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Length of the Sum and Product of Algebraic Numbers

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Abstract—In the present paper, we consider products of lengths of algebraic numbers whose sum or product is a chosen algebraic number. These products are used to construct a new height function for algebraic numbers. With the help of this function, a metric on the set of all algebraic numbers, which induces the discrete topology, is introduced.

Key words: *algebraic number, length function, height function, Mahler measure, Lehmer conjecture.*

1. INTRODUCTION

Let the symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \overline{\mathbb{Q}}, \overline{\mathbb{Q}}^*$ denote the set of positive integers, the ring of integers, the field of rationals, the field of algebraic numbers, and the multiplicative group of nonzero algebraic numbers, respectively. Denote by $L(\alpha)$ the *length* of an algebraic number α, i.e., the sum of the absolute values of the coefficients in the minimal polynomial of α in $\mathbb{Z}[x]$. The function $L(f)$ is a very convenient height function on the set of all polynomials with complex coefficients, because the inequalities $L(f+g) \le L(f) + L(g)$ and $L(fg) \le L(f)L(g)$ always hold. However, these properties of the length $L(\alpha)$ fail to hold on the set of all algebraic numbers $\overline{\mathbb{Q}}$.

In [1, 2], new height functions for algebraic numbers were considered (these functions use the Mahler measure and the ordinary height of an algebraic number). Applying these functions, one can construct a metric on some quotient groups of the group $\overline{\mathbb{Q}}^*$. Recall that by the *Mahler measure* of an algebraic number α one means the product of the leading coefficient of its minimal polynomial in $\mathbb{Z}[x]$ and of all roots of the polynomial whose absolute value exceeds one. Denote the Mahler measure of α by $M(\alpha)$ and the logarithmic Weil height of a number α by $h(\alpha) = (1/d)\log M(\alpha)$, where $d = \deg \alpha$. Earlier, Schmidt [3] noticed that $h(\alpha/\beta)$ is a metric (on the quotient group $\overline{\mathbb{Q}}^*/\Omega$) defining a distance between $\alpha\Omega$ and $\beta\Omega$, where Ω is the multiplicative group of all roots of unity. We had shown in [1] how to construct a metric on the same set by using the so-called metric Mahler measure. The topology on $\overline{\mathbb{Q}}^*/\Omega$ thus obtained is discrete if and only if the Lehmer conjecture[1] is valid. Later on, the same idea was used to construct a metric height based on the ordinary height of an algebraic number (see [2, 4]).

The aim of the present paper is to construct a metric by using the length L. The construction seems to be a most natural one, because the distance is introduced on $\overline{\mathbb{Q}}$ rather than on some quotient group of $\overline{\mathbb{Q}}^*$. Indeed, for any $\alpha \in \overline{\mathbb{Q}}$, we introduce $\mathcal{L}(\alpha)$ by the formula

$$\mathcal{L}(\alpha) = \min L(\alpha_1)L(\alpha_2)\cdots L(\alpha_m),$$

[1] *Translator's note*: This conjecture claims that the set of nontrivial values of the Mahler measure is bounded away from 1.

where the minimum is taken over any $m \in \mathbb{N}$ and any $\alpha_1, \alpha_2, \dots, \alpha_m \in \overline{\mathbb{Q}}$ such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = \alpha.$$

One can readily see that, in this case, the function

$$\varrho(\alpha, \beta) = \log \mathcal{L}(\alpha - \beta)$$

defines a distance on $\overline{\mathbb{Q}}$. It is clear that this metric induces a discrete topology, because the distance between two distinct algebraic numbers is $\geq \log 2$.

