Heights of sums of roots of unity

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Introduction

In this talk I'll compare the structure of the set of values of three height functions on different algebraic numbers:

Part 1. Pisot numbers: the numbers themselves

Part 2. Sums of roots of unity (= cyclotomic integers): the Cassels height;

Part 3. All algebraic numbers: the Mahler measure.

The results on Pisot numbers are 'classical'.
The new results on the Cassels height are joint work with James McKee (Royal Holloway) and Byeong-Kweon Oh (National University, Seoul).
The results on Mahler measure are partial and speculative.

Part 1: Pisot numbers

Recall: a Pisot number is a real algebraic integer > 1 whose other conjugates all lie in $|z| < 1$.

Examples

$\alpha_0 := 1.3247\cdots$, with minimal polynomial $z^3 - z - 1$;

$\varphi := \frac{1}{2}(1 + \sqrt{5}) = 1.6180\cdots$,
with minimal polynomial $z^2 - z - 1$;

2, with minimal polynomial $z - 2$. (!)

Pisot numbers were discovered by Thue (1912), then Hardy (1916). In the late 1930's Pisot and Vijayaraghavan considered the set S of all Pisot numbers.

Theorem (Salem(1944))

The set S is a closed subset of the real line.

What does S look like?

Limits of Pisot numbers

Denote by $S^{(1)}$ the set of limit points of S (its so-called *derived set*), and for $k \geq 2$ let $S^{(k)}$ be the derived set of $S^{(k-1)}$.

Facts:

$\alpha_0 = 1.3247\cdots$ is the smallest Pisot number (Siegel 1945);
$\varphi = 1.6180\cdots$ is the smallest element of $S^{(1)}$ (Dufresnoy and Pisot 1955);
2 is the smallest element of $S^{(2)}$(Grandet-Hugot 1965).

So this is what the start of S looks like, with $\bullet \in S \setminus S^{(1)}$, $\blacksquare \in S^{(1)} \setminus S^{(2)}$, $\blacklozenge = 2$.

$\bullet\bullet\cdots\blacksquare\cdots\bullet\bullet\bullet\bullet\cdots\blacksquare\cdots\bullet\bullet\bullet\bullet\cdots\blacksquare\cdots\bullet\bullet\bullet\bullet\cdots\blacksquare\cdots\blacksquare\cdots\blacksquare\cdots\blacklozenge$

Furthermore, the least element of $S^{(k)}$ is $> k^{1/2}$ (Boyd 1979).

Limits of Pisot numbers (continued 1)

How are convergent sequences of Pisot numbers obtained?

Theorem

Suppose that $M_\alpha(z)$ *is the minimal polynomial of a Pisot number, and that* $A(z)$ *is an integer polynomial with* $A(0) > 0$ *and* $|M_\alpha(z)| > |A(z)|$ *on* $|z| = 1$*. Then* $z^n M_\alpha(z) \pm A(z)$ *is the minimal polynomial of a Pisot number* $\alpha_{(n,\pm)}$ *say, with, as* $n \to \infty$*,* $\alpha_{(n,+)} \to \alpha$ *from below, and* $\alpha_{(n,-)} \to \alpha$ *from above.*

This result comes from two applications of Rouché's Theorem:

'If analytic functions f and g satisfy $|f| > |g|$ on a circle in $\mathbb{C}$ then f and $f + g$ have the same number of zeros inside the circle.'

Firstly, we apply Rouché with $f = z^n M_\alpha$ and $g = \pm A$, first to the unit circle $|z| = 1$. This shows that $z^n M_\alpha(z) \pm A(z)$ has all zeros except one in $|z| < 1$, so is the minimal polynomial of a Pisot number.

Limits of Pisot numbers (continued 2)

Secondly, we apply it to a circle of radius ε with centre α. This shows that, for k sufficiently large, that $z^n M_\alpha(z) \pm A(z)$ has a zero within that circle. Hence $\alpha_{(n,\pm)} \to \alpha$ as $n \to \infty$.
More generally, for $k \geq 2$ can construct elements of $S^{(k)}$ that are limits, from both sides, of elements of $S^{(k-1)}$. In fact all elements of $S^{(k)}$ have this property (Boyd and Mauldin 1996).

