THE INVOLUTION IN THE ALGEBRAIC K-THEORY OF SPACES

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The primary purpose of this paper is to study the canonical involution on the algebraic K-theory of spaces functor A(X). From a technical point of view the main result is that several ways of defining such an involution lead in fact to the same result.

The secondary purpose is to establish that the involution on A(X) relates nicely to involutions on related functors, specifically the algebraic K-theory of rings, and concordance spaces. Technically this follows simply by comparing the latter involutions to suitable models of the involution on A(X). In particular there results the expected fact that the involution on concordance spaces corresponds to the involution on the algebraic K-theory of rings.

This is certainly a desirable result, and indeed several applications of it have already been published, cf. [2], [3], [5]. For example, it is possible to obtain some numerical information on the homotopy type of the diffeomorphism groups of some manifolds.

Aside from this technical result, the study of the involution on A(X) also has some interest of its own. We obtain another proof of the theorem that stable homotopy splits off the algebraic K-theory of spaces.

Here is a summary of the contents of the paper.

In § 1 a concept of equivariant Spanier-Whitehead duality is discussed. Here the word 'equivariant' refers to the homotopy theory of spaces over a fixed space.

Using this concept of duality a model for A(X) is developed which lends itself to a natural definition of an involution. Namely, on the level of the categories of spaces used to define A(X) the involution corresponds to the transition from a space over X to its equivariant Spanier-Whitehead dual.

It will be convenient later on to have a different description of duality available. Namely, instead of considering spaces over a fixed space one can equivalently use simplicial sets with an action of a simplicial group. The translation into this framework is given in the second part of § 1.

The concept of Spanier-Whitehead duality that we need is a version of Ranicki duality for simplicial groups. Its relation to the usual concept of equivariant Spanier-Whitehead duality is briefly discussed in the appendix to § 1, cf. also [20].

In § 2 it is described how an involution on A(X) may be defined in various ways. It is shown that these definitions lead to the same involution up to homotopy (cf. cor. 2.10., and the remark after prop. 2.5.).

Using the 'manifold model' [18], the relation of A(X) with the concordance space functor is described, and it is shown that the involutions on both functors correspond to each other.

Using the model of A(X) developed in § 1, the involution is compared with that on the algebraic K-theory of rings (cf. prop. 2.11.).

In § 3 the description of A(X) obtained in § 1 is adapted to give another proof of the splitting theorem: The canonical map $\Omega^\infty S^\infty(X_+) \longrightarrow A(X)$ from the stable homotopy of X to the algebraic K-theory splits up to homotopy. In fact, a very direct description of a splitting map $A(X) \longrightarrow \Omega^\infty S^\infty(X_+)$ is given.

A summary of the contents of paragraphs 1 and 2 has been published in [15].

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§ 1. Duality in equivariant homotopy theory

In this paragraph we discuss a concept of Spanier-Whitehead duality appropriate in equivariant homotopy theory, meaning the homotopy theory of spaces (= simplicial sets) parametrized by a simplicial set X. The motivation for this is that A(X), the *algebraic K-theory* of X may be defined in terms of certain categories of simplicial sets over X, cf. [16]. Actually we consider two equivalent formulations of this duality, one involving simplicial sets over X, the other one employing simplicial sets with an action of the loop group of X. Both versions are used in § 2.

We introduce some language and notations. If X is a connected simplicial set let $\mathcal{R}(X)$ denote the category of *retractive simplicial sets over* X, i.e. an object is a triple (Y,r,s) consisting of a simplicial set Y, a retraction $r:Y \to X$, and a section s of r. A morphism from (Y,r,s) to (Y',r',s') is a map $f:Y \to Y'$ such that $r'f = r$ and $fs = s'$. An h-*equivalence* is a morphism in $\mathcal{R}(X)$ which is a weak homotopy equivalence. Let $h\mathcal{R}(X)$ denote the category of h-equivalences; $\mathcal{R}_f(X)$ is the subcategory of $\mathcal{R}(X)$ of those objects (Y,r,s) satisfying that Y-s(X) contains only finitely many non-degenerate simplices. The category $\mathcal{R}_{hf}(X)$ is the category of *homotopy finite* objects, i.e. it is the full subcategory of $\mathcal{R}(X)$ of those objects which can be related to an object of $\mathcal{R}_f(X)$ by a finite chain of h-equivalences. Let $h\mathcal{R}_{hf}(X)$ be the intersection of $h\mathcal{R}(X)$ and $\mathcal{R}_{hf}(X)$, and similarly with $h\mathcal{R}_f(X)$. We will be interested in certain subcategories of $h\mathcal{R}(X)$: let $h\mathcal{R}^\ell_k(X)$ denote the connected component of $h\mathcal{R}(X)$ containing the objects

$$X \cup_{\underbrace{\partial D^\ell \cup \ldots \cup \partial D^\ell}_{k}} \underbrace{D^\ell \cup \ldots \cup D^\ell}_{k} \longrightarrow X$$

Indeed, all such objects are in the same connected component of $h\mathcal{R}(X)$, regardless of the attaching maps.

There is an external pairing

$$\begin{aligned} {}_X\wedge_{X'} : \mathcal{R}(X) \times \mathcal{R}(X') &\longrightarrow \mathcal{R}(X\times X') \\ (Y\,,\,Y') &\longmapsto Y\times Y' \cup_{X\times Y' \cup Y\times X'} X\times X' \,. \end{aligned}$$

This pairing is natural in X and X', and associative up to canonical isomorphism. The properties of this pairing may be conveniently summarized by saying that it defines a *bi-exact functor* in the sense of [19]. It also preserves finiteness (resp. finiteness up to homotopy), where these terms are defined with regard to the categories $\mathcal{R}_f(X)$ (resp. $\mathcal{R}_{hf}(X)$).

Notation: For typographical reasons we shall simply write $Y \wedge Y'$ instead of $Y_X\wedge_{X'}Y'$ if there is no risk of confusion. A special case of this pairing is given by the *fibrewise suspension over* X, defined as

$$\Sigma^n_X(Y) = S^n \;{}_*\wedge_X\; Y \,,$$

where S^n denotes a pointed simplicial set representing the n-sphere.

We can now give the fundamental

Definition:
$$A(X) = \mathbb{Z} \times \left|\varinjlim_{k,\ell} h\mathcal{R}^\ell_k(X)\right|^+ .$$

The maps in the direct system are given by Σ_X in the ℓ-variable, and by wedge with an ℓ-sphere in the k-variable; "+" denotes Quillen's construction to abelianize the fundamental group.

To define the concept of Spanier-Whitehead duality in the context of retractive spaces, we fix a d-spherical fibration ξ over X, (i.e. fibre$(|\xi| \to |X|) \simeq S^d$) with a given section. Let Th(ξ) denote an object of $\mathcal{R}(X\times X)$ satisfying

(i) $\mathrm{Th}(\xi) \rightleftarrows X\times X$ is in the same component of $h\mathcal{R}(X\times X)$ as the object

$$X \times X \cup_X \xi \rightleftarrows X \times X$$

where the maps in the push-out are given by the diagonal map, and the section of ξ;

(ii) $\mathrm{Th}(\xi) \to X\times X$ is a (Kan) fibration;

(iii) There is a map $\iota:\mathrm{Th}(\xi) \to \mathrm{Th}(\xi)$ covering the flip map $X\times X \to X\times X$ and such that ι^2 = identity.

Such a space will be called a *Thom space* of ξ. Note that $\mathrm{Th}(\xi)/X^2$ is essentially the Thom space of ξ in the usual sense.

We will also need the suspensions of Th(ξ). Since these are not automatically Kan fibrations again, we use the following modification. There is a functorial way of turning a map $A \to B$ of simplicial sets into a Kan fibration. This can be done by using a relative version of Kan's functor Ex^∞, cf. [6]. We continue to denote this functor Ex^∞. Define

$$\mathrm{Th}_n(\xi) = Ex^\infty(\Sigma^n_{X^2}(\mathrm{Th}(\xi))), \quad \iota_n = Ex^\infty(\Sigma^n_{X^2}(\iota)).$$

Of course, there always exists a Thom space in this sense: e.g. choose $Th(\xi) = Ex^{\infty}(X^2 \cup_X \xi)$. It will however be convenient later on not to be restricted to one choice of a Thom space.

Spanier-Whitehead duality is defined with respect to a chosen space $Th(\xi)$. Let (Y,r,s) (resp. Y',r',s') be an object of $R(X)$. An n-*duality map* is a map in $R(X\times X)$

$$u: \quad Y \wedge Y' \longrightarrow Th_{n-d}(\xi)$$

satisfying that the induced map

$$\alpha_u : H_q(Y',X;\mathbb{Z}[\pi_1 X]) \longrightarrow H^{n-q}(Y,X;\mathbb{Z}[\pi_1 X])$$
$$z \longmapsto u^*(t)/z$$

is an isomorphism for all q. Here $t \in H^n(Th_{n-d}(\xi),X^2;\mathbb{Z}[\pi_1 X]')$ is a class mapping to a generator of $H^n(p^*Th_{n-d}(\xi),\tilde{X}\times\tilde{X};\mathbb{Z}[\pi_1 X]) \approx \mathbb{Z}[\pi_1 X]$ under the canonical map, where '$\sim$' denotes the universal covering, $p: \tilde{X}\times\tilde{X} \longrightarrow X\times X$ is the canonical projection, and $\mathbb{Z}[\pi_1 X]'$ denotes the right $\mathbb{Z}[\pi_1 X^2]$-module $\mathbb{Z}[\pi_1 X]$ with $\pi_1 X^2$-action given by $x.(g,h) = h^{-1}xg$, $(g,h) \in (\pi_1 X)^2$, $x \in \mathbb{Z}[\pi_1 X]$.
Similarly we require that $\alpha_{u'}$ gives an isomorphism, where u' is the composition of u with the flip map $Y\wedge Y' \to Y'\wedge Y$.
It turns out that this definition gives the correct notion of duality in the context of manifolds and for the purpose of K-theory.

Before stating some elementary facts about duality, we have to mention a technical point. Since in our definition of Thom spaces we insisted on the Kan condition we cannot just identify $Th_{n+1}(\xi)$ with the suspension of $Th_n(\xi)$. These spaces are of course homotopy equivalent, but it is desirable to have a specific homotopy equivalences such that the following diagrams commute.

$$\begin{array}{ccc}
S^1{}_*\wedge_*(S^n{}_*\wedge_{X^2}Th(\xi)) & \xrightarrow{\approx} & S^{n+1}{}_*\wedge_{X^2}Th(\xi) \\
\downarrow & & \downarrow \\
S^1{}_*\wedge_{X^2}Th_n(\xi) & \xrightarrow{f_n} & Th_{n+1}(\xi)
\end{array}$$

$$\begin{array}{ccccc}
S^1{}_*\wedge_*(S^n{}_*\wedge_{X^2}Th(\xi)) & \xrightarrow{\approx} & (S^n{}_*\wedge_*S^1){}_*\wedge_{X^2}Th(\xi) & \xrightarrow{\approx} & S^{n+1}{}_*\wedge_{X^2}Th(\xi) \\
\downarrow & & & & \downarrow \\
S^1{}_*\wedge_{X^2}Th_n(\xi) & & \xrightarrow{g_n} & & Th_{n+1}(\xi)
\end{array} .$$

The vertical arrows are the natural inclusions induced from the map $Y \to Ex^{\infty}(Y)$.

The reason for considering the maps g_n which permute the suspension coordinates is to ensure compatibility with certain constructions to be performed later on. The existence of the maps f_n (resp. g_n) follows from the Kan condition on $Th_{n+1}(\xi)$.

Lemma 1.1. Let $u\colon Y \wedge Y' \to Th_{n-d}(\xi)$ be an n-duality map. Then

(i) $u'\colon Y' \wedge Y \to Th_{n-d}(\xi)$ is also an n-duality map, where

$u'(y' \wedge y) = \iota_{n-d}(u(y \wedge y'))$,

(ii) $\Sigma_\ell u\colon \Sigma_X(Y) \wedge Y' = (S^1 {}_*\wedge_X Y) \wedge Y' \to S^1 {}_*\wedge_X 2Th_{n-d}(\xi) \xrightarrow{f_n} Th_{n-d+1}(\xi)$ and

$\Sigma_r u\colon Y \wedge \Sigma_X(Y') \to S^1 {}_*\wedge_X 2Th_{n-d}(\xi) \xrightarrow{g_n} Th_{n-d+1}(\xi)$

are (n+1)-duality maps. □

We now want to investigate the dependence of duality on the spherical fibration ξ. Let $\mathcal{R}_{fib}(X)$ denote the subcategory of those objects of $\mathcal{R}(X)$ satisfying that the structural retraction is a fibration. There is an operation

$$\bullet\colon \mathcal{R}_{fib}(X) \times \mathcal{R}(X) \longrightarrow \mathcal{R}(X)$$

$$(\xi\ ,\ Y) \longmapsto \xi \bullet Y := \xi \times_X Y \cup_{\xi \cup_X Y} X\ .$$

We list some of its properties:

(i) If $Y \in \mathcal{R}_{fib}(X)$, then $\xi \bullet Y$ is (up to a dimension shift) the fibrewise join over X of ξ and Y.

(ii) If $Y \in \mathcal{R}_f(X)$, and fibre$(\xi \to X)$ is finite, then $\xi \bullet Y \in \mathcal{R}_f(X)$.

(iii) $\xi \bullet -\colon h\mathcal{R}(X) \longrightarrow h\mathcal{R}(X)$ is an exact functor in the sense of [16].

(iv) The operation is compatible with the external pairing:

$$(\xi \bullet Y) \wedge (\eta \bullet Y') \approx (\xi \wedge \eta) \bullet (Y \wedge Y'),\text{ where}$$

$\xi, \eta \in \mathcal{R}_{fib}(X)$, $Y, Y' \in \mathcal{R}(X)$.

(v) If $\xi = X \times S^r$, then $\xi \bullet Y = \Sigma^r_X(Y)$, the r-fold fibrewise suspension over X.

There is a kind of Thom isomorphism in this setting.
Let ξ be a d-spherical fibration ($d \geq 2$) over X as before, Y an object of $\mathcal{R}_{hf}(X)$ satisfying that $\pi_i X \xrightarrow{\approx} \pi_i Y$, $i = 0,1$. Let $t \in H^d(\xi, X)$ be a Thom class of ξ.

Lemma 1.2. There are isomorphisms for all $q \geq 0$

$$\varphi_t\colon H^q(Y,X;\mathbb{Z}[\pi_1 X]) \longrightarrow H^{q+d}(\xi\bullet Y,Y;\mathbb{Z}[\pi_1 X]),\ \varphi_t(a) = t \cup a$$

$$\psi_t\colon H_{q+d}(\xi\bullet Y,X;\mathbb{Z}[\pi_1 X]) \longrightarrow H_q(Y,X;\mathbb{Z}[\pi_1 X]),\ \psi_t(b) = t \cap b\ .$$

Proof: Without loss of generality assume that $Y \in R_f(X)$. Then one can find a filtration $Y_0 = X \subset Y_1 \subset \dots \subset Y_k = Y$, such that $Y_i/Y_{i-1} \simeq X \vee \bigvee S^i$. Now by property (iii) above, $\xi\cdot -$ is an exact functor. So by a five-lemma argument one immediately reduces to the case that $Y = X \vee S^0 = X_+$. But in this case the assertion of the lemma is obvious. □

If $Th(\xi)$ is a Thom space of ξ, and η is another spherical fibration (with a section), then by properties (iii) and (iv) above, $(\eta \wedge \varepsilon) \cdot Th(\xi)$ is a Thom space of $\eta \cdot \xi$, $(\varepsilon = X \times S^0)$; in shorthand notation $(\eta \wedge \varepsilon) \cdot Th(\xi) = Th(\eta \cdot \xi)$.

Corollary 1.3. Let $u: Y \wedge Y' \to Th_{n-d}(\xi)$ be an n-duality map with respect to ξ, and let η be a d'-spherical fibration (orientable and with a section). Then the map

$$(\eta \wedge \varepsilon) \cdot u: \eta \cdot Y \wedge Y' \to (\eta \wedge \varepsilon) \cdot Th_{n-d}(\xi) = Th_{n-d-d'}(\eta \cdot \xi)$$

is an (n+d')-duality map with respect to $\eta \cdot \xi$.

Proof: We have a commutative diagram

$$\begin{array}{ccc} H_q(\eta \cdot Y, X; \mathbb{Z}[\pi_1 X]) & \longrightarrow & H^{n-q+d'}(Y', X; \mathbb{Z}[\pi_1 X]) \\ \downarrow \approx & & \downarrow = \\ H_{q-d'}(Y, X; \mathbb{Z}[\pi_1 X]) & \longrightarrow & H^{n-q+d'}(Y', X; \mathbb{Z}[\pi_1 X]) \end{array}$$

where the vertical map on the left is the isomorphism of lemma 1.2., and the horizontal maps are given by slant product with a Thom class of ξ (resp. $\eta \cdot \xi$). By assumption the lower of these maps is an isomorphism, hence so is the upper one. Interchanging the roles of homology and cohomology gives another commutative diagram for which the same argument applies. □

Let ξ be as before. Define a category $\mathcal{DR}^n_{Th(\xi)}(X)$, in which an object is given by a triple (Y, Y', u), where Y (resp. Y') is an object of $R_{hf}(X)$, subject to the technical condition that the inclusion of X in Y (resp. Y') induces an isomorphism on π_0 and π_1, and $u: Y \wedge Y' \to Th_{n-d}(\xi)$ is an n-duality map.