It is sometimes more convenient to consider representations of α in the form of products of algebraic numbers rather than in the form of their sums. To define a distance function, we subtract one from the length and define a multiplicative analog of this metric on the quotient group $\overline{\mathbb{Q}}^*/\mathcal{U}$, where $\mathcal{U}$ is a multiplicative group of all roots of unity of degree 2^m, $m = 0, 1, 2, \dots$. Let us define the value $\mathcal{L}^*(\alpha)$ by the formula

$$\mathcal{L}^*(\alpha) = \min\bigl(L(\alpha_1) - 1\bigr)\bigl(L(\alpha_2) - 1\bigr) \cdots \bigl(L(\alpha_m) - 1\bigr),$$

where the minimum is taken over all $m \in \mathbb{N}$ and $\alpha_1, \alpha_2, \dots, \alpha_m \in \overline{\mathbb{Q}}$ such that $\alpha_1\alpha_2 \cdots \alpha_m = \alpha$. In this case, the function

$$\varrho^*(\alpha\mathcal{U}, \beta\mathcal{U}) = \log \mathcal{L}^*\left(\frac{\alpha}{\beta}\right)$$

defines a distance on $\overline{\mathbb{Q}}^*/\mathcal{U}$. (Indeed, the triangle inequality is obvious, and the condition $L(\alpha) = 2$ is equivalent to the condition $\alpha \in \mathcal{U}$.) It is of importance that this metric also induces the discrete topology, because the distance between two distinct cosets $\alpha\mathcal{U}$ and $\beta\mathcal{U}$, where $\alpha, \beta \neq 0$ and $\alpha/\beta \notin \mathcal{U}$, is always $\geq \log 2$.

2. MAIN RESULTS

It is clear that $\mathcal{L}(\alpha) \leq L(\alpha)$ and $\mathcal{L}^*(\alpha) \leq L(\alpha) - 1$. How small can the quantities $\mathcal{L}(\alpha)$ and $\mathcal{L}^*(\alpha)$ be? The example given by the number $\beta = (-2)^{-1/d} - 1/N$, where $d \in \mathbb{N}$ and N is a large odd positive integer, shows that

$$\mathcal{L}(\beta) \leq L((-2)^{-1/d})L\left(-\frac{1}{N}\right) = 3(N+1).$$

Moreover, $L(\beta) = 2(N+1)^d + N^d$, and hence the number $\mathcal{L}(\alpha)$ can be significantly less than $L(\alpha)$. The following theorem gives bounds for the quantities $\mathcal{L}(\alpha)$ and $\mathcal{L}^*(\alpha)$.

Theorem 1. *If α is an algebraic number of degree d, then*

$$L(\alpha)^{1/d} \leq \mathcal{L}(\alpha) \leq L(\alpha), \qquad \bigl(L(\alpha) - 1\bigr)^{1/d} \leq \mathcal{L}^*(\alpha) \leq L(\alpha) - 1.$$

Moreover, $\mathcal{L}(\alpha) \geq \lceil\overline{\alpha}\rceil + 1$ and $\mathcal{L}^*(\alpha) \geq \lceil\overline{\alpha}\rceil$.

Here $\lceil\overline{\alpha}\rceil$ stands for the maximal absolute value among those of all conjugates of the number α over the field $\mathbb{Q}$.

As was already noted above, the inequalities $\mathcal{L}(\alpha) \leq L(\alpha)$ and $\mathcal{L}^*(\alpha) \leq L(\alpha) - 1$ are obvious. The inequality $\mathcal{L}(\alpha) \geq L(\alpha)^{1/d}$ follows from the first part of the following theorem.

Theorem 2. *If $m \in \mathbb{N}$, $\alpha_1, \alpha_2, \dots, \alpha_m \in \overline{\mathbb{Q}}$, and $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_m$ is a number of degree d, then*

$$L(\alpha_1 + \alpha_2 + \dots + \alpha_m)^{1/d} \le L(\alpha_1)L(\alpha_2)\cdots L(\alpha_m).$$

If $\alpha = \alpha_1\alpha_2\cdots\alpha_m$ is a number of degree d, then

$$2^{1-1/d}L(\alpha_1\alpha_2\cdots\alpha_m)^{1/d} \le L(\alpha_1)L(\alpha_2)\cdots L(\alpha_m).$$

The proofs of Theorems 1 and 2 are presented in Sec. 4.

