THE POINCARÉ DUALITY THEOREM AND ITS CONVERSE I.

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FLOER CENTER OF GEOMETRY

Festive Opening Colloquium

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Local to global and, if possible, global to local

- There are many theorems in TOPOLOGY of the type

  $$\text{local input} \Longrightarrow \text{global output}$$

- Theorems of the type

  $$\text{global input} \Longrightarrow \text{local output}$$

  are even more interesting, and correspondingly harder to prove! This frequently requires ALGEBRA.
- *Algebra is a pact one makes with the devil!*
  (Sir Michael Atiyah)
- *I rather think that algebra is the song that the angels sing!*
  (Barry Mazur)
- *One thing I've learned about algebra ... don't take it too seriously* (Peanuts cartoon)

Poincaré duality and its converse

- The Poincaré duality of an n-dimensional topological manifold M
$$H^*(M) \cong H_{n-*}(M)$$
is a local $\Longrightarrow$ global theorem.
- **Theorem** Let $n \geqslant 5$. A space X with n-dimensional Poincaré duality $H^*(X) \cong H_{n-*}(X)$ is homotopy equivalent to an n-dimensional topological manifold if and only if X has sufficient local Poincaré duality.
- Modern take on central result of the Browder-Novikov-Sullivan-Wall high-dimensional surgery theory for differentiable and PL manifolds, and its Kirby-Siebenmann extension to topological manifolds (1962-1970)
- Will explain "sufficient" over the course of the lectures!

The Seifert-van Kampen Theorem and its converse

- Local $\Longrightarrow$ global. The fundamental group of a union

$$X = X_1 \cup_Y X_2 \ , \ Y = X_1 \cap X_2$$

is an amalgamated free product

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \ .$$

- Global $\Longrightarrow$ local. Let $n \geqslant 6$. If X is an n-dimensional manifold such that $\pi_1(X) = G_1 *_H G_2$ then $X = X_1 \cup_Y X_2$ for codimension 0 submanifolds $X_1, X_2 \subset X$ with

$$\partial X_1 = \partial X_2 = Y = (n-1)\text{-dimensional manifold} \ ,$$
$$\pi_1(X_1) = G_1 \ , \ \pi_1(X_2) = G_2 \ , \ \pi_1(Y) = H \ .$$

The Vietoris Theorem and its converses

- **Theorem** If $f : X \to Y$ is a surjection of compact metric spaces such that for each $y \in Y$ the restriction
$$f| \ : \ f^{-1}(y) \to \{y\}$$
induces an isomorphisms in homology
$$H_*(f^{-1}(y)) \ \cong \ H_*(\{y\})$$
then f induces isomorphisms in homology
$$f_* \ : \ H_*(X) \ \cong \ H_*(Y) \ .$$
- Local input: each $f^{-1}(y)$ ($y \in Y$) is acyclic
$$\widetilde{H}_*(f^{-1}(y)) \ = \ 0 \ .$$
- Global output: f_* is an isomorphism.
- Would like to have converses of the Vietoris theorem! For example, under what conditions is a homotopy equivalence homotopic to a homeomorphism?

Manifolds and homology manifolds

- An n-**dimensional topological manifold** is a topological space M such that each $x \in M$ has an open neighbourhood homeomorphic to $\mathbb{R}^n$.
- An n-**dimensional homology manifold** is a topological space M such that the local homology groups of M at each $x \in M$ are isomorphic to the local homology groups of $\mathbb{R}^n$ at 0

$$H_*(M, M \backslash \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{if } * \neq n \end{cases}$$

- A topological manifold is a homology manifold.
- A homology manifold need not be a topological manifold.
- Will only consider compact M which can be realized as a subspace $M \subset \mathbb{R}^{n+k}$ for some large $k \geqslant 0$, i.e. a compact ENR. This is automatically the case for topological manifolds.

The triangulation of manifolds

- A **triangulation** of a space X is a simplicial complex K together with a homeomorphism

$$X \cong |K|$$

with $|K|$ the polyhedron of K.
- X is compact if and only if K is finite.
- Triangulation of n-dimensional topological manifolds:
  - Exists and is unique for $n \leqslant 3$
  - Known: may not exist for $n = 4$
  - Unknown: if exists for $n \geqslant 5$
    (Update: now known. Manolescu 2013: Nontriangulable topological manifolds in each dimension $n \geqslant 5$)
  - Differentiable and PL manifolds are triangulated for all $n \geqslant 0$
- Triangulation of n-dimensional homology manifolds:
  - Exists and is unique for $n \leqslant 3$
  - Known: may not exist for $n \geqslant 4$.

The naked homeomorphism

- *Poincaré, for one, was emphatic about the importance of the naked homeomorphism - when writing philosophically - yet his memoirs treat DIFF or PL manifolds only.*
  in L. Siebenmann's 1970 ICM lecture on topological manifolds.
- ... *topological manifolds bear the simplest possible relation to their underlying homotopy types. This is a broad statement worth testing.* (ibid.)
- Will describe how surgery theory manufactures the homotopy theory of topological manifolds of dimension > 4 from Poincaré duality spaces and chain complexes.
- Poincaré duality is the most important property of the algebraic topology of manifolds.

