Course Syllabus for Math 281: Algebraic K-theory and Manifold Topology

Course Description

Let M and N be smooth closed manifolds of dimension n. An *h-cobordism* from M to N is a compact smooth manifold B of dimension $(n+1)$ with boundary $\partial B \simeq M \amalg N$ having the property that the inclusion maps $M \hookrightarrow B \hookleftarrow N$ are homotopy equivalences. If $n \geq 5$ and the manifold M is simply connected, then the celebrated *h-cobordism theorem* of Smale asserts that B is diffeomorphic to a product $M \times [0,1]$ (and, in particular, M is diffeomorphic to N).

If M is not simply connected, then it is generally not true that any h-cobordism B from M to N is diffeomorphic to a product $M \times [0,1]$. In fact, one can introduce an algebraic invariant $\tau(B)$ (called the *Whitehead torsion* of B) belonging to an abelian group $Wh(M)$ (called the *Whitehead group* of M). The invariant $\tau(B)$ vanishes whenever B is diffeomorphic to a product $M \times [0,1]$, and the converse holds provided that the dimension of M is greater than 5 (this statement is known as the *s-cobordism theorem*). The Whitehead group $Wh(M)$ depends only on the fundamental group of M and vanishes when M is simply connected, so that the s-cobordism theorem implies the h-cobordism theorem.

For many purposes, it is useful to know whether there is an analogue of the s-cobordism theorem for *families* of manifolds. Fix a compact smooth n-dimensional manifold M, a finite cell complex X, and suppose we given a fiber bundle $B \to X$ where each of the fibers B_x is an h-cobordism from M to some other n-manifold N_x (where the smooth structures vary continuously with x). Under what circumstances can we deduce that B is equivalent to a product $M \times [0,1] \times X$ (so that the fiber bundle $\{N_x\}_{x \in X}$ is trivial)? In this course, we will study an analogue of the Whitehead torsion which is adapted to the parametrized setting:

- In place of the Whitehead group $W(M)$, we will consider an infinite loop space $\mathbf{Wh}(M)$ called the (smooth) *Whitehead space* of M, with $\pi_0 \mathbf{Wh}(M) = Wh(M)$.
- To every fiber bundle $B \to X$ as above, one can associate a map $\tau(B) : X \to \mathbf{Wh}(M)$ which is well-defined up to homotopy. When X is a point, the homotopy class of this map determines an element of $\pi_0 \mathbf{Wh}(M) \simeq Wh(M)$ which agrees with the classical Whitehead torsion of the bordism B.
- The map $\tau(B)$ is nullhomotopic whenever B is equivalent to a product $M \times [0,1] \times X$. One of our main objectives in this course will be to prove the following converse: if $\tau(B)$ is nullhomotopic, then the fiber bundles

$$B \times [0,1]^k \to X \qquad M \times [0,1]^{k+1} \times X$$

are equivalent for $k \gg 0$.

Meeting Time and Place MWF at 12, Science Center 310.

Office Hours Wednesday 2-3, or by appointment.

Texts Course notes will be provided on the course webpage. Some portions of this course will follow the book "Spaces of PL Manifolds and Categories of Simple Maps" by Waldhausen, Jahren, and Rognes. Pointers to relevant literature will be provided as the course proceeds.

Course Website http://www.math.harvard.edu/∼lurie/281.html

Prerequisites Familiarity with the machinery of modern algebraic topology (simplicial sets, spectra, ...). Several other topics (such as piecewise linear topology, microbundles, immersion theory, the language of quasi-categories) will receive a cursory review as we need them. A high level of mathematical sophistication will be assumed.

Possible Topics

- Wall's Finiteness Obstruction
- Simple homotopy equivalences and Whitehead torsion.
- Polyhedra and simple maps; "higher" simple homotopy theory
- Waldhausen's algebraic K-theory of spaces
- Assembly; the "parametrized index theorem" of Dwyer-Weiss-Williams.
- Regular neighborhood theory.
- Piecewise linear manifolds, microbundles, and immersion theory
- The parametrized (stable) s-cobordism theorem
- Concordance spaces and the Hatcher spectral sequence
- Hilbert cube manifolds and infinite-dimensional topology

Grading Undergraduates or graduate students wishing to take this course for a grade should speak with the instructor.

Overview (Lecture 1)

August 27, 2014

Let M and N be smooth closed manifolds of dimension n. An *h-cobordism* from M to N is a compact smooth manifold B of dimension $(n+1)$ with boundary $\partial B \simeq M \amalg N$ having the property that the inclusion maps $M \hookrightarrow B \hookleftarrow N$ are homotopy equivalences. If $n \geq 5$ and the manifold M is simply connected, then the celebrated *h-cobordism theorem* of Smale asserts that B is diffeomorphic to a product $M \times [0,1]$ (and, in particular, M is diffeomorphic to N). This theorem has many pleasant consequences.

Application 1 (Generalized Poincare Conjecture)**.** Let X be a smooth manifold of dimension d which has the homotopy type of a sphere S^d. Choose a pair of points $x, y \in X$ with $x \neq y$, and let X^0 be the manifold obtained from X by removing the interiors of nonoverlapping small disks around x and y. Then X^0 is a manifold with boundary $S^{d-1} \amalg S^{d-1}$. If $d \geq 3$, then removing the points x and y from X (or small disks around them) does not change the fundamental group of X, so that X^0 is simply connected. A simple calculation with homology shows that the inclusions of each boundary component into X^0 is a homotopy equivalence: that is, X^0 is an h-cobordism from S^{d-1} to itself. Applying the h-cobordism theorem, we deduce that M^0 is diffeomorphic to a product $S^{d-1} \times [0,1]$. The original manifold X can be recovered (up to homeomorphism) from X^0 by adjoining a cone on each of its boundary components. It follows that X is homeomorphic to the sphere S^d.

Remark 2. Beware that the manifold X need not be diffeomorphic to S^d: the identification of X^0 with $S^{d-1} \times [0,1]$ may not be compatible with the identification of ∂X^0 with $S^{d-1} \amalg S^{d-1}$.

In the non-simply connected case, it is generally not true that an h-cobordism from M to N is diffeomorphic to a product $M \times [0,1]$. To guarantee this, one needs a stronger hypothesis on the inclusion $M \hookrightarrow B$. Before introducing this hypothesis, we need to embark on a slight digression.

Definition 3. Let X be a finite simplicial complex. Suppose that there is a simplex $\sigma \subseteq X$ containing a face $\sigma_0 \subseteq \sigma$ such that σ is not contained in any larger simplex of X, and σ_0 is not contained in any larger simplex other than σ. Let $Y \subseteq X$ be the subcomplex obtained by removing the interiors of σ and σ_0. Then the inclusion $\iota : Y \hookrightarrow X$ is a homotopy equivalence. In this situation, we will say that ι is an *elementary expansion*. Note that Y is a retract of X; a retraction of X onto Y will be called an *elementary collapse*.

Definition 4. Let $f : Y \to X$ be a map between finite simplicial complexes. We will say that f is a *simple homotopy equivalence* if it is homotopic to a finite composition of elementary expansions and elementary collapses.

Any compact smooth manifold can be regarded as a finite simplicial complex (by choosing a triangulation), so it makes sense to talk about simple homotopy equivalence between smooth manifolds. One then has the following result:

Theorem 5 (s-Cobordism Theorem)**.** *Let B be an h-cobordism theorem between smooth manifolds M and N of dimension ≥ 5. Then B is diffeomorphic to a product $M \times [0,1]$ if and only if the inclusion map $M \hookrightarrow B$ is a simple homotopy equivalence.*

One of our main goals in this course is to formulate and prove a *parametrized* version of Theorem 5. Let us now outline the path we will take.

1 Theme 1: Higher Simple Homotopy Theory

The subject of "higher" simple homotopy theory begins with the following:

Question 6. Let B be a finite simplicial complex and suppose we are given a fibration $q : E \to B$. Under what conditions can we find another finite simplicial complex E' and a fibration $q' : E' \to B$ which is fiber-homotopy equivalent to q?

There is an obvious necessary condition: if $q : E \to B$ is fiber-homotopy equivalent to a fibration between finite simplicial complexes, then each fiber $\{E_b\}_{b \in B}$ must itself have the homotopy type of a finite complex. Let us say that a fibration $q : E \to B$ is *homotopy finite* if it satisfies this condition, and *finite* if the total space E is itself a finite complex.

For a fixed base B, there is a bijective correspondence between equivalence classes of homotopy finite fibrations $q : E \to B$ and homotopy classes of maps

$$B \to \amalg \operatorname{BAut}(X),$$

where the coproduct is taken over all homotopy equivalence classes of finite simplicial complexes X, and $\operatorname{Aut}(X)$ denotes the space of homotopy equivalences of X with itself. In other words, the coproduct $\amalg \operatorname{BAut}(X)$ is a *classifying space* for homotopy finite fibrations.

Following Hatcher ([2]), we will introduce another space $\mathcal{M}$ which enjoys the following analogous property: for any finite simplicial complex B, there is a bijective correspondence between homotopy classes of maps from B into $\mathcal{M}$ and equivalence classes of finite fibrations $q : E \to B$ where E is also a finite simplicial complex (here the right notion of equivalence is *concordance*; we will return to this point later). We can think of $\mathcal{M}$ as a sort of "moduli space of finite simplicial complexes." Every finite simplicial complex X will determine a point $[X] \in \mathcal{M}$, and one can show that two points $[X], [Y] \in \mathcal{M}$ belong to the same connected component of $\mathcal{M}$ if and only if there is a simple homotopy equivalence from X to Y.

2 Theme 2: Simple Homotopy Theory and Algebraic K-Theory

To apply the s-cobordism theorem in practice, one needs to address the following:

Question 7. Let $f : X \to Y$ be a homotopy equivalence of finite simplicial complexes (or manifolds). When is f is a simple homotopy equivalence?

This is a classical question which was answered by J.H.C. Whitehead. To each homotopy equivalence $f : X \to Y$ as above, one can associate an algebraic invariant $\tau(f)$ called the *Whitehead torsion* of f, which belongs to a certain abelian group $\operatorname{Wh}(X)$ called the *Whitehead group* of X. Whitehead proved that $\tau(f)$ vanishes if and only if f is a simple homotopy equivalence. Moreover, this is automatic in many cases: for example, if X is simply connected then the entire group $\operatorname{Wh}(X)$ is trivial (which is why one does not need to consider simple homotopy theory in the statement of the classical h-cobordism theorem).

One of our goals in this course will be to study "higher" analogues of the Whitehead torsion, which can be used to answer the following generalization of Question 7:

Question 8. What is the relationship between the spaces $\mathcal{M}$ and $\amalg \operatorname{BAut}(X)$? How close is the map θ to being a homotopy equivalence?

To address Question 8, Waldhausen introduced the subject that is now known as *Waldhausen K-theory*, or the *algebraic K-theory of spaces*. This theory associates to every space X a spectrum $A(X)$, called the *A-theory spectrum of* X. This construction is functorial in X. Consequently, to any space X one can associate an *assembly map*

$$A(*) \wedge X_+ \to A(X).$$

We will denote the cofiber of this map by $\mathbf{Wh}(X)$ and refer to it as the *(topological) Whitehead spectrum* of X. It is a generalization of the Whitehead group in the sense that there is a canonical isomorphism $\operatorname{Wh}(X) \simeq \pi_1 \mathbf{Wh}(X)$. One of the main theorems of this course will be the following result of Waldhausen:

Theorem 9. *Let Y be a finite simplicial complex. Then the homotopy fiber of the map $\theta : \mathcal{M} \to \amalg \mathrm{BAut}(X)$ taken at the point $[Y] \in \mathcal{M}$ is canonically homotopy equivalent to $\Omega^{\infty+1}\mathbf{Wh}(Y)$.*

Remark 10. Any homotopy equivalence of finite simplicial complexes $f : Y \to Z$ determines a path between the images of $[Y]$ and $[Z]$ in the $\amalg \mathrm{BAut}(X)$, and therefore allows us to lift $[Z]$ to a point η of the homotopy fiber

$$\mathcal{M} \times_{\amalg \mathrm{BAut}(X)} \{[Y]\}.$$

Under the bijection $\pi_0(\mathcal{M} \times_{\amalg \mathrm{BAut}(X)} \{[Y]\}) \simeq \pi_1 \mathbf{Wh}(Y)$ supplied by Theorem 9, the homotopy class of the point η corresponds to the Whitehead torsion $\tau(f)$. Consequently, the statement that $\tau(f) = 0$ if and only if f is a simple homotopy equivalence can be regarded as a special case of Theorem 9.

Theorem 9 can be regarded as a definitive answer to Questions 8: it provides a purely "algebraic" description of the homotopy fibers of the map θ. Of course, to deduce any concrete consequences from this, one would need to understand the spectra $A(X)$: this is a very difficult problem about which much is known, but one which we will *not* consider in this course.

3 Theme 3: Simple Homotopy Theory as Stabilized Manifold Theory

There are several variants of Question 6 that one could consider:

Question 11. Let B be a finite simplicial complex and let $q : E \to B$ be a fibration. Under what conditions is q fiber-homotopy equivalent to a map $q' : E' \to B$ of finite simplicial complexes which exhibits E' as a *fiber bundle* over B?

At first glance, it would seem that Question 11 should have a very different answer than Question 8. Suppose that x and y are two points which belong to the same path component of B. If $q : E \to B$ is a fibration, then we can conclude that the fibers E_x and E_y are homotopy equivalent. If $q : E \to B$ is a finite fibration, then higher simple homotopy theory will tell us that E_x and E_y are related by a *simple* homotopy equivalence. But if $q : E \to B$ is a fiber bundle, then it follows E_x and E_y are homeomorphic: a dramatically stronger conclusion.

Somewhat surprisingly, it turns out that the answers to Questions 6 and 11 are the same:

Theorem 12. *Let $q : E \to B$ be a fibration between finite simplicial complexes. Then q is fiber-homotopy equivalent to a fiber bundle $q' : E' \to B$. Moreover, one can arrange that the fibers of q' are (piecewise-linear) manifolds (with boundary).*

In fact, one can be more precise. For each integer $d \geq 0$, one can define a classifying space $\mathcal{M}_d$ for framed (piecewise-linear) manifolds with boundary. There are stabilization maps

$$\mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \cdots$$

given by forming the product with the interval $[0, 1]$. Another of our goals in this course will be to prove the following result of Chapman:

Theorem 13. *The direct limit $\varinjlim \mathcal{M}_d$ is homotopy equivalent to $\mathcal{M}$.*

Remark 14. Let K be a finite simplicial complex. Then K is always homotopy equivalent to a framed (piecewise-linear) manifold M (with boundary). To find M, one can choose a piecewise-linear embedding of K into some Euclidean space $\mathbb{R}^n$ and take a regular neighborhood of K (with respect to a sufficiently fine triangulation of $\mathbb{R}^n$). Theorem 13 says roughly that it is possible to perform this construction "in families."

4 Theme 4: The Classification of Manifolds

Fix an integer $d \geq 0$. For each closed d-manifold M, let $\mathrm{Homeo}(M)$ denote the group of homeomorphisms of M with itself. We will refer to the disjoint union (taken over all homeomorphism classes of closed d-manifolds)

$$\mathcal{N} = \coprod_M \mathrm{BHomeo}(M)$$

as the *classifying space for manifolds of dimension d.* For any nice space B, there is a bijection between homotopy classes of maps from B into $\mathcal{N}$ and fiber bundles over B whose fibers are compact d-manifolds.

Remark 15. In this overview, we will focus our attention on topological manifolds and homeomorphisms, rather than smooth manifolds and diffeomorphisms. All results have analogues in the smooth setting, but many of the statements are a bit more complicated.

One basic problem in high-dimensional topology is to understand the homotopy type of the classifying space $\mathcal{N}$ (equivalently, to understand the homotopy type of the homeomorphism groups $\mathrm{Homeo}(M)$). As a first step, one can approximate $\mathcal{N}$ by another space $\mathcal{N}^h$, which classifies manifolds "up to h-cobordism" rather than up to homeomorphism. This approximation is much easier to understand: for $d \geq 5$, there is a completely algebraic description of $\mathcal{N}^h$ in terms of algebraic L-theory. Consequently, the problem of understanding the classifying space $\mathcal{N}$ is reduced to the problem of understanding the homotopy fibers of the map $\mathcal{N} \to \mathcal{N}^h$. This requires us to analyze the problem of "straightening out" h-cobordisms:

Question 16. Let M be a compact manifold and let $q : E \to B$ be a fiber bundle, where each fiber E_b is an h-cobordism from M to some other manifold N_b. When is E isomorphic to a product $M \times [0,1] \times B$?

When the base B is a point, the s-cobordism theorem gives a complete answer to Question 16 (at least for $\dim(M) \geq 5$): the h-cobordism E is a product if and only if the inclusion $f : M \hookrightarrow E$ is a simple homotopy equivalence, or equivalently if the Whitehead torsion $\tau(f)$ vanishes. Another of our main objectives in this course is to study a *parametrized* s-cobordism theorem:

Theorem 17 (Parametrized Stable s-Cobordism Theorem)**.** *Let $q : E \to B$ be as in Question 16, where B is a finite simplicial complex. Then there is an obstruction $\tau(E) \in \mathbf{Wh}(M)^{-1}(B)$ which vanishes if and only if there is a fiberwise equivalence*

$$E \times [0,1]^k \simeq M \times [0,1]^{k+1} \times B$$

for $k \gg 0$.

Remark 18. The obstruction $\tau(E)$ is given by the homotopy class of the map

$$B \to \Omega^{\infty+1}\mathbf{Wh}(M) \simeq \mathcal{M} \times_{\amalg \mathrm{BAut}(X)} \{[M]\}$$

classifying the finite fibration $E \to B$ and its fiber-homotopy equivalence to the constant bundle $M \times B \to B$).

Warning 19. As stated, Theorem 17 does not imply the (topological) s-cobordism theorem. In the special case where B is a point, it asserts that an h-cobordism E with vanishing Whitehead torsion becomes homeomorphic to $M \times [0,1]$ after forming a product with $[0,1]^k$ for $k \gg 0$. The s-cobordism theorem says that we can take $k = 0$ provided that $\dim(M) \geq 5$.

One might hope for a common generalization which asserts that if $\tau(E) = 0$ and $\dim(M)$ is sufficiently large (compared with $\dim(B)$), then E is equivalent to $M \times [0,1] \times B$.

5 Theme 5: Infinite-Dimensional Topology

Theorems 12 and 17 suggest that higher simple homotopy theory can be viewed as a "stabilized" theory of manifolds where we allow the dimension to grow. It turns out that many aspects of the theory can be elucidated by considering manifolds of infinite dimension, such as the Hilbert cube

$$Q = [0,1] \times [0,1] \times \cdots .$$

For example, one has the following theorem of Chapman:

Theorem 20. *Let $f : X \to Y$ be a map of finite simplicial complexes. Then f is a simple homotopy equivalence if and only if the induced map $X \times Q \to Y \times Q$ is homotopic to a homeomorphism.*

Theorem 20 greatly expands the usefulness of simple homotopy theory. For example, it implies that the Whitehead torsion is a *topological* invariant, which is not *a priori* clear from its definition (and was not known before Theorem 20).

If time permits, we'll discuss the proof of Theorem 20 and some related results (many of which are also due to Chapman) which imply that the moduli space $\mathcal{M}$ admits several "infinite-dimensional" descriptions:

- It is a classifying space for fibrations whose fibers are compact ANRs.
- It is classifying space for fiber bundles whose fibers are compact and locally homeomorphic to Q (such spaces are called *Hilbert cube manifolds*).

The Wall Finiteness Obstruction (Lecture 2)

August 28, 2014

We begin with the following:

Question 1. Let M be a compact manifold. Does M have the homotopy type of a finite CW complex?

Of course, if M is triangulable, then it is actually *homeomorphic* to a finite simplicial complex. This provides an affirmative answer 1 when M is smooth (since any smooth manifold can be triangulated).

For general topological manifolds, Question 1 is not so easy to answer. As a starting point, we note that any (paracompact) topological manifold M has the homotopy type of a (possibly infinite) CW complex X: this follows from the theory of absolute neighborhood retracts, which we will review in a subsequent lecture. Of course, we should not expect that X can be chosen finite in the case where M is noncompact (for a counterexample, consider the "surface of infinite genus").

Let us fix a homotopy equivalence $f : M \to X$, where X is a CW complex. If M is compact, then $f(M)$ is contained in a finite subcomplex $X_0 \subseteq X$.

Exercise 2. Prove this.

Let g be a homotopy inverse to f. Then the composite map

$$X \stackrel{fg}{\to} X_0 \hookrightarrow X$$

is homotopic to the identity map from M to itself. This motivates the following:

Definition 3. Let X be a CW complex (or, more generally, a space with the homotopy type of a CW complex). We say that X is *finitely dominated* if it is a retract (in the homotopy category of CW complexes) of a finite CW complex Y. In other words, X is finitely dominated if there exists a finite CW complex Y and a pair of maps

$$i : X \to Y \qquad r : Y \to X$$

such that $r \circ i$ is homotopic to the identity map from X to itself.

Of course, every finite CW complex X is finitely dominated: we can take $Y = X$ and the maps i and r to be the identity. More generally, if X is homotopy equivalent to a finite CW complex, then X is finitely dominated.

Question 4. Let X be a finitely dominated CW complex. Does X have the homotopy type of a finite CW complex?