Note that if $d = 1$, i.e., $\alpha \in \mathbb{Q}$, then it follows from Theorem 1 that $\mathcal{L}(\alpha) = L(\alpha)$ and $\mathcal{L}^*(\alpha) = L(\alpha) - 1$. For $d = 1$, Theorem 2 implies the following corollary.

Corollary. *If a number $r \in \mathbb{Q}$ is represented in the form of a sum or a product of arbitrary algebraic numbers, then the product of their lengths does not exceed the length of the number r.*

We face an interesting question: To what extent are the bounds

$$\mathcal{L}(\alpha) \ge L(\alpha)^{1/d} \qquad \text{and} \qquad \mathcal{L}^*(\alpha) \ge (L(\alpha) - 1)^{1/d}$$

sharp? Our example of the number β shows that if the inequality $\mathcal{L}(\alpha) \ge c_d L(\alpha)^{1/d}$ holds, then $c_d \le 3^{1-1/d}$. Moreover, if d is a power of 2, then the polynomial $x^d + 1$ is irreducible in the ring $\mathbb{Z}[x]$. For each of the roots $\zeta \in \mathcal{U}$ of this polynomial, we have the inequality $\mathcal{L}(\zeta) \le L(\zeta) = 2$. Hence $c_d \le 2^{1-1/d}$ if d is a power of 2. It is also clear that

$$\mathcal{L}^*(\zeta) \le L(\zeta) - 1 = 1 \qquad \text{and} \qquad \mathcal{L}^*(\zeta) \ge (L(\zeta) - 1)^{1/d} = 1.$$

Thus, the inequality $\mathcal{L}^*(\alpha) \ge (L(\alpha) - 1)^{1/d}$ in Theorem 1 is sharp if d is an integer power of 2. In the other cases, the example $\gamma = 2^{1/d}$ shows that $c_d^* \le 2^{1-1/d}$ if the inequality $\mathcal{L}^*(\alpha) \ge c_d^*(L(\alpha) - 1)^{1/d}$ holds. Apparently, the inequality $\mathcal{L}(\alpha) \ge c_d L(\alpha)^{1/d}$, where $c_d = 2^{1-1/d}$ for $d = 2^s$ and $c_d = 3^{1-1/d}$ for $d \ne 2^s$, and also the inequality $\mathcal{L}^*(\alpha) \ge c_d^*(L(\alpha) - 1)^{1/d}$, where $c_d^* = 2^{1-1/d}$ for $d \ne 2^s$, are valid.

3. AUXILIARY RESULTS

Lemma 1. *If $\alpha \in \overline{\mathbb{Q}}^*$, then $M(\alpha) \le L(\alpha) - 1$.*

Proof. Let

$$a_d x^d + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$$

be the basic polynomial of the number α. By the Gonçalves inequality (see, e.g., [5, p. 244] or [6], where a simpler version of this inequality is presented), we have

$$\big(M(\alpha) + |a_d a_0| M(\alpha)^{-1}\big)^2 - 2|a_d a_0| = M(\alpha)^2 + |a_d a_0|^2 M(\alpha)^{-2} \le |a_d|^2 + \dots + |a_1|^2 + |a_0|^2.$$

We write $S = |a_{d-1}|^2 + \dots + |a_1|^2$. It follows from the inequality that

$$M(\alpha) \le \frac{1}{2}\big(\sqrt{S + (|a_d| + |a_0|)^2} + \sqrt{S + (|a_d| - |a_0|)^2}\,\big).$$

By the inequality

$$\sqrt{S} \le |a_{d-1}| + \dots + |a_1| = L(\alpha) - |a_d| - |a_0|,$$

we can readily see that the first square root does not exceed $L(\alpha)$ and the other does not exceed the value

$$\sqrt{S} + \big||a_d| - |a_0|\big| \le L(\alpha) - |a_d| - |a_0| + \big||a_d| - |a_0|\big| = L(\alpha) - 2\min\big\{|a_d|, |a_0|\big\}.$$

