Nahm's Equations, The Bogomolny Equations, And An Application To Geometric Langlands

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I will be talking about the gauge theory approach to the geometric Langlands correspondence.

But instead of any sort of overview, I am going to focus on just one result and explain what is involved in interpreting it in terms of supersymmetric quantum gauge theory.

(See V. Ginzburg, "Perverse Sheaves On A Loop Group And Geometric Langlands Duality,'' alg-geom/9511007; also I. Mirkovic and K. Vilonen, "Perverse Sheaves On Affine Grassmannians and Geometric Langlands Duality," alg-geom/9911050)

Let us start with a (simple) Lie group G and its Langlands or Goddard-Nuyts-Olive dual group $G^\vee$. If we write T and $T^\vee$ for the respective maximal tori, then the basic relation between them is that

$$\mathrm{Hom}(T^\vee, U(1)) = \mathrm{Hom}(U(1), T)$$

(I first heard of these things from Atiyah in visiting Oxford in December, 1977.)

Now let $R^\vee$ be an irreducible representation of $G^\vee$. Its highest weight is a homomorphism

$$\rho^\vee : T^\vee \to U(1)$$

which corresponds to a homomorphism in the opposite direction

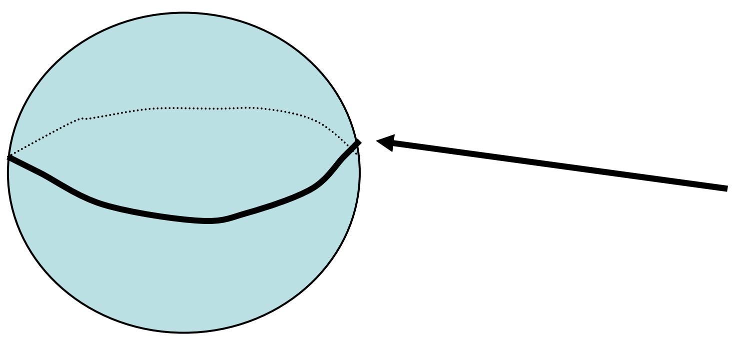
$$\rho : U(1) \to T$$

We can use

$$\rho : S^1 \cong U(1) \rightarrow T \subset G$$

as a “clutching function” to define a holomorphic $G_{\mathbf{C}}$ bundle over

$$\mathbf{CP}^1 \cong S^2$$



$$S^1 \subset S^2$$

The bundle E we make this way is trivial if we delete a single point, p , say the south pole, from $\mathbf{CP}^1$. So E is “a Hecke modification at p of the trivial bundle over $\mathbf{CP}^1$ ”, by which we mean simply a $G_{\mathbf{C}}$ bundle $E \to \mathbf{CP}^1$ together with a trivialization ϕ of $E|_{\mathbf{CP}^1 \backslash p}$

(that is, of the restriction of E to the complement of p)

Example: Suppose that

$$G^{\vee} = SL(N, \mathbf{C})$$ Then

$$G = PGL(N, \mathbf{C})$$

Let us think of a $G-$bundle as a rank N complex vector bundle V but with an equivalence relation

$$V \cong V \otimes \mathcal{L}$$

for any complex line bundle $\mathcal{L}$

Now let us take the representation $R^{\vee}$ to be the obvious $N-$dimensional representation of $G^{\vee} = SL(N, \mathbf{C})$

In this case, we can describe V as follows. Let z be a local coordinate at the south pole on $\mathbf{CP}^1$. V differs from the trivial bundle $\mathbf{C}^N \times \mathbf{CP}^1 \to \mathbf{CP}^1$ as follows.

For some one-dimensional subspace $S \subset \mathbf{C}^N$, a section of V takes the form, near the south pole,

$$v = a + \frac{s}{z}$$

where a, s are holomorphic near $z = 0$.
and $s(0) \in S$

Clearly, what we have constructed depends on the choice of S, so we really have a family of possible V' s, parametrized by $\mathbf{CP}^{N-1}$

Something like this happens for any choice of a representation $R^\vee$ of the dual group $G^\vee$. To this representation, we associate the clutching function

$$\rho : U(1) \to T \subset G$$

via which we construct a whole family $\mathcal{N}(\rho)$ of possible Hecke modifications of the trivial bundle – at a given point on a curve.