The original statement of Poincaré duality

- Analysis Situs and its Five Supplements (1892–1904)
- 228 ANALYSIS SITUS.

  Donc

  $$P_p = P_{h-p}.$$

  Par conséquent, pour une variété fermée, les nombres de Betti également distants des extrêmes sont égaux.

  Ce théorème n'a, je crois, jamais été énoncé; il était cependant connu de plusieurs personnes qui en ont même fait des applications.
- Originally proved for a differentiable manifold M, but long since established for topological and homology manifolds.
- $h = n$, the dimension of M.
- $P_p = \dim_{\mathbb{Z}} H_p(M)$, the pth Betti number of M.
- Happy birthday! 2011 is the 100th anniversary of Brouwer's proof that homeomorphic manifolds have the same dimension. Also true for homology manifolds.

Orientation

- A **local fundamental class** of an n-dimensional homology manifold M at $x \in M$ is a choice of generator

$$[M]_x \in \{1, -1\} \subset H_n(M, M \backslash \{x\}) \;=\; \mathbb{Z} \;.$$

- The local Poincaré duality isomorphisms are defined by

$$[M]_x \cap - \;:\; H^*(\{x\}) \;\cong\; H_{n-*}(M, M \backslash \{x\}) \;.$$

- An n-dimensional homology manifold M is **orientable** if there exists a fundamental homology class $[M] \in H_n(M)$ such that for each $x \in M$ the image

$$[M]_x \in H_n(M, M \backslash \{x\}) \;=\; \mathbb{Z}$$

  is a local fundamental class.
- We shall only consider manifolds which are orientable, together with a choice of fundamental class $[M] \in H_n(M)$.

Poincaré duality in modern terminology

- **Theorem** For an n-dimensional manifold M the cap products with the orientation $[M] \in H_n(M)$ are Poincaré duality isomorphisms

$$[M] \cap - \ : \ H^*(M) \ \cong \ H_{n-*}(M) \ .$$

- **Idea of proof** Glue together the local Poincaré duality isomorphisms

$$[M]_x \cap - \ : \ H^*(\{x\}) \ \cong \ H_{n-*}(M, M \backslash \{x\}) \ (x \in M)$$

to obtain the global Poincaré duality isomorphisms

$$[M] \cap - \ = \ \varprojlim_{x \in M} [M]_x \cap - \ :$$

$$H^*(M) \ = \ \varprojlim_{x \in M} H^*(\{x\}) \ \cong \ H_{n-*}(M) \ = \ \varprojlim_{x \in M} H_{n-*}(M, M \backslash \{x\})$$

- Need to work on the chain level, rather than directly with homology.

Poincaré duality spaces

- **Definition** An n-**dimensional Poincaré duality space** X is a finite CW complex X with a homology class $[X] \in H_n(X)$ such that cap product with $[X]$ defines Poincaré duality isomorphism

$$[X] \cap - \ : \ H^*(X; \mathbb{Z}[\pi_1(X)]) \cong H_{n-*}(X; \mathbb{Z}[\pi_1(X)]) \ .$$

- In the simply-connected case $\pi_1(X) = \{1\}$ just

$$[X] \cap - \ : \ H^*(X) \cong H_{n-*}(X) \ .$$

- Homotopy invariant: any finite CW complex homotopy equivalent to an n-dimensional Poincaré duality space is an n-dimensional Poincaré duality space.
- A triangulable n-dimensional homology manifold is an n-dimensional Poincaré duality space.
- A nontriangulable n-dimensional homology manifold is homotopy equivalent to an n-dimensional Poincaré duality space.

Floer's Diplom thesis

- Floer's 1982 Bochum Diplom thesis (under the supervision of Ralph Stöcker) was on the homotopy-theoretic classification of $(n-1)$-connected $(2n+1)$-dimensional Poincaré duality spaces for $n > 1$.
- http://www.maths.ed.ac.uk/˜aar/papers/floer.pdf

Klassifikation hochzusammenhängender Poincaré-Räume

Andreas Floer

Diplomarbeit

Ruhr-Universität Bochum

Abteilung für Mathematik

1982

Manifold structures in the homotopy type of a Poincaré duality space

- (Existence) When is an n-dimensional Poincaré duality space homotopy equivalent to an n-dimensional topological manifold?
- (Uniqueness) When is a homotopy equivalence of n-dimensional topological manifolds homotopic to a homeomorphism?
- There are also versions of these questions for differentiable and PL manifolds, and also for homology manifolds.
- But it is the topological manifold version which is the most interesting! Both intrinsically, and because most susceptible to algebra, at least for $n > 4$.

Surfaces

- Surface = 2-dimensional topological manifold.
- Every orientable surface is homeomorphic to the standard surface Σ_g of genus $g \geqslant 0$.
- Every 2-dimensional Poincaré duality space is homotopy equivalent to a surface.
- A homotopy equivalence of surfaces is homotopic to a homeomorphism.
- In general, the analogous statements for false for n-dimensional manifolds with $n > 2$.