A morphism $(Y, Y', u) \to (Z, Z', v)$ is a pair of morphisms in $R_{hf}(X)$

$$f: Y \to Z, \quad f': Z' \to Y'$$

such that the diagram

$$\begin{array}{ccc} Y \wedge Z' & \xrightarrow{f \wedge id} & Z \wedge Z' \\ \downarrow id \wedge f' & & \downarrow v \\ Y \wedge Y' & \xrightarrow[u]{} & Th_{n-d}(\xi) \end{array}$$

commutes.

A morphism (f,f') in $\mathcal{DR}^n_{Th(\xi)}(X)$ is called an h-equivalence if f and f' are h-equivalences. The subcategory of h-equivalences will be denoted $h\mathcal{DR}^n_{Th(\xi)}(X)$.

The category $h\mathcal{DR}^n_{Th(\xi)}(X)$ does not essentially depend on the particular choice of the space $Th(\xi)$. Namely, suppose $Th'(\xi)$ is another model for the Thom space of ξ in the sense defined above. The conditions on such a space imply that there is a fibre homotopy equivalence $Th(\xi) \xrightarrow{\simeq} Th'(\xi)$.

Lemma 1.4. A fibre homotopy equivalence α: $Th(\xi) \longrightarrow Th'(\xi)$ induces a functor $\bar{\alpha}$: $h\mathcal{DR}^n_{Th(\xi)}(X) \to h\mathcal{DR}^n_{Th'(\xi)}(X)$ which is a homotopy equivalence.

Proof: It is clear that α induces such a functor. To see that this functor is a homotopy equivalence choose an inverse of α, α': $Th'(\xi) \longrightarrow Th(\xi)$. This defines a functor $\bar{\alpha}'$ in the other direction. Let g: $Th(\xi) \wedge I_+ \to Th(\xi)$ be a homotopy over X from $\alpha'\alpha$ to id. Define an endofunctor f of $h\mathcal{DR}^n_{Th(\xi)}(X)$ by

$$(Y,Y',u) \longmapsto (Y \,{}_X\!\wedge_* I_+, Y', g\circ(u \,{}_X\!\wedge_* \mathrm{id})) .$$

There are two natural transformations

$$\mathrm{id} \longrightarrow f \longleftarrow \overline{\alpha'\alpha} .$$

These provide the required homotopy $\bar{\alpha}'\bar{\alpha} \simeq$ id, cf. [11]. Similarly, $\bar{\alpha}\bar{\alpha}' \simeq$ id. □

In view of this lemma the choice of the Thom space $Th(\xi)$ does not really matter. In the following we will always assume that a definite choice of $Th(\xi)$ has been made. To simplify the notation we will usually write $h\mathcal{DR}^n_\xi(X)$ instead of the more precise $h\mathcal{DR}^n_{Th(\xi)}(X)$ whenever there is no danger of confusion.

By lemma 1.1. there are two suspension functors

$$\Sigma_\ell: \mathcal{DR}^n_\xi(X) \longrightarrow \mathcal{DR}^{n+1}_\xi(X),\ (Y,Y',u) \mapsto (\Sigma_X Y, Y', \Sigma_\ell u), \text{ resp.}$$

$$\Sigma_r: \mathcal{DR}^n_\xi(X) \longrightarrow \mathcal{DR}^{n+1}_\xi(X),\ (Y,Y',u) \mapsto (Y, \Sigma_X Y', \Sigma_r u) .$$

Stably the category $\mathcal{DR}^n_\xi(X)$ does not depend on the spherical fibration ξ. Namely, in view of cor. 1.3.

$$(Y,Y',u) \longmapsto (\xi\cdot Y, Y', (\xi \wedge \varepsilon)\cdot u)$$

defines a functor

$$\varphi_\xi : h\mathcal{DR}^n_\varepsilon(X) \longrightarrow h\mathcal{DR}^{n+d}_\xi(X),$$

where $\varepsilon = X \times S^0$ is the trivial spherical fibration.

Lemma 1.5. The functor φ_ξ induces a weak homotopy equivalence

$$\varinjlim_{\Sigma_\ell} h\mathcal{DR}^n_\varepsilon(X) \longrightarrow \varinjlim_{\Sigma_\ell} h\mathcal{DR}^n_\xi(X) .$$

Proof: First assume that X is finite up to homotopy. Then we can find an inverse η of ξ such that $\xi\cdot\eta \simeq X \times S^r$. Multiplication with η defines a functor

$$\varphi_\eta : h\mathcal{DR}^n_\xi(X) \longrightarrow h\mathcal{DR}^{n+r-d}_\varepsilon(X) .$$

The composite $\varphi_\xi\varphi_\eta$ (resp. $\varphi_\eta\varphi_\xi$) is the same up to homotopy as the r-fold suspension

$$h\mathcal{DR}^n_\varepsilon(X) \longrightarrow h\mathcal{DR}^{n+r}_\varepsilon(X) \quad (\text{resp. } h\mathcal{DR}^n_\xi(X) \longrightarrow h\mathcal{DR}^{n+r}_\xi(X))$$

(cf. property (v) of the operation •). The general case of the lemma follows by a direct limit argument. □

Remark: There is an analogous assertion with Σ_ℓ replaced by Σ_r throughout. □

Since $\Sigma_\ell\Sigma_r = \Sigma_r\Sigma_\ell$ it makes sense to talk about the limit

$$\lim_{\substack{\rightarrow \\ \Sigma_\ell, \Sigma_r}} h\mathcal{DR}^n_\varepsilon(X).$$

We have

Proposition 1.6. The forgetful functor

$$\delta : \lim_{\substack{\rightarrow \\ \Sigma_\ell, \Sigma_r}} h\mathcal{DR}^n_\varepsilon(X) \longrightarrow \lim_{\substack{\rightarrow \\ \Sigma_X}} h\mathcal{R}_{hf}(X)$$

$$(Y, Y', u) \longmapsto Y$$

is a weak homotopy equivalence.

Proof: This will be proved below after prop. 1.15. □

Remark: By imposing a condition on the homotopy type as in the definition of the categories $h\mathcal{R}^\ell_k(X)$ one defines categories $h\mathcal{DR}^{\ell,m}_k(X)$, i.e. an object is a triple (Y,Y',u), where Y (resp. Y') has the homotopy type of a wedge of X and k spheres of dimension ℓ (resp. m). (The spherical fibration ξ is suppressed in the notation of these categories).

The map δ restricts to a weak homotopy equivalence

$$\delta : \lim_{\substack{\rightarrow \\ \ell, m}} h\mathcal{DR}^{\ell,m}_k(X) \longrightarrow \lim_{\substack{\rightarrow \\ \ell}} h\mathcal{R}^\ell_k(X) .$$ □

A different setting for the duality just described is provided by using simplicial sets with a group action instead of retractive simplicial sets. The group in question is the *loop group* of X, cf. [7]. This setting is sometimes more convenient to work in. We have to give a few definitions first.

Let G be a simplicial group. $\mathcal{U}(G)$ is the category of pointed simplicial sets with right (simplicial) G-action. $\mathcal{U}_f(G)$ is the subcategory of those G-sets which are free (in the pointed sense, i.e. $xg = x$ implies $g = 1$ or $x = *$) and finitely generated over G, i.e. they are generated as a G-set by finitely many simplices. An h-equivalence is a G-map which is a weak homotopy equivalence of the underlying simplicial sets, $h\mathcal{U}(G)$ is the subcategory of h-equivalences, $\mathcal{U}_{hf}(G)$ is the subcategory of $\mathcal{U}(G)$ of those G-sets which are related to objects of $\mathcal{U}_f(G)$ by a finite chain of homotopy equivalences; $h\mathcal{U}_{hf}(G) := \mathcal{U}_{hf}(G) \cap h\mathcal{U}(G)$.

Let M and M' denote objects of $\mathcal{U}(G)$. An n-*duality map* is a pointed right (G×G)-map

$$u: M \wedge M' \longrightarrow Ex^{\infty}(S^n \wedge G_+)$$

satisfying that it induces an isomorphism of $\mathbb{Z}[\pi_0 G]$-modules

$$\alpha_u: H_q^G(M') \longrightarrow H_G^{n-q}(M)$$

$$z \longmapsto u^*(t)/z, \quad 0\leq q\leq n,$$

where $t \in H^n_{GxG}(Ex^{\infty}(S^n\wedge G_+);\mathbb{Z}[\pi_0 G]')$ is a class mapping to a generator of $H^n_{GxG_0}(Ex^{\infty}(S^n\wedge G_+);\mathbb{Z}[\pi_0 G]) \approx \mathbb{Z}[\pi_0 G]$ under the canonical map, where G_0 denotes the identity component of G, and $\mathbb{Z}[\pi_0 G]'$ denotes the $\mathbb{Z}[\pi_0 G^2]$-module $\mathbb{Z}[\pi_0 G]$ with $\pi_0 G^2$-action given by $x.(a,b) = b^{-1}xa$, $(a,b) \in (\pi_0 G)^2$, $x \in \mathbb{Z}[\pi_0 G]$.

Here we consider $S^n\wedge G_+$ as a right simplicial (G×G)-set via $(x\wedge g)\cdot(h,k)=x\wedge k^{-1}gh$, $x \in S^n$, $g,h,k \in G$. This induces a G×G-action on $Ex^{\infty}(S^n\wedge G_+)$.

Similarly we ask that $\alpha_{u'}$ is an isomorphism, where u' is defined as the composite

$$M'\wedge M \xrightarrow{\approx} M\wedge M' \xrightarrow{u} Ex^{\infty}(S^n\wedge G_+) \xrightarrow{\iota_n} Ex^{\infty}(S^n\wedge G_+)$$

and the map ι_n is induced by $g \mapsto g^{-1}$.

By definition $H_*^G(M;A) = H_*(Mx^G E,*x^G E;A)$, $H_G^*(M;A) = H^*(Mx^G E,*x^G E;A)$, where E is a universal G-bundle, and A is a π_0G-module.

Ex^{∞} denotes the functor of [6] which turns a simplicial set into a Kan simplicial set.

Note that in case that M and M' are finite (up to homotopy) the second condition on a duality map is implied by the first, and vice versa.

We call M' an n-*dual* of M if there exists an n-duality map $u: M \wedge M' \longrightarrow Ex^{\infty}(S^n \wedge G_+)$.

Example 1.7. The map $\mu: (S^k \wedge G_+) \wedge (S^{n-k} \wedge G_+) \longrightarrow S^n \wedge G_+ \longrightarrow Ex^{\infty}(S^n \wedge G_+)$ induced from the map $G \times G \longrightarrow G$, $(g,h) \longmapsto h^{-1}g$ is an n-duality map. □

Just as in the case of retractive spaces, duality is compatible with suspension. The rigorous statement is as follows. Choose sequences a_n and b_n of homotopy equivalences such that the following diagrams commute.

$$\begin{array}{ccc} S^1 \wedge Ex^{\infty}(S^n \wedge G_+) & \xrightarrow{a_n} & Ex^{\infty}(S^{n+1} \wedge G_+) \\ \uparrow & & \uparrow \\ S^1 \wedge S^n \wedge G_+ & \xrightarrow{\approx} & S^{n+1} \wedge G_+ \end{array} \qquad \begin{array}{ccc} Ex^{\infty}(S^n \wedge G_+) \wedge S^1 & \xrightarrow{b_n} & Ex^{\infty}(S^{n+1} \wedge G_+) \\ \uparrow & & \uparrow \\ S^n \wedge G_+ \wedge S^1 \xrightarrow{\approx} S^n \wedge S^1 \wedge G_+ & \xrightarrow{\approx} & S^{n+1} \wedge G_+ \end{array}$$

Here the vertical arrows denote the canonical inclusions. The maps a_n, b_n exist by the Kan condition on $Ex^{\infty}(S^{n+1} \wedge G_+)$.

We now can state the analogue of lemma 1.1.

Lemma 1.8. Let $u\colon M \wedge M' \longrightarrow Ex^{\infty}(S^n \wedge G_+)$ be an n-duality map; $\iota_n\colon Ex^{\infty}(S^n \wedge G_+) \longrightarrow Ex^{\infty}(S^n \wedge G_+)$ denotes the inversion map induced by $g \longmapsto g^{-1}$. Then

(i) $u'\colon M \wedge M \longrightarrow Ex^{\infty}(S^n \wedge G_+)$, $m' \wedge m \longmapsto \iota_n u(m \wedge m')$ is an n-duality map;

(ii) $\Sigma_{\ell}(u)\colon (S^1 \wedge M) \wedge M' \xrightarrow{id \wedge u} S^1 \wedge Ex^{\infty}(S^n \wedge G_+) \xrightarrow{a_n} Ex^{\infty}(S^{n+1} \wedge G_+)$

and

$\Sigma_r(u)\colon M \wedge (S^1 \wedge M') \xrightarrow{\approx} (M \wedge M') \wedge S^1 \xrightarrow{u \wedge id} Ex^{\infty}(S^n \wedge G_+) \wedge S^1 \xrightarrow{b_n} Ex^{\infty}(S^{n+1} \wedge G_+)$

are (n+1)-duality maps. □

We want to give a function space description of duality now, analogous to that of [14]. Let $F^n_G(M)$ denote the simplicial set of pointed right G-equivariant maps from M to $Ex^{\infty}(S^n \wedge G_+)$. G acts freely (pointed) from the left on this function space. Convert this to a right action using $g \longmapsto g^{-1}$. $F^n_G(M)$ is a Kan simplicial set, since $Ex^{\infty}(S^n \wedge G_+)$ satisfies the Kan condition.

The evaluation map

$$e\colon F^n_G(M) \wedge M \longrightarrow Ex^{\infty}(S^n \wedge G_+)$$

induces a map

$$\alpha_e\colon H^G_q(F^n_G(M)) \longrightarrow H^{n-q}_G(M).$$

Let $M \in \mathcal{U}_f(G)$ be of G-dimension k.

Lemma 1.9. The map α_e is an isomorphism in the range $0 \leq q \leq 2(n-k) - 1$.

Proof: By induction. The assertion is trivially true in the case $M = *$. For $M = G_+$ we have $F^n_G(M) = Map_*(S^0, S^n \wedge G_+) = S^n \wedge G_+$, and the evaluation map $G_+ \wedge S^n \wedge G_+ \xrightarrow{\approx} S^n \wedge G_+ \wedge G_+ \longrightarrow S^n \wedge G_+$ is a special case of the map of example 1.7.. So α_e is an isomorphism in that case. Since M was supposed to be finite, it has a G-skeleton filtration $* = M_0 \subset M_1 \subset \ldots \subset M_k = M$, such that we have cofibration sequences

$$M_{i-1} \rightarrowtail M_i \twoheadrightarrow \bigvee_{\alpha} S^i \wedge G_+,$$

$\alpha \in$ some finite index set. The general case then follows by a five lemma argument and the fact that the canonical map $S^{n-i} \wedge G_+ \longrightarrow \Omega^i(S^n \wedge G_+)$ is $(2(n-i) - 1)$-connected. □

Corollary 1.10. Let M be an object of $\mathcal{U}_f(G)$. Suppose that $u\colon M \wedge M' \longrightarrow Ex^{\infty}(S^n \wedge G_+)$ is an n-duality map. Let $n > \dim M$. Then the map

$$\hat{u}\colon M' \longrightarrow F^n_G(M)$$

adjoint to u is $(2(n-\dim M) - 1)$-connected.

Proof: There is a commutative diagram

$$\begin{array}{ccc} M \wedge M' & \xrightarrow{u} & \\ {\scriptstyle id \wedge \hat{u}}\downarrow & & Ex^{\infty}(S^n \wedge G_+) \\ M \wedge F^n_G(M) & \xrightarrow{e} & \end{array}$$

which implies another one

$$\begin{array}{ccc} H^G_q(M') & \xrightarrow{\alpha_u} & \\ \downarrow & & H^{n-q}_G(M) \\ H^G_q(F^n_G(M)) & \xrightarrow{\alpha_e} & \end{array}$$

By assumption α_u is an isomorphism for all q; α_e is an isomorphism in a certain range by lemma 1.9. Hence so is $\hat{u}$. □

<u>*Remark:*</u> If M is finite up to homotopy only, define dim M to be the least of all dimensions of finite G-sets which can be related to M by finite chains of homotopy equivalences. Then the assertion of the corollary also holds in this case.

Suppose that we are given two n-duality maps $u: M \wedge M' \longrightarrow Ex^{\infty}(S^n \wedge G_+)$ (resp. $v: N \wedge N' \longrightarrow Ex^{\infty}(S^n \wedge G_+)$). Let $f: M \longrightarrow N$ denote a morphism in $\mathcal{U}(G)$. If there exists a morphism $f': N' \longrightarrow M'$ such that the following diagram commutes up to homotopy

$$\begin{array}{ccc} M \wedge N' & \xrightarrow{f \wedge id} & N \wedge N' \\ {\scriptstyle id \wedge f'}\downarrow & & \downarrow{\scriptstyle v} \\ M \wedge M' & \xrightarrow{u} & Ex^{\infty}(S^n \wedge G_+) \end{array}$$

then f' will be called an *n-dual of* f.

Suppose that M (resp. N') is homotopy equivalent to a G-set of G-dimension at most k (resp. n). Let $n > 2k+1$. Further suppose that M' satisfies the Kan condition.

<u>*Lemma 1.11.*</u> In this situation there exists an n-dual of f.

