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Torsion in L-groups

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Introduction.

Let $L_n^h(\pi,w)$ denote the Wall group for the homotopy equivalence problem, π a finitely presented group and $w: \pi \to \{\pm 1\}$ a homomorphism. These groups figure in many geometric problems, and their rank is the same as that of other related surgery groups which have been computed in many important cases. Their torsion, for π finite, reflects the subtle relation between signature and discriminant of quadratic forms, discussed below. This paper contributes two calculations and a sample application to manifolds with finite fundamental group.

Theorem A. *The torsion of $L_{2k}^h(Z_{2^r})$ is a vector space over Z_2 of dimension $[2(2^{r-2}+2)/3]-[r/2]-\varepsilon$, $\varepsilon = 1$ if k is even and 0 if k is odd.*

(In Theorem A, $[x]$ = greatest integer in x.)

Theorem B. *$L_{2k+1}^h(Z_{2^r},-)$ is a vector space over Z_2 of dimension 2^{r-3}, $r \geq 3$, and zero for $r = 1,2$.*

Since the rank of $L_{2k}^h(Z_{2^r})$ is well-known (see [W1]), Theorem A completely determines this group. Taylor and Oliver, and Milgram have obtained at least Theorem A independently. Their method involves use of the computation of projective L-groups and a study of the relation of L^h to these via an exact sequence whose third term is a cohomology group of Z_2 with coefficients in $K_0(\pi)$. In this paper, we apply some of the algebraic

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results obtained in the course of our study of non-linear similarity. We use these to analyze the sequence (4.1) in [Sh] ("Rothenberg's sequence") relating L^s-groups (computed for cyclic groups in [W1]) to L^h-groups. We believe the independence lemma we use on units in the group ring $Z[Z_{2^r}]$ may be of independent interest.

Here is an application of the methods and results of this paper to the problem of providing a Poincaré Duality (PD) space with a manifold structure. Let X be a finite complex. Then X is called a PD space of dimension m if there exists a class $[X] \in H_m(X)$ such that for all i

$$\cap[X]:H^i(X;B) \to H_{n-i}(X,B)$$

is an isomorphism for every local coefficient system B over X. A closed manifold is a Poincaré duality space. If X is <u>connected</u>, then X will be a PD space if and only if there exists $[X] \in H_m(X)$ with $\cap[X]$ an isomorphism for the local coefficient system (i.e. the $Z\pi_1X$-module) $B = Z\pi_1X$.

Now suppose X is connected and $\pi_1(X) = G$ is a finite group and $m = 2k$. Then an invariant $\chi(X) \in R(G)$ is defined (in [W2, §13B], this invariant is denoted $\sigma(G,X)$.) To define it, (compare §3) let $\rho:G \to U(n)$ be an irreducible complex representation. Then $\mathbb{C}^n$ becomes a local coefficient system over X, and the isomorphism

$$(\cap[X])^{-1}:H_k(X;\mathbb{C}^n) \to H^k(X,\mathbb{C}^n) = H_k(X;\mathbb{C}^n)^*$$

is easily seen to be the adjoint of a unimodular $(-1)^k$ Hermitian form β over the complex numbers. Let

$$\sigma_\rho(X) = \text{signature of} \begin{cases} \beta & k \text{ even} \\ \sqrt{-1}\,\beta & k \text{ odd} \end{cases} .$$

Then let

$$\chi(X) = \sum_{\rho} \sigma_{\rho}(X)\rho,$$

the sum over irreducible representations. By a theorem of Atiyah-Bott, as extended by Wall to the PL and topological case, if X has the homotopy of a compact manifold, smooth, PL, or even topological, then $\chi(X)$ is a multiple of the regular representation.

A stable orientable linear, PL, or topological Euclidean space bundle (or block bundle) ξ over X is called reducible if its Thom class is spherical. If X has the homotopy type of manifold, such a bundle exists, namely, the stable normal bundle of the manifold.

In §6, the calculations of this paper will be applied to obtain a precise list of invariants for finding a manifold structure on a PD space X with $\pi_1 X = Z_{2^r}$. One consequence is:

Theorem C. Let X be a connected finite complex, and suppose that X is a PD space, of dimension $2k$ with $\pi_1 X = Z_{2^r}$, satisfying the following:

(i) $\chi(X)$ is a multiple of the regular representation,

(ii) there is a reducible (PL or TOP) bundle over X.

(iii) X has the homotopy type of the two-fold cover $\hat{Y}$ of a finite complex Y with $\pi_1 Y = Z_{2^{r+1}}$, and the image of $[X]$ in $H_{2k}(\hat{Y})$ is in the image of the transfer (from $H_{2k}(Y)$).

Then X has the homotopy type of a PL or TOP manifold (with the given reducible bundle as normal bundle).

We leave a smooth version to the reader. For k even, the usual condition on the L-polynomial of the bundle is needed to kill the simply-connected part of a surgery obstruction. For k odd, one has the usual Arf-invariant difficulties.

§1. A basis for $H^0(Wh(Z_{2^r}))$.

The group ring of Z_{2^r} is the ring $R = R_r = Z[T|T^{2^r} = 1]$, and this ring has the involution $p(T)^- = p(T^{-1})$. This involution induces one on the Whitehead group of Z_{2^r}, and $H^0(Wh(Z_{2^r}))$ denotes the cohomology of Z_2 with respect to this involution; i.e. elements satisfying $x = \bar{x}$, modulo elements of the form $x+\bar{x}$. When necessary for clarity, the generator T of Z_{2^r} will be denoted T_r, and we suppose $T_r = (T_{r+1})^2$. A unit $u(T) \in R^{\times}_{(r)}$, with $u(T) = u(T^{-1})$ represents an element of $H^0(Wh(Z_{2^r}))$. In this section we will give some units, $U^{(r)}_{m,s,i}$, $3 \leq s \leq m \leq r$, $i \equiv 1 (\text{mod } 4)$, $1 \leq i < 2^{s-1}$, which represent a basis of the Z_2-vector space $H^0(Wh(Z_{2^r}))$.

To define our units, we first set, for $r \geq 3$, $i \equiv 1(\text{mod } 4)$

$U^{(r)}_{r,r,i} = (\sum_{j=0}^{2^{r-1}} T^{ij})(\sum_{j=0}^{2^{r-1}} T^j) - (1+2^{r-2})(\sum_{j=0}^{2^r-1} T^j)$. To see that $U_{r,r,i}$ is a self-conjugate unit, consider the fibered square

$$\text{(1.1)} \qquad \begin{array}{ccc} R_r & \xrightarrow{\gamma_r} & Z[T|1+T+\ldots+T^{2^r-1} = 0] \\ \downarrow a_r & & \downarrow \bar{a}_r \\ Z & \longrightarrow & Z/2^rZ \end{array},$$

$a_r(\sum a_i T^i) = \sum a_i$. Then $a_r(U_{r,r,i}) = 1$ and also

$\gamma_r(U_{r,r,i}) = \left(\dfrac{T^{2^{r-1}+i}-1}{T^i - 1}\right)\left(\dfrac{T^{2^{r-1}+1}-1}{T - 1}\right)$, since $(T^i)^{2^{r-1}+1} = T^{2^{r-1}i+i} = T^{2^r+i}$

if i is odd. An element in R_r is determined by its image under a_r and γ_r, and is a unit if and only if these images are. But

$\rho^{(r)}_{j,k} = \left(\dfrac{T^j-1}{T^k-1}\right)$ is always a unit in $Z[T|1+T+\ldots+T^{2^r-1} = 0]$ if j is odd (a "cyclotomic unit") and $\overline{\rho}_{j,k} = T^{k-j}\rho_{j,k}$. It follows that $U_{r,r,i}$ is a self conjugate unit of R_r.

