlacing Theorem. See [GR, Theorem 9.1.1]). *If a graph G has eigenvalues $\lambda_1 \leqslant \dots \leqslant \lambda_n$, and a vertex of G is deleted to produce a graph H with eigenvalues $\mu_1 \leqslant \dots \leqslant \mu_{n-1}$, then the eigenvalues of G and H interlace, namely*

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{n-1} \leqslant \lambda_n .$$

We denote the largest eigenvalue of a graph G, called its *index*, by $\lambda(G)$. We call a graph that has all its eigenvalues in the interval $[-2,2]$ a *cyclotomic graph*. Connected graphs that have index at most 2 have been classified, and in fact all are cyclotomic.

Lemma 3 ([Smi], [Neu]—see also [CR, Theorem 2.1]). *The connected cyclotomic graphs are precisely the induced subgraphs of the graphs $\tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$, and those of the $(n+1)$-vertex graphs $\tilde{A}_n$ $(n \geqslant 2)$, $\tilde{D}_n$ $(n \geqslant 4)$, as in Figure 1.*

Clearly, general cyclotomic graphs are then graphs all of whose connected components are cyclotomic.

An *internal path* of a graph G is a sequence of vertices $x_1, \dots, x_k$ of G such that all vertices (except possibly x_1 and x_k) are distinct, x_i is adjacent to x_{i+1} for $i = 1, \dots, k-1$, x_1 and x_k have

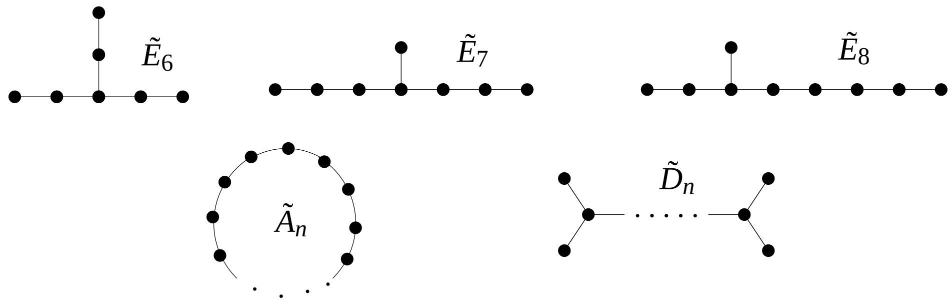


FIGURE 1. The maximal connected cyclotomic graphs $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{A}_n (n \geqslant 2)$ and $\tilde{D}_n (n \geqslant 4)$. The number of vertices is one more than the subscript.

degree at least 3, while $x_2, \ldots, x_{k-1}$ have degree 2. An *internal edge* is an edge on an internal path.

Lemma 4. (i) *Suppose that the connected graph G has G' as a proper subgraph. Then $\lambda(G') < \lambda(G)$.*

(ii) *Suppose that G^* is a graph obtained from a connected graph G by subdividing an internal edge. Then $\lambda(G^*) \leqslant \lambda(G)$, with equality if and only if $G = \tilde{D}_n$ for some $n \geqslant 5$.*

For the proof, see [GR, Theorem 8.8.1(b)] and [HS, Proposition 2.4]. Note that on subdividing a (noninternal) edge of $\tilde{A}_n$, $\lambda(G) = 2$ does not change. For any other connected graph G, if we subdivide a noninternal edge of G to get a graph G^*, then G is (isomorphic to) a subgraph of G^*, so that, by (i), $\lambda(G^*) > \lambda(G)$.

Lemma 5 (See [CR, Theorem 1.3] and references therein)**.** *Every vertex of a graph G has degree at most $\lambda(G)^2$.*

Proof. Suppose G has a vertex of degree $d \geqslant 1$ (the result is trivial if every vertex has degree 0) Then the star subgraph G' of G on that vertex and its adjacent vertices has $\lambda(G') = \sqrt{d}$. This follows from the fact that its "quotient" (see Section 6) is $(z+1-zd/(z+1))^{-1}$, so that it has two distinct eigenvalues $\pm\lambda$ satisfying $\lambda^2 = (\sqrt{z}+1/\sqrt{z})^2 = d$. (If $d \geqslant 2$, then 0 is also an eigenvalue.) By Lemma 4(i), $\lambda(G') \leqslant \lambda(G)$, giving the result. □

3. SALEM GRAPHS

Let G be a graph on n vertices, and let $\chi_G(x)$ be its characteristic polynomial (the characteristic polynomial of the adjacency matrix of G).

When G is nonbipartite, we define the *reciprocal polynomial* of G, denoted $R_G(x)$, by

$$R_G(z) = z^n \chi_G(z + 1/z)\,.$$

By construction R_G is indeed a reciprocal polynomial, its roots coming in pairs, each root $\beta = \alpha + 1/\alpha$ of χ_G corresponding to the (multiset) pair $\{\alpha, 1/\alpha\}$ of roots of R_G.

When G is bipartite, the reciprocal polynomial $R_G(x)$ is defined by

$$R_G(z) = z^{n/2} \chi_G(\sqrt{z} + 1/\sqrt{z})\,.$$

In this case $\chi_G(-x) = (-1)^n \chi_G(x)$: the characteristic polynomial is either even or odd. From this one readily sees that $R_G(z)$ is indeed a polynomial, and the correspondence this time is between the pairs $\{\beta, -\beta\}$ and $\{\alpha, 1/\alpha\}$, where $\beta = \sqrt{\alpha} + 1/\sqrt{\alpha}$. (We may suppose that the branch of the square root is chosen such that $\beta \geqslant 0$).

As the roots of χ_G are all real, in both cases the roots of R_G are either real, or lie on the unit circle: if $\beta > 2$ then the above correspondence is with a pair $\{\alpha, 1/\alpha\}$, both positive; if $\beta \in [-2, 2]$ then it is with the pair $\{\alpha, 1/\alpha = \overline{\alpha}\}$, both of modulus 1.

Proposition 6. *For a cyclotomic graph G, R_G is indeed a cyclotomic polynomial. For a nontrivial Salem graph G, $\tau(G)$ is indeed a Salem number.*

Proof. From the above discussion, for G a cyclotomic graph, R_G has all its roots of modulus 1 and so is a cyclotomic polynomial, by Kronecker's Theorem.

We now take G to be a Salem graph, with index $\lambda = \lambda(G)$. We can construct its reciprocal polynomial, R_G, and $\tau = \tau(G)$ is a root of this. Moreover, λ is the only root of P_G that is greater than 2, so that apart from λ and possibly $-\lambda$ all the roots of P_G lie in the real interval $[-2, 2]$. As noted above, such roots of P_G correspond to roots of R_G that have modulus 1; λ (respectively the pair $\pm\lambda$) corresponds to the pair of real roots τ, $1/\tau$, with $\tau > 1$. The minimal polynomial of τ, call it m_τ, is a factor of R_G. Its roots include τ and $1/\tau$. Were λ (respectively λ^2) to be a rational integer—cases excluded in the definition—then these would be the only roots of m_τ, and τ would be a reciprocal quadratic Pisot number. As this is not the case, m_τ has at least one root with modulus 1, and exactly one root (τ) with modulus greater than 1, so τ is a Salem number. □

Many of the results that follow are most readily stated using Salem graphs, although our real interest is only in nontrivial Salem graphs. It is an easy matter, however, to check from the definition whether or not a particular Salem graph is trivial.

While we are able in Section 7 to describe all Salem trees, we are not at present able to do the same for Salem graphs. However, the following result greatly restricts the kinds of graphs that can be Salem graphs. It is an essential ingredient in the proof of Theorem 1.

Proposition 7. *Let G be a connected graph having index $\lambda > 2$ and second largest eigenvalue at most 2. Then*

(a) *The vertices $V(G)$ of G can be partitioned as $V(G) = M \cup A \cup H$, in such a way that*
- *The induced subgraph $G|_M$ is one of the 18 graphs of* [CR, Theorem 2.3] *minimal with respect to the property of having index greater than 2; it has only one eigenvalue greater than 2;*
- *The set A consists of all vertices of $G - M$ adjacent in G to some vertex of M;*
- *The induced subgraph $G|_H$ is cyclotomic.*

(b) *G has at most $B := 10(3\lambda^4 + \lambda^2 + 1)$ vertices of degree greater than 2, and at most $\lambda^2 B$ vertices of degree 1.*

Proof. Such a graph G has a minimal vertex-deleted induced subgraph $G|_M$ with index greater than 2, given by [CR, Theorem 2.3]; $G|_M$ can be one of 18 graphs, each with at most 10 vertices. Note that $G|_M$ has only one eigenvalue greater than 2, as when a vertex is removed from $G|_M$ the resulting graph has, by minimality, index at most 2. Hence, by Lemma 2, $G|_M$ cannot have more than one eigenvalue greater than 2.

Now let A be the set of vertices in $V(G)-M$ adjacent in G to a vertex of M. Then, by interlacing, the induced subgraph G' on $V(G)-A$ has at most one eigenvalue greater than 2, which must be the index of $G|_M$. Hence the other components of G' must together form a cyclotomic graph, H say. By definition, there are no edges in G having one endvertex in M and the other in H.

As the index of G is λ, the maximum degree of a vertex of G is bounded by λ^2, by Lemma 5. Applying this to the vertices of M, we see that there are at most $10\lambda^2$ edges with one endvertex in M and the other in A. Thus the size $\#A$ of A is at most $10\lambda^2$. Now, applying the degree bound λ^2 to the vertices of A, we similarly get the upper bound $\lambda^2\#A$ for the number of edges with one endvertex in A and the other in H. These edges are adjacent to at most $\lambda^2\#A$ vertices in H of degree >2 in G. Now every connected cyclotomic graph contains at most two vertices of degree greater than 2 (in fact only the type $\tilde{D}_n$, as in Figure 1, having two). Also, since every connected component of H has at least one such edge incident in it, the number of such components is at most $\lambda^2\#A$. This gives at most another $2\lambda^2\#A$ vertices of degree >2 in H that are not adjacent to a vertex of A. Adding up, we see that the total number of vertices of degree >2 is at most $\#M+\#A+\lambda^2\#A+2\lambda^2\#A \leqslant 10(3\lambda^4+\lambda^2+1)$.

To bound the number of vertices of degree 1, we associate to each such vertex the nearest (in the obvious sense) vertex of degree greater than 2, and then use the fact that these latter vertices have degree at most $\lambda(G)^2$, by Lemma 5. □

On the positive side, the next results enable us to construct many Salem graphs. Our first result does this for bipartite Salem graphs.

Theorem 8. (a) *Suppose that G is a noncyclotomic bipartite graph and such that the induced subgraph on $V(G)-\{v\}$ is cyclotomic. Then G is a Salem graph.*

(b) *Suppose that G is a noncyclotomic bipartite graph, with the property that for each minimal induced subgraph M of G the complementary induced subgraph $G|_{V(G)-V(M)}$ is cyclotomic. Then G is a Salem graph.*

Here the "minimal" graph M is as in Proposition 7: a minimal vertex-deleted subgraph with index greater than 2.