Thus,

$$M(\alpha) \le L(\alpha) - \min\big\{|a_d|, |a_0|\big\} \le L(\alpha) - 1. \quad \square$$

Lemma 2. *If $r \in \mathbb{Q}$ and if $\alpha \in \overline{\mathbb{Q}}$ is of degree d, then*

$$L(r+\alpha)^{1/d} \le L(r)L(\alpha)^{1/d} \qquad \textit{and} \qquad L(r\alpha)^{1/d} \le M(r)L(\alpha)^{1/d}.$$

Proof. Since $L(0) = M(0) = 1$, the inequalities are obvious if $r = 0$ or $\alpha = 0$. Let $r = u/v \ne 0$, where u and v are coprime integers. In this case, $L(r) = |u| + |v|$ and $M(r) = \max\{|u|, |v|\}$.

If $a_d x^d + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$ is the basic polynomial of the number α, then $u/v + \alpha$ is a root of the polynomial

$$Q(x) = a_d(vx-u)^d + \dots + a_1 v^{d-1}(vx-u) + a_0 v^d \in \mathbb{Z}[x].$$

Since $\deg(u/v+\alpha) = \deg\alpha = d$, it follows that the basic polynomial of the number $u/v+\alpha$ is either $Q(x)$ or a divisor of $Q(x)$ of degree d. Therefore,

$$\begin{aligned} L\Big(\frac{u}{v}+\alpha\Big) &\le L(Q) \le |a_d|\big(|v|+|u|\big)^d + \dots + |a_1|\,|v|^{d-1}\big(|v|+|u|\big) + |a_0|\,|v|^d \\ &\le \big(|a_d| + \dots + |a_1| + |a_0|\big)\big(|v|+|u|\big)^d = L(\alpha)L\Big(\frac{u}{v}\Big)^d. \end{aligned}$$

Extracting the root of degree d, we obtain the first inequality.

In the other case, the number $u\alpha/v$ is a root of the polynomial

$$R(x) = a_d v^d x^d + \dots + a_1 v u^{d-1} x + a_0 u^d \in \mathbb{Z}[x].$$

Hence, as above,

$$\begin{aligned} L\Big(\frac{u\alpha}{v}\Big) &\le L(R) \le |a_d|\,|v|^d + \dots + |a_1|\,|v|\,|u|^{d-1} + |a_0|\,|u|^d \\ &\le \big(|a_d| + \dots + |a_1| + |a_0|\big)\max\big\{|u|, |v|\big\}^d = L(\alpha)M\Big(\frac{u}{v}\Big)^d. \end{aligned}$$

Extracting the root of degree d, we obtain the other inequality. $\square$

Lemma 3. *If $\alpha \in \overline{\mathbb{Q}}$, then $L(\alpha) \ge \lceil\overline{\alpha}\rceil + 1$.*

Proof. If α and α' are conjugate algebraic numbers, i.e., if both numbers are roots of the basic polynomial for α, $a_d x^d + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$, then $L(\alpha) = L(\alpha')$. Therefore, it suffices to prove the inequality

$$L(\alpha) \ge |\alpha| + 1.$$

This is obvious for $\alpha = 0$ and for $0 < |\alpha| \le 1$. Suppose that $|\alpha| > 1$. It follows from the relation $-a_d = a_{d-1}/\alpha + a_{d-2}/\alpha^2 + \dots + a_0/\alpha^d$ that

$$|a_d| \le \left|\frac{a_{d-1}}{\alpha}\right| + \left|\frac{a_{d-2}}{\alpha^2}\right| + \dots + \left|\frac{a_0}{\alpha^d}\right| \le \frac{|a_{d-1}| + \dots + |a_0|}{|\alpha|}.$$

This implies that

$$L(\alpha) = |a_d| + |a_{d-1}| + \dots + |a_0| \ge |a_d| + |a_d|\,|\alpha| \ge |\alpha| + 1.$$