Bundle theories

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	spaces	bundles	classifying spaces
differentiable	manifolds	vector bundles	BO $\pi_*(BO)$ infinite
topological	manifolds	topological bundles	$BTOP$ $\pi_*(BTOP)$ infinite
homotopy theory	Poincaré duality spaces	spherical fibrations	BG $\pi_*(BG) = \pi^S_{*-1}$ finite

- Forgetful maps downwards. Difference between the first two rows = finite (but non-zero) = exotic spheres (Milnor).
- An n-dimensional differentiable manifold M has a tangent bundle $\tau_M : M \to BO(n)$ and a stable normal bundle $\nu_M : M \to BO$.
- Similarly for a topological manifold M, with $BTOP(n)$.
- An n-dimensional Poincaré duality space X has a Spivak normal fibration $\nu_X : X \to BG$.

The Hirzebruch signature theorem

- The **signature** of a $4k$-dimensional Poincaré duality space X is

$$\sigma(X) \ = \ \text{signature}(H^{2k}(X), \text{intersection form}) \in \mathbb{Z}$$

- The **Hirzebruch $\mathcal{L}$-genus** of a vector bundle η over a space X is a certain polynomial $\mathcal{L}(\eta) \in H^{4*}(X;\mathbb{Q})$ in the Pontrjagin classes $p_*(\eta) \in H^{4*}(M)$.
- **Signature Theorem** (1953) If M is a $4k$-dimensional differentiable manifold then

$$\sigma(M) \ = \ \langle \mathcal{L}(\tau_M), [M] \rangle \in \mathbb{Z}$$

- There have been many extensions of the theorem since 1953!

The Browder converse of the Hirzebruch signature theorem

- **Theorem** (Browder, 1962) For $k > 1$ a simply-connected $4k$-dimensional Poincaré duality space X is homotopy equivalent to a $4k$-dimensional differentiable manifold M if and only if $\nu_X : X \to BG$ lifts to a vector bundle $\eta : X \to BO$ such that
$$\sigma(X) \ = \ \langle \mathcal{L}(-\eta), [X] \rangle \in \mathbb{Z} \ .$$
- Novikov (1962) initiated the complementary theory of necessary and sufficient conditions for a homotopy equivalence of simply-connected differentiable manifolds to be homotopic to a diffeomorphism.
- Many developments in the last 50 years, including versions for topological manifolds and homeomorphisms.

The Browder-Novikov-Sullivan-Wall surgery theory I.

- Is an n-dimensional Poincaré duality space X homotopy equivalent to an n-dimensional topological manifold?
- The surgery theory provides a 2-stage obstruction for $n > 4$, working outside of X, involving normal maps $(f, b) : M \to X$ from manifolds M, with b a bundle map.
- Primary obstruction in the topological K-theory of vector bundles to the existence of a normal map $(f, b) : M \to X$.
- Secondary obstruction $\sigma(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ in the Wall surgery obstruction group, depending on the choice of (f, b) in resolving the primary obstruction. The algebraic L-groups defined algebraically using quadratic forms over $\mathbb{Z}[\pi_1(X)]$.
- The mixture of topological K-theory and algebraic L-theory not very satisfactory!

The Browder-Novikov-Sullivan-Wall surgery theory II.

- Is a homotopy equivalence $f : M \to N$ of n-dimensional topological manifolds homotopic to a homeomorphism?
- For $n > 4$ similar 2-stage obstruction theory for deciding if f is homotopic to a homeomorphism.
- The mapping cylinder of f

$$L \;=\; M \times [0,1] \cup_{(x,1)\sim f(x)} N$$

  defines an $(n+1)$-dimensional Poincaré pair $(L, M \sqcup N)$ with manifold boundary. The 2-stage obstruction for uniqueness is the 2-stage obstruction for relative existence.
- Again, the mixture of topological K-theory and algebraic L-theory not very satisfactory!

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The total surgery obstruction
I. Existence of manifold structures

- The $\mathbb{S}$-**groups** $\mathbb{S}_*(X)$ are $\mathbb{Z}$-graded abelian groups defined for any space X. A map $f : X \to Y$ induces $f_* : \mathbb{S}_*(X) \to \mathbb{S}_*(Y)$. If f is a homotopy equivalence, then f_* is an isomorphism
- The **total surgery obstruction** $s(X) \in \mathbb{S}_n(X)$ of an n-dimensional Poincaré duality space X with the following properties.
- If $f : X \to Y$ is a homotopy equivalence of n-dimensional Poincaré duality spaces then $f_* s(X) = s(Y) \in \mathbb{S}_n(Y)$.
- If X is an n-dimensional homology manifold then $s(X) = 0 \in \mathbb{S}_n(X)$.
- **Main Theorem** If $n \geqslant 5$ and $s(X) = 0 \in \mathbb{S}_n(X)$ then X is homotopy equivalent to an n-dimensional topological manifold.
- Global input $\Longrightarrow$ local output.
- Proof by Browder-Novikov-Sullivan-Wall theory.