The order type of S

Recall start of S:

$$\bullet\bullet\cdots\blacksquare\cdots\bullet\bullet\bullet\bullet\cdots\blacksquare\cdots\bullet\bullet\bullet\bullet\cdots\blacksquare\cdots\bullet\bullet\bullet\bullet\cdots\blacksquare\cdots\blacksquare\cdots\blacksquare\cdots\blacklozenge$$

We can now describe the order type of S, i.e., which ordinal describes its topological structure.
Let ρ be the order type of $\mathbb{N} := \{1, 2, 3 \cdots\}$, and ρ^* be its reverse order type. Then the order type of S, up to halfway between its first and second limit point, is $a_1 := \rho + 1 + \rho^*$. Then this pattern is repeated up to the first element 2 of $S^{(2)}$, so that the order type of $S \cap [1, 2]$ is $a_1\rho + 1$.
Defining $a_{n+1} := a_n\rho + 1 + (a_n\rho)^*$, get that the order type of S is $\sum_{n=1}^{\infty} a_n$.

Thue sets

Recalling that Axel Thue was the discoverer of the Pisot numbers, we define a *Thue set* T to be a subset of the positive real line with the following properties:

(i) The set T is a closed subset of $\mathbb{R}_+$;

(ii) For $k \geq 1$ the kth derived set $T^{(k)}$ is nonempty, and every element of it is a limit from both sides of elements of $T^{(k-1)}$;

(iii) $t_k := \min\{t \mid t \in T^{(k)}\} \to \infty$ as $k \to \infty$.

Note that all derived sets $T^{(k)}$ of a Thue set are also Thue sets.

So the set S of Pisot numbers is a Thue set.

Indeed, all derived sets $S^{(k)}$ are Thue sets!

Part 2: Cassels heights of cyclotomic integers

A *cyclotomic integer* is an algebraic integer β that can be written as a sum of roots of unity. Any such β lies in $\mathbb{Z}[\omega_n]$ for some n, where ω_n is a primitive nth root of unity, and it is well known that $\mathbb{Z}[\omega_n]$ is the ring of integers of the field $\mathbb{Q}(\omega_n)$.

If $\beta_1 = \beta, \beta_2, \ldots, \beta_d$ are the Galois conjugates of β, define $\mathscr{M}(\beta)$ by

$$\mathscr{M}(\beta) = \frac{1}{d} \sum_{j=1}^{d} |\beta_j|^2. \qquad \text{(Cassels 1969)}$$

Let us call this value the *Cassels height of* β. Because the $|\beta_j|^2$ are the conjugates of $|\beta|^2$, $\mathscr{M}(\beta)$ is rational, with denominator dividing d.

From the AM-GM inequality: $\mathscr{M}(\beta) \geq 1$ for $\beta \neq 0$.

Let

$$\mathscr{C} = \{\mathscr{M}(\beta) \mid \beta \text{ a nonzero cyclotomic integer}\}.$$

The nine smallest elements of $\mathscr{C}$ are

$$1, \tfrac{3}{2}, \tfrac{5}{3}, \tfrac{7}{4}, \tfrac{9}{5}, \tfrac{11}{6}, \tfrac{15}{8}, \tfrac{17}{9}, \tfrac{19}{10}.$$

A limit point of $\mathscr{C}$

Easily calculate that

$$\mathscr{M}(1+\omega_n) = 2\left(1+\frac{\mu(n)}{\varphi(n)}\right).$$

Restricting n to being squarefree, and letting $n \to \infty$, we see that $\mathscr{M}(1+\omega_n) \to 2$ from above or from below, depending on whether it has an even or odd number of prime factors.

Thus 2 is a limit point of $\mathscr{C}$.

$\mathscr{M}$ of sums of pth roots of unity

Given an odd prime p, let

$$\mathscr{C}_p = \{\mathscr{M}(\beta) \mid \beta \in \mathbb{Z}[\omega_p]\},$$

where ω_p is a primitive pth root of unity.

Theorem (McKee,Oh,S. 2020 [3])

For all primes $p \geq 5$ the set $\mathscr{C}_p$ is given by

$$\mathscr{C}_p = \left\{ \frac{1}{p'} \left(\tfrac{1}{2} s(p-s) + rp \right) \mid s = 0, 1, \ldots, p' \text{ and } r \geq 0 \right\}.$$

Here $p' := (p-1)/2$.