We can use part (a) of the theorem to construct Salem graphs. Take a forest of cyclotomic bipartite graphs (that is, any graph of Lemma 3 except an odd cycle $\tilde{A}_{2n}$), and colour the vertices black or red, with adjacent vertices differently coloured. Join some (as few or as many as you like) of the black vertices to a new red vertex. Of course, one may as well take enough such edges to make G connected. This construction gives the most general bipartite, connected graph such that removing the vertex v produces a graph with all eigenvalues in $[-2,2]$. This result is an extension of Theorem 16(a) below, which is for trees. Theorem 16(b) gives a construction for more Salem trees.

In 2001 Piroska Lakatos [L2] proved a special case of Theorem 8 where the components $G|_{V(G)-\{v\}}$ consisted of paths, joined in G at one or both endvertices to v.

Proof. The proof of (a) is immediate from Lemma 2.

Part (b) comes straight from a result of D. Powers—see [CS, p. 456]. This states that if the vertices of a graph G are partitioned as $V(G)=V_1\cup V_2$ with $G|_{V_i}(i=1,2)$ having indices

$\lambda^{(i)}(i=1,2)$, then the second-largest eigenvalue of G is at most $\max_{V_1\cup V_2=V(G)}\min(\lambda^{(1)},\lambda^{(2)})$. It is clear that we may restrict consideration to $G|_{V_1}$ that are minimal, which gives the result. □

The next result gives a construction for some nonbipartite Salem graphs.

Theorem 9. *Suppose that G is a noncyclotomic nonbipartite graph containing a vertex v such that the induced subgraph on $V(G)-\{v\}$ is cyclotomic. Suppose also that G is a line graph. Then G is a Salem graph.*

Proof. Recall that a line graph L is obtained from another graph H by defining the vertices of L to be the edges of H, with two vertices of L adjacent if and only if the corresponding edges of H are incident at a common vertex of H. It is known that line graphs have least eigenvalue at least -2 ([GR, Chapter 12]). The proof of this follows easily from the fact that the adjacency matrix A of L is given by $A+2I=B^TB$, where B is the incidence matrix of H. Further, by Lemma 2, G has one eigenvalue $\lambda(G)>2$. □

To use this result constructively, first note that all paths and cycles are line graphs, as well as being cyclotomic. Then take any graph H consisting of one or two connected components, each of which is a path or cycle, and add to H an extra edge joining any two distinct nonadjacent vertices. Then the line graph of this augmented graph, if not again cyclotomic, will be a nonbipartite Salem graph.

4. Lemmas on reciprocal polynomials of graphs

For the proof of Theorem 1, we shall need to consider special families of graphs, obtained by adding paths to a graph. Here we establish the general structure of the reciprocal polynomials of such families, and show how in certain cases one can retrieve a Pisot number from a sequence of graph Salem numbers.

Throughout this section, reciprocal polynomials will be written as functions of a variable z, and we conveniently treat the bipartite and nonbipartite cases together by writing $y=\sqrt{z}$ if the graph is bipartite, and $y=z$ otherwise.

Lemma 10. *Let G be a graph with a distinguished vertex v. For each $m\geq 0$, let G_m be the graph obtained by attaching one endvertex of an m-vertex path to the vertex v (so G_m has m more vertices than G).*

Let $R_m(z)$ be the reciprocal polynomial of G_m. Then for $m\geqslant 2$ we have

$$(y^2-1)R_m(z)=y^{2m}P(z)-P^*(z)\,,$$

for some monic integer polynomial $P(z)$ that depends on G and v but not on m, and with $P^*(z)=z^{\deg P}P(1/z)$.

Proof. Let $\chi_m(\lambda)$ be the characteristic polynomial of G_m. Then expanding this determinant along the row corresponding to the vertex at the "loose" endvertex of the attached path (that which is not v) we get (for $m\geqslant 2$)

$$\chi_m=\lambda\chi_{m-1}-\chi_{m-2}\,.$$

Recognising this as a Chebyshev recurrence, or using induction, we get (on replacing λ by $y+1/y$ and multiplying through by the appropriate power of y)

$$R_m(z) = \frac{y^{2(t+1)}-1}{y^2-1}R_{m-t}(z) - \frac{y^{2t}-1}{y^2-1}y^2R_{m-t-1}(z)$$

for any t between 1 and $m-1$. Taking $t = m-1$ gives

$$R_m(z) = \frac{y^{2m}-1}{y^2-1}R_1(z) - \frac{y^{2(m-1)}-1}{y^2-1}y^2R_0(z)\,.$$

Putting $P(z) = R_1(z) - R_0(z)$ we are done. □

An easy induction extends this lemma to deal with any number of added pendant paths.

Lemma 11. *Let G be a graph, and $(v_1,\ldots,v_k)$ a list of (not necessarily distinct) vertices of G. Let $G_{m_1,\ldots,m_k}$ be the graph obtained by attaching one endvertex of a new m_i-vertex path to vertex v_i (so $G_{m_1,\ldots,m_k}$ has $m_1+\ldots+m_k$ more vertices than G). Let $R_{m_1,\ldots,m_k}(z)$ be the reciprocal polynomial of $G_{m_1,\ldots,m_k}$. Then if all the m_i are $\geqslant 2$ we have*

$$(y^2-1)^kR_{m_1,\ldots,m_k}(z) = \sum_{\varepsilon_1,\ldots,\varepsilon_k\in\{0,1\}} y^{2\sum\varepsilon_i m_i}P_{(\varepsilon_1,\ldots,\varepsilon_k)}(z)\,,$$

for some integer polynomials $P_{(\varepsilon_1,\ldots,\varepsilon_k)}(z)$ that depend on G and $(v_1,\ldots,v_k)$ but not on $m_1,\ldots,m_k$.

With notation as in the Lemma, we refer to $P_{(1,\ldots,1)}$ as the *leading polynomial* of $R_{m_1,\ldots,m_k}$. Given any $\varepsilon > 0$, if we then take all the m_i large enough, the number of zeros of $R_{m_1,\ldots,m_k}$ in the region $|z| \geqslant 1+\varepsilon$ is equal to the number of zeros of its leading polynomial in that region.

Lemma 12. *Suppose that G is connected and that G_m (as in Lemma 10) is a Salem graph for all sufficiently large m. Then G_m is a nontrivial Salem graph for all sufficiently large m. Furthermore $P(z)$, the leading polynomial of R_m, is a product of a Pisot polynomial (with Pisot number θ as its root, say), a power of z, and perhaps a cyclotomic polynomial. Moreover the Salem numbers $\tau_m := \tau(G_m)$ converge to θ as $m\to\infty$.*

Proof. Preserving the notation of Lemma 10, we have $(y^2-1)R_m(z) = y^{2m}P(z) - P^*(z)$. We suppose that m is large enough that the only roots of $R_m(z)$ are τ_m, its conjugates, and perhaps some roots of unity. By Lemma 4, the τ_m are strictly increasing, so in particular they have modulus $\geqslant 1+\varepsilon$ for all sufficiently small positive ε. From the remark preceding this Lemma, we deduce that $P(z)$ has exactly one root outside the closed unit disc, θ say. Applying Rouché's Theorem on the boundary of an arbitrarily small disc centred on θ we deduce that, for all large enough m, R_m and P have the same number of zeros (namely one) within that disc, and hence $\tau_m \to \theta$ as $m\to\infty$. Since the eigenvalues of trivial Salem graphs form a discrete set, we can discard the at most finite number of trivial Salem graphs in our sequence, and so assume that all our Salem graphs are nontrivial, so that the τ_m are Salem numbers.

It remains to prove that θ is a Pisot number. The only alternative would be that θ is a Salem number. But then θ would also be a root of $P^*(z)$, so would be a root of $R_m(z)$ for all m, giving $\tau_m = \theta$ for all m. This contradicts the fact that the τ_m are strictly increasing as m increases. □

It is interesting to note that the Pisot number θ in Lemma 12 cannot be a reciprocal quadratic Pisot number, the proof showing that it is not conjugate to $1/\theta$.

Corollary 13. *With notation as in Lemma 11, suppose further that G is connected and that $G_{m_1,\dots,m_k}$ is a Salem graph for all sufficiently large $m_1,\dots,m_k$. Then $G_{m_1,\dots,m_k}$ is a nontrivial Salem graph for but finitely many $(m_1,\dots,m_k)$. Furthermore $P_{(1,\dots,1)}(z)$, the leading polynomial of $R_{m_1,\dots,m_k}(z)$, is the product of the minimal polynomial of some Pisot number (θ, say), a power of z, and perhaps a cyclotomic polynomial.*

Moreover, if we all let the m_i tend to infinity in any manner (one at a time, in bunches, or all together, perhaps at varying rates), the Salem numbers $\tau_{m_1,\dots,m_k} = \tau(G_{m_1,\dots,m_k})$ tend to θ.

Proof. Throughout we suppose that the m_i are all sufficiently large that all the graphs under consideration are Salem graphs. As in the proof of the previous lemma, we may assume that these are nontrivial, so that the $\tau_{m_1,\dots,m_k}$ are Salem numbers. Fixing $m_2,\dots,m_k$ (all large enough), and letting $m_1 \to \infty$, we apply Lemma 12 to deduce that $\tau_{m_1,\dots,m_k}$ tends to a Pisot number, say $\theta_{\infty,m_2,\dots,m_k}$, that is a root of

$$\sum_{\varepsilon_2,\dots,\varepsilon_k\in\{0,1\}} y^{2\sum_{i\ge 2}\varepsilon_i m_i} P_{(1,\varepsilon_2,\dots,\varepsilon_k)}(z)\,.$$

Now we let $m_2 \to \infty$, and we get a sequence of Pisot numbers that converge to the unique root of

$$\sum_{\varepsilon_3,\dots,\varepsilon_k\in\{0,1\}} y^{2\sum_{i\ge 3}\varepsilon_i m_i} P_{(1,1,\varepsilon_3,\dots,\varepsilon_k)}(z)\,.$$

outside the closed unit disc. Since the set of Pisot numbers is closed ([Sa]), this number, $\theta_{\infty,\infty,m_3,\dots,m_k}$, must be a Pisot number.

Similarly we let the remaining $m_i \to \infty$, producing a Pisot number $\theta = \theta_{\infty,\dots,\infty}$ that is the unique root of $P_{(1,1,\dots,1)}$ outside the closed unit disc. Hence $P_{(1,1,\dots,1)}$ has the desired form.

