MAHLER MEASURES OF POLYNOMIALS THAT ARE SUMS OF A BOUNDED NUMBER OF MONOMIALS

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Abstract. We study Laurent polynomials in any number of variables that are sums of at most k monomials. We first show that the Mahler measure of such a polynomial is at least $h/2^{k-2}$, where h is the height of the polynomial. Next, restricting to such polynomials having integer coefficients, we show that the set of logarithmic Mahler measures of the elements of this restricted set is a closed subset of the nonnegative real line, with 0 being an isolated point of the set. In the final section, we discuss the extent to which such an integer polynomial of Mahler measure 1 is determined by its k coefficients.

1. Statement of results

For a polynomial $f(z) \in \mathbb{C}[z]$, we denote by $m(f)$ its *logarithmic Mahler measure*

$$m(f) = \int_0^1 \log |f(e^{2\pi i t})| dt, \tag{1}$$

and write $M(f) = \exp(m(f))$ for the (classical) *Mahler measure* of f. Although first defined by D.H. Lehmer [7], its systematic study was initiated by Kurt Mahler [8, 9, 10].

Let $h(f)$ denote the *height* of f (the maximum modulus of its coefficients). Our first result relates these two quantities.

Theorem 1. *For an integer $k \geq 2$, let*

$$f(x) = a_1 z^{n_1} + \cdots + a_{k-1} z^{n_{k-1}} + a_k \in \mathbb{C}[z] \text{ with } n_1 > n_2 > \cdots > n_{k-1} > 0 \tag{2}$$

be a nonzero polynomial. Then

$$M(f) \geq \frac{h(f)}{2^{k-2}}.$$

The example $(z+1)^{k-1}$ shows that the constant $1/2^{k-2}$ in this inequality cannot be improved to any number bigger than $1/\binom{k-1}{\lfloor (k-2)/2 \rfloor}$ (which $\sim \sqrt{2\pi k}/2^k$ as $k \to \infty$).

The inequality for the special case $(n_1, n_2 \ldots, n_{k-1}) = (k-1, k-2, \ldots, 1)$ (i.e., a polynomial of degree $k-1$) follows from a result of Mahler [10, equation (6)].

In the other direction, we have from (1) the trivial bound $M(f) \leq k h(f)$.

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Corollary 1. *Given $k \geq 1$, there are only finitely many possible choices for integers $a_1, \ldots, a_k$ such that $M(f) = 1$ for some $f(x) = a_1 z^{n_1} + \cdots + a_{k-1} z^{n_{k-1}} + a_k$ and any choice of distinct integer exponents $n_1, n_2, \ldots, n_{k-1}$.*

This corollary leaves open the question of whether, for fixed $a_1, \ldots, a_k$, the number of choices for the exponents n_i is finite or infinite. This is discussed in Section 5.

Theorem 1 in fact holds for Laurent polynomials in several variables, as the next result states. Since it follows quite easily from the one-variable case, we decided to relegate this general case to a corollary. Recall that the logarithmic Mahler measure in the general case is defined as

$$m(F) = \int_0^1 \cdots \int_0^1 \log |F(e^{2\pi i t_1}, \ldots, e^{2\pi i t_k})| dt_1 \cdots dt_k \tag{3}$$

Again, $M(F) := \exp(m(F))$.

In [1], David Boyd studied the set $\mathcal{L}$ of Mahler measures of polynomials F in any number of variables having integer coefficients. He conjectured that $\mathcal{L}$ is a closed subset of $\mathbb{R}$. Our Theorem 2 below is a result in the direction of this conjecture, but where we restrict the polynomials F under consideration to be the sum of at most k monomials. In [13, Theorem 3], the second author proved another restricted closure result of this kind, where the restriction was, instead, to integer polynomials F of bounded length (sum of the moduli of its coefficients).

Boyd's conjecture is a far-reaching generalisation of a question of D.H. Lehmer [7], who asked whether there exists an absolute constant $C > 1$ with the property that, for integer polynomials f in one variable, either $M(f) = 1$ or $M(f) \geq C$.