Now we define the units $U^r_{r,s,i}$, $3 \leq s < r$, $1 \leq i < 2^{s-1}$, $i \equiv 1 (\mathrm{mod}\ 4)$. Assume by induction that $U^{(r-1)}_{r-1,s,i}$ has been defined, is self-conjugate, has image 1 under a_{r-1}, and that

$$\gamma_{r-1}(U^{(r-1)}_{r-1,s,i}) = T^{\delta}_{r-1} \prod_J \rho^{(r-1)}_{j,k}$$

is a product of T^{δ}_{r-1} and cyclotomic units, with $\sum_J (k-j) \equiv 2\delta \ (\mathrm{mod}\ 2^{r-1})$. Note that

$$\overline{a}_r(\rho^{(r)}_{j,k}) \equiv \overline{a}_{r-1}(\rho^{(r-1)}_{j,k}) \bmod 2^{r-1}.$$

Hence $\overline{a}_r(T^{\delta}_r \prod_J \rho^{(r)}_{j,k}) \equiv 1 + \varepsilon \cdot 2^{r-1} \bmod 2^r$, where $\varepsilon = 0$ or 1. Hence there is a unique element $U^{(r)}_{r,s,i} \in R_r$ with $a_r(U^{(r)}_{r,s,i}) = 1$ and

$$\gamma_r(U^{(r)}_{r,s,i}) = \left(\frac{T^{2^{r-2}}(T^{2^{r-1}+1}-1)}{(T-1)} \right)^{\varepsilon} T^{\delta} \prod_J \rho^{(r)}_{j,k};$$

clearly $U_{r,s,i}$ will be a self-conjugate unit, with the appropriate image under γ_r to continue the inductive definition.

Finally, if $s \leq m < r$ write

$$U^{(m)}_{m,s,i} = \sum_{i=0}^{2^m-1} b_i T^i_m$$

and then let

$$U^{(r)}_{m,s,i} = \sum_{i=0}^{2^m-1} b_i T^{2^{r-m}i}.$$

This inductive definition was given for convenience only. An explicit formula can be given as follows: Let $\varepsilon_s(k)$, $k = 0,1,2,\ldots$ be the unique sequence of zeroes and ones with

$$(1+2^{s-1}) \prod_{t=s}^{m} (1+2^{t-1})^{\varepsilon_s(t-s)} \equiv 1 (\mathrm{mod}\ 2^m).$$

Let

$$\lambda(m,s) = \frac{1}{2^m}\,[(1+2^{s-1}) \prod_{t=s}^{m} (1+2^{t-1})^{\varepsilon(t-s)} - 1].$$

Let

$$\omega(m,s,i) = -\sum_{t=s}^{m} \varepsilon(t-s)2^{t-2} - 2^{s-2}i.$$

Then

$$U_{m,s,i} = T^{2^{r-m}\omega(m,s,i)} \Big(\sum_{j=0}^{2^{s}-1} T^{2^{r-m}ij}\Big) \prod_{t=s}^{m} \Big(\sum_{j=0}^{2^{t}-1} T^{2^{r-m}j}\Big)^{\varepsilon(t-s)}$$

$$- \lambda(m,s)\Big(\sum_{j=0}^{2^{m}-1} T^{2^{r-m}j}\Big).$$

Let $I_r:R_{r-1} \to R_r$ and $\pi_r:R_r \to R_{r-1}$ be ring homorphisms (preserving 1) with $I_r(T_{r-1}) = T_r^2$ and $\pi_r(T_r) = T_{r-1}$. Let $\tau_r:R_r \to R_{r-1}$, the transfer, be defined as follows: if $u(T_r) \in R_r$, then $u(T_r)u(-T_r) = v(T_r^2) = v(T_{r-1})$. Set $\tau_r(u(T_r)) = v(T_r^2)$. Obviously, τ_r carries units to units.

(1.2) <u>Proposition.</u> Let $3 \leq s \leq m \leq r$. Then

$$I_r(U^{(r-1)}_{m,s,i}) = U^{(r)}_{m,s,i} \quad \text{if} \quad m < r;$$

$$\pi_r(U^{(r)}_{m,s,i}) = T_{r-1}^{-\varepsilon_s(m-s)2^{r-2}}\, U^{(r-1)}_{m-1,s,i} \quad \text{if} \quad s < m;$$

$$\pi_r(U^{(r)}_{m,m,i}) = 1;$$

$$\tau_r(U^{(r)}_{m,s,i}) = (U^{(r-1)}_{m,s,i})^2 \quad \text{if} \quad m < r;$$

$$\tau_r(U^{(r)}_{r,r,i}) = 1; \text{ and}$$

$$\tau_r(U^{(r)}_{r,s,i}) = T_{r-1}^{-\varepsilon_s(r-s)2^{r-2}}\, U^{(r-1)}_{r-1,s,i} \quad \text{if} \quad s < r.$$

<u>Proof.</u> The statement about I_r is clear, and so is that about π_r, from the inductive definition. To prove the 2nd & 3rd statement about τ_r, write

$$\gamma_r(U^{(r)}_{r,s,i}) = T^{\delta} \prod_J \rho^{(r)}_{j,k}$$

as above, with $2\delta = \sum_J (k-j) \bmod 2^r$ again, and since $i \equiv 1 (\mathrm{mod}\ 4)$, j and k will always be $\equiv 1(\mathrm{mod}\ 4)$ also; hence δ is even. It is not difficult to show that $\tilde{\tau}_r : Z[T_r | 1+T_r+\ldots+T_r^{2^{r-1}} = 0] \to Z[T_{r-1} | 1+T_{r-1}+\ldots+T_{r-1}^{2^{r-1}-1} = 0]$ is also defined, with $\tilde{\tau}_r\gamma_r = \gamma_{r-1}\tau_r$. A quick calculation gives $\tilde{\tau}_r(\rho^{(r)}_{j,k}) = \rho^{(r-1)}_{j,k}$; and $\tilde{\tau}_r(T^{\delta}_r) = T^{\delta}_{r-1}$. Hence (note $\rho^{(r-1)}_{2^{r-1}+i,i} = 1$)

$$\gamma_{r-1}(\tau_r U^{(r)}_{r,s,i}) = \begin{cases} \gamma_{r-1}(T_{r-1}^{-\varepsilon_s(r-s)2^{r-2}} U^{(r-1)}_{r-1,s,i}) & s < r \\ 1 & s = r\ . \end{cases}$$

From (1.1), it follows that on units γ_{r-1} is a monomorphism (with image those units having image $\equiv\pm 1$ in $Z/2^{r-1}$), for $r \geq 3$, which yields the result. Finally, the first statement concerning τ_r follows from the observation that $\tau_r I_r(x) = x^2$.

(1.3) Theorem. The elements $U^{(r)}_{m,s,i}$, $3 \leq s \leq m \leq r$, $i \equiv 1(\mathrm{mod}\ 4)$, and $1 \leq i < 2^{s-1}$, represent a basis for the Z_2-vector space $H^0(Wh(Z_{2^r}))$.

Since $Wh(Z_{2^r}) \cong R^{\times}_r/\{\pm T^i\}$ and the involution on $Wh(Z_{2^r})$ is actually trivial [B], (1.3) can be restated as follows:

(1.3)' The units $U^{(r)}_{m,s,i}$ and the trivial units -1 and T form a basis for $R^{\times}_r/(R^{\times}_r)^2$.

To prove these results, we first recall that

$$\dim_{Z_2} H^0(Wh(Z_{2^r})) = 2^{r-1}-r.$$

Since there are exactly $2^{r-1}-r$ units $U_{m,s,i}$, $3 \leq s \leq m \leq r$, $i \equiv 1(\mathrm{mod}\ 4)$, $1 \leq i < 2^{s-1}$, it suffices to check the independent of these units. The obvious inductive argument, together with Prop. 1.2 (the statements concerning π_r), shows that it suffices to prove that the units $U^{(r)}_{m,m,i}$, $3 \leq m \leq r$, $1 \leq i < 2^{m-1}$, $i \equiv 1 (\mathrm{mod}\ 4)$ are independent modulo squares and trivial units. This will follow easily from the independence lemma of the next section.