This furthermore has a natural compactification $\overline{\mathcal{N}}(\rho)$ and geometric Langlands duality associates the representation $R^{\vee}$ of the dual group to the cohomology of $\overline{\mathcal{N}}(\rho)$. Let us see how this works in our example.

In our example, $R^\vee$ was the $N-$dimensional representation of $SL(N,\mathbf{C})$, and $\mathcal{N}(\rho)$ (which in this example needs no compactification) is $\mathbf{CP}^{N-1}$. Not coincidentally, the cohomology of $\mathbf{CP}^{N-1}$ is of rank N, the dimension of $R^\vee$.

The generators of the cohomology are in degrees $0,2,4,\dots,2N-2$

Let us shift the degrees by $-(N-1)$ so that they are symmetrically placed around zero. Then we see that the degrees of the cohomology classes are the eigenvalues of the diagonal matrix

$$h = \begin{pmatrix} N-1 & & \dots & \\ & N-3 & \dots & \\ & & \ddots & \\ & & \dots & -(N-1) \end{pmatrix}$$

with the indicated eigenvalues.

One may recognize this matrix; it is an element of the Lie algebra of

$$G^\vee = SL(N, \mathbf{C})$$

that generates the maximal torus of, in the terminology of Kostant, a “principal SL_2 subgroup” of $G^\vee$

This is the general state of affairs: the grading by degree of the cohomology of $\overline{\mathcal{N}}(\rho)$ corresponds to the action of the Lie algebra element h on $R^\vee$

The nilpotent “raising operator” of the principal SL_2 also plays a role.

We recall that in a well-known paper, Atiyah and Bott used gauge theory to define certain universal cohomology classes on any family of $G_{\mathbf{C}}-$ bundles over a Riemann surface C. The definition applies immediately to $\overline{\mathcal{N}}(\rho)$, which is a family of $G_{\mathbf{C}}-$ bundles over $\mathbf{CP}^1$ (Hecke modifications of a trivial bundle).

For now, for simplicity, I'll just mention a special case – a two-dimensional class x that (for $SL(N,\mathrm{C})$) can also be constructed as the first Chern class of the "determinant line bundle."

Under the identification of the representation $R^\vee$ with the cohomology of $\overline{\mathcal{N}}(\rho)$, the element x of the cohomology corresonds to the raising operator of the principal SL_2

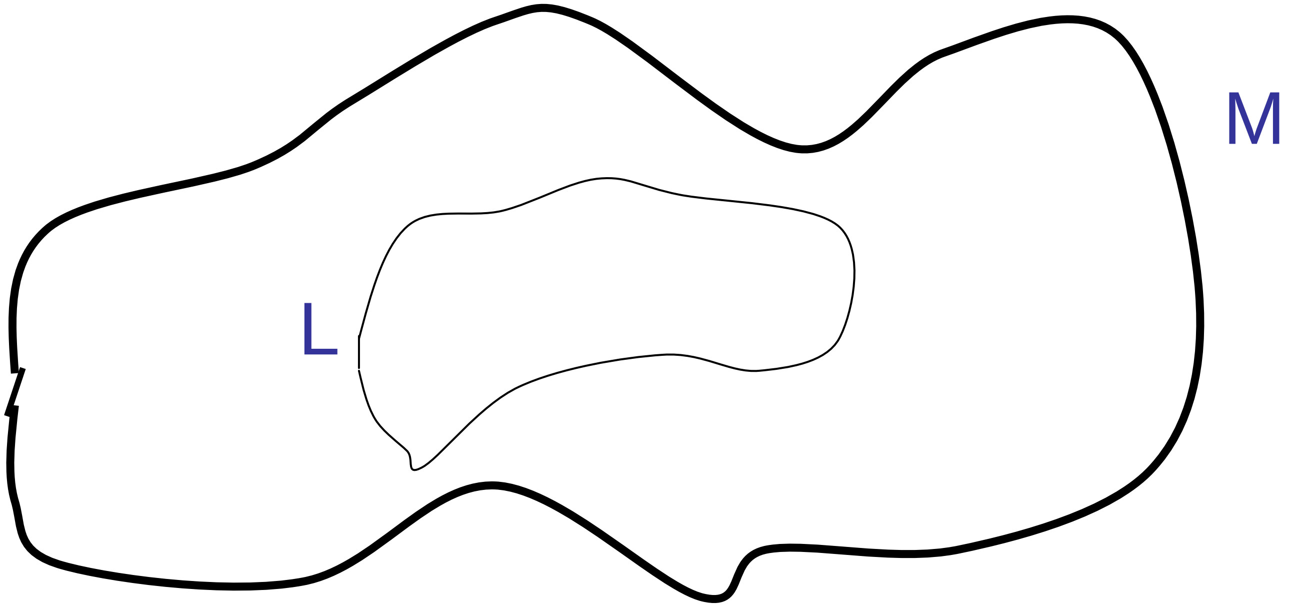
In other words, in our example, it corresponds to