Proof: The condition on a dual map $f': N' \longrightarrow M'$ is equivalent to asking if there exists an arrow $N' \longrightarrow M'$ such that the following diagram commutes up to homotopy.

$$\begin{array}{ccc} M' & \xrightarrow{\hat{u}} & F^n_G(M) \\ \uparrow & & \uparrow{\scriptstyle f^*} \\ N' & \xrightarrow{\hat{v}} & F^n_G(N) \end{array}$$

where $\hat{u}$ (resp. $\hat{v}$) denotes the adjoint of u (resp. v).
First assume that N' is n-dimensional (not just up to homotopy). By cor. 1.10. the map $\hat{u}$ is $(2(n-k)-1)$-connected. Hence, since M' is Kan, one can construct a lifting

up to homotopy as indicated by the broken arrow. Furthermore, this lifting is unique up to homotopy. In the general case, where N' is only homotopy equivalent to an n-dimensional G-set, one can still find a Kan simplicial set $\bar{N}$, and an n-dimensional G-set $\tilde{N}$ such that $N' \overset{\simeq}{\rightarrowtail} \bar{N} \overset{\simeq}{\leftarrowtail} \tilde{N}$.
Now one first finds a map $\tilde{N} \longrightarrow M'$; this extends to $\bar{N} \longrightarrow M'$ by the Kan condition on M'. Restricting this map to N' gives the desired map. □

Our next goal is to show that for every object of $\mathcal{U}_{hf}(G)$ there exists an n-dual if n is sufficiently large. To show this we need the following technical lemma.

Lemma 1.12. Let $M \xrightarrow{f} N \xrightarrow{g} N \underset{f}{\cup} CM$ be a cofibration sequence of objects of $\mathcal{U}_{hf}(G)$. (CM denotes the cone on M.) Let M' (resp. N') denote an n-dual of M (resp. N), and let $f'\colon N' \longrightarrow M'$ denote an n-dual of f. Then there exists an (n+1)-duality map

$$w\colon N \underset{f}{\cup} CM \wedge M' \underset{f'}{\cup} CN' \longrightarrow Ex^{\infty}(S^{n+1} \wedge G_+).$$

Proof: Let $u\colon M \wedge M' \longrightarrow Ex^{\infty}(S^n \wedge G_+)$ (resp. $v\colon N \wedge N' \longrightarrow Ex^{\infty}(S^n \wedge G_+)$) denote an n-duality map. The map w is constructed as follows.

Let α denote the composite

$$\Sigma(M \wedge N') \xrightarrow{\mu} (\Sigma M \wedge N') \vee (M \wedge \Sigma N') \xrightarrow{(id \wedge f', f \wedge -id)} (\Sigma M \wedge M') \vee (N \wedge \Sigma N'),$$

where μ denotes the comultiplication, and −id is a homotopy inverse. By definition of f' the following composite map is nullhomotopic:

$$\Sigma(M \wedge N') \xrightarrow{\alpha} (\Sigma M \wedge M') \vee (N \wedge \Sigma N') \xrightarrow{(\Sigma_\ell u, \Sigma_r v)} Ex^{\infty}(S^{n+1} \wedge G_+) .$$

Hence the map $(\Sigma_\ell u, \Sigma_r v)$ may be extended to a map

$$\bar{w}\colon (\Sigma M \wedge M') \wedge (N \wedge \Sigma N') \underset{\alpha}{\cup} C\Sigma(M \wedge N') \longrightarrow Ex^{\infty}(S^{n+1} \wedge G_+).$$

The left-hand side is isomorphic to $((N \underset{f}{\cup} CM) \wedge (M' \underset{f'}{\cup} CN'))/N \wedge M'$.
Let w denote the composite

$$N \underset{f}{\cup} CM \wedge M' \underset{f'}{\cup} CN' \longrightarrow\!\!\!\!\rightarrow ((N \underset{f}{\cup} CM) \wedge (M' \underset{f'}{\cup} CN'))/N \wedge M' \xrightarrow{\bar{w}} Ex^{\infty}(S^{n+1} \wedge G_+) .$$

By construction of w the following diagrams commute up to homotopy.

$$\begin{array}{ccc}
N \underset{f}{\cup} CM \wedge M' & \xrightarrow{h \wedge id} & \Sigma M \wedge M' \\
\downarrow{\scriptstyle id \wedge g'} & & \downarrow{\scriptstyle \Sigma_\ell u} \\
N \underset{f}{\cup} CM \wedge M' \underset{f'}{\cup} CN' & \xrightarrow{w} & Ex^{\infty}(S^{n+1} \wedge G_+)
\end{array}$$

$$\begin{array}{ccc}
N \wedge M' \underset{f'}{\cup} CN' & \xrightarrow{id \wedge h'} & N \wedge \Sigma N' \\
\downarrow{\scriptstyle g \wedge id} & & \downarrow{\scriptstyle \Sigma_r v} \\
N \underset{f}{\cup} CM \wedge M' \underset{f'}{\cup} CN' & \xrightarrow{w} & Ex^{\infty}(S^{n+1} \wedge G_+)
\end{array} .$$

Here $h\colon N \cup_f CM \longrightarrow M$ (resp. $h'\colon M' \cup_{f'} CN' \longrightarrow N'$) denotes the canonical map in the cofibre sequence of f (resp. f').

These diagrams imply the following commutative diagram of (co-)homology groups.

$$\begin{array}{ccccccccc}
H_q^G(N') & \longrightarrow & H_q^G(M') & \longrightarrow & H_q^G(M' \cup_{f'} CN') & \longrightarrow & H_q^G(\Sigma N') & \longrightarrow & H_q^G(\Sigma M') \\
\downarrow{\scriptstyle \alpha_{\Sigma_\ell v}} & & \downarrow{\scriptstyle \alpha_{\Sigma_\ell u}} & & \downarrow{\scriptstyle \alpha_w} & & \downarrow{\scriptstyle \alpha_{\Sigma_r v}} & & \downarrow{\scriptstyle \alpha_{\Sigma_r u}} \\
H_G^p(\Sigma N) & \longrightarrow & H_G^p(\Sigma M) & \longrightarrow & H_G^p(N \cup_f CM) & \longrightarrow & H_G^p(N) & \longrightarrow & H_G^p(M)
\end{array}$$

$(p = n+1-q)$.

Since by assumption u and v are duality maps, all the vertical arrows except possibly α_w are isomorphisms. Hence by the five lemma so is α_w, as was to be shown. □

Proposition 1.13. Let M be an object of $\mathcal{U}_{hf}(G)$. There exists an n-dual of M, if n is sufficiently large.

Proof: We say that an object N of $\mathcal{U}_{hf}(G)$ is obtained from M by *attaching of a* k-*cell* if N is isomorphic to the pushout of the following diagram of pointed G-maps

$$M \longleftarrow \partial\Delta^k \quad G_+ \rightarrowtail \Delta^k \wedge G_+$$

where $\partial\Delta^k$ denotes the boundary of the k-simplex, and the map on the right is the natural inclusion.

Since M is finite up to homotopy one can find a Kan simplicial set $\bar{M}$, and a finite G-set $\tilde{M}$ such that $M \overset{\simeq}{\rightarrowtail} \bar{M} \overset{\simeq}{\leftarrowtail} \tilde{M}$.

Suppose we have found an n-duality map $\tilde{M} \wedge M' \longrightarrow Ex^\infty(S^n \wedge G_+)$. Then we may extend this to a map $\bar{M} \wedge M' \longrightarrow Ex^\infty(S^n \wedge G_+)$, since the inclusion $\tilde{M} \wedge M' \longrightarrow \bar{M} \wedge M'$ is a homotopy equivalence, and because $Ex^\infty(S^n \wedge G_+)$ satisfies the Kan condition. This extended map is clearly also an n-duality map. Restricting this map to the subspace $M \wedge M'$ finally gives a duality map for M. This argument shows that there is no loss of generality in assuming M to be finite.

Now any finite object of $\mathcal{U}(G)$ may be obtained from the base point by attaching of a finite number of cells. Hence an n-dual of M can be constructed inductively, the inductive step being provided by lemma 1.12. □

Define a category $\mathcal{DU}^n(G)$ in which an object is a triple (M,M',u), where M and M' are objects of $\mathcal{U}_{hf}(G)$, and $u\colon M \wedge M' \longrightarrow Ex^\infty(S^n \wedge G_+)$ is an n-duality map. We add the technical condition that $\pi_i M = \pi_i M' = 0$, $i = 0,1$. A morphism from (M,M',u) to (N,N',v) is a pair of morphisms in $\mathcal{U}(G)$, $f\colon M \longrightarrow N$, and $f'\colon N' \longrightarrow M'$ such that the following diagram commutes

$$\begin{array}{ccc}
N' \wedge M & \xrightarrow{\ \mathrm{id} \wedge f\ } & N' \wedge N \\
{\scriptstyle f' \wedge \mathrm{id}}\downarrow & & \downarrow{\scriptstyle v} \\
M' \wedge M & \xrightarrow[\ u\ ]{} & Ex^\infty(S^n \wedge G_+)
\end{array}.$$

A morphism (f,f') is called an h-equivalence if both f and f' are h-equivalences. By lemma 1.8. there are two suspension functors Σ_ℓ (resp. Σ_r): $\mathcal{DU}^n(G) \longrightarrow \mathcal{DU}^{n+1}(G)$ given by suspending M (resp. M').

We are now going to compare the two settings for duality. Let X denote a connected simplicial set, and let G be its loop group in the sense of [7]. Let E be a universal G-bundle. There is an adjoint functor pair (cf. [16]):

$$\Phi_X: h\mathcal{R}(X) \longrightarrow h\mathcal{U}(G)$$

$$(Y,r,s) \longmapsto Y \times_X E/E$$

$$\Psi_G: h\mathcal{U}(G) \longrightarrow h\mathcal{R}(X)$$

$$M \longmapsto (M\times^G E \rightleftarrows * \times^G E).$$

Let ε again denote the trivial fibration $X \times S^o$. The spaces $\Psi_{G^2}(Ex^\infty(S^n \wedge G_+))$ can be used as Thom spaces $Th_n(\varepsilon)$ in the sense defined above.
Here G_+ is considered as an object of $\mathcal{U}(G \times G)$.
Define a functor

$$D\Phi : h\mathcal{DR}^n_\varepsilon(X) \longrightarrow h\mathcal{DU}^n(G)$$

$$(Y,Y',u) \longmapsto (\Phi_X(Y),\Phi_X(Y'),u'),$$

where u' is the composite

$$\Phi_X(Y') \wedge \Phi_X(Y) \longrightarrow \Phi_{X^2}(Th_n(\varepsilon)) = \Phi_{X^2}\Psi_{G^2}(Ex^\infty(S^n \wedge G_+)) \longrightarrow Ex^\infty(S^n \wedge G_+) .$$

Similarly,

$$D\Psi : h\mathcal{DU}^n(G) \longrightarrow h\mathcal{DR}^n_\varepsilon(X)$$

$$(u: M \wedge M' \longrightarrow Ex^\infty(S^n \wedge G_+) \longrightarrow (\Psi_G(M) \wedge \Psi_G(M') \longrightarrow \Psi_{G^2}(Ex^\infty(S^n \wedge G_+) = Th_n(\varepsilon)).$$

Proposition 1.14. DΦ and DΨ are mutually inverse homotopy equivalences.

Proof: We first remark that DΦ and DΨ are not adjoint. Let f: $h\mathcal{DR}^n_\varepsilon(X) \longrightarrow h\mathcal{DR}^n_\varepsilon(X)$ be given by $(Y,Y',u) \longrightarrow (\Psi\Phi(Y),Y,\bar{u})$, where $\bar{u}$: $\Psi\Phi(Y) \wedge Y' \longrightarrow \Psi\Phi(Y \wedge Y') \xrightarrow{\Psi\Phi(u)} \Psi\Phi\Psi(Ex^\infty(S^n \wedge G_+)) \longrightarrow (Ex^\infty(S^n \wedge G_+))$ and Φ (resp. Ψ) is short for Φ_X (resp. Ψ_G). Similarly, f' is the corresponding endofunctor of $h\mathcal{DR}^n_\varepsilon(X)$ defined by a condition on Y'. There is a natural transformation from the identity to f, and another one from f' to the identity. Since D$\Psi \cdot$ DΦ = f'f, the composite D$\Psi \cdot$ DΦ is therefore homotopic to the identity; similarly with the other composition. □

This proposition shows that both settings for duality are actually equivalent. In the next proposition it is shown that the choice of duality data is in fact a "contractible choice".

Proposition 1.15. The forgetful functor

$$\varepsilon : \varinjlim_{\Sigma_\ell,\Sigma_r} h\mathcal{DU}^n(G) \longrightarrow \varinjlim_{\Sigma} h\mathcal{U}_{hf}(G)$$

$$(M,M',u) \longmapsto M$$

is a weak homotopy equivalence.

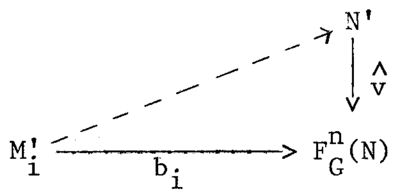
Proof: To prove the assertion we need a stronger finiteness condition on the objects of $h\mathcal{DU}^n(G)$. We adapt an argument of [19] to show that this condition may be assumed without loss of generality.

Let $h\mathcal{DU}^n(G)'$ denote the full subcategory of $h\mathcal{DU}^n(G)$ of those objects (M,M',u) which satisfy that M' is actually finite, not just finite up to homotopy.
The inclusion $h\mathcal{DU}^n(G)' \subset h\mathcal{DU}^n(G)$ is a homotopy equivalence. To see this we introduce two further subcategories. Namely, let $h\mathcal{DU}^n(G)_{Kan}$ denote the full subcategory consisting of those objects which satisfy that (the underlying simplicial set of) M' is a Kan simplicial set; let $h\mathcal{DU}^n(G)''$ be the full subcategory of those objects of $h\mathcal{DU}^n(G)$ which lie in either $h\mathcal{DU}^n(G)'$ or $h\mathcal{DU}^n(G)_{Kan}$. The inclusion $h\mathcal{DU}^n(G)'' \subset h\mathcal{DU}^n(G)$ is a homotopy equivalence. This may be seen from the existence of the functor Ex^∞. Ex^∞ extends to a functor $h\mathcal{DU}^n(G) \longrightarrow h\mathcal{DU}^n(G)_{Kan}$. There is a natural transformation $Ex^\infty \longrightarrow Id$, given by the canonical map $M' \longrightarrow Ex^\infty M'$. This shows that $h\mathcal{DU}^n(G)_{Kan} \subset h\mathcal{DU}^n(G)$ is a homotopy equivalence. By the same argument $h\mathcal{DU}^n(G)'' \subset h\mathcal{DU}^n(G)$ is a homotopy equivalence.

The next step is to show that the inclusion $i\colon h\mathcal{DU}^n(G)' \longrightarrow h\mathcal{DU}^n(G)''$ is a homotopy equivalence.
We use Quillen's theorem A, cf. [10]. So we have to show that the right fibre $(M,M',u)/i$ over any object (M,M',u) of $h\mathcal{DU}^n(G)''$ is contractible.
It suffices to prove that any finite diagram $\mathcal{D} \longrightarrow (M,M',u)/i$ in the fibre is contractible.
Let $(f_i,f_i')\colon (M,M',u) \longrightarrow (M_i,M_i',u_i)$ represent such a diagram. If M' is already finite there is nothing to prove since the diagram then has an obvious initial object. So assume M' is Kan. Since M' is also finite up to homotopy, one can find a diagram $M \overset{\simeq}{\rightarrowtail} \bar{M} \overset{\simeq}{\leftarrowtail} M'$, where $\tilde{M}$ is finite and $\bar{M}$ is obtained from $\tilde{M}$ by filling horns. Since the M_i' are all finite, one can find another subset $\approx\!\!M$ of $\bar{M}$ containing the images $f_i'(M_i')$ of all the simplicial sets M_i' and which can be obtained from $\approx\!\!M$ by filling finitely many (G)-horns. Hence $\approx\!\!M$ is itself finite (as a G-set).
Since M' is a Kan set one can find a retraction $\bar{M} \longrightarrow M'$. This gives a map $q\colon \approx\!\!M \longrightarrow M'$.
Define a duality map $\approx\!\!u$ as the composite $M \wedge \approx\!\!M \longrightarrow M \wedge M' \xrightarrow{u} Ex^\infty(S^n \wedge G_+)$. Then the object $((M,\approx\!\!M,u);\ (id,q)\colon (M,M',u) \longrightarrow (M,\approx\!\!M,\approx\!\!u))$ of the fibre is an initial object for the diagram $\mathcal{D}$. Hence the diagram is contractibe, whence i is a homotopy equivalence. This in turn implies that the inclusion $h\mathcal{DU}^n(G)' \subset h\mathcal{DU}^n(G)$ is a homotopy equivalence.