We will see that the answer to Question 4 is "no" in general, but "yes" in many cases (for example, when X is simply connected). Moreover, the *topological* question of deciding whether or not X is finitely dominated will be reduced to an *algebraic* one (the vanishing of a certain K-theory class).

Remark 5. It is not true in general that every finitely dominated space X has the homotopy type of a finite CW complex. Nevertheless, it can be shown that every compact manifold has this property: that is, Question 1 has an affirmative answer, though we will not establish that in this lecture (a proof is given in Kirby-Siebenmann; if time permits, we'll discuss a stronger result later in this course).

We begin by summarizing some of the finiteness properties enjoyed by finitely dominated spaces.

Lemma 6. *Let X be a finitely dominated space. Then:*

(a) *The set $\pi_0 X$ is finite.*

(b) *For each base point $x \in X$, the group $\pi_1(X, x)$ is finitely presented.*

(c) *Let $x \in X$ and let $X^\circ \subseteq X$ be the path component of x. For each abelian group V with an action of the fundamental group $\pi_1(X, x)$, let $\mathrm{H}^*(X^\circ; V)$ denote the cohomology of X° with coefficients in the local system determined by V. Then the construction $V \mapsto \mathrm{H}^*(X^\circ; V)$ commutes with filtered direct limits.*

(d) *For each $x \in X$ as above, there exists an integer n such that $\mathrm{H}^*(X^\circ; V) \simeq 0$ for $* > n$ and any representation V of $\pi_1(X, x)$.*

Proof. Choose finite CW complex Y and a map $i : X \to Y$ which admits a left homotopy inverse r. Note that $\pi_0 Y$ is finite and that the map $\pi_0 X \to \pi_0 Y$ is injective (it has a left inverse given by r). This proves (a). To prove (b), we note that for each $x \in X$ the induced map $i_* : \pi_1(X, x) \to \pi_1(Y, y)$ is split injective. Since Y is a finite complex, the group $\pi_1(Y, y)$ is finitely presented. It follows that the group $\pi_1(X, x)$ is finitely presented, which proves (b). Let Y° be the path component of y. By composition with a left inverse to i_*, we see that representations V of $\pi_1(X, x)$ can be extended functorially to representations of $\pi_1(Y, y)$, which we can regard as local systems on Y°. Using cellular cochains to compute $\mathrm{H}^*(Y^\circ; V)$, we see immediately that the construction $V \mapsto \mathrm{H}^*(Y^\circ; V)$ commutes with filtered direct limits. We can functorially identify $\mathrm{H}^*(X^\circ; V)$ with a direct summand of $\mathrm{H}^*(Y^\circ; V)$, so the construction $V \mapsto \mathrm{H}^*(X^\circ; V)$ also commutes with filtered direct limits. This proves (c), and assertion (d) follows if we take $n = \dim(Y)$. □

Remark 7. Lemma 6 actually *characterizes* finitely dominated spaces; we will give a proof at the end of this lecture.

We next show that conditions (a) through (c) guarantee that X behaves approximately like a finite-dimensional CW complex.

Proposition 8. *Let X be a CW complex which satisfies conditions (a), (b), and (c) of Lemma 6. For each integer $n \geq 0$, there exists a finite CW complex Z of dimension $< n$ and an $(n-1)$-connected map $f : Z \to X$.*

We will deduce Proposition 8 from the following more precise statement:

Lemma 9. *Let X be a CW complex which satisfies conditions (a), (b), and (c) of Lemma 6. Suppose we are given an $(n-1)$-connected map $f : Z \to X$, where Z is a finite CW complex. Then there exists another finite CW complex Z', obtained from Z by attaching finitely many n-cells, and an n-connected map $f' : Z' \to X$ extending f. In particular, we have $\dim(Z') \leq \max\{n, \dim(Z)\}$.*

For the remainder of this lecture, it will be convenient to always assume that the space X is connected (we can always handle disconnected spaces by considering each connected component separately). Fix a base point $x \in X$, let $G = \pi_1(X, x)$, and let $\widetilde{X}$ be a universal cover of X (so that G acts on $\widetilde{X}$ by deck transformations).

Proof of Lemma 9. If $n = 0$, then we can either take $Z' = Z$ (if Z is nonempty) or $Z' = *$ (if Z is empty).

We next consider the case $n = 1$. If Z is not connected, then we first enlarge Z by adding 1-cells connecting the different components of Z (the map f extends continuously over this enlargement by virtue of our assumption that X is connected). Without loss of generality, we may assume that there exists a 0-cell $z \in Z$ such that $f(z) = x$. We let Z' be obtained from Z by attaching several loops based at the point z, and we define f so that it carries these loops to generators of the group $G = \pi_1(X, x)$. Since G is finitely generated, this only requires finitely many 1-cells.

We now consider the case $n = 2$. Let $z \in Z$ be as above, and consider the group homomorphism $\phi : \pi_1(Z, z) \to G$ induced by f. Since $\pi_1(Z, z)$ is finitely generated and G is finitely presented, the kernel $\ker(\phi)$ is generated as a normal subgroup by finitely many elements of $\pi_1(Z, z)$. Each of these elements can be represented by a loop in the 1-skeleton of Z. We first enlarge Z by attaching 2-cells along each of these loops (since the corresponding elements of $\pi_1(Z, z)$ are annihilated by ϕ, the function f will extend continuously). We may therefore reduce to the case where ϕ is an isomorphism.

We can now treat all of the cases $n \geq 2$ in a uniform manner. For any abelian group V with an action of G, we let $\mathrm{H}_*(X, Z; V)$ and $\mathrm{H}^*(X, Z; V)$ denote the homology and cohomology of X relative to Z with coefficients in V. Since f is $(n-1)$-connected, the groups $\mathrm{H}_*(X, Z; \mathbf{Z}[G])$ vanish for $* < n$. It follows from the universal coefficient theorem that we have canonical isomorphisms

$$\mathrm{H}^n(X, Z; V) \simeq \mathrm{Hom}_{\mathbf{Z}[G]}(\mathrm{H}_n(X, Z; \mathbf{Z}[G]), V).$$

Since X satisfies condition (c) of Lemma 6 and Z is finite CW complex, the construction $V \mapsto \mathrm{H}^n(X, Z; V)$ commutes with filtered direct limits (exercise!)

It follows that the construction

$$V \mapsto \mathrm{Hom}_{\mathbf{Z}[G]}(\mathrm{H}_n(X, Z_n; \mathbf{Z}[G]), V)$$

also commutes with filtered direct limits. In particular, $\mathrm{H}_n(X, Z; \mathbf{Z}[G])$ is finitely generated as a module over $\mathbf{Z}[G]$.

Set $\widetilde{Z} = Z \times_X \widetilde{X}$. Applying the relative Hurewicz theorem to the map $\widetilde{Z} \to \widetilde{X}$, we deduce that the Hurewicz map

$$\pi_n(X, Z) \simeq \pi_n(\widetilde{X}, \widetilde{Z}) \to \mathrm{H}_n(\widetilde{X}, \widetilde{Z}; \mathbf{Z}) \simeq \mathrm{H}_n(X, Z; \mathbf{Z}[G])$$

is an isomorphism. Consequently, the group $\pi_n(X, Z)$ is finitely generated as a $\mathbf{Z}[G]$-module. Each element of $\pi_n(X, Z)$ supplies a recipe for attaching an n-cell to Z and extending the definition of f over that n-cell. Without loss of generality, we may assume that the relevant attaching maps factor through the $(n-1)$-skeleton of Z. Let Z' be the CW complex obtained from Z by attaching n-cells corresponding to a set of generators of $\pi_n(X, Z)$, so that f extends to an n-connected map $f' : Z' \to X$. $\square$

For any space X satisfying conditions (a), (b), and (c) of Lemma 6, Lemma 9 allows us to construct a sequence of better and better approximations to X. It is condition (d) that will allow us to stop this construction.

Definition 10. Let X be a CW complex and let $n \geq 2$ be an integer. We will say that X has *homotopy dimension* $\leq n$ if it satisfies condition (d) of Lemma 6: that is, if $\mathrm{H}^*(X; \mathcal{L})$ vanishes for $* > n$ and any local system of abelian groups $\mathcal{L}$ on X.

Remark 11. Definition 10 makes sense for any value of n, but is not really the right condition when $n = 0$ and $n = 1$: in those cases, one should also require vanishing for "nonabelian" cohomology.

Lemma 12. *Let X be a CW complex satisfying the conditions of Lemma 6. Let Z be a finite CW complex of dimension $\leq n-1$ and let $f : Z \to X$ be an $(n-1)$-connected map. If X has homotopy dimension $\leq n$, then the homology group $\mathrm{H}_n(X, Z; \mathbf{Z}[G])$ is a finitely generated projective $\mathbf{Z}[G]$-module.*

Proof. Let V be any abelian group with an action of G, which determines local systems on X and Z which we will also denote by V. Since Z is $(n-1)$-dimensional, the local cohomology groups $\mathrm{H}^*(Z; V)$ vanish for $* \geq n$. Using the exact sequence

$$\mathrm{H}^{*-1}(Z; V) \to \mathrm{H}^*(X, Z; V) \to \mathrm{H}^*(X; V),$$

we see that the groups $\mathrm{H}^*(X, Z; V)$ vanish for $* > n$. Any exact sequence of representations $0 \to V' \to V \to V'' \to 0$ gives rise to a long exact sequence

$$\mathrm{H}^n(X, Z; V') \to \mathrm{H}^n(X, Z; V) \to \mathrm{H}^n(X, Z; V'') \to \mathrm{H}^{n+1}(X, Z; V') \simeq 0$$

It follows that the construction

$$V \mapsto \mathrm{H}^n(X, Z; V) \simeq \mathrm{Hom}_{\mathbf{Z}[G]}(\mathrm{H}_n(X, Z; \mathbf{Z}[G]); V)$$

is a *right exact* functor of V, so that $\mathrm{H}_n(X, Z_{n-1}; \mathbf{Z}[G])$ is a projective $\mathbf{Z}[G]$-module. As in the proof of Lemma 9, it is finitely generated because the construction $V \mapsto \mathrm{H}^n(X, Z; V)$ commutes with filtered direct limits. □

Remark 13. In the situation of Lemma 12, the relative homology groups $\mathrm{H}_*(X, Z; \mathbf{Z}[G])$ vanish for $* \neq n$. For $* < n$, this follows from our connectivity assumption on the map $f : Z \to X$. On the other hand, suppose there were some $m > n$ for which $\mathrm{H}_m(X, Z; \mathbf{Z}[G]) \neq 0$. Choose m as small as possible and set $A = \mathrm{H}_m(X, Z; \mathbf{Z}[G])$. Using the projectivity of $\mathrm{H}_n(X, Z; \mathbf{Z}[G])$, the universal coefficient formula gives

$$\mathrm{H}^m(X, Z; A) = \mathrm{Hom}_{\mathbf{Z}[G]}(\mathrm{H}_m(X, Z); \mathbf{Z}[G]), A) = \mathrm{Hom}_{\mathbf{Z}[G]}(A, A) \neq 0.$$

This is impossible, since $\mathrm{H}^m(X; A)$ and $\mathrm{H}^{m-1}(Z; A)$ both vanish.

Of course, the projective module $P = \mathrm{H}_n(X, Z; \mathbf{Z}[G])$ depends on the choice of $(n-1)$-connected map $f : Z \to X$. For example, we could enlarge the CW complex Z by adjoining some several $(n-1)$-spheres which map trivially to X; this would have the effect of replacing P by a direct sum $P \oplus \mathbf{Z}[G]^r$, where r is the number of additional spheres. This motivates the following:

Definition 14. Let R be a ring and let P be a finitely generated R-module. We say that P is *stably free* if $P \oplus R^a$ is a free R-module for some $a \geq 0$.

Proposition 15. *Let $n \geq 3$ and let X be a space which satisfies the conditions of Lemma 6 which is of homotopy dimension $\leq n$. Choose a finite CW complex Z of dimension $< n$ and an $(n-1)$-connected map $f : Z \to X$. Then the following conditions are equivalent:*

- (i) *The space X is homotopy equivalent to a finite CW complex of dimension $\leq n$.*
- (ii) *The projective module $P = \mathrm{H}_n(X, Z; \mathbf{Z}[G])$ is stably free.*

Proof. Suppose first that P is stably free. As indicated above, we can then alter the definition of Z (attaching some extra spheres) to arrange that P is actually free. In this case, we repeat the construction of Lemma 9 but slightly more carefully: we choose a basis for $\mathrm{H}_n(X, Z; \mathbf{Z}[G]) \simeq \pi_n(X, Z)$ and attach only n-cells corresponding to those basis elements. This produces a map $f' : Z' \to X$ which is an isomorphism on fundamental groups where the relative homology $\mathrm{H}_*(X, Z; \mathbf{Z}[G]) \simeq \mathrm{H}_*(\widetilde{X}, \widetilde{Z}'; \mathbf{Z})$ vanishes, so that f' is a homotopy equivalence by Whitehead's theorem.

Conversely, suppose that X is homotopy equivalent to a finite CW complex of dimension n. Without loss of generality we may assume that the map f is cellular, so that the relative homology $\mathrm{H}_n(X, Z; \mathbf{Z}[G])$ can be computed by a cellular chain complex of finitely generated free $\mathbf{Z}[G]$-modules

$$0 \to \mathbf{Z}[G]^{r_n} \to \mathbf{Z}[G]^{r_{n-1}} \to \cdots \to \mathbf{Z}[G]^{r_0} \to 0.$$

Since this complex is acyclic away from the top degree, it is split exact: that is, it has the form

$$0 \to Q_n \oplus Q_{n-1} \to Q_{n-1} \oplus Q_{n-2} \to \cdots \to Q_1 \oplus Q_0 \to Q_0 \to 0.$$

It follows by induction on i that each Q_i is stably free; in particular $P = Q_n$ is stably free. □

Remark 16. I believe it is an open question whether Proposition 15 is also valid for $n = 2$ (the proof given above certainly does not apply).

Corollary 17. *Let X be a finitely dominated space which is simply connected. Then X has the homotopy type of a finite CW complex.*

Proof. Every finitely generated projective $\mathbf{Z}$-module is free. $\square$

Proposition 15 allows us to quantify the failure of finitely dominated spaces to be homotopy equivalent to finite CW complexes.

Definition 18. Let R be a ring. We let $K_0(R)$ denote the Grothendieck group of projective R-modules: that is, the free abelian group generated by symbols $[P]$, where P is a projective R-module, modulo the relations

$$[P] = [P'] + [P'']$$

when there exists an isomorphism $P \simeq P' \oplus P''$.

The construction $n \mapsto n[R]$ determines a group homomorphism $\mathbf{Z} \to K_0(R)$. We let $\widetilde{K}_0(R)$ denote the cokernel of this homomorphism. We refer to $\widetilde{K}_0(R)$ as the *reduced K-group of R*.

Proposition 19. *Let X be a finitely dominated space of homotopy dimension $\leq n$. Let Z be a finite CW complex of dimension $< n$ and let $f : Z \to X$ be an $(n-1)$-connected map. Then the image of the class $[\mathrm{H}_n(X, Z; \mathbf{Z}[G])]$ in the reduced K-group $\widetilde{K}_0(\mathbf{Z}[G])$ does not depend on the choice of Z or f.*

Proof. Let Z' be another finite CW complex of dimension $< n$ equipped with an $(n-1)$-connected map $f' : Z' \to X$. We wish to show that $[\mathrm{H}_n(X, Z; \mathbf{Z}[G])] = [\mathrm{H}_n(X, Z'; \mathbf{Z}[G])]$ in the group $\widetilde{K}_0(\mathbf{Z}[G])$.

Starting with the map $Z \amalg Z' \to X$ and repeatedly applying Lemma 9, we obtain a homotopy commutative diagram

$$\begin{tikzcd}
Z \arrow[dr] \arrow[drr, "f"] & & \\
& Z'' \arrow[r] & X \\
Z'. \arrow[ur] \arrow[urr, "f'"'] & &
\end{tikzcd}$$

It will therefore suffice to show that we have equalities

$$[\mathrm{H}_n(X, Z; \mathbf{Z}[G])] = [\mathrm{H}_n(X, Z''; \mathbf{Z}[G])] \qquad [\mathrm{H}_n(X, Z'; \mathbf{Z}[G])] = [\mathrm{H}_n(X, Z'; \mathbf{Z}[G])].$$

In other words, we may replace Z' by Z'' and thereby reduce to the case where the map f factors as a composition

$$Z \xrightarrow{g} Z' \xrightarrow{f'} X.$$

Note that the map g is automatically $(n-2)$-connected. Using Lemma 9, we see that g factors as a composition

$$Z \xrightarrow{g'} Z^{+} \xrightarrow{g''} Z'$$

where g'' is $(n-1)$-connected and Z^{+} is obtained from Z by attaching finitely many $(n-1)$-cells. Replacing g by g' or g'', we are reduced to two special cases:

(a) The map g is $(n-1)$-connected. In this case, we have a short exact sequence

$$0 \to \mathrm{H}_n(Z', Z; \mathbf{Z}[G]) \to \mathrm{H}_n(X, Z; \mathbf{Z}[G]) \to \mathrm{H}_n(X, Z'; \mathbf{Z}[G]) \to 0$$

(the exactness on the left follows from Remark 13). This sequence splits (since $\mathrm{H}_n(X, Z'; \mathbf{Z}[G])$ is projective), so we have

$$[\mathrm{H}_n(X, Z; \mathbf{Z}[G])] = [\mathrm{H}_n(X, Z'; \mathbf{Z}[G])] + [\mathrm{H}_n(Z', Z; \mathbf{Z}[G])].$$

It will therefore suffice to show that the class $[\mathrm{H}_n(Z', Z; \mathbf{Z}[G])]$ vanishes in $\widetilde{K}_0(\mathbf{Z}[G])$. This follows from Proposition 15, since Z is a finite CW complex.

(*b*) The CW complex Z' is obtained from Z by attaching $(n-1)$-cells. In this case, we have an exact sequence

$$0 \to \mathrm{H}_n(X, Z; \mathbf{Z}[G]) \to \mathrm{H}_n(X, Z'; \mathbf{Z}[G]) \to \mathrm{H}_{n-1}(Z', Z; \mathbf{Z}[G]) \to 0$$

which gives

$$\begin{aligned}
[\mathrm{H}_n(X, Z'; \mathbf{Z}[G])] &= [\mathrm{H}_n(X, Z; \mathbf{Z}[G])] + [\mathrm{H}_{n-1}(Z', Z; \mathbf{Z}[G])] \\
&= [\mathrm{H}_n(X, Z; \mathbf{Z}[G])] + [\mathbf{Z}[G]^r] \\
&= [\mathrm{H}_n(X, Z; \mathbf{Z}[G])].
\end{aligned}$$

□

Proposition 19 motivates the following:

Definition 20. Let X be a finitely dominated space. Choose an integer $n \geq 2$ such that X has homotopy dimension $\leq n$, a CW complex Z of dimension $< n$, and an $(n-1)$-connected map $f : Z \to X$. The *Wall finiteness obstruction* of X is the element

$$w(X) = (-1)^n [\mathrm{H}_n(X, Z; \mathbf{Z}[G])] \in \widetilde{K}_0(\mathbf{Z}[G]).$$

Proposition 21. *Let X be a finitely dominated space. Then the Wall finiteness obstruction $w(X)$ is well-defined.*

Proof. We have already seen that $w(X)$ does not depend on the map $f : Z \to X$. We now check that it is independent of n. Let us temporarily denote $w(X)$ by $w_n(X)$ to emphasize its hypothetical dependence on n. Choose any integer $n \geq 2$ such that X has homotopy dimension $\leq n$; we will show that $w_n(X) = w_{n+1}(X)$. To prove this, choose a CW complex Z of dimension $< n$ and an $(n-1)$-connected map $f : Z \to X$. Using Lemma 9, we can extend f to an n-connected map $f' : Z' \to X$ where Z' is obtained from Z by attaching finitely many n-cells. We then have a (split) short exact sequence

$$0 \to \mathrm{H}_{n+1}(X, Z'; \mathbf{Z}[G]) \to \mathrm{H}_n(Z, Z'; \mathbf{Z}[G]) \to \mathrm{H}_n(X, Z; \mathbf{Z}[G]) \to 0$$

which gives the relation

$$[\mathrm{H}_n(X, Z; \mathbf{Z}[G])] + [\mathrm{H}_{n+1}(X, Z'; \mathbf{Z}[G])] = [\mathrm{H}_n(Z, Z'; \mathbf{Z}[G])] = 0 \in \widetilde{K}_0(\mathbf{Z}[G]).$$

□

By virtue of Proposition 15, the finiteness obstruction $w(X)$ is zero if and only if X has the homotopy type of a finite CW complex.

We conclude this lecture by tying up a few loose ends. We start with a converse to Lemma 6.

Proposition 22. *Let X be a CW complex satisfying conditions* (a) *through* (d) *of Lemma 6. Then X is finitely dominated.*

Proof Sketch. Write X as a union of finite subcomplexes X_α. It will suffice to show that the identity map $\mathrm{id} : X \to X$ is homotopic to a map which factors through some X_α. We can replace each of the inclusions $X_\alpha \to X$ by a fibration $p_\alpha : E_\alpha \to X$; we wish to show that one of these inclusions has a section.