$$f = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

with 1’s just above the main diagonal.

There are more facts of this nature, but instead of explaining them all, what I really want to do is to explain how one can understand them using gauge theory.

Consider a four-manifold M with an embedded one-manifold L:



We want to study gauge theory on M, modified in some way along L.

One “classical” modification is to suppose that L is the trajectory of a “charged particle” in the representation $R^{\vee}$ of the gauge group $G^{\vee}$. Mathematically we achieve this by including in the “path integral” a factor of the holonomy around L of the connection A – with a trace in the chosen representation:

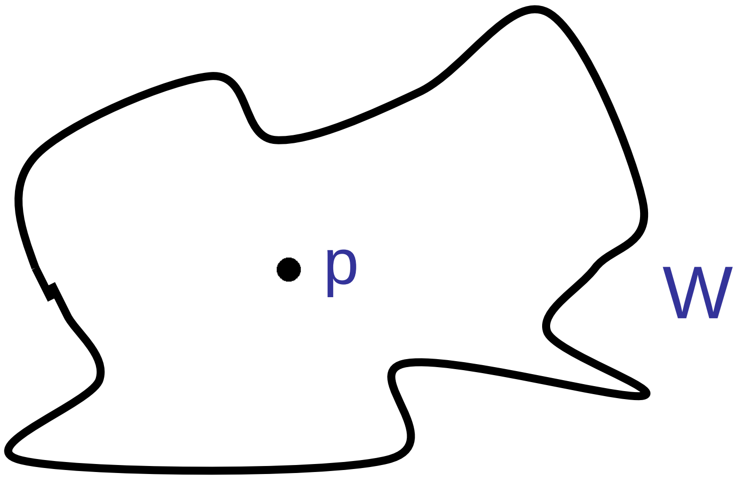
$$\mathrm{Tr}_{R^{\vee}} \mathrm{Hol}(A; L)$$

It was essentially shown by ‘t Hooft nearly 30 years ago that the dual operation in G gauge theory is to modify the theory by requiring the fields to have a certain type of singularity along L – this singularity (which mathematically was first studied by Kronheimer in the context of the Bogomolny equations) gives a way to study Hecke modifications via gauge theory.

For the moment, we postpone any details.

To study the representation $R^\vee$, separated from some of the wonders of four dimensions, it is convenient to take $M = W \times \mathbf{R}$, where W is a three-manifold and $\mathbf{R}$ parametrizes the "time." We similarly take $L = p \times \mathbf{R}$ where p is a point in W

The rest of this lecture is based on a "Hamiltonian" point of view in which we only talk about W:



We want to make a simple choice of W so as to study the representation $R^{\vee}$ and not the wonders of three-manifolds.

