

SEMISIMPLE QUANTUM COHOMOLOGY AND BLOW-UPS

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ABSTRACT. Using results of Gathmann, we prove the following theorem: If a smooth projective variety X has generically semisimple (p, p) -quantum cohomology, then the same is true for the blow-up of X at any number of points. This is a successful test for a modified version of Dubrovin's conjecture from the International Congress of Mathematicians (ICM) 1998.

1. INTRODUCTION

This paper is motivated by a conjecture proposed by Boris Dubrovin in his talk at the International Congress of Mathematicians (ICM) in Berlin 1998. It claims that the quantum cohomology of a projective variety X is generically semisimple if and only if its bounded derived category $D^b(X)$ of coherent sheaves is generated by an exceptional collection. We discuss here a modification of this conjecture proposed in [BM01] and show its compatibility with blowing up at a point.

Quantum multiplication gives (roughly speaking) a commutative associative multiplication $\circ_\omega: H^*(X) \otimes H^*(X) \rightarrow H^*(X)$ depending on a parameter $\omega \in H^*(X)$. Semisimplicity of quantum cohomology means that for generic parameters ω , the resulting algebra is semisimple. More precisely, quantum cohomology produces a formal Frobenius supermanifold whose underlying manifold is the completion at the point zero of $H^*(X)$. We call a Frobenius manifold generically semisimple if it is purely even and the spectral cover map $\text{Spec}(\mathcal{TM}, \circ) \rightarrow \mathcal{M}$ is unramified over a general fibre. Generically semisimple Frobenius manifolds are particularly well understood. There exist two independent classifications of their germs, due to Dubrovin and Manin. Both identify a germ via a finite set of invariants. As mirror symmetry statements include an isomorphism of Frobenius manifolds, this means that in the semisimple case one will have to control only this finite set of invariants.

In [BM01], it was proven that the even-dimensional part H^{ev} of quantum cohomology cannot be semisimple unless $h^{p,q} = 0$ for all $p \neq q, p+q \equiv 0 \pmod{2}$. On the other hand, the subspace $\bigoplus_p H^{p,p}(X)$ gives rise to a Frobenius submanifold. This suggested the following modification of Dubrovin's conjecture: The Frobenius submanifold of (p, p) -cohomology is semisimple if and only if there exists an exceptional collection of length $\text{rk} \bigoplus_p H^{p,p}(X)$.

A consequence of this modified conjecture is the following: *Whenever X has semisimple (p, p) -quantum cohomology, the same is true for the blow-up of X at any number of points.* We prove this in Theorem 3.1.1.

We would like to point out that our result suggests another small change of the formulation of Dubrovin's conjecture. Dubrovin assumed that being

Fano is an additional necessary condition for semisimple quantum cohomology. However, as our result holds for the blow-up at an arbitrary number of points, it yields many non-Fano counter-examples. We suggest to just drop any reference to X being Fano from the conjecture.

Acknowledgements. I am grateful to Andreas Gathmann who made helpful comments on an early version of this paper, in particular referring me to J. Hu's results. I am indebted to my advisor Yuri I. Manin for continued inspiring and encouraging guidance and support.

2. DEFINITIONS AND NOTATIONS

Let X be a smooth projective variety over \mathbb{C} . By $H_X := \bigoplus H^{p,p}(X, \mathbb{C})$, we denote the space of (p, p) -cohomology. Let $\Delta_0, \dots, \Delta_m, \Delta_{m+1}, \dots, \Delta_r$ be a homogeneous basis of H_X , such that Δ_0 is the unit, and $\Delta_{m+1}, \dots, \Delta_r$ are a basis of $H^{1,1}(X)$.

We denote the correlators in the quantum cohomology of X by

$$\langle \Delta_{i_1} \dots \Delta_{i_n} \rangle_\beta.$$

This is the number of appropriately counted stable maps

$$f: (\mathcal{C}, y_1, \dots, y_n) \rightarrow X$$

where \mathcal{C} is a semi-stable curve of genus zero, y_1, \dots, y_n are marked points on \mathcal{C} , the fundamental class of \mathcal{C} is mapped to β under f , and $\Delta_{i_1}, \dots, \Delta_{i_n}$ are cohomology classes representing conditions for the images of the marked points. In the case of $\beta = 0$ it is artificially defined to be zero if $n \neq 3$, and equal to $\int_X \Delta_{i_1} \cup \Delta_{i_2} \cup \Delta_{i_3}$ if $n = 3$.

Such a correlator vanishes unless

$$(1) \quad k(\beta) := (c_1(X), \beta) = 3 - \dim X + \sum \left(\frac{|\Delta_{i_j}|}{2} - 1 \right)$$

where $|\Delta_{i_j}|$ are the degrees of the cohomology classes.

Before writing down the potential of quantum cohomology and the resulting product, we will define the ring that it lives in. Let $\{x_k | k \leq m\}$ be the dual coordinates of $H_X/H^{1,1}(X)$ corresponding to the homogeneous basis $\{\Delta_k\}$. Instead of dual coordinates in $H^{1,1}(X)$, we want to consider exponentiated coordinates. This is done most elegantly by adjoining a formal coordinate q^β for effective classes $\beta \in H_2(X, \mathbb{Z})/\text{torsion}$ with $q^{\beta_1 + \beta_2} = q^{\beta_1} q^{\beta_2}$. Now let

$$F_X = \mathbb{Q}[[x_k, q]]$$

be the completion of the polynomial ring generated by x_k and monomials q^β with β effective.