Finally we note that in whatever manner the m_i tend to infinity, $P_{(1,1,\dots,1)}$ eventually dominates outside the unit circle, and a Rouché argument near θ shows that the Salem numbers converge to θ. $\square$

Lemma 14. *Let G be a graph with two (perhaps equal) distinguished vertices v_1 and v_2. Let $G^{(m_1,m_2)}$ be the graph obtained by identifying the endvertices of a new (m_1+m_2+3)-vertex path with vertices v_1 and v_2 (so that $G^{(m_1,m_2)}$ has m_1+m_2+1 more vertices than G). Let $R^{(m_1,m_2)}$ be the reciprocal polynomial of $G^{(m_1,m_2)}$.*

Removing the appropriate vertex (w say) from the new path, we get the graph G_{m_1,m_2} (in the notation of Lemma 11), with reciprocal polynomial R_{m_1,m_2}.

Then

$$R^{(m_1,m_2)} = (y^2-1)R_{m_1,m_2}(z) + Q_{m_1,m_2}(z)\,,$$

where Q_{m_1,m_2} has negligible degree compared to R_{m_1,m_2}, in the sense that

$$\deg(Q_{m_1,m_2})/\deg(R_{m_1,m_2}) \to 0$$

as $\min(m_1,m_2) \to \infty$.

With the natural extension of our previous notion of a leading polynomial, this Lemma implies that $R^{(m_1,m_2)}$ has the same leading polynomial as R_{m_1,m_2}.

Proof. Expanding $\chi_{G^{(m_1,m_2)}} = \det(\lambda I - \text{adjacency matrix of } G^{(m_1,m_2)})$ along the row corresponding to the vertex w, we get

$$\chi_{G^{(m_1,m_2)}} = \lambda\chi_{G_{m_1,m_2}} - \chi_{G_{m_1-1,m_2}} - \chi_{G_{m_1,m_2-1}} + Q_1(\lambda)\,,$$

where Q_1, and also Q_2, Q_3, Q_4 below, have negligible degree compared to the other polynomials in the equation where they appear.

Substituting $\lambda = y + 1/y$ and multiplying by the appropriate power of y gives

$$R^{(m_1,m_2)} = (y^2+1)R_{m_1,m_2}(z) - y^2 R_{m_1-1,m_2}(z) - y^2 R_{m_1,m_2-1}(z) + Q_2(z)\,.$$

Applying Lemma 11 for the case $k = 2$, we get

$$\begin{aligned}(y^2-1)^2 R^{(m_1,m_2)}(z) =& P_{(1,1)}(z)\left\{(y^2+1)y^{2(m_1+m_2)} - y^{2+2(m_1-1+m_2)} - y^{2+2(m_1+m_2-1)}\right\} + Q_3(z)\\ =& y^{2(m_1+m_2)}(y^2-1)P_{(1,1)}(z) + Q_3(z)\,.\end{aligned}$$

Comparing with

$$(y^2-1)^2 R_{m_1,m_2}(z) = y^{2(m_1+m_2)} P_{(1,1)}(z) + Q_4(z)\,,$$

we get the advertised result. □

5. PROOF OF THEOREM 1

Proof. Consider an infinite sequence of nontrivial Salem graphs G, for which the Salem numbers $\tau(G)$ tend to a limit. We are interested in limit points of the set T_{graph}, so we may suppose, by moving to a subsequence, that our sequence has no constant subsequence; moreover we can suppose that the graphs are either all bipartite, or all nonbipartite. Indeed we shall suppose that they are all nonbipartite, and leave the trivial modifications for the bipartite case to the reader. These Salem numbers are bounded above, and hence so are the indices of their graphs. Hence Proposition 7 gives an upper bound on the number of vertices of degree not equal to 2 of these Salem graphs, and Lemma 5 gives an upper bound on the degrees of vertices that each such graph can have. Now, the set of all multigraphs with at most B_1 vertices each of which is of degree at most B_2 is finite. Thus, on associating to each Salem graph in the sequence the multigraph with no vertices of degree 2 having that Salem graph as a subdivision (that is, placing extra vertices of degree 2 along edges of the multigraph retrieves the Salem graph), we obtain only finitely many different multigraphs. Hence, by replacing the sequence of Salem graphs by a subsequence, if necessary, we can assume that all Salem graphs in the sequence are associated to the same multigraph, M say. Now label the edges of M by $e_1, \ldots, e_m$ say. Each edge e_j corresponds to a path, of length $\ell_{j,n}$ say, on the n-th Salem graph of the sequence, joining two vertices of degree not equal to 2.

Now consider the sequence $\{\ell_{1,n}\}$. If it is bounded, it has an infinite constant subsequence. Otherwise, it has a subsequence tending monotonically to infinity. Hence, on taking a suitable subsequence, we can assume that $\{\ell_{1,n}\}$ has one or other of these properties. Furthermore, since any infinite subsequence of a sequence having one of these properties inherits that property, we can take further infinite subsequences without losing that property. Thus we do the same successively for $\{\ell_{2,n}\}$, then $\{\ell_{3,n}\}, \{\ell_{4,n}\}, \ldots, \{\ell_{m,n}\}$. The effect is that we can assume that every sequence $\{\ell_{j,n}\}$ is either constant or tends to infinity monotonically. Those that are constant can simply be incorporated into M (now allowing it to have vertices of degree 2), so that we can in fact assume that they all tend to infinity monotonically.

Let us suppose that our sequence of Salem graphs, $\{G_r\}$ has s increasingly-subdivided internal edges, and t pendant-increasing edges. Form another set of graphs by removing a vertex from the middle (or near middle) of each increasingly-subdivided edge of each G_r, leaving $2s+t$ pendant-increasing edges. We shall use K_r to denote a graph in this sequence, with $n_1, \ldots, n_{2s+t}$ for the lengths of its pendant-increasing edges.

Claim: for *any* sufficiently large $n_1, \ldots, n_{2s+t}$, we have a Salem graph. For (i) we soon exclude all cyclotomic graphs from the list given in Section 6; and (ii) we can never have more than one eigenvalue that is > 2, otherwise, by adding vertices to reach one of our G_r we would find a Salem graph with more than one eigenvalue > 2, using Lemma 2; and (iii) we can never have an eigenvalue that is < -2, by similar reasoning.

Now we apply Corollary 13 to deduce that the limit of our sequence of Salem numbers coming from the K_r is a Pisot number. (Note that K_r need not be connected. All but one component will be cyclotomic, and the noncyclotomic component produces our Pisot number (the others merely contribute cyclotomic factors to the leading polynomial).) Finally, by Lemma 14 this limiting Pisot number is also the limit of the original sequence of Salem numbers.

The last sentence of Theorem 1 follows immediately. $\square$

Examining the proof, we see that the number m of lengthening paths attached to the noncyclotomic growing component tells us that the limiting Pisot number is in the m-th derived set of T_{graph}, and so in the $(m-1)$-th derived set of S_{graph}.

6. Cyclotomic rooted trees

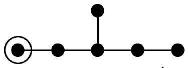
If T is a rooted tree, by which we of course mean a tree with a distinguished vertex r say, its *root*, then T' will denote the rooted forest (set of rooted trees) $T - \{r\}$, the root of each tree in T' being its vertex that is adjacent (in T) to r.

The *quotient* of a rooted tree is the rational function $q_T = \frac{\prod_i R_i(z)}{R_T(z)}$, where R_T is the reciprocal polynomial of the tree, and the R_i are the reciprocal polynomials of its rooted subtrees, the trees of T'. We define the ν-*value* $\nu(T)$ of a tree T to be $q_T(1)$, allowing $\nu(T) = \infty$ if q_T has a pole at 1. Note that by Lemma 2, all zeros and poles of $(z-1)q_T$ are simple. The poles correspond to a subset of the distinct eigenvalues of T via $\lambda = \sqrt{z} + 1/\sqrt{z}$.

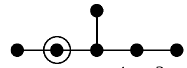
In this section we use Lemma 3 to list all rooted cyclotomic trees, along with their quotients and ν-values. These will be used in the following section (Theorem 16) to show how to construct all Salem trees.

In our list, each entry for a tree T contains the following: a name for T, based on Coxeter graph notation; a picture of T, with the root circled; its quotient $q_T(z)$ and ν-values $\nu(T) = q_T(1)$. Here $\Phi_n = \Phi_n(z)$ is the n-th cyclotomic polynomial.

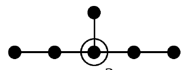
First, the rooted trees that are proper subtrees of $\tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$, but not subtrees of any $\tilde{D}_n$:

$E_6(1)$ 

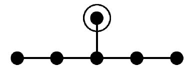
$$\frac{\Phi_2\Phi_8}{\Phi_3\Phi_{12}} = \frac{(z+1)(z^4+1)}{(z^2+z+1)(z^4-z^2+1)}, \nu = \frac{4}{3}$$

$E_6(2)$ 

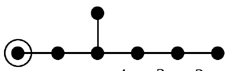
$$\frac{\Phi_2\Phi_5}{\Phi_3\Phi_{12}} = \frac{(z+1)(z^4+z^3+z^2+z+1)}{(z^2+z+1)(z^4-z^2+1)}, \nu = \frac{10}{3}$$

$E_6(3)$ 

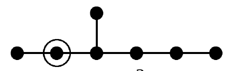
$$\frac{\Phi_2\Phi_3}{\Phi_{12}} = \frac{(z+1)(z^2+z+1)}{z^4-z^2+1}, \nu = 6$$

$E_6(4)$ 

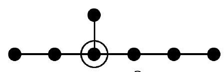
$$\frac{\Phi_2\Phi_6}{\Phi_{12}} = \frac{(z+1)(z^2-z+1)}{z^4-z^2+1}, \nu = 2$$

$E_7(1)$ 

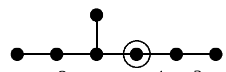
$$\frac{\Phi_2\Phi_{10}}{\Phi_{18}} = \frac{(z+1)(z^4-z^3+z^2-z+1)}{z^6-z^3+1}, \nu = 2$$

$E_7(2)$ 

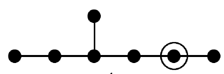
$$\frac{\Phi_2\Phi_3\Phi_6}{\Phi_{18}} = \frac{(z+1)(z^2+z+1)(z^2-z+1)}{z^6-z^3+1}, \nu = 6$$

$E_7(3)$ 

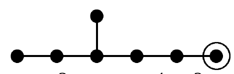
$$\frac{\Phi_2\Phi_3\Phi_4}{\Phi_{18}} = \frac{(z+1)(z^2+z+1)(z^2+1)}{z^6-z^3+1}, \nu = 12$$

$E_7(4)$ 

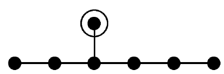
$$\frac{\Phi_3\Phi_5}{\Phi_2\Phi_{18}} = \frac{(z^2+z+1)(z^4+z^3+z^2+z+1)}{(z+1)(z^6-z^3+1)}, \nu = \frac{15}{2}$$

$E_7(5)$ 

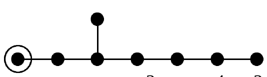
$$\frac{\Phi_2\Phi_8}{\Phi_{18}} = \frac{(z+1)(z^4+1)}{z^6-z^3+1}, \nu = 4$$