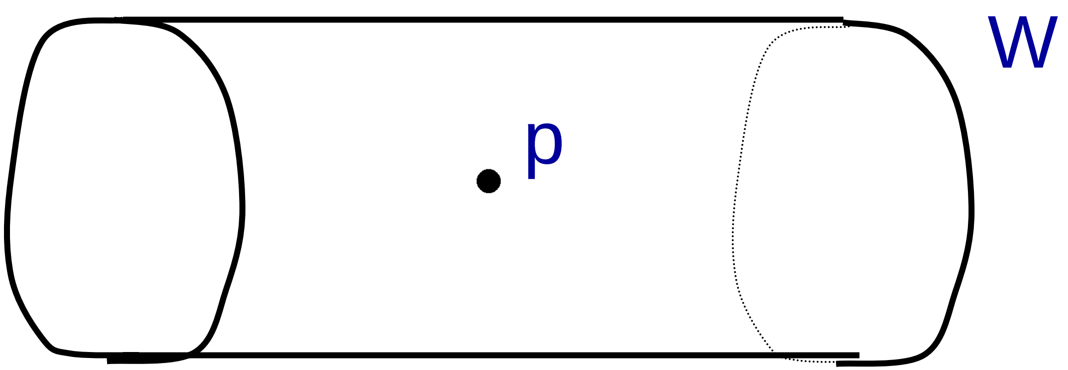
One might think that the simplest three-manifold is S^3 -- we don't have to worry about non-trivial flat connections. But there is a snag: the trivial flat connection on S^3 has non-trivial automorphisms, which cause a complication.

We can avoid these automorphisms if we take W to be a three-manifold with boundary, with Dirichlet boundary conditions. The gauge group then acts freely on the space of connections.

What is the simplest three-manifold with boundary? A three-ball B^3 is an obvious contender, but it turns out to be easier to take

$$W = S^2 \times I$$

where I is a unit interval.



Now we need boundary conditions at the two ends, which we need to pick so that

(i) gauge transformations act freely; (ii) there are no non-trivial flat connections.

Here (i) fails if we use Neumann at both ends (i.e. free boundary conditions), and

(ii) fails if we use Dirichlet at both ends

(Dirichlet means that the connection is trivialized on the boundary, and gauge transformations equal 1 there).

So we pick, say, Neumann on the left and Dirichlet on the right.

Now what are we going to do on the three-manifold W?

The answer – assuming we mean to study the twisted topological field theory relevant to geometric Langlands – is that we study the Bogomolny equations

of gauge group G, with a certain singularity at the point p. On the G side, these are the conditions for supersymmetry in this twisted theory.

The Bogomolny equations are certain equations of mathematical physics for a pair (A, ϕ) where A is a connection on a G-bundle $E \to W$, and ϕ is a section of $\mathrm{ad}(E)$. The equations read

$$F = \star D_A \phi$$

where F is the curvature.

These equations have been extensively studied by Atiyah, Hitchin, and other mathematicians.

If we take any gauge field at all on $W = C \times I$ for any Riemann surface C (in our case $C = S^2$), we can restrict to $C \times y$ for some $y \in I$ and – since any connection on a Riemann surface determines a holomorphic structure – we get a holomorphic G-bundle $E_y \to C$

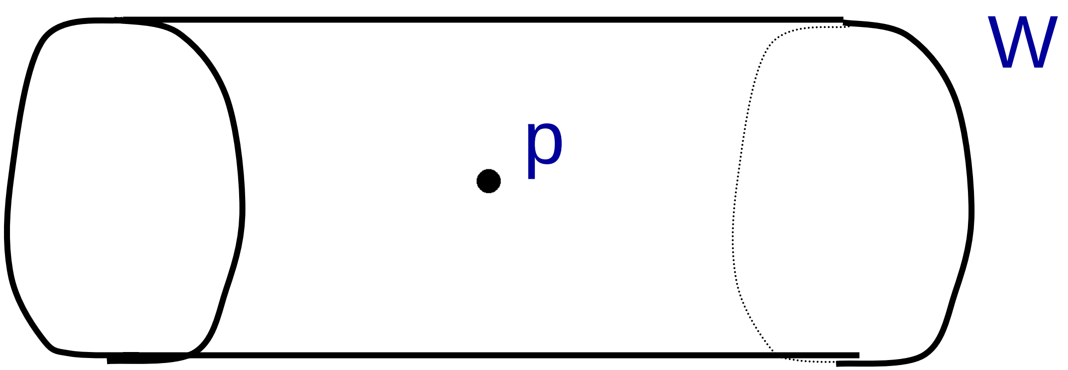
One of the special properties of the Bogomolny equations is that, if (A, ϕ) obeys those equations, then E_y is independent of y (in a canonical way).

In our case, there is a point $p = s_0 \times y_0$

with $s_0 \in S^2 = C, \; y_0 \in I$

where the Bogomolny equations are not obeyed (as there is a singularity).