We consider F_X as the structure ring of the formal Frobenius manifold associated to H_X . The vector space H_X acts on F_X as a space of derivations: $\Delta_k, k \leq m$ acts as $\frac{\partial}{\partial x_k}$, and the divisorial classes $\Delta_k, k > m$ act via $q^\beta \mapsto (\Delta_k, \beta) q^\beta$. Hence we can formally consider H_X as the space of horizontal tangent fields of the formal Frobenius manifold \mathcal{M} , and $F_X \otimes H_X$ as its tangent bundle $\mathcal{T}\mathcal{M}$.

The flat structure of this formal manifold is given by the Poincaré pairing g on H_X . Given the flat metric, the whole structure of a formal Frobenius manifold is an algebra structure on $F_X \otimes H_X$ over F_X given by the third partial derivatives of a potential $\Phi \in F_X$:

$$g(\Delta_i \circ \Delta_j, \Delta_k) = \Delta_i \Delta_j \Delta_k \Phi$$

To be able to consistently work only with exponentiated coordinates on $H^{1,1}$, we slightly deviate from this definition: We use only the non-classical part

$$\Phi_X = \sum_{\beta \neq 0} \langle e^{\sum_{k \leq m} x_k \Delta_k} \rangle_{\beta} q^{\beta}.$$

of the Gromov-Witten potential (it is a consequence of the divisor axiom that it makes sense to write Φ_X in this way), and define the product via $g(\Delta_i \circ \Delta_j, \Delta_k) = g(\Delta_i \cup \Delta_j, \Delta_k) + \Delta_i \Delta_j \Delta_k \Phi_X$. The choice of the ring F_X is governed by the two properties that it has to contain Φ_X , and that H_X has to act on it as a vector space of derivations. This is enough to ensure that all standard constructions associated to a Frobenius manifold are defined over F_X .

Explicitly, the multiplication is given by

$$(2) \quad \Delta_i \circ \Delta_j = \Delta_i \cup \Delta_j + \sum_{\beta \neq 0} \sum_{k \neq 0} \langle \Delta_i \Delta_j \Delta_k e^{\sum_{k \leq m} x_k \Delta_k} \rangle_{\beta} \Delta^k q^{\beta}$$

where Δ^k are the elements of the basis dual to (Δ_k) with respect to the Poincaré pairing. The multiplication endows $F_X \otimes H_X$ with the structure of a commutative, associative algebra with $1 \otimes \Delta_0$ being the unit.

We call the whole structure of the formal Frobenius manifold on F_X and H_X *reduced quantum cohomology*. The map of rings

$$(3) \quad F_X \rightarrow F_X \otimes H_X, \quad f \mapsto f \otimes \Delta_0$$

is the formal replacement of the spectral cover map $\text{Spec}(\mathcal{T}\mathcal{M}, \circ) \rightarrow \mathcal{M}$ of a non-formal Frobenius manifold.

Definition 2.0.1. *X has semisimple reduced quantum cohomology if the spectral cover map (3) is generically unramified.*

More concretely, semisimplicity over a geometric point $F_X \rightarrow k$ of F_X means that after base change to k , the ring $k \otimes H_X$ with the quantum product is isomorphic to k^{r+1} with component-wise multiplication. Generic semisimplicity means that this is true for a dense open subset in the set of k -valued points of F_X .

Finally, we recall the definition of the Euler field of quantum cohomology. It is given by

$$\mathcal{E} = -c_1(X) + \sum_{k \leq m} \left(1 - \frac{|\Delta_k|}{2} \right) x_k \Delta_k.$$

It induces a grading on F_X and $F_X \otimes H_X$ by its Lie derivative. E.g., a vector field is homogeneous of degree d if $\text{Lie}_{\mathcal{E}}(X) = [\mathcal{E}, X] = dX$. It is clear that the Poincaré pairing is homogenous of degree $(2 - \dim X)$ by the induced Lie derivative on $(H_X^*)^{\otimes 2}$. Further, from the dimension axiom (1) it follows that Φ_X is homogeneous of degree $(3 - \dim X)$. It a purely formal consequence

of these two facts that the multiplication \circ is homogeneous of degree 1 with respect to \mathcal{E} (see [Man99, section I.2]).

3. SEMISIMPLE QUANTUM COHOMOLOGY AND BLOW-UPS

3.1. Motivation. So let us now assume that the variety X satisfies the modified version of Dubrovin's conjecture, i.e. that it has both an exceptional collection of length $\text{rk} \bigoplus_p H^{p,p}(X)$, and semisimple reduced quantum cohomology. Let \tilde{X} be its blow-up at some points. By remark 4.4.2, this is a test for the modified version of Dubrovin's conjecture 4.2.2: We know that \tilde{X} has an exceptional system of desired length, so it should have semisimple reduced quantum cohomology as well:

Theorem 3.1.1. *Let $\tilde{X} \rightarrow X$ be the blow-up of a smooth projective variety X at any number of closed points.*

If the reduced quantum cohomology of X is generically semisimple, then the same is true for \tilde{X} .

In the case of dimension two, Del Pezzo surfaces were treated in [BM01], where the results of [GP98] on their quantum cohomology were used. The generalization presented here uses instead the results in Andreas Gathmann's paper [Gat01], with an improvement from the later paper [Hu00] by J. Hu. The essential idea is a variant of the idea used in [BM01]: a partial compactification of the spectral cover map where the exponentiated coordinate of an exceptional class vanishes. However, in our case, this is only possible after base change to a finite cover of the spectral cover map.