For $p = 3$, $\mathscr{C}_3$ is easily seen to be the set of integers N with prime factorisation of the form $N = \prod_q q^{e_q}$, where e_q is even for all primes $q \equiv 2 \pmod 3$.

Universal quadratic polynomials

For the proof in the case $p = 5$ we need to prove the universality of two ternary quadratic polynomials.

Proposition

Both of the quadratic polynomials

$$a^2 + ab + b^2 + c^2 + a + b + c$$

and

$$a^2 + b^2 + c^2 + ab + bc + ca + a + b + c$$

represent all positive integers for integer values of their variables (i.e., they are **universal***).*

Of course it would be interesting to study $\mathscr{C}_n := \{\mathscr{M}(\beta) \mid \beta \in \mathbb{Z}[\omega_n]\}$ for n composite, too.

Closure, additivity of $\mathscr{C}$

Th set $\mathscr{C}$ of Cassels heights has an interesting structure. In 2009 Stan and Zaharescu [5, Theorem 4] proved the following results concerning $\mathscr{C}$:

(i) **Closure.** The set $\mathscr{C}$ is a closed subset of $\mathbb{Q}$. (See also [2, Theorem 9.1.1]).

(ii) **Additivity.** The set $\mathscr{C}$ is closed under addition.

The kth derived set of $\mathscr{C}$

We extend (i) and (ii) to obtain the following results, connecting the kth derived set $\mathscr{C}^{(k)}$ of $\mathscr{C}$ and the Minkowski sumset

$$(k+1)\mathscr{C} := \{c_1 + c_2 + \cdots + c_{k+1} \mid c_1, c_2, \ldots, c_{k+1} \in \mathscr{C}\}. \quad (1)$$

Theorem

For $k \geq 1$ the kth derived set $\mathscr{C}^{(k)}$ of $\mathscr{C}$ is equal to the sumset $(k+1)\mathscr{C}$.Furthermore every element of $\mathscr{C}^{(k)}$ is a limit from both sides of elements of $\mathscr{C}^{(k-1)}$.

The following is an immediate consequence.

Corollary

The smallest element of $\mathscr{C}^{(k)}$ $(k \geq 0)$ is $k+1$. Furthermore, a stronger version of additivity holds, namely that $\mathscr{C}^{(k)} + \mathscr{C}^{(\ell)} = \mathscr{C}^{(k+\ell+1)}$ $(k, \ell \geq 0)$.

Idea of proof

The proof of the theorem is a generalisation of the following result:

Proposition

Let J be an infinite increasing sequence of positive integers, and γ_1 and γ_2 be nonzero cyclotomic integers. Then

$$\lim_{\substack{\ell\to\infty \\ j\in J}} \mathscr{M}(\gamma_1+\omega_j\gamma_2)=\mathscr{M}(\gamma_1)+\mathscr{M}(\gamma_2).$$

Also, J can be chosen so that infinitely many of the values $\mathscr{M}(\gamma_1+\omega_j\gamma_2)$ are distinct, so that $\mathscr{M}(\gamma_1)+\mathscr{M}(\gamma_2)$ is a genuine limit point of the sequence $\{\mathscr{M}(\gamma_1+\omega_j\gamma_2)\}_{j\in J}$.
Furthermore, J can be chosen so that the limit is approached either from above or from below.

A consequence of a result of Loxton

The following result is also important in the proof:
For a given cyclotomic integer β with $\mathscr{M}(\beta) \leq B$ there is a bound N such that β can be expressed as the sum of at most N roots of unity.

Corollary

The set $\mathscr{C}$ is a Thue set.

Since all derived sets of a Thue set are again Thue sets, all the derived sets $\mathscr{C}^{(k)}$ for $k \geq 1$ are also Thue sets.