$E_7(6)$ 

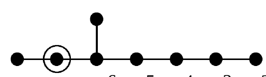
$$\frac{\Phi_3\Phi_{12}}{\Phi_2\Phi_{18}} = \frac{(z^2+z+1)(z^4-z^2+1)}{(z+1)(z^6-z^3+1)}, \nu = \frac{3}{2}$$

$E_7(7)$ 

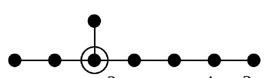
$$\frac{\Phi_7}{\Phi_2\Phi_{18}} = \frac{z^6+z^5+z^4+z^3+z^2+z+1}{(z+1)(z^6-z^3+1)}, \nu = \frac{7}{2}$$

$E_8(1)$ 

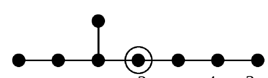
$$\frac{\Phi_2\Phi_4\Phi_{12}}{\Phi_{30}} = \frac{(z+1)(z^2+1)(z^4-z^2+1)}{z^8+z^7-z^5-z^4-z^3+z+1}, \nu = 4$$

$E_8(2)$ 

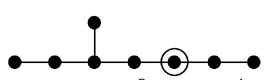
$$\frac{\Phi_2\Phi_7}{\Phi_{30}} = \frac{(z+1)(z^6+z^5+z^4+z^3+z^2+z+1)}{z^8+z^7-z^5-z^4-z^3+z+1}, \nu = 14$$

$E_8(3)$ 

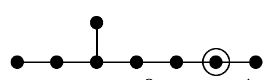
$$\frac{\Phi_2\Phi_3\Phi_5}{\Phi_{30}} = \frac{(z+1)(z^2+z+1)(z^4+z^3+z^2+z+1)}{z^8+z^7-z^5-z^4-z^3+z+1}, \nu = 30$$

$E_8(4)$ 

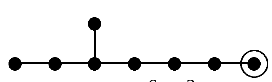
$$\frac{\Phi_2\Phi_4\Phi_5}{\Phi_{30}} = \frac{(z+1)(z^2+1)(z^4+z^3+z^2+z+1)}{z^8+z^7-z^5-z^4-z^3+z+1}, \nu = 20$$

$E_8(5)$ 

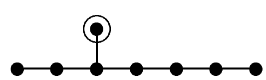
$$\frac{\Phi_2\Phi_3\Phi_8}{\Phi_{30}} = \frac{(z+1)(z^2+z+1)(z^4+1)}{z^8+z^7-z^5-z^4-z^3+z+1}, \nu = 12$$

$E_8(6)$ 

$$\frac{\Phi_2\Phi_3\Phi_{12}}{\Phi_{30}} = \frac{(z+1)(z^2+z+1)(z^4-z^2+1)}{z^8+z^7-z^5-z^4-z^3+z+1}, \nu = 6$$

$E_8(7)$ 

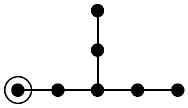
$$\frac{\Phi_2\Phi_{18}}{\Phi_{30}} = \frac{(z+1)(z^6-z^3+1)}{z^8+z^7-z^5-z^4-z^3+z+1}, \nu = 2$$

$E_8(8)$ 

$$\frac{\Phi_2\Phi_4\Phi_8}{\Phi_{30}} = \frac{(z+1)(z^2+1)(z^4+1)}{z^8+z^7-z^5-z^4-z^3+z+1}, \nu = 8$$

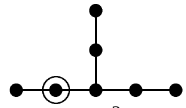
Next, the rooted versions of $\tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$. Note that all their quotients have a pole at $z = 1$, so that $\nu = \infty$ for all these trees.

$\tilde{E}_6(1)$



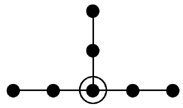
$$\frac{\Phi_{12}}{\Phi_1^2\Phi_2\Phi_3} = \frac{z^4-z^2+1}{(z-1)^2(z+1)(z^2+z+1)}$$

$\tilde{E}_6(2)$



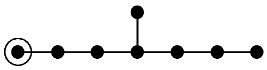
$$\frac{\Phi_2\Phi_6}{\Phi_1^2\Phi_3} = \frac{(z+1)(z^2-z+1)}{(z-1)^2(z^2+z+1)}$$

$\tilde{E}_6(3)$



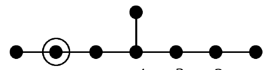
$$\frac{\Phi_3}{\Phi_1^2\Phi_2} = \frac{z^2+z+1}{(z-1)^2(z+1)}$$

$\tilde{E}_7(1)$



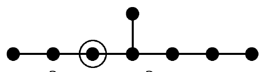
$$\frac{\Phi_{18}}{\Phi_1^2\Phi_2\Phi_3\Phi_4} = \frac{z^6-z^3+1}{(z-1)^2(z+1)(z^2+z+1)(z^2+1)}$$

$\tilde{E}_7(2)$



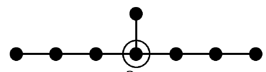
$$\frac{\Phi_2\Phi_{10}}{\Phi_1^2\Phi_3\Phi_4} = \frac{(z+1)(z^4-z^3+z^2-z+1)}{(z-1)^2(z^2+z+1)(z^2+1)}$$

$\tilde{E}_7(3)$



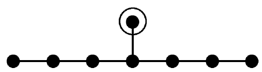
$$\frac{\Phi_3\Phi_6}{\Phi_1^2\Phi_2\Phi_4} = \frac{(z^2+z+1)(z^2-z+1)}{(z-1)^2(z+1)(z^2+1)}$$

$\tilde{E}_7(4)$



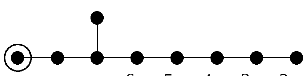
$$\frac{\Phi_2\Phi_4}{\Phi_1^2\Phi_3} = \frac{(z+1)(z^2+1)}{(z-1)^2(z^2+z+1)}$$

$\tilde{E}_7(5)$



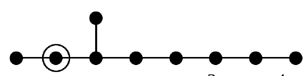
$$\frac{\Phi_8}{\Phi_1^2\Phi_2\Phi_3} = \frac{z^4+1}{(z-1)^2(z+1)(z^2+z+1)}$$

$\tilde{E}_8(1)$



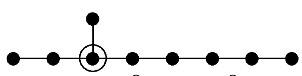
$$\frac{\Phi_2\Phi_{14}}{\Phi_1^2\Phi_3\Phi_5} = \frac{(z+1)(z^6-z^5+z^4-z^3+z^2-z+1)}{(z-1)^2(z^2+z+1)(z^4+z^3+z^2+z+1)}$$

$\tilde{E}_8(2)$



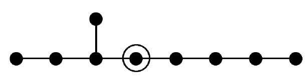
$$\frac{\Phi_2\Phi_4\Phi_8}{\Phi_1^2\Phi_3\Phi_5} = \frac{(z+1)(z^2+1)(z^4+1)}{(z-1)^2(z^2+z+1)(z^4+z^3+z^2+z+1)}$$

$\tilde{E}_8(3)$



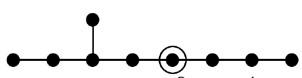
$$\frac{\Phi_2\Phi_3\Phi_6}{\Phi_1^2\Phi_5} = \frac{(z+1)(z^2+z+1)(z^2-z+1)}{(z-1)^2(z^4+z^3+z^2+z+1)}$$

$\tilde{E}_8(4)$



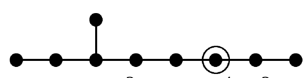
$$\frac{\Phi_5}{\Phi_1^2\Phi_2\Phi_3} = \frac{z^4+z^3+z^2+z+1}{(z-1)^2(z+1)(z^2+z+1)}$$

$\tilde{E}_8(5)$



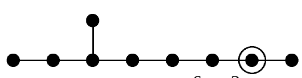
$$\frac{\Phi_2\Phi_4\Phi_8}{\Phi_1^2\Phi_3\Phi_5} = \frac{(z+1)(z^2+1)(z^4+1)}{(z-1)^2(z^2+z+1)(z^4+z^3+z^2+z+1)}$$

$\tilde{E}_8(6)$



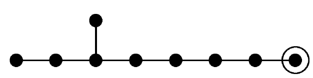
$$\frac{\Phi_3\Phi_{12}}{\Phi_1^2\Phi_2\Phi_5} = \frac{(z^2+z+1)(z^4-z^2+1)}{(z-1)^2(z+1)(z^4+z^3+z^2+z+1)}$$

$\tilde{E}_8(7)$



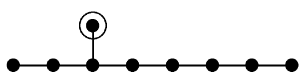
$$\frac{\Phi_2\Phi_{18}}{\Phi_1^2\Phi_3\Phi_5} = \frac{(z+1)(z^6-z^3+1)}{(z-1)^2(z^2+z+1)(z^4+z^3+z^2+z+1)}$$

$\tilde{E}_8(8)$



$$\frac{\Phi_{30}}{\Phi_1^2\Phi_2\Phi_3\Phi_5} = \frac{z^8+z^7-z^5-z^4-z^3+z+1}{(z-1)^2(z+1)(z^2+z+1)(z^4+z^3+z^2+z+1)}$$

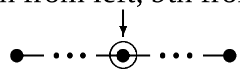
$\tilde{E}_8(9)$



$$\frac{\Phi_9}{\Phi_1^2\Phi_2\Phi_5} = \frac{z^6+z^3+1}{(z-1)^2(z+1)(z^4+z^3+z^2+z+1)}$$

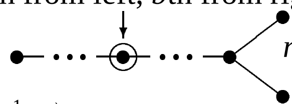
Then, the five infinite families:

(ath from left, bth from right)

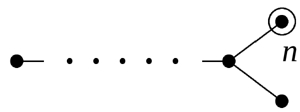
$A_n(a,b)$  $n = a+b-1$ vertices, $1 \leqslant a \leqslant b$

$\frac{(z^a-1)(z^b-1)}{(z-1)(z^{a+b}-1)}, \nu = \frac{ab}{a+b}$

(ath from left, bth from right)

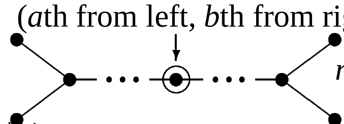
$D_n(a,b)$  $n = a+b$ vertices, $a \geqslant 1$, $b \geqslant 2$

$\frac{(z^a-1)(z^{b-1}+1)}{(z-1)(z^{a+b-1}+1)}, \nu = a$

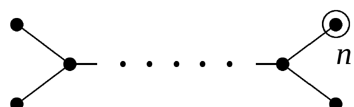
$D_n(0)$  n vertices, $n \geqslant 5$ (note that $D_4(0) = D_4(1,3)$)

$\frac{z^n-1}{(z-1)(z+1)(z^{n-1}+1)}, \nu = \frac{n}{4}$

(ath from left, bth from right)

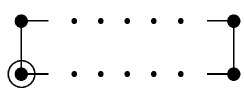
$\tilde{D}_n(a,b)$  $n+1 = a+b+1$ vertices, $2 \leqslant a \leqslant b$

$\frac{(z^{a-1}+1)(z^{b-1}+1)}{(z-1)(z^{a+b-2}-1)}, \nu = \infty$

$\tilde{D}_n(0)$  $n+1$ vertices, $n \geqslant 4$

$\frac{z^{n-1}+1}{(z-1)(z+1)(z^{n-2}-1)}, \nu = \infty$

We also note in passing the rooted even cycles:

$\tilde{A}_{2n-1}$  $2n$ vertices, $n \geqslant 2$

$\frac{z^n+1}{(z-1)(z^n-1)}, \nu = \infty$

As we have seen in Theorem 8, these can be used for constructing bipartite Salem graphs, but obviously not Salem trees.