3.2. More notations. We want to compare the reduced quantum cohomology of \tilde{X} with that of X . We may and will restrict ourselves to the blow-up $j: \tilde{X} \rightarrow X$ of a single point. For the pull-back $j^*: H^*(X) \rightarrow H^*(\tilde{X})$ and the push-forward $j_*: H^*(\tilde{X}) \rightarrow H^*(X)$ we have the identity $j_*j^* = \text{id}_{H^*(X)}$. Hence $H^*(\tilde{X}) = j^*(H^*(X)) \oplus \ker j_*$ canonically. We will identify $j^*(H^*(X))$ with $H^*(X)$ from now on and get a canonical decomposition $H_{\tilde{X}} = H_X \oplus H_E$ with $H_E = \bigoplus_{1 \leq k \leq n-1} \mathbb{C} \cdot E^k$, where E is the exceptional divisor of j . The dual coordinates $(x_k) =: \underline{x}$ on $H_X/H^{1,1}(X)$ get extended via coordinates $(x_2^E, \dots, x_{n-1}^E) =: \underline{x}^E$ to dual coordinates of $H_{\tilde{X}}/H^{1,1}(\tilde{X})$. Let $E' \in H_2(\tilde{X})$ be the class of a line in the exceptional divisor $E \cong \mathbb{P}^{n-1}$. From Poincaré duality and the decomposition of $H^*(\tilde{X})$, we get a corresponding decomposition $H_2(\tilde{X}, \mathbb{Z}) = H_2(X, \mathbb{Z}) \oplus \mathbb{Z} \cdot E'$ in homology, where we have identified $H_2(X)$ with its image via the dual of j_* . With this identification, the cone of effective curves in X is a subcone of the effective cone in \tilde{X} . Hence F_X is a subring of $F_{\tilde{X}}$. We will call elements $\beta \in H_2(X) \subset H_2(\tilde{X})$ non-exceptional, and $\beta \in \mathbb{Z}E'$ purely exceptional.

We can also view F_X as a quotient of $F_{\tilde{X}}$: Let I be the completion of the subspace in $F_{\tilde{X}}$ generated by monomials $\underline{x}^a \cdot (\underline{x}^E)^b q^{\tilde{\beta}}$ with $\underline{b} \neq (0, \dots, 0)$ or $\tilde{\beta} \notin H_2(X)$. Then evidently $F_X = F_{\tilde{X}}/I$. But note that I is *not* an ideal, as there are effective classes $\tilde{\beta}_1, \tilde{\beta}_2 \in H_2(\tilde{X}) \setminus H_2(X)$ whose sum $\tilde{\beta}_1 + \tilde{\beta}_2$ is in $H_2(X)$.

Also, it is not true that $F_X \otimes H_X$ is a subring of $F_{\tilde{X}} \otimes H_{\tilde{X}}$. The next section will summarize the results of [Gat01] that will enable us to study the relation between the two reduced quantum cohomology rings.

3.3. Gathmann's results.

Theorem 3.3.1. *The following assertions relate the Gromov-Witten invariants of \tilde{X} to those of X (which we will denote by $\langle \dots \rangle_{\tilde{\beta}}^{\tilde{X}}$ and $\langle \dots \rangle_{\beta}^X$, respectively):*

- (1) (a) *Let $\beta \in H_2(\tilde{X})$ be any non-exceptional homology class—so β is any element of $H_2(X)$ —, and let T_1, \dots, T_m be any number of non-exceptional classes in $H^*(X) \subset H^*(\tilde{X})$, which we can identify with their preimages in $H^*(X)$. Then it does not matter whether we compute the following Gromov-Witten invariants with respect to \tilde{X} or X :*

$$\langle T_1 \otimes \dots \otimes T_m \rangle_{\tilde{\beta}}^{\tilde{X}} = \langle T_1 \otimes \dots \otimes T_m \rangle_{\beta}^X.$$

- (b) *Consider the Gromov-Witten invariants $\langle T_1 \otimes \dots \otimes T_m \rangle_{\tilde{\beta}}^{\tilde{X}}$ with $\tilde{\beta}$ being purely exceptional, i. e. $\tilde{\beta} = d \cdot E'$. If any of the cohomology classes T_1, \dots, T_m are non-exceptional, the invariant is zero. All invariants involving only exceptional cohomology classes can be computed recursively from the following:*

$$\langle E^{n-1} \otimes E^{n-1} \rangle_{E'}^{\tilde{X}} = 1.$$

They depend only on n .

- (2) (a) *Using the associativity relations, it is possible to compute all Gromov-Witten invariants of \tilde{X} from those mentioned above in 1a and 1b.*
- (b) *Vanishing of mixed classes: Write $\tilde{\beta} \in H_2(\tilde{X})$ in the form $\tilde{\beta} = \beta + d \cdot E'$ where β is the non-exceptional part; assume that $\beta \neq 0$. Let T_1, \dots, T_m be non-exceptional cohomology classes. Let l be a non-negative integer, and let $2 \leq k_1, \dots, k_l \leq n - 1$ be integers satisfying*

$$(k_1 - 1) + \dots + (k_l - 1) < (d + 1)(n - 1).$$

Unless we have both $d = 0$ and $l = 0$, this implies the vanishing of

$$\langle T_1 \otimes \dots \otimes T_m \otimes E^{k_1} \otimes \dots \otimes E^{k_l} \rangle_{\tilde{\beta}} = 0.$$

Proof. The statement in no. 1a is proven by J. Hu in [Hu00, Theorem 1.2]. This is lemma 2.2 in [Gat01]; since the proof of this lemma is the only place where Gathmann uses the convexity of X (see remark 2.3 in that paper), we can drop this assumption from his theorems.