For each integer m, let $\tau_{\leq m} E_\alpha$ denote the mth stage in the relative Postnikov tower of E_α over X (so that we have a fibration $p_{\alpha,m} : \tau_{\leq m} E_\alpha \to X$ whose fibers have no homotopy groups above m). Suppose we are given a section s_m of some $p_{\alpha,m}$. Note that if $m \geq 1$, then the fiber product

$$(\tau_{\leq m+1} E_\alpha) \times_{E_\alpha} X$$

is a fibration over whose fibers have the form $K(A_{x,\alpha}, m+1)$, where $x \mapsto A_{x,\alpha}$ is a local system of abelian groups on X. Consequently, the obstruction to lifting s_m to a section of $p_{\alpha,m+1}$ is measured by a cohomology class $\eta(s_m) \in \mathrm{H}^{m+2}(X; A_\alpha)$. If $m+2$ is larger than the homotopy dimension of X (which is finite by assumption), then $\eta(s_m)$ automatically vanishes, so any section of $p_{\alpha,m}$ can be lifted to a section of p_α.

We will complete the proof by showing that for every integer m, there exists an index α such that $p_{\alpha,m}$ admits a section. The proof proceeds by induction on m. Suppose first that $m \geq 1$ and that we are given a section s_m as above. We claim that it is possible to choose $\beta \geq \alpha$ such that the image of $\eta(s_m)$ vanishes in $\mathrm{H}^{m+2}(X; A_\beta)$. In fact, we claim that the direct limit $\varinjlim_{\beta \geq \alpha} \mathrm{H}^{m+2}(X; A_\beta)$ vanishes. Since X satisfies condition (c) of Lemma 6, it will suffice to show that the direct limit $\varinjlim_{\beta \geq \alpha} A_\beta$ vanishes as a local system of abelian groups on X. This follows immediately from the fact that X is a homotopy colimit of the diagram $\{E_\alpha\}$.

It remains to treat small values of m. Let us begin with the case $m = 0$, so that each $\tau_{\leq m} E_\alpha$ can be regarded as a covering space of X. Let S_α denote the fiber over the base point $x \in X$, so that each S_α is a set with an action of the group G. To choose a section of $\tau_{\leq m} E_\alpha$, we must show that S_α contains an element which is fixed by G. Because G is finitely generated, passage to G-invariants commutes with filtered direct limits. It will therefore suffice to show that the direct limit $\varinjlim S_\alpha$ contains an element which is fixed by G. This is clear, since $\varinjlim S_\alpha$ consists of a single point (it is π_0 of the homotopy fiber of the identity map $X \to X$)

We conclude by treating the case $m = 1$. Let us assume that there exists an index α and that we have chosen a section of the map $\tau_{\leq 0} E_\alpha \to X$. For each $\beta \geq \alpha$, let E'_β denote the fiber product $\tau_{\leq 1} E_\beta \times_{\tau_{\leq 0} E_\beta} X$. The projection map $q_\beta : E'_\beta \to X$ is a fibration whose fibers have the form $K(\Pi; 1)$. Let G_β denote the fundamental group of E'_β, so that we have $\varinjlim G_\beta \simeq G$. Since G is finitely presented, it follows that the natural map $G_\beta \to G$ admits a section for β sufficiently large. We are therefore reduced to finding a section of the induced map $E'_\beta \times_{BG_\beta} BG \to X$. This is a fibration whose fibers are of the form $K(\Pi, 1)$ where Π is abelian, and is therefore classified by an element of $\mathrm{H}^2(X; \mathcal{L}_\beta)$ for some local system of abelian groups $\mathcal{L}_\beta$ on X. As before, we have $\varinjlim_\beta \mathcal{L}_\beta = 0$ so (by virtue of condition (c)) the direct limit $\varinjlim_\beta \mathrm{H}^2(X; \mathcal{L}_\beta)$ vanishes, and therefore the fibration is trivial for β sufficiently large. $\square$

Remark 23. Let G be a finitely presented group. Then every class $\eta \in \widetilde{K}_0(\mathbf{Z}[G])$ arises as the Wall finiteness obstruction of some finitely dominated space X with $\pi_1 X = G$. To see this, we first choose a connected finite 2-dimensional CW complex X_0 with $\pi_1 X_0 = G$. Let $\eta = [P]$ where P is a finitely generated projective $\mathbf{Z}[G]$-module, so that P appears as a direct summand of some free $\mathbf{Z}[G]$-module $\mathbf{Z}[G]^r$. Then P is the image of an idempotent map $e : \mathbf{Z}[G]^r \to \mathbf{Z}[G]^r$. Choose an even integer $n \geq 2$ and let Y be the CW complex obtained from X_0 by adding r n-cells with trivial attaching maps. Using the relative Hurewicz theorem we deduce that the relative homotopy group $\pi_n(Y, X_0)$ is isomorphic to the free module $\mathbf{Z}[G]^r$. We can therefore choose a map $\overline{e} : Y \to Y$ which is the identity on X_0 and which induces the idempotent endomorphism e of $\mathrm{H}_n(Y, X_0; \mathbf{Z}[G]) \simeq \mathbf{Z}[G]^r$. Using the fact that $\overline{e}$ is an idempotent in the homotopy category, one can show that the homotopy colimit X of the diagram

$$Y \xrightarrow{\overline{e}} Y \xrightarrow{\overline{e}} Y \xrightarrow{\overline{e}} \cdots$$

satisfies the conditions of Lemma 6 and is therefore a finitely dominated space of homotopy dimension $\leq n$; a simple calculation shows that the composite map $X_0 \hookrightarrow Y \to X$ is $(n-1)$-connected and that the relative homology $\mathrm{H}_n(X, X_0; \mathbf{Z}[G])$ is isomorphic to P.

Whitehead Torsion (Lecture 3)

September 10, 2014

Let X be a space with the homotopy type of a CW complex. In the previous lecture, we studied the question of whether or not that CW could be chosen to be *finite*. More precisely, we saw that if a connected space X is finitely dominated (meaning that it behaves cohomologically like a finite CW complex), then there is an obstruction $\eta \in \widetilde{K}_0(\pi_1 X)$ which vanishes if and only if X has the homotopy type of a finite CW complex Y.

In this lecture, we will study the question of uniqueness. Suppose we are given two finite CW complexes Y and Y' equipped with homotopy equivalences

$$f : Y \to X \qquad g : X \to Y'.$$

Then $g \circ f$ is a homotopy equivalence from Y to Y'. One can ask if this homotopy equivalence can be "witnessed" entirely in the world of finite CW complexes.

In what follows, we use the term *CW complex* to refer to a space Y with a *specified* decomposition into open cells. For each integer $n \geq -1$, we let Y^n denote the n-skeleton of Y. Recall that a map of CW complexes $f : X \to Y$ is *cellular* if it carries each X^n into Y^n.

Construction 1. Let D^n denote the closed unit ball of dimension n and let $S^{n-1} = \partial D^n$ denote its boundary. We will regard S^{n-1} as decomposed into hemispheres S^{n-1}_- and S^{n-1}_+ which meet along the "equator" $S^{n-2} = S^{n-1}_- \cap S^{n-1}_+$.

Let Y be a CW complex equipped with a map $f : (S^{n-1}_-, S^{n-2}) \to (Y^{n-1}, Y^{n-2})$. Then the pushout $Y \amalg_{S^{n-1}_-} D^n$ has the structure of a CW complex which is obtained from Y by adding two more cells: an $(n-1)$-cell given by the image of the interior of S^{n-1}_+ (attached via the map $f|_{S^{n-2}} : S^{n-2} \to Y^{n-2}$) and an n-cell given by the image of the interior of D^n attached via the map

$$S^{n-1} = S^{n-1}_- \amalg_{S^{n-2}} S^{n-1}_+ \to Y^{n-1} \amalg_{S^{n-2}} S^{n-1}_+.$$

In this case, we will refer to the CW complex $Y \amalg_{S^{n-1}_-} D^n$ as an *elementary expansion* of Y, and to the inclusion map $Y \hookrightarrow Y \amalg_{S^{n-1}_-} D^n$ as an *elementary expansion*.

The hemisphere $S^{n-1}_- \subseteq D^n$ is a retract (even a deformation retract) of D^n. Composition with any retraction induces a (cellular) $c : Y \amalg_{S^{n-1}_-} D^n \to Y$, which we will refer to as an *elementary collapse*. Note that the homotopy class of c does not depend on the choice of retraction $D^n \to S^{n-1}_-$

Definition 2. Let $f : X \to Y$ be a map of CW complexes. We will say that f is a *simple homotopy equivalence* if it is homotopic to a finite composition

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_n = Y,$$

where each f_i is either an elementary expansion or an elementary collapse.

We say that two finite CW complexes are *simple homotopy equivalent* if there exists a simple homotopy equivalence between them.

Example 3. Let X and Y be finite CW complexes and let $f : X \to Y$ be a continuous map. We let $M(f) = (X \times [0,1]) \amalg_{X \times \{1\}} Y$ denote the mapping cylinder of f. If f is a cellular map, then we can regard $M(f)$ as a finite CW complex (taking the cells of $M(f)$ to be the cells of Y together with cells of the form $e \times \{0\}$ and $e \times (0,1)$, where e is a cell of X). The inclusion $Y \hookrightarrow M(f)$ is always a simple homotopy equivalence: in fact, it can be obtained by a finite sequence of elementary expansions which simultaneously add pairs of cells $e \times \{0\}$ and $e \times (0,1)$ (where we add cells in order of increasing dimension).

Note that the map f is homotopic to a composition

$$X \simeq X \times \{0\} \xhookrightarrow{\iota} M(f) \xrightarrow{r} Y,$$

where r is the canonical retraction from $M(f)$ onto Y (which is homotopy inverse to the inclusion $Y \hookrightarrow M(f)$, and can be obtained by composing a finite sequence elementary collapses). It follows that f is a simple homotopy equivalence if and only if ι is a simple homotopy equivalence. Consequently, when we are studying the question of whether or not some map f is a simple homotopy equivalence, there is no real loss of generality in assuming that f is the inclusion of a subcomplex.

It is easy to see that any simple homotopy equivalence is a homotopy equivalence. One can ask whether the converse holds:

Question 4. Let $f : X \to Y$ be a homotopy equivalence between finite CW complexes. Is f a simple homotopy equivalence? If not, how can we tell?

To address Question 4, we will introduce an algebraic invariant (called the *Whitehead torsion*) which vanishes for simple homotopy equivalences, but not for all homotopy equivalences. First, we need a brief digression.

Definition 5. Let R be a ring (not necessarily commutative). For each integer $n \geq 0$, we let $\mathrm{GL}_n(R)$ denote the group of automorphisms of R^n as a right R-module (equivalently, the group of invertible n-by-n matrices with coefficients in R). Every automorphism α of R^n extends to an automorphism $\alpha \oplus \mathrm{id}_R$ of R^{n+1}; this construction yields inclusions

$$\mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}_2(R) \hookrightarrow \mathrm{GL}_3(R) \hookrightarrow \cdots$$

We let $\mathrm{GL}_\infty(R)$ denote the direct limit of this sequence, and we define $K_1(R)$ to be the abelianization of $\mathrm{GL}_\infty(R)$.

Remark 6. Let R be a commutative ring. For every integer n, the determinant gives a group homomorphism

$$\det : \mathrm{GL}_n(R) \to R^\times.$$

These maps are compatible as n varies and therefore determine a group homomorphism $\det : K_1(R) \to R^\times$. This map is split surjective (split by the canonical map $\mathrm{GL}_1(R) \to \mathrm{GL}_\infty(R) \to K_1(R)$). This map can be shown to be an isomorphism when R is a field or $R = \mathbf{Z}$, but it is not an isomorphism in general.

By construction, for any ring R we have a canonical homomorphism $\mathrm{GL}_n(R) \to K_1(R)$, which one can think of as a kind of "universal determinant".

Exercise 7. For every unit $x \in R^\times$, let $[x]$ denote the image of x under the composite map $\mathrm{GL}_1(R) \to \mathrm{GL}_\infty(R) \to K_1(R)$. Suppose that $\sigma \in \mathrm{GL}_n(R)$ is a permutation matrix. Show that the image of σ in $K_1(R)$ is given by $[\epsilon]$, where $\epsilon = \pm 1$ is the sign of the permutation σ.

Exercise 8. Let $g \in \mathrm{GL}_n(R)$. Let us say that g is *potentially upper triangular* if there exists a decomposition of R^n as a direct sum $P_1 \oplus P_2 \oplus \cdots \oplus P_m$ such that for each $x \in P_i$, we have

$$g(x) \in x + P_1 + \cdots + P_{i-1}.$$

Show that if g is potentially upper triangular, then the image of g in $K_1(R)$ vanishes.

Let R be a ring. A *based chain complex over* R is a bounded chain complex of R-modules

$$\cdots \to F_n \xrightarrow{d} F_{n-1} \xrightarrow{d} F_{n-2} \to \cdots$$

together with a choice of *unordered* basis for each F_m (so that each F_m is a free R-module). In this case, we let $\chi(F_*)$ denote the sum $\sum(-1)^m r_m$, where r_m denotes the cardinality of the basis of F_m. We will refer to $\chi(F_*)$ as the *Euler characteristic* of (F_*, d).

Warning 9. If R is a nonzero commutative ring, then the Euler characteristic $\chi(F_*)$ is independent of the choice of basis of the modules F_*. For a general noncommutative ring R, this need not be the case.

Exercise 10. Let (F_*, d) be a finite based chain complex which is *acyclic*: that is, the homology of (F_*, d) vanishes. Show that if R admits a nonzero homomorphism to a commutative ring, then $\chi(F_*, d) = 0$.

Let (F_*, d) be a based chain complex over R which is *acyclic*. Since each F_m is a free R-module, it then follows that the identity map $\mathrm{id} : F_* \to F_*$ is chain homotopic to zero: that is, there exists a map $h : F_* \to F_{*+1}$ satisfying $dh + hd = \mathrm{id}$. We let $F_{\text{even}} = \bigoplus_n F_{2n}$ and $F_{\text{odd}} = \bigoplus_n F_{2n+1}$.

Lemma 11. *In the situation above, the map* $d + h : F_{\text{even}} \to F_{\text{odd}}$ *is an isomorphism.*

Proof. We have $(d + h)(d + h) = d^2 + dh + hd + h^2 = \mathrm{id} + h^2$, which has an inverse given by the sum $1 - h^2 + h^4 - h^6 + \cdots$ (note that this sum is actually finite, since the chain complex F_* is bounded and h increases degrees). □

The specification of a basis for each F_m determines isomorphisms

$$F_{\text{even}} \simeq R^a \qquad F_{\text{odd}} \simeq R^b$$

for some integers $a, b \geq 0$, which are well-defined up to the action of permutation matrices.

Definition 12. Let $\widetilde{K}_1(R)$ denote the quotient of $K_1(R)$ by the subgroup $[\pm 1]$. If (F_*, d) is an acyclic based complex with $\chi(F_*) = 0$, we define the *torsion* of (F_*, d) to be the image of $d + h \in \mathrm{GL}_a(R)$ under the map $\mathrm{GL}_a(R) \to \mathrm{GL}_\infty(R) \to \widetilde{K}_1(R)$; by virtue of Exercise 7, this does not depend on the ordering of the basis elements of F_*. We will denote the torsion of (F_*, d) by $\tau(F_*)$.

Lemma 13. *In the situation of Definition 12, the torsion* $\tau(F_*)$ *is well-defined: that is, it does not depend on the choice of nullhomotopy* h.

Proof. Any other nullhomotopy of (F_*, d) has the form $h+e$, where $e : F_* \to F_{*+1}$ is a map satisfying $de+ed = 0$. We have already seen that $(d+h)^2 = 1+h^2$, so that we have $(d+h)^{-1} = (d+h)(1-h^2+h^4-h^6+\cdots)$. Multiplying by $(d + h + e)$, we obtain

$$\begin{aligned}
(d+h+e)(d+h)^{-1} &= 1 + e(d+h)^{-1} \\
&= 1 + e(d+h)(1 - h^2 + h^4 + \cdots) \\
&= 1 + ed + \text{ degree ¿ 0 }.
\end{aligned}$$

Since (F_*, d) is a bounded acyclic chain complex of free modules, it is split exact: in particular, each F_n contains the group $Z_n = \ker(d : F_n \to F_{n-1})$ as a direct summand. Note that ed annihilates the group Z_n, and that $ed = -de$ carries F_n into Z_n. It follows that any map of the form $1 + ed +$ degree ¿ 0 is potentially upper triangular when regarded as an automorphism of F_{even}, and therefore has vanishing image in $K_1(R)$ (Exercise 8). □

Exercise 14. Suppose we are given a short exact sequence of finite based chain complexes

$$0 \to (F'_*, d') \to (F_*, d) \to (F''_*, d'') \to 0.$$

Assume that the chosen basis for each F_m consists of the images of the basis elements of F'_m together with preimages of the basis elements of each F''_m. Show that:

(a) If (F'_*, d') and (F''_*, d'') are acyclic, then (F_*, d) is acyclic.

(b) If $\chi(F'_*, d') = \chi(F''_*, d'') = 0$, then $\chi(F_*, d) = 0$.

(c) If conditions (a) and (b) hold, then $\tau(F_*, d) = \tau(F'_*, d')\tau(F''_*, d'')$ in $\widetilde{K}_1(R)$.

Definition 15. Let $f : X_* \to Y_*$ be a map of chain complexes over a ring R. The *mapping cone of* f is defined to be the chain complex

$$C(f)_* = X_{*-1} \oplus Y_*$$

with differential $d(x, y) = (-dx, f(x) + dy)$ Note that if X_* and Y_* are based complexes, then we can regard $C(f)_*$ as a based complex (where we fix some convention for how our bases should be ordered; we will not worry about this point).

Suppose that we have $\chi(X_*, d) = \chi(Y_*, d)$ and that f is a quasi-isomorphism (that is, it induces an isomorphism on homology). Then $\chi(C(f)_*, d) = 0$ and $C(f)_*$ is acyclic. We define the *torsion of* f to be the element $\tau(f) = \tau(C(f)_*, d) \in K_1(R)$.

Example 16. Let (F_*, d) be an acyclic based complex with $\chi(F_*) = 0$, and let f be the identity map from F_* to itself. Then the mapping cone $C(f)_*$ has an explicit nullhomotopy given by $(x, y) \mapsto (y, 0)$. Let us identify $C(f)_{\text{even}}$ and $C(f)_{\text{odd}}$ with F_*, so that $d + h$ is given by

$$(x, y) \mapsto (y - dx, x + dy).$$

This map is given by a permutation matrix modulo the filtration by degree, so we have $\tau(f) = 1 \in \widetilde{K}_1(R)$.

Let us now explain how to apply the preceding ideas. Suppose that X and Y are finite CW complexes and that we are given a homotopy equivalence $f : X \to Y$. For simplicity, we will assume that X and Y are connected (otherwise, we can analyze each connected component separately). We fix a base point $x \in X$ and set $G = \pi_1(X, x) \simeq \pi_1(Y, f(x))$. Let $\widetilde{Y}$ be a universal cover of Y and let $\widetilde{X} = X \times_Y \widetilde{Y}$ be the corresponding universal cover of X, so that G acts on $\widetilde{X}$ and $\widetilde{Y}$ by deck transformations. Let us further assume that f is a cellular map. Then f induces a map of cellular chain complexes

$$\lambda : C_*(\widetilde{X}; \mathbf{Z}) \to C_*(\widetilde{Y}; \mathbf{Z}).$$

Note that we can regard $C_*(\widetilde{X}; \mathbf{Z})$ and $C_*(\widetilde{Y}; \mathbf{Z})$ as chain complexes of free $\mathbf{Z}[G]$-modules, with basis elements in bijection with the cells of X and Y respectively. Since f is a homotopy equivalence, the map λ is a quasi-isomorphism. We may therefore consider the torsion $\tau(\lambda) \in \widetilde{K}_1(\mathbf{Z}[G])$. However, it is not quite well-defined: in order to extract an element of $C_*(\widetilde{X}; \mathbf{Z})$ from a cell $e \subseteq X$, we need to choose a cell of $\widetilde{X}$ lying over e (which is ambiguous up to the action of G) and an orientation of the cell e (which is ambiguous up to a sign). This motivates the following:

Definition 17. Let G be a group. The *Whitehead group* $\mathrm{Wh}(G)$ of G is the quotient of $K_1(\mathbf{Z}[G])$ by elements of the form $[\pm g]$, where $g \in G$.

If $f : X \to Y$ is a cellular homotopy equivalence of connected finite CW complexes, we define the *Whitehead torsion* $\tau(f) \in \mathrm{Wh}(G)$ to be the image in $\mathrm{Wh}(G)$ of the torsion of the induced map

$$\lambda : C_*(\widetilde{X}; \mathbf{Z}) \to C_*(\widetilde{Y}; \mathbf{Z}).$$

We will continue our discussion of the Whitehead torsion in the next lecture.

Whitehead Torsion, Part II (Lecture 4)

September 9, 2014

In this lecture, we will continue our discussion of the Whitehead torsion of a homotopy equivalence $f : X \to Y$ between finite CW complexes. In the previous lecture, we gave a definition in the special case where f is cellular. To remove this hypothesis, we need the following:

Proposition 1. *Let X and Y be connected finite CW complexes and suppose we are given cellular homotopy equivalences $f, g : X \to Y$. If f and g are homotopic, then $\tau(f) = \tau(g) \in \mathrm{Wh}(\pi_1 X)$.*

Lemma 2. *Suppose we are given quasi-isomorphisms $f : (X_*, d) \to (Y_*, d)$ and $g : (Y_*, d) \to (Z_*, d)$ between finite based complexes with $\chi(X_*, d) = \chi(Y_*, d) = \chi(Z_*, d)$. Then*

$$\tau(g \circ f) = \tau(g)\tau(f)$$

in $\widetilde{K}_1(R)$.