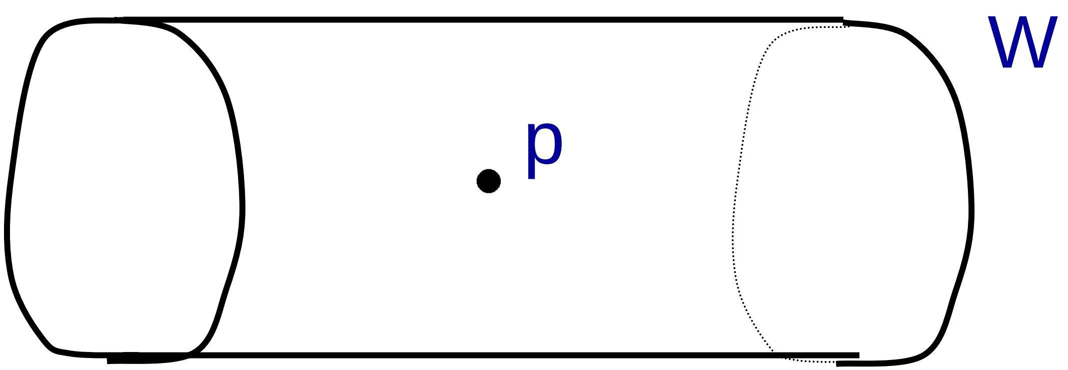
In crossing $y = y_0$, E_y jumps



Even when we cross $y = y_0$, the holomorphic type of the bundle E_y does not change if one omits the point s_0 from C -- as the Bogomolny equations are obeyed away from s_0.

The jump in E_y across $y = y_0$ is very special -- E_y undergoes a Hecke modification of type ρ. (The type is encoded in the required singularity of the solution of the Bogomolny equations.)

Now suppose we have Dirichlet boundary conditions on the right – trivializing E_y at that boundary – and Neumann boundary conditions at the other end – so any E_y is allowed there. The result is that the space of solutions of the Bogomolny eqns



is our friend $\overline{\mathcal{N}}(\rho)$, the compactified space of Hecke modifications.

(The compactification arises in gauge theory from monopole bubbling, which is analogous to instanton bubbling.)

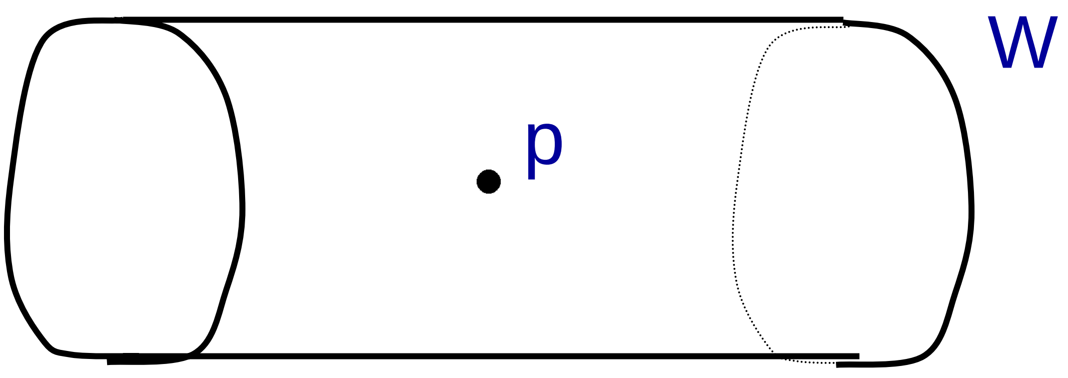
In topological field theory, the space of physical states is the cohomology of the moduli space. So in the present case, the space of physical states is the cohomology of $\bar{\mathcal{N}}(\rho)$.

So far everything I have said is essentially in my original paper with A. Kapustin on geometric Langlands via gauge theory.

But now I will continue the story in a different way using some more recent results with D. Gaiotto – see our papers **arXiv:0807.3720, arXiv:0804.2902** on the action of electric-magnetic duality on boundary conditions in gauge theory.

To learn something new, about the space of physical states, we need to look at the dual description involving $G^\vee$ gauge theory. Some things are simpler

than what we've met so far, and some are less simple.