§2. The independence lemma.

(2.1) Suppose there exist units $v \in Z[Z_{2^r}]^x$ and $u \in Z[Z_{2^{m-1}}]^x$ and integers δ_i, $i \equiv 1(\mathrm{mod}\ 4)$, $1 \leq i < 2^{m-1}$ and ℓ, so that

$$(2.1.1) \quad \prod_i (U^{(r)}_{m,m,i})^{\delta_i} = \pm T^{\ell} uv^2.$$

Then $\delta_i \equiv 0(\mathrm{mod}\ 2)$ for all i.

In (2.1.1), u is identified with its image under the inclusion $I_r I_{r-1} \cdots I_m$ (recall $T_m = (T_r)^{2^{r-m}}$). The proof of (1.3) or (1.3)' is completed with (2.1) and decreasing induction on m.

To prove (2.1), assume (2.1.1) is satisfied and apply the composite $\pi_r \pi_{r-1} \cdots \pi_2$, projecting R_r to R_1, and then map to $R_1 \otimes Z_2 = Z_2[T_1 | T_1^2 = 0]$. Under this map, it follows from (1.2) that the left side of (2.1.1) maps to 1. Clearly $Z[Z_{2^{m-1}}]$ will map to Z_2, and so the unit u maps to 1 also. If v maps to $a+bT_1$, then v^2 maps to $(a+b) \in Z_2$ also. Hence ℓ must be even. Absorbing the sign into u and $T^{\ell/2}$ into v, we may therefore replace (2.1.1) with

$$(2.2) \quad \prod_i (U^{(r)}_{m,m,i})^{\delta_i} = uv^2.$$

Next, we wish to reduce to the case $m = r$. Suppose $m < r$, and let $\rho: R_r \to R_r$ be the involution $(T = T_r)$

$$\rho(\sum_i a_i T^i) = \sum_i (-1)^i a_i T^i.$$

Then ρ has the fixed point set R_{r-1} $(= I_r(R_{r-1}))$. Hence, if ρ is applied to (2.1.1), the same equation is obtained, except that v is replaced by $\rho(v)$. Hence

$$(v/\rho(v))^2 = 1.$$

But the torsion of $Z[Z_{2^r}]^{\times}$ consists entirely of the trivial units $\{\pm T^i\}$ ([B], see also [M], [W2]). So

$$v/\rho(v) = \pm 1 \quad \text{or} \quad \pm T^{2^{r-1}}.$$

Now map to $R_r \otimes Z_2 = Z_2[T \mid T^{2^r} = 0]$. In this ring, v and $\rho(v)$ become equal, i.e. $v/\rho(v)$ maps to 1, where as $T^{2^{r-1}} \neq 1$ even mod 2. Thus $v/\rho(v) = \pm 1$; i.e. $v = \pm\rho(v)$. Hence either v or Tv is in $Z[Z_{2^{r-1}}]^{\times}$, as $\rho(T) = -1$, so $\rho(Tv) = -T\rho(v)$. Replacing v with Tv if necessary, we therefore obtain an equation of the form (2.11), but with r replaced by $(r-1)$. (In fact, for the case $v = -\rho(v)$, in this equation in $Z[Z_{2^{r-1}}]$, $\ell = 1$. It follows from the argument preceding 2.2 that $\rho(v) = -v$ could not occur.)

Next we wish to derive from (2.2) an equation of the form of (5.1) of [CS1, §5]. First apply $\gamma = \gamma_r$ to both sides of (2.1); we then obtain, with $q = 2^{r-2}$,

$$(2.3) \qquad \prod_i \left(\frac{T^{2q+i}-1}{T^i - 1}\right)^{\delta_i} \left(\frac{T^{2q+1}-1}{T - 1}\right)^{\delta_i} = \gamma(uv^2);$$

the product is over i with $1 \leq i < 2q$ <u>and</u> $i \equiv 1 \pmod 4$. This is just an equation of the form of (5.1) of [CS1], with minor notational changes.

For a odd, let

$$f_a:\{0,1,\ldots 2^r-1\} \to \{0,1\} = Z/2Z$$

be defined by $f_a(x) = 1$ if the least non-negative residue of $ax \bmod 2^r$ is between 1 and 2^{r-1}, and $f_a(x) = 0$ otherwise. In §5 of [CS1] it is shown that (2.3) implies the vanishing of a certain cohomology class $\chi \in H^1(R_r^x) = H^1(Z_2,R_r^x)$, with respect to the involution ρ defined above. From §7 of [CS1], it follows that the vanishing of this class implies the following equation of functions to $Z/2Z$ (see also the last equation in the 2nd complete paragraph on page 341 of [CS1]):

$$\sum_i \delta_i f_i + (\sum_i \delta_i) f_1 + (\ell/2)(f_1+f_{2q+1}) = 0.$$

In this case, $\ell = 2(\sum_i \delta_i)$. Again, all sums are over i with $1 \le i < 2^{r-1}$ and $i \equiv 1 (\bmod 4)$. Hence we obtain

$$(2.4) \qquad (\sum_i \delta_i f_i) + (\sum_i \delta_i) f_{2q+1} = 0 \quad (q = 2^{r-2}).$$

However, 2^r is a <u>tempered</u> <u>number</u> ([CS2], see also [CS1,3,4]). Hence all relations among the functions f_a are consequences of the "obvious" ones:

$$f_a + f_{2q+a} = f_1 + f_{2q+1}.$$

It follows that if $\delta_i \not\equiv 0 (\bmod 2)$, for $1 < i < 2q$, (2.4) would have a term involving f_{2q+i} as well; since the sum is over i between 1 and $2q$, this does not occur. Hence $\delta_i \equiv 0 (\bmod 2)$ for $1 < i < 2q$, and (2.4) becomes $\delta_1(f_1+f_{2q+1}) = 0$. This obviously implies $\delta_1 \equiv 0 (\bmod 2)$ also (as $f_1(1) + f_{2q+1}(1) = 1$). This completes the proof of (2.1).

§3. Signatures and determinants.

Let G be a finite group and let ρ be an irreducible complex representation, $\rho: G \to U(n)$. Let $\alpha = (H, \phi, \mu)$ be a $(-1)^k$ Hermitian unimodular[1] (quadratic) form over $Z[G]$, H a (stably) free $Z[G]$-module, representing an element $[\alpha]$ in $L_{2k}^h(G)$. Let $\alpha_{\mathbb{C}} = \alpha \otimes_{Z[G]} \mathbb{C}^n$, a $(-1)^k$-Hermitian form over the complex numbers, where $\mathbb{C}^n$ has a $Z[G]$-module structure via ρ. Then let

$$\sigma_\rho(\alpha) = \text{signature of} \begin{cases} \alpha_{\mathbb{C}} & k \text{ even} \\ \sqrt{-1}\,\alpha_{\mathbb{C}} & k \text{ odd} \end{cases}$$

Let $R(G)$ denote the complex representation ring of G; then a well-defined homomorphism (the multisignature)

$$\chi: L_{2k}^h(G) \to R(G)$$

is defined by

$$\chi([\alpha]) = \sum_\rho \sigma_\rho(\alpha)\rho,$$

where the sum is over irreducible representations. Let

$$\lambda: L_{2k}^s(G) \to L_{2k}^h(G)$$

be the natural map, $L_{2k}^s(G)$ the obstruction group for the surgery problem to obtain a simple homotopy equivalence. According to [W1], the following holds:

[1] Throughout this paper "unimodular" is used to mean that the adjoint $\text{Ad}\,\phi: H \to H^*$ is an isomorphism.

