Some Nilpotent Complexes

Shmuel Weinberger

In this note we construct some nilpotent complexes whose existences were unknown. (A CW complex X is nilpotent if $\pi_1(X)$ is, and $\pi_i(X)$ are nilpotent $\pi_1 X$ modules.

Proposition 1: For any finite nilpotent group π, there is a three dimensional nilpotent complex with fundamental group π.

Proposition 2: There are nilpotent finite Poincaré complexes with $\pi_1 = \mathbb{Z}_{15}$ which are not simple Poincaré complexes in the sense of (Wa) Wall.

For Proposition 1, it is not hard to see that dim $X \geq 3$ (if π is nontrivial) and P. Kahn asked if one can arrange for dim $X = 3$, for then by crossing with a torus one obtains for abelian groups nilpotent spaces of smallest possible dimension with the given group as $\pi_1((BK))$.

For Proposition 2, note that in [CW] it is shown that nilpotent finite Poincaré complexes with π_1 an odd p-group are simple.

Lemma. If $\pi_1 X$ is nilpotent and acts nilpotently on $H_i(\tilde{X})$, then X is nilpotent.

Proof. Quite easy from localization theory. See eg. [We].

Proof of Proposition 1: Nilpotent groups are products of their p-Sylow subgroups, $\pi = \Pi\pi_p$. The version of Wall finiteness obstruction theory given in [We] shows that $S^3 \times K(\pi_p,1)$ has the $\mathbb{Z}[\frac{1}{p}][\pi_p]$ homology type of a finite 3-dimensional complex M_p. Now just Zabrodsky mix the M_p's. This produces a finitely dominated three dimensional homologically nilpotent complex as desired. Q.E.D.

Problem: Can this complex be taken finite?

Proof of Proposition 2: Recall from [KM] that $\tilde{K}_0(\mathbb{Z}_{15}) = \mathbb{Z}_2$. As a result

$$L^p_{odd}(\mathbb{Z}_{15}) \to H(\mathbb{Z}_2;K_0(\mathbb{Z}_{15})) \to L^h_{even}(\mathbb{Z}_{15})$$
$$\| \qquad\qquad \|$$
$$0 \qquad\qquad \mathbb{Z}_2$$

$L^h_{even}(\mathbb{Z}_{15})$ has an element of order 2 which we will construct shortly. Since L^s is torsion free [Wa 2] this element is detected by $H(\mathbb{Z}_2;Wh(\mathbb{Z}_{15}))$ in the Rothenberg sequence.

Let M be the nontrivial module $\mathbb{Z}_9$ over $\mathbb{Z}_{15}$. M is nilpotent, and has a resolution $0 \to P \to \mathbb{Z}(\mathbb{Z}_{15}) \to M \to 0$ where P is the nontrivial element of $K_0(\mathbb{Z}_{15})$. $P \oplus P^*$ admits a hyperbolic form which is the desired element of L^h. Better yet, consider the form (= as an element of L^h) on $N \oplus N^*$ where $N = [\mathbb{Z}_{15}] \oplus P^*$. This has a free hyperbolic pair in it based on $P \oplus P^* \subset N$.

Apply the proof of the Wall realization theorem to $S^2 \times L^3_{15}$ and the form on $N \oplus N^*$ and glue in copies of $D^3 \times L^3_{15}$. It is easy to see that the result of surgering the geometric spheres corresponding to $P \oplus P^*$ produces the desired complex. Q.E.D.

Remark: Mislin has also used the module M in his work on finiteness obstructions for nilpotent complexes.

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