First of all, there are no Bogomolny equations to worry about. The supersymmetric equations on the dual side just tell us to look for flat connections.

This is also simple, since we have chosen W and the boundary conditions so that there are not any nontrivial flat connections, and also no gauge theory automorphisms to worry about.

(These statements hold despite some subtleties about the boundary conditions that we come to momentarily.)

There is also no singularity at the point p; instead, there is a copy of the representation $R^\vee$ sitting at the point p. Since there is nothing else that has to be quantized (we've arranged that there are no nontrivial flat connections) the physical Hilbert space is just a copy of $R^\vee$.

That is the basic reason that there is a correspondence $R^\vee \leftrightarrow H^*(\bar{\mathcal{N}}(\rho))$

However, I haven't yet said anything that would account for the appearance of the principal SL_2 subgroup. There is a trick here. Bogomolny's equations do not come in on the $G^\vee$ side. But some other equations of mathematical physics do come in – Nahm's equations. These are equations for a triple $\vec{\sigma}$ valued in $\mathfrak{g}^\vee \otimes \mathbf{R}^3$ (where $\mathfrak{g}^\vee$ is the Lie algebra of $G^\vee$).

Nahm's equations read

$$\frac{d\vec{\sigma}}{dy} + \vec{\sigma} \times \vec{\sigma} = 0$$

Nahm related them originally to the Bogomolny equations by an analog of the ADHM transform for instantons. Many other applications have been found since then, and the equations have been studied by many mathematicians, for example

Kronheimer (1989,1990)

Atiyah and Bielawski, math/0110112

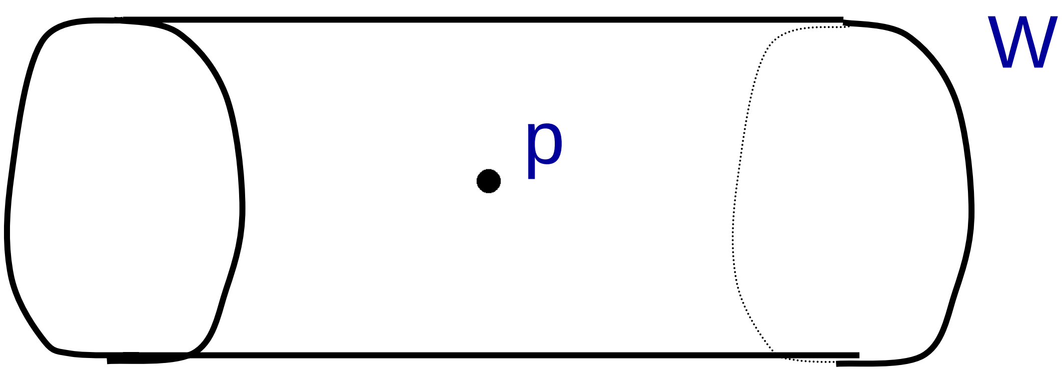
The gauge theory machinery that leads to geometric Langlands is a little elaborate – it involves a twisted version of N=4 supersymmetric Yang-Mills theory.

Part of this machinery is a triple of fields $\vec{\sigma}$ as before, and on the $G^\vee$ side, supersymmetry leads to Nahm's equations.

Often (for instance in analyzing the dual of an irreducible flat connection) these fields do not play a prominent role, because the boundary conditions set them to zero.

In the present case, that is not so.

To find the right boundary conditions, we start on the G side (Dirichlet on the right, Neumann on the left) and apply electric-magnetic duality. However, the action of duality on boundary conditions is subtle.



In abelian gauge theory, duality simply exchanges Dirichlet and Neumann, but in nonabelian gauge theory, there is more to it (Gaiotto and EW, 2008)

The dual of Neumann boundary conditions is a modification of Dirichlet in which the field $\vec{\sigma}$ that obeys Nahm's equations plays an important role.