The total surgery obstruction
II. Uniqueness of manifold structures

- The **total surgery obstruction** of a homotopy equivalence $h : N \to M$ of n-dimensional topological manifolds is an element $s(h) \in \mathbb{S}_{n+1}(M)$ with the following properties.
- If the point inverses $h^{-1}(x) \subset N$ ($x \in M$) are acyclic

$$h| \ : \ H_*(h^{-1}(x)) \ \cong \ H_*(\{x\})$$

  then $s(h) = 0 \in \mathbb{S}_{n+1}(M)$.
- If $n \geqslant 5$ and $s(h) = 0 \in \mathbb{S}_{n+1}(M)$ then h is homotopic to a homeomorphism. (Need also Whitehead torsion $\tau(h) = 0$). Every $s \in \mathbb{S}_{n+1}(M)$ is $s = s(h)$ for some h.
- Global input $\Longrightarrow$ local output.
- (A.R.) **The total surgery obstruction** (Proc. 1978 Aarhus Topology Conference, Springer Lecture Notes)

The Wall surgery obstruction

- In 1969 C.T.C. Wall constructed the **surgery obstruction groups** $L_n(A)$ of a ring with involution A, using quadratic structures on f.g. free A-modules.
- 4-periodic: $L_n(A) = L_{n+4}(A)$
- $L_0(A) =$ Witt group of quadratic forms over A.
- $L_1(A) =$ stable automorphism group of quadratic forms over A.
- $L_2(A) =$ Witt group of symplectic-quadratic forms over A.
- $L_3(A) =$ stable automorphism group of symplectic-quadratic forms over A.
- A normal map $(f, b) : M \to X$ from an n-dimensional manifold M to an n-dimensional Poincaré duality space X has a surgery obstruction $\sigma(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ such that $\sigma(f, b) = 0$ if (and for $n \geqslant 5$ only if) (f, b) is normal bordant to a homotopy equivalence.

Local $\Longrightarrow$ global in surgery theory

- The algebraic L-groups $L_*(\mathbb{Z}[\pi_1(X)])$ depend only on the fundamental group $\pi_1(X)$ of a space X, so are global.
- The Witt groups of sheaves of quadratic forms over X define the generalized homology groups $H_*(X;\mathbf{L}(\mathbb{Z}))$, which are local. Here $\mathbf{L}(\mathbb{Z})$ is a spectrum with

$$\pi_*(\mathbf{L}(\mathbf{Z})) \ = \ L_*(\mathbb{Z}) \ = \ \mathbb{Z}, 0, \mathbb{Z}_2, 0, \ldots \ \text{(4-periodic)}$$

  the simply-connected surgery obstruction groups.
- For any space X there is an exact sequence

$$\cdots \to H_n(X;\mathbf{L}(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \\ \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbf{L}(\mathbb{Z})) \to \ldots$$

- A is the local $\Longrightarrow$ global **assembly** map in L-theory. Originally defined geometrically by Quinn.
- The $\mathbb{S}$-groups $\mathbb{S}_*(X)$ measure the failure of A to be an isomorphism.

The failure of local Poincaré duality

- Let X be an n-dimensional Poincaré duality space. The failure of local Poincaré duality at $x \in X$ is measured by the groups $K_*(X,x)$ in the exact sequences

$$\cdots \to H^{n-r-1}(\{x\}) \xrightarrow{[X]_x \cap -} H_{r+1}(X, X\backslash\{x\}) \to K_r(X,x) \to H^{n-r}(\{x\}) \to \dots$$

- X is a homology manifold if and only if $K_*(X,x) = 0$ $(x \in X)$.
- Roughly speaking, the total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ is the cobordism class of a sheaf over X of chain complexes with quadratic Poincaré duality over $\mathbb{Z}$ with $K_*(X,x)$ the stalk at $x \in X$.
- Chain complex with quadratic Poincaré duality
  = chain complex with quadratic structure
  = generalization of quadratic form.

Bringing in the sheaves

(From *The Night of the Hunter*)

- The book
A.R. **Algebraic L-theory and manifolds** (CUP, 1992) developed the theory for simplicial complexes K, with an assembly map

$$A \ : \ \{(\mathbb{Z}, K)\text{-modules}\} \to \{\mathbb{Z}[\pi_1(K)]\text{-modules}\}$$

to provide the passage from local to global in algebra. This is sufficient for applications, since every Poincaré duality space is homotopy equivalent to one which is triangulated.
- Unfortunately, have not yet been able to develop the necessary sheaf theory. However, the paper
A.R.+Michael Weiss **On the construction and topological invariance of the Pontryagin classes** (Geometriae Dedicata 2010) points in the right direction!

Rings with involution

- An **involution** on a ring A is a function

$$A \to A \; ; \; a \mapsto \overline{a}$$

such that

$$\overline{a+b} \;=\; \overline{a}+\overline{b}\;,\;\; \overline{ab} \;=\; \overline{b}\,\overline{a}\;,\;\; \overline{\overline{a}} \;=\; a \;(a,b \in A)\;.$$

- **Example 1** A commutative ring A, with $\overline{a} = a$.
- **Example 2** A group ring $A = \mathbb{Z}[\pi]$ with $\overline{g} = g^{-1}$ $(g \in \pi)$.
- Regard a left A-module P as a right A-module with

$$P \times A \to P \; ; \; (x,a) \mapsto \overline{a}x \;.$$

- The tensor product of left A-modules P, Q is the abelian group defined by

$$P \otimes_A Q \;=\; P \otimes_{\mathbb{Z}} Q/\{ax \otimes y - x \otimes \overline{a}y \,|\, a \in A, x \in P, y \in Q\}$$

with transposition isomorphism

$$P \otimes_A Q \to Q \otimes_A P \; ; \; x \otimes y \mapsto y \otimes x \;.$$

Duality over a ring with involution

- The **dual** of a left A-module P is the left A-module

$$P^* = \text{Hom}_A(P, A) , \; A \times P^* \to P^* \; ; \; (a, f) \mapsto (x \mapsto f(x)\bar{a}) \; .$$

- The natural A-module morphism

$$P \to P^{**} \; ; \; x \mapsto (f \mapsto \overline{f(x)})$$

is an isomorphism for f.g. free P.