We are now reduced to proving that the restriction of the map ε of the proposition

to the subcategory $\varinjlim h\mathcal{D}\mathcal{U}^n(G)'$ is a homotopy equivalence.
Again we use theorem A of Quillen. Let N be an object of $\varinjlim h\mathcal{U}_{hf}(G)$. We have to show that the right fibre N/ε is contractible. Let $\mathcal{D}: I \to N/\varepsilon$ be a finite diagram in the fibre. $\mathcal{D}$ is represented by $(M_i, M_i^!, u_i: M_i \wedge M_i^! \to Ex^\infty(S^n \wedge G_+);\ a_i: N \xrightarrow{\simeq} M_i)_{i \in I}$.
By proposition 1.13. there exists an m-dual of N for some large m. We may assume that n is large, and in particular that there exists an n-dual of N. Let $v: N \wedge N' \to Ex^\infty(S^n \wedge G_+)$ denote an n-duality map with $N' \in \mathcal{U}_{hf}(G)$. Assume that N' is a Kan set.
Consider the following diagram



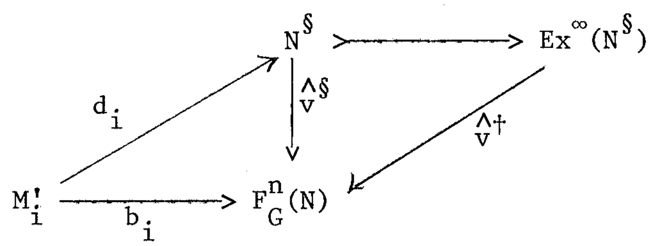
where $\hat{v}$ denotes the adjoint of v, and b_i is the composite

$$M_i^! \xrightarrow{\hat{u}_i} F_G^n(M_i) \xrightarrow{a_i^*} F_G^n(N) .$$

Here $\hat{u}_i$ is the adjoint of u_i and a_i^* is induced from a_i.
Let $n > 2 \max \{\dim M_i, \dim N\}$. Further assume that $\dim M_i^! \leqq n$. By coro. 1.10. the map $\hat{v}$ is $(2(n-\dim N) - 1)$-connected, hence by assumption on n it is at least n-connected.
Since $\dim M_i^! < n$, and N' was assumed to be Kan, by obstruction theory therefore there exist liftings up to homotopy $c_i: M_i^! \to N'$ of b_i as indicated by the broken arrow in the diagram. For each i choose a specific homotopy $h_i: M_i \times \Delta^1 \to F_G^n(N)$ such that $h_i | M_i^! \times 0 = \hat{v}c_i$, and $h_i | M_i^! \times 1 = b_i$.
The map b_i is at least n-connected since a_i^* is a homotopy equivalence and $\hat{u}_i$ is n-connected. Since $M_i^!$ and N' have no G-homology in dimensions $>n$, and by the Hurewicz theorem, c_i is therefore a homotopy equivalence.
Let $N^\S$ denote the pushout of the following diagram

$$N' \xleftarrow{\amalg c_i} \amalg M_i^! \times 0 \longrightarrow \amalg M_i^! \times \Delta^1 .$$

There is a map $\hat{v}^\S: N^\S \to F_G^n(N)$ given by $\hat{v}$ on N', and h_i on $M_i^! \times \Delta^1$. Further there is a commutative diagram



where $d_i: M_i^! = M_i^! \times 1 \to M_i^! \times \Delta^1 \to N^\S$, and $\hat{v}^\dagger$ is some extension of $\hat{v}^\S$ (which exists by the Kan condition on $F_G^n(N)$).

Since $Ex^\infty(N^\S)$ is Kan, the inclusion $\bigcup_i d_i(M_i^!) \subset Ex^\infty(N^\S)$ factors as

$$\bigcup_i d_i(M_i^!) \longrightarrow \bar{N} \xrightarrow{\simeq} Ex^\infty(N^\S),$$

where the second arrow is a homotopy equivalence and $\bar{N}$ is obtained from $\bigcup_i d_i(M_i^!)$ by attaching of finitely many G-cells. Now this union is a finite G-set. Therefore $\bar{N}$ is also finite. The map

$$f_i\colon M_i^! \xrightarrow{d_i} \bigcup_i d_i(M_i^!) \longrightarrow \bar{N}$$

is a homotopy equivalence, since its composition with $\bar{N} \xrightarrow{\simeq} Ex^\infty(N^\S)$ is one.
Let $\hat{v}$ denote the composite $\bar{N} \longrightarrow Ex^\infty(N^\S) \xrightarrow{\hat{v}^\dagger} F_G^n(N)$ and let $\bar{v}\colon N \wedge \bar{N} \longrightarrow Ex^\infty(S^n \wedge G_+)$ denote the adjoint of this map. By construction $\bar{v}$ is an n-duality map. The object $(N,\bar{N},\bar{v})$ of $h\mathcal{DU}^n(G)'$ maps to the diagram $(M_i,M_i^!,u_i,a_i)$ by (a_i,f_i). Hence $((N,\bar{N},\bar{v});\ id\colon N \longrightarrow N)$ is a cone point for the diagram $\mathcal{D}$.
This proves that any finite diagram in N/ε is nullhomotopic, as was to be shown. □

Remark: Just as in the case of the categories $\mathcal{R}(X)$ (resp. $\mathcal{DR}_\varepsilon^n(X)$) certain subcategories $\mathcal{U}_k^\ell(G)$ (resp. $\mathcal{DU}_k^{\ell,m}(G)$) of $\mathcal{U}(G)$ (resp. $\mathcal{DU}^n(G)$) may be defined. By restriction to a connected component one obtains from proposition 1.15. another homotopy equivalence

$$\lim_{\substack{\rightarrow \\ \ell,m}} h\mathcal{DU}_k^{\ell,m}(G) \longrightarrow \lim_{\substack{\rightarrow \\ \ell}} h\mathcal{U}_k(G). \qquad \square$$

Proof of prop. 1.6.: There is a commutative diagram

$$\begin{array}{ccc} h\mathcal{DR}_\varepsilon^n(X) & \xrightarrow{D\Phi} & h\mathcal{DU}^n(G) \\ \delta\downarrow & & \downarrow\varepsilon \\ h\mathcal{R}_{hf}(X) & \xrightarrow{\Phi} & h\mathcal{U}_{hf}(G) \end{array} .$$

Φ is a homotopy equivalence because it has an adjoint, $D\Phi$ is a homotopy equivalence by prop. 1.14.; ε becomes a homotopy equivalence after passing to the limit by prop. 1.15., therefore so does δ, as was to be shown. □

As a corollary to proposition 1.6. and 1.15. one obtains the following descriptions of A(X).

<u>*Corollary 1.16.*</u>

$$\begin{aligned} A(X) &\simeq \mathbf{Z} \times \Big|\lim_{\substack{\rightarrow \\ k,\ell,m}} h\mathcal{DR}_k^{\ell,m}(X)\Big|^+ \\ &\simeq \mathbf{Z} \times \Big|\lim_{\substack{\rightarrow \\ k,\ell,m}} h\mathcal{DU}_k^{\ell,m}(G)\Big|^+ \end{aligned}$$

(G = G(X)). □

Appendix

In this appendix we first prove that the categories $\mathcal{DR}^n_\xi(X)$ are covariantly functorial in X. Then we use this result to compare the concept of Spanier-Whitehead duality developed in this paragraph with other concepts of duality.

Let ξ_i denote a d-spherical fibration (with a section), $i = 1,2$. Let $\bar{f}: \xi_1 \longrightarrow \xi_2$ be a map covering $f: X_1 \longrightarrow X_2$; $\bar{f}$ induces a map $X_1^2 \cup_{X_1} \xi_1 \longrightarrow X_2^2 \cup_{X_2} \xi_2$, and hence a map $\bar{\bar{f}}: Th(\xi_1) \longrightarrow Th(\xi_2)$.

Proposition 1.17. In this situation there is a functor

$$f_*: \mathcal{DR}^n_{\xi_1}(X_1) \longrightarrow \mathcal{DR}^n_{\xi_2}(X_2)$$

given by

$$(Y_1, Y_1', u_1) \longmapsto (Y_1 \underset{X_1}{\cup} X_2,\ Y_1' \underset{X_1}{\cup} X_2, \bar{u}_1)$$

where $\bar{u}_1$ denotes the composite

$$(Y_1 \underset{X_1}{\cup} X_2) \wedge (Y_1' \underset{X_1}{\cup} X_2) = (Y_1 \wedge Y_1') \underset{X_1^2}{\cup} X_2^2 \longrightarrow Th_{n-d}(\xi_1) \underset{X_1^2}{\cup} X_2^2 \longrightarrow Th_{n-d}(\xi_2)$$

the last map being induced by $\bar{\bar{f}}$.

Proof: The fact which requires proof is that $\bar{u}_1$ indeed defines an n-duality map. Let $G_i = \pi_1 X_i$, $i = 1,2$. We have to show that the map

$$\bar{u}_1: H_q(Y_1' \underset{X_1}{\cup} X_2, X_2; \mathbb{Z}[G_2]) \longrightarrow H^{n-q}(Y_1 \underset{X_1}{\cup} X_2, X_2; \mathbb{Z}[G_2])$$

given by slant product with a certain class t in $H^d(Th(\xi_1), X_1^2; \mathbb{Z}[G_1])$ is an isomorphism for all $q \leqq n$.

Let $C_* = C_*(\tilde{Y}_1, \tilde{X}_1)$ (resp. $C_*' = C_*(\tilde{Y}_1', \tilde{X}_1)$) denote the chain complex of the universal cover of the pair (Y_1, X_1) (resp. $Y_1', X_1)$). Define $D_* = Hom_{\mathbb{Z}[G_1]}(C_{n-*}, \mathbb{Z}[G_1])$. The chain complexes C_*, C_*', and D_* consist of free $\mathbb{Z}[G_1]$-modules of finite rank. There is a (degree 0) map $C_*' \longrightarrow D_*$, given by $c' \longrightarrow z/c'$, where $c' \in C_*'$, and z is a cocycle in $Hom_{\mathbb{Z}[G_1 \times G_1]}(C_n((Y_1,X_1) \times (Y_1',X_1)); \mathbb{Z}[G_1])$ representing the image under u_1^* of the class t in $H^n(Y_1 \wedge Y_1', X_1 x X_1; \mathbb{Z}[G_1])$. Using this map define a map of double complexes

$$\varphi_{u_1}: F_* \underset{G_1}{\otimes} C' \longrightarrow F_* \underset{G_1}{\otimes} D_*$$

where F_* denotes a free $\mathbb{Z}[G_1]$-resolution of the G_1-module $\mathbb{Z}[G_2]$. There a spectral sequences associated to these double complexes which are given by

$$E^1_{p,q} = F_p \underset{G_1}{\otimes} H_q(C_*') = F_p \underset{G_1}{\otimes} H_q(Y_1', X_1; \mathbb{Z}[G_1])$$

$$\Longrightarrow H_{p+q}(C_* \underset{G_1}{\otimes} \mathbb{Z}[G_2]) = H_{p+q}(Y_1 \underset{X_1}{\cup} X_2, X_2; \mathbb{Z}[G_2]),$$

and

$$E'^{1}_{p,q} = F_p \underset{G_1}{\otimes} H_q(D_*) = F_p \underset{G_1}{\otimes} H^{n-q}(\mathrm{Hom}_{\mathbb{Z}[G_1]}(C_*, \mathbb{Z}[G_1]))$$

$$= F_p \underset{G_1}{\otimes} H^{n-q}(Y_1,X_1; \mathbb{Z}[G_1])$$

$$\Longrightarrow H_{p+q}(D_* \underset{G_1}{\otimes} \mathbb{Z}[G_2]) = H^{n-(p+q)}(Y_1 \underset{X_1}{\cup} X_2,X_2; \mathbb{Z}[G_2]).$$

By assumption u_1 is an n-duality map, therefore φ_{u_1} induces an isomorphism $E^{1}_{p,q} \xrightarrow{\approx} E'^{1}_{p,q}$. Hence we obtain an isomorphism of the abutments as was to be shown. □

The proposition has the following immediate consequence. Given an n-duality map $u: Y \wedge Y' \longrightarrow Th_{n-d}(\varepsilon)$ in $\mathcal{DR}^n_\varepsilon(X)$ the induced map

$$\bar{u}: Y/X \wedge Y'/X \longrightarrow Th_{n-d}(\varepsilon)/X^2 \simeq S^n \wedge X_+ \longrightarrow S^n$$

is an ordinary n-duality map.

Indeed, this is the special case $X_1 = X$, $X_2 = *$ of the proposition.

The concept of duality is closely linked with some sort of *finiteness condition*. The definition of duality used in this paper requires that the cofibre of the inclusion $X \to Y$ is finite (at least up to homotopy). In view of this the concept of duality defined here might be called 'cofibre-wise' duality.
A different finiteness condition would be to ask that the fibre of the retraction $Y \to X$ be finite. The ensuing concept of duality would accordingly have to be called 'fibre-wise' duality. In the theory of 'fibre-wise' Spanier-Whitehead duality one starts with a space Y over X, satisfying that the structural map $Y \to X$ is a fibration with fibre homotopy equivalent to a finite complex. The Spanier-Whitehead dual is then defined by taking the ordinary dual of each fibre, cf. [8], [9], [20].
In contrast with prop. 1.17. it turns out that this kind of duality depends contravariantly on the base space. Moreover, one has an 'operation' of 'fibre-wise' duality on 'cofibre-wise' duality. This is induced from the action of the category $\mathcal{R}_{fib}(X)$ on $\mathcal{R}(X)$, cf. the definition after lemma 1.1.

The concepts of cofibre-wise duality and of fibre-wise duality are different in general: Even if both kinds of duals are defined, they may not coincide. As an example consider the case where X is a compact n-dimensional manifold with boundary ∂. Let $Y = X \times S^0$. Y is a space over X via the projection. Under these circumstances both concepts of duality are defined. A fibre-wise n-dual is given by $X \times S^n$, whereas a cofibre-wise n-dual is given by $X \underset{\partial}{\cup} X$, the double of X, as one may see from the geometric description of duality given below in lemma 2.8.

§ 2. The canonical involution in the algebraic K-theory of spaces

One of the reasons for studying algebraic K-theory is that it gives information about the *concordance spaces* (or synonymously: pseudo-isotopy spaces) of manifolds.

Reacall the definition. Let X be a compact manifold with boundary ∂X. Let $C(X) = \mathrm{Aut}(X \times [0,1] \text{ rel } X \times 0 \cup \partial X \times [0,1])$, where Aut(...) denotes CAT-automorphisms of X. There is a canonical involution $\iota: C(X) \longrightarrow C(X)$ given by

$$\iota(f) = (\mathrm{id} \times r) \circ f \circ (\mathrm{id} \times r) \circ ((f|X \times 1)^{-1} \times \mathrm{id}),$$

where $r: [0,1] \longrightarrow [0,1]$ denotes the reflection at the midpoint. This involution on $C(X)$ gives (after localizing away from 2) a splitting up to homotopy into the two eigenspaces of ι:

$$C(X) \simeq C(X)^s \times C(X)^a,$$

where $C(X)^s$ (resp. $C(X)^a$) denotes the symmetric (resp. anti-symmetric) part of $C(X)$. The interest in this splitting comes from the fact that the factors have a meaning of their own, and can be treated with different methods, cf. [4].
There is a stabilization map

$$\Sigma: C(X) \longrightarrow C(X \times I)$$

given by product with the interval (or rather a technical modification of this, because of the condition of standard behaviour at the boundary). The *stable concordance space* of X is defined as $\underline{C}(X) = \lim\limits_{\overrightarrow{k}} C(X \times I^k)$. By [4], $\underline{C}(X)$ is a homotopy functor. It is this space that can be related to the algebraic K-theory of X. Following [18], we shall describe this relationship. Let X denote a compact manifold of dimension d with boundary ∂X; I denotes the interval [a,b]. A *partition* is a triple (M,F,N), where M is a compact codimension zero submanifold of $X \times I$, N is the closure of the complement of M, and $F = M \cap N$. F is to be standard in a neighborhood of $\partial X \times I$, i.e. there exists a number $t \in I$ such that F equals $X \times t$ in this neighborhood.
Let $P(X)$ denote the simplicial set in which a p-simplex is a (CAT-) locally trivial family of partitions parametrized by the p-simplex Δ^p. Let $H(X)$ denote the simplicial subset of $P(X)$ defined by the condition that M is an h-cobordism rel. boundary between $X \times 0$ and F. $H(X)$ is called the h-*cobordism-space* of X.

Proposition 2.1. $H(X)$ is a classifying space for $C(X)$.

Proof: To prove this, we construct a free action of the simplicial group $C(X)$ on some contractible space $E(X)$ such that the orbit space is the component of $H(X)$ containing the trivial cobordism. Let $E(X)$ denote the space (=simplicial set) of embeddings $X \times [0,1] \longrightarrow X \times [a,b]$ restricting to the identity on $X \times a$. This is a space of collars and hence is contractible. $C(X)$ acts on $E(X)$ by composition of maps. $E(X)$ may also be viewed as the space of trivial h-cobordisms together with a given trivialization. The action of $C(X)$ then just changes the trivialization. Hence the orbit space is indeed the 0-component of $H(X)$, as was to be shown. □

There is an obvious involution on the space $H(X)$, given ty turning a partition upside down. One may ask what is the relation between the involutions on $C(X)$ and $H(X)$.

Proposition 2.2. The involutions on $C(X)$ and on $H(X) = BC(X)$ agree up to homotopy.