Nahm's equations on the half-line $y \geq 0$ have a singular solution with

$$\vec{\sigma} = \frac{\vec{t}}{y}$$

where $\vec{t}$ are any three elements of the $\mathfrak{g}^\vee$ Lie algebra that obey the commutation relations of $\mathfrak{sl}(2)$

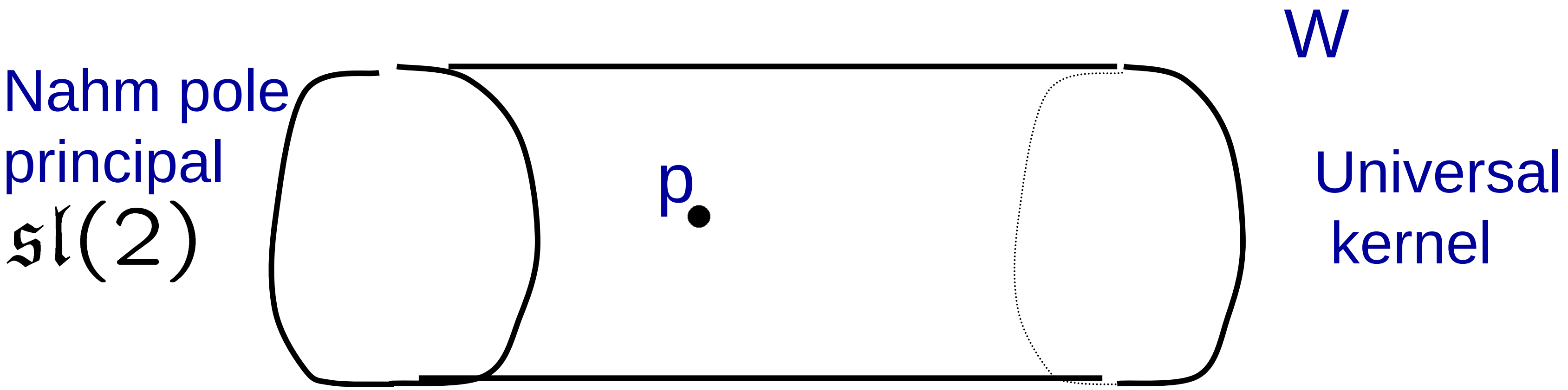
Thus, we get such a solution for each choice of embedding $\tau : \mathfrak{sl}(2) \to \mathfrak{g}^\vee$

Each choice of τ gives a supersymmetric boundary condition: one requires $\vec{\sigma}$ to have the indicated singularity at $y \to 0$

It turns out that ordinary Neumann boundary are dual to Dirichlet boundary conditions modified by a “Nahm pole” associated with the principal $\mathfrak{sl}(2)$ embedding.

And ordinary Dirichlet boundary conditions have an even more unusual dual, related to the “universal kernel” of geometric Langlands, expressed in quantum field theory language.

So the dual picture looks like this:



For the questions we are discussing today,

the important boundary condition is the one that uses the Nahm pole. (There is a crucial detail that I won't explain: globally the solution of Nahm's equations obeying the boundary conditions is unique.)

Now we can address the following question:

The grading of $H^*(\bar{\mathcal{N}}(\rho))$ by the degree of a cohomology class, what does it correspond to on the $G^\vee$ side?

To answer this question, we need to know that the degree of a cohomology class corresponds to a symmetry of the twisted N=4 super Yang-Mills theory that is usually called “ghost number” in the context of this sort of topological field theory.

On the $G^\vee$ side, this symmetry acts by rotating two of the three scalar fields $\vec{\sigma}$.

How can this give a symmetry of the boundary conditions at the left of the picture, where $\vec{\sigma}$ is required to have a pole?

The answer is that the actual symmetry is generated by the sum of an ordinary rotation of two components of $\vec{\sigma}$, plus a gauge transformation that rotates them back.

Since the boundary condition involves a pole associated with a principal embedding $\tau : \mathfrak{sl}(2) \longrightarrow \mathfrak{g}^{\vee}$, the relevant gauge transformation generates the maximal torus of a principal $\mathfrak{sl}(2)$.

We identify the space of physical states with the representation $R^\vee$; the rotation of $\vec{\sigma}$ does not act on $R^\vee$, but the maximal torus of the principal $\mathfrak{sl}(2)$ does.

This is why duality maps the grading by the degree of a cohomology class to grading by the action of that maximal torus, as we saw by hand in an example at the beginning of the lecture.