(3.1) Theorem. Let G be cyclic. Then the composite $\chi\lambda$ has the image $\{4(\rho+(-1)^k\bar{\rho}) \mid \rho \in R(G)\}$, is a monomorphism if k is even, and has kernel isomorphic to Z_2 for k odd (detected by the Arf invariant).

Now assume G is abelian (so that an irreducible representation is 1-dimensional). Let $(H,\phi,\mu) = \alpha$ be as above. Then, upon choice of a basis for H,

$$\det \alpha \in Z[G]^{\times}$$

is defined; a change of basis will multiply $\det \alpha$ by a unit of the form $x\bar{x}$.

(3.2) Proposition. Let G be abelian. Suppose α is a $(-1)^k$-Hermitian unimodular form over $Z[G]$ of rank $2r$. Let ρ be an irreducible representation of G. Then $\sigma_\rho(\alpha) \equiv 0 (\mathrm{mod}\ 2)$, and $\sigma_\rho(\alpha) \equiv 0 (\mathrm{mod}\ 4)$ if and only if $(-1)^{r(k+1)}\rho(\det \alpha) > 0$.

Remark. Since ρ is irreducible, $\rho: G \to S^1 \subset \mathbb{C}$, and extends to a homomorphism $Z[G] \to \mathbb{C}$, also denoted ρ. Since α is $(-1)^k$-Hermitian and H has even rank, $(\det \alpha)^- = \det \alpha$. Hence $\rho(\det \alpha)$ is real. Since $\rho(x\bar{x}) = \rho(x)\rho(x)^- > 0$, the sign of $\rho(\det \alpha)$ is unaffected by a change of basis.

This result is nothing more than a simple consequence of an old formula for computing the signature of a Hermitian form over $\mathbb{C}$; see e.g. [J]. Given a Hermitian form over $\mathbb{C}$, one can find a basis so that the (determinants of) the sequence of principle minors, ordered by size starting with zero, contains no successive zeroes. By convention, the determinant of the 0×0 minor is 1. The signature of the form is then given as $P-C$, where P is the number of permanences of sign and C the number of changes in the signs sequence of the principal minors. The sign of a zero is chosen arbitrarily.

$$\begin{array}{ccc} C^*(\tilde{X}) & \xrightarrow{\cap\xi} & C_*(\tilde{X}) \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h_*} \\ C^*(\tilde{M}) & \xrightarrow{\cap\eta} & C_*(\tilde{M}) \end{array} \quad ,$$

η representing [M]. Since $\cap\eta$ is a simple equivalence, it follows that $\overline{\Delta}(X) = \tau(h)+\tau(h)^-$; i.e. $\Delta(X)$ is trivial in $H^0(Wh(Z_{2^r}))$.

To prove the converse, suppose ξ is reducible. Then, by the well-known transversality arguments, there is a degree one normal map

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

into X, with surgery obstruction $\sigma(f,b) \in L^h_{2k}(Z_{2^r})$. Further, the following formulas hold:

$$\chi(\sigma(f,b)) = \chi(M) - \chi(X) \quad \text{and}$$

$$d(\sigma(f,b)) = \Delta(X) \qquad \text{(see (4.1))}.$$

These can be proven by standard arguments of surgery theory.

Let ρ denote the regular representation. Then the first equation and (ii) imply that

$$\chi(\sigma(f,b)) = q\rho,$$

where q is an integer. On the other hand, if k is even the coefficient of the trivial representation in $\chi(\sigma(f,b))$ is just the difference of the signatures $I(M)-I(X) = 8t$; hence $q = 8t$. Hence we may replace M by its connected sum with $|t|$ copies of a P.L. manifold of signature $8t/|t|$, to kill $\chi(\sigma(f,b))$. If k is odd, since $\chi(\sigma(f,b)) = -\chi(\sigma(f,b))^-$, $q = 0$ automatically. Hence we may assume $\chi(\sigma(f,b)) = 0$.

Now apply this to $\alpha \otimes_{Z[G]} \mathbb{C} = \alpha_{\mathbb{C}}$ or $\sqrt{-T}\,\alpha_{\mathbb{C}}$ as above. Then $P+C = 2r$. Hence $\sigma_\rho(\alpha)$ is also even. For k even, the sign of the largest principal minor is just that of $\rho(\det \alpha)$, and for k odd it is $(-1)^r \rho(\det \alpha) = \det(\sqrt{-T}\,\alpha_{\mathbb{C}})$. If the sign of the largest minor is positive, then C must be even. Hence $P-C = P+C-2C = 2r-2C$ will be divisible by 4 if and only if r is even. Similarly, if the last sign is negative, $P-C$ will be divisible by 4 if and only if r is odd. The result follows.

A result similar to (3.1) for cyclic groups of odd order (and slightly misstated) is stated in [W2] and was used there in the classification of fake lens spaces (see also [BPW]).

(3.3) Proposition. For $3 \leq s \leq m \leq r$, $1 \leq i < 2^{s-1}$, $i \equiv 1 (\text{mod } 4)$, there is a $(-1)^k$-Hermitian unimodular form $\alpha^{(r)}_{m,s,i}$, representing an element of $L^h_{2k}(Z_{2^r})$, with

$$\det(\alpha_{m,s,i}) = T^{2^{r-1}\varepsilon_s(m+1-s)} U^{(r)}_{m,s,i},$$

with respect to a suitable basis. $(T = T_r)$.

Notes: 1. In particular,

$$\det(\alpha_{m,m,i}) = T^{2^{r-1}} U^{(r)}_{m,m,i}.$$

2. For k even, it follows that $\text{rank}(\alpha_{m,s,i}) \equiv 0 (\text{mod } 4)$, applying 3.2 to the trivial representation. Recall that a unimodular even form over Z has signature $\equiv 0(8)$.

(3.4) Lemma. Let $x \in R_r^\times$, with $x = \bar{x}$ and $a(x) = 1$ for k odd. Then there is a $(-1)^k$ symmetric unimodular form β with

$$\det \beta = x \text{ or } T^{2^{r-1}} x,$$

with respect to a suitable choice of basis.

Proof: According to [W1, §], there is a short exact sequence

$$(3.5) \qquad 0 \to L^s_{2k}(Z_{2^r}) \xrightarrow{\lambda} L^h_{2k}(Z_{2^r}) \xrightarrow{d} Wh(Z_{2^r}) \otimes Z_2 \to 0.$$

Of course, $Wh(Z_{2^r}) \otimes Z_2 = H^0(Wh(Z_{2^r}))$. There is well known surjective determinant map (actually an isomorphism in this case [B])

$$Wh(Z_{2^r}) \to R_r^\times/\{\pm T^i\} \to 1,$$

and that the composition of the map induced on H^0 with the appropriate map of the above exact sequence is a surjective homomorphism

$$d: L^h_{2k}(Z_{2^r}) \to R_r^\times/\{\pm T^i y\bar{y} | y \in R_r^\times\},$$

with $d[\alpha] = [\det \alpha]$.

The determinant of a form can be multiplied by $y\bar{y}$ merely by changing basis (even by multiplying a single basis element by y). Hence there exists a form β with $\det(\beta) = \pm T^i x$, with respect to a suitable choice of basis. Further, $\det \beta = (\det \beta)^-$, since β is Hermitian or skew-Hermitian of even rank. (To compute the rank, pass all the way to $Z/2Z$, to obtain a symmetric unimodular form with $x \cdot x \equiv 0$. Such a form always has even rank.) Hence $T^i = T^{-i}$, thus $i = 0$ or 2^{r-1}. So $\det \beta = \pm x$ or $\pm T^{2^{r-1}} x$.

If k is even and the minus sign appears, just replace β by its orthogonal sum with a kernel; i.e. with $\kappa = (k, \phi, \mu)$, where ϕ has the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to some basis. Clearly this will change the sign.

Suppose k is odd. Then $\beta \otimes_{R_r} Z$ will be a unimodular skew form over the integers. It is well-known that such a form is a sum of kernels; in this case a kernel will have matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence $\det(\beta \otimes_{R^r} Z) = +1$.