Proof: Define a map $\varphi\colon C(X) \longrightarrow E(X)$ by $\varphi(f)(x,t) = \alpha(f(x,t/2))$, where $\alpha\colon X\times[0,1] \longrightarrow X\times[a,b]$ is the canonical linear isomorphism. Similarly, let $\psi\colon C(X) \longrightarrow E(X)$ take f to $\psi(f)\colon\ (x,t) \longmapsto \alpha(\iota(f)(x,t/2)$. There is a natural map $p\colon E(X) \longrightarrow H(X)$, defined by forgetting the product structure of a collar:

$$f\colon X\times[0,1] \longrightarrow X\times[a,b] \longrightarrow (\mathrm{im}(f),\ldots)\ .$$

Let $\iota'\colon H(X) \longrightarrow H(X)$ denote the involution on $H(X)$. We obtain a pull-back diagram

$$\begin{array}{ccc} C(X) & \xrightarrow{\ \psi\ } & E(X) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\iota' p} \\ E(X) & \xrightarrow[\ p\ ]{} & H(X) \end{array} .$$

Clearly the involution on $C(X)$ can be described by interchanging the corners of the diagram and applying the involution ι' on $H(X)$. In view of prop. 2.1. the diagram is also homotopy cartesian. This proves the proposition because from the diagram one obtains homotopy equivalences

$$E(X) \times_{H(X)} E(X) \xrightarrow{\simeq} E(X) \times_{H(X)} H(X)^I \times_{H(X)} E(X) \xleftarrow{\simeq} \Omega H(X)$$

which are compatible with the involutions if the middle term is given the involution defined by

$(f,w\colon I \longrightarrow H(X),g) \longmapsto (g,\ \iota'\circ w\circ r,f)$, $r\colon I \longrightarrow I$ the reflection map.

□

On the simplicial set of partitions $P(X)$ define a partial ordering by letting $(M,F,N) < (M',F',N')$ if firstly M is contained in M', and secondly the maps

$$F' \longrightarrow M' - (M - F) \longleftarrow F$$

are homotopy equivalences. This defines a simplicial partially ordered set, and hence a simplicial category which will be denoted $hP(X)$. We have a particular partition given by attaching k trivial m-handles to $X\times[a,a']$ in such a way that the complementary (d-m)-handles are trivially attached to $X\times[b,b']$, $a<a'<b'<b$. Let $hP_k^{\ell,m}(X)$ be the connected component of $hP(X)$ containing this particular partition, $(\ell = d-m)$.

An (anti-)involution on $hP(X)$ is defined by the contravariant functor

$$T'\colon hP(X) \longrightarrow hP(X),\quad (M,F,N) \longmapsto (N^*,F^*,M^*),$$

where M^* (resp. N^*) is the image of M (resp. N) under the map $\mathrm{id}\times r\colon X\times I \longrightarrow X\times I$. It restricts to a contravariant functor

$$T': h\mathcal{P}_k^{\ell,m}(X) \longrightarrow h\mathcal{P}_k^{m,\ell}(X) .$$

In [18] it is proved that the categories $h\mathcal{P}_k^{\ell,m}(X)$ approximate $A(X)$. To fit these approximations together one needs a stabilization process. There are two ways to stabilize a partition (M,F,N), namely taking the lower (resp. upper) part to its product with an interval. These disagree because of the condition of standard behaviour near the boundary. We have to consider a technical modification of the various spaces of partitions. We fix some standard choices. Let $X' \subset \mathrm{Int}\, X$ be a submanifold of X such that $\mathrm{Cl}(X-X')$ is a collar on ∂X. Similarly, let J denote an interval containing two subintervals J', J'' such that $J' \subset \mathrm{Int}\, J$, $J'' \subset \mathrm{Int}\, J'$, further let $[a',b']$ be a symmetric subinterval of I.

Let $\underline{\mathcal{P}}(X)$ be the simplicial subset of $\mathcal{P}(X)$ of those partitions satisfying that

$$F \subset X \times [a',b'] \;;\; F \cap (X-X') \times I = (X-X') \times a'.$$

The inclusion $\underline{\mathcal{P}}(X) \subset \mathcal{P}(X)$ (resp. $h\underline{\mathcal{P}}(X) \subset h\mathcal{P}(X)$) is a homotopy equivalence. Define the *lower stabilization* as the map

$$\sigma_\ell : h\underline{\mathcal{P}}(X) \longrightarrow h\underline{\mathcal{P}}(X \times J)$$

which takes the lower part of a partition (M,F,N) to

$$M \times J' \cup X \times [a,a'] \times J \subset X \times I \times J.$$

The upper part of a partition is mapped by σ_ℓ to the fibrewise suspension of N considered as a space over X.

The *upper stabilization* is the map

$$\sigma_u : h\underline{\mathcal{P}}(X) \longrightarrow h\underline{\mathcal{P}}(X \times J)$$

defined by

$$M \longmapsto M \times J' \cup X' \times [a,b'] \times \mathrm{Cl}(J'-J'') \cup X \times [a,a'] \times J \subset X \times I \times J.$$

The involution T' does not restrict to a map $\underline{\mathcal{P}}(X) \longrightarrow \underline{\mathcal{P}}(X)$ (because of the standard behaviour near the boundary). So a slight modification of T' is necessary. Choose a map $j: \mathcal{P}(X) \longrightarrow \underline{\mathcal{P}}(X)$ homotopy inverse to the natural inclusion $i: \underline{\mathcal{P}}(X) \longrightarrow \mathcal{P}(X)$. Letting $T = jT'i$ define a map $T: \underline{\mathcal{P}}(X) \longrightarrow \underline{\mathcal{P}}(X)$ (resp. a contravariant functor $T: h\underline{\mathcal{P}}(X) \longrightarrow h\underline{\mathcal{P}}(X)$), and one verifies

Lemma 2.3. (i) T is an involution up to homotopy, i.e. $T^2 \simeq \mathrm{id}$;
(ii) $\sigma_u T \simeq T\sigma_\ell$; (iii) $T\sigma_u \simeq \sigma_\ell T$. □

Now consider the limit $\lim\limits_{\substack{\rightarrow\\ \ell,m}} h\underline{\mathcal{P}}(X \times J^{\ell+m-d})$ where the maps in the direct system are given by σ_ℓ (resp. σ_u). Using a mapping cylinder argument, T can be defined as a map

$$\lim_{\substack{\rightarrow\\ \ell,m}} h\underline{\mathcal{P}}(X \times J^{\ell+m-d}) \longrightarrow \lim_{\substack{\rightarrow\\ \ell,m}} h\underline{\mathcal{P}}(X \times J^{\ell+m-d})$$

and from lemma 2.3. we have

Lemma 2.4. T is a weak involution in the sense that the restriction of T^2 to any compactum is homotopic to the restriction of the identity; in particular, T induces an involution on homotopy groups. □

In [18] it is proved that a connected component of A(X) can be obtained by performing the + construction on the space

$$\left| \lim_{\substack{\rightarrow \\ k,\ell,m}} h\underline{P}_k^{\ell,m}(X \times J^{\ell+m-d}) \right| .$$

Hence lemma 2.4. provides a weak involution on A(X). We continue to denote this involution with the letter T.

The relation between algebraic K-theory and concordance spaces may be described by a certain commutative diagram

$$\begin{array}{ccc} \lim\limits_{\substack{\rightarrow \\ \ell,m}} \underline{H}(X \times J^{\ell+m-d}) & \longrightarrow & \lim\limits_{\substack{\rightarrow \\ k,\ell,m}} \underline{P}_k^{\ell,m}(X \times J^{\ell+m-d}) \\ \downarrow & & \downarrow \\ \lim\limits_{\substack{\rightarrow \\ \ell,m}} h\underline{H}(X \times J^{\ell+m-d}) & \longrightarrow & \lim\limits_{\substack{\rightarrow \\ k,\ell,m}} h\underline{P}_k^{\ell,m}(X \times J^{\ell+m-d}) \end{array}$$

where $\underline{H}(\ldots)$ denotes the intersection of $H(\ldots)$ with $\underline{P}(\ldots)$, $h\underline{H}(\ldots)$ is the simplicial subcategory of $h\underline{P}(\ldots)$ with $\underline{H}(\ldots)$ as simplicial set of objects. The vertical maps of the diagram are the natural inclusions, and the horizontal maps are given by the identification of $\underline{H}(\ldots)$ (resp. $h\underline{H}(\ldots)$) with $\underline{P}_o^{\ell,m}(\ldots)$ (resp. $h\underline{P}_o^{\ell,m}(\ldots)$). In [18] it is shown that after performing the + construction the diagram is homotopy cartesian in a range of dimensions. Further the term in the lower left corner is contractible.

Each of the terms in the diagram has a description in terms of spaces of partitions. Further the operation of turning a partition upside down gives an involution on each of these spaces. The vertical maps in the diagram are compatible with the involution by definition of the involution on the categories $h\underline{P}(X)$, resp. $h\underline{H}(X)$. The horizontal maps are given by the canonical map from a member of a direct system to its limit. They are compatible with the involution because the involution is defined on each of the spaces in the direct system.

Our next goal will be to relate the involution T to another one defined in a quite different manner. We return to the setting of § 1. The description of A(X) given there (in particular cor. 1.14.) provides a natural involution on A(X) in a straightforward way. The details are as follows.

Let X be a simplicial set, $\xi \longrightarrow X$ an (orientable) d-spherical fibration (with a section); Th(ξ) is a Thom space of ξ in the sense of § 1. Let ρ_n denote the self map of S^n given by the following permutation of factors:

$$S^n \equiv S_1^1 \wedge S_2^1 \wedge \ldots \wedge S_n^1 \xrightarrow{\approx} S_n^1 \wedge S_{n-1}^1 \wedge \ldots \wedge S_1^1 \equiv S^n.$$

Define a contravariant functor

$$\tau_{\xi,n}: h\mathcal{DR}^n_\xi(X) \longrightarrow h\mathcal{DR}^n_\xi(X)$$
$$(Y,Y',u) \longmapsto (Y',Y,\tau_{\xi,n}(u)),$$

where $\tau_{\xi,n}(u)$: $Y' \wedge Y \xrightarrow{\approx} Y \wedge Y' \xrightarrow{u} S^{n-d} \wedge Th(\xi) \xrightarrow{\rho_{n-d} \wedge \iota} S^{n-d} \wedge Th(\xi) = Th_{n-d}(\xi)$.

By lemma 1.1. this is a duality map again. Clearly, $\tau^2_{\xi,n}$ = id. The map ι in this definition ensures that $\tau_{\xi,n}$ is a map over $X \times X$, and the map ρ_{n-d} is introduced to guarantee compatibility with the suspension functors. Indeed, one easily verifies that

(i) $\Sigma_\ell \tau_{\xi,n} = \tau_{\xi,n+1}\Sigma_r$, (ii) $\Sigma_r \tau_{\xi,n} = \tau_{\xi,n+1}\Sigma_\ell$.

Therefore one has a well-defined functor

$$\tau_\xi: \varinjlim_{\Sigma_\ell,\Sigma_r} h\mathcal{DR}^n_\xi(X) \longrightarrow \varinjlim_{\Sigma_\ell,\Sigma_r} h\mathcal{DR}^n_\xi(X)^{op} ,$$

and in particular τ restricts to a functor

$$\tau_\xi: \varinjlim_{\ell,m} h\mathcal{DR}^{\ell,m}_k(X) \longrightarrow \varinjlim_{\ell,m} h\mathcal{DR}^{\ell,m}_k(X)^{op} .$$

By cor. 1.16. the categories $h\mathcal{DR}^{\ell,m}_k(X)$ approximate A(X). So one finally obtains an involution τ_ξ on A(X) depending on the spherical fibration ξ. By prop. 1.17. this involution is natural for maps of pairs $(X,\xi) \longrightarrow (X',\xi')$, where $\xi \longrightarrow \xi'$ covers $X \longrightarrow X'$.

We now want to investigate the dependence of the involution τ_ξ on the spherical fibration ξ.

Recall from § 1 that there is an operation of (spherical) fibrations on spaces over X, given by $(Y,\xi) \longmapsto \xi \cdot Y$. Let $\xi\cdot$: $A(X) \longrightarrow A(X)$ denote the map induced from this operation. Let ξ^{-1} denote an inverse of ξ; ε is the trivial spherical fibration $X \times S^0 \longrightarrow X$.

Proposition 2.5. The following diagram commutes up to homotopy:

$$\begin{array}{ccc} A(X) & \xrightarrow{\tau_\varepsilon} & A(X) \\ {\scriptstyle \xi^{-1}\cdot}\downarrow & \searrow^{\tau_\xi} & \downarrow{\scriptstyle \xi\cdot} \\ A(X) & \xrightarrow[\tau_\varepsilon]{} & A(X) \end{array}$$

Proof: We prove that the right triangle commutes up to homotopy. The proof for the other triangle is entirely analogous. By cor. 1.3. there are maps φ_ξ (resp. ψ_ξ): $h\mathcal{DR}^n_\varepsilon(X) \longrightarrow h\mathcal{DR}^{n+d}_\xi(X)$ given by

$$(Y,Y',u) \longmapsto (\xi \cdot Y, Y', (\xi \wedge \varepsilon) \cdot u)$$

(resp. $(Y,Y',u) \longmapsto (Y, \xi \cdot Y', (\varepsilon \wedge \xi) \cdot u)$).

Let $\delta_\xi\colon h\mathcal{DR}^n_\xi(X) \longrightarrow h\mathcal{R}_{hf}(X)$ denote the forgetful functor $(Y,Y',u) \longmapsto Y$. There is a commutative diagram

$$\begin{array}{ccccccc}
h\mathcal{R}_{hf}(X) & \xleftarrow{\ \delta_\varepsilon\ } & h\mathcal{DR}^n_\varepsilon(X) & \xrightarrow{\ \tau_\varepsilon\ } & h\mathcal{DR}^n_\varepsilon(X) & \xrightarrow{\ \delta_\varepsilon\ } & h\mathcal{R}_{hf}(X) \\
\| & & \downarrow{\scriptstyle \psi_\xi} & & \downarrow{\scriptstyle \varphi_\xi} & & \downarrow{\scriptstyle \xi\cdot} \\
h\mathcal{R}_{hf}(X) & \xleftarrow[\ \delta_\xi\ ]{} & h\mathcal{DR}^{n+d}_\xi(X) & \xrightarrow[\ \tau_\xi\ ]{} & h\mathcal{DR}^{n+d}_\xi(X) & \xrightarrow[\ \delta_\xi\ ]{} & h\mathcal{R}_{hf}(X)
\end{array} .$$

Restricting to the connected components of 'spherical objects', and passing to the limit gives homotopy equivalences δ_ε (resp. δ_ξ) by prop. 1.6. together with lemma 1.5., and also φ_ξ (resp. ψ_ξ) by lemma 1.5. The upper row of the diagram represents the involution τ_ε on A(X), the lower row represents τ_ξ. This proves the proposition. □

Of course, an involution on A(X) can also be defined using simplicial sets with group action. It is induced by the contravariant functor, also denoted τ_n,

$$h\mathcal{DU}^n(G) \longrightarrow h\mathcal{DU}^n(G), \qquad (M,M',u) \longmapsto (M',M,\tau_n(u)),$$

where

$$\tau_n(u)\colon M' \wedge M \xrightarrow{\approx} M \wedge M' \xrightarrow{u} Ex^\infty(S^n \wedge G_+) \xrightarrow{Ex^\infty(\rho_n \wedge \iota)} Ex^\infty(S^n \wedge G_+),$$

$$(\iota\colon G_+ \longrightarrow G_+,\ g \longmapsto g^{-1}).$$

One easily verifies that the functors $D\Phi$ and $D\Psi$ are equivariant with respect to this involution. By prop. 1.14. this involution is therefore the same, up to homotopy, as the involution τ_ε defined just before.

In the following it will be convenient to have a slightly different description of the categories $h\mathcal{DR}^n_\xi(X)$ available. Namely, instead of working with simplicial sets one could as well use spaces having the homotopy type of CW complexes and continuous maps throughout to define the categories $h\mathcal{DR}^n_\xi(X)$. Geometric realization induces a functor $h\mathcal{DR}^n_\xi(X) \longrightarrow h\mathcal{DR}^n_\xi(|X|)$ which is a weak homotopy equivalence.
In the following the symbol $h\mathcal{DR}^n_\xi(X)$ will have either of these two meanings depending on whether X is a simplicial set or a topological space.

To compare the involution defined on $h\underline{P}(X)$ with that defined on $h\mathcal{DR}^n_\xi(X)$ one has to relate both categories. Now $h\mathcal{DR}^n_\xi(X)$ is a category while $h\underline{P}(X)$ is a simplicial category. So to compare both one has to make $h\mathcal{DR}^n_\xi(X)$ a simplicial category as well.
Let $h\mathcal{DR}^n_\xi(X)_p$ denote the category with objects locally trivial p-parameter families Y,Y' of objects of $h\mathcal{R}_{hf}(X)$ together with a p-parameter family of n-duality maps

$$\begin{array}{ccc}
Y \wedge Y' & \longrightarrow & Th_{n-d}(\xi) \times \Delta^p \\
& \searrow \quad \swarrow & \\
& \Delta^p &
\end{array} .$$

Similarly, morphisms are given by p-parameter families of morphisms of $h\mathcal{DR}^n_\xi(X)$.
(Here $Y \wedge Y'$ denotes a p-parameter version of the fibrewise smash product over X,

namely

$$Y \wedge Y' = Y \times_{\Delta^p} Y' \cup_{(X\times\Delta^p)\times_{\Delta^p} Y' \cup Y \times_{\Delta^p} (X\times\Delta^p)} X \times \Delta^p \times X \quad .)$$

The categories $hDR^n_\xi(X)_p$ assemble to a simplicial category which is denoted $hDR^n_\xi(X)_\bullet$. Forgetting part of the structure we also have a simplicial category $hR_{hf}(X)_\bullet$. Identifying $hDR^n_\xi(X)$ with $hDR^n_\xi(X)_o$, the total degeneracy map gives an inclusion $hDR^n_\xi(X) \longrightarrow hDR^n_\xi(X)_\bullet$, and we have

Lemma 2.6. The inclusion $hDR^n_\xi(X) \longrightarrow hDR^n_\xi(X)_\bullet$ is a weak homotopy equivalence.