Note that, since $\tilde{E}_8(2)$ and $\tilde{E}_8(5)$ have the same quotient, we can readily construct different Salem trees having the same quotient, and hence corresponding to the same Salem number.

For each of the Salem quotients $S(z)$ catalogued above, we observe in passing that $(z-1)S(z)$ is an interlacing quotient, as defined in [MS2]. This is an easy consequence of the Interlacing Theorem (Lemma 2).

7. A COMPLETE DESCRIPTION OF SALEM TREES

In this section we consider the case of those (of course bipartite) Salem graphs defined by trees. As before, if T is a rooted tree, then T' will denote the rooted forest obtained by deleting the root r of T, with the root of each subtree being the vertex that (in T) is adjacent to r. The quotient of a rooted forest is defined to be the sum of the quotients of its rooted trees. For rooted trees T_1 and T_2, we define the rooted tree T_1+T_2 to be the tree obtained by joining the roots of T_1 and T_2 by an edge, and making the root of T_1 its root.

Lemma 15. (i) [MRS, Corollary 4] *For a rooted tree T with rooted subtrees $T'=\{T_i\}$, its quotient q_T is given recursively by*

$$q_T=\frac{1}{z+1-zq_{T'}}=\frac{1}{z+1-z\sum_i q_{T_i}},$$

with $q_\bullet = 1/(z+1)$ for the single-vertex tree $\bullet$.

(ii) *For the rooted tree T_1+T_2 we have*

$$q_{T_1+T_2}=\frac{q_{T_1}}{1-zq_{T_1}q_{T_2}}=\frac{z+1-zq_{T_2'}}{(z+1-zq_{T_1'})(z+1-zq_{T_2'})-z}.$$

Proof of (ii). Applying (i) to T_1+T_2 and then to T_1 gives

$$q_{T_1+T_2}=\frac{1}{z+1-zq_{T_1'}-zq_{T_2}}=\frac{1}{1/q_{T_1}-zq_{T_2}}=\frac{q_{T_1}}{1-zq_{T_1}q_{T_2}}.$$

Now applying (i) again to both T_1 and T_2 gives the alternative formula. □

Note that (i) implies that $\nu(T)=1/(2-\nu(T'))$, with $\nu(T')=\sum_i \nu(T_i)$.

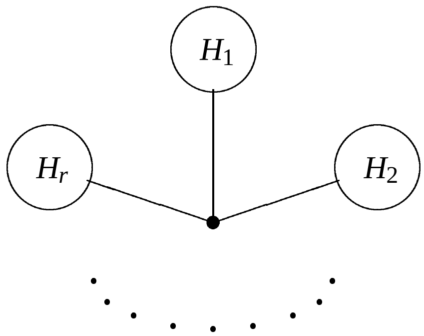
The next Theorem describes all Salem trees. For an alternative approach to a generalisation of this topic, see Neumaier [Neu, Theorem 2.6].

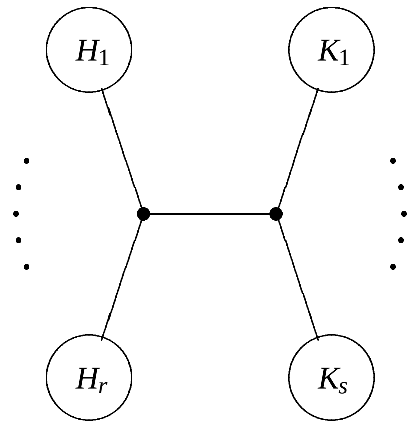
Theorem 16. (a) *Suppose that T is a rooted tree with $\nu(T')>2$, for which the forest T' is a collection of cyclotomic trees. Then T is a Salem tree. (If $\nu(T')\leqslant 2$ then T is again an cyclotomic tree.)*

(b) *Suppose that T_1 and T_2 are Salem trees of type (a) with $(\nu(T_1')-2)(\nu(T_2')-2)\leqslant 1$. Then T_1+T_2 is a Salem tree. (If $(\nu(T_1')-2)(\nu(T_2')-2)>1$ then the reciprocal polynomial of T_1+T_2 has two roots outside the unit circle.)*

(c) *Every Salem tree is of type (a) or type (b).*

In case (a) of Theorem 16, there is a single central vertex joined to r cyclotomic subtrees H_1, $\dots$, H_r, while in case (b) we have a central edge with each endvertex joined to one or more cyclotomic subtrees $H_1,\dots,H_r,K_1,\dots,K_s$:





Proof. (a) Take $\varepsilon > 0$ such that R_T does not vanish on the interval $I = (1, 1+\varepsilon)$. Since T' is cyclotomic, $R_{T'} > 0$ on $(1, \infty)$, and hence in particular $R_{T'} > 0$ on I. Since $\nu(T') > 2$, $q_T(1) = 1/(2-\nu(T')) < 0$, so $R_{T'}/R_T < 0$ on I. Hence $R_T < 0$ on I. Since $R_T(z) \to \infty$ as $z \to \infty$, R_T has at least one root on $(1, \infty)$. By interlacing, R_T cannot have more than one root on $(1, \infty)$, since $R_{T'}$ has none. This gives the first result.

(b) Let $T = T_1 + T_2$. Take $\varepsilon > 0$ such that neither R_T nor $R_{T'}$ vanish on $I = (1, 1+\varepsilon)$. Now T' is the forest $\{T_1', T_2\}$, so that $R_{T'}$ has one root on $(1, \infty)$, a root of R_{T_2}. By interlacing, R_T has one or two roots on that interval.

On I, $R_{T'} < 0$, and $z + 1 - zq_{T_2'} < 0$, since T_2 is a Salem tree of type (a). If $(\nu(T_1') - 2)(\nu(T_2') - 2) < 1$, then $R_{T'}/R_T > 0$ on I, so $R_T < 0$ on I. Since $R_T(z) \to \infty$ as $z \to \infty$, R_T has an odd number of roots, hence exactly one, on $(1, \infty)$. (On the other hand, if $(\nu(T_1') - 2)(\nu(T_2') - 2) > 1$, then $R_{T'}/R_T < 0$ on I, so $R_T > 0$ on I, and then R_T has an even number, and hence two, roots on $(1, \infty)$.)

The only delicate case is if $(\nu(T_1') - 2)(\nu(T_2') - 2) - 1 = 0$. Define the rational function $f(z) = (z + 1 - zq_{T_1'})(z + 1 - zq_{T_2'}) - z$, so that $q_T = (z + 1 - zq_{T_2'})/f(z)$. We need to identify the sign of $f(z)$ on I. Putting $x = \sqrt{z} + 1/\sqrt{z}$, the equation $f(z) = 0$ transforms to $\psi(x) := (\sqrt{z}q_{T_1'} - x)(\sqrt{z}q_{T_2'} - x) = 1$. Interlacing implies that each $\sqrt{z}q_{T_i'}$ (which is a function of x) is decreasing between successive poles, and hence so too is each factor of $\psi(x)$. But since T_1 and T_2 are Salem trees of type (a), each factor of $\psi(x)$ is positive at $x = 2$; hence $\psi(x) < 1$ as x approaches 2 from above; hence $f(z) < 0$ on I. Now, as before, we have $R_{T'}/R_T > 0$ on I, so $R_T < 0$ on I, and the now familiar argument shows that R_T has exactly one root on $(1, \infty)$.

(c) Suppose that T is a tree such that R_T has one root > 1 but is not of type (a). Pick any vertex t_0 of T. Then, by interlacing, $T - \{t_0\}$ has one component, T_1 say, that is a Salem tree, the other components being cyclotomic. Let t_1 be the root of T_1 (the vertex adjacent to t_0 in T). Now replace t_0 by t_1 and repeat the argument, obtaining a new vertex t_2. If

$t_2 = t_0$ then we are finished. Otherwise, we repeat the argument, obtaining a walk on T, using vertices $t_0, t_1, t_2, \cdots$. Since T has no cycles, any walk in T must eventually double back on itself, so that some t_i equals t_{i-2}. Then T is of the form $T_1 + T_2$, where T_1 and T_2 are of type (a), with roots t_{i-1} and t_i.

$\square$

Note that while in case (a) T is a rooted tree with the property that removal of a single vertex gives a forest of cyclotomic trees, in case (b) the tree $T_1 + T_2$ has the property that removal of the edge joining the roots of T_1 and T_2, with its incident vertices, also gives a forest of cyclotomic trees.

Theorem 16 is a restriction of Theorem 8 (above) to trees. However, it is stronger, as we are able to say precisely which trees are Salem trees. Theorem 16 also shows how to construct all Salem trees. To construct trees of type (a), we take any collection of rooted cyclotomic trees, as listed in Section 6, the sum of whose ν-values exceeds 2. For trees of type (b), we take two such collections whose ν-values sum to s_1 and s_2 say, with $s_1 \geqslant s_2 > 2$, subject to the additional constraint that $(s_1 - 2)(s_2 - 2) \leqslant 1$. A check on possible sums of ν-values reveals that the smallest possible value for $s_2 > 2$ is $85/42$, coming from the tree $T(1,2,6)$, using the labelling of Figure 7. This implies the upper bound $s_1 \leqslant 44$ for s_1. Of course, when $s_2 > 85/42$, the upper bound for s_1 will be smaller. Note too that the condition $s_1 \geqslant s_2$ implies that $s_2 \leqslant 3$.

The first examples of Salem numbers of trace below -1 were obtained using the construction in Theorem 16(a) (see [MS1]). The smallest known degree for a Salem number of trace below -1 coming from a graph is of degree 460, obtained when $T' = \{A_{70}(1,69), D_{196}(182,14), D_{232}(220,12)\}$ in Theorem 16(a). Much smaller degrees have been obtained by other means, and the minimal degree is known to be 20 ([MS1]). It is also now known that all integers occur as traces of Salem numbers ([MS2]).