The other equations follow trivially from statements in lemma 2.4 and proposition 3.1 in [Gat01]. \square

3.4. Proof of Theorem 3.1.1. Let us first restate Gathmann's results in terms of the potentials Φ_X and $\Phi_{\tilde{X}}$: We can write $\Phi_{\tilde{X}}$ as

$$(4) \quad \Phi_{\tilde{X}} = \Phi_X + \Phi_{\text{pure}} + \Phi_{\text{mixed}}$$

where Φ_X is the sum coming from all non-exceptional $\tilde{\beta} = \beta$ and non-exceptional cohomology classes (coinciding with the potential of X by no. 1a), Φ_{pure} is the sum coming from all correlators with $\tilde{\beta}$ being purely exceptional (i. e. a positive multiple of E'), and Φ_{mixed} the sum from correlators for mixed homology classes $\tilde{\beta} = \beta + d \cdot E'$ with $0 \neq \beta \in H_2(X)$ and $d \neq 0$.

Now let $\tilde{\mathcal{E}}$ and \mathcal{E} be the Euler fields of \tilde{X} and X , respectively. Let us consider the grading induced by

$$\tilde{\mathcal{E}} - \mathcal{E} = (n-1)E + \sum_{2 \leq k \leq n-1} (1-k)x_k^E E^k.$$

Lemma 3.4.1. *With respect to $\tilde{\mathcal{E}} - \mathcal{E}$, the potential Φ_{pure} is homogeneous of degree $3 - n$, and Φ_{mixed} only has summands of degree less than or equal to $1 - n$.*

Proof. The assertion about Φ_{pure} is just the dimension axiom (1) of \tilde{X} , as $\mathcal{E}\Phi_{\text{pure}} = 0$. The statement about Φ_{mixed} is equivalent to Gathmann's vanishing result, theorem 3.3.1 no. 2b. \square

Let $J \triangleleft F_{\tilde{X}}$ be the ideal generated by x_2^E, \dots, x_{n-1}^E . We will show that the spectral cover map of \tilde{X} is already generically semisimple when restricted to the fibre

$$(5) \quad F_{\tilde{X}}/J \rightarrow H_{\tilde{X}} \otimes F_{\tilde{X}}/J.$$

Write a monomial $q^{\tilde{\beta}}$ in $F_{\tilde{X}}$ as $q^{\tilde{\beta}} = Q^{-d}q^{\beta}$ if $\tilde{\beta} = \beta + d \cdot E'$ with $\beta \in H_2(X)$. We make the base change to the cover given by adjoining $Z := \sqrt[n-1]{Q}$. More precisely, we first localize¹ at Q^{-1} and adjoin a $(n-1)$ -th root of Q : We consider $R := (F_{\tilde{X}}/J)[Q][Z]/(Z^{n-1} - Q)$.

On the other hand, consider the subring B of R that consists of power series in which Z only appears with non-negative degrees.² Then R is a completion of the localisation $B[Z^{-1}]$ of B . We claim that the quantum product "extends" to a product over B .

We define M as the free B -submodule of $B \otimes H^*(\tilde{X})$ generated by

$$\langle H^*(X), ZE, Z^2E^2, \dots, Z^{n-1}E^{n-1} = QE^{n-1} \rangle.$$

More invariantly, B is the completed subspace of R generated by monomials with non-positive degree with respect to $\tilde{\mathcal{E}} - \mathcal{E}$. And M is the submodule of $B \otimes H_{\tilde{X}}$ generated by $B \otimes H_X$ and all elements of strictly negative degree in $B \otimes H_E$.

¹Note that Q itself is not an element of $F_{\tilde{X}}$.

²The ring B is neither $F_X[[Z]]$ nor $F_X[Z]$; it is a different completion of $F_X[Z]$.

Lemma 3.4.2. • *The quantum product restricts to M , i. e. $M \circ M \subseteq M$, and there is the following cartesian digram:*

$$\begin{array}{ccc} B & \longrightarrow & M \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \otimes H^*(\tilde{X}) \end{array}$$

- *Consider the push-out with respect to $B \rightarrow B/(Z) = F_X$. Then the spectral cover map decomposes as*

$$\begin{array}{ccc} B & \longrightarrow & M \\ \downarrow & & \downarrow \\ F_X & \longrightarrow & (F_X \otimes H_X) \oplus F_X[z]/(z^{n-1} - (-1)^{n-1}) \end{array}$$

where the product on $F_X \otimes H_X$ is the quantum product of X .

First, we show how to derive Theorem 3.1.1 from the above lemma. By the induction hypothesis, $F_X \rightarrow F_X \otimes H_X$ is generically semisimple. The second part of the lemma then tells us that the map $B \rightarrow M$ is generically semisimple over the fibre of $Z = 0$.

E. g. by the criterion [EGA, IV, Proposition 17.3.6] of unramifiedness, it is clear that semisimplicity is an open condition for finite flat maps. Hence, also $B \rightarrow M$ is generically semisimple. The same is then true for its completed localization $(F_{\tilde{X}}/J)[Q][Z]/(Z^{n-1}-Q)$. It is also evident that the finite extension $(F_{\tilde{X}}/J)[Q] \rightarrow (F_{\tilde{X}}/J)[Q][Z]/(Z^{n-1}-Q)$ cannot change generic semisimplicity. Hence the spectral cover map (5) must be generically semisimple (as its localization at Q is). And again by openness of semisimplicity, it also holds for the full reduced quantum cohomology of \tilde{X} .