It follows that under the augmentation $a_r : R_r \to Z$, $a_r(\det \beta) = +1$. Hence, since $a_r(x) = 1$, the minus sign is impossible.

Proof of 3.3. First apply (3.4), but with r replaced by $r+1$, to obtain a $(-1)^k$-Hermitian unimodular form β over R_{r+1} with (for a suitable basis)

$$\det \beta = U^{(r+1)}_{m+1,s,i} \quad \text{or} \quad T^{2^r}_{r+1} U^{(r+1)}_{m+1,s,i}.$$

The map $\pi_{r+1} : R_{r+1} \to R_r = Z[Z_{2^r}]$ provides an R_{r+1}-module structure on R_r, and it is not hard to see that

$$\det(\beta \otimes_{R_{r+1}} R_r) = \pi_{r+1}(\det \beta).$$

Let $\alpha_{m,s,i} = \beta \otimes_{R_{r+1}} R_r$. Since $\pi_r(T^{2^r}_{r+1}) = 1$, that $\alpha_{m,s,i}$ has the desired determinant now follows from (1.2).

§4. The image of the multisignature (Proof of Theorem A).

Let $R_{2k}(G)$, a Z_2 vector space, be the quotient of the group elements $2(\rho + (-1)^k \bar{\rho})$, $\rho \in R(G)$, by those of the form $4(\rho + (-1)^k \bar{\rho})$. Then by (3.1), (3.2) and the exact sequence (3.5), there is a diagram[1]

$$(4.1) \qquad \begin{array}{ccc} L^h_{2k}(Z_{2^r}) & \xrightarrow{\chi^{(r)}} & \{2(\rho + (-1)^k \bar{\rho}) \mid \rho \in R(Z_{2^r})\} \\ \downarrow d^{(r)} & \searrow \sigma^{(r)} & \downarrow \omega^{(r)} \\ H^0(Wh(Z_{2^r})) & \xrightarrow{\chi_2^{(r)}} & R_{2k}(Z_{2^r}) . \end{array}$$

[1] Recall $\overline{\chi(\alpha)} = (-1)^k \chi(\alpha)$.

Here $\chi^{(r)}$ is the multisignature, $d^{(r)}$ the determinant map d above with $H^0(Wh(Z_{2^r})$ identified with $R_r^{\times}/\{\pm T^i y^2 | y \in R_r^{\times}\}$; recall that $\bar{y} = T^j y$ some j, if $y \in R_r^{\times}$ [B][W2, §14]). Let $\sigma^{(r)} = \omega^{(r)}\chi^{(r)} = \chi_2^{(r)} d^{(r)}$.

(4.2) <u>Theorem</u>. <u>The elements</u> $\sigma^{(r)}(\alpha_{m,s,i})$ (<u>see</u> 3.3), <u>with</u> $3 \leq s \leq m \leq r$, $1 \leq i < 2^{s-1}$, $i \equiv 1 (\mathrm{mod}\ 4)$, and $s \leq 2m-r$, <u>form a basis</u> (<u>over</u> Z_2) <u>for the image of</u> $\sigma^{(r)}$.

(4.3) <u>Corollary</u>. $\mathrm{Dim}_{Z_2}(\mathrm{Im}\ \sigma^{(r)}) = [2/3(2^{r-1}-1)]-[(r-1)/2]$.

The corollary follows by just computing the number of indices m,s,i with $s \leq 2m-r$. It can be restated as follows (see 4.1):

(4.4) <u>Corollary</u>. $\mathrm{Dim}_{Z_2}(\chi^{(r)}(L_{2k}^h(Z_{2^r}))/\chi^{(r)}\lambda(L_{2k}^s(Z_{2^r}))) =$

$$= [2/3(2^{r-1}-1)]-[(r-1)/2].$$

Recall $\lambda = \lambda^{(r)}$ from (3.1). In view of (3.1), (3.5), and the fact that $\dim H^0(Wh(Z_{2^r})) = 2^{r-1}-r$, it follows easily that for k even

$$\dim \mathrm{Torsion}(L_{2k}^h(Z_{2^r})) = (2^{r-1}-r) - \dim(\mathrm{Im}\ \chi/\mathrm{Im}\ \chi\lambda),$$

and one more than this for k odd.

Clearly the right side is just

$$[2/3(2^{r-2}+2]-[r/2]-1$$

which implies Theorem A.

The rest of this section is devoted to the proof of (4.2). Let t_r be the representation of Z_{2^r} to $\mathbb{C}$ determined by

$$t_r(T_r) = e^{2\pi i/2^r}.$$

$$(\pi_r)_!:R_2(Z_{2^r}) \to R_2(Z_{2^{r-1}}),$$

$$(I_r)_!:R_2(Z_{2^{r-1}}) \to R_2(Z_{2^r}),$$

$$(\tau_r)_!:R_2(Z_{2^r}) \to R_2(Z_{2^{r-1}}),$$

with $(\pi_r)_!\sigma^{(r)} = \sigma^{(r-1)}(\pi_r)_*$, etc.

Let $W^{(r)}_{m,s} \subset R_2(Z_{2^r})$ be the span of the elements $\sigma^{(r)}(\alpha_{m,s,i})$, $1 \leq i < 2^{s-1}$, $i \equiv 1 (\mathrm{mod}\ 4)$.

(4.6) <u>Proposition.</u> $W^{(r)}_{r,r} = \ker(\pi_r)_! \cap \ker(\tau_r)_!$, <u>and</u> $\{\sigma^{(r)}(\alpha_{r,r,i}) | 1 \leq i < 2^{r-1},\ i \equiv 1(4)\}$ <u>is a basis for it.</u>

(<u>Note</u>: We write $\sigma^{(r)}(\alpha_{r,r,i})$ for $\sigma^{(r)}$ applied to the equivalence class of it, and similarly for χ, χ_2, etc..)

<u>Proof.</u> From (3.3), $\det(\alpha_{r,r,i}) = T^{2^{2^{r-1}}} U_{r,r,i}$. By (1.2), $\tau_r(T^{2^{2^{r-1}}} U_{r,r,i}) = \pi_r(T^{2^{2^{r-1}}} U_{r,r,i}) = 1$. Hence $(\pi_r)_!\sigma(\alpha_{r,r,i}) =$ $= \sigma^{(r-1)}(\pi_r)_*(\alpha_{r,r,i}) = \chi_2^{(r)}\pi_r(\det \alpha_{r,r,i}) = \chi^{(r)}_{(2)}(1) = 0$. Similarly, $(\tau_r)_!\sigma(\alpha_{r,r,i}) = 0$. Hence $W_{r,r} \subset \ker(\pi_r)_! \cap \ker(\tau_r)_!$.

However, $\ker(\pi_r)_! \cap \ker(\tau_r)_!$ is precisely the elements of the form $\sum_{i\ \mathrm{odd}} \gamma_i t_r^i$ $(\gamma_i \in 2Z/4Z)$, $\gamma_i = \gamma_{2^r-i}$ and $\gamma_i = \gamma_{2^{r-1}+i}$. Hence this Z_2-vector space has dimension 2^{r-3}, $r \geq 3$ (and 0, $r = 1$ or 2) therefore it suffices to prove that the elements $\sigma(\alpha_{r,r,i})$ are linearly independent. This will be done using (3.2).