We now state our generalisation of Theorem 1. In it, we write $\mathbf{z}_\ell = (z_1, \ldots, z_\ell)$.

Corollary 2. *Let $F(\mathbf{z}_\ell) \in \mathbb{C}[\mathbf{z}_\ell]$ be a nonzero Laurent polynomial in $\ell \geq 1$ variables that is the sum of k monomials. Then*

$$M(F) \geq \frac{h(F)}{2^{k-2}}.$$

Corollary 2 is an essential ingredient in our next result. For this, we fix $k \geq 1$ and consider the set $\mathcal{H}_k$ of Laurent polynomials $F(\mathbf{z}_\ell) = F(z_1, \ldots, z_\ell)$ for all $\ell \geq 1$ with integer coefficients that are the sum of at most k monomials. So such an F is of the form

$$F(\mathbf{z}_\ell) = \sum_{\mathbf{j} \in J} c(\mathbf{j}) \mathbf{z}_\ell^{\mathbf{j}},$$

where $J \subset \mathbb{Z}^\ell$ has k elements, $\mathbf{z}_\ell^{\mathbf{j}} = z_1^{j_1} \cdots z_\ell^{j_\ell}$, where $\mathbf{j} = (j_1, \ldots, j_\ell)$, and the $c(\mathbf{j})$'s are integers, some of which could be 0. The number of variables ℓ defining F is unspecified, and can be arbitrarily large. We let $m(\mathcal{H}_k)$ denote the set $\{m(F) : F \in \mathcal{H}_k\}$.

Theorem 2. *The set $m(\mathcal{H}_k)$ is a closed subset of $\mathbb{R}_{\geq 0}$. Furthermore, 0 is an isolated point of $m(\mathcal{H}_k)$.*

In fact the isolation of 0 in $m(\mathcal{H}_k)$ has been essentially known for some time, indeed with explicit lower bounds for the size of the gap between 0 and the rest of the set. The first such bound was given for one-variable polynomials by Dobrowolski, Lawton and Schinzel [3]. This was improved by Dobrowolski in [4] and later improved further in [5], where it was shown that for noncyclotomic $f \in \mathbb{Z}[z]$

$$M(f) \geq 1 + \frac{1}{\exp(a3^{\lfloor (k-2)/4 \rfloor} k^2 \log k)},$$

where $a < 0.785$. Arguing as in the proof of Corollary 2 below shows that the gap holds for polynomials in several variables too, and so applies to all $m(F)$ in $m(\mathcal{H}_k) \backslash \{0\}$.

2. Proof of Theorem 1 and Corollary 1

Proof. We first prove the theorem by induction under the restriction that all $a_1, \ldots, a_k$ are assumed to be nonzero. We employ two well-known facts:

(i) $M(f) = M(f^*)$ where $f^*(z) = z^n f(z^{-1})$, with $n = \deg f$. This immediately follows from (1).
(ii) $M(f) \geq M(\frac{1}{n} f')$. This was proved by Mahler in [9] .

For the base case $k = 2$ of our induction, we have $M(f) = \max\{|a_1|, |a_2|\} = h(f)$, as required.

Suppose now that the conclusion of the (restricted) theorem is true for some $k \geq 2$, and suppose that f has $k+1$ nonzero terms, that is, $f(x) = a_1 z^{n_1} + \cdots + a_k z^{n_k} + a_{k+1}$. Then $f^*(z) = a_{k+1} z^{n_1} + a_k z^{n_1 - n_k} + \cdots + a_1$. Because the a_i are assumed nonzero, both f and f^* have degree n_1. Suppose that $h(f) = |a_i|$ for some i, $(1 \leq i \leq k+1)$. Then $h(\frac{1}{n_1} f') \geq \frac{n_i}{n_1} h(f)$, and $h(\frac{1}{n_1}(f^*)') \geq \frac{n_1 - n_i}{n_1} h(f)$. Clearly $\max\{\frac{n_i}{n_1}, \frac{n_1 - n_i}{n_1}\} \geq \frac{1}{2}$, with f' and $(f^*)'$ having k terms each. Hence, by (i), (ii) and the induction hypothesis

$$M(f) \geq \max\left\{ M\left(\tfrac{1}{n_1} f'\right), M\left(\tfrac{1}{n_1}(f^*)'\right)\right\} \geq \tfrac{1}{2}\frac{h(f)}{2^{k-2}},$$

which completes the inductive step, and the induction argument.