Proof: Indeed, this follows from the fact that the maps $hDR^n_\xi(X) \longrightarrow hDR^n_\xi(X)_k$ are weak homotopy equivalences for all k. □

Now let X denote a compact (orientable) manifold of dimension d. There is a map $h\underline{P}(X) \longrightarrow hR(X)_\bullet$ given by $(M,F,N) \longmapsto M$, where M is considered as a space over $X \times a$. We want to lift this to a map $h\underline{P}(X) \longrightarrow hDR^n_\xi(X)_\bullet$ (for suitable ξ and n) in such a way that it is compatible with the involution on both terms. To do so one associates to a partition (M,F,N) a duality map as follows. Let $M' = M - F$, $N' = N - F$. The inclusion

$$i: M' \times N' \longrightarrow (X \times [a,b])^2 - \text{diagonal}$$

induces a map over $X \times X$

$$j: M' \wedge N' \longrightarrow ((X \times [a,b])^2 - \Delta) \cup_{X^2 \times [a,b] \times b \,\cup\, X^2 \times a \times [a,b]} X^2$$

(Δ = diagonal).

If (M,F,N) is a p-parameter family of partitions, one replaces the product $M' \times N'$ by the fibre product $M' \times_{\Delta^p} N'$, and $M' \wedge N'$ by the p-parameter version of the smash-product defined above.

Let Z denote the target of the map j. Z is an object of $R(X^2)$ by the obvious projection map, and the inclusion given by

$$X \times X \longrightarrow X \times a \times X \times b \longrightarrow Z \,.$$

Let ξ denote the tangent microbundle of X (resp. an $\mathbb{R}^n$-bundle to which it corresponds by the Kister-Mazur theorem). Let ξ^+ denote the fibrewise one-point-compactification of ξ. ξ^+ is an orientable d-spherical fibration with a section. Convert the map $Z \longrightarrow X^2$ into a fibration $Z' \longrightarrow X^2$.

Lemma 2.7. $Z' \longrightarrow X^2$ is a Thom space of ξ^+ in the sense of § 1.

Proof: We first show that $Z \longrightarrow X^2$ is in the same component of $hR(X \times X)$ as the object $X \times X \cup_X \xi^+$. We represent ξ^+ as follows. Let η denote a neighborhood of the diagonal in $X \times X$ which is an $\mathbb{R}^n$-bundle; η' is a smaller neighborhood satisfying the same requirement. Then $\xi^+ = \eta \cup_{\eta-\eta'} X$. We have the following chain of homotopy equivalences

$$Z' \xleftrightarrow{\simeq} Z \xleftarrow{\simeq} (X \times [a,b])^2 - \Delta \xleftarrow{\simeq} (X \times [a,b])^2 - U$$

$U = \eta' \times \Delta'$, Δ' = diagonal of $[a,b] \times [a,b]$)

$$= (X^2 - \eta') \times [a,b]^2 \cup_{(X^2-\eta') \times ([a,b]^2 - \Delta')} X^2 \times ([a,b]^2 - \Delta')$$

$$\xrightarrow{\simeq} (X^2 - \eta') \cup_{(X^2-\eta') \times S^o} X^2 \times S^o = X^2 \cup_{X^2 - \eta'} X^2$$

$$= X^2 \cup_{(X^2-\eta') - (X^2-\eta)} X^2 - (X^2 - \eta) = X^2 \cup_{\eta-\eta'} \eta \; .$$

Now the inclusion $\eta - \eta' \longrightarrow X^2$ is homotopic to the composite $\eta - \eta' \xrightarrow{pr} X \xrightarrow{\Delta} X^2$, as one sees from the diagram

$$\begin{array}{ccccc} \eta-\eta' & \longrightarrow & \eta & \longrightarrow & X^2 \\ & {\scriptstyle pr}\searrow & \downarrow{\scriptstyle pr} & \nearrow{\scriptstyle \Delta} & \\ & & X & & \end{array}$$

where the left triangle is commutative, and the right triangle commutes up to homotopy since $\eta \xrightarrow{pr} X$ is a homotopy equivalence. Therefore $X^2 \cup_{\eta-\eta'} \eta$ is in the same connected component of $h\mathcal{R}(X^2)$ as the object

$$\text{pushout } (X^2 \longleftarrow X \longleftarrow \eta - \eta' \longrightarrow \eta) =$$

$$X^2 \cup_X (X \cup_{\eta-\eta'} \eta) = X^2 \cup_X \xi^+ \; .$$

Define a map $\iota \colon Z \longrightarrow Z$ by $(x,x',s,t) \longmapsto (x',x,r(t),r(s))$, where $s,t \in [a,b]$ and $r\colon [a,b] \to [a,b]$ is the reflection map. Clearly, $\iota^2 = id$. It induces a map $\iota'\colon Z' \longrightarrow Z'$ with the same property. Therefore Z' has all properties required of a Thom space of ξ^+. □

Lemma 2.8. The map j is a d-duality map.

Proof: Let $t \in H^{d+1}((X \times [a,b])^2, (X \times [a,b])^2 - \Delta)$ be a Thom class of the tangent microbundle of $X \times [a,b]$. The exact sequence of the triple $((X \times [a,b])^2, Z, X^2)$ identifies t with a generator t' of $H^d(Z, X^2)$. There is a commutative diagram ($q \leqq d$)

$$\begin{array}{ccc} H_q(N', X \times b) & \xrightarrow{\alpha_j} & H^{d-q}(M', X \times a) \approx H^{d-q}(M, X \times a) \\ \downarrow \approx & & \downarrow \approx \\ H_q(X \times [a,b] - M, X \times [a,b] - X \times [a,b)) & \xrightarrow[\gamma_t]{} & H^{d+1-q}(X \times [a,b), M) \end{array}$$

where $\alpha_j(z) = j^*(t')/z$. (All homology groups have $\mathbb{Z}[\pi_1 X]$-coefficients.) The vertical isomorphism on the right comes from the exact sequence of the triple $(X \times [a,b], M, X \times a)$. The bottom map γ_t is the usual Alexander duality isomorphism. (One has to be a little careful, since the assumptions of the duality theorem are not quite satisfied here, e.g. $X \times [a,b)$ is not compact, and, more seriously, M is not contained in the interior of $X \times [a,b]$. But in the special situation at hand this does not affect the result because the intersection of M with the boundary of $X \times [a,b]$ is homotopy equivalent to X.) Hence α_j is an isomorphism as asserted.

The last two lemmas provide a map

$$f\colon hP(X) \longrightarrow hDR^d_{\xi^+}(X).$$
$$(M,F,N) \longmapsto (M',N',j)$$

which is compatible with the involutions T' (resp. τ). Of course, f restricts to a map

$$f\colon hP^{\ell,m}_k(X) \longrightarrow hDR^{\ell,m}_k(X). \quad (\ell+m=d).$$

We would like to stabilize this map with respect to dimension. In order to do so one first has to replace the categories $hP^{\ell,m}_k(X)$ by $h\underline{P}^{\ell,m}_{\underline{k}}(X)$. Secondly, one modifies the suspension maps on the catefories $hDR^n_\xi(X)$. Namely, let

$$\Sigma'_\ell\colon DR^n_\xi(X) \longrightarrow DR^n_\xi(X\times J')$$
$$(Y,Y',u' \longmapsto (Y\times J' \cup_{Y\times\partial J'} X\times\partial J', Y'\times J', \ldots),$$

and similarly with Σ_r. (J' denotes some interval). We obtain a diagram

$$\begin{array}{ccc} h\underline{P}^{\ell,m}_{\underline{k}}(X) & \xrightarrow{\quad f \quad} & hDR^{\ell,m}_k(X). \\ {\scriptstyle\sigma_\ell}\downarrow & & \downarrow{\scriptstyle\Sigma_r} \\ h\underline{P}^{\ell,m}_{\underline{k}}(X\times J) & \xrightarrow[\quad f' \quad]{} & hDR^{\ell,m}_k(X\times J'). \end{array}$$

where f' takes a partition (M,F,N) in $h\underline{P}(X\times J)$ to

$$(M'-F\cup_{X\times J} X\times J',\ N'-F\cup_{X\times J} X\times J', \ldots),\ \text{and}$$

and $X\times J \xrightarrow{\approx} X\times J'$ is given by some fixed isomorphism $J \xrightarrow{\approx} J'$. This diagram commutes up to homotopy since there is a natural transformation $\Sigma_r f \longrightarrow f'\sigma_\ell$ which is given by

$$M\times J' \longmapsto M\times J' \cup X\times[a,a']\times J$$

$$N\times J' \cup_{N\times\partial J'} X\times\partial J' \longmapsto N\times J \cup_{N\times Cl(J-J')} X\times\partial J' \quad .$$

There is a similar diagram with σ_ℓ (resp. Σ_r) replaced by σ_u (resp. Σ_ℓ).

Hence one obtains a map in the limit

$$f\colon \varinjlim_{k,\ell,m} h\underline{P}^{\ell,m}_{\underline{k}}(X\times J^{\ell+m-d}) \longrightarrow \varinjlim_{k,\ell,m} hDR^{\ell,m}_k(X\times J'^{\,\ell+m-d}).$$

which is well-defined up to homotopy. Standard mapping cylinder arguments now show that f is compatible with the involutions up to weak homotopy, i.e. the restrictions of τf (resp. fT) to any compactum are homotopic.

Lemma 2.9. The map f is a weak homotopy equivalence.

Proof: Composing f with the forgetful map

$$g:\ \lim_{\substack{\to\\ k,\ell,m}} h\mathcal{D}\mathcal{R}_k^{\ell,m}(X\times J^{\prime\,\ell+m-d})_\bullet \longrightarrow \lim_{\substack{\to\\ k,\ell,m}} h\mathcal{R}_k^{\ell}(X\times J^{\prime\,\ell+m-d})_\bullet$$

gives (up to a minor modification) the map proved to be a homotopy equivalence in [18, prop. 5.4.]. The map g is a homotopy equivalence by prop. 1.6. and lemma 2.6. Hence f is also a homotopy equivalence, as was to be shown. □

Corollary 2.10. The involutions defined by T and τ on A(X) agree up to weak homotopy. □

Our next goal is to show that the involution on A(X) gives upon 'linearization' the usual involution on the K-theory of (group) rings. We first have to explain the meaning of this statement. Let R be a ring. We define the K-theory of R to be

$$K(R) = \mathbb{Z} \times BGl(R)^+ ,$$

that is, we replace the class group by $\mathbb{Z}$ in order to make the analogy with A(X) more transparent. Let R be equipped with an anti-involution $\bar{\ }: R \to R$. For a typical example let $R = \mathbb{Z}[G]$, the group ring of a group G, and the anti-involution being defined by $g \mapsto g^{-1}$, $g \in G$. There is an induced involution on $Gl_k(R)$ given by $A \mapsto (\bar{A}^t)^{-1}$, the conjugate transpose inverse of A. This defines the usual involution on K(R).

There is a canonical map $A(X) \longrightarrow K(R)$, called 'linearization', where $R = \mathbb{Z}[\pi_1 X]$, cf. [16]. In order to define this map we use a slightly different description of K(R). Namely, let $iso\mathcal{F}_k(R)$ denote the category of free (right) R-modules of rank k and their isomorphisms. The canonical inclusion $Gl_k(R) \longrightarrow iso\mathcal{F}_k(R)$ is an equivalent of categories. This allows one to define

$$K(R) = \mathbb{Z} \times |\lim_{\substack{\to\\ k}} iso\mathcal{F}_k(R)|^+ .$$

The linearization map is induced by the functors

$$h\mathcal{U}_k^{\ell}(G) \longrightarrow iso\mathcal{F}_k(R)$$
$$M \longmapsto H_\ell^G(M),$$

$k,\ell \geqq 0$, $G = G(X)$, the loop group of X.

Proposition 2.11. The linearization map $A(X) \to K(\mathbb{Z}[\pi_1 X])$ is equivariant with respect to the involution on both terms.

Proof: Let $R = \mathbb{Z}[\pi_1 X] = \mathbb{Z}[\pi_0 G]$. Let $iso\mathcal{D}\mathcal{F}_k(R)$ denote the category of triples (A,A',u), where A and A' are free (right) R-modules of rank k, and $u: A \otimes A' \to R$ is an $R\otimes R$-map defining a non-singular pairing. (R is a right $R\otimes R$-module by letting $r.(s\otimes t) = \bar{t}\, r\, s$.) A morphism $(A,A',u) \to (B,B',v)$ is a pair of isomorphisms $f: A \to B$, $f': B' \to A'$, such that $u(f'\otimes id) = v(id\otimes f)$. The category $iso\mathcal{D}\mathcal{F}_k(R)$ has an involution defined by $(A,A',u) \mapsto (A',A,\bar{u})$. The canonical inclusion $Gl_k(R) \to iso\mathcal{D}\mathcal{F}_k(R)$ is compatible with the involutions on both terms. Further there is a functor

$$h\mathcal{DU}_k^{\ell,m}(G) \longrightarrow iso\mathcal{DF}_k(R)$$
$$(M,M',u) \longmapsto (H_\ell^G(M),\ H_m^G(M'),\ H_{\ell+m}^G(u)).$$

The category $iso\mathcal{DF}_k(R)$ was designed in exactly such a way as to make this map equivariant with respect to the involutions. Altogether we obtain a commutative diagram

$$\begin{array}{ccccc} h\mathcal{DU}_k^{\ell,m}(G) & \longrightarrow & iso\mathcal{DF}_k(R) & \longleftarrow & Gl_k(R) \\ \downarrow & & \downarrow & & \| \\ h\mathcal{U}_k^{\ell}(G) & \longrightarrow & iso\mathcal{F}_k(R) & \longleftarrow & Gl_k(R). \end{array}$$

The middle vertical map is an equivalence of categories; the vertical map on the left becomes a homotopy equivalence after passing to the limit with respect to ℓ and m by prop. 1.13. Therefore the upper left arrow is an approximation to the linearization map. We have seen above that the arrows in the upper row of the diagram preserve the involution. This proves the proposition. □

Remark: The linearization map considered above is actually a special case of a more general natural transformation from the K-theory of spaces to the K-theory of simplicial rings in the sense of [16]. The K-theory of a simplicial ring R can be defined from the category of free simplicial R-modules in a way formally quite similar to the construction of A(X) from the category of free pointed simplicial G-sets. The natural transformation is then given by the map $A(X) \longrightarrow K(\mathbb{Z}[G])$ (G=simplicial loop group of X), which associates to a free pointed simplicial G-set M the simplicial $\mathbb{Z}[G]$-module $\tilde{\mathbb{Z}}[M]$, the underlying simplicial abelian group of which is freely generated by the non-basepoint elements of M. Now in the context of simplicial $\mathbb{Z}[G]$-modules the concept of duality can be defined in complete analogy to the 'non-linear' case, and $K(\mathbb{Z}[G])$ can be constructed from a larger category of $\mathbb{Z}[G]$-modules by including duality data. This again leads to an involution on $K(\mathbb{Z}[G])$, which by its very construction is compatible with that on A(X) via the linearization map.
The composition of the linearization map $A(X) \longrightarrow K(\mathbb{Z}[G])$ with the map $K(\mathbb{Z}[G]) \longrightarrow K(\mathbb{Z}[\pi_0 G]) = K(\mathbb{Z}[\pi_1 X])$ induced from the connected component map $G \longrightarrow \pi_0 G$ is identical with the map of proposition 2.11.

§ 3. The splitting theorem

In this section we apply the concept of duality developed in the previous sections to give another proof of the splitting theorem, [17], [18]:

Theorem: The canonical map $\Omega^\infty S^\infty(X_+) \longrightarrow A(X)$ is a coretraction up to weak homotopy.

The theorem will be proved by constructing a splitting map $A(X) \longrightarrow \Omega^\infty S^\infty(X_+)$. To make the proof more transparent we give an informal preview of the argument.

Recall the category $h\mathcal{DR}^n(X)$ from § 1. We agree that duality is taken with respect to the trivial spherical fibration $\varepsilon = X \times S^0$ if no spherical fibration is mentioned explicitly. This category approximates $A(X)$ in a sense which was made precise there. To define the splitting map, the category $h\mathcal{DR}^n(X)$ will have to be replaced by a certain simplicial topological space. This is done in two steps. First a simplicial set $DR^n(X)_\bullet$ is constructed together with a chain of homotopy equivalences

$$h\mathcal{DR}^n(X) \longrightarrow h\mathcal{DR}^n(X)_\bullet \longleftarrow DR^n(X)_\bullet$$

where $h\mathcal{DR}^n(X)_\bullet$ is a certain simplicial category combining both $h\mathcal{DR}^n(X)$ and $DR^n(X)_\bullet$. In a second step each simplex of $DR^n(X)_\bullet$ is replaced by a certain contractible space. This gives a simplicial topological space, which is denoted $\underline{DR}^n(X)_\bullet$. It is on this space that the splitting map is defined.