Let $\zeta = t_r(T)$, and let

Then $R(Z_{2^r}) = Z[t_r | t_r^{2^r} = 1]$. Let

$$(I_r)_*: L_{2k}^h(Z_{2^{r-1}}) \to L_{2k}^h(Z_{2^r}),$$

$$(\pi_r)_*: L_{2k}^h(Z_{2^r}) \to L_{2k}^h(Z_{2^{r-1}}),$$

$$(\tau_r)_*: L_{2k}^h(Z_{2^r}) \to L_{2k}^h(Z_{2^{r-1}}),$$

be the indicated induced maps. $(I_r)_*$ and $(\pi_r)_*$ are just induced by the maps I_r and π_r on group rings, and $(\tau_r)_*$ is by just the transfer map of surgery theory. The maps I_r, π_r and τ_r also induce maps between the quotients $H^0(Wh(Z_{2^r})) = R_r^x/\{\pm T^i y^2\}$ and $R_{r-1}^x/\{\pm T_{r-1}^i y^2\}$, and the obvious diagrams involving all these maps and d commute.

(4.5) Proposition. Let $x \in L_{2k}^h(Z_{2^{r-1}})$ and $y \in L_{2k}^h(Z_{2^r})$. Assume that

$$\sigma^{(r-1)}(x) = \sum_0^{2^{r-1}-1} \gamma_i t_{(r-1)}^i \quad \text{and}$$

$$\sigma^{(r)}(y) = \sum_0^{2^r-1} \delta_i t_{(r)}^i \quad (\gamma_i, \delta_i \in 2Z/4Z).$$

Then the following hold:

(4.5.1) $$\sigma^{(r)}((I_r)_* x) = \sum_0^{2^{r-1}-1} \gamma_i (t_r^i + t_r^{2^{r-1}+i})$$

(4.5.2) $$\sigma^{(r-1)}((\pi_r)_* y) = \sum_0^{2^{r-1}-1} \delta_{2i} t_{r-1}^i; \quad \text{and}$$

(4.5.3) $$\sigma^{(r-1)}((\tau_r)_* y) = \sum_0^{2^{r-1}-1} (\delta_i + \delta_{2^{r-1}+i}) t_{(r-1)}^i.$$

These formulas follow from similar formulas for χ, whose proofs we leave to the reader. These formulas obviously provide maps

$$\xi_{ij} = t^j(T^{2^{r-1}} U_{r,r,i}). \quad \text{Then}$$

$$\xi_{ij} = \frac{-(\zeta^{ij}+1)(\zeta^j+1)}{(\zeta^{ij}-1)(\zeta^j-1)} \quad ; \quad \text{hence}$$

$$\xi_{ij} = \frac{-(\zeta^{ij}-\zeta^{-ij})(\zeta^j-\zeta^{-j})}{(\zeta^{ij}-2+\zeta^{-ij})(\zeta^j-2+\zeta^{-j})} .$$

Clearly the denominator is a positive real number. Let σ_{ij} be the coefficient of t^j in $\sigma^{(r)}(\alpha_{r,r,i})$.

Now $\zeta^k-\zeta^{-k} = (\sqrt{-1})z$, where z is real and $z > 0$ if and only if the least positive residue of $k \bmod 2^r$ is less than 2^{r-1}. Hence, for $2^{r-1} \leq j < 2^r$, $j \equiv 1 (\bmod 4)$,

$$\xi_{ij} < 0 \text{ if and only if } f_j(i) = 1.$$

Hence, by (3.2), $\sigma_{ij} = 2f_j(i) \pmod 4$ for $2^{r-1} \leq j < 2^r$, and in fact

$$\sigma(\alpha_{r,r,i}) = \sum_j 2f_j(i)(t^j+t^{-j}+t^{2^{r-1}+j}+t^{2^{r-1}-j})$$

the sum over $2^{r-1} \leq j < 2^r$, $j \equiv 1 (\bmod 4)$.

(4.7) <u>Lemma</u> ([CS2]). <u>The matrix</u> $(f_j(i))$, $1 \leq i,j < 2^{r-1}$, $1 \equiv j \equiv 1 (\bmod 4)$, <u>is non-singular over</u> Z_2.

This lemma clearly implies the independence of the elements $\sigma(\alpha_{r,r,i})$, and this completes the proof.

(4.8) <u>Proposition</u>. $\ker(\pi_r)_! \cap \text{Image}(I_r)_! \subset W_{r,r}$.

<u>Proof</u>. It is obvious that $(\tau_r)_!(I_r)_! = 0$. Hence

$(\ker \pi_r)_! \cap \text{Im}(I_r)_! \subset (\ker \pi_r)_! \cap (\ker \tau_r)_! = W_{r,r}$, by 4.6.

(4.9) <u>Proposition</u>. <u>For</u> $k,t \geq 0$ <u>and</u> $2k+t \leq r-3$,

$$W^{(r)}_{r-k-t,r-2k-t} \subset \sum_{\ell=0}^{k} W^{(r)}_{r-\ell,r-2\ell}.$$

Proof. By induction on k. Suppose $k = 0$. Then we must show that $\sigma(\alpha_{m,m,i}) \in W^{(r)}_{r,r}$ for $m < r$. But

$$\sigma^{(r)}(\alpha_{m,m,i}) = \chi^{(r)}_2(\det \alpha_{m,m,i}) = \chi^{(r)}_2(T^{2^{r-1}} U^{(r)}_{m,m,i})$$
$$= \chi^{(r)}_2(I_r(T^{2^{r-2}}_{r-1} U^{(r-1)}_{m,m,i}) = (I_r)_! \chi^{(r-1)}_2(\det \alpha^{(r-1)}_{m,m,i}))$$
$$= (I_r)_! \sigma^{(r-1)}(\alpha^{(r-1)}_{m,m,i}).$$

So $\sigma^{(r)}(\alpha^{(r)}_{m,m,i}) \in \operatorname{Im}(I_r)_!$.

Since $\pi_r(T^{2^{r-1}} U^{(r)}_{m,m,i}) = 1$, a similar argument implies that $\sigma(\alpha_{m,m,i}) \in \ker(\pi_r)_!$, and then the case $k = 0$ follows from (4.8).

Suppose $k > 0$. We claim that

$$(4.10) \qquad (\pi_r)_!(W^{(r)}_{m,s}) = \begin{cases} W^{(r-1)}_{m-1,s} & s < m \\ 0 & s = m \end{cases}.$$

In fact, if $s < m$, $(\pi_r)_!(\sigma^{(r)}_{m,s,i}) = (\pi_r)_!(\chi^{(r)}_2(T^{2^{r-1}} \varepsilon U^{(r)}_{m,s,i})) =$
$= \chi^{(r-1)}_2(\pi_r(T^{2^{r-1}} \varepsilon U^{(r)}_{m,s,i})) = \chi^{(r-1)}_2(T^{2^{r-2}} \varepsilon_s{}^{(m-s)} U^{(r-1)}_{m-1,s,i}) =$
$= \chi^{(r-1)}_{(2)}(\det \alpha^{(r-1)}_{m-1,s,i}) = \sigma^{(r-1)}(\alpha^{(r-1)}_{m-1,s,i})$, and similarly one gets 0 if $m = s$.

Similarly, one shows that for $m < r$, $s < m$,

$$(4.11) \qquad W^{(r-1)}_{m-1,s} = (I_{r-1})_!(W^{(r-2)}_{m-1,s}).$$

Hence

$$(\pi_r)_!(W^{(r)}_{r-k-t,r-2k-t}) = (I_{r-1})_!(W^{(r-2)}_{(r-2)-h-t,(r-2)-2h-t}),$$

where $h = k-1$. By induction,

$$W^{(r-2)}_{(r-2)-h-t,(r-2)-2h-t} \subset \sum_{e=0}^{h} W^{(r-2)}_{(r-2)-e,(r-2)-2e}.$$

But, as we have just seen (with $h = e$ and $t = 0$),

$$(I_{r-1})_!(W^{(r-2)}_{(r-2)-e,(r-2)-2e}) = (\pi_r)_!(W^{(r)}_{r-(e+1),r-2(e+1)}).$$

Hence

$$(\pi_r)_!(W^{(r)}_{r-k-t,r-2k-t}) \subset (\pi_r)_!(\sum_1^k W^{(r)}_{r-\ell,r-2\ell}), \text{ i.e.}$$

(4.12) $$W^{(r)}_{r-k-t,r-2k-t} \subset \sum_1^k W^{(r)}_{r-\ell,r-2\ell} + \ker(\pi_r)_!.$$

For $1 \leq \ell$, $W^{(r)}_{r-\ell,r-2\ell} = (I_r)_!(W^{(r-1)}_{r-\ell,r-2\ell})$, and similarly $W^{(r)}_{r-k-t,r-2k-t} \subset \text{Image}(I_r)_!$. Hence these all lie in $\ker(\tau_r)_!$, as $(\tau_r)_!(I_r)_! = 0$. Therefore in (4.12), $(\ker \pi_r)_!$ can be replaced by $(\ker \pi_r)_! \cap (\ker \tau_r)_!$, which equals $W^{(r)}_{r,r}$ by (4.6). This completes the proof of (4.9).