Now we can do the general case. If some of the a_i can be 0, then $f(z)$ is of the form $z^j f_1(z)$, where $j \geq 0$ and f_1 is of the form (2), but with k_1 nonzero terms, where $0 < k_1 \leq k$. Then, using (1),

$$M(f(z)) = M(z^j f_1(z)) = M(f_1(z))$$

and, since $h(f_1) = h(f)$ we have

$$M(f) = M(f_1) \geq \frac{h(f_1)}{2^{k_1 - 2}} \geq \frac{h(f)}{2^{k-2}}.$$

□

Corollary 1 now follows straight from the theorem, because any such f must have height at most 2^{k-2}, giving at most $(2^{k-1}+1)^k$ possible choices for $a_1, \ldots, a_k$.

3. Proof of Corollary 2

For the Proof of Corollary 2, we need the following simple result.

Lemma 1. *Let $\mathbf{r}_n = (1, n, n^2, \ldots, n^{\ell-1}) \in \mathbb{Z}^\ell$. Then for any finite set V of nonzero vectors in $\mathbb{R}^\ell$ there is an integer N such that for each $n > N$ the vector $\mathbf{r}_n$ is not orthogonal to any vector $\mathbf{v} \in V$.*

Proof. Write $\mathbf{v} \in V$ in the form $\mathbf{v} = (v_1, \ldots, v_j, 0, \ldots, 0)$ say, where $v_j \neq 0$ and $j \leq \ell$. If $j = 1$ then $|\mathbf{v} \cdot \mathbf{r}_n| = |v_1| > 0$, so assume $j \geq 2$. Then

$$\begin{aligned} |\mathbf{v} \cdot \mathbf{r}_n| &= \left| \sum_{i=1}^{j} v_i n^{i-1} \right| \\ &\geq |v_j| \left(n^{j-1} - n^{j-2} \left(\sum_{i=1}^{j-1} |v_i/v_j| \right) \right) \\ &> 0 \text{ for } n > \sum_{i=1}^{j-1} |v_i/v_j|, = N_{\mathbf{v}} \text{ say .} \end{aligned}$$

Now take $N = \max_{\mathbf{v} \in V} N_{\mathbf{v}}$. □

Following [12], given a fixed integer $s \geq 1$ and a polynomial F in s variables, $\ell \geq 0$ and an $\ell \times s$ matrix $A = (a_{ij}) \in \mathbb{Z}^{\ell \times s}$, define the s-tuple $\mathbf{z}_\ell^A$ by

$$\mathbf{z}_\ell^A = (z_1, \ldots, z_\ell)^A = (z_1^{a_{11}} \cdots z_\ell^{a_{\ell 1}}, \ldots, z_1^{a_{1s}} \cdots z_\ell^{a_{\ell s}})$$

(which is $(1, 1, \ldots, 1) \in \mathbb{Z}^s$ when $\ell = 0$) and $F_A(\mathbf{z}_\ell) = F(\mathbf{z}_\ell^A)$, a polynomial in ℓ variables $z_1, \ldots, z_\ell$. Then $m(F_A)$ is defined by (3) with $F = F_A$ and $s = \ell$. Denote by $\mathcal{P}(F)$ the set $\{F_A : A \in \mathbb{Z}^{\ell \times s}, \ell \geq 0\}$, and by $\mathcal{M}(F)$ the set $\{m(F_A) : F_A \in \mathcal{P}(F), F_A \neq 0\}$.