- For A-modules P, Q the abelian group morphisms

$$P^* \otimes_A Q \to \text{Hom}_A(P, Q) \; ; \; f \otimes y \mapsto (x \mapsto \overline{f(x)}y) \; ,$$

$$* : \text{Hom}_A(P, Q) \to \text{Hom}_A(Q^*, P^*); f \mapsto (f^* : g \mapsto (x \mapsto g(f(x))))$$

are isomorphisms for f.g. free P, Q.

Quadratic forms on chain complexes I.

- A.R. **The algebraic theory of surgery I., II.** (1980, Proc. LMS)
- The *n*-**dual** of a f.g. free A-module chain complex

$$C \ : \ \cdots \to C_r \xrightarrow{d} C_{r-1} \to \cdots \to C_1 \xrightarrow{d} C_0 \to \ldots$$

  is the f.g. free A-module chain complex

$$C^{n-*} \ : \ \cdots \to C^0 \xrightarrow{d^*} C^1 \to \cdots \to C^{r-1} \xrightarrow{d^*} C^r \to \ldots$$

  with $C^r = C_r^*$.
- An ‘algebraic Poincaré complex’ is a f.g. free A-module chain complex C with a chain equivalence $C^{n-*} \simeq C$ satisfying extra conditions. There are two flavours: **symmetric** and **quadratic**. Will ignore the difference today, using **algebraic** for both!

Quadratic forms on chain complexes II.

- For any f.g. free A-module chain complex C there is defined an isomorphism of A-module chain complexes

$$C \otimes_A C \to \mathrm{Hom}_A(C^{-*}, C) \; ; \; x \otimes y \mapsto (f \mapsto \overline{f(x)}.y) \; .$$

  The homology group

$$H_n(C \otimes_A C) \; = \; H_0(\mathrm{Hom}_A(C^{n-*}, C))$$

  is the group of chain homotopy classes of chain maps $\phi : C^{n-*} \to C$.
- The action of $T \in \mathbb{Z}_2$ by the **transposition involution**

$$T \; : \; C \otimes_A C \to C \otimes_A C \; ; \; x \otimes y \mapsto (-)^{pq} y \otimes x \; (x \in C_p, y \in C_q)$$

  corresponds to the **duality involution**

$$T \; : \; \mathrm{Hom}_A(C^{-*}, C) \to \mathrm{Hom}_A(C^{-*}, C) \; ; \; f \mapsto (-)^{pq} f^* \; ,$$

$$(f : C^p \to C_q) \mapsto ((-)^{pq} f^* : C^q \to C_p) \; , \; y(f^*(x)) = x(f(y)) \; .$$

Algebraic Poincaré cobordism

- An n-**dimensional algebraic Poincaré complex over** A (C, ϕ) is an n-dimensional f.g. free A-module chain complex C together with a chain equivalence $\phi : C^{n-*} \to C$ such that there exists a chain homotopy $T\phi \simeq \phi : C^{n-*} \to C$.
- If $1/2 \notin A$ need additional structure: either symmetric or quadratic.
- A cobordism $(L; M, M')$ of n-dimensional manifolds has Poincaré-Lefschetz duality

$$[L] \cap - : \; H^{n+1-*}(L, M) \; \cong \; H_*(L, M') \; .$$

- **Proposition** (Mishchenko, R., 1970's) The Wall group $L_n(A)$ is the group of cobordism classes of n-dimensional algebraic Poincaré complexes (C, ϕ) over A, with $(C, \phi) \sim (C', \phi')$ if $C \oplus C' \subset D$ for an $(n+1)$-dimensional f.g. free A-module chain complex D such that $H^{n+1-*}(D, C) \cong H_*(D, C')$.

The polyhedron of a simplicial complex

- A **simplicial complex** K is a collection of finite subsets $\sigma \subseteq K^{(0)}$ of an ordered **vertex set** $K^{(0)}$ such that:
  (a) $v \in K$ for each $v \in K^{(0)}$,
  (b) if $\sigma \in K$ and $\tau \subseteq \sigma$ then $\tau \in K$.
- The **dimension** of $\sigma \in K$ is

$$|\sigma| = (\text{no. of vertices in } \sigma) - 1$$

Let $K^{(n)}$ denote the set of n-simplexes in K.

- The **polyhedron** of K is the usual identification space

$$|K| = (\coprod_{n=0}^{\infty} \Delta^n \times K^{(n)})/\sim$$

with Δ^n the convex hull of $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$.