Proof.[of the lemma] We want to analyze the behaviour of multiplication with respect to the grading of $\tilde{\mathcal{E}} - \mathcal{E}$. We decompose the quantum product $\circ_{\tilde{X}}$, understood as a bilinear map $(B \otimes H_{\tilde{X}}) \otimes (B \otimes H_{\tilde{X}}) \rightarrow B \otimes H_{\tilde{X}}$, into a sum $\circ_{\tilde{X}} = \circ_X + \circ_{\text{class}}^E + \circ_{\text{pure}} + \circ_{\text{mixed}}$ according to the decomposition of $\Phi_{\tilde{X}}$ in (4); we have written \circ_{class}^E for the classial cup product of exceptional classes $E^i \circ_{\text{class}}^E E^j = E^{i+j}$ for $0 \leq i, j \leq n-1$ and $i > 0$ or $j > 0$. So for example \circ_{pure} is defined by $\tilde{g}(U \circ_{\text{pure}} V, W) = UVW\Phi_{\text{pure}}$ with \tilde{g} as the Poincaré pairing on \tilde{X} .

We claim that \circ_X , \circ_{pure} and \circ_{mixed} are of degree 0, 1 and ≤ -1 , respectively.

This is clear for \circ_X and follows with standard Euler field computations from the assertions in lemma 3.4.1 (compare with the computations in [Man99, section I.2]):

Take a homogeneous component Φ_d of degree d of any of the two relevant potentials, and \circ_d the corresponding component of the multiplication. Let U ,

V and W be vector fields of degree u , v and w , respectively:

$$\begin{aligned}
(\tilde{\mathcal{E}} - \mathcal{E})\tilde{g}(U \circ_d V, W) &= (\tilde{\mathcal{E}} - \mathcal{E})UVW\Phi_d \\
&= [\tilde{\mathcal{E}} - \mathcal{E}, U]VW\Phi_d + U[\tilde{\mathcal{E}} - \mathcal{E}, V]W\Phi_d \\
(6) \quad &+ UV[\tilde{\mathcal{E}} - \mathcal{E}, W]\Phi_d + UVW(\tilde{\mathcal{E}} - \mathcal{E})\Phi_d \\
&= (u + v + w + d)UVW\Phi_d \\
&= (u + v + w + d)\tilde{g}(U \circ_d V, W)
\end{aligned}$$

Now write $\tilde{g} = g + g^E$ where g is the Poincaré pairing of X and g^E the pairing of exceptional classes $g^E(E^i, E^k) = \delta_{i+j, n}(-1)^{n-1}$. Then g is of degree zero, and g^E of degree $2 - n$ with respect to $\tilde{\mathcal{E}} - \mathcal{E}$. Let $\circ_d = \circ_d^0 + \circ_d^E$ accordingly. Then $U \circ_d V = U \circ_d^0 V + U \circ_d^E V$ is just the decomposition of $U \circ_d V$ in the orthogonal sum $H_{\tilde{X}} = H_X \oplus H_E$; in particular, $U \circ_d V$ is homogeneous if and only if $U \circ_d^0 V$ and $U \circ_d^E V$ are. So we have:

$$\begin{aligned}
(\tilde{\mathcal{E}} - \mathcal{E})\tilde{g}(U \circ_d^0 V, W) &= (\tilde{\mathcal{E}} - \mathcal{E})g(U \circ_d^0 V, W) \\
&= g([\tilde{\mathcal{E}} - \mathcal{E}, U \circ_d^0 V], W) + g(U \circ_d^0 V, [\tilde{\mathcal{E}} - \mathcal{E}, W]) \\
&= g([\tilde{\mathcal{E}} - \mathcal{E}, U \circ_d^0 V], W) + wg(U \circ_d^0 V, W) \\
(\tilde{\mathcal{E}} - \mathcal{E})\tilde{g}(U \circ_d^E V, W) &= (\tilde{\mathcal{E}} - \mathcal{E})g^E(U \circ_d^E V, W) \\
&= \text{Lie}_{\tilde{\mathcal{E}} - \mathcal{E}}(g^E)(U \circ_d^E V, W) \\
&\quad + g([\tilde{\mathcal{E}} - \mathcal{E}, U \circ_d^E V], W) + g(U \circ_d^E V, [\tilde{\mathcal{E}} - \mathcal{E}, W]) \\
&= g((\tilde{\mathcal{E}} - \mathcal{E})(U \circ_d^E V), W) + (2 - n + w)\tilde{g}(U \circ_d^E V, W).
\end{aligned}$$

Comparing with (6), we see that $U \circ_d^0 V$ is of degree $u + v + d$, and $U \circ_d^E V$ of degree $u + v + d + n - 2$, in other words, \circ_d^0 has degree d and \circ_d^E degree $d + n - 2$. Hence, the claim about the degree of \circ_{mixed} is obvious, and the one about \circ_{pure} follows from the additional fact the the derivative of Φ_{pure} in H_X -direction is zero, so that \circ_{pure}^0 is zero.

It is clear that M is closed with respect to \circ_X and \circ_{class}^E . That it is also closed under the multiplication \circ_{mixed} follows directly by degree reasons from the description of M in terms of degrees. With respect to \circ_{pure} we can argue via degrees if we additionally note that $H_X \circ_{\text{pure}} H_{\tilde{X}} = 0$.