7.1. **Earlier results.** Theorem 16(a) generalises [MRS, Corollary 9], which gave the same construction, but only for starlike trees. In 1988 Floyd and Plotnick [FP, Theorem 5.1], without using graphs but using an unpublished result of Cannon, showed how to construct Salem numbers in a way equivalent to our construction using star-like trees. The same construction was also published by Cannon and Wagreich [CW, Proposition 5.2] and Parry [P, Corollary 1.8] in 1992. In 1999 Piroska Lakatos [L1, Theorem 1.2] deduced essentially the same star-like tree construction from results of A'Campo and Pena on Coxeter transformations. Also, in 2001 Eriko Hironaka [Hi, Proposition 2.1] produced an equivalent construction, in the context of knot theory, as the Alexander polynomial of a pretzel knot.

8. Pisot graphs

As we have seen in Section 5, a graph Pisot number is a limit of graph Salem numbers whose graphs may be assumed to come from a family obtained by taking a certain multigraph, and assuming that some of its edges have an increasing number of subdivisions. We use this family to define a graph having bi-coloured vertices: we start with the multigraph, with black vertices. For every increasingly subdivided pendant edge, we change the colour of the pendant vertex to white, while for an increasing internal edge we subdivide it with two white vertices. Thus a single white vertex represents a pendant-increasing edge, while a pair of adjacent white vertices represents an increasing internal edge. These Pisot graphs in fact represent a sequence of Salem numbers tending to the Pisot number. Now, we have seen in the proof of Theorem 1 that the limit point of the Salem numbers corresponding to a Salem graph with an increasing internal edge is the same as that of the graph when this edge is broken in the middle. Hence for any Pisot graph we can remove any edge joining two white vertices without changing the corresponding Pisot number. (Doing this may disconnect the graph, in which case only one of the connected components corresponds to the Pisot number.) It follows that every graph Pisot number has a graph all of whose white vertices are pendant (have degree 1).

For Pisot graphs that are trees (*Pisot trees*), and furthermore have all white vertices pendant, we can define their quotients by direct extension of the quotient of an ordinary tree (that is, one without white vertices, as in Section 7). Now from Section 6 the path $A_n(1,n)$ has quotient $(z^n-1)/(z^{n+1}-1)$, which, for $z>1$ tends to $1/z$ as $n\to\infty$. Thus, following [MRS, p. 315], we can take the quotient of a white vertex $\circ$ to be $1/z$, and then calculate the quotient of these trees in the same way as for ordinary trees. The irreducible factor of its denominator with a root in $|z|>1$ then gives the minimal polynomial of the Pisot number.

FIGURE 2. Pisot graphs of the smallest Pisot number (minimal polynomial z^3-z-1), and for the smallest limit point of Pisot numbers (minimal polynomial z^2-z-1). See end of Section 8.

For instance, for the two Pisot trees in Figure 2, take their roots to be the central vertex. Then we can use Lemma 15(i) to compute the quotient of the left-hand one to be

$$\frac{1}{z+1-z\left(\frac{1}{z}+\frac{z-1}{z^2-1}+\frac{z^2-1}{z^3-1}\right)}=\frac{(z+1)(z^2+z+1)}{z(z^3-z-1)},$$

so that the corresponding Pisot number has minimal polynomial z^3-z-1. Similarly, the right-hand one has quotient $\frac{z+1}{z^2-z-1}$, with minimal polynomial z^2-z-1.

9. SMALL ELEMENTS OF THE DERIVED SETS OF PISOT NUMBERS

In this section we give a proof of a graphical version of the following result of Bertin [Be]. Recall that the (1st) derived set of a given real set is the set of limit points of the set, while for $k\geqslant 2$ its k-th derived set is the set of limit points of its $(k-1)$-th derived set.

Theorem 17. *Let $k\in\mathbb{N}$. Then $(k+\sqrt{k^2+4})/2$ belongs to the $(2k-1)$-th derived set of the set S_{graph} of graph Pisot numbers, while $k+1$ belongs to the $(2k)$-th derived set of S_{graph}.*

Bertin's result was that these numbers belonged to the corresponding derived set of S, rather than that of S_{graph}. They are the smallest known elements of the relevant derived set of S.

FIGURE 3. The subtrees used to make small elements of the derived sets of the set of graph Pisot numbers. Their Pisot quotients are $1/z$ and $1/(z-1)$ respectively. They give such elements as a limit of increasing graph Pisot numbers. See Theorem 17.

Proof. The proof consists simply of exhibiting two families of trees containing $2k$ and $2k+1$ white vertices respectively, and showing that their reciprocal polynomials are z^2-kz-1 and $z-(k+1)$. From the discussion above, this will show that their zeros in $|z|>1$, namely those given in the statement of the Theorem, are in the $(2k-1)$-th and $(2k)$-th derived set of the set of Pisot numbers, respectively. For the graph with $2k$ white vertices we take k of the 3-vertex graphs shown in Figure 3 joined to a central vertex, while for the graph with $2k+1$ vertices we take the same graph with one extra white vertex joined to the central vertex (the other graph shown in this figure). The result is shown in Figure 4 for $k=5$. We can use Lemma 15, extended to include trees containing an infinite path. This shows that the tree has quotient $(z+1-kz/(z-1))^{-1} = (z-1)/(z^2-kz-1)$ when it has $2k$ white vertices, and quotient $(z+1-z(k/(z-1)+1/z))^{-1} = (z-1)/(z(z-(k+1)))$ when it has $2k+1$ white vertices. The poles of these quotients give the required Pisot numbers.

□

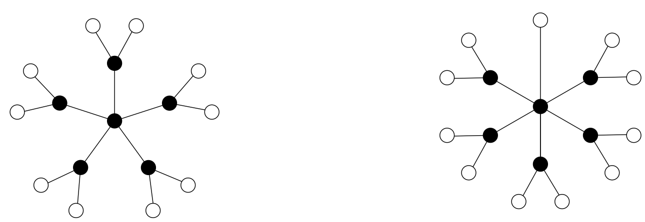


FIGURE 4. The infinite graphs showing that $(k+\sqrt{k^2+4})/2$ belongs to the $(2k-1)$-th derived set of the set of graph Pisot numbers (left, $k=5$ shown), and that $k+1$ belongs to the $(2k)$-th derived set (right). Here increasing sequences are produced—see Theorem 17.

The graphs of Figure 4 show how the elements of the derived sets are limits from below of elements of S_{graph}. We can also show that they are limits from above, using the 5- and 11-vertex graphs of Figure 5 to construct Pisot graphs showing these numbers to be elements of the relevant derived set by showing them to be limit points from above rather than below. The graphs in Figure 6 are examples of this construction. Further, one could construct graphs using a mixture of subgraphs from Figures 3 and 5. Thus, if we distinguished two types of limit point depending on whether the point was a limit from below or from above, we could define two types of derived set, and hence, by iteration, an (n_-, n_+)-derived set of S_{graph}. This mixed construction would produce elements of these sets.

FIGURE 5. The subtrees used to make small elements of the derived sets of the set of graph Pisot numbers. Their Pisot quotients are $1/z$ (left) and $1/(z-1)$ (right). They give such elements as a limit of decreasing graph Pisot numbers. See the remarks after Theorem 17.

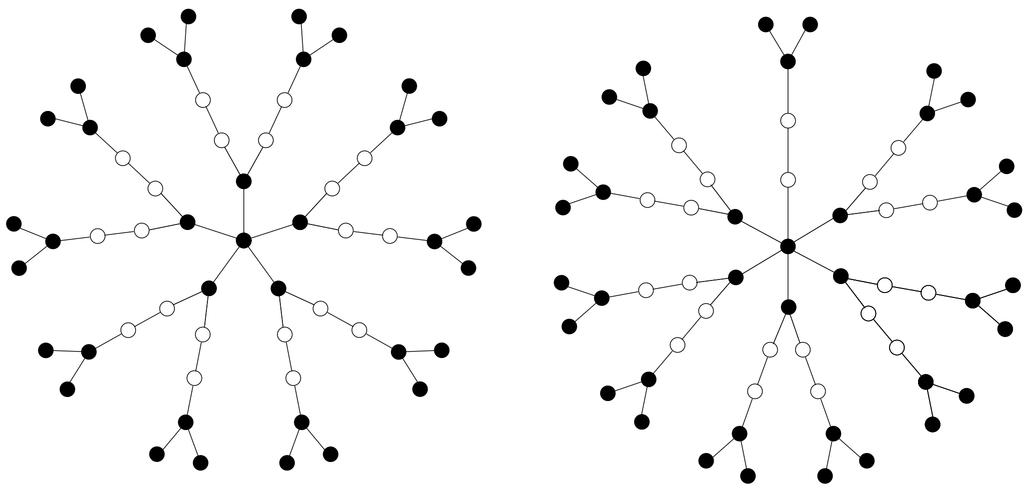


FIGURE 6. The infinite graphs showing that $(k+\sqrt{k^2+4})/2$ belongs to the $(2k-1)$-th derived set of the set of graph Pisot numbers (left, $k=5$ shown), and that $k+1$ belongs to the $(2k)$-th derived set (right). Here decreasing sequences are produced—see the remarks after Theorem 17.

10. THE MAHLER MEASURE OF GRAPHS

In this section we find (Theorem 19) all the graphs of Mahler measure less than $\rho := \frac{1}{2}(1+\sqrt{5})$. Our definition of Mahler measure for graphs—see below—seems natural. This is because we then obtain as a corollary that the strong version of "Lehmer's Conjecture", which states that τ_1 is the smallest Mahler measure greater than 1 of any algebraic number, is true for graphs:

Corollary 18. *The Mahler measure of a graph is either* 1 *or at least* $\tau_1 = 1.176280818\cdots$, *the largest real zero of Lehmer's polynomial* $L(z) = z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1$. *Among connected graphs, this minimum Mahler measure is attained only for the graph* $T(1,2,6)$ *defined in Figure 7 (= the Coxeter graph* E_{10}*).*

We define the *Mahler measure* $M(G)$ of an n-vertex graph G to be $M(z^n\chi_G(z+1/z))$, where χ_G is the characteristic polynomial of its adjacency matrix, and M of a polynomial also denotes its Mahler measure. Recall that for a monic polynomial $P(z) = \prod_i (z-\alpha_i)$ its Mahler measure is defined to be $M(P) = \prod_i \max(1, |\alpha_i|)$. When G is bipartite, $M(G)$ is also the Mahler measure of its reciprocal polynomial $R_G(z) = z^{n/2}\chi_G(\sqrt{z}+1/\sqrt{z})$. This is because then $M(z^n\chi_G(z+1/z)) = M(R_G(z^2))$.

The graphs having Mahler measure 1 are precisely the cyclotomic graphs.

It turns out that the connected graphs of smallest Mahler measure bigger than 1 are all trees. Using the notation of [CR], define the trees $T(a,b,c)$ and $Q(a,b,c)$ as in Figure 7.