The simplicial chain complex

- The **simplicial chain complex** $C(K)$ has

$$d \ : \ C(K)_n = \mathbb{Z}[K^{(n)}] \to C(K)_{n-1} = \mathbb{Z}[K^{(n-1)}] \ ;$$

$$(v_0 v_1 \dots v_n) \mapsto \sum_{i=0}^{n} (-)^i (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$$
$$(v_0 < v_1 < \dots < v_n)$$

- The homology and cohomology groups of the polyhedron are the same as those of the simplicial complex

$$H_*(|K|) \ = \ H_*(K) \ = \ H_*(C(K)) \ ,$$
$$H^*(|K|) \ = \ H^*(K) \ = \ H^*(C(K)) \ .$$

- For any simplicial complexes K, L $H_n(|K| \times |L|)$ is the group of chain homotopy classes of chain maps $C(K)^{n-*} \to C(L)$.

Polyhedral Poincaré complexes

- A **triangulated** n**-dimensional Poincaré space** is a finite simplicial complex K with universal cover $\widetilde{K}$ and a homology class $[K] \in H_n(K)$ satisfying the equivalent conditions:
- (a) the cap products

$$[K] \cap - \ : \ H^{n-*}(\widetilde{K}) \ = \ H_*(C(\widetilde{K})^{n-*}) \to H_*(\widetilde{K})$$

are $\mathbb{Z}[\pi_1(K)]$-module isomorphisms.
- (b) The image $\Delta[K] \in H_n(X)$ under the diagonal map

$$\Delta \ : \ |K| \to X = |\widetilde{K}| \times_{\pi_1(K)} |\widetilde{K}| \ ; \ x \mapsto (\widetilde{x}, \widetilde{x})$$

is a chain homotopy class of $\mathbb{Z}[\pi_1(K)]$–module chain equivalences $\phi = \Delta[K] : C(\widetilde{K})^{n-*} \to C(\widetilde{K})$.
- (c) The cap product $[X] \cap - : H^n(X) \to H_n(X)$ is an isomorphism, with $\Delta[K]^* \in H^n(X)$ a $\mathbb{Z}[\pi_1(K)]$-module chain homotopy inverse $\phi^{-1} : C(\widetilde{K}) \to C(\widetilde{K})^{n-*}$.
- $(C(\widetilde{K}), \phi)$ is an n-dimensional algebraic Poincaré complex over $\mathbb{Z}[\pi_1(K)]$.

Dual cells

- The **barycentric subdivision** of K is the simplicial complex K' with $K'^{(0)} = K$ and
$$K'^{(n)} \ = \ \{(\sigma_0, \sigma_1, \dots, \sigma_n) \,|\, \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n\} \ .$$
Homeomorphic polyhedron $|K'| \cong |K|$.
- The **dual cells** of K are the contractible subcomplexes
$$D(\sigma) \ = \ \{(\sigma_0, \sigma_1, \dots, \sigma_n) \in K' \,|\, \sigma_0 \subseteq \sigma\} \subseteq K' \ .$$
- The **boundary** of the dual cell $D(\sigma)$ is
$$\partial D(\sigma) \ = \ \{(\sigma_0, \sigma_1, \dots, \sigma_n) \in D(\sigma) \,|\, \sigma_0 \neq \sigma\} \ .$$
- **Proposition** The local homology groups of $|K|$ at $x \in |K|$ are the homology groups of the dual cells relative to boundaries
$$H_*(|K|, |K| \backslash \{x\}) \ = \ H_{*-|\sigma|}(D(\sigma), \partial D(\sigma)) \, (x \in \text{interior}(\sigma), \sigma \in K) \ .$$
For each $\sigma \in K$ and $x \in \text{interior}(\sigma)$ there are natural maps
$$\partial_\sigma \ : \ H_*(|K|) = H_*(K) \to H_*(|K|, |K| \backslash \{x\}) = H_{*-|\sigma|}(D(\sigma), \partial D(\sigma))$$

The $(\mathbb{Z}, K)$-category I. Modules

- A.R.+M.Weiss **Chain complexes and assembly** Math. Z. (1999)
- A $(\mathbb{Z}, K)$-**module** is a f.g. free $\mathbb{Z}$-module M with splitting
$$M = \sum_{\sigma \in K} M(\sigma) .$$
- A **morphism** of $(\mathbb{Z}, K)$-modules $f : M \to N$ is a $\mathbb{Z}$-module morphism such that
$$f(M(\sigma)) \subseteq \sum_{\tau \geqslant \sigma} N(\tau) \ (\sigma \in K) .$$
- **Proposition** A $(\mathbb{Z}, K)$-module morphism $f : M \to N$ is an isomorphism if and only if each
$$f(\sigma, \sigma) \ : \ M(\sigma) \to N(\sigma) \ (\sigma \in K)$$
is a $\mathbb{Z}$-module isomorphism.