Proof of 4.2. By the previous proposition (4.9), the elements $\sigma(\alpha_{m,s,i})$, $3 \leq s \leq m \leq r$, $1 \leq i < 2^{s-1}$, $i \equiv 1 \pmod 4$, and $s \leq 2m-r$ generate the image of σ. (To see this, note that if $m = r-k-t$, $s = m-2k-t$, then $s = 2m-r+t$.) Therefore it will suffice to prove their linear independence. For $r = 3$, this is a consequence of (4.6). We argue by induction on r.

As in the proof of (4.9), (compare (4.10) and (4.11)), we have

(4.13) $$(\pi_r)_!\sigma(\alpha^{(r)}_{m,s,i}) = \begin{cases} 0 & \text{if } m = s \\ \sigma(\alpha^{(r-1)}_{r-1,s,i}) & \text{if } m = r,\ s < m \\ (I_{r-1})_!\sigma(\alpha^{(r-2)}_{m-1,s,i}) & \text{if } m < r,\ s < m. \end{cases}$$

It is obvious that $(I_{r-1})_!$ is a monomorphism. It then follows easily from the inductive hypothesis, (4.6), and (4.13) that the elements $\sigma(\alpha^{(r)}_{m,s,i})$ with either $m = s = r$ or $s < m < r$ and $s \leq 2m-r$ $(= 2(m-1)-(r-2))$ are linearly independent.

On the other hand

$$(\tau_r)_! \sigma(\alpha^{(r)}_{m,s,i}) = \begin{cases} 0 & \text{if } r = m = s \text{ or } m < r \\ \sigma(\alpha^{(r-1)}_{r-1,s,i}) & \text{if } m = r \text{ and } s < m. \end{cases}$$

For example, for $s < r$, $(\tau_r)_! \sigma(\alpha^{(r)}_{r,s,i}) = (\tau_r)_! (\chi_2^{(r)}(T^{2^{r-1}} \varepsilon U^{(r)}_{r,s,i})) =$

$= \chi_2^{(r-1)}(\tau_r(T^{2^{r-1}} \varepsilon U^{(r)}_{r,s,i})) = \chi_2^{(r-1)}(T^{2^{r-2}}_{r-1} \varepsilon_S(r-s) U^{(r-1)}_{r-1,s,i})$

$= \chi_2^{(r-1)}(\det \alpha^{(r-1)}_{r-1,s,i}) = \sigma(\alpha^{(r-1)}_{r-1,s,i})$, and the other cases are argued similarly, using (1.2).

So the elements $\sigma(\alpha^{(r)}_{m,s,i})$ with $m = s = r$ or with $s < m < r$ and $s \leq 2m-r$ map to 0 under $(\tau_r)_!$, whereas the elements $\sigma(\alpha^{(r)}_{r,s,i})$, $m < r$, map to the elements $\sigma(\alpha^{(r-1)}_{r-1,s,i})$, which, since $s \leq (r-1) = 2(r-1)-(r-1)$, are linearly independent by induction. This accounts for all the elements $\sigma(\alpha^{(r)}_{m,s,i})$ with $s \leq 2m-r$ and so completes the proof.

§5. Proof of Theorem B.

Consider the exact sequence [Sh, 4.1]

$$L^s_{4k+1}(Z_{2^r},-) \to L^h_{2k+1}(Z_{2^r},-) \to H^1(Wh(Z_{2^r})) \to$$
$$\to L^s_{4k}(Z_{2^r},-) \xrightarrow{\lambda} L^h_{4k}(Z_{2^r},-).$$

However, in this case the cohomology $H^1(Wh(Z_{2^r}))$ is taken with respect to the involution induced by

$$(\sum_i a_i T^i)^* = (\sum_i (-1)^i a_i T^{-i})$$

on the level of the group ring $R_r = Z[Z_{2^r}]$.

According to [W1], $L^s_{4k+1}(Z_{2^r},-) = 0$. Also from [W1], it follows that is a monomorphism, since $L^s_{4k}(Z_{2^r},-)$ can be detected by multisignatures and Arf invariants, whose definition extends to $L^h_{4k}(Z_{2^r})$ as well. Hence

$$L^h_{2k+1}(Z_{2^r},-) \cong H^1(Wh(Z_{2^r})),$$

the homology taken with respect to the above-mentioned involution. For $r = 1,2$, $Wh(Z_{2^r}) = 0$, so assume $r \geq 3$.

Let $S_r = \{u \in R_r^X | u = \bar{u}$ and $a_r(u) = 1\}$. Then there is a short exact sequence

$$1 \to \{1,T^{2^{r-1}}\} \to S_r \to Wh(Z_{2^r}) \to 0.$$

To see this, just recall again [B] (compare [W2, §14]) that $Wh(Z_{2^r}) = R_r^X/\{\pm T^i\}$ and the involution on $Wh(Z_{2^r})$ induced by $^-$ is trivial. Hence every element of $Wh(Z_{2^r})$ is represented by $u \in R_r^X$ with $a_r(u) = 1$ and

$$\bar{u} = \pm T^i u,$$

some i. Since $a(\bar{u}) = a(u) = 1$, the sign is positive. Project to $Z[Z_2]$; in this ring the involution $^-$ maps to the identity. It follows that $i = 2j$. Clearly $T^j u \in S_r$ and represents the same element of $Wh(Z_{2^r})$. Hence the map from S_r is surjective, and the kernel is easily identified.

Passing to cohomology, we obtain a long exact sequence

$$H^0(S_r) \to H^0(Wh(Z_{2^r})) \to \{1,T^{2^{r-1}}\} \to H^1(S_r) \to$$
$$\to H^1(Wh(Z_{2^r})) \to \{1,T^{2^{r-1}}\}$$
$$\to H^2(S_r) \to H^2(Wh(Z_{2^r})).$$

As above, every element of $Wh(Z_{2^r})$ has a representative $u \in R_r^\times$ with $u = \bar{u}$; hence

$$u(T)^* = u(-T^{-1}) = u(-T).$$

Hence an element of $H^0(Wh(Z_{2^r}))$ will be represented by a unit u with $u(T) = T^j u(-T)$. Pass to $Z_2 \otimes R_r$; this equation then implies $T^j = 1$; i.e. $u(T) = u(-T)$. Hence u represents an element of $H^0(S_r)$, and so the map from $H^0(S_r)$ to $H^0(Wh(Z_{2^r}))$ is surjective, and similarly for H^2 as $H^2 = H^0$ for cohomology of Z_2. Hence