Theorem 19. *If G is a connected graph whose Mahler measure $M(G)$ lies in the interval $(1,\rho)$ then G is one of the following trees:*

- $G = T(a,b,c)$ *for* $a \leqslant b \leqslant c$ *and*
  $a = 1, b = 2, c \geqslant 6$
  $a = 1, b \geqslant 3, c \geqslant 4$
  $a = 2, b = 2, c \geqslant 3$
  $a = 2, b = 3, c = 3$

  or
- $G = Q(a,b,c)$ *for* $a \leqslant c$ *and*
  $a = 2, b \geqslant 1, c = 3$
  $a = 2, b \geqslant 3, 4 \leqslant c \leqslant b+1$
  $a = 3, 4 \leqslant b \leqslant 13, c = 3$
  $a = 3, 5 \leqslant b \leqslant 10, c = 4$
  $a = 3, 7 \leqslant b \leqslant 9, c = 5$
  $a = 3, 8 \leqslant b \leqslant 9, c = 6$
  $a = 4, 7 \leqslant b \leqslant 8, c = 4.$

All these graphs G are all Salem graphs, apart from $Q(3,13,3)$, whose polynomial has two zeros on $(1,2)$, so that all $M(G)$ apart from $M(Q(3,13,3))$ are Salem numbers. Also, the set of limit points of the set of all $M(G)$ in $(1,\rho]$ consists of the graph Pisot number ρ and the graph Pisot numbers that are zeros of $z^k(z^2-z-1)+1$ for $k = 2,3,\ldots$, which approach ρ as $k \to \infty$.

Furthermore, the only $M(G) < 1.3$ are $M(T(1,2,c))$ for $c = 6,7,8,9,10$, these values increasing with c. (Also $M(T(1,2,9)) = M(T(1,3,4))$.)

In [Hi, Theorem 1.1], Hironaka shows essentially that Lehmer's number $\tau_1 = M(T(1,2,6))$ is the smallest Mahler measure of a starlike tree. The graph Pisot numbers in the Theorem have been shown by Hoffman ([Ho]) in 1972 to be limit points of transformed graph indices, which is equivalent to our representation of them as limits of graph Salem numbers.

It is clear how to extend the theorem to nonconnected graphs: since $\tau_1^3 > \rho$, for such a graph G to have $M(G) \in (1,\rho)$, one or two connected components must be as described in the theorem, with all other connected components cyclotomic. Using the results of the Theorem, it is an easy exercise to check the possibilities.

Proof. The proof depends heavily on results of Brouwer and Neumaier [BN] and Cvetković, Doob and Gutman [CDG], as described conveniently by Cvetković and Rowlinson in their survey paper [CR, Theorem 2.4]. These results tell us precisely whch connected graphs have largest eigenvalue in the interval $(2, \sqrt{2+\sqrt{5}}] = (2, 2.058\cdots]$. They are all trees of the form

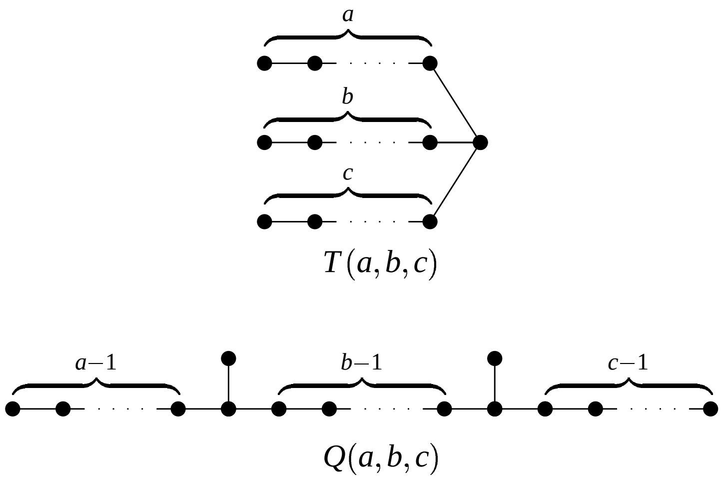


FIGURE 7. The trees $T(a,b,c)$ and $Q(a,b,c)$

$T(a,b,c)$ or $Q(a,b,c)$. Those of the form $T(a,b,c)$ are precisely those given in the statement of the theorem. As they are starlike trees, they have, by [MRS, Lemma 8], one eigenvalue $\lambda > 2$, and so their reciprocal polynomial $R_{T(a,b,c)}$ has a single zero β on $(1,\infty)$ with $\beta^{1/2}+\beta^{-1/2}=\lambda$, and $M(T(a,b,c))=\beta$. Since $\rho^{1/2}+\rho^{-1/2}=\sqrt{2+\sqrt{5}}$, we have $\beta \in (1,\rho]$.

From the previous paragraph it is clear that

- all graphs G with exactly one eigenvalue in $(2,\sqrt{2+\sqrt{5}}]$ have $M(G)\in(1,\rho)$;
- only graphs G with largest eigenvalue in $(2,\sqrt{2+\sqrt{5}}]$ can have $M(G)\in(1,\rho)$.

It remains to see which of the graphs $Q(a,b,c)$ having largest eigenvalue in this interval actually do have $M(G)\in(1,\rho]$. The graphs of this type given in the Theorem are all those with one eigenvalue in $(2,\sqrt{2+\sqrt{5}}]$, along with $Q(3,13,3)$ which, although having two eigenvalues greater than 2, nevertheless does have $M(G)<\rho$. The other graphs with largest eigenvalue in $(2,\sqrt{2+\sqrt{5}}]$ are, from the theorem cited above:

$Q(3,b,3)$ for $b \geqslant 14$,
$Q(3,b,4)$ for $b \geqslant 11$,
$Q(3,b,5)$ for $b \geqslant 10$,
$Q(3,b,6)$ for $b \geqslant 10$,
$Q(3,b,c)$ for $b \geqslant c+2, c \geqslant 7$,
$Q(4,b,4)$ for $b \geqslant 9$,
$Q(4,b,5)$ for $b \geqslant 8$,
$Q(4,b,c)$ for $b \geqslant c+4, c \geqslant 6$,
$Q(a,b,c)$ for $a \geqslant 5, b \geqslant a+c, c \geqslant 5$.

We must show that none of these trees G have $M(G)\leqslant\rho$. We can reduce this infinite list to a small finite one by the following simple observation. Suppose we remove the k-th vertex from the central path of the tree $Q(a,b,c)$, splitting it into $T(1,a-1,k-1)$ and $T(1,c-1,b-1-k)$.

By interlacing we have, for $k=2,\dots,b-2$,

$$M(Q(a,b,c)) \geqslant M(T(1,a-1,k-1)) \times M(T(1,c-1,b-1-k)). \quad (1)$$

Now

$$\begin{aligned} M(T(1,2,6)) =& 1.176280818\cdots, \\ M(T(1,2,9)) =& M(T(1,3,4)) = 1.280638156\cdots, \\ M(T(1,3,6)) =& M(T(1,4,4)) = 1.401268368\cdots, \end{aligned}$$

from which we have that both of $M(T(1,2,6)) \times M(T(1,3,6)) = M(T(1,2,6)) \times M(T(1,4,4))$ and $M(T(1,2,9))^2 = M(T(1,3,4))^2$ are greater than ρ. Since $M(T(a,b,c))$ is, when greater than 1, an increasing function of a, b and c separately, and, of course, independent of the order of a,b,c, we can show that all but 18 of the above $Q(a,b,c)$ have $M(Q(a,b,c)) > \rho$. Applying (1), we have

- $M(Q(3,b,3)) \geqslant M(T(1,2,9)) \times M(T(1,2,b-11)) > \rho$ for $b \geqslant 20$. Cases $b=14,\dots,19$ must be checked individually.
- $M(Q(3,b,4)) \geqslant M(T(1,2,6)) \times M(T(1,3,b-8)) > \rho$ for $b \geqslant 14$. Check $b=11,12,13$ individually.
- $M(Q(3,b,5)) \geqslant M(T(1,2,6)) \times M(T(1,4,b-8)) > \rho$ for $b \geqslant 12$. Check $b=10,11$.
- $M(Q(3,b,6)) \geqslant M(T(1,2,6)) \times M(T(1,5,b-8)) > M(T(1,2,6)) \times M(T(1,4,b-8)) > \rho$ for $b \geqslant 12$. Check $b=10,11$.
- $M(Q(3,b,7)) \geqslant M(T(1,2,6)) \times M(T(1,6,b-8)) > \rho$ for $b \geqslant 11$. Check $b=9,10$.
- $M(Q(3,b,8)) \geqslant M(T(1,2,6)) \times M(T(1,7,b-8)) > M(T(1,2,6)) \times M(T(1,6,b-8)) > \rho$ for $b \geqslant 11$. Check $b=10$.
- For $c \geqslant 9$, $M(Q(3,b,c)) \geqslant M(T(1,2,6)) \times M(T(1,c-1,b-8)) > \rho$ for $b \geqslant 11$.
- $M(Q(4,b,4)) \geqslant M(T(1,3,4)) \times M(T(1,3,b-6)) > \rho$ for $b \geqslant 10$. Check $b=9$.
- $M(Q(4,b,5)) \geqslant M(T(1,3,4)) \times M(T(1,4,b-6)) > \rho$ for $b \geqslant 9$. Check $b=8$.
- For $c \geqslant 6$, $M(Q(4,b,c)) \geqslant M(T(1,3,4)) \times M(T(1,c-1,b-6)) > M(T(1,3,4)) \times M(T(1,4,b-6)) > \rho$ for $b \geqslant 9$.
- For $a \geqslant 5, c \geqslant 5$, $M(Q(a,b,c)) \geqslant M(T(1,a-1,3)) \times M(T(1,c-1,b-5)) \geqslant M(T(1,4,3)) \times M(T(1,4,b-5)) > \rho$ for $b \geqslant 8$.

We remark that it is straightforward, with computer assistance, using Lemma 15, to make the checks required in the proof. Denoting by $q_k(a,b,c)$ the quotient of $Q(a,b,c)$ with root at the k-th vertex of the central path, and by $t(a,b,c)$ the quotient of $T(a,b,c)$ having root at the endvertex of the c-path, this lemma tells us that

$$\begin{aligned} q_k(a,b,c) =& (z+1-z(t(1,a-1,k-1)+t(1,c-1,b-1-k)))^{-1}, \\ t(a,b,c) =& (z+1-zt(a,b,c-1))^{-1} \end{aligned}$$

with $t(a,b,0) = \frac{(z^{a+1}-1)(z^{b+1}-1)}{(z-1)(z^{a+b+2}-1)}$, the quotient of the rooted path $A_{a+b+1}(a+1,b+1)$. Then the denominator of $q_k(a,b,c)$ gives the reciprocal polynomial of $Q(a,b,c)$, at least up to a cyclotomic factor (one can show using Lemma 4(i) and Lemma 15(i) that all roots >1 of the reciprocal polynomial of $Q(a,b,c)$ are indeed poles of its quotient).

Concerning the limit points of $M(G)\cap[1,\rho)$, one can check that

- $M(T(1,b,c)) \to M(z^b(z^2-z-1)+1)$ as $c\to\infty$;
- $M(T(2,2,c)) \to \rho$ as $c\to\infty$;
- For $c\geqslant 3$, $M(Q(2,b,c)) \to M(z^{c-1}(z^2-z-1)+1)$ as $b\to\infty$.