Part 3: The set L of Mahler measures of integer polynomials

Let $k \geq 1$, $\mathbf{z}_k := (z_1, \ldots, z_k)$ and $F(\mathbf{z}_k)$ be a nonzero Laurent polynomial with integer coefficients. Then its Mahler measure $M(F)$ is defined as

$$M(F) = \exp\left\{ \int_0^1 \cdots \int_0^1 \log |F(e^{2\pi i t_1}, \ldots, e^{2\pi i t_k})| dt_1 \cdots dt_k \right\}. \tag{2}$$

If F is a 1-variable polynomial, say $F(z) = \prod_j (z - \alpha_j)$, then

$$M(F) = \prod_{j : |\alpha_j| \geq 1} |\alpha_j|.$$

What does the set L of all Mahler measures of such polynomials look like?

In 1981 Boyd [1] conjectured that L is closed.

Boyd also showed that

$$\log M(z^n + z + 1) = \log M(z_1 + z_2 + 1) + \frac{c(n)}{n^2} + O\left(\frac{1}{n^3}\right),$$

where

$$c(n) := \begin{cases} -\frac{\pi\sqrt{3}}{6} & \textit{if} \quad n \equiv 1 \pmod 3 \\ \frac{\pi\sqrt{3}}{18} & \textit{otherwise} \end{cases}$$

Thus the 2-variable Mahler measure $M(z_1 + z_2 + 1)$ is a limit from both sides of 1-variable Mahler measures.
This is essentially the only proven known example of this phenomenon! But there is strong evidence for more structure in L, in the light of another result of Boyd and Lawton:

Theorem

Given a k-variable polynomial $F(z_1, \ldots, z_k)$ and an infinite sequence of integer vectors $(r_1^{(n)}, \ldots, r_k^{(n)})$, then

$$M(F(z^{r_1^{(n)}}, \ldots, z^{r_k^{(n)}})) \to M(F(z_1, \ldots, z_k))$$

as $n \to \infty$ provided that the length of the shortest nonzero integer k-vector orthogonal to $(r_1^{(n)}, \ldots, r_k^{(n)})$ tends to ∞ as $n \to \infty$.

The missing ingredient in this result is that we don't know that the difference

$$M(F(z^{r_1^{(n)}}, \ldots, z^{r_k^{(n)}})) - M(F(z_1, \ldots, z_k))$$

takes both signs infinitely often.

Or indeed that it is not zero infinitely often.

Example

In fact, for some F it *can* be zero infinitely often:

Let $F(z_1, z_2) := z_1 + z_2 - 2$. Then $M(F) = 2$ and that if $\mathbf{r}^{(n)} \in \mathbb{Z}^2$ has positive components for all n then $M\left(F_{\mathbf{r}^{(n)}}(z)\right) = 2$ for all n.

Also, can show by Rouché's Theorem that for $\mathbf{r}^{(n)} = (1, -n)$ the numbers $\{M\left(F_{\mathbf{r}^{(n)}}(z)\right)\}_{n \in \mathbb{N}}$ form a strictly increasing sequence of Pisot numbers, with limit 2.

The closure of the set of all $M(F(z^{r_1}, \ldots, z^{r_k}))$

This closure can be described explicitly, as follows:
Given $F(z_1, \ldots, z_k)$, an integer $\ell \geq 0$ and an $\ell \times k$ integer matrix $A = (a_{ij})$, define the k-tuple $\mathbf{z}_\ell^A$ by

$$\mathbf{z}_\ell^A := (z_1, \ldots, z_\ell)^A := (z_1^{a_{11}} \cdots z_\ell^{a_{\ell 1}}, \ldots, z_1^{a_{1k}} \cdots z_\ell^{a_{\ell k}}) \qquad (3)$$

and $F_A(\mathbf{z}_\ell) = F(\mathbf{z}_\ell^A)$, a polynomial in ℓ variables $z_1, \ldots, z_\ell$.
Further define

$$\mathcal{M}(F) := \{M(F_A) : A \in \mathbb{Z}^{\ell \times k}, \ell \geq 0, F_A \neq 0\}, \qquad (4)$$

Theorem (S. 2018 [4])

The set $\mathcal{M}(F)$ is the closure of the set of all 1-variable polynomials $M(F(z^{r_1}, \ldots, z^{r_k}))$.

I tentatively conjecture that the set L of all Mahler measures of polynomials in any number of variables and having integer coefficients also forms a Thue set.

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