Proof. We define a based chain complex (W_*, d) by the formula

$$W_* = X_{*-1} \oplus Y_* \oplus Y_{*-1} \oplus Z_*$$

$$d(x, y, y', z) = (-dx, f(x) + dy + y', -dy', g(y') + dz).$$

Then (W_*, d) contains $(C(f)_*, d)$ as a based subcomplex with quotient $(C(g)_*, d)$, so an Exercise from the previous lecture gives $\tau(W_*, d) = \tau(g)\tau(f)$. We now choose a new basis for each W_* by replacing each basis element of $y \in Y_*$ by $(0, y, 0, g(y))$; this is an upper triangular change of coordinates and therefore does not affect the torsion $\tau(W_*, d)$. Now the construction $(y', y) \mapsto (0, y, y', g(y))$ identifies $C(\mathrm{id}_Y)_*$ with a based subcomplex of W_* having quotient $C(-g \circ f)_*$. Applying the same Exercise again we get

$$\tau(W_*, d) = \tau(\mathrm{id}_Y)\tau(-g \circ f) = \tau(g \circ f).$$

$\square$

Remark 3. Suppose that $f : X \to Y$ is the inclusion of X as a subcomplex of Y. Let $\lambda : C_*(\widetilde{X}; \mathbf{Z}) \to C_*(\widetilde{Y}; \mathbf{Z})$ be as above. Then the mapping cone $C(\lambda)_*$ contains the mapping cone $C(\mathrm{id}_{C_*(\widetilde{X};|Z|)})_*$ as a based subcomplex, and the quotient is the relative cellular chain complex $C_*(\widetilde{Y}, \widetilde{X}; \mathbf{Z})$. It follows that the Whitehead torsion of f can be computed as (the image in $\mathrm{Wh}(\pi_1 X)$ of the torsion of the acyclic complex $C_*(\widetilde{Y}, \widetilde{X}; \mathbf{Z})$.

Proof of Proposition 1. Choose a homotopy $h : X \times [0, 1] \to Y$ from $f = h_0$ to $g = h_1$. We may assume without loss of generality that h is cellular. Then the Whitehead torsion $\tau(h)$ is well-defined; we will prove that $\tau(f) = \tau(h) = \tau(g)$. Note that f is given by the composition

$$X \times \{0\} \xhookrightarrow{i} X \times [0, 1] \xrightarrow{h} Y.$$

Then $\tau(f) = \tau(h)\tau(i)$ in $\mathrm{Wh}(\pi_1 X)$ (Lemma 2). It will therefore suffice to show that $\tau(i)$ vanishes. Using Remark 3, we can identify $\tau(i)$ with the torsion of the relative cellular chain complex

$$C_*(\widetilde{X} \times [0,1], \widetilde{X} \times \{0\}; \mathbf{Z}),$$

which vanishes (we saw this in the previous lecture). □

If $f : X \to Y$ is *any* homotopy equivalence between connected finite CW complexes, we define $\tau(f) = \tau(f_0)$ where f_0 is a cellular map which is homotopic to f. By virtue of Proposition 1, this definition is independent of the choice of f_0.

Proposition 4. *Let $f : X \to Y$ and $g : Y \to Z$ be homotopy equivalences between connected finite CW complexes, all having fundamental group G. Then $\tau(gf) = \tau(g)\tau(f)$ in $\mathrm{Wh}(G)$.*

Proof. This follows immediately from Lemma 2. □

Corollary 5. *Let $f : X \to Y$ be a simple homotopy equivalence between finite CW complexes. Then $\tau(f) = 1$.*

Proof. Using Proposition 4, we can reduce to the case where f is an elementary expansion. In this case, $\tau(f)$ is the torsion of the relative cellular chain complex $C_*(\widetilde{Y}, \widetilde{X}; \mathbf{Z}[G])$ which has the form

$$\cdots \to 0 \to \mathbf{Z}[G] \xrightarrow{\pm g} \mathbf{Z}[G] \to 0.$$

□

Remark 6. Let X be a finite connected CW complex and set $G = \pi_1 X$. Then every element $\eta \in \mathrm{Wh}(G)$ can be realized as the Whitehead torsion of a homotopy equivalence $f : X \to Y$. To see this, choose any matrix $M \in \mathrm{GL}_n(\mathbf{Z}[G])$. Fix an integer $k \geq 2$ and let X' be the CW complex obtained from X by attaching n copies of S^k at some base point $x \in X$, so we have an evident retraction $r : X' \to X$. Applying the relative Hurewicz theorem to the map of universal covers $\widetilde{X}' \to \widetilde{X}$, we obtain a canonical isomorphism $\pi_{k+1}(X, X') \simeq \mathbf{Z}[G]^n$. Consequently, the matrix M provides the data for attaching n copies of D^{k+1} to X' in such a way that that the retraction r extends over the resulting CW complex Y. The inclusion $X \hookrightarrow Y$ is an isomorphism on fundemental groups, and the relative chain complex of the inclusion of universal covers is given by

$$\cdots \to 0 \to \mathbf{Z}[G]^n \xrightarrow{M} \mathbf{Z}[G]^n \to 0 \to \cdots.$$

Since M is invertible, we conclude that the inclusion $f : X \to Y$ is a homotopy equivalence and that $\tau(f) \in \mathrm{Wh}(G)$ is represented by the matrix $M^{\pm 1}$ (depending on the parity of k)

The Whitehead groups $\mathrm{Wh}(G)$ are generally nonzero:

Example 7. Let G be an abelian group. Then the determinant homomorphism $K_1(\mathbf{Z}[G]) \to \mathbf{Z}[G]^\times$, which induces a surjective map

$$\mathrm{Wh}(G) \to (\mathbf{Z}[G]^\times)/\{\pm g\}_{g \in G}.$$

The group on the right generally does not vanish. For example, if $G = \mathbf{Z}/5\mathbf{Z}$, then $\mathbf{Z}[G] \simeq \mathbf{Z}[t]/(t^5 - 1)$ contains a unit $1 - t^2 - t^3$ (with inverse $1 - t - t^4$) which is not of the form $\pm t^i$.

Combined with Remark 6, this supplies a negative answer to the question raised in the previous lecture: there exist homotopy equivalences between finite CW complexes with nonvanishing torsion, and such homotopy equivalences cannot be simple. However, it turns out that the Whitehead torsion is the only obstruction:

Theorem 8 (Whitehead)**.** *Let $f : X \to Y$ be a homotopy equivalence between connected finite CW complexes with $\tau(f) = 1 \in \mathrm{Wh}(G)$, where $G = \pi_1 X$. Then f is a simple homotopy equivalence.*

Example 9. One can show that the determinant map $K_1(\mathbf{Z}) \to \mathbf{Z}^\times = \{\pm 1\}$ is an isomorphism, so that the Whitehead group $\mathrm{Wh}(G)$ vanishes when G is the trivial group. Theorem 8 then implies that any homotopy equivalence between *simply connected* finite CW complexes is a simple homotopy equivalence.

Example 10. A nontrivial theorem of Bass, Heller, and Swan asserts that the Whitehead group $\mathrm{Wh}(\mathbf{Z}^d)$ is trivial for each integer d. Together with the s-cobordism theorem, this implies that every h-cobordism from a torus T^d to another manifold M is isomorphic to a product $T^d \times [0,1]$.

For use in the proof of Theorem 8, we include the following example of a simple homotopy equivalence:

Example 11. Let X be a finite CW complex, and suppose we are given a pair of maps

$$f, g : S^{n-1} \to X^{n-1}.$$

Let Y and Z be the CW complexes obtained from X by attaching n-cells along f and g, respectively. Then Y and Z are simple homotopy equivalent. To see this, choose a homotopy $h : S^{n-1} \times [0,1] \to X^n$, and let W be the cell complex obtained from $Y \amalg_X Z$ by attaching an $(n+1)$-cell along the induced map

$$D^n \amalg_{S^{n-1}\times\{0\}} (S^{n-1} \times [0,1]) \amalg_{S^{n-1}\times\{1\}} D^n \to Y \amalg_X Z.$$

Then the inclusions $Y \hookrightarrow W \hookleftarrow Z$ are both elementary expansions.

Let us conclude this lecture by sketching a proof of Theorem 8. Let $f : X \to Y$ be a homotopy equivalence of finite CW complexes such that $\tau(f) = 1$; we wish to show that f is a simple homotopy equivalence. Without loss of generality we may assume f is cellular. Replacing Y by the mapping cylinder $M(f)$, we can assume that f is the inclusion of a subcomplex.

Fix a cell e of minimal possible dimension which belongs to Y but not to X; we will regard this cell as the image of a map g from a hemisphere S^n_- into Y which carries the equator $S^{n-1} \subseteq S^n_-$ into X^{n-1}. Since the inclusion f is a homotopy equivalence, the map g is homotopic to a map from the disk into X via a homotopy which is fixed on S^{n-1}; we may regard this homotopy as defining a map $\overline{g} : D^{n+1} \to Y$ carrying S^n_+ into X. Let us identify D^{n+1} with the lower hemisphere S^{n+1}_- of an $(n+1)$-sphere S^{n+1}, and let Y' denote the elementary expansion of Y given by $Y \amalg_{S^{n+1}_-} D^{n+2}$; we will denote the interior of D^{n+2} by $e' \subseteq Y'$.

Let X' be the subcomplex of Y' given by the union of X and the upper hemisphere S^{n+1}_+. Then X' is an elementary expansion of X. The cells of Y' that do not belong to X' are almost exactly the same as the cells of Y that do not belong to X: the only exception is that Y' has a new cell e' of dimension $n+2$, and that the cell $e \subseteq Y$ now belongs to X'. Replacing the inclusion $X \hookrightarrow Y$ by $X' \hookrightarrow Y'$, we have "traded up" an n-cell for an $(n+2)$-cell. Repeating this process finitely many times, we can reduce to the case where Y is obtained from X by adding only cells of dimension n and $n+1$ for some $n \geq 2$. Let us denote the cells of dimension n by $e_1, \ldots, e_m$ and the cells of dimension $(n+1)$ by $e'_1, \ldots, e'_m$ (note that the number of $(n+1)$-cells is necessarily equal to the number of n-cells, since f is a homotopy equivalence).

Let Y_0 be the subcomplex of Y obtained from X by attaching only the n-cells. We have a long exact sequence

$$\pi_{n+1}(Y, X) \to \pi_{n+1}(Y, Y_0) \xrightarrow{M} \pi_n(Y_0, X) \to \pi_n(Y, X).$$

Since the inclusion $X \hookrightarrow Y$ is a homotopy equivalence, the groups $\pi_n(Y,X)$ and $\pi_{n+1}(Y,X)$ are trivial, so that M is an isomorphism. Using the relative Hurewicz theorem, we see that $\pi_n(Y_0, X) \simeq \mathrm{H}_n(Y_0, X; \mathbf{Z}[G])$ is a free module $\mathbf{Z}[G]^m$. Moreover, it *almost* has a canonical basis: each of the cells e_i determines a generator of $\pi_n(Y_0, X)$ which is ambiguous up to a sign (due to orientation issues) and to the action of G (due to base point issues). Similarly, the cells e'_i determine a basis for $\pi_{n+1}(Y, Y_0) \simeq \mathbf{Z}[G]^m$ which is ambiguous up to the action of $\pm G$. Then M can be regarded as an element of $\mathrm{GL}_m(\mathbf{Z}[G])$, and the image of M in $\mathrm{Wh}(G)$ is given by $\tau(f)^{\pm 1}$ (where the sign depends on the parity of n). Since $\tau(f) = 1$, it is possible to choose bases as above so that M belongs to the commutator subgroup of $\mathrm{GL}_m(\mathbf{Z}[G])$.

Let $x \in X$ be a base point. Let Y^+ denote the elementary expansion

$$Y^+ = ((Y \amalg_{\{x\}} D^{n+1}) \amalg_{\{x\}} D^{n+1}) \amalg \cdots$$

obtained from Y by attaching m copies of the disk D^{n+1} at the base point of X; here we regard Y^+ as obtained from Y by adding m cells of dimension n (the boundaries of the new disks) and m cells of dimension $(n+1)$ (the interiors of the new disks). Replacing Y by Y^+ has the effect of replacing m by $2m$ and M by the matrix

$$M^+ = \begin{bmatrix} M & 0 \\ 0 & \mathrm{id} \end{bmatrix}.$$

Since M belongs to the commutator subgroup of $\mathrm{GL}_m(\mathbf{Z}[G])$, the matrix M^+ can be written as a product of matrices of the form

$$\begin{bmatrix} \mathrm{id} & X \\ 0 & \mathrm{id} \end{bmatrix} \text{ or } \begin{bmatrix} \mathrm{id} & 0 \\ Y & \mathrm{id} \end{bmatrix}$$

(see Remark 12 below). Replacing Y by Y^+, m by $2m$, and M by M^+, we may reduce to the case where M has the form

$$M_1 \cdots M_k$$

where each M_i is either upper-triangular or lower-triangular.

To complete the proof, it will suffice to show that for a homotopy equivalence $f : X \to Y$ with associated matrix M as above and any matrix U which is either upper (or lower) triangular, we can find another homotopy equivalence $f' : X \to Y'$ with associated matrix MU for which the induced homotopy equivalence $Y \simeq Y'$ is simple. To see this, consider the filtration

$$Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_m = Y$$

where $Y_i = Y_0 \cup e'_1 \cup \cdots \cup e'_i$. Using Example 11, we see that the simple homotopy type of Y_i is unchanged if we modify the attaching map $\partial\, e'_i \to Y_0$ by an arbitrary homotopy within Y_{i-1}; by means of such modifications we can multiply M by any upper-triangular matrix that we like.

Remark 12 (Whitehead's Lemma)**.** Let R be any ring, and let $H \subseteq \mathrm{GL}_2(R)$ be the subgroup generated by matrices of the form

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}.$$

For every invertible element $g \in R$, the calculation

$$\begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -g^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & g \\ -g^{-1} & 0 \end{bmatrix}$$

shows that $\begin{bmatrix} 0 & g \\ -g^{-1} & 0 \end{bmatrix} \in H$.

If g and h are invertible elements of R, we have

$$\begin{aligned} \begin{bmatrix} ghg^{-1}h^{-1} & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & g \\ -g^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & h^{-1} \\ -h & 0 \end{bmatrix} \begin{bmatrix} 0 & (hg)^{-1} \\ -hg & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &\in H. \end{aligned}$$

Replacing R by the ring of n-by-n matrices over R, we see that any element of the commutator subgroup of $\mathrm{GL}_n(R)$ can be written (in $\mathrm{GL}_{2n}(R)$) as a product of matrices of the form

$$\begin{bmatrix} \mathrm{id} & X \\ 0 & \mathrm{id} \end{bmatrix} \text{ and } \begin{bmatrix} \mathrm{id} & 0 \\ Y & \mathrm{id} \end{bmatrix}.$$

Cell-Like Maps (Lecture 5)

September 15, 2014

In the last two lectures, we discussed the notion of a *simple* homotopy equivalences between finite CW complexes. *A priori*, the question of whether or not a map $f : X \to Y$ is a simple homotopy equivalence depends on the specified cell decompositions of X and Y. This motivates the following:

Question 1. Let $f : X \to Y$ be a homeomorphism of finite CW complexes. Is f a simple homotopy equivalence?

Chapman gave an affirmative action to Question 1 using techniques of infinite-dimensional topology. The following special case is much easier:

Proposition 2. *Let $f : X \to Y$ be a homeomorphism of finite CW complexes with the property that for every cell $e \subseteq Y$, the inverse image $f^{-1}e$ is a union of cells. Then f is a simple homotopy equivalence.*

We postpone the proof; we will discuss a stronger result at the end of this lecture. Let us instead discuss some applications of Proposition 2.

Definition 3. Let X be a topological space. A *simple homotopy structure* on X is an equivalence class of homotopy equivalences $f : X \to Y$ (where Y is a finite CW complex), where we regard $f : X \to Y$ and $f' : X \to Y'$ as equivalent if the composition $f' \circ g$ is a simple homotopy equivalence, where $g : Y \to X$ denotes a homotopy inverse to f.

If X and X' are topological spaces equipped with simple homotopy structures represented by homotopy equivalences $f : X \to Y$ and $f' : X' \to Y'$, then we will say that a map $h : X \to X'$ is a *simple homotopy equivalence* if the composition $f' \circ h \circ g : Y \to Y'$ is a simple homotopy equivalence, where g is a homotopy inverse to f.

Example 4. Let X be a topological space. Any cell decomposition which exhibits X as a finite CW complex determines a simple homotopy structure on X.

We now discuss some examples of topological spaces which can be endowed with simple homotopy structures in a natural way.

Definition 5. Let K be a subset of a Euclidean space $\mathbb{R}^n$. We will say that K is a *linear simplex* if it can be written as the convex hull of a finite subset $\{x_1, \ldots, x_k\} \subset \mathbb{R}^n$ which are independent in the sense that if $\sum c_i x_i = 0 \in \mathbb{R}^n$ and $\sum c_i = 0 \in \mathbb{R}$, then each c_i vanishes (equivalently, if the convex hull of $\{x_1, \ldots, x_k\}$ has dimension exactly $(k-1)$).

We will say that K is a *polyhedron* if, for every point $x \in K$, there exists a finite number of linear simplices $\sigma_i \subseteq K$ such that the union $\bigcup_i \sigma_i$ contains a neighborhood of X.

Remark 6. Any open subset of a polyhedron in $\mathbb{R}^n$ is again a polyhedron.

Remark 7. Every polyhedron $K \subseteq \mathbb{R}^n$ admits a *PL triangulation*: that is, we can find a collection of linear simplices $S = \{\sigma_i \subseteq K\}$ with the following properties:

(1) Any face of a simplex belonging to S also belongs to S.

(2) Any nonempty intersection of any two simplices of S is a face of each.

(3) The union of the simplices σ_i is K.

Moreover, one can show that any two PL triangulations of K admit a common refinement.

Remark 8. Let X be a polyhedron. The following conditions are equivalent:

- As a topological space, X is compact.
- Every PL triangulation of X involves only finitely many simplices.
- There exists a PL triangulation of X involving only finitely many simplices.

If these conditions are satisfied, we will say that X is a *finite polyhedron*.

Example 9. Let X be a finite polyhedron. Then any PL triangulation of X determines a simple homotopy structure on X. By virtue of Remark 7 and Proposition 2, this simple homotopy structure is independent of the choice of PL triangulation.

Definition 10. Let $K \subseteq \mathbb{R}^n$ be a polyhedron. We will say that a map $f : K \to \mathbb{R}^m$ is *linear* if it is the restriction of an affine map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will say that f is *piecewise linear* (PL) if there exists a triangulation $\{\sigma_i \subseteq K\}$ such that each of the restrictions $f|_{\sigma_i}$ is linear.

If $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ are polyhedra, we say that a map $f : K \to L$ is *piecewise linear* if the underlying map $f : K \to \mathbb{R}^m$ is piecewise linear.

Remark 11. Let $f : K \to L$ be a piecewise linear homeomorphism between polyhedra. Then the inverse map $f^{-1} : L \to K$ is again piecewise linear. To see this, choose any triangulation of K such that the restriction of f to each simplex of the triangulation is linear. Taking the image under f, we obtain a triangulation of L such that the restriction of f^{-1} to each simplex is linear.

Remark 12. Any piecewise linear homeomorphism of polyhedra $f : X \to Y$ is a simple homotopy equivalence (with respect to the simple homotopy structures of Example 9).

Remark 13. The collection of all polyhedra can be organized into a category, where the morphisms are given by piecewise linear maps. This allows us to think about polyhedra *abstractly*, without reference to an embedding into a Euclidean space: a pair of polyhedra $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ can be isomorphic even if $n \neq m$.

Example 14. Let M be a compact smooth manifold. Then M admits a *Whitehead triangulation*: that is, there is a finite simplicial complex K equipped with a homeomorphism $K \to M$ which is differentiable (with injective differential) on each simplex of K. This determines a simple homotopy structure on M which is *independent* of the choice of Whitehead triangulation (this follows from the fact that any pair of Whitehead triangulations $f : K \to M$ and $g : L \to M$, one can find arbitrarily close Whitehead triangulations $f' : K \to M$ and $g' : L \to M$ such that the homeomorphism $g'^{-1} \circ f' : K \to L$ is piecewise linear).

By virtue of Example 14, it makes sense to ask if a map between compact smooth manifolds is a simple homotopy equivalence (as in the statement of the s-cobordism theorem). Let us now turn to Proposition 2 itself.

Proof of Proposition 2. Without loss of generality we may assume that Y is connected. The assumption that $f^{-1}e$ is a union of cells for each open cell $e \subseteq Y$ implies that the inverse homeomorphism $f^{-1} : Y \to X$ is cellular. Let $\widetilde{Y}$ be a universal cover of Y, and for every space Z with a map $Z \to Y$ we let $\widetilde{Z}$ denote the fiber product $Z \times_Y \widetilde{Y}$ (this is a covering space of Z which may or may not be connected). Set $G = \pi_1 Y$ so that G acts on $\widetilde{Y}$ by deck transformations. We will prove that f is a simple homotopy equivalence by showing that f^{-1} induces a map of cellular chain complexes $C_*(\widetilde{Y}; \mathbf{Z}) \to C_*(\widetilde{X}; \mathbf{Z})$ with vanishing Whitehead torsion. We will deduce this from the following more general assertion:

(∗) For every subcomplex $Y_0 \subseteq Y$ with inverse image $X_0 \subseteq X$, the torsion of the induced map of cellular chain complexes $C_*(\widetilde{Y}_0; \mathbf{Z}) \to C_*(\widetilde{X}_0; \mathbf{Z})$ vanishes in $\mathrm{Wh}(G)$.