Of course, by Salem's classical construction, $M(z^b(z^2-z-1)+1)\to\rho$ as $b\to\infty$. Note too that $M(z^2(z^2-z-1)+1) = M(z^3-z-1)$, the smallest Pisot number.

□

From the proof, and the fact that all Pisot numbers in $[1,\rho)$ are known (see Bertin *et al* [BDGPS, p.133]) we have the following.

Corollary 20. *The only graph Pisot numbers in $[1,\rho)$ are the roots of $z^n(z^2-z-1)+1$ for $n\geqslant 2$. The other Pisot numbers in this interval, namely the roots of $z^6-2z^5+z^4-z^2+z-1$ and of $z^n(z^2-z-1)+z^2-1$ for $n\geqslant 2$, are not graph Pisot numbers.*

11. Small Salem numbers from graphs

The notation τ_n indicates the nth Salem number in Mossinghoff's table [M], listing all 47 known Salem numbers that are smaller than 1.3. (This is an update of that in [Bo].)

We have seen that the only numbers in this list that are elements of T_{graph} are τ_1, τ_7, τ_{19}, τ_{23}, and τ_{41}. On the other hand, if we apply the construction in Theorem 16(b) with $T_1 = T_2$, then from the explicit formula in Lemma 15(ii) we see that the Salem number produced is automatically the square of a smaller Salem number. (If $q_{T_1} = q/p$ in lowest terms, then from Lemma 15(ii) we have that the squarefree part of $R_{T_1+T_1}$ is $f(z) = q(z)^2 - zp(z)^2$. Now $f(z^2) = (q(z^2)-zp(z^2))(q(z^2)+zp(z^2))$, and the gcd of these two factors divides z. Hence, if $\tau\neq 0$ and $f(\tau)=0$, then $\sqrt{\tau}$ and $-\sqrt{\tau}$ are roots of the different factors of $f(z^2)$ and are not algebraic conjugates. In particular, if τ is a Salem number then so is $\sqrt{\tau}$. We can apply this construction regardless of the value of the quotient of T_1.) In this way we can produce τ_2^2, τ_3^2, τ_5^2, τ_{12}^2, τ_{21}^2, τ_{23}^2, and τ_{41}^2 as elements of T_{graph}.

These results, and a wider search for small powers of small Salem numbers, are recorded in the following table. A list of cyclotomic graphs indicates the components of T' in the construction of Theorem 16(a); two lists separated by a semi-colon indicate the components of T_1' and T_2' in the construction of Theorem 16(b).

Salem number	Cyclotomic graphs
τ_1	$D_9(0)$
τ_1^2	$D_{11}(3,8)$
τ_1^6	$E_7(1),\tilde{D}_4(0);A_5(2,4)$
τ_1^8	$E_6(1),A_2(1,2);E_7(5),\tilde{E}_6(3)$
τ_2^2	$E_8(7);E_8(7)$
τ_3^2	$E_7(6);E_7(6)$
τ_3^5	$A_1(1,1),A_9(2,8);D_{15}(8,7)$
τ_4^5	$E_6(4),D_7(1,6);D_{13}(3,10)$
τ_5^2	$E_6(1);E_6(1)$
τ_5^3	$E_6(1);\tilde{E}_8(7)$
τ_5^4	$E_6(4),D_{18}(12,6)$
τ_5^5	$A_4(1,4),A_4(1,4);D_4(1,3),D_8(1,7)$
τ_5^6	$A_1(1,1),A_3(2,2);D_6(2,4),D_8(4,4)$
τ_7	$D_{10}(0)$
τ_7^4	$E_6(1),A_1(1,1);E_6(1),A_1(1,1)$
τ_7^5	$A_7(2,6);D_4(1,3),\tilde{D}_{10}(5,5)$
τ_7^6	$E_7(3),D_7(4,3);D_9(1,8)$
τ_{10}^3	$E_8(8);D_8(0)$
τ_{12}^2	$D_5(0);D_5(0)$
τ_{12}^3	$E_7(5);E_7(6)$
τ_{12}^5	$E_7(4),\tilde{E}_6(1);A_7(3,5)$
τ_{15}^2	$D_{18}(6,12)$
τ_{15}^4	$A_1(1,1),D_{10}(0);A_1(1,1),D_{10}(0)$
τ_{16}^4	$E_7(1),D_9(1,8),D_8(2,6)$
τ_{19}	$D_{11}(0)$
τ_{19}^3	$\tilde{E}_8(8);D_4(2,2)$
τ_{19}^4	$E_6(4),A_1(1,1);E_6(4),A_1(1,1)$
τ_{19}^5	$\tilde{E}_6(2),A_3(2,2);A_3(1,3),D_6(1,5)$
τ_{21}^2	$E_7(1);E_7(1)$
τ_{21}^5	$E_6(3),A_4(2,3);A_6(1,6)$
τ_{23}	$E_8(1)$
τ_{23}^2	$\tilde{E}_8(6)$
τ_{23}^3	$E_7(2);D_6(1,5)$
τ_{23}^4	$\tilde{E}_7(3),D_{12}(9,3)$
τ_{35}^4	$E_6(4),E_7(1);A_2(1,2),A_6(1,6)$
τ_{41}	$D_{13}(0)$
τ_{41}^2	$D_6(0);D_6(0)$
τ_{41}^3	$A_7(2,6);D_{10}(5,5)$
τ_{41}^4	$A_2(1,2),A_2(1,2);A_6(2,5),D_5(1,4)$

REFERENCES

[Be] Bertin, Marie-Jos´e. Ensembles d´eriv´es des ensembles $\Sigma_{q,n}$ et de l'ensemble S des PV-nombres. *Bull. Sci. Math. (2)* 104 (1980), 3–17.

[BDGPS] Bertin, M.-J.; Decomps-Guilloux, A.; Grandet-Hugot, M.; Pathiaux-Delefosse, M.; Schreiber, J.-P. *Pisot and Salem numbers.* Birkhäuser Verlag, Basel, 1992.

[Bo] Boyd, David W. Small Salem numbers. *Duke Math. J.* 44 (1977), 315–328.

[BN] Brouwer, A. E.; Neumaier, A. The graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$. *Linear Algebra Appl.* 114/115 (1989), 273–276.

[CDG] Cvetkovi´c, Dragoš; Doob, Michael; Gutman, Ivan. On graphs whose spectral radius does not exceed $(2+\sqrt{5})^{1/2}$. *Ars Combin.* 14 (1982), 225–239.

[CR] Cvetkovi´c, D.; Rowlinson, P. The largest eigenvalue of a graph: a survey. *Linear and Multilinear Algebra* 28 (1990), 3–33.

[CS] Cvetkovic, Dragos; Simic, Slobodan. The second largest eigenvalue of a graph (a survey). *Filomat* 9, 449–472 (1995).

[CW] Cannon, J. W.; Wagreich, Ph. Growth functions of surface groups. *Math. Ann.* 293 (1992), 239–257.

[F] Floyd, William J. Growth of planar Coxeter groups, P.V. numbers, and Salem numbers. *Math. Ann.* 293 (1992), 475–483.

[FP] Floyd, William J.; Plotnick, Steven P. Symmetries of planar growth functions. *Invent. Math.* 93 (1988), 501–543.

[GR] Godsil, C.;Royle, G. *Algebraic Graph Theory*, Graduate Texts in Mathematics 207, Springer (2000).

[Hi] Hironaka, Eriko. The Lehmer polynomial and pretzel links. *Canad. Math. Bull.* 44 (2001), 440–451. Erratum: *Canad. Math. Bull.* 45 (2002), 231.

[Ho] Hoffman, Alan J. On limit points of spectral radii of non-negative symmetric integral matrices, in: *Graph theory and applications* (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972), 165–172. Lecture Notes in Math., Vol. 303, Springer, Berlin, 1972.

[HS] Hoffman, Alan J.; Smith, John Howard. On the spectral radii of topologically equivalent graphs, in: *Recent advances in graph theory* (Proc. Second Czechoslovak Sympos., Prague, 1974), 273–281. Academia, Prague, 1975.

[L1] Lakatos, Piroska. On the spectral radius of Coxeter transformations of trees. *Publ. Math. Debrecen* 54 (1999), 181–187.

[L2] Lakatos, Piroska. Salem numbers, PV numbers and spectral radii of Coxeter transformations. *C. R. Math. Acad. Sci. Soc. R. Can.* 23 (2001), 71–77.

[M] Mossinghoff, M. List of small Salem numbers, `http://www.cecm.sfu.ca/~mjm/Lehmer/lists/SalemList.html`

[McK] McKee,J.F. Families of Pisot numbers with negative trace, *Acta Arith.* 93 (2000), 373–385.

[MRS] McKee, J.F.;Rowlinson, P.; Smyth, C.J. Salem numbers and Pisot numbers from stars, in: *Number theory in progress*, Vol. I, K. Győry, H. Iwaniec and J. Urbanowicz (eds.), de Gruyter, Berlin (1999), 309–319.

[MS1] McKee, J.F.; Smyth, C.J. Salem numbers of trace -2 and traces of totally positive algebraic integers. Proc. 6th Algorithmic Number Theory Symposium, (University of Vermont 13 - 18 June 2004), Lecture Notes in Comput. Sci. 3076 (Springer, New York, 2004), 327–337.

[MS2] McKee, J.F.; Smyth, C.J. There are Salem numbers of every trace, *Bull. London Math. Soc.* 37 (2005), 25-36.

[Neu] Neumaier, A. The second largest eigenvalue of a tree, *Linear Algebra and its Applications* 46 (1982), 9–25.

[P] Parry, Walter. Growth series of Coxeter groups and Salem numbers. *J. Algebra* 154 (1993), 406–415.

[Sa] Salem, R. A remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan, *Duke Math. J.* 11 (1944), 103–108.

[Smi] Smith, John H. Some properties of the spectrum of a graph, in: *Combinatorial Structures and their Applications* (Proc. Calgary Internat. Conf., Calgary, Alta., 1969), 1970, 403–406, Gordon and Breach, New York.

[Sm1] Smyth, C.J. Totally positive algebraic integers of small trace, *Annales de l'Institute Fourier de l'Univ. de Grenoble*, 34 (1984), 1–28.
[Sm2] Smyth, C.J. Salem numbers of negative trace, *Mathematics of Computation* 69 (2000), 827–838.

DEPARTMENT OF MATHEMATICS, ROYAL HOLLOWAY, UNIVERSITY OF LONDON, EGHAM HILL, EGHAM, SURREY TW20 0EX, UK
E-mail address: James.McKee@rhul.ac.uk

SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, KING'S BUILDINGS, MAYFIELD ROAD, EDINBURGH EH9 3JZ, SCOTLAND, U.K.
E-mail address: C.Smyth@ed.ac.uk