To show that the map constructed is a retraction up to homotopy we consider the following diagram: (For simplicity let $X = *$)

$$\begin{array}{ccccc} & & \underline{DR}^n(*)_\bullet & \longrightarrow & \Omega^\infty S^\infty \\ & \nearrow & \downarrow \simeq & & \\ B\Sigma_\infty & \longrightarrow & DR^n(*)_\bullet & & \end{array}$$

The lower horizontal arrow represents the inclusion $\Omega^\infty S^\infty \longrightarrow A(*)$ before the + construction. It is shown that this arrow may be lifted as indicated by the broken arrow, and further that the composite $B\Sigma_\infty \longrightarrow \Omega^\infty S^\infty$ agrees up to homotopy with a map described by Segal, [12]. Hence, after performing the + construction it gives a weak homotopy equivalence.
To make precise the way these spaces approximate $A(*)$ (resp. $A(X)$) one has to stabilize them in various ways.

Finally let us mention that from the description of the splitting map given here it is not clear how this map is related to the splittings constructed in [17] and [18].

We start by giving the precise definitions now. Define a simplicial set $R(X)_\bullet$ by stipulating that a p-simplex be given by an object $Y \underset{s}{\overset{r}{\rightleftarrows}} X \times \Delta^p$ of $\mathcal{R}_{hf}(X \times \Delta^p)$ such that for each face inclusion $\alpha\colon \Delta^q \subset \Delta^p$ the following conditions are satisfied:

(i) $\alpha^*Y := a^{-1}(\Delta^q)$ is an object of $\mathcal{R}_{hf}(X \times \Delta^q)$, where a is the composite $Y \xrightarrow{r} X \times \Delta^p \xrightarrow{pr} \Delta^p$;

(ii) the following diagram commutes

$$\begin{array}{ccc} X \times \Delta^q & \xrightarrow{\ id \times \alpha\ } & X \times \Delta^p \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ \alpha^*Y & \longrightarrow & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\ X \times \Delta^q & \xrightarrow{\ id \times \alpha\ } & X \times \Delta^p \end{array} ;$$

(iii) $\alpha^*Y \rightarrowtail Y$ is a weak homotopy equivalence.

The face maps are given by the obvious restriction maps. This definition can be modified to include duality data. Concretely, let $DR(X)_\bullet$ denote the simplicial set a p-simplex of which is given by a tuple (Y,Y',u), where Y and Y' are objects of $\mathcal{R}_{hf}(X)_p$ and

$$u\colon (Y \cup_{X\times\Delta^p} X) \wedge (Y' \cup_{X\times\Delta^p} X) \longrightarrow Th_n(\varepsilon)$$

is an n-duality map in $\mathcal{R}(X)$. (Hence, for each $\alpha\colon \Delta^q \subset \Delta^p$ the restriction of u to $Y_\alpha := \alpha^*Y \cup_{X\times\Delta^q} X$ is also an n-duality map.)

The simplicial set $DR^n(X)_\bullet$ and the category $h\mathcal{DR}^n(X)$ described earlier can be combined into a simplicial category $h\mathcal{DR}^n(X)_\bullet$. By definition the objects of $h\mathcal{DR}^n(X)_p$ are given by $DR^n(X)_p$. Let (Y,Y',u) (resp. (Z,Z',v)) denote a simplex of $DR^n(X)_p$. A morphism $(Y,Y',u) \longrightarrow (Z,Z',v)$ in $h\mathcal{DR}^n(X)_p$ is a pair of weak homotopy equivalences $f\colon Y \longrightarrow Z$, $f'\colon Z' \longrightarrow Y'$ satisfying that the following diagrams commute

(i)

$$\begin{array}{ccccc} & & X\times\Delta^p & & \\ & \swarrow & & \searrow & \\ Y & & \xrightarrow{\;f\;} & & Z \\ & \searrow & & \swarrow & \\ & & X & & \end{array} \quad , \quad \begin{array}{ccccc} & & X\times\Delta^p & & \\ & \swarrow & & \searrow & \\ Z' & & \xrightarrow{\;f'\;} & & Y' \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

(ii) for each $\;:\Delta^q \subset \Delta^p$

$$\begin{array}{ccc} \alpha^*Y & \xrightarrow{\;f|\alpha^*Y\;} & \alpha^*Z \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\;f\;} & Z \end{array} \qquad \begin{array}{ccc} \alpha^*Z' & \xrightarrow{\;f'|\alpha^*Z'\;} & \alpha^*Y' \\ \downarrow & & \downarrow \\ Z' & \xrightarrow{\;f'\;} & Y' \end{array}$$

$$\begin{array}{ccc} Y_\alpha \wedge Z'_\alpha & \xrightarrow{\;f_\alpha \wedge id\;} & Z_\alpha \wedge Z'_\alpha \\ {\scriptstyle id\wedge f'_\alpha}\downarrow & & \downarrow{\scriptstyle v_\alpha} \\ Y_\alpha \wedge Y'_\alpha & \xrightarrow[\;u_\alpha\;]{} & Th_n(\varepsilon) \end{array} .$$

The simplicial category $h\mathcal{DR}^n(X)_\bullet$ contains both $h\mathcal{DR}^n(X) = h\mathcal{DR}^n(X)_o$ and $DR^n(X)_\bullet$ being its simplicial set of objects.

Lemma 3.1. The inclusions

$$DR^n(X)_\bullet \longrightarrow h\mathcal{DR}^n(X)_\bullet \longleftarrow h\mathcal{DR}^n(X)$$

are weak homotopy equivalences.

Proof: Let $s\colon h\mathcal{DR}^n(X) \longrightarrow h\mathcal{DR}^n(X)_m$ denote the degeneracy map; let d be the map in the other direction given by restriction to the m-th vertex of Δ^m. Define a functor f from $h\mathcal{DR}^n(X)_m$ to itself by mapping $(Y \xrightarrow{p} \Delta^m,\ Y' \xrightarrow{p'} \Delta^m, \ldots)$ to

$(p^{-1}(v_m) \times \Delta^m \xrightarrow{pr_2} \Delta^m,\ Y' \xrightarrow{p'} \Delta^m, \ldots)$. (Here v_m denotes the m-th vertex of Δ^m). Similarly f' is the endofunctor of $h\mathcal{DR}^n(X)_m$ given by

$$(Y \xrightarrow{p} \Delta^m,\ Y' \xrightarrow{p'} \Delta^m, \ldots) \longmapsto (Y \xrightarrow{p} \Delta^m,\ p^{-1}(v_m) \times \Delta^m \xrightarrow{pr_2} \Delta^m, \ldots).$$

Clearly sd = f'f. Define another functor g: $h\mathcal{DR}^n(X)_m \longrightarrow h\mathcal{DR}^n(X)_m$ by $(Y \longrightarrow \Delta^m,\ Y' \longrightarrow \Delta^m, \ldots) \longmapsto (Y \times \Delta^m \xrightarrow{pr} \Delta^m,\ Y' \longrightarrow \Delta^m)$.
There are natural transformations given by the inclusion $p^{-1}(v_m) \times \Delta^m \longrightarrow Y \times \Delta^m$, resp. by the map $Y \xrightarrow{(id,p)} Y \times \Delta^m$. This proves that f is homotopic to the identity. By a similar argument the functor f' is homotopic to the identity. Since ds = id anyway this proves that s is a homotopy equivalence. This is true for every m, so by the realization lemma (cf. e.g. [16]) the right arrow in the lemma is a homotopy equivalence.

To show that the left arrow is a homotopy equivalence we employ a variant of Quillen's theorem A, [10]. Let i. denote the arrow in question. We show that the left fibre $i./((Z,Z',v);[m])$ over a fixed object in degree m is contractible. Since $DR^n(X).$ is a simplicial set the fibre will be a simplicial set, too, rather than a simplicial category. A p-simplex of the fibre is given by

$$(a\colon [p] \to [m];\ (Y,Y',u) \in h\mathcal{DR}^n(X)_p,\ (\alpha,\alpha')\colon (Y,Y'u) \xrightarrow{\simeq} a^*(Z,Z',v)).$$

There is a simplicial subset F. of the fibre defined by the condition that $Y' = a^*Z'$, and the structure map $a^*Z' \to Y'$ is the identity. In fact, F. is a deformation retract of $i./((Z,Z',v);[m])$. To see this let j: F. $\longrightarrow i./((Z,Z',v);[m])$ denote the inclusion map. There is an obvious retraction

$$k\colon i./((Z,Z',v);[m]) \longrightarrow F.$$

given by

$$(a;\ (Y,Y',u),\ (\alpha,\alpha')\colon (Y,Y') \xrightarrow{\simeq} (a^*Z,a^*Z'))$$
$$\longmapsto (a;\ (Y,a^*Z'),\ (\alpha,id)\colon (Y,a^*Z') \xrightarrow{\simeq} (a^*Z,a^*Z')).$$

We describe a simplicial homotopy from the identity map on $i./((Z,Z',v);[m])$ to the composite jk by specifying a family of maps $h_q\colon i./((Z,Z',v);[m])_p \longrightarrow i./((Z,Z',v);[m])_{p+1}$, $q = 0,\ldots,p$.

Let x be a p-simplex of $i./((Z,Z',v);[m])$ as described above. Let $T(\alpha')$ denote the mapping cylinder of the map $\alpha'\colon a^*Z' \longrightarrow Y'$. There are canonical maps over $\Delta^p \times \Delta^1$ $\beta\colon T(\alpha') \longrightarrow Y' \times \Delta^1$, and $\gamma\colon a^*Z' \times \Delta^1 \longrightarrow T(\alpha')$. The map h_q is defined to take the p-simplex x to the (p+1)-simplex of $i./((Z,Z',v);[m])$ given by

$$(a\sigma_q\colon [p+1] \longrightarrow [m], (\varphi_q^*(Y \times \Delta^1), \varphi_q^*(T(\alpha')), u_q),$$
$$(\alpha_q,\alpha_q')\colon (\varphi_q^*(Y \times \Delta^1), \varphi_q^*(T(\alpha')), u_q) \longrightarrow (a\sigma_q)^*(Z,Z',v)),$$

where (i) $\varphi_q\colon \Delta^{p+1} \longrightarrow \Delta^p \times \Delta^1$ are the characteristic maps of the non-degenerate (p+1)-simplices of $\Delta^p \times \Delta^1$, $q = 0,\ldots,p$.

(ii) σ_q are the surjective maps $[p+1] \longrightarrow [p]$, $q = 0,\dots,p$;

(iii) u_q is the composite $\varphi_q^*(Y \times \Delta^1) \wedge \varphi_q^*(T(\alpha')) \xrightarrow{\;\simeq\;}$
$(Y \times \Delta^1) \wedge T(\alpha') \xrightarrow{\text{id} \wedge \beta} (Y \times \Delta^1) \wedge (Y' \times \Delta^1) \longrightarrow$
$Y \wedge Y' \xrightarrow{u} Th_{n-d}(\varepsilon)$;

(iv) $\alpha_q \colon \varphi_q^*(Y \times \Delta^1) \xrightarrow{\varphi_q^*(\beta \times \text{id})} \varphi_q^*(a^*Z \times \Delta^1) = (a\sigma_q)^*Z$
$\alpha_q' \colon (a\sigma_q)^*(Z') = \varphi_q^*(Z' \times \Delta^1) \xrightarrow{\varphi_q^*(\gamma \times \text{id})} \varphi_q^*(T(\alpha'))$.

One checks that the maps h_q assemble to a simplicial homotopy from the identity map on $i./((Z,Z',v);[m])$ to the map kj. Hence F. is a deformation retract of $i./((Z,Z',v);[m])$. By an argument which is very similar, one proves that the canonical map $F. \to \Delta^m$ is a homotopy equivalence. Hence $i./((Z,Z',v);[m])$ is contractible, and by an application of theorem A we conclude that i. is a homotopy equivalence as asserted.

□

Remark: The homotopy equivalences of the lemma restrict to homotopy equivalences of the subcategories $DR_k^{\ell,m}(X).$ (resp. $h\mathcal{DR}_k^{\ell,m}(X).$) defined by restricting the homotopy type of the spaces involved.

There is a (left) stabilization map

$$DR_k^{\ell,m}(X). \longrightarrow DR_k^{\ell+1,m}(X).$$

given by

$$(Y,Y',u) \longmapsto ((S^1 \times \Delta^p) \underset{\Delta^p}{\wedge}{}_{X\times\Delta^p} Y, Y', \dots).$$

Similarly, there is a right stabilization map, by suspending Y', and finally, in the k-variable, one stabilizes by taking the wedge sum with an ℓ-sphere (resp. m-sphere).

In view of this remark, the algebraic K-theory of X may now be described using the simplicial sets $DR_k^{\ell,m}(X).$ in the following way:

<u>*Corollary 3.2.*</u> $$A(X) \simeq \mathbb{Z} \times |\lim_{\substack{\to \\ k,\ell,m}} DR_k^{\ell,m}(X).|^+$$ □

Let us now specialize our arguments to the case X = point. The general case will be delt with afterwards.

Recall that an n-duality map was defined to be a pointed map $u \colon Y \wedge Y' \to S^n$ satisfying a certain non-singularity condition. It is also possible to describe this duality by a certain map $v \colon S^n \to Y \wedge Y'$. Namely, given the map u, define

$$w \colon Y \wedge Y' \wedge Y \wedge Y' \longrightarrow S^n \wedge S^n$$

by

$$(a \wedge a' \wedge b \wedge b') \longmapsto (u(a \wedge b') \wedge u(b \wedge a')) .$$

It is easy to check that this defines a 2n-duality pairing. Define v to be the dual of u with respect to the duality w. Equivalently, v is characterized by the condition that the following diagram commutes up to homotopy

$$(*)\qquad \begin{array}{ccc} S^n & \xrightarrow{\quad v \quad} & Y \wedge Y' \\ {\scriptstyle id^{\#}}\downarrow & & \downarrow{\scriptstyle w^{\#}} \\ Map(S^n, S^{2n}) & \xrightarrow{\quad u^* \quad} & Map(Y \wedge Y', S^{2n}) \end{array} .$$

Here Map(A,B) denotes the space of pointed maps from A to B. The vertical arrows in the diagram are given by the adjoint of the identity (resp. the adjoint of w).

Let * denote the one-point space. We are going to construct a certain simplicial *space* from the simplicial set DR(*). by including further duality data.
Let x be a 0-simplex of $DR^n(*)$. which is represented by the n-duality map $u: Y \wedge Y' \longrightarrow S^n$. Let E_x denote the space of pointed maps

$$S^n \longrightarrow \underset{\leftarrow}{holim}(Y \wedge Y' \xrightarrow{w^{\#}} Map(Y \wedge Y', S^{2n}))$$

satisfying that the following diagram commutes

$$\begin{array}{ccc} S^n & \longrightarrow & \underset{\leftarrow}{holim}(Y \wedge Y' \xrightarrow{w^{\#}} Map(Y \wedge Y', S^{2n})) \\ \downarrow & & \downarrow \\ Map(S^n, S^{2n}) & \longrightarrow & Map(Y \wedge Y', S^{2n}) \end{array} .$$

In other words, a point of E_x is given by a map $v: S^n \longrightarrow Y \wedge Y'$ making diagram (*) commute up to homotopy, together with a specific homotopy commutativity $h: S^n \wedge Y \wedge Y' \wedge [0,1]_+ \longrightarrow S^{2n}$ between the maps

$$S^n \wedge Y \wedge Y' \xrightarrow{v \wedge id} Y \wedge Y' \wedge Y \wedge Y' \xrightarrow{\quad w \quad} S^{2n}$$

and

$$S^n \wedge Y \wedge Y' \xrightarrow{id \wedge u} S^n \wedge S^n =\!=\!=\!= S^{2n} .$$

The space E_x is the same up to homotopy as

$$\Omega^n(\text{fibre}(Y \wedge Y' \longrightarrow Map(Y \wedge Y', S^{2n}))) .$$

The map $w^{\#}$ is (2n-1)-connected. Therefore, E_x is (n-1)-connected. Similarly, if x denotes a p-simplex of $DR^n(*)_p$ represented by Y,Y' and the n-duality map $u: Y/\Delta^p \wedge Y'/\Delta^p \longrightarrow S^n$, we let E_x denote the space of pointed maps

$$S^n \wedge \Delta^p_+ \longrightarrow \underset{\leftarrow}{holim}(Y/\Delta^p \wedge Y/\Delta^p \xrightarrow{w^{\#}} Map(Y/\Delta^p \wedge Y'/\wedge^p, S^{2n}))$$

satisfying that

(i) the following diagram commutes

$$\begin{array}{ccc}
S^n \wedge \Delta^p_+ & \longrightarrow & \operatorname{holim}_{\leftarrow}(Y/\Delta^p \wedge Y'/\Delta^p \xrightarrow{w^{\#}} \operatorname{Map}(Y/\Delta^p \wedge Y'/\Delta^p, S^{2n})) \\
\downarrow \scriptstyle{pr} & & \downarrow \\
S_n & & \\
\downarrow \scriptstyle{id^{\#}} & & \\
\operatorname{Map}(S^n, S^{2n}) & \xrightarrow{u^*} & \operatorname{Map}(Y/\Delta^p \wedge Y'/\Lambda^p, S^{2n})
\end{array}$$

(ii) the map $S^n \wedge \Delta^p_+ \longrightarrow \operatorname{holim}_{\leftarrow}(Y/\Delta^p \wedge Y'/\Delta^p \xrightarrow{w^{\#}} \operatorname{Map}(Y/\Delta^p \wedge Y'/\Delta^p, S^{2n})$ $\longrightarrow Y/\Delta^p \wedge Y'/\Delta^p$ is a map in the category $\mathcal{R}(*)_\circ$, i.e. for each face inclusion $\alpha\colon \Delta^q \subset \Delta^p$ there is a commutative diagram

$$\begin{array}{ccc}
S^n \wedge \Delta^q_+ & \longrightarrow & Y_\alpha \wedge Y'_\alpha \\
\downarrow \scriptstyle{id \wedge \alpha} & & \downarrow \\
S^n \wedge \Delta^p_+ & \longrightarrow & Y/\Delta^p \wedge Y'/\Delta^p
\end{array} \quad .$$

Again E_x is an $(n-1)$-connected space. For every p let $\underline{DR}^n(*)_p$ be the disjoint union of the spaces E_x for all $x \in DR^n(*)_p$. In view of condition (ii) this defines a simplicial space $DR^n(*)_\circ$. There is a canonical map

$$\underline{DR}^n(*)_\circ \longrightarrow DR^n(*)_\bullet$$
$$E_x \longmapsto x \quad .$$

This map is $(n-1)$-connected, since it is $(n-1)$-connected in each simplicial degree.