The proof of the lemma is complete. $\square$

4. LOWER BOUNDS FOR PRODUCTS OF LENGTHS

Let us begin with the proof of the inequality

$$L(\alpha_1+\alpha_2+\dots+\alpha_m)^{1/d} \le L(\alpha_1)L(\alpha_2)\cdots L(\alpha_m),$$

where $\alpha=\alpha_1+\alpha_2+\dots+\alpha_m$ is an algebraic number of degree d. If at least one of the numbers $\alpha_1,\alpha_2,\dots,\alpha_m$ is rational, then, using the inequality $L(r+\alpha)^{1/d} \le L(r)L(\alpha)^{1/d}$ given in Lemma 2, we can apply the induction on m. We therefore, assume that $m \ge 2$ and $\alpha_1,\alpha_2,\dots,\alpha_m \in \overline{\mathbb{Q}}\setminus\mathbb{Q}$. Denote the degrees of the numbers $\alpha_1,\alpha_2,\dots,\alpha_m$ by $d_1,d_2,\dots,d_m \ge 2$. Using the well-known inequality $L(\alpha) \le 2^d M(\alpha)$ and the inequality

$$h(\alpha_1+\alpha_2+\dots+\alpha_m) \le h(\alpha_1)+h(\alpha_2)+\dots+h(\alpha_m)+\log m$$

(see, e.g., [7, Lemma 3.7]), we see that

$$\begin{aligned} L(\alpha_1+\alpha_2+\dots+\alpha_m)^{1/d} &\le 2M(\alpha_1+\alpha_2+\dots+\alpha_m)^{1/d} \\ &\le 2mM(\alpha_1)^{1/d_1}M(\alpha_2)^{1/d_2}\cdots M(\alpha_m)^{1/d_m}. \end{aligned}$$

However, by Lemma 1,

$$M(\alpha_j)^{1/d_j} \le M(\alpha_j)^{1/2} \le \bigl(L(\alpha_j)-1\bigr)^{1/2} \le \frac{L(\alpha_j)}{2}$$

for any $j=1,2,\dots,m$. Hence

$$L(\alpha_1+\alpha_2+\dots+\alpha_m)^{1/d} \le \frac{2m}{2^m}L(\alpha_1)L(\alpha_2)\cdots L(\alpha_m),$$

which proves the desired inequality, since $2m \le 2^m$ for $m \ge 2$.

To prove the inequality

$$2^{1-1/d}L(\alpha_1\alpha_2\cdots\alpha_m)^{1/d} \le L(\alpha_1)L(\alpha_2)\cdots L(\alpha_m),$$

where $\alpha=\alpha_1\alpha_2\cdots\alpha_m$ is an algebraic number of degree d, we note first that this inequality is obvious if $\alpha=0$ or $m=1$. If $\alpha\ne 0$, then all numbers $\alpha_1,\alpha_2,\dots,\alpha_m$ are also nonzero. If at least one of the numbers $\alpha_1,\alpha_2,\dots,\alpha_m$, say, α_m, is rational, then

$$L(\alpha_1\alpha_2\cdots\alpha_m)^{1/d} \le M(\alpha_m)L(\alpha_1\cdots\alpha_{m-1})^{1/d} \le \bigl(L(\alpha_m)-1\bigr)L(\alpha_1\alpha_2\cdots\alpha_{m-1})^{1/d}$$

by Lemmas 1 and 2. Hence

$$2^{1-1/d}L(\alpha_1\alpha_2\cdots\alpha_m)^{1/d} < 2^{1-1/d}L(\alpha_m)L(\alpha_1\alpha_2\cdots\alpha_{m-1})^{1/d},$$

and the desired inequality can readily be obtained by induction on m. We therefore, assume that $m\ge 2$ and $\alpha_1,\alpha_2,\dots,\alpha_m$ are algebraic numbers of degrees $d_1,d_2,\dots,d_m \ge 2$, respectively. In this case, using the inequalities $L(\alpha)\le 2^d M(\alpha)$ and

$$h(\alpha_1\alpha_2\cdots\alpha_m) \le h(\alpha_1)+h(\alpha_2)+\dots+h(\alpha_m)$$