We proceed by induction on the number of cells in Y_0. Let us therefore assume that (∗) is known for a subcomplex $Y_0 \subseteq Y$, and see what happens when we add one more cell to obtain another subcomplex $Y_1 \subseteq Y$. As we saw in the previous lecture, Whitehead torsion is multiplicative in short exact sequences. To carry out the inductive step, it will suffice to show that torsion of the map of relative cellular cochain complexes

$$\theta : C_*(\widetilde{Y}_1, \widetilde{_0}; \mathbf{Z}) \to C_*(\widetilde{X}_1, \widetilde{X}_0; \mathbf{Z})$$

vanishes in $\mathrm{Wh}(G)$. Let $e \subseteq Y$ be the open cell of Y_1 which does not belong to Y_0. Then e is simply connected, so the map $\widetilde{e} \to e$ admits a section. A choice of section (and orientation of e) determines a one-element basis for $C_*(\widetilde{Y}_1, \widetilde{Y}_0; \mathbf{Z})$ as a $\mathbf{Z}[G]$-module, and by choosing cells of $\widetilde{X}_1 - \widetilde{X}_0$ which belong to the inverse image of the section we obtain a basis for $C_*(\widetilde{X}_1, \widetilde{X}_0; \mathbf{Z})$ as a $\mathbf{Z}[G]$-module (ambiguous up to signs) with respect to which the differential on $C_*(\widetilde{X}_1, \widetilde{X}_0; \mathbf{Z})$ and the map θ can be represented by matrices with integral coefficients (rather than coefficients in $\mathbf{Z}[G]$). It follows that the image of $\tau(\theta)$ in $\mathrm{Wh}(G)$ factors through the map $\widetilde{K}_1(\mathbf{Z}) = \mathrm{Wh}(*) \to \mathrm{Wh}(G)$, and therefore vanishes because the group $\widetilde{K}_1(\mathbf{Z}) = K_1(\mathbf{Z})/\{[\pm 1]\}$ is trivial. □

Our next goal is to prove an analogue of Proposition 2 where we weaken the hypothesis that f is a homeomorphism. First, we need to introduce some terminology.

Definition 15. Let X be a topological space. We will say that X has *trivial shape* if X is nonempty and, for every CW complex Y, every map $f : X \to Y$ is homotopic to a constant map.

Example 16. Any contractible topological space X has trivial shape. The converse holds if X has the homotopy type of a CW complex.

Let us collect up some consequences of Definition 15:

Proposition 17. *Let X be a paracompact topological space with trivial shape. Then:*

(1) *The space X is connected.*

(2) *Every locally constant sheaf (of sets) on X is constant.*

(3) *For every abelian group A, we have*

$$\mathrm{H}^*(X; \underline{A}) \simeq \begin{cases} A & if * = 0 \\ 0 & otherwise. \end{cases}$$

Here cohomology means sheaf cohomology with coefficients in the constant sheaf $\underline{A}$ on X (which does not agree with singular cohomology in this generality).

Proof. If X is not connected, then there exists a map $f : X \to S^0$ which is not homotopic to a constant map. This proves (1). To prove (2), suppose that $\mathcal{F}$ Is a locally constant sheaf of sets on X. Pick a point $x \in X$ and set $S = \mathcal{F}_x$. Since X is connected, every stalk of $\mathcal{F}$ is (noncanonically) isomorphic to S. Let G denote the permutation group of S (regarded as a discrete group) and let BG denote its classifying space. Since X is paracompact, isomorphism classes of locally constant sheaves of sets on X with stalks isomorphic to S are in bijection with homotopy classes of maps $f : X \to \mathrm{BG}$. Since any such map $f : X \to \mathrm{BG}$ is homotopic to a constant map, any such locally constant sheaf is actually constant.

To prove (3), we note that since X is paracompact, we can identify $\mathrm{H}^n(X; \underline{A})$ with the set of homotopy classes of maps from X into the Eilenberg-MacLane space $K(A, n)$. Since X has trivial shape, this can be identified with the set of connected components $\pi_0 K(A, n) \simeq \begin{cases} A & \mathrm{if} * = 0 \\ 0 & \text{otherwise.} \end{cases}$. □

Remark 18. Definition 15 is perhaps only appropriate when the topological space X is paracompact; for many purposes, it is the conclusions of Proposition 17 which are important. For us, this will be irrelevant: we will only be interested in the case where X is a compact Hausdorff space.

Definition 19. Let $f : X \to Y$ be a map of Hausdorff spaces. We will say that f is *cell-like* if it satisfies the following conditions:

(1) The map f is closed.

(2) For each point $y \in Y$, the fiber $X_y = f^{-1}\{y\}$ is compact.

(3) For each point $y \in Y$, the fiber X_y has trivial shape.

Remark 20. Conditions (1) and (2) of Definition 19 say that the map $f : X \to Y$ is *proper*.

Warning 21. Definition 19 is generally only behaves well if we make some additional assumptions on the spaces X and Y involved: for example, if we assume they are absolute neighborhood retracts or that they are paracompact and finite-dimensional. See Remark **??** below.

Example 22. Any homeomorphism of Hausdorff spaces is a cell-like map.

Proposition 23. *Let f be a cell-like map of CW complexes. Then f is a homotopy equivalence.*

Proof. We first argue that f is an equivalence on fundamental groupoids. For this, it suffices to show that the pullback functor f^* induces an equivalence from the category of locally constant sheaves (of sets) on Y to the category of locally constant sheaves (of sets) on X. This amounts to two assertions:

(a) For every locally constant sheaf $\mathcal{F}$ on Y, the unit map

$$\mathcal{F} \to f_* f^* \mathcal{F}$$

is an isomorphism.

(b) For every locally constant sheaf $\mathcal{G}$ on X, the pushforward $f_* \mathcal{G}$ is locally constant and the counit map $f^* f_* \mathcal{G} \to \mathcal{G}$ is an isomorphism.

Note that if $\mathcal{G}$ is a locally constant sheaf on X. Fix a point $x \in X$, let $S = \mathcal{G}_x$ be the stalk of $\mathcal{G}$ at the point x, and let $y = f(x)$. Since the fiber X_y is connected, we can choose an open neighborhood V of X_y such that the stalk $\mathcal{G}_{x'}$ is (noncanonically) isomorphic to S for each $x' \in U$. As in the proof of Proposition 17, this implies that $\mathcal{G}\,|_U$ is classified by a map $U \to \mathrm{BG}$, where G is the group of permutations of S. This map is nullhomotopic on the fiber X_y and therefore on some neighborhood of X_y. Since f is proper, we may assume without loss of generality that such a neighborhood has the form $f^{-1}V$ for some open set $V \subseteq Y$ containing the point y. It follows that $\mathcal{G}\,|_{f^{-1}(V)}$ is isomorphic to the constant sheaf $\underline{S}$ with the value S. Since f is a proper map, the stalk of $f_* \mathcal{G}$ at any point $y' \in V$ can be identified with the set of continuous maps from $X_{y'}$ into S (where S has the discrete topology). Since each $X_{y'}$ s connected, it follows that $(f_* \mathcal{G})|_V$ can also be identified with the constant sheaf having the value S. This proves (b). Using the properness of f, we see that assertion (a') reduces to the following:

(a') For every set S, the canonical map $S \to \mathrm{Hom}(X_y, S)$ is a bijection.

This follows immediately from the connectedness of X_y.

To complete the proof that f is a homotopy equivalence, it will suffice to show that for every local system of abelian groups $\mathcal{A}$ on Y, the canonical map

$$\theta : \mathrm{H}^*(Y; \mathcal{A}) \to \mathrm{H}^*(X; f^* \mathcal{A})$$

is an isomorphism. Since X and Y are CW complexes, we can regard $\mathcal{A}$ as a locally constant sheaf on Y and identify the domain and codomain of θ with *sheaf* cohomology. To prove this, it suffices to prove the stronger *local* assertion that the unit map $\mathcal{A} \to Rf_* f^* \mathcal{A}$ is an isomorphism in the derived category of sheaves on Y. Since f is proper, the cohomologies of the stalks of $Rf_* f^* \mathcal{A}$ are given by the sheaf cohomology groups $\mathrm{H}^*(X_y; \mathcal{A}_y)$, so that the desired result follows from Proposition 17. $\square$

Proposition 23 does not need the full strength of the assumption that X and Y are CW complexes: it is enough to know that X and Y have the homotopy type of CW complexes (so that homotopy equivalences are detected by Whitehead's theorem), that local systems can be identified with locally constant sheaves, and that singular cohomology agrees with sheaf cohomology. Consequently, it remains valid if we assume only that X and Y are open subsets of CW complexes or absolute neighborhood retracts (ANRs).

Corollary 24. *Let $f : X \to Y$ be a proper map of CW complexes (or ANRs). Then f is cell-like if and only if, for every open set $U \subseteq Y$, the induced map $f^{-1}U \to U$ is a homotopy equivalence.*

Proof. The "only if" direction follows from Proposition 23. Conversely, suppose that for each $U \subseteq Y$ the map $f^{-1}U \to U$ is a homotopy equivalence. Pick a point $y \in Y$, a CW complex Z, and a map $g : X_y \to Z$; we wish to prove that g is nullhomotopic. Then g factors through a finite subcomplex of Z; we may therefore assume without loss of generality that Z is finite. We can extend g to a continuous map $\overline{g} : V \to Z$ for some open set $V \subseteq X$ containing X_y (this follows from the fact that the finite CW complex Z is an absolute neighborhood retract). Because f is proper, V contains a set of the form $f^{-1}(U)$ where U is a neighborhood of y in Y. Using the local contractibility of Y, we may further assume that U is contractible. Then $f^{-1}(U)$ is also contractible. The map g factors as a composition

$$X_y \hookrightarrow f^{-1}U \xrightarrow{\overline{g}} Z,$$

and is therefore nullhomotopic. □

We will prove the following result in the next lecture:

Proposition 25. *Let $f : X \to Y$ be a map of finite CW complexes. Assume that f is cell-like, cellular, and that for every cell $e \subseteq Y$, the inverse image $f^{-1}e$ is a union of cells. Then f is a simple homotopy equivalence.*

Remark 26. In the statement of Proposition 25, the second and third hypothesis on f are not necessary. However, the proof requires infinite-dimensional methods.

Concordance of Polyhedra (Lecture 6)

September 15, 2014

We begin by tying up a few loose ends from the previous lecture. We begin with a few remarks about cell-like maps:

Example 1. Let $f : X \to Y$ be a proper map of Hausdorff spaces, and let $M(f) = (X \times [0,1]) \amalg_{X \times \{1\}} Y$ be its mapping cylinder. Then the canonical retraction $r : M(f) \to Y$ (given on $X \times [0,1]$ by $(x,t) \mapsto f(x)$) is a cell-like map: the inverse image of each point $y \in Y$ homeomorphic to the cone on $f^{-1}\{y\}$, and therefore contractible.

Example 2. Let Y be a CW complex, and suppose we are given a cellular map $f : D^n \to Y$. Then the mapping cylinder $M(f) = (D^n \times [0,1]) \amalg_{D^n \times \{1\}} Y$ is an elementary expansion of Y, and the map $M(f) \to Y$ of Example 1 is the associated elementary collapse. It follows that any elementary collapse is a cell-like map. Consequently, any simple homotopy equivalence of CW complexes can be obtained by composing cell-like maps and their homotopy inverses.

Warning 3. In the previous lecture, we showed that a proper map $f : X \to Y$ of CW complexes is cell-like if and only if, for each $U \subseteq Y$, the induced map $f^{-1}U \to U$ is a homotopy equivalence. It follows that the collection of cell-like maps between CW complexes is closed under composition. Beware that the collection of cell-like maps is not closed under composition in general (one can avoid this problem by slightly modifying the definition of cell-like map, but this will not be important for us in what follows).

We now recall the result promised in the previous lecture:

Proposition 4. *Let $f : X \to Y$ be a map of finite CW complexes. Assume that f is cell-like, cellular, and that for every cell $e \subseteq Y$, the inverse image $f^{-1}e$ is a union of cells. Then f is a simple homotopy equivalence.*

Example 5. Let $f : X \to Y$ be a piecewise linear map of polyhedra. Then one can always choose triangulations τ_X and τ_Y of X and Y respectively that are *compatible with* f in the following sense:

(1) The map f carries each vertex of the triangulation τ_X to a vertex of the triangulation τ_Y.

(2) The map f is linear on each simplex of the triangulation τ_X (and therefore maps each simplex of τ_X linearly onto a simplex of τ_Y).

The triangulations τ_X and τ_Y determine CW structures on X and Y for which the induced map $f : X \to Y$ is cellular, and for which the inverse image $f^{-1}e$ is a union of cells for each cell $e \subseteq Y$. Consequently, any cell-like piecewise-linear map of polyhedra $f : X \to Y$ can be considered to satisfy the requirements of Proposition 4.

Proof of Proposition 4. Without loss of generality we may assume that Y is connected. The map f is a homotopy equivalence by Proposition **??**. To show that it is a simple homotopy equivalence, it will suffice to show that the Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1 Y)$ vanishes.

Let $\widetilde{Y}$ be a universal cover of Y, and for every space Z with a map $Z \to Y$ we let $\widetilde{Z}$ denote the fiber product $Z \times_Y \widetilde{Y}$ (this is a covering space of Z which may or may not be connected). Set $G = \pi_1 Y$ so that

G acts on $\widetilde{Y}$ by deck transformations. We wish to show that the torsion of the quasi-isomorphism of cellular chain complexes $C_*(\widetilde{X}; \mathbf{Z}) \to C_*(\widetilde{Y}; \mathbf{Z})$ vanishes in $\mathrm{Wh}(G)$. We will deduce this from the following more general assertion:

$(*)$ For every subcomplex $Y_0 \subseteq Y$ with inverse image $X_0 \subseteq X$, the torsion of the induced map of cellular chain complexes $C_*(\widetilde{X}_0; \mathbf{Z}) \to C_*(\widetilde{Y}_0; \mathbf{Z})$ vanishes in $\mathrm{Wh}(G)$.

We proceed by induction on the number of cells in Y_0. Let us therefore assume that $(*)$ is known for a subcomplex $Y_0 \subseteq Y$, and see what happens when we add one more cell to obtain another subcomplex $Y_1 \subseteq Y$. As we saw in the previous lecture, Whitehead torsion is multiplicative in short exact sequences. To carry out the inductive step, it will suffice to show that torsion of the map of relative cellular cochain complexes

$$\theta : C_*(\widetilde{X}_1, \widetilde{X}_0; \mathbf{Z}) \to C_*(\widetilde{Y}_1, \widetilde{Y}_0; \mathbf{Z})$$

vanishes in $\mathrm{Wh}(G)$. Let $e \subseteq Y$ be the open cell of Y_1 which does not belong to Y_0. Then e is simply connected, so the map $\widetilde{e} \to e$ admits a section. A choice of section (and orientation of e) determines a one-element basis for $C_*(\widetilde{Y}_1, \widetilde{Y}_0; \mathbf{Z})$ as a $\mathbf{Z}[G]$-module, and by choosing cells of $\widetilde{X}_1 - \widetilde{X}_0$ which belong to the inverse image of the section we obtain a basis for $C_*(\widetilde{X}_1, \widetilde{X}_0; \mathbf{Z})$ as a $\mathbf{Z}[G]$-module (ambiguous up to signs) with respect to which the differential on $C_*(\widetilde{X}_1, \widetilde{X}_0; \mathbf{Z})$ and the map θ can be represented by matrices with integral coefficients (rather than coefficients in $\mathbf{Z}[G]$). It follows that the image of $\tau(\theta)$ in $\mathrm{Wh}(G)$ factors through the map $\widetilde{K}_1(\mathbf{Z}) = \mathrm{Wh}(*) \to \mathrm{Wh}(G)$, and therefore vanishes because the group $\widetilde{K}_1(\mathbf{Z}) = K_1(\mathbf{Z})/\{[\pm 1]\}$ is trivial. $\square$

Recall that a map of topological spaces $q : E \to B$ is said to be a *fibration* if it has the homotopy lifting property: that is, for any map $f : X \to E$ and any homotopy h from $(q \circ f)$ to another map $g : X \to B$, there exists a homotopy $\overline{h} : X \times [0,1] \to E$ with $\overline{h}|_{X \times \{0\}} = f$ and $q \circ \overline{h} = h$.

Let $q : E \to B$ be a fibration, and let $p : [0,1] \to B$ be a continuous path which begins at a point $b = p(0)$ and ends at a point $b' = p(1)$. The construction

$$(e, t) \mapsto p(t)$$

determines a homotopy from the constant map $E_b \to \{b\} \subseteq B$ to the constant map $E_b \to \{b'\} \subseteq B$. Applying the homotopy lifting property, we deduce that there is a continuous map $\overline{h} : E_b \times [0,1] \to E$ such that $q(\overline{h}(e,t)) = p(t)$. The restriction $\overline{h}|_{E_b \times \{1\}}$ is a continuous map from E_b to $E_{b'}$. We will denote this map by $p_! : E_b \to E_{b'}$; one can show that up to homotopy, it depends only on the homotopy class of the path p (and not on the choice of map $\overline{h}$). The map $p_!$ is always a homotopy equivalence: it has a homotopy inverse given by $p'_!$, where $p' : [0,1] \to B$ is given by $p'(t) = p(1-t)$.

In the special case where $B = [0,1]$ and we take p to be the identity map, we see that any fibration $q : E \to [0,1]$ determines a homotopy equivalence $p_! : E_0 \to E_1$. Our objective in this course is to study what further constraints on $p_!$ are imposed by the condition that the total space E be "finite" in some sense.

Definition 6. Let X and Y be finite polyhedra. A *concordance* from X to Y is a (piecewise linear) fibration of finite polyhedra $q : E \to [0,1]$ together with PL homeomorphisms $X \simeq q^{-1}\{0\}$ and $Y \simeq q^{-1}\{1\}$. We will say that X and Y are *concordant* if there exists a concordance from X to Y.

Note that every concordance from X to Y determines a homotopy equivalence $p_! : X \to Y$, which is well-defined up to homotopy. We can now state our next main result:

Theorem 7. *Let X and Y be finite polyhedra and let $f : X \to Y$ be a homotopy equivalence. The following conditions are equivalent:*

(1) *The map f is a simple homotopy equivalence.*

(2) *There exists a concordance $q : E \to [0,1]$ from X to Y such that the induced homotopy equivalence $X \simeq q^{-1}\{0\} \to q^{-1}\{1\} \simeq Y$ is homotopic to f.*

Corollary 8. *Let X and Y be finite polyhedra. Then X and Y are simple homotopy equivalent if and only if they are concordant.*

We will prove the implication (2) $\Rightarrow$ (1) in this lecture, saving the reverse implication for later. Let us suppose that we have chosen triangulations τ_E and τ_B of E and B which are compatible in the sense of Example **??**. The triangulation τ_B is just a partition of B into subintervals. We may therefore reduce to the case where the triangulation τ_B consists of only a single interval (and its endpoints 0 and 1).

For every simplex σ of τ_E which is not contained in either E_0 or E_1, we let $\sigma_0 = \sigma \cap E_0$ and $\sigma_1 = \sigma \cap E_1$. Let $E_{01} \subseteq E_0 \times E_1$ denote the subset given by those products $\sigma_0 \times \sigma_1$, where σ ranges over those simplices of τ_E which are contained in neither E_0 nor E_1. Then E_{01} is equipped with projection maps $h_0 : E_{01} \to E_0$ and $h_1 : E_{01} \to E_1$. By construction, for every point $x \in E_{01}$, the line segment joining $h_0(x)$ to $h_1(x)$ is contained in E. We can therefore define a "straight-line" homotopy $h : E_{01} \times [0,1] \to E$ by the formula

$$h(x,t) = (1-t)h_0(x) + th_1(x).$$

Exercise 9. Show that the homotopy h above induces a homeomorphism of topological spaces

$$E_0 \amalg_{E_{01} \times \{0\}} (E_{01} \times [0,1]) \amalg_{E_{01} \times \{1\}} E_1 \to E.$$

In other words, if $M(h_0)$ and $M(h_1)$ denote the mapping cylinders of h_0 and h_1 respectively, then E is homeomorphic to the pushout $M(h_0) \amalg_{E_{01}} M(h_1)$.

Warning 10. The homotopy h is usually *not* piecewise-linear. The description of E as a pushout $M(h_0) \amalg_{E_{01}} M(h_1)$ is valid only in the category of topological spaces, not in the category of polyhedra.

The above argument proves the following:

Proposition 11. *Let E be a finite polyhedron equipped with a piecewise linear map $E \to [0,1]$. Then it is possible to subdivide $[0,1]$ into finitely many subintervals such that the inverse image of each subinterval $[t,t']$ is homeomorphic either to the mapping cylinder of a PL map $E_t \to E_{t'}$ or to the (reversed) mapping cylinder of a PL map $E_{t'} \to E_t$.*

We will deduce the implication (2) $\Rightarrow$ (1) of Theorem 6 from Proposition 10 together with the following:

Proposition 12. *Let $f : X \to Y$ be a continuous map of compact Hausdorff spaces and let $M(f) = (X \times [0,1]) \amalg_{X \times \{1\}} Y$ denote the mapping cylinder of f. If the projection map $q : M(f) \to [0,1]$ is a fibration, then f is a cell-like map.*

We will later see that the converse of Proposition 11 is true as well, provided that the spaces X and Y are reasonably nice.