Assembly

- Let $p : \widetilde{K} \to K$ be the universal cover of a connected simplicial complex K. The **assembly** functor

$$A \ : \ \{(\mathbb{Z}, K)\text{-modules}\} \to \{\text{f.g. free } \mathbb{Z}[\pi_1(K)]\text{-modules}\}$$

  is defined by

$$A(M) \ = \ \sum_{\widetilde{\sigma} \in \widetilde{K}} M(p(\widetilde{\sigma})) \ .$$

- Local $\Longrightarrow$ global.
- **Example** For finite K the simplicial chain complex $C(K')$ is a $(\mathbb{Z}, K)$-module chain complex with

$$C(K')(\sigma) \ = \ C(D(\sigma), \partial D(\sigma)) \ (\sigma \in K)$$

  The assembly is the simplicial $\mathbb{Z}[\pi_1(K)]$-module chain complex of $\widetilde{K}'$

$$A(C(K')) \ = \ C(\widetilde{K}') \ .$$

The algebraic Vietoris theorem

- Let $f : L \to K'$ be a simplicial map with K, L finite.
- Regard $C(L)$ as a $(\mathbb{Z}, K)$-module chain complex by

$$C(L)(\sigma) \ = \ C(f^{-1}D(\sigma), f^{-1}\partial D(\sigma)) \ (\sigma \in K) \ .$$

- **Proposition** f has acyclic point inverses if and only if

$$f \ : \ C(L) \to C(K')$$

  is a $(\mathbb{Z}, K)$-module chain equivalence.
- **Corollary** If f has acyclic point inverses then

$$\widetilde{f} \ : \ C(\widetilde{L}) \to C(\widetilde{K}')$$

  is a $\mathbb{Z}[\pi_1(K)]$-module chain equivalence

The $(\mathbb{Z}, K)$-category II. Products

- The **product** of $(\mathbb{Z}, K)$-modules A, B is the $(\mathbb{Z}, K)$-module

$$A \otimes_{(\mathbb{Z},K)} B \;=\; \sum_{\lambda,\mu \in K, \lambda\cap\mu \neq \emptyset} A(\lambda) \otimes_{\mathbb{Z}} B(\mu) \subseteq A \otimes_{\mathbb{Z}} B \text{ with}$$

$$(A \otimes_{(\mathbb{Z},K)} B)(\sigma) \;=\; \sum_{\lambda,\mu \in K, \lambda\cap\mu = \sigma} A(\lambda) \otimes_{\mathbb{Z}} B(\mu) \; .$$

- **Example** For simplicial maps $f : L \to K'$, $g : M \to K'$ the pullback polyhedron

$$L \times_K M \;=\; \{(x,y) \in |L| \times |M| \,|\, f(x) = g(y) \in |K|\}$$

has homology

$$H_*(L \times_K M) \;=\; H_*(C(L) \otimes_{(\mathbb{Z},K)} C(M))$$

with

$$C(L)(\sigma) \;=\; C(f^{-1}D(\sigma), f^{-1}\partial D(\sigma)) \; ,$$

$$C(M)(\sigma) \;=\; C(g^{-1}D(\sigma), g^{-1}\partial D(\sigma)) \; .$$

The $(\mathbb{Z}, K)$-category III. Duality

- The **dual** of a $(\mathbb{Z}, K)$-module M is the $(\mathbb{Z}, K)$-module chain complex TM with

$$TM(\sigma)_r = \begin{cases} \sum\limits_{\tau \geqslant \sigma} M(\tau)^* & \text{if } r = -|\sigma| \\ 0 & \text{otherwise.} \end{cases}$$

- The dual of a $(\mathbb{Z}, K)$-module chain complex C is a $(\mathbb{Z}, K)$-module chain complex TC. Analogue of Verdier duality for sheaves.
- **Example** The dual of $C(K')$ is $(\mathbb{Z}, K)$-equivalent to the cochain complex of K

$$TC(K') \simeq C(K)^{-*}\ ,\ C(K)^r(\sigma) = \begin{cases} \mathbb{Z} & \text{if } r = -|\sigma| \\ 0 & \text{otherwise.} \end{cases}$$

- For any $(\mathbb{Z}, K)$-module chain complexes C, D

$$H_*(C \otimes_{(\mathbb{Z},K)} D) = H_*(\mathrm{Hom}_{(\mathbb{Z},K)}(TC, D))\ .$$

The assembly map

- **Proposition** (i) The generalized homology group $H_n(K;\mathbf{L}(\mathbb{Z}))$ is the cobordism group of n-dimensional algebraic Poincaré complexes $(C,\phi : TC_{*-n}\to C)$ in the $(\mathbb{Z},K)$-module category.
- (ii) The assembly functor

  $$A \ : \ \{(\mathbb{Z},K)\text{-modules}\}\to\{\mathbb{Z}[\pi_1(K)]\text{-modules}\}$$

  induces assembly maps in algebraic L-theory

  $$A \ : \ H_n(K;\mathbf{L}(\mathbb{Z}))\to L_n(\mathbb{Z}[\pi_1(K)])$$

- (iii) $\mathbb{S}_n(K)$ is the cobordism group of $(n-1)$-dimensional algebraic Poincaré complexes (C,ϕ) in the $(\mathbb{Z},K)$-module category such that the assembly $A(C)$ is a contractible f.g. free $\mathbb{Z}[\pi_1(K)]$-module chain complex, $H_*(A(C))=0$.

From local to global Poincaré duality, and back again!