In the case $\ell = 1$, and with A replaced by $\mathbf{r} = (r_1, \ldots, r_s)$, we have $z^{\mathbf{r}} = (z^{r_1}, \ldots, z^{r_s})$ and $F_{\mathbf{r}}(z) = F(z^{\mathbf{r}})$.

We also need the following.

Proposition 1. *Let $\ell \geq 1$, $n \geq 1$, and $\mathbf{r}_n = (1, n, n^2, \ldots, n^{\ell-1})$, as in Lemma 1. Then for any Laurent polynomial $F(\mathbf{z}_\ell)$ in ℓ variables $\mathbf{z}_\ell = (z_1, \ldots, z_\ell)$ we have $m(F_{\mathbf{r}_n}(z)) \to m(F(\mathbf{z}_\ell))$ as $n \to \infty$. Furthermore, for n sufficiently large, $h(F_{\mathbf{r}_n}) = h(F)$.*

Proof. The first part follows from results of Boyd [2, p. 118] and Lawton [6] respectively; see also [13, Lemma 13 and Proposition 14]. Next, note that F is the sum of k monomials of the form $c(\mathbf{j})\mathbf{z}_\ell^{\mathbf{j}}$, so that $F_{\mathbf{r}}$ is the sum of k monomials of the form $c(\mathbf{j})(z^{\mathbf{r}})^{\mathbf{j}} = a_i z^{t_i}$ for some i, where $\mathbf{j} \in J$. We now take $\mathbf{r} = \mathbf{r}_n$, and apply Lemma 1 to the set V of all nonzero differences $\mathbf{j} - \mathbf{j}'$ between elements of J. The lemma then guarantees that, for n sufficiently large, the t_i are distinct, so that $F_{\mathbf{r}}$ and F have the same coefficients. In particular, $h(F_{\mathbf{r}}) = h(F)$. □

Proof of Corollary 2. This now follows from Theorem 1, using the fact, from Proposition 1, that, for any $\varepsilon > 0$, F has the same height and the same number of monomials as some one-variable polynomial $F_{\mathbf{r}}$ with $|m(F_{\mathbf{r}}) - m(F)| < \varepsilon$. □

4. Proof of Theorem 2

Proof of Theorem 2. Throughout, $k \geq 2$ is fixed, while $\ell \geq 0$ can vary. Take any $F \in \mathcal{H}_k$, with $F(\mathbf{z}_\ell) = \sum_{\mathbf{j}\in J} c(\mathbf{j})\mathbf{z}_\ell^{\mathbf{j}}$, say, where J is a k-element subset of $\mathbb{Z}^\ell$. Then $F \in \mathcal{P}(a_1z_1 + \cdots + a_kz_k)$ for some integers a_i, where $\{c(\mathbf{j})\}_{\mathbf{j}\in J} = \{a_i\}_{i=1,\ldots,k}$ as multisets. (Again, some a_i's could be 0.) Conversely, every element of $\mathcal{P}(a_1z_1+\cdots+a_kz_k)$ is a sum of k monomials. (Note that because monomial terms may combine to form a single monomial term, or indeed vanish, the resulting a_i's for some polynomials in $\mathcal{P}(a_1z_1 + \cdots + a_kz_k)$ may be different from the a_i's that we started with. Because of this, the height of some such polynomials may be larger or smaller than the height $\max_{i=1}^k |a_i|$ of $a_1z_1+\cdots+a_kz_k$. This does not matter, however!)

Next, take some bound $B > 0$ and consider all F such that $m(F) \leq B$. Then, by Corollary 2,

$$h(F) \leq 2^{k-2}e^B,$$

so that there are only finitely many choices for the integers a_i. So $m(F)$ belongs to the union – call it U_B – of finitely many sets $\mathcal{M}(a_1z_1+\cdots+a_kz_k) := m(\mathcal{P}(a_1z_1+\cdots+a_kz_k))$, intersected with the interval $[0, B]$. Thus U_B is closed since, by [13, Theorem 1], each set $\mathcal{M}(a_1z_1+\cdots+a_kz_k)$ is closed. Note that the finite number of sets comprising U_B depends on k and on B, but not on F. Finally, we see that $m(\mathcal{H}_k)$ is closed. This is because any convergent sequence in $m(\mathcal{H}_k)$, being bounded, belongs, with its limit point, to U_B for some B.