(see [7, Property 3.3]), we can readily see that

$$L(\alpha_1\alpha_2\cdots\alpha_m)^{1/d} \le 2M(\alpha_1\alpha_2\cdots\alpha_m)^{1/d} \le 2M(\alpha_1)^{1/d_1}M(\alpha_2)^{1/d_2}\cdots M(\alpha_m)^{1/d_m}.$$

As above,

$$M(\alpha_j)^{1/d_j} \le M(\alpha_j)^{1/2} \le \bigl(L(\alpha_j)-1\bigr)^{1/2}$$

for any $j=1,2,\dots,m$, and, therefore,

$$L(\alpha_1\alpha_2\cdots\alpha_m)^{1/d} \le 2\Bigl(\bigl(L(\alpha_1)-1\bigr)\bigl(L(\alpha_2)-1\bigr)\cdots\bigl(L(\alpha_m)-1\bigr)\Bigr)^{1/2}$$

for any $m \ge 2$ and any $\alpha_1,\alpha_2,\dots,\alpha_m \in \overline{\mathbb{Q}}\setminus\mathbb{Q}$. The desired inequality follows now from the estimates

$$\bigl(L(\alpha_j)-1\bigr)^{1/2} \le \frac{L(\alpha_j)}{2},$$

where $j=1,2,\dots,m$, $m\ge 2$, because

$$2^{1-1/d}\cdot 2\cdot 2^{-m} = 2^{2-m-1/d} < 1.$$

Thus, the proof of Theorem 2 is complete. It follows from the first inequality of Theorem 2 and from the definition of $\mathcal{L}(\alpha)$ that $L(\alpha)^{1/d} \le \mathcal{L}(\alpha) \le L(\alpha)$. The inequality $\mathcal{L}^*(\alpha) \le L(\alpha)-1$ is obvious. Since $\mathcal{L}^*(\alpha\zeta)=\mathcal{L}^*(\alpha)$ for any $\zeta\in\mathcal{U}$, it follows that, in order to prove the inequality $\mathcal{L}^*(\alpha) \ge (L(\alpha)-1)^{1/d}$, it suffices to establish the inequality

$$\bigl(L(\alpha_1\alpha_2\cdots\alpha_m)-1\bigr)^{1/d} \le \bigl(L(\alpha_1)-1\bigr)\bigl(L(\alpha_2)-1\bigr)\cdots\bigl(L(\alpha_m)-1\bigr)$$

for any $\alpha_1,\alpha_2,\dots,\alpha_m\notin\mathcal{U}$. Here $\alpha=\alpha_1\alpha_2\cdots\alpha_m$ is an algebraic number of degree d. This inequality is obvious if $\alpha=0$ or $m=1$. Moreover, as above, without loss of generality we can assume that all numbers $\alpha_1,\alpha_2,\dots,\alpha_m$ are irrational, because if $\alpha_m\in\mathbb{Q}$, then

$$\bigl(L(\alpha_1\alpha_2\cdots\alpha_m)-1\bigr)^{1/d} < L(\alpha_1\alpha_2\cdots\alpha_m)^{1/d} \le \bigl(L(\alpha_m)-1\bigr)L(\alpha_1\alpha_2\cdots\alpha_{m-1})^{1/d}.$$

However, the inequality $L(\alpha_1\alpha_2\cdots\alpha_m)^{1/d} \le 2((L(\alpha_1)-1)\cdots(L(\alpha_m)-1))^{1/2}$ had been already proved under these assumptions. Since $\alpha_j\notin\mathcal{U}\cup\{0\}$, it follows that $L(\alpha_j)\ge 3$ for any index $j=1,2,\dots,m$. Hence $(L(\alpha_j)-1)^{1/2} \le (L(\alpha_j)-1)/\sqrt{2}$, and

$$\bigl(L(\alpha_1\alpha_2\cdots\alpha_m)-1\bigr)^{1/d} < L(\alpha_1\alpha_2\cdots\alpha_m)^{1/d} \le 2^{1-m/2}\bigl(L(\alpha_1)-1\bigr)\cdots\bigl(L(\alpha_m)-1\bigr).$$

This proves the desired inequality, because $m\ge 2$.