- For any simplicial complex K

$$H_n(K) \ = \ H_n(\mathrm{Hom}_{(\mathbb{Z},K)}(TC(K'),C(K'))) \ .$$

The cap product with any homology class $[K] \in H_n(K)$ is a $(\mathbb{Z},K)$-module chain map

$$\phi \ = \ [K] \cap - : TC(K')_{*-n} \to C(K')$$

with diagonal components

$$\begin{aligned}\phi(\sigma,\sigma) = \partial_\sigma[K] \cap - : TC(K')_{*-n}(\sigma) = C(D(\sigma))^{n-*-|\sigma|} \\ \to C(K')(\sigma) = C(D(\sigma),\partial D(\sigma)) \ (\sigma \in K) \ ,\end{aligned}$$

with assembly

$$[K] \cap - : TC(\widetilde{K}')_{*-n} \simeq C(\widetilde{K})^{n-*} \to C(\widetilde{K}') \simeq C(\widetilde{K}) \ .$$

- K is a homology manifold if and only if $[K] \cap -$ is a $(\mathbb{Z},K)$-module chain equivalence. This is essentially Poincaré's original proof of duality!

The total surgery obstruction

- The **total surgery obstruction** of a polyhedral n-dimensional Poincaré duality space K is the cobordism class

$$s(K) \;=\; (\mathcal{C}(\phi)_{*+1}, \psi) \in \mathbb{S}_n(K) \;,$$

  with $\mathcal{C}(\phi)$ the $\mathbb{Z}[\pi_1(K)]$-contractible algebraic mapping cone of the $(\mathbb{Z}, K)$-module chain map

$$\phi \;=\; [K] \cap - : TC(K')_{n-*} \to C(K') \;.$$

- The image

$$t(K) \;=\; [s(K)] \in H_{n-1}(K; \mathbf{L}(\mathbb{Z}))$$

  is such that $t(K) = 0$ if and only if there exists a normal map $(f, b) : M \to |K|$, M an n-dimensional topological manifold.
- $s(K) = 0$ if and only if there exists a normal map (f, b) with surgery obstruction $\sigma(f, b) = 0 \in L_n(\mathbb{Z}[\pi_1(K)])$.
- For $n \geqslant 5$ $s(K) = 0$ if and only if $|K|$ is homotopy equivalent to an n-dimensional topological manifold, by B-N-S-W theory.

The symmetric signature

- The **symmetric signature** of a triangulated n-dimensional Poincaré space K is the algebraic Poincaré cobordism class

$$\sigma(K) \;=\; (C(\widetilde{K}),\phi) \in L_n(\mathbb{Z}[\pi_1(K)]) \;.$$

- The symmetric signature is a homotopy invariant, generalizing the signature.
- Modulo 2-torsion, the total surgery obstruction is the image

$$s(K) \;=\; [\sigma(K)] \in \mathrm{im}(L_n(\mathbb{Z}[\pi_1(K)]) \to \mathbb{S}_n(K)) \;.$$

- **Theorem** (A.R., 1992) Modulo 2-torsion, if $n \geqslant 5$ $|K|$ is homotopy equivalent to an n-dimensional topological manifold if and only if $s(K) = 0 \in \mathbb{S}_n(K)$, if and only if

$$\sigma(K) \in \mathrm{im}(A : H_n(K;\mathbf{L}(\mathbb{Z})) \to L_n(\mathbb{Z}[\pi_1(K)])) \;.$$

- For $n = 4k$, $\pi_1(K) = \{1\}$ this is just Browder's converse of the Hirzebruch signature theorem.

The homotopy types of topological manifolds

- For $n \geqslant 5$ the homotopy types of n-dimensional topological manifolds M fit into a fibre square

$$\begin{array}{ccc} \text{topological manifolds} & \longrightarrow & \text{cobordism of local APC's} \\ \downarrow & & \downarrow A \\ \text{PD spaces} & \longrightarrow & \text{cobordism of global APC's} \end{array}$$

  with PD = Poincaré duality, APC = algebraic Poincaré complexes, A = assembly.
- Local = in the $(\mathbb{Z}, K)$-module category, for a finite simplicial complex K with a surjection $|K| \to M$ with acyclic point inverses, and $\pi_1(|K|) \cong \pi_1(M)$,
- Global = in the $\mathbb{Z}[\pi]$-module category, $\pi = \pi_1(|K|) = \pi_1(M)$.

Three conjectures

- The **Novikov conjecture** (1969) on the homotopy invariance of the higher signatures of manifolds with fundamental group π is equivalent to the injectivity of the local $\Longrightarrow$ global assembly map $1 \otimes A : H_*(B\pi; \mathbf{L}(\mathbb{Z})) \otimes \mathbb{Q} \to L_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$.
History and survey of the Novikov conjecture.
- The **Borel conjecture** (1953) on the existence and rigidity of topological manifold structures on aspherical Poincaré complexes $B\pi$ is essentially equivalent to the assembly map $A : H_*(B\pi; \mathbf{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi])$ being an isomorphism, so that local $\Longleftrightarrow$ global.
1953 letter from Borel to Serre.
- The **Farrell-Jones conjecture** (1982) that a generalized assembly map from equivariant homology to the L-theory of $\mathbb{Z}[\pi]$ is an isomorphism for all groups π.

Conclusion

- Starting with Novikov himself, many authors in the last 40 years have proved many special cases of the Novikov, Borel and Farrell-Jones conjectures, using a wide variety of algebraic, geometric and analytic methods.
- Some (though not all) have used the algebraic L-theory assembly map defined here.
- There is still much work to be done to understand the relationship between all these methods of proof, and maybe even prove new results!