Finally, to show that 0 is an isolated point of $m(\mathcal{H}_k)$, note that, by [13, Theorem 2] it is an isolated point of every $\mathcal{M}(a_1z_1 + \cdots + a_kz_k)$ that contains 0. Hence, since $0 \in U_B$ for every $B > 0$, it is an isolated point of U_B and therefore also of $m(\mathcal{H}_k)$.

□

5. Products of cyclotomic polynomials that have the same coefficients

In this section we address the question of whether two or more integer polynomials having Mahler measure 1 (and so being products of cyclotomic polynomials $\Phi_n(z)$) can have the same set of k nonzero coefficients. We restrict our attention to the case where all the coefficients a_i are 1. This already indicates what can happen.

Let $k \geq 2$ and define

$$S = \{(n_1,\ldots,n_{k-1}) \in \mathbb{Z}^{k-1} \mid n_1 > n_2 > \cdots > n_{k-1} > 0 \text{ with } \gcd(n_1,\ldots,n_{k-1}) = 1\}.$$

Proposition 2. *For* $\mathbf{n} \in S$ *define* $f_{\mathbf{n}}(x) = z^{n_1} + \cdots + z^{n_{k-1}} + 1$. *Let*

$$S_c = \{\mathbf{n} \in S \mid M(f_{\mathbf{n}}) = 1\}.$$

The set S_c *is finite if and only if* k *is a prime number.*

Since for instance $\Phi_5(z)$ and $\Phi_5(z)\Phi_6(z)$ have the same nonzero coefficients, S_c can, however, contain more than one element for k prime.

Proof. Consider first the case of composite k.
Suppose that $k = st$, where integers s and t are greater than 1. Let $g(z) = \sum_{j=0}^{s-1} z^j$ and

$h(x) = \sum_{j=0}^{t-1} z^j$. If m and l are arbitrary integers greater than 1 and such that $\gcd(m, l) = \gcd(m, t!) = \gcd(l, s!) = 1$ then it is not difficult to check that $g(z^m)h(z^l) = f_{\mathbf{n}}(z)$ for some $\mathbf{n} \in S$. Since $M(g(z^m)) = M(h(z^l)) = 1$, we see that in fact $f_{\mathbf{n}}(z) \in S_{\rm c}$, and so $S_{\rm c}$ is infinite.

Now suppose that $k = p$ is prime.
Let $\mathbf{n} \in S_{\rm c}$ and consider $f_{\mathbf{n}}(z) = z^{n_1} + \cdots + z^{n_{k-1}} + 1$. By Kronecker's Theorem, $f_{\mathbf{n}}$ is a product of cyclotomic polynomials. Further, $f(1) = p$. However the value of a cyclotomic polynomial at 1 is $\Phi_m(1) = 1$ if m is divisible by two distinct primes or $\Phi_m(1) = q$ if m is a power of a single prime q. Thus, for some n, Φ_{p^n} divides f.

We claim that $n = 1$. To show this, we use a theorem of Mann [11] which, in our notation, takes the form of the following lemma.

Lemma 2. *Let $f(z) = \sum_{i=1}^{k-1} a_i z^{n_i} + a_k \in \mathbb{Z}[z]$, with $(n_1, \ldots, n_{k-1}) \in S$ and where the coefficients a_i, $1 \le i \le k$ nonzero. If a cyclotomic polynomial Φ divides f but Φ does not divide any proper subsum of $\sum_{i=1}^{k-1} a_i z^{n_i} + a_k$ then $\Phi = \Phi_q$, where q is squarefree and composed entirely of primes less than or equal to k.*

In the case of $f_{\mathbf{n}}$, a proper subsum defines a polynomial g such that $g(1)$ counts its number of monomials. Hence $g(1) < p$, and consequently g cannot be divisible by Φ_{p^n}. By Lemma 2, p^n is squarefree, so $n = 1$, as claimed.