So we have proven $M \circ M \subseteq M$, and it remains to analyze the product on $M/ZM \cong F_X \otimes H_{\tilde{X}}$. Note that all elements in M of degree ≤ -2 are mapped to zero in this quotient.

It is clear that \circ_X induces the quantum product of X on the subspace $F_X \otimes H_X$ and is zero on H_E . We already noted that $H_X \circ_{\text{pure}} H_{\tilde{X}} = 0$. Also, $Y_1 \circ_{\text{mixed}} Y_2$ is always zero if Y_1 or Y_2 is in $M \cap B \otimes H_E$ for degree reasons.

We investigate the product with ZE . For this we can ignore \circ_X and \circ_{mixed} . The classical part contributes $ZE \circ_{\text{class}}^E (ZE)^i = (ZE)^{i+1}$ for $0 \leq i \leq n - 1$. For \circ_{pure} we finally have to use the explicit multiplication formula:

$$ZE \circ_{\text{pure}} (ZE)^i = (-1)^{n-1} \sum_{d>0} \sum_j \langle EE^i E^j \rangle_{dE'} Z^{i+1} E^{n-j} Q^{-d}.$$

By the dimension axiom, this can only be non-zero if $(n-1)d = 3-n+(1-1)+(i-1)+(j-1)$, or, equivalently, $(n-1)(d+1) = i+j$. This is only possible for $d=1$ and $i=j=n-1$, where we have $\langle EE^{n-1}E^{n-1} \rangle_{E'} = -\langle E^{n-1}E^{n-1} \rangle_{E'} = -1$. We thus get

$$ZE \circ (ZE)^i = \begin{cases} Z^{i+1}E^{i+1} & \text{if } i \leq n-2 \\ (-1)^n ZE & \text{if } i = n-1. \end{cases}$$

Let $Y := (-1)^n QE^{n-1} = (-1)^n Z^{n-1}E^{n-1}$. As a consequence of the last equation, multiplication by Y in the ring M/ZM is the identity on $(M \cap B \otimes H_E)/ZM \cong F_X \otimes H_E$. In particular, Y is an idempotent and gives a splitting of $M/ZM \cong F_X \otimes H_E \oplus K$ into the image $F_X \otimes H_E$ and the kernel K of $Y \circ$. The algebra structure on $F_X \otimes H_E$ is isomorphic to $F_X[z]/(z^{n-1} - (-1)^{n-1})$ via $z \mapsto ZE$.

The kernel is generated by $\Delta_1, \dots, \Delta_m, \Delta_0 - Y$, and $\Delta_0 - Y$ is its unit. Mapping each element in K to its degree zero component, we get an isomorphism $K \rightarrow F_X \otimes H_X$ that maps the multiplication on K isomorphically to its degree zero component \circ_X , and the lemma is proven. \square

3.5. Further Questions. The first example where our theorem applies is the case of $X = \mathbb{P}^n$. For $n=2$, this yields the semisimplicity of quantum cohomology for all Del Pezzo surfaces as proven earlier in [BM01]. Further, semisimplicity has been established in [TX97] by Tian and Xu, using results of Beauville (see [Bea97]), for low degree complete intersections in \mathbb{P}^n .

Generally speaking, once the three-point Gromov-Witten correlators are known, and thus generators and relations for the small quantum cohomology ring, it is an exercise purely in commutative algebra to check generic semisimplicity in small quantum cohomology. For example, using Batyrev's formula for Fano toric varieties [Bat93] and its explicit version for the projectivization of splitting bundles over \mathbb{P}^n given in [AM00], semisimplicity can be shown to hold for these bundles. In the recent preprint [Cio04], generic semisimplicity was systematically studied for three-dimensional Fano manifolds, thus verifying Dubrovin's conjecture for 36 Fano threefolds out of 59 having purely even-dimensional cohomology.

Of course, our theorem 3.1.1 covers only the first part of Dubrovin's conjecture. It would be very encouraging if it was possible to show his statement on Stokes matrices in a similar way. To my knowledge, the only case where this part has been checked is the case of projective spaces (cf. [Guz99]).

Revisiting Gathmann's algorithm to compute the invariants of \tilde{X} (Theorem 3.3.1, no. 2a), we notice that all the initial data it uses is already determined by the multiplication in the special fibre $Z=0$ of our partially compactified spectral cover map. In other words, the Frobenius manifold on $F_{\tilde{X}}$ and $H_{\tilde{X}}$ is already determined by the structure at $Z=0$.

Yet our construction does not yield a Frobenius structure at the divisor $Z=0$. If there was a formalism of Frobenius manifolds with singularities along divisors, and if there was a way to extend Dubrovin's Stokes matrices to these divisorial Frobenius manifolds, this might also lead to an elegant treatment of Stokes matrices of blow-ups.

Also, one would like to extend the method to the case of the blow-up along a subvariety, analogously to Orlov's Theorem 4.4.1. The next-trivial case of the blow-up along a fibre $\{x_0\} \times Y$ in a product $X \times Y$ follows from our result and the discussion of products in section 4.3.

4. EXCEPTIONAL SYSTEMS AND DUBROVIN'S CONJECTURE

In this section, we briefly review Dubrovin's conjecture and its modified version, and explain how our theorem fits into this context.

4.1. Exceptional systems in triangulated categories. We consider a triangulated category \mathcal{C} . We assume that it is linear over a ground field \mathbb{C} .