Proof of Proposition 11. Fix a point $y \in Y$. We wish to show that the fiber $X_y = f^{-1}\{y\}$ has trivial shape. In other words, we wish to show that every map from X_y to a CW complex Z is nullhomotopic. Any such map extends over a neighborhood $U \subseteq X$ containing X_y; it will therefore suffice to show that the inclusion $X_y \hookrightarrow U$ is nullhomotopic. Since f is a proper map, U contains an open set of the form $f^{-1}V$ where V is a neighborhood of y in Y. We may therefore assume without loss of generality that $U = f^{-1}V$.

Let $C \subseteq M(f)$ denote the mapping cylinder of the projection $X_y \to \{y\}$ and let $W \subseteq M(f)$ be the mapping cylinder of the projection $U \to V$. Since q is a fibration, we can choose a homotopy $h : C \times [0,1] \to M(f)$ such that $h(c,0) = c$ and $q(h(c,t)) = \max\{q(c), 1-t\}$. Since $h|_{C \times \{0\}}$ factors through W, it follows that there exists $\epsilon > 0$ such that $h|_{C \times [0,\epsilon]}$ factors through W. Let $g : X_y \to f^{-1}(V)$ be the map characterized by

$$h(x, 1-\epsilon, \epsilon) = (g(x), 1-\epsilon).$$

Restricting h to $(X_y \times \{1-\epsilon\}) \times [0,\epsilon]$, we see that g is homotopic to the inclusion $X_y \hookrightarrow f^{-1}(V)$. On the other hand, by restricting h to $((X_y \times [1-\epsilon, 1]) \amalg_{X_y \times \{1\}} \{y\}) \times \{\epsilon\} \subseteq C \times [0,1]$, we see that g is nullhomotopic. It follows that the inclusion $X_y \hookrightarrow f^{-1}(V)$ is nullhomotopic, as desired. $\square$

Higher Simple Homotopy Theory (Lecture 7)

September 17, 2014

Recall that finite polyhedra X and Y are *concordant* if there is a piecewise-linear fibration $q : E \to [0,1]$ with $X \simeq q^{-1}\{0\}$ and $Y \simeq q^{-1}\{1\}$. In the last lecture, we asserted that X and Y are simply homotopy equivalent if and only if they are concordant, and proved the "if" direction. Our goal in this lecture is to use this fact as a starting point for the study of "higher" simple homotopy theory, following ideas of Hatcher.

For any finite polyhedron B, we can contemplate piecewise-linear fibrations $q : E \to B$ (where E is also a finite polyhedron). Our first goal is to construct a universal example of such a fibration, so that the base B can be regarded as a classifying space for PL fibrations. It is not clear that such a classifying space exists in the setting of finite polyhedra, but we can give an *almost* tautological construction of one in the setting of simplicial sets.

Definition 1. For each integer n, let Δ^n denote the topological simplex of dimension n and let $\mathcal{M}_n$ denote the set of all finite polyhedra $E \subseteq \Delta^n \times \mathbb{R}^\infty$ for which the projection map $E \to \Delta^n$ is a fibration.

Note that for any linear map of simplices $\alpha : \Delta^m \to \Delta^n$, the construction $E \mapsto E \times_{\Delta^n} \Delta^m$ defines a map of sets $\alpha^* : \mathcal{M}_n \to \mathcal{M}_m$. In particular, we can regard the construction $[n] \mapsto \mathcal{M}_n$ as a simplicial set, which we will denote by $\mathcal{M}$.

Before we analyze the simplicial set $\mathcal{M}$, we need a few general facts about the relationship between polyhedra and simplicial sets.

Remark 2. Let K_0, K_1, and K_{01} be polyhedra, and suppose we are given piecewise linear embeddings

$$K_0 \overset{i_0}{\hookleftarrow} K_{01} \overset{i_1}{\hookrightarrow} K_1.$$

Then the pushout $K_0 \amalg_{K_{01}} K_1$ exists in the category of polyhedra: that is, we can regard endow $K_0 \amalg_{K_{01}} K_1$ with the structure of a polyhedron, where a map $K_0 \amalg_{K_{01}} K_1 \to L$ is piecewise linear if and only if its restriction to K_0 and K_1 is piecewise linear.

Beware that this need not be true if i_0 is not an embedding, even if i_1 is an embedding. This is often a technical nuisance.

Example 3. Let X be a finite simplicial set. We say that X is *nonsingular* if every simplex $\sigma : \Delta^n \to X$ is either degenerate (meaning that it factors through Δ^m for $m < n$) or is a monomorphism of simplicial sets (in particular, all the faces of σ are again nondegenerate).

For any nonsingular finite simplicial set X, the geometric realization $|X|$ can be regarded as a finite polyhedron. More precisely, there is a unique PL structure on $|X|$ having the property that for every nondegenerate n-simplex of X, the associated map $\Delta^n \to |X|$ is piecewise linear (this follows by invoking Remark 2 repeatedly).

In what follows, we will often not distinguish between a (finite nonsingular) simplicial set X and the polyhedron $|X|$. For example, we use the symbol Δ^n to denote both the n-simplex as a simplicial set and the topological n-simplex, and apply similar considerations to the boundary $\partial\,\Delta^n$ and the horns $\Lambda^n_i \subseteq \Delta^n$.

We will also need the following technical fact, whose proof we omit (see Lemma 2.7.12 of [1]):

Proposition 4. *Let $q : E \to B$ be a map of finite polyhedra. The following conditions are equivalent:*

(1) *The map q is a fibration.*

(2) *For every triangulation of B and every simplex σ of the triangulation, the induced map $E \times_B \sigma \to \sigma$ is a fibration.*

(3) *There exists a triangulation of B such that, for every simplex σ of the triangulation, the induced map $E \times_B \sigma \to \sigma$ is a fibration.*

Corollary 5. *Let B be a finite nonsingular simplicial set. Then* $\mathrm{Hom}(B, \mathcal{M})$ *can be identified with the set of finite polyhedra $E \subseteq |B| \times \mathbb{R}^\infty$ for which the projection map $E \to |B|$ is a fibration.*

Proof. The geometric realization $|B|$ admits a triangulation for which each simplex is contained in the image of some simplex of B (beware that the nondegenerate simplices of B do not generally themselves determine a triangulation of $|B|$, unless one is liberal with the meaning of the word "triangulation"). □

Corollary 6. *The simplicial set $\mathcal{M}$ is a Kan complex.*

Proof. Suppose we are given a map $f_0 : \Lambda^n_i \to \mathcal{M}$, given by a polyhedron $E \subseteq |\Lambda^n_i| \times \mathbb{R}^\infty$ for which the projection $E \to |\Lambda^n_i|$ is a fibration. Choose a piecewise linear retraction $r : |\Delta^n| \to |\Lambda^n_i|$, and define $\overline{E} = E \times_{|\Lambda^n_i|} |\Delta^n|$. Then $\overline{E}$ can be identified with a map $f : \Delta^n \to \mathcal{M}$ extending f_0. □

We next investigate the role of the Kan complex $\mathcal{M}$ as a "classifying space."

Exercise 7. Let B be a finite polyhedron. Suppose we are given fibrations of finite polyhedra $f : X \to B$, $g : Y \to B$. We will say that f and g are *concordant* if there exists a fibration of finite polyhedra $h : Z \to B \times [0, 1]$ for which the inverse image of $B \times \{0\}$ is isomorphic to X and the inverse image of $B \times \{1\}$ is isomorphic to Y. Show that concordance is an equivalence relation.

Let B be a finite nonsingular simplicial set. Any map $f : B \to \mathcal{M}$ determines a fibration of finite polyhedra $E_f \to |B|$, and any homotopy between maps $f, g : |B| \to \mathcal{M}$ determines a concordance from E_f to E_g. We therefore obtain a well-defined map from the set $[B, \mathcal{M}]$ of homotopy classes of maps from B into $\mathcal{M}$ to the set of concordance classes of fibrations over $|B|$.

Proposition 8. *This map is bijective.*

Proof. To prove surjectivity, it suffices to note that for any map of finite polyhedra $X \to |B|$, we can choose a compatible PL embedding of X into $|B| \times \mathbb{R}^\infty$.

To prove injectivity, it suffices to show that if $X \subseteq |B| \times \mathbb{R}^\infty$ and $Y \subseteq |B| \times \mathbb{R}^\infty$ are polyhedra fibered over $|B|$ and we are given any concordance $Z \to |B \times \Delta^1|$ from X to Y, then we can choose a PL embedding of Z into $|B \times \Delta^1| \times \mathbb{R}^\infty$ which is compatible with the given embeddings on X and Y. □

References

[1] Waldhausen, F., Jahren, B. and J. Rognes. *Spaces of PL Manifolds and Categories of Simple Maps.*

Fibrations of Polyhedra (Lecture 8)

September 20, 2014

In the previous lecture, we introduced a simplicial set $\mathcal{M}$ which parametrizes fibrations in the piecewise-linear category. To analyze $\mathcal{M}$, we consider the following:

Question 1. Let $q : E \to B$ be a piecewise-linear map of finite polyhedra. When is q a fibration?

To address this question, let us choose compatible triangulations τ_E and τ_B of E and B respectively, with vertex sets V_E and V_B. Then q maps each simplex of τ_E linearly onto a simplex of τ_B.

For each vertex $b \in V_B$, we let $E_b = q^{-1}\{b\}$ denote the fiber over b, so that τ_E induces a triangulation of E_b. More generally, suppose that σ is an n-simplex of τ_B with vertices $\{b_0, \ldots, b_n\}$. For every simplex $\overline{\sigma} \in \tau_E$ with $q(\overline{\sigma}) = \sigma$, let

$$\overline{\sigma}_i = \overline{\sigma} \cap E_{b_i}.$$

Let $E_\sigma \subseteq E_{b_0} \times \cdots \times E_{b_n}$ denote the union

$$\bigcup \overline{\sigma}_0 \times \cdots \times \overline{\sigma}_n$$

where the union is taken over all simplices $\overline{\sigma}$ of τ_E with $q(\overline{\sigma}) = \sigma$. Note that for every such $\overline{\sigma}$, there is a canonical map

$$f_\sigma : \overline{\sigma}_0 \times \cdots \times \overline{\sigma}_n \times \Delta^n \to \overline{\sigma} \subseteq E$$

given by

$$(x_0, \ldots, x_n, t_0, \ldots, t_n) \mapsto \sum t_i x_i.$$

Note that if $\sigma' \subseteq \sigma$ in τ_B, then there is a canonical projection map $E_\sigma \to E_{\sigma'}$ which fits into a commutative diagram

$$\begin{tikzcd}
 & E_\sigma \times \sigma' \arrow[dl] \arrow[dr] & \\
E_{\sigma'} \times \sigma' \arrow[dr, "f_{\sigma'}"] & & E_\sigma \times \sigma \arrow[dl, "f_\sigma"] \\
 & E. &
\end{tikzcd}$$

Exercise 2. The preceding maps can be assembled to a homeomorphism of topological spaces

$$\varinjlim_{\sigma' \subseteq \sigma} E_\sigma \times \sigma' \simeq E.$$

In other words, the polyhedron E can be realized as the *coend* of the contravariant functor $\sigma \mapsto E_\sigma$ (from the partially ordered set τ_B to topological spaces) against the covariant functor $\sigma \mapsto \sigma$ (from τ_B to topological spaces).

Alternatively, this result can be interpreted as saying that the polyhedron E is the homotopy colimit of the functor $\sigma \mapsto E_\sigma$.

Warning 3. The homeomorphism

$$\varinjlim_{\sigma' \subseteq \sigma} E_\sigma \times \sigma' \simeq E.$$

is generally not piecewise-linear (in fact, the colimit on the right hand side generally does not exist in the category of polyhedra). This is visible in the definition of the maps f_σ: the construction $(x_0, \ldots, x_n, t_0, \ldots, t_n) \mapsto \sum t_i x_i$ is quadratic, not linear.

Remark 4. It follows from Exercise 2 that for every point $b \in B$, the fiber $E_b = q^{-1}\{b\}$ is homeomorphic to E_σ where σ is the unique simplex of τ_B whose interior contains b.

It follows from Exercise 2 that E can be recovered (as a topological space) from the triangulation τ_B and the contravariant functor $\sigma \mapsto E_\sigma$. We can therefore address Question 1 as follows:

Theorem 5. *Let $q : E \to B$ be as above. Then q is a fibration if and only if, for every inclusion $\sigma' \subseteq \sigma$ in τ_B, the induced map $E_\sigma \to E_{\sigma'}$ is a cell-like map.*

The "only if" direction we have already proven: for each of the maps $\rho : E_\sigma \to E_{\sigma'}$, the mapping cylinder $M(\rho)$ can be realized as the fiber product $[0,1] \times_B E$ (where the path $[0,1] \to B$ is any straight line joining a point in the interior of σ to a point in the interior of σ'), so that if q is a fibration then $M(\rho) \to [0,1]$ is also a fibration and therefore ρ is cell-like. Our goal in this lecture is to prove the converse, following the argument given in [1].

Remark 6. In the previous lecture, we asserted without proof that a map of finite polyhedra $q : E \to B$ is a fibration if and only if it is a fibration over each simplex of a triangulation τ_B of B. This follows immediately from Theorem 5 (since we can always pass to a refinement of τ_B for which there is a compatible triangulation of E).

The other direction Theorem 5 is an immediate consequence of the following slightly more general statement:

Theorem 7. *Let B be a finite polyhedron with a triangulation τ_B. Suppose we are given a contravariant functor $\sigma \mapsto E_\sigma$ from τ_B to topological spaces. Assume that each E_σ is a compact ANR (for example, a finite polyhedron) and that each inclusion $\sigma' \subseteq \sigma$ induces a cell-like map $E_\sigma \to E_{\sigma'}$. Then the canonical map*

$$\mathrm{hocolim}_{\sigma \in \tau_B} E_\sigma = \varinjlim_{\sigma' \subseteq \sigma} E_\sigma \times \sigma' \to B$$

is a fibration.

Our proof will proceed by induction on the dimension of the polyhedron B. Recall that the condition that q be a fibration can be tested locally on B. Consequently, it will suffice to show that every point $b \in B$ has an open neighborhood U for which the induced map $E \times_B U \to U$ is a fibration. Passing to a subdivision of τ_B, we can arrange that b is a vertex of B.

Recall that the *closed star* C of b is the union of those simplices of τ_B which contain the vertex b, and the *link* L of b is the union of those simplices of τ_B which are contained in C but do not contain b. Then C can be identified with the cone $(L \times [0,1]) \amalg_{L \times \{1\}} *$, and the open star $C - L \simeq (L \times (0,1]) \amalg_{L \times \{1\}} *$ is an open neighborhood of b.

For each simplex σ of L, let $\sigma^+ \subseteq C$ denote the simplex spanned by σ and b, and set $E' = \mathrm{hocolim}_{\sigma \subseteq L} E_{\sigma^+}$. Since the link L has dimension smaller than the dimension of B, it follows from the inductive hypothesis that the canonical map $E' \to L$ is a fibration. Moreover, the maps $E_{\sigma^+} \to E_b$ assemble to give a cell-like map $E' \to \mathrm{hocolim}_{\sigma \subseteq L} E_b = L \times E_b$ of spaces fibered over L. Unwinding the definitions, we see that the fiber product $(C - L) \times_B E$ is homeomorphic to

$$(E' \times (0,1]) \amalg_{E' \times \{1\}} E_b.$$

It will therefore suffice to prove the following:

Proposition 8. *Let L be a finite polyhedron. Suppose we are given compact ANRs X and Y and a cell-like map $X \to Y \times L$ which induces a fibration $X \to L$. Then the induced map*

$$(X \times [0,1]) \amalg_{X \times \{1\}} Y \to (L \times [0,1]) \amalg_{L \times \{1\}} \{*\} = C(L)$$

is also a fibration.

The main ingredient we will need is the following lemma, which we will prove at the end of this lecture:

Lemma 9. *In the situation of Proposition 8, let $F : (X \times [0,1]) \amalg_{X \times \{1\}} Y \to Y \times C(L)$ be the canonical map. Then there exists a map $G : Y \times C(L) \to (X \times [0,1]) \amalg_{X \times \{1\}} Y$ and a homotopy H from the identity to $G \circ F$ which preserves fibers over $C(L)$ and is the identity on Y.*

Set $Z = (X \times [0,1]) \amalg_{X \times \{1\}} Y$. Proposition 8 asserts that every lifting problem of the form

$$\begin{tikzcd} A \times \{0\} \arrow[r, "\overline{p}_0"] \arrow[d] & Z \arrow[d, "\phi"] \\ A \times [0,1] \arrow[r, "p"] \arrow[ur, dashed, "\overline{p}"] & C(L) \end{tikzcd}$$

has a solution. In this case, $\overline{p}_0$ determines a point $x \in Z$. There are two cases to consider.

Case (1) We have $p(0) = *$, so that x belongs to the fiber $\phi^{-1}\{*\} = Y$. In this case, we can define $\overline{p}$ by the formula $\overline{p}(t) = G(x, p(t))$.

Case (2) We have $p(0) \neq *$. In this case, there is a real number $0 < \theta \leq 1$ such that p carries the interval $[0, \theta]$ into the open set $C(L) - \{*\}$. We know ϕ restricts to a fibration $X \times [0,1) \to C(L) - \{*\}$, so that $p|_{[0,\theta]}$ can be lifted to a path $\overline{p}' : [0, \theta] \to X \times [0,1) \subseteq Z$. We can then define $\overline{p}$ by the formula

$$\overline{p}(t) = \begin{cases} H(\overline{p}'(t), \frac{t}{\theta}) & \text{if } t \leq \theta \\ G(\overline{p}(\theta), p(t)) & \text{if } t \geq \theta. \end{cases}$$

Let us now consider the case of a general parameter space A. We may assume without loss of generality that A is a metric space (in fact, it suffices to treat the universal case where $A = Z \times_{C(L)} C(L)^{[0,1]}$, which is metrizable). Let $A_1 = \{a \in A : p(a, 0) = *\}$ and let $A_2 = A - A_1$. We would like to construct the map $\overline{p}$ by applying the preceding recipes separately to A_1 and A_2. The main difference is that we will not regard θ as a constant, but instead as a continuous function $\theta : A_2 \to (0, 1]$. We will arrange that θ has the following property:

(a) We have $p(a, t) \neq *$ for $a \in A_2$ and $t \leq \theta(a)$.

Assume that θ satisfies (a) and define $B_\theta = \{(a, t) \in A_2 \times [0,1] : t \leq \theta(a)\}$. Using the fact that the map $X \times [0,1) \to C(L) - \{*\}$ is a fibration), we can extend $\overline{p}_0$ to a partially defined homotopy $\overline{p}'_\theta : B_\theta \subseteq Z$ extending $\overline{p}_0|_{A_2}$ and lying over $p|_{B_\theta}$. We can then define maps

$$\phi_1 : A_1 \times [0,1] \to Z \qquad \phi_2 : A_2 \times [0,1] \to Z$$

by the formulae

$$\phi_1(a,t) = G(\overline{p}_0(a), p(t))$$

$$\phi_2(a,t) = \begin{cases} H(\overline{p}'_\theta(a,t), \frac{t}{\theta(a)}) & \text{if } t \leq \theta(a) \\ G(\overline{p}'_\theta(a,t), p(a,t)) & \text{if } t \geq \theta(a). \end{cases}$$

Let $\overline{p} : A \times [0,1] \to Z$ be the map given on $A_i \times [0,1]$ by ϕ_i. To complete the proof, it will suffice to show that θ and $\overline{p}'_\theta$ can be chosen so that $\overline{p}$ is continuous.

Let us now assume:

(b) The map θ extends to a continuous map $\overline{\theta} : A \to [0,1]$ with $\overline{\theta}|_{A_1} = 0$.

Remark 10. To choose the map $\overline{\theta}$, let us equip the product $A \times [0,1]$ with the taxi-cab metric, and set $K = p^{-1}\{*\} \subseteq A \times [0,1]$. We can then define $\overline{\theta}$ by the formula

$$\overline{\theta}(a) = \min\{1, \frac{1}{2} d((a,0), K)\}.$$

Let $\overline{B}_\theta^{\leq} = \{(a,t) \in A \times [0,1] : t \leq \overline{\theta}(a)\}$, let $\overline{B}_\theta^{\geq} = \{(a,t) \in A \times [0,1] : t \geq \overline{\theta}(a)\}$. Assume that $\overline{p}|\overline{B}_\theta^{\leq}$ is continuous. Then the map $h : A \to Z$ given by $h(a) = \overline{p}(a, \overline{\theta}(a))$ is continuous. The restriction of $\overline{p}$ to $\overline{B}_\theta^{\geq}$ is given by

$$\overline{p}(a,t) = G(h(a), p(a,t)),$$

and is therefore continuous. It follows that $\overline{p}$ is continuous, as desired. We are therefore reduced to proving that we can choose $\overline{\theta}$ so that $\overline{p}|\overline{B}_\theta^{\leq}$ is continuous.