To complete the proof of Theorem 1, it remains to show that $\mathcal{L}(\alpha) \ge \lceil\overline{\alpha}\rceil + 1$ and $\mathcal{L}^*(\alpha) \ge \lceil\overline{\alpha}\rceil$. If

$$\mathcal{L}(\alpha) = L(\alpha_1)L(\alpha_2)\cdots L(\alpha_m), \qquad \text{where} \quad \alpha=\alpha_1+\alpha_2+\dots+\alpha_m,$$

then the assumption that every number conjugate to α, say, α', can be represented as the sum of numbers conjugate to $\alpha_1,\alpha_2,\dots,\alpha_m$, respectively, implies the relation $\mathcal{L}(\alpha)=\mathcal{L}(\alpha')$. Similarly, $\mathcal{L}^*(\alpha)=\mathcal{L}^*(\alpha')$.

Therefore, to prove the first inequality, it suffices to show that $\mathcal{L}(\alpha)\ge|\alpha|+1$. To this end, we use Lemma 3 and apply the representation $\alpha=\alpha_1+\alpha_2+\dots+\alpha_m$. We obtain

$$\begin{aligned}\mathcal{L}(\alpha) = L(\alpha_1)L(\alpha_2)\cdots L(\alpha_m) &\ge \bigl(|\alpha_1|+1\bigr)\bigl(|\alpha_2|+1\bigr)\cdots\bigl(|\alpha_m|+1\bigr)\\ &\ge |\alpha_1+\alpha_2+\dots+\alpha_m|+1 = |\alpha|+1.\end{aligned}$$

The inequality $\mathcal{L}^*(\alpha) \ge \lceil\overline{\alpha}\rceil$ follows from the inequality $\mathcal{L}^*(\alpha)\ge|\alpha|$, which can be established in a similar way; namely, if

$$\mathcal{L}^*(\alpha) = (L(\alpha_1)-1)(L(\alpha_2)-1)\cdots(L(\alpha_m)-1), \qquad \text{where} \quad \alpha=\alpha_1\alpha_2\cdots\alpha_m,$$

then

$$\mathcal{L}^*(\alpha) = \bigl(L(\alpha_1)-1\bigr)\bigl(L(\alpha_2)-1\bigr)\cdots\bigl(L(\alpha_m)-1\bigr) \ge |\alpha_1|\,|\alpha_2|\cdots|\alpha_m| = |\alpha|.$$

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REFERENCES

1. A. Dubickas and C. J. Smyth, “On the metric Mahler measure,” *J. Number Theory*, **86** (2001), 368–387.
2. A. Dubickas and C. J. Smyth, “On metric heights,” *Period. Math. Hungar.*, **42** (2003), 135–155.
3. W. Schmidt, “Heights of points on subvarieties of $\mathbb{G}_m^n$,” *Contemp. Math.*, **210** (1998), 97–99.
4. A. Dubickas, “Asymptotic density of surds with stable height,” *Acta Appl. Math.*, **78** (2003), 99–102.
5. A. Schinzel, *Polynomials with Special Regard to Reducibility*, Cambridge Univ. Press, Cambridge, 2000.
6. N. I. Fel′dman, *Approximations of Algebraic Numbers* [in Russian], Moskov. Gos. Univ., Moscow, 1981.
7. M. Waldschmidt, *Diophantine Approximation on Linear Algebraic Groups*, Springer-Verlag, Berlin–New York, 2000.

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