Definition 4.1.1. • *An exceptional object in \mathcal{C} is an object \mathcal{E} such that the endomorphism complex of \mathcal{E} is concentrated in degree zero and equal to \mathbb{C} :*

$$\mathbf{RHom}^\bullet(\mathcal{E}, \mathcal{E}) = \mathbb{C}[0]$$

- *An exceptional collection is a sequence $\mathcal{E}_0, \dots, \mathcal{E}_m$ of exceptional objects, such that for all $i > j$ we have no morphisms from \mathcal{E}_i to \mathcal{E}_j :*

$$\mathbf{RHom}^\bullet(\mathcal{E}_i, \mathcal{E}_j) = 0 \quad \text{if } i > j$$

- *An exceptional collection of objects is called a complete exceptional collection (or exceptional system), if the objects $\mathcal{E}_0, \dots, \mathcal{E}_m$ generate \mathcal{C} as a triangulated category: The smallest subcategory of \mathcal{C} that contains all \mathcal{E}_i , and is closed under isomorphisms, shifts and cones, is \mathcal{C} itself.*

The first example is the bounded derived category $D^b(\mathbb{P}^n)$ on a projective space with the series of sheaves $\mathcal{O}(i), \mathcal{O}(i+1), \dots, \mathcal{O}(i+n)$ (for any i). Exceptional systems were studied extensively by a group at the Moscow University, see e. g. the collection of papers in [Rud90].

More generally, exceptional systems exist on flag varieties; other examples include quadrics in \mathbb{P}^n and projective bundles over a variety for which the existence of an exceptional system is already known.

4.2. Dubrovin's conjecture. On the other side of Dubrovin's conjecture we consider the Frobenius manifold M associated (as in [Man99] or [Dub99]) to the quantum cohomology of X . As already mentioned in the introduction, Dubrovin's conjecture relates generic semisimplicity of M to the existence of an exceptional system:

Conjecture 4.2.1. [Dub98] *Let X be a projective variety.*

The quantum cohomology of X is generically semisimple if and only if there exists an exceptional system in its derived category $D^b(X)$.

In further claims of his conjecture, he relates invariants of M to characteristics of the exceptional system: The so-called *Stokes matrix* S of the Frobenius manifold should have entries $S_{ij} = \chi(\mathcal{E}_i, \mathcal{E}_j)$. We almost completely omit these parts of his conjecture in our discussion.

An expectation underlying Dubrovin's conjecture is that the mirror partner of such a variety X will be the unfolding of a function with isolated singularities. The quantum cohomology should be isomorphic to a Frobenius manifold

structure on the base space of the unfolding, as established by Barannikov for projective spaces, cf. [Bar01].

If X has cohomology with Hodge indices other than (p, p) , it can neither have an exceptional system, nor can the Frobenius manifold of its quantum cohomology be semisimple:

- To make sense of all parts of Dubrovin's conjecture, an exceptional collection should have length $\text{rk } H^{\text{ev}}(X)$. But the length of an exceptional collection is bounded by the rank of $N^*(X)$, the group of algebraic cycles modulo numerical equivalence.³ And this is bound by $\text{rk } N^*(X) \leq \text{rk } H_X$.
- The subspace $\bigoplus_p H^{p,p}(X) \subset H^*(X)$ gives rise to a Frobenius submanifold M' of M ; this is the Frobenius manifold we constructed in section 2. This is a maximal Frobenius submanifold of M that has a chance of being semisimple ([BM01, Theorem 1.8.1]).

This suggested the following modification:

Conjecture 4.2.2. [BM01] *The variety X has generically semisimple reduced quantum cohomology (i. e. M' is generically semisimple) if and only if there exists an exceptional collection of length $\text{rk } \bigoplus_p H^{p,p}(X)$ in $D^b(X)$.*

4.3. Products. It follows easily from well-known facts that Dubrovin's conjecture is compatible with products, i. e. when it is true for two varieties X, Y , it will also hold for their product $X \times Y$.

Theorem 4.3.1. *Let $\mathcal{E}_0, \dots, \mathcal{E}_m$ be an exceptional system on X , and $\mathcal{F}_0, \dots, \mathcal{F}_{m'}$ one on Y . Then $(\mathcal{E}_{i_k} \boxtimes \mathcal{F}_{j_k})_k$ forms an exceptional system on $X \times Y$, where $(i_k, j_k)_k$ indexes the set $\{1, \dots, m\} \times \{1, \dots, m'\}$ in any order such that $i_k < i_{k'}$ or $j_k < j_{k'}$ for all $k < k'$.*

This follows from the Leray spectral sequence computing the Ext-groups on $X \times Y$. It also shows that the Stokes matrix of the exceptional system on $X \times Y$ is the tensor product of the Stokes matrices on X and Y :

$$\chi(\mathcal{E}_{i_k} \boxtimes \mathcal{F}_{j_k}, \mathcal{E}_{i_{k'}} \boxtimes \mathcal{F}_{j_{k'}}) = \chi(\mathcal{E}_{i_k}, \mathcal{E}_{i_{k'}}) \cdot \chi(\mathcal{F}_{j_k}, \mathcal{F}_{j_{k'}})$$

The corresponding statements hold for quantum cohomology: Let M and M' be the Frobenius manifolds associated to the quantum cohomology of X and Y , respectively. The Frobenius manifold of the quantum cohomology of $X \times Y$ is the tensor product $M \otimes M'$ ([KM96], [Beh99], [Kau96]). A pair of semisimple points in M and M' yields a semisimple point in $M \otimes M'$, and the Stokes matrix of the tensor product is the tensor product of the Stokes matrices of M and M' ([Dub99, Lemma 4.10]). It is also clear that the same holds for the reduced quantum cohomology on H_X, H_Y and $H_X \otimes H_Y$.