Note the map $(a,t) \mapsto (a, \overline{\theta}(a)t)$ determines a proper surjection $\pi : A \times [0,1] \to \overline{B}_\theta^{\leq}$. Consequently, it will suffice to show that $\overline{p} \circ \pi : A \times [0,1] \to Z$ is continuous. Let $r : A \times [0,1] \to Z$ be the map given by

$$r(a,t) = \begin{cases} \overline{p}_0(a) & \text{if } a \in A_1 \\ \overline{p}'(a, t\theta(a)) & \text{if } a \in A_2. \end{cases}$$

Then $(\overline{p} \circ \pi)(a,t) = H(r_\theta(a,t), t)$. It will therefore suffice to show that we can arrange that r is continuous. To prove this, let us choose a metric d_Z on the space Z and define $K' = \{(a,t) \in B_\theta : d_Z(\overline{p}'_\theta(a,t), \overline{p}_0(a)) \geq \theta(a)\}$. Let $\theta' : A_2 \to [0,1]$ be defined by the formula

$$\theta'(a) = \min\{\theta(t), d((a,0), K')\}.$$

Then $\theta' \leq \theta$, so θ' also extends to a continuous map $\overline{\theta}' : A \to [0,1]$ satisfying $\overline{\theta}'|_{A_1} = 0$. Replacing θ by θ' and $\overline{p}'_\theta$ with $\overline{p}'_\theta|_{B_{\theta'}}$, we can assume that the function r satisfies $d_Z(r(a,t), r(a,0)) \leq \theta(a)$ for $a \in A_2$. It follows that if we are given a sequence of points (a_i, t_i) in $A_2 \times [0,1]$ which approach a limit (a,t) in $A_1 \times [0,1]$, then we have

$$\lim r(a_i, t_i) = \lim r(a_i, 0) = r(a, 0) = r(a,t),$$

so that r is continuous as desired.

It remains to prove Lemma 9. The proof will require some careful estimates. From this point forward, we will fix a metric d on Y. We will employ the following abuse of notation: given any space Y' with a map $\pi : Y' \to Y$ and any pair of points $a, b \in Y'$, we set $d(a,b) = d(\pi(a), \pi(b))$ (note that this is generally *not* a metric on Y'; for example, points belonging to the same fiber of π have distance zero from one another). The cases of interest to are $Y' = Y \times L$ and $Y' = X$.

Given a map $Y' \to Y$ is as above, we will say that a path $p : [0,1] \to Y'$ is *ϵ-small* if, for every pair $s, t \in [0,1]$, the distance $d(p(s), p(t))$ is less than ϵ. More generally, given a topological space S and a homotopy $h : S \times [0,1] \to Y'$, we will say that h is *ϵ-small* if the paths $h|_{\{s\} \times [0,1]}$ are ϵ-small for each $s \in S$.

Let $f : X \to Y \times L$ denote the projection map. The main technical ingredient we will need is the following:

Proposition 11. *For each $\epsilon > 0$, there exists a map $g_\epsilon : Y \times L \to X$ and an ϵ-small homotopy $h_\epsilon : X \times [0,1] \to X$ from id_X to $g_\epsilon \circ f$, where g_ϵ and h_ϵ are compatible with projection to L.*

Let us assume Proposition 11 for the moment and show that it leads to a proof of Lemma 9.

Remark 12. In the situation of Proposition 11, it follows from the existence of the homotopy h_ϵ that we have $d(y, (f \circ g_\epsilon)(y)) < \epsilon$ for each $y \in Y \times L$. In other words, the maps g_ϵ are *approximately* sections to f.

Remark 13. We have assumed that Y is a compact ANR, so there exists an embedding of Y into a Banach space B and a retraction $r : U \to Y$, where U is an open neighborhood of Y in B. Fix a real number $\epsilon > 0$. For sufficiently small δ, any pair of points $y, y' \in Y$ with $d(y, y') < \delta$ have the property that the interval joining y to y' in B belongs entirely to U, so that the construction

$$p_{y,y'} : [0,1] \to Y$$

$$p_{y,y'}(t) = r((1-t)y + ty')$$

determines a continuous path from y to y' in Y. The path $p_{y,y'}$ depends continuously on y and y'. It follows that the function $(y, y') \mapsto \sup\{d(p_{y,y'}(s), p_{y,y'}(t))\}$ is also a continuous function, which vanishes when $y = y'$. Shrinking δ if necessary, we may assume that $\delta < \epsilon$ and that if $d(y, y') < \delta$ then the path $p_{y,y'}$ is ϵ-small. In this case, we will say that δ *is small compared to* ϵ and write $\delta \ll \epsilon$.

Remark 14. Suppose that $\delta \ll \epsilon$. It follows from Remark 12 that for each $y \in Y \times L$, there is an ϵ-small path joining y to $(f \circ g_\delta)(y)$, which depends continuously on y. These paths can be assembled to a $k_{\delta,\epsilon} : Y \times L \times [0,1] \to Y \times L$. which is compatible with the projection to L. In particular, we see that g_δ is a right homotopy inverse to f (it is also a left homotopy inverse, by virtue of the existence of the homotopy h_ϵ.

Remark 15. Suppose that $2\delta \ll \epsilon$. For every point $x \in X$, the constructions

$$t \mapsto F(h_\delta(x,t)) \qquad t \mapsto k_{\delta,\epsilon}(f(x), t)$$

determine δ-small paths from $f(x)$ to $(f \circ g_\delta \circ f)(x)$. Using the triangle inequality, we see that the distance between these paths is at most 2δ. It follows that there is an ϵ-small homotopy

$$v_{\delta,\epsilon} : X \times [0,1] \times [0,1] \to Y \times L$$

from $F \circ h_\delta$ to $k_{\delta,\epsilon} \circ (F \times \mathrm{id}_{[0,1]})$.

We are now ready to construct the map G appearing in the statement of Lemma 9. Choose a sequence of positive real numbers $\epsilon_0, \epsilon_1, \ldots$ with $\epsilon_0 \leq 1$ and $2\epsilon_{n+1} \ll \epsilon_n$ (from which it follows that $\epsilon_n \leq \frac{1}{2^n}$ for all n). We define a continuous map $G^\circ : Y \times L \times \mathbb{R}_{\geq 0} \to X$ by the formula

$$G^\circ(y,t) = \begin{cases} g_{\epsilon_n}(k_{\epsilon_{n+1},\epsilon_n}(y, 2(t-n))) & \text{if } n \leq t \leq n + \frac{1}{2} \\ h_{\epsilon_n}(g_{\epsilon_{n+1}}(y), 2(n+1-t)) & \text{if } n + \frac{1}{2} \leq t \leq n+1. \end{cases}.$$

Let us identify $\mathbb{R}_{\geq 0}$ with the half-open interval $[0,1)$, so that the construction

$$(y,t) \mapsto (G^\circ(y), t)$$

determines a continuous map $Y \times L \times [0,1) \to X \times [0,1)$. We claim that map admits a continuous extension $G : Y \times L \times [0,1] \to Z$ whose restriction to $Y \times L \times \{1\}$ is given by the projection onto the first factor. To prove this, it suffices to show that for every sequence of points (y_i, t_i) in $Y \times L \times [0, \infty)$ where the y_i converge to some point $y \in Y \times L$ and the t_i converge to ∞, the sequence of points $f(G^\circ(y_i, t_i))$ converge to y in $Y \times L$. This is clear: note that if $n \leq t \leq n + \frac{1}{2}$, then

$$\begin{aligned} d(y, f(G^\circ(y,t))) &\leq d(y, k_{\epsilon_{n+1},\epsilon_n}(y, 2(t-n))) + d(k_{\epsilon_{n+1},\epsilon_n}(y, 2(t-n))), (f \circ g_{\epsilon_n})(k_{\epsilon_{n+1},\epsilon_n}(y, 2(t-n))) \\ &\leq 2\epsilon_n, \end{aligned}$$

while for $n + \frac{1}{2} \leq t \leq n+1$ we have

$$\begin{aligned} d(y, f(G^\circ(y,t))) &\leq d(y, f(G^\circ(y, n+1))) + d(f(G^\circ(y, n+1)), f(G^\circ(y,t)) \\ &= d(y, fg_{\epsilon_{n+1}}(y)) + d(f(G^\circ(y, n+1)), f(G^\circ(y,t)) \\ &\leq \epsilon_{n+1} + \epsilon_n. \end{aligned}$$

To construct the homotopy H, we begin by considering a map $T : X \times [0,\infty) \to X$ given by the formula

$$T(x,t) = \begin{cases} g_{\epsilon_n}(f(h_{\epsilon_{n+1}}(x, 2(t-n)))) & \text{if } n \leq t \leq n + \frac{1}{2} \\ h_{\epsilon_n}(g_{\epsilon_{n+1}}(f(x)), 2(n+1-t)) & \text{if } n + \frac{1}{2} \leq t \leq n+1. \end{cases}$$

There is a canonical homotopy from the projection map $\pi : X \times [0,\infty) \to X$ to T, which carries a pair $(x,t) \in X \times [0,\infty)$ to the path

$$s \mapsto \begin{cases} h_{\epsilon_n}(h_{\epsilon_{n+1}}(x, 2(t-n)s), s) & \text{if } n \leq t \leq n + \frac{1}{2} \\ h_{\epsilon_n}(h_{\epsilon_{n+1}}(x, s), 2(n+1-t)s) & \text{if } n + \frac{1}{2} \leq t \leq n+1. \end{cases}$$

The maps $v_{\epsilon_{n+1},\epsilon_n}$ of Remark 15 can be assembled to a homotopy from T to the map $(x,t) \mapsto G(f(x),t)$. Concatenating these homotopies, we obtain a map

$$H^\circ : X \times [0,\infty) \times [0,1] \to X \times [0,\infty).$$

We claim that (after identifying $[0,\infty)$ with $[0,1)$) H° extends continuously to a homotopy $H : Z \times [0,1] \to Z$ from id_Z to G which is trivial on the closed subset $Y \subseteq Z$. To prove this, we must show that if $\{x_i\}$ is a sequence of points of X whose images in Y converge to a point y and $\{t_i\}$ is a sequence of positive real numbers which converges to ∞, then the paths $(\pi_Y \circ f \circ H)|_{\{(x_i,t_i)\} \times [0,1]}$ converge to the constant path based at the point y, which is a consequence of the following elementary lemma which we leave to the reader:

Lemma 16. *Let $\{p_i : [0,1] \to Y\}_{i \geq 0}$ be a sequence of continuous paths in Y. Assume that:*

- (a) *For each $\epsilon > 0$, the paths p_i are ϵ-small for almost all i.*
- (b) *The sequence of points $\{p_i(0)\}_{i \geq 0}$ converges to a point $y \in Y$.*

Then the paths p_i converge to the constant path $[0,1] \to \{y\} \hookrightarrow Y$.

We now turn to the proof of Proposition 11. In the case where L is a single point, we have the following:

Proposition 17. *Let $f : X \to Y$ be a surjective map of compact ANRs. The following conditions are equivalent:*

1. *The map f is cell-like.*
2. *For every $\epsilon > 0$, there exists a map $g : Y \to X$ and an ϵ-small homotopy $h : X \times [0,1] \to X$ from the identity map to $g \circ f$ (recall that all distances are measured with respect to some metric on Y).*

Let us assume Proposition 17 for a moment, and see how it leads to a proof of Proposition 11. Choose a metric on L. Given a cell-like map $f : X \to Y \times L$ and any $\epsilon > 0$, Proposition 17 guarantees the existence of a map $g' : Y \times L \to X$ and a homotopy $h' : X \times [0,1] \to X$ from id_X to $g' \circ f$ such that the homotopy $f \circ h'$ is ϵ-small both in Y *and* in L. We wish to show that we can arrange that g' and h' commute with the projection to L. We will deduce this from the following:

Lemma 18. *Fix $\delta > 0$. For each $\epsilon > 0$, let $U \subseteq X \times L$ be the open set consisting of those points (x,v) such that the distance from v to the image of x (measured with respect to the metric on L) is $< \epsilon$. For ϵ sufficiently small, there exists a map $r : U \to X$ satisfying the following conditions:*

1. *The map r commutes with the projection to L.*
2. *If $x \in X$ and v is its image in L, then $r(x,v) = x$.*
3. *For all $(x,v) \in X$, there is an δ-small path from $r(x,v)$ to x.*

If ϵ is chosen small enough to satisfy the requirements of Lemma 18, then we can set

$$g(y) = r(g'(y), \pi_L g'(y)) \qquad h(x,t) = r(h'(x,t), \pi_L h'(x,t))$$

where $\pi_L : X \to L$ is the projection map. It then follows from the triangle inequality that the homotopy h is $(\epsilon + 2\delta)$-small; Proposition 11 then follows choosing δ and ϵ sufficiently small.

Proof of Lemma 18. Since the map $\pi_L : X \to L$ is a fibration, we can choose a path lifting function $u : X \times_L L^{[0,1]} \to X^{[0,1]}$. Let us identify X with its image in $X \times_L L^{[0,1]}$ (that is, the set of pairs (x,c) where $c : [0,1] \to L$ is the constant path based at $\pi_L(x)$). Without loss of generality, we can assume that the restriction $u|_X$ is the diagonal embedding $X \hookrightarrow X^{[0,1]}$.

Applying the discussion of Remark 13 to the space L, we see that if ϵ is sufficiently small, then any two points $v, v' \in L$ at distance $< \epsilon$ can be joined by a path $p_{v,v'}$ which depends continuously on v and v'. We can then define $r : U \to X$ by the formula

$$r(x,v) = u(x, p_{\pi_L(x),v})(1).$$

It is easy to see that r satisfies conditions (1) and (2), and condition (3) can be ensured by shrinking ϵ if necessary. □

Proof of Proposition 17. We first show that (2) $\Rightarrow$ (1) (we don't actually need this implication, but it is a pleasant characterization of the class of cell-like maps). We will show that each fiber X_y of f has trivial shape. Since X_y is nonempty, it suffices to show that for any CW complex S, any map $f_0 : X_y \to S$ is nullhomotopic. The map f_0 extends continuously to a map $f : V \to S$, where V is some open neighborhood of X_y in X. It will therefore suffice to show that the inclusion $X_y \hookrightarrow V$ is nullhomotopic. Choose ϵ small enough that $V \subseteq f^{-1}B_\epsilon(y)$, where $B_\epsilon(y)$ denotes a ball of radius ϵ about Y. Assumption (2) implies that there exists a map $g : Y \to X$ and an ϵ-small homotopy from id_X to $g \circ f$. This restricts to a homotopy from the inclusion map $X_y \hookrightarrow V$ to a constant map.

We now consider the interesting direction: the implication (1) $\Rightarrow$ (2). Since Y is compact, we can cover Y by finitely many balls of radius ϵ; let us denote those balls by $\{U_i\}_{i \in I}$. For every nonempty subset $J \subseteq I$, set $U_J = \bigcap_{i \in J} U_i$. For every chain $J_0 \subseteq \cdots \subseteq J_m$ of nonempty subsets of I, we will construct a map

$$G_{\vec{J}} : \Delta^m \times U_{J_m} \to f^{-1}U_{J_0}$$

and a homotopy

$$H_{\vec{J}} : \Delta^m \times f^{-1}U_{J_m} \to f^{-1}U_{J_0}$$

from the identity to $G_{\vec{J}} \circ f$. Moreover, we will choose these maps to be compatible with one another in the sense that if $\vec{J}' = (J'_0 \subseteq \cdots \subseteq J'_{m'})$ is another chain of nonempty subsets of I which is contained in $\vec{J}$, then $H_{\vec{J}}$ and $H_{\vec{J}'}$ agree on $\Delta^{m'} \times f^{-1}U_{J_m}$ (which implies that $G_{\vec{J}}$ and $G_{\vec{J}'}$ agree on $\Delta^{m'} \times U_{J_m}$). The construction proceeds by induction on the size of $\vec{J}$; at each stage, we are forced to extend a map over the inclusion

$$i : (\partial\,\Delta^m \times M) \amalg_{(\partial\,\Delta^m \times f^{-1}U_{J_m})} (\Delta^m \times f^{-1}U_{J_m}) \hookrightarrow \Delta^m \times M$$

where M denotes the mapping cylinder of the projection $f^{-1}U_{J_m} \to U_{J_m}$. To show that this extension is possible, it suffices to show that i admits a left inverse. This follows from the fact $f^{-1}U_{J_m}$ is a deformation retract of M (since the projection $f^{1}U_{J_m} \to U_{J_m}$ is a homotopy equivalence by virtue of our assumption that f is cell-like).

Let P denote the partially ordered set of nonempty subsets of I and let Δ denote the nerve of P. Then Δ is a topological simplex with vertices corresponding to the elements of I, and its presentation as the nerve of P gives a triangulation of Δ (given by barycentric subdivision) with one m-simplex $\sigma_{\vec{J}}$ for every chain $\vec{J} = (J_0 \subseteq \cdots \subseteq J_m)$ as above.

Choose a partition of unity $\{\lambda_i\}_{i \in I}$ on Y having the property that for each index i, the *closure* of the support of λ_i is contained in the open set U_i. We can regard the λ_i as defining a continuous map $\lambda : Y \to \Delta$.

Moreover, for each $y \in Y$ there exists a chain $\vec{J} = (J_0 \subseteq \cdots \subseteq J_m)$ such that $\lambda(y) \in \sigma_{\vec{J}} \simeq \Delta^m$ and $y \in U_{J_m}$. We define $G : Y \to X$ by the formula $G(y) = G_{\vec{J}}(y, \lambda(y))$. Similarly, for $x \in X$ with $f(x) = y$, we set $H(x,t) = H_{\vec{J}}(x, \lambda(y), t)$. It is not difficult to see that G and H are well-defined and have the desired properties. $\square$

References

[1] Hatcher, A. *Higher Simple Homotopy Theory.*

Fibrations of Simplicial Sets (Lecture 9)

September 22, 2014

In the previous lecture, we gave a concrete criterion for a PL map of finite polyhedra $f : E \to B$ to be a fibration. Our goal in this lecture is to translate that criterion into a more combinatorial language.

Notation 1. Given a partially ordered set P, we let $\mathrm{N}(P)$ denote the nerve of P (regarded as a simplicial set): the n-simplices of $\mathrm{N}(P)$ are chains $a_0 \leq \cdots \leq a_n$ of elements of P.

Let X be a finite nonsingular simplicial set. We let $\Sigma(X)$ denote the collection of all nondegenerate simplices of X. We regard $\Sigma(X)$ as a partially ordered set by identifying each simplex σ with the simplicial subset of X given by the image of the map $\sigma : \Delta^n \to X$. We let $\mathrm{Sd}(X)$ denote the nerve of the partially ordered set $\Sigma(X)$. We will refer to $\mathrm{Sd}(X)$ as the *subdivision* of X.

Remark 2. Every map of nonsingular simplicial sets $f : X \to Y$ induces a map $\Sigma(X) \to \Sigma(Y)$, which carries each simplex $\sigma \subseteq X$ to its image in Y. We will denote this image by $f(\sigma)$. Beware that there is some potential for confusion here: if σ is an n-simplex of X, then $f(\sigma)$ need not be the image of σ under the induced map $X_n \to Y_n$ (since the latter image might be degenerate in Y).

Exercise 3. Let X be a finite nonsingular simplicial set. For each simplex $\sigma \in \Sigma(X)$, let v_σ be a point of $|X|$ which belongs to the interior of $|\sigma|$. Show that there is a unique homeomorphism $\alpha_X : |\mathrm{Sd}(X)| \to |X|$ which carries each vertex $\sigma \in \Sigma(X)$ to the point v_σ and carries each simplex of $|\mathrm{Sd}(X)|$ linearly to a simplex of $|X|$.

Remark 4. Let $f : X \to Y$ be a map of nonsingular simplicial sets. Suppose that we have chosen a vertex $v_\sigma \in |X|$ belonging to the interior of each simplex $\sigma \in \Sigma(X)$, and similarly we have chosen a vertex $w_\tau \in |Y|$ in the interior of each simplex $\tau \in \Sigma(Y)$. Suppose further that the v_σ and w_τ are *compatible* in the sense that the map $|f|$ carries each v_σ to the point $w_{f(\sigma)}$ (note that it is always possible to arrange this: in fact, if the points w_τ have been chosen we can always choose vertices v_σ which are compatible with them). Then the diagram of spaces

$$\begin{tikzcd} |\mathrm{Sd}(X)| \arrow[r, "\alpha_X"] \arrow[d, "|\mathrm{Sd}(f)|"] & |X| \arrow[d, "|f|"] \\ |\mathrm{Sd}(Y)| \arrow[r, "\alpha_Y"] & |Y| \end{tikzcd}$$

commutes.

Warning 5. For any nonsingular simplicial set X, we can make a canonical choice of the points v_σ by taking each v_σ to be the barycenter of $|\sigma|$. The homeomorphisms $\alpha_X : |\mathrm{Sd}(X)| \to |X|$ obtained in this way are functorial for embeddings of nonsingular simplicial sets, but not for general maps of nonsingular simplicial sets.

Definition 6. Let $f : X \to Y$ be a map of finite simplicial sets. We will say that f is *cell-like* if the induced map of topological spaces $|f| : |X| \to |Y|$ is cell-like.