At the beginning of the lecture, we mentioned another point – a certain two-dimensional characteristic class ("the first Chern class of the determinant line bundle") maps to the raising operator of the principal $\mathfrak{sl}(2)$ To understand this we must represent the universal cohomology classes of Atiyah and Bott by quantum field operators, determine how they transform under duality, and then determine how their images act on the representation $R^\vee$

In view of the time, I will be rather brief.

Donaldson, in defining the Donaldson polynomials, adapted the definitions of Atiyah and Bott to four dimensions, and in that context the universal cohomology classes were interpreted in terms of gauge theory about 20 years ago.

For example, the two-dimensional class that I mentioned (which is the key class in Donaldson's work) is represented by a local quantum field operator that takes values in two-forms on a four-manifold M.

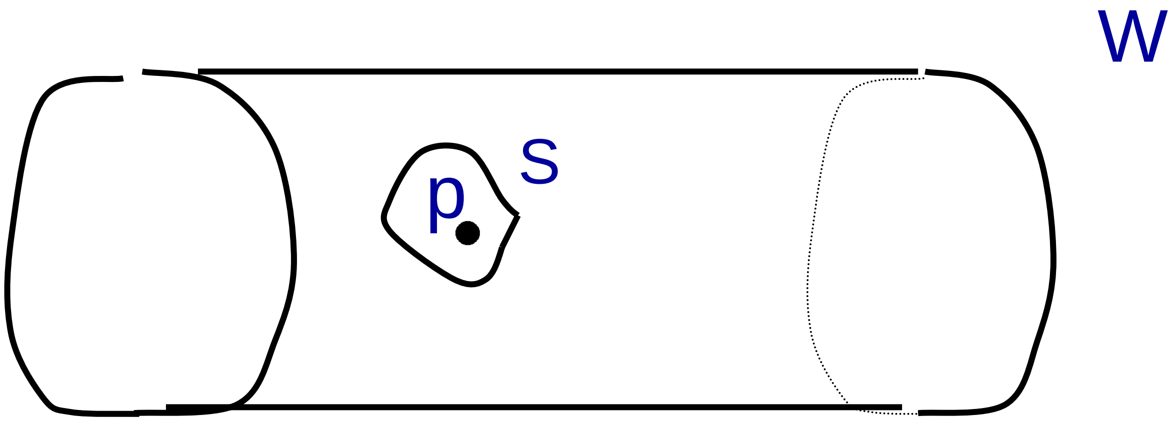
The usual formula is

$$\mathcal{O}_{(2)} = \mathrm{Tr}\,(\sigma F + \psi \wedge \psi)$$

where here σ is a complex linear combination of our three fields $\vec{\sigma}$

(This physical formula was interpreted in differential geometry by Atiyah and Jeffrey.)

To get a cohomology class on moduli space



we simply integrate this two-form on a small two-sphere S in W that “links” the point p.

The dual picture is just the same thing with

$$\mathcal{O}_{(2)} = \mathrm{Tr}\ (\sigma F + \psi \wedge \psi)$$

replaced by the dual operator

$$\tilde{\mathcal{O}}_{(2)} = \mathrm{Tr}\ (\sigma \star F + \psi \wedge \psi)$$

One evaluates this using for $\star F$ the electric field produced by the point charge in the representation $R^\vee$

Upon doing this, we find that the two-dimensional cohomology class of $\bar{\mathcal{N}}(\rho)$ maps to the "raising operator" of the principal $\mathfrak{sl}(2)$.

There is an analogous story for all of the universal characteristic classes.

One last thing that I should perhaps say is that this subject can be approached from quite a different angle. The naïve Dirichlet boundary condition in the $G^\vee$ theory corresponds, in the usual language, to a "brane" (object in the derived category of coherent sheaves) whose support is at the trivial flat connection. Its dual is rather complicated (as analyzed by Gaiotto and EW). This boundary condition can be generalized to incorporate an arbitrary homomorphism $\tau : \mathfrak{sl}(2) \to \mathfrak{g}^\vee$

that is included via a Nahm pole; this plays the role of Arthur's $SL(2)$ in the classical Langlands program. The dual becomes simpler as τ becomes "bigger" and the easiest result is for the principal $\mathfrak{sl}(2)$ embedding, where the dual is given by the simplest Neumann boundary conditions. (Further recent developments on this by E. Frenkel and S. Gukov.)

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