Next, we need the following result.

Theorem 3 (Dobrowolski [5, Theorem 2 and Corollary 1])**.** *Let $f(z) = \sum_{i=1}^{k} a_i z^{n_i} \in \mathbb{Z}[z]$, $f(0) \neq 0$, be a polynomial with k nonzero coefficients. There are positive constants c_1 and c_2, depending only on k, and polynomials $f_0, f_2 \in \mathbb{Z}[z]$ such that if*

$$\deg f_c \ge \left(1 - \frac{1}{c_1}\right) \deg f$$

then either

$$f(z) = f_0(z^l), \qquad \text{where } \deg f_0 \le c_2, \tag{4}$$

or

$$f(z) = \left(\prod_i \Phi_{q_i}(z^{l_i})\right) f_2(z), \text{ where } \min_i\{l_i\} \ge \max\left\{\frac{1}{2c_1} \deg f, \deg f_2\right\}. \tag{5}$$

Furthermore then $f_2(z) = \pm \sum_{i=j}^{k} a_i z^{n_i}$ for some j with $1 < j \le k$.

In this theorem Φ_q is the q-th cyclotomic polynomial, while $f_{\rm c}$ is the product of all cyclotomic polynomials dividing f.

Now we apply Theorem 3. If equation (4) of its conclusion occurs then, with our restriction on S, $\deg f \le c_2$. If equation (5) occurs, then either $\deg f \le 2c_1$ or $\min\{l_i\} \ge 2$. In the latter case Φ_p must divide f_2, where f_2 is a proper subsum of f. Hence $f_2(1) < p$, contradicting $\Phi_p \mid f_2$. Thus in all admissible cases the degree of f is bounded by a constant depending only on k. Therefore $S_{\rm c}$ is finite. □

References

[1] Boyd, David W., Speculations concerning the range of Mahler's measure. Canad. Math. Bull. **24** (1981), no. 4, 453–469.

[2] ———, Kronecker's theorem and Lehmer's problem for polynomials in several variables. J. Number Theory **13** (1981), no. 1, 116–121.

[3] Dobrowolski, E. ; Lawton, W. ; Schinzel, A., On a problem of Lehmer. Studies in pure mathematics, 135–144, Birkhäuser, Basel, 1983.

[4] Dobrowolski, Edward, Mahler's measure of a polynomial in function of the number of its coefficients. Canad. Math. Bull. **34** (1991), no. 2, 186–195.

[5] ———, Mahler's measure of a polynomial in terms of the number of its monomials. Acta Arith. **123** (2006), no. 3, 201–231.

[6] Lawton, Wayne, A problem of Boyd concerning geometric means of polynomials. J. Number Theory **16** (1983), no. 3, 356–362.

[7] Lehmer, D. H., Factorization of certain cyclotomic functions. Ann. of Math. (2) **34** (1933), no. 3, 461–479.

[8] Mahler, K., An application of Jensen's formula to polynomials. Mathematika **7** (1960), 98–100.

[9] ———, On the zeros of the derivative of a polynomial, Proc. Roy. Soc. London Ser. A **264** (1961), 145–154.

[10] ———, On some inequalities for polynomials in several variables. J. London Math. Soc. **37** (1962), 341–344.

[11] Mann, H.B., On linear relation between roots of unity, Mathematika **12** (1965), 107–117.

[12] Schinzel, A., Polynomials with special regard to reducibility. With an appendix by Umberto Zannier. Encyclopedia of Mathematics and its Applications, 77. Cambridge University Press, Cambridge, 2000.

[13] Smyth, C. J., Closed sets of Mahler measures, arXiv:1606.04338 [math.NT].

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