(5.1) $$L^h_{2k+1}(Z_2,-) \cong H^1(S_r)/\{1,T^{2^{r-1}}\}.$$

By definition, $H^1(S_r)$ consists of elements u of S_r with $u(T) = u(-T)^{-1}$, modulo those of the form $v(T)v(-T)^{-1}$, for some $v \in S_r$. But $u(T) = u(-T)^{-1}$ if and only if $\tau_r(u(T)) = u(T)u(-T) = 1$. Hence the inclusion $K = (\ker \tau_r) \cap S_r \subset S_r$ induces a <u>surjective</u> map

$$K/K^2 = H^1(K) \to H^1(S_r).$$

Now suppose $u \in R_r$ and $u^2 \in K$. Then $\tau_r(u^2) = 1$. So $u(T)^2 u(-T)^2 = 1$. Hence, since the torsion of $R_r^\times$, consists entirely of trivial units, $u(T)u(-T) = \pm 1$ or $\pm T^{2^{r-1}}$. Hence

$$u^2 = \pm T^\varepsilon u(T)/u(-T)^{-1}, \quad \varepsilon = 0 \text{ or } 2^{r-1}.$$

By application of a_r, it is clear that the sign must be positive. Hence the previous map induces a surjective map

$$\omega: K/(R_r^\times)^2 \cap K \to H^1(S_r)/\{1,T^{2^{r-1}}\}.$$

The next step is to apply (1.3)' and (1.2). Since the torsion of $R_r^\times$ consists entirely of the trivial units, it follows from (1.3)' that the units

$U^{(r)}_{m,s,i}$ generate a free (abelian) subgroup V of R^x_r of rank $2^{r-1}-r$, i.e. they are linearly independent. From (1.2) it then follows that $V \cap K$ has the basis (as a free abelian group) $\{U^{(r)}_{r,r,i} | 1 \leq i < 2^{r-1},\ i \equiv 1 (\mathrm{mod}\ 4)\}$. It then follows by (1.3)' that these units represent a basis for the image of K in $R^x_r/(R^x_r)^2$ as a Z_2-vector space. Since $K/K \cap (R^x_r)^2 \subset R^x_r/(R^x_r)^2$, it finally follows that $K/K \cap (R^x_r)^2$ has as a basis the elements represented the elements $U^{(r)}_{r,r,i}$, $1 \leq i < 2^{r-1}$, and in particular, has dimension 2^{r-3}.

Now suppose $\omega(x)$ is trivial. Let x be represented by a product

$$\prod_i (U_{r,r,i})^{\delta_i},$$

with $\delta_i = 0$ or 1, $1 \leq i < 2^{r-1}$. Then

$$\prod_i (U_{r,r,i})^{\delta_i} = T^{\varepsilon}(u(T)/u(-T)),$$

$\varepsilon = 0$ or 2^{r-1}. Let $v(T) = u(T)u(-T) = \tau_r(u(T))$; $v(T) \in R_{r-1}$ $(= I_r(R_{r-1})) \subset R_r$. Then

$$\prod_i (U_{r,r,i})^{\delta_i} = (T^{\varepsilon/2}/u(-T))^2 v(T).$$

Hence by (2.1), $\delta_i = 0$ for all i; i.e. x is trivial. Hence ω is an isomorphism. By (5.1), this proves Theorem B.

Finally here is an exercise for the reader:

<u>Prove that</u> $\mathrm{Tor}(L^h_{2k}(Z_{2^r},-)) = \mathrm{Tor}(L^h_{2k}(Z_{2^{r-1}}))$.

§6. Smoothing Poincaré Complexes

Let X be a connected Poincare Duality space of dimension $2k$, with $\pi_1 X = G$. Let $\xi \in C_{2k}(X)$ be a cycle representing $[X]$. Let $C_*(X;Z[G])$

be the chain complex $C_*(\tilde{X})$ of the universal covering space, and let $C^*(X;Z[G])$ be the corresponding co-chain complex (just the usual co-chains of $\tilde{X}$ if G is finite). Then

$$\cap\xi : C^*(X;Z[G]) \to C_{2k-*}(X;Z[G])$$

is a chain equivalence which, up to chain homotopy, depends only upon X. The cells of X determine preferred bases of these chain and co-chain complexes. Hence $\cap\xi$ has a torsion in $Wh(G)$ which depends only on X. Denote this element $\bar{\Delta}(X)$. Then $\bar{\Delta}(X) = \bar{\Delta}(X)^-$, and so $\bar{\Delta}(X)$ represents an element $\Delta(X) \in H^0(Wh(G))$.

Now suppose $G = Z_{2^r}$. Then, by (1.3), $\Delta(X)$ has a unique representative of the form

$$\prod_{m,s,i} (U_{m,s,i})^{\delta_{m,s,i}},$$

where the product is over $3 \leq s \leq m \leq r$, $i \equiv 1 (\mathrm{mod}\ 4)$, $1 \leq i < 2^{s-1}$, and $\delta_{m,s,i} = 0$ or 1. Let

$$\Delta_{m,s,i}(X) = \delta_{m,s,i}.$$

(6.1) <u>Theorem</u>. <u>The connected Poincaré duality space</u> X <u>of dimension</u> $2k \geq 6$ <u>with</u> $\pi_1 X = Z_{2^r}$ <u>has the homotopy type of a</u> PL (<u>or</u> TOP) <u>manifold if and only if all of the following hold</u>:

(i) <u>there is a reducible</u> PL (<u>or</u> TOP) <u>bundle over</u> X;

(ii) $\chi(X)$ <u>is a multiple of the regular representation</u>;

(iii) $\Delta_{m,s,i}(X) = 0$ <u>for</u> $3 \leq s \leq m \leq r$, $i \equiv 1 (\mathrm{mod}\ 4)$, $1 \leq i < 2^{s-1}$, <u>and</u> $s > 2m-r$.

<u>Proof</u>. Necessity of (i) and (ii) has already been explained. For (iii), if M is a manifold and $h: M \to X$ a homotopy equivalence, then one considers the diagram

It follows from (4.2) (see also (4.1)) that $\Delta_{m,s,i}(X) = 0$ for $s \leq 2m-r$. Hence, by (iii), $\Delta_{m,s,i}(X) = 0$ for all m,s,i; i.e. $\Delta(X)$ is trivial. Hence by (3.5), $\sigma(f,b)$ actually is in the image of $L^s_{2k}(Z_{2^r})$. Hence, by (3.1), if k is even $\sigma(f,b) = 0$. If k is odd, $\sigma(f,b)$ can be killed by replacing M with its connected sum with a Kervaire manifold. So a normal map (f,b) with $\sigma(f,b) = 0$ is obtained, hence (f,b) is normally cobordant to a homotopy equivalence, which completes the proof.

Proof of Theorem C. Let X be as in Theorem C, and let $h:X \to \hat{Y}$ be a homotopy equivalence. Let $[Y] \in H_{2k}(Y)$, with transfer $h_*[X] \in H_{2k}(\hat{Y})$. Then h induces a homotopy equivalence $\tilde{h}:\tilde{X} \to \tilde{Y}$, and it is not hard to check that $\tilde{h}_*([X] \cap z) = \tilde{h}^*([Y] \cap \tilde{h}_* z)$. It follows that

$$\cap[Y]:H^i(\tilde{Y}) \to H_{2k-i}(\tilde{Y})$$

is an isomorphism for all i, and hence Y is a Poincaré Duality space.

Hence the invariant $\Delta(Y) \in H^0(Wh(Z_{2^{r+1}}))$ is defined. It is not hard to see that

$$\tau_{r+1}\Delta(Y) = \Delta(X),$$

here τ_{r+1} denotes the map induced on H^0 by the transfer. By (1.3)

$$\Delta(Y) \equiv \Pi(U^{(r+1)}_{m,s,i})^{\lambda_{m,s,i}}.$$

Hence $\Delta(X) \equiv \Pi\tau_{r+1}(U^{(r+1)}_{m,s,i})^{\lambda_{m,s,i}}$. Note that squares of self-conjugate units are trivial in $H^0(Wh(Z_{2^r}))$. Hence it follows from 1.2 that $\Delta_{m,s,i}(X) = 0$ for $m \neq r$. In particular, $\Delta_{m,s,i}(X) = 0$ for $s > 2m-r$, $3 \leq s \leq m \leq r$. Hence Theorem (6.1) applies to conclude that X has the homotopy type of a manifold.

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