Hence, Dubrovin's conjecture follows for the product if it holds for $X \times Y$. And in cases where $H_X \otimes H_Y = H_{X \times Y}$, i. e. $\bigoplus_p H^{p,p}(X) \otimes \bigoplus_p H^{p,p}(Y) = \bigoplus_p H^{p,p}(X \times Y)$, the same holds for the modified conjecture 4.2.2.

³From the Hirzebruch-Riemann-Roch theorem, it follows easily that the Chern characters of the exceptional objects are linearly independent.

4.4. Complete exceptional systems and blow-ups.

Theorem 4.4.1. [Orl92] *Let Y be a smooth subvariety of the smooth projective variety X . Let $\rho: \tilde{X} \rightarrow X$ be the blow-up of X along Y .*

If both Y and X have an exceptional system, then the same is true for \tilde{X} .

Consider the case where Y is a point: Let $\mathbb{P}^{n-1} \cong E \subset \tilde{X}$ be the exceptional divisor (n is the dimension of X). If $\mathcal{E}_0, \dots, \mathcal{E}_r$ is a given exceptional system in $D^b(X)$, then $\mathcal{O}_E(-n+1), \dots, \mathcal{O}_E(-2), \mathcal{O}_E(-1), \rho^*(\mathcal{E}_0), \dots, \rho^*(\mathcal{E}_r)$ is an exceptional system in $D^b(\tilde{X})$. Hence, the following holds:

Remark 4.4.2. *If X has an exceptional collection of length $\text{rk } H_X$, then the analogous statement is true for the blow-up of X at any number of points.*

REFERENCES

- [AM00] Vicenze Ancona and Marco Maggesi. On the quantum cohomology of Fano bundles over projective spaces. 2000. math.AG/0012046.
- [Bar01] S. Barannikov. Semi-infinite Hodge structures and mirror symmetry for projective spaces. 2001. math.AG/0108148.
- [Bat93] Victor V. Batyrev. Quantum cohomology rings of toric manifolds. *Astérisque*, (218):9–34, 1993. alg-geom/9310004.
- [Bea97] Arnaud Beauville. Quantum cohomology of complete intersections. In *R.C.P. 25, Vol. 48*, volume 1997/42 of *Prépubl. Inst. Rech. Math. Av.*, pages 57–68. Univ. Louis Pasteur, Strasbourg, 1997. alg-geom/9501008.
- [Beh99] K. Behrend. The product formula for Gromov-Witten invariants. *J. Algebraic Geom.*, 8(3):529–541, 1999. alg-geom/9710014.
- [BM01] A. Bayer and Yu. I. Manin. (Semi)simple exercises in quantum cohomology. 2001. math.AG/0103164.
- [Cio04] Gianni Ciolli. On the quantum cohomology of some Fano threefolds and a conjecture of Dubrovin. 2004. math.AG/0403300.
- [Dub98] Boris Dubrovin. Geometry and analytic theory of Frobenius manifolds. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, pages 315–326 (electronic). 1998. math.AG/9807034.
- [Dub99] Boris Dubrovin. Painlevé transcendents in two-dimensional topological field theory. In *The Painlevé property*, CRM Ser. Math. Phys., pages 287–412. Springer, New York, 1999. math.AG/9803107.
- [EGA] Alexandre Grothendieck. Elements de géométrie algébrique. *Inst. Hautes Études Sci. Publ. Math.*, 4, 8, 11, 17, 20, 24, 28, 32, 1960–1967.
- [Gat01] Andreas Gathmann. Gromov-Witten invariants of blow-ups. *J. Algebraic Geom.*, 10(3):399–432, 2001. math.AG/9804043.
- [GP98] L. Göttsche and R. Pandharipande. The quantum cohomology of blow-ups of \mathbf{P}^2 and enumerative geometry. *J. Differential Geom.*, 48(1):61–90, 1998. alg-geom/9611012.
- [Guz99] Davide Guzzetti. Stokes matrices and monodromy of the quantum cohomology of projective spaces. *Comm. Math. Phys.*, 207(2):341–383, 1999. math.AG/9904099.
- [Hu00] J. Hu. Gromov-Witten invariants of blow-ups along points and curves. *Math. Z.*, 233(4):709–739, 2000. math.AG/9810081.
- [Kau96] Ralph Kaufmann. The intersection form in $H^*(\overline{\mathcal{M}}_{0n})$ and the explicit Künneth formula in quantum cohomology. *Internat. Math. Res. Notices*, (19):929–952, 1996. alg-geom/9608026.
- [KM96] M. Kontsevich and Yu. Manin. Quantum cohomology of a product. *Invent. Math.*, 124(1-3):313–339, 1996. With an appendix by R. Kaufmann.
- [Man99] Yu. I. Manin. *Frobenius manifolds, quantum cohomology, and moduli spaces*, volume 47 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1999.

- [Orl92] D. O. Orlov. Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Izv. Ross. Akad. Nauk Ser. Mat.*, 56(4):852–862, 1992.
- [Rud90] A. N. Rudakov. *Helices and vector bundles*, volume 148 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 1990.
- [TX97] Gang Tian and Geng Xu. On the semi-simplicity of the quantum cohomology algebras of complete intersections. *Math. Res. Lett.*, 4(4):481–488, 1997. alg-geom/9611035.

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