Of course, one can again restrict the homotopy type of the spaces involved in the construction of these simplicial sets. Thus one obtains simplicial spaces $\underline{DR}_k^{\ell,m}(*)_\circ$. There are three stabilization maps. For example, stabilization with respect to ℓ is given by

$$\underline{DR}_k^{\ell,m}(*)_p \longrightarrow \underline{DR}_k^{\ell+1,m}(*)_p$$

$$(S^n \wedge \Delta^p_+ \to Y/\Delta^p \wedge Y'/\Delta^p \to S^n) \longmapsto (S^{n+1} \wedge \Delta^p_+ \to \Sigma(Y)/\Delta^p \wedge Y'/\Delta^p \to S^{n+1}).$$

One therefore has a well-defined map

$$\lim_{\substack{\rightarrow \\ \ell,m}} \underline{DR}_k^{\ell,m}(*)_\circ \longrightarrow \lim_{\substack{\rightarrow \\ \ell,m}} DR_k^{\ell,m}(*)_\bullet \quad .$$

This map is a weak homotopy equivalence since by the above it is the limit of $(\ell+m-1)$-connected maps. Hence one obtains still another description of $A(*)$:

Proposition 3.3. $A(*) \simeq \mathbb{Z} \times \left| \lim_{\substack{\rightarrow \\ k,\ell,m}} \underline{DR}_k^{\ell,m}(*)_\circ \right|^+$ □

We now give a description of the map $\Omega^\infty S^\infty \longrightarrow A(*)$. By the theorem of Barratt-Priddy-Quillen-Segal, cf. [13], there is a weak homotopy equivalence $B\Sigma_\infty^+ \simeq \Omega^\infty S^\infty_{(o)}$, where Σ_∞ denotes the infinite symmetric group, and $\Omega^\infty S^\infty_{(o)}$ is the 0-component of stable homotopy. We define a map $B\Sigma_\infty \longrightarrow \varinjlim_n DR^n(*)$. which induces the map $\Omega^\infty S^\infty \longrightarrow A(*)$ in view of the description of $A(*)$ afforded by cor. 3.2.. We need a suitable model of $B\Sigma_\infty$.

Consider the following *configuration space*.

Let $C_k'^{\ell,m} = ((\mathbb{R}^\ell)^k - \text{(fat) diagonal}) \times ((\mathbb{R}^m)^k - \text{(fat) diagonal})$. (Recall that the fat diagonal is defined by the condition that at least two vectors of a k-tuple of vectors are identical.) This space is $(\min(\ell,m)-2)$-connected. The symmetric group Σ_k acts freely on $C_k'^{\ell,m}$ via the diagonal action. Let $C_k^{\ell,m}$ be the orbit space of this action. It follows that the spaces $C_k^{\ell,m}$ approximate $B\Sigma_k$. The space $C_k^{\ell,m}$ contains as a deformation retract the space $D_k^{\ell,m}$ defined by thickening the points of a configuration in $C_k^{\ell,m}$ to unit discs, one in $\mathbb{R}^\ell$, the other one in $\mathbb{R}^m$.

Let $c = (\alpha_i\colon D^\ell \longrightarrow \mathbb{R}^\ell,\ \alpha_i'\colon D^m \longrightarrow \mathbb{R}^m)_{i\in I}$ be a point in $D_k^{\ell,m}$. (I denotes an index set of cardinality k.) To this point there can be associated a map $\varphi_c\colon S^{\ell+m} = \mathbb{R}^{\ell+m} \cup \{\infty\} \longrightarrow S^{\ell+m}$ of degree k as follows. Choose a fixed degree 1 map $f\colon (D^\ell \times D^m, \partial) \longrightarrow S^{\ell+m}$. Define

$$\varphi_c(x) = \begin{cases} f((\alpha_i \times \alpha_i')^{-1}(x)) & \text{if } x \in (\alpha_i \times \alpha_i')(D^\ell \times D^m) \\ * & \text{otherwise.} \end{cases}$$

Taking c to φ_c defines a map $\varphi\colon D_k^{\ell,m} \longrightarrow \Omega^{\ell+m}S^{\ell+m}_{(k)}$. The subscript 'k' on the right refers to the component of degree k maps.

<u>*Lemma 3.4.*</u> The map φ induces a map

$$(\varinjlim_k D_k^{\ell,m})^+ \longrightarrow \Omega^{\ell+m}S^{\ell+m}_{(o)}$$

which is $(\min(\ell,m)-2)$-connected.

Proof: Consider the usual configuration space of k disjoint particles in $\mathbb{R}^\ell$. Denote this space by the symbol D_k^ℓ. There is a canonical diagonal map $D_k^\ell \longrightarrow D_k^{\ell,\ell}$, which is $(\ell-2)$-connected. More generally one can define a map $D_k^\ell \longrightarrow D_k^{\ell,m}$ if $m \geqq \ell$. Further there is a map $D_k^\ell \longrightarrow \Omega^\ell S^\ell_{(k)}$ and a commutative diagram $(m \geqq \ell)$

$$\begin{array}{ccc} D_k^\ell & \longrightarrow & \Omega^\ell S^\ell_{(k)} \\ \downarrow & & \downarrow \\ D_k^{\ell,m} & \longrightarrow & \Omega^{\ell+m}S^{\ell+m}_{(k)} \end{array}.$$

It is proved in [12] that the upper horizontal map induces a weak homotopy equivalence $(\varinjlim_k D_k^\ell)^+ \longrightarrow \varinjlim_k \Omega^\ell S^\ell_{(k)} \longrightarrow \Omega^\ell S^\ell_{(o)}$. The vertical maps in the diagram are $(\ell-2)$-connected in the case that $m \geqq \ell$. The other case follows since $D_k^{\ell,m} \simeq D_k^{m,\ell}$. □

Let $S(D_k^{\ell,m})$ denote the singular complex of $D_k^{\ell,m}$. Define a map

$$S(D_k^{\ell,m}) \longrightarrow DR_k^{\ell,m}(*).$$

in the following way. Let $c = (\alpha_i\colon \Delta^p \times D^\ell \longrightarrow \mathbf{R}^\ell,\ \alpha_i^!\colon \Delta^p \times D^m \longrightarrow \mathbf{R}^m)_{i\in I}$ represent a p-simplex of $S(D_k^{\ell,m})$. Consider the spaces $S^\ell \wedge I_+$ (resp. $S^m \wedge I_+$), where I is considered as a discrete topological space. There is an obvious duality pairing

$$u_c\colon (S^\ell \wedge I_+) \wedge (S^m \wedge I_+) \longrightarrow S^{\ell+m}$$

which is induced by the map $I \times I \longrightarrow S^o$ which takes exactly the complement of the diagonal to the base point of S^o. Associating to the configuration c the tuple consisting of the spaces $(S^\ell \wedge I_+) \times \Delta^p$ (resp. $(S^m \wedge I_+) \times \Delta^p$) and the canonical duality of these spaces induced from u_c defines the required map. We would like to lift this map to the simplicial space $\underline{DR}^{\ell+m}(*)$. . Let $f\colon (D^\ell \times D^m, \partial) \longrightarrow (S^\ell \wedge S^m, *)$ be as before, and let

$$v_c'\colon S^{\ell+m} \wedge \Delta_+^p \longrightarrow S^\ell \wedge S^m \wedge I_+ \wedge \Delta_+^p$$

$$x \wedge s \longmapsto \begin{cases} f(\alpha_i \times \alpha_i^!)^{-1}(x) \wedge i \wedge s & \text{if } x \in (\alpha_i \times \alpha_i^!)(\{s\} \times D^\ell \times \{s\} \times D^m) \\ * & \text{otherwise} \end{cases} .$$

(Again, $S^{\ell+m}$ is regarded as $\mathbf{R}^\ell \times \mathbf{R}^m \cup \{\infty\}$.)
Let v_c denote the composite of v_c' with the diagonal map

$$S^\ell \wedge S^m \wedge I_+ \wedge \Delta_+^p \longrightarrow (S^\ell \wedge I_+ \wedge \Delta_+^p) \wedge (S^m \wedge I_+ \wedge \Delta_+^p).$$

It is clear that v_c is a duality map and that v_c is dual to u_c in the sense defined above, at least up to a sign depending on the parity of m. Since we eventually have to pass to the limit with respect to ℓ and m anyway, we may assume m even without essential loss of generality.

Next we have to define a certain homotopy

$$h_c\colon (S^{\ell+m} \wedge \Delta_+^p) \wedge (S^\ell \wedge I_+ \wedge \Delta_+^p) \wedge (S^m \wedge I_+ \wedge \Delta_+^p) \wedge [0,1]_+ \longrightarrow S^{2(\ell+m)}$$

which is part of the data of a point in $\underline{DR}^{\ell+m}(*)$. .
We proceed as follows. A point $(\alpha_i, \alpha_i^!)$ of a configuration c determines a map $\alpha_i \times \alpha_i^!\colon D^\ell \times \Delta^p \times D^m \times \Delta^p \longrightarrow \mathbf{R}^{\ell+m}$ and hence a map of degree 1 $S^\ell \wedge \Delta_+^p \wedge S^m \wedge \Delta_+^p \to S^{\ell+m}$.
Letting the radius of the disc D^ℓ (resp. D^m) grow to infinity defines a canonical homotopy between this map and the projection $S^\ell \wedge \Delta_+^p \wedge S^m \wedge \Delta_+^p \to S^{\ell+m}$. For each $i \in I$ define

$$h_i\colon S^{\ell+m} \wedge \Delta_+^p \wedge S^\ell \wedge \Delta_+^p \wedge S^m \wedge \Delta_+^p \wedge [0,1]_+ \longrightarrow S^{\ell+m} \wedge S^{\ell+m}$$

to be the projection on the first two factors, and on the other factors the homotopy determined by $\alpha_i, \alpha_i^!$ just described. Define the map

$$h_c'\colon S^{\ell+m} \wedge \Delta_+^p \wedge S^\ell \wedge I_+ \wedge \Delta_+^p \wedge S^m \wedge I_+ \wedge \Delta_+^p \wedge [0,1]_+ \longrightarrow S^{\ell+m} \wedge S^{\ell+m}$$

$$(x \wedge s \wedge x' \wedge i \wedge s' \wedge x'' \wedge j \wedge s'' \wedge t) \longmapsto \begin{cases} * & \text{if } i \neq j \\ h_i(x \wedge s \wedge x' \wedge s' \wedge x'' \wedge s'' \wedge t) & \text{if } i = j. \end{cases}$$

Compose the homotopy h'_c with some standard homotopy between the map

$$S^{\ell} \wedge S^{m} \wedge S^{\ell} \wedge S^{m} \longrightarrow S^{\ell} \wedge S^{m} \wedge S^{\ell} \wedge S^{m}$$

$$(x \wedge x' \wedge y \wedge y') \longmapsto (x \wedge y' \wedge y \wedge x')$$

and the identity map. Such a homotopy exists because of our assumption on the parity of m. This defines the required homotopy h_c .

Taking the configuration c to (u_c, v_c, h_c) gives a map

$$(\dagger) \qquad s: \mathcal{S}(D_k^{\ell,m}) \longrightarrow \underline{DR}_k^{\ell,m}(*). \quad .$$

Let $D_k := \lim\limits_{\overrightarrow{\ell,m}} D_k^{\ell,m}$. By lemma 3.4. this is a classifying space for Σ_k. Let $\mathcal{S}(D_k)$ denote the singular complex of D_k. Passing to the limit with respect to ℓ and m in $(\dagger)$ hence gives a map

$$B\Sigma_k \longrightarrow \lim_{\overrightarrow{\ell,m}} \underline{DR}_k^{\ell,m}(*).$$

which, after passing to the limit in k and performing the + construction gives the map $\Omega^{\infty}S^{\infty} \longrightarrow A(*)$.

We now describe a splitting of this map. Associate to any p-simplex (u_c, v_c, h_c) of $DR_k^{\ell,m}(*)$. the composite

$$u_c\, v_c: S^{\ell+m} \wedge \Delta_+^p \longrightarrow S^{\ell+m} \quad .$$

This defines a map

$$DR_k^{\ell,m}(*)_p \longrightarrow \mathrm{Map}(S^{\ell+m} \wedge \Delta_+^p,\ S^{\ell+m})_{(k)}$$

resp.

$$r: DR_k^{\ell,m}(*). \longrightarrow \mathrm{Map}(S^{\ell+m} \wedge \Delta_+^{\bullet},\ S^{\ell+m})_{(k)} \quad .$$

(Again, $\mathrm{Map}(\ldots)_{(k)}$ denotes the component of degree k maps).

The simplicial space on the right is the singular complex of $\Omega^{\ell+m}S^{\ell+m} = \mathrm{Map}(S^{\ell+m}, S^{\ell+m})$ considered as a simplicial *space* in the natural way. Checking the restriction of r to the subspace $\mathcal{S}(D_k^{\ell,m})$ of $\underline{DR}_k^{\ell,m}(*)$. immediately reveals that this is nothing else but the map φ described above. Hence one obtains:

Lemma 3.5. There is a commutative diagram

$$\begin{array}{ccc}
\mathcal{S}(D_k^{\ell,m}) & \xrightarrow{\quad \varphi \quad} & \mathcal{S}(\Omega^{\ell+m}S^{\ell+m}{}_{(k)}) \\
\Big\downarrow s & & \Big\downarrow \\
\underline{DR}_k^{\ell,m}(*). & \xrightarrow{\quad r \quad} & \mathcal{S}(\Omega^{\ell+m}S^{\ell+m}{}_{(k)}).
\end{array} \quad ,$$

where $\mathcal{S}(\Omega^{\ell+m}S^{\ell+m}).$ denotes the singular complex considered as a simplicial space, and the vertical map on the right is the natural inclusion.

□

After passing to the limit with respect to k, ℓ, and m and applying the + construction, the map φ becomes a homotopy equivalence by lemma 3.4. The vertical arrow on the right is a homotopy equivalence anyway, and the + construction does not change the terms on the right. This proves that the map

$$\Big|\lim_{\substack{\rightarrow\\ k,\ell,m}} \underline{DR}_k^{\ell,m}(*).\Big|^+ \longrightarrow \Big|\lim_{\vec{k}} S(\Omega^\infty S^\infty_{(k)}).\Big| \simeq \Omega^\infty S^\infty_{(o)}$$

is a retraction up to weak homotopy, and hence the theorem of this paragraph in the case $X = *$.

The modifications required for the general case are straightforward: To a duality

$$(Y \cup_{X\times\Delta^p} X) \wedge (Y' \cup_{X\times\Delta^p} X) \longrightarrow Th_n(\varepsilon) \quad \text{in } DR^n(X)_p$$

there is associated another duality in $DR^n(*)_p$, which is given by

$$Y/X\times\Delta^p) \wedge Y'(X\times\Delta^p) \longrightarrow Th_n(\varepsilon)/X^2 = S^n \wedge X_+ \longrightarrow S^n$$

(cf. prop. 1.17.). This defines a map $DR^n(X). \longrightarrow DR^n(*).$. Define the simplicial space $\underline{DR}^n(X).$ as the pull-back of the following diagram.

$$\begin{array}{ccc} \underline{DR}^n(X). & \longrightarrow & \underline{DR}^n(*). \\ \downarrow & & \downarrow \\ DR^n(X). & \longrightarrow & DR^n(*). \end{array}$$

Similarly define $\underline{DR}_k^{\ell,m}(X).$. Hence a p-simplex of $\underline{DR}^n(X).$ (resp. $\underline{DR}_k^{\ell,m}(X).$) consists of a certain n-duality (resp. $(\ell+m)$-duality)

$$(Y \cup_{X\times\Delta^p} X) \wedge (Y' \cup_{X\times\Delta^p} X) \longrightarrow Th_n(\varepsilon)$$

over Δ^p, together with a map $S^n \wedge \Delta^p_+ \rightarrow Y/X\times\Delta^p \wedge Y'/X\times\Delta^p$, and additional data. Associating to such a p-simplex the composite

$$S^n \wedge \Delta^p_+ \longrightarrow Y/X\times\Delta^p \wedge Y'/X\times\Delta^p \longrightarrow Th(\varepsilon)/X^2 = S^n \wedge X_+$$

defines a map

$$\underline{DR}_k^{\ell,m}(X). \longrightarrow Map(S^{\ell+m}\wedge\Delta^{\bullet}_+, S^{\ell+m}\wedge X_+) = S(\Omega^{\ell+m}S^{\ell+m}(X_+)).$$

By the same argument as in the case $X = *$ this map is shown to be a retraction up to homotopy after passing to the limit with respect to k,ℓ,m, and performing the + construction.

This ends the proof of the theorem. □

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