Proposition 7. *Let $f : X \to Y$ be a map of finite nonsingular simplicial sets. The following conditions are equivalent:*

(1) *The map f is cell-like.*

(2) *For every simplex $\sigma \in \Sigma(Y)$, the partially ordered set $\{\tau \in \Sigma(X) : f(\tau) = \sigma\}$ has weakly contractible nerve.*

(3) *The induced map of partially ordered set $\Sigma(X) \to \Sigma(Y)$ is left cofinal: in other words, for every simplex $\sigma \in \Sigma(Y)$, the partially ordered set $\{\tau \in \Sigma(X) : \sigma \subseteq f(\tau)\}$ has weakly contractible nerve.*

Proof. Let σ be a nondegenerate simplex of Y, and fix a point v in the interior of $|\sigma|$. We can extend $\{v\}$ to a choice of one point in the interior of each simplex of Y, and then make compatible choices for X to obtain a commutative diagram

$$\begin{tikzcd} |\operatorname{Sd}(X)| \arrow[r, "\alpha_X"] \arrow[d, "|\operatorname{Sd}(f)|"] & |X| \arrow[d, "|f|"] \\ |\operatorname{Sd}(Y)| \arrow[r, "\alpha_Y"] & |Y|. \end{tikzcd}$$

It follows that the inverse image $|f|^{-1}\{v\}$ is homeomorphic to the geometric realization of the nerve of the partially ordered set $\{\tau \in \Sigma(X) : f(\tau) = \sigma\}$. Consequently, f is cell-like if and only if every such nerves is weakly contractible, which proves that (1) and (2) are equivalent. The equivalence of (2) and (3) follows from the fact that the inclusion of partially ordered sets

$$\{\tau \in \Sigma(X) : f(\tau) = \sigma\} \hookrightarrow \{\tau \in \Sigma(X) : \sigma \subseteq f(\tau)\}$$

has a right adjoint (given by forming the intersection with $f^{-1}\sigma$). □

We next make Proposition 7 more explicit for the nerve of a map of partially ordered sets.

Definition 8. Let $f : P \to Q$ be a map of partially ordered sets. We will say that f is a *Cartesian fibration* if, for every pair of elements $q' \leq q$ in Q and every element $p \in P$ with $f(p) = q$, the set $\{a \in P : a \leq p \text{ and } f(a) \leq q'\}$ has a largest element p' satisfying $f(p') = q'$.

In this case, the construction $p \mapsto p'$ determines a map of partially ordered sets $P_q = f^{-1}\{q\} \to f^{-1}\{q'\} = P_{q'}$. These maps are transitive in the evident sense, so that we can regard $q \mapsto P_q$ as a (contravariant) functor from Q to the category of partially ordered sets.

Conversely, suppose we are given a contravariant functor from Q to the category of partially ordered sets which assigns to each $q \in Q$ a partially ordered set P_q and to each $q' \leq q$ a map $\alpha_{q',q} : P_q \to P_{q'}$. We can then regard $P = \bigcup_{q \in Q} P_q$ as a partially ordered set by declaring $p' \leq p$ for $p' \in P_{q'}$ and $p \in P_q$ if and only if $q' \leq q$ and $p' \leq \alpha_{q',q} p$. Then the evident map $P \to Q$ is a Cartesian fibration of partially ordered sets.

Exercise 9. Check that the constructions outlined in Definition 8 are mutually inverse, so that the data of a partially ordered set P with a Cartesian fibration $P \to Q$ is equivalent to the data of a contravariant functor from Q to the category of partially ordered sets.

Example 10. Let $f : X \to Y$ be a map of nonsingular simplicial sets. Then the induced map $\Sigma(X) \to \Sigma(Y)$ is a Cartesian fibrations of partially ordered sets: for every simplex $\sigma \subseteq X$ and every facet $\tau \subseteq f(\sigma)$, the collection of those simplices of X which are contained in σ and lie over τ has a maximal element, given by $\tau \cap f^{-1}(\sigma)$.

If $f : P \to Q$ is a Cartesian fibration of simplicial sets, then the nerve $\mathrm{N}(P)$ can be identified with the homotopy colimit of the diagram $q \mapsto \mathrm{N}(P_q)$. In particular, if each fiber $\mathrm{N}(P_q)$ is weakly contractible, then the map $\mathrm{N}(P) \to \mathrm{N}(Q)$ is a weak homotopy equivalence. In fact, we can be more precise:

Corollary 11. *Let $f : P \to Q$ be a Cartesian fibration of partially ordered sets. The following conditions are equivalent:*

(1) *For each $q \in Q$, the fiber $P_q = f^{-1}\{q\}$ has weakly contractible nerve.*

(2) *The map of simplicial sets* $\mathrm{N}(P) \to \mathrm{N}(Q)$ *is cell-like.*

Proof. The implication (2) $\Rightarrow$ (1) is obvious (and does not require f to be a Cartesian fibration). Let us prove the converse. Let σ be a nondegenerate simplex of $\mathrm{N}(Q)$, given by a chain $\{q_0 < \cdots < q_m\}$. We let S_σ denote the collection of nondegenerate simplices of $\mathrm{N}(P)$ which lie over σ: that is, the collection of chains $\{p_0 < \cdots < p_n\}$ in P whose image in Q is σ. We wish to show that $\mathrm{N}(S_\sigma)$ is weakly contractible. Note that if $\sigma' \subseteq \sigma$, then the construction $\tau \mapsto \tau \cap f^{-1}\sigma'$ determines a map of partially ordered sets $S_\sigma \to S_{\sigma'}$. This map is a Cartesian fibration (exercise!)

Let us now fix a simplex $\sigma = \{q_0 < \cdots < q_m\}$. For $0 \leq i \leq m$, set $\sigma_i = \{q_i < q_{i+1} < \cdots < q_m\}$, so that we have a tower of Cartesian fibrations

$$S_\sigma = S_{\sigma_0} \to \cdots \to S_{\sigma_m}.$$

The simplicial set $\mathrm{N}(S_{\sigma_m}) \simeq \mathrm{Sd}(\mathrm{N}(P_{q_m}))$ is weakly contractible by virtue of assumption (1). To complete the proof, it will suffice to show that each of the maps $\theta : S_{\sigma_i} \to S_{\sigma_{i+1}}$ induces a weak homotopy equivalence of nerves. To prove this, it suffices to show that each fiber of θ has weakly contractible nerve. Fix a simplex $\tau = \{p_0 < \ldots < p_n\}$ in P lying over σ_{i+1}. Unwinding the definitions, we see that $\theta^{-1}\{\tau\}$ is the partially ordered set of nonempty chains in $R = \{p \in P : f(p) = q_i \text{ and } p \leq p_0\}$. That is, we have $\mathrm{N}(\theta^{-1}\{\tau\}) \simeq \mathrm{Sd}\,\mathrm{N}(R)$. It will therefore suffice to show that $\mathrm{N}(R)$ is weakly contractible. In fact, R has a largest element by virtue of our assumption that f is a Cartesian fibration. □

Remark 12. In the situation of Corollary 11, the assumption that f is a Cartesian fibration implies that the inclusion maps $f^{-1}\{q\} \hookrightarrow \{p \in P : q \leq f(p)\}$ admits a right adjoint, and therefore induces a weak homotopy equivalence. Consequently, the map $\mathrm{N}(P) \to \mathrm{N}(Q)$ is cell-like if and only if f is left cofinal.

We now ask when a map $f : X \to Y$ of finite nonsingular simplicial sets induces a fibration $|f| : |X| \to |Y|$. For every simplex $\sigma \in \Sigma(Y)$, let X_σ be the nerve of the partially ordered set $\{\tau \in \Sigma(X) : f(\tau) = \sigma\}$. Using a variation of the constructions sketched in the previous lecture, we see that $|X|$ can be identified with the homotopy colimit of the diagram $\sigma \mapsto |X_\sigma|$: that is, it is obtained by gluing together the products $|X_\sigma| \times |\sigma|$ (this is actually a slight generalization of the situation described in the previous lecture, because the simplices of X need not determine a triangulation of $|X|$: the intersection of two simplices need not be a simplex). Here the transition maps $X_\sigma \mapsto X_{\sigma'}$ are induced by maps of partially ordered sets given by $\tau \mapsto \tau \cap f^{-1}\{\sigma'\}$. Since these maps are Cartesian fibrations, we obtain the following:

Proposition 13. *Let $f : X \to Y$ be a map of finite nonsingular simplicial sets. The following conditions are equivalent:*

(a) *The map $|f| : |X| \to |Y|$ is a fibration.*

(b) *For every inclusion $\sigma' \subseteq \sigma$ in $\Sigma(Y)$ and every simplex $\tau' \in \Sigma(X)$ with $f(\tau') = \sigma'$, the partially ordered set $\{\tau \in \Sigma(X) : f(\tau) = \sigma \text{ and } \tau' = \tau \cap f^{-1}\sigma'\}$ has weakly contractible nerve.*

Remark 14. Condition (b) of Proposition 13 can be regarded as a combinatorial version of the path lifting property of fibrations for paths which begin in the interior of σ' and then enter the interior of σ.

Combinatorial Models for $\mathcal{M}$ (Lecture 10)

September 24, 2014

Let $f : X \to Y$ be a map of finite nonsingular simplicial sets. In the previous lecture, we showed that the induced map $|f| : |X| \to |Y|$ is a fibration if and only if it satisfies the following combinatorial path lifting condition: for every simplex $\sigma_0 \in \Sigma(X)$ with image $\tau_0 = f(\sigma_0)$ in $\Sigma(Y)$ and every simplex $\tau \in \Sigma(Y)$ containing τ_0, the partially ordered set $\{\sigma \in \Sigma(X) : f(\sigma) = \tau \text{ and } \sigma_0 = \sigma \cap f^{-1}\tau_0\}$ is weakly contractible. Our first objective in this lecture is to apply this criterion in the special case where f arises from a map of partially ordered sets.

Proposition 1. *Let $f : P \to Q$ be a map of finite partially ordered sets. Assume that f is a Cartesian fibration. The following conditions are equivalent:*

(1) *The induced map $|\operatorname{N}(P)| \to |\operatorname{N}(Q)|$ is a fibration.*

(2) *For every $q' \leq q$ in Q, the induced map of fibers $P_q \to P_{q'}$ is left cofinal.*

Proof. Let $\operatorname{Chain}(P) = \Sigma(\operatorname{N}(P))$ denote the partially ordered set of nonempty chains in P, and similarly for Q. Fix a simplex $\sigma_0 \in \operatorname{Chain}(P)$ having image $\tau_0 \in \operatorname{Chain}(Q)$. For each chain τ in $\operatorname{Chain}(Q)$ containing τ_0, let

$$S_\tau = \{\sigma \in \operatorname{Chain}(P) | f(\sigma) = \tau \text{ and } \sigma \cap f^{-1}\tau_0 = \sigma_0\}.$$

The criterion of the previous lecture shows that (1) holds if and only if each of the partially ordered sets S_τ is weakly contractible.

Assume first that (1) is satisfied. Let $q' < q$ in Q; we wish to show that the map $P_q \to P_{q'}$ is left cofinal. For this, pick $p' \in P_{q'}$; we need to show that $T = \{p \in P_q : p' \leq p\}$ is weakly contractible. Taking $\sigma_0 = \{p'\}$, $\tau_0 = \{q'\}$, and $\tau = \{q', q\}$, we see that the set S_τ above can be identified with $\operatorname{Chain}(T)$. Condition (1) implies that S_τ is weakly contractible, so that T is likewise weakly contractible (since $\operatorname{N}(S_\tau) \simeq \operatorname{Sd}\operatorname{N}(T)$).

We now prove that (2) $\Rightarrow$ (1). Choose $\sigma_0 \in \operatorname{Chain}(P)$ and $\tau \in \operatorname{Chain}(Q)$ as above; we wish to show that S_τ is weakly contractible. If $\tau = \tau_0$, then there is nothing to prove. Otherwise, choose an element q which belongs to $\tau - \tau_0$ and set $\tau' = \tau - \{q\}$. Note that if we are given a simplex $\tau' \in \operatorname{Chain}(Q)$ with $\tau_0 \subseteq \tau' \subseteq \tau$, then the construction $\sigma \mapsto \sigma \cap f^{-1}\tau'$ determines a Cartesian fibration $S_\tau \to S_{\tau'}$. Proceeding inductively, we may assume that $S_{\tau'}$ is weakly contractible; it then suffices to show that the fibers of the map $S_\tau \to S_{\tau'}$ are weakly contractible. Replacing τ_0 by τ' we may reduce to the case where τ is obtained from τ_0 by adding a single element q.

Then τ corresponds to a chain

$$\{q_{-m} < \ldots < q_{-1} < q < q_1 < \cdots < q_n\}$$

where either m or n could be zero (but not both), and σ_0 corresponds to a chain $\{p_0 < \ldots < p_k\}$ lying over $\{q_{-m} < \ldots < q_{-1} < q_1 < \cdots < q_n\}$. Let p_- be the largest element of σ_0 which lies over q_{-1} (if $m \neq 0$) and let p_+ be the largest element of σ_0 which lies over q_1 (if $n \neq 0$). Unwinding the definitions, we see that $\operatorname{N}(S_\tau)$ can be identified with the subdivision of the nerve of the poset

$$\begin{cases} \{p \in P_q : p \leq p_+\} \text{ if } m = 0 \\ \{p \in P_q : p_- \leq p\} \text{ if } n = 0 \\ \{p \in P_q : p_- \leq p \leq p_+\} \text{ if } m, n \neq 0. \end{cases}$$

Since f is a Cartesian fibration, the partially ordered sets of the first and third type have largest elements. It will therefore suffice to consider the case where $n = 0$. Let $\alpha : P_q \to P_{q_{-1}}$ be the map induced by the inequality $q_{-1} < q$. Then the poset in question can be identified with $\{p \in P_q : \alpha(p) \geq p_-\}$, which is weakly contractible by virtue of our assumption that α is left cofinal. $\square$

Remark 2. It follows from Proposition 1 that if $f : X \to Y$ is a cell-like PL map of polyhedra, then X and Y are concordant (set $Q = \{0 < 1\}$ and apply Proposition 1 to the posets of simplices for compatible triangulations of X and Y). This *almost* proves that concordance of polyhedra is equivalent to simple homotopy equivalence (it would supply a complete proof if we had worked in the category of polyhedra at the outset, and considered only elementary expansions and elementary contractions in piecewise linear setting).

Let us now put Proposition 1 to work.

Definition 3. Let $\mathcal{C}_{\mathrm{cof}}$ denote the category whose objects are finite partially ordered sets and whose morphisms are left cofinal maps of partially ordered sets.

Consider the simplicial set $\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$. By definition, an n-simplex σ of $\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$ is a diagram of left cofinal maps

$$P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \leftarrow P_n$$

of finite partially ordered sets. In this case, the disjoint union $P(\sigma) = \bigcup_{0 \leq i \leq n} P_i$ can be regarded as a partially ordered set equipped with a Cartesian fibration $P(\sigma) \to [n]$ which satisfies the hypotheses of Proposition 1. It follows that the induced map $|\mathrm{N}(P(\sigma))| \to |\mathrm{N}([n])| \simeq \Delta^n$ is a piecewise linear fibration.

Let $\widetilde{\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})}$ be the simplicial set whose n-simplices consist of n-simplices σ of $\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$ together with a PL embedding $|\mathrm{N}(P(\sigma))| \hookrightarrow \mathbb{R}^\infty$. There is an evident projection map $\phi : \widetilde{\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})} \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$ given by forgetting the embeddings into $\mathbb{R}^\infty$, which is easily seen to be a trivial Kan fibration. We also have a map $\psi : \widetilde{\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})}) \to \mathcal{M}$ which is given by forgetting the diagram

$$P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_n$$

and remembering only the image of the map $|P(\sigma)| \to \Delta^n \times \mathbb{R}^\infty$. Composing ψ with a section of ϕ and using the canonical isomorphism $\mathcal{M} \simeq \mathcal{M}^{\mathrm{op}}$, we obtain a map of simplicial sets $\mathrm{N}(\mathcal{C}_{\mathrm{cof}}) \to \mathcal{M}$ which is well-defined up to homotopy.

Variant 4. Let $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}$ denote the category whose objects are finite nonsingular simplicial sets and whose morphisms are cell-like maps. The construction $X \mapsto \Sigma(X)$ determines a functor $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}} \to \mathcal{C}_{\mathrm{cof}}$. Composing with the construction of Definition 3, we obtain a map of simplicial sets $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) \to \mathcal{M}$.

Variant 5. Let $\mathcal{C}_{\mathrm{cell}}$ denote the subcategory of $\mathcal{C}_{\mathrm{cof}}$ whose objects are finite partially ordered sets and whose morphisms are Cartesian fibrations with weakly contractible fibers (note that any such map is left cofinal). Then the functor

$$\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}} \to \mathcal{C}_{\mathrm{cof}}$$

$$X \mapsto \Sigma(X)$$

factors through $\mathcal{C}_{\mathrm{cell}}$.

We can now state the first main theorem of this course:

Theorem 6. *The maps of simplicial sets*

$$\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) \xrightarrow{\alpha} \mathrm{N}(\mathcal{C}_{\mathrm{cell}}) \xrightarrow{\beta} \mathrm{N}(\mathcal{C}_{\mathrm{cof}}) \xrightarrow{\gamma} \mathcal{M}$$

are all weak homotopy equivalences.

Theorem 6 supplies several different "purely combinatorial" definitions of simple homotopy theory.

We begin by discussing the easy parts of Theorem 6.

Proposition 7. *The map $\alpha : \mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}}) \to \mathrm{N}(\mathcal{C}_{\mathrm{cell}})$ is a weak homotopy equivalence.*

Proof. In the previous lecture, we saw that a Cartesian fibration $P \to Q$ with weakly contractible fibers induces a cell-lie map $\mathrm{N}(P) \to \mathrm{N}(Q)$. Consequently, the construction $P \mapsto \mathrm{N}(P)$ can be regarded as a functor from $\mathcal{C}_{\mathrm{cell}}$ to $\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}}$. This functor defines a map of simplicial sets $\alpha' : \mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}}) \to \mathrm{N}(\mathcal{C}_{\mathrm{cell}})$. We will show that α' is homotopy inverse to α.

Consider first the composition $\alpha' \circ \alpha$, which is given by taking the nerve of the subdivision functor $\mathrm{Sd} : \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}} \to \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}}$. For any nonsingular simplicial set X, there is a canonical map $\mathrm{Sd}(X) \to X$, which carries a nondegenerate n-simplex of $\mathrm{Sd}(X)$ given by a chain

$$\sigma_0 \subseteq \cdots \subseteq \sigma_n$$

to the n-simplex of X given by the composite map

$$\Delta^n \xrightarrow{f} \sigma_n \to X,$$

where f carries the ith vertex of Δ^n to the last vertex of σ_i. We claim that this map is cell-like (so that it determines a homotopy from $\alpha' \circ \alpha$ to the identity). Invoking the criterion of the previous lecture, we are reduced to showing that the map of posets $\Sigma(\mathrm{Sd}(X)) \to \Sigma(X)$ has weakly contractible fibers. Unwinding the definitions, we see that the inverse image of a simplex $\sigma \in \Sigma(X)$ can be identified with the partially ordered set S of chains

$$\vec{\tau} = \{\tau_0 \subset \sigma_1 \subset \cdots \subset \tau_n\}$$

in $\Sigma(X)$ which have the property that $\sigma \subseteq \tau_n$ and the vertices of σ are precisely those vertices of τ_n which occur as the final vertex of some τ_i.

Let d be the dimension of σ and for $0 \leq i \leq d$ let $\sigma_i \subseteq \sigma$ be the facet spanned by the first $(i+1)$-vertices of σ. Let $S_i \subseteq S$ be the subset consisting of those chains $\vec{\tau}$ which satisfy $\tau_j = \sigma_j$ for $j \leq i$, and let $S'_i \subseteq S_i$ be the further subset consisting of those chains $\vec{\tau}$ where $\sigma_{i+1} \subseteq \tau_{i+1}$; by convention, we let $S'_{-1} = S$. Note that each S_i is a deformation retract of S'_{i-1} (by the construction $\vec{\tau} \mapsto \vec{\tau} \cup \{\sigma_i\}$) and that each S'_i is a deformation retract of S_i (by the construction $\vec{\tau} \mapsto \{\tau_j : j \leq i \text{ or } \sigma_{i+1} \subseteq \tau_j\}$. It follows that S is weakly homotopy equivalent to S_d and therefore weakly contractible (since S_d has a smallest element given by the chain $\{\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_d\}$.

Let us now consider the other composition $\alpha \circ \alpha'$, which is induced by the functor $\mathcal{C}_{\mathrm{cell}} \to \mathcal{C}_{\mathrm{cell}}$ given by $P \mapsto \mathrm{Chain}(P)$. We claim that this induces a map $\mathrm{N}(\mathcal{C}_{\mathrm{cell}}) \to \mathrm{N}(\mathcal{C}_{\mathrm{cell}})$ which is homotopic to the identity. To see this, we assign to each finite partially ordered set P another finite partially ordered set $T(P) = \{(p, \sigma) \in P \times \mathrm{Chain}(P) : (\forall p' \in \sigma) p \leq p'\}$. It is not difficult to see that T determines a functor from $\mathcal{C}_{\mathrm{cell}}$ to itself and that the projection maps

$$P \leftarrow T(P) \to \mathrm{Chain}(P)$$

are Cartesian fibrations with weakly contractible fibers (those fibers are posets of the form $\{q \in P : q \leq p\}$ and $\mathrm{Chain}(\{q \in P : q \geq p\})$ for some $p \in P$, respectively). □

Proposition 8. *The map $\beta : \mathrm{N}(\mathcal{C}_{\mathrm{cell}}) \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}})$ is a weak homotopy equivalence.*

The proof of Proposition 8 is a bit more involved. First, we recall that the subdivision $\mathrm{Sd}(X)$ can be defined for an arbitrary simplicial set by setting

$$\mathrm{Sd}(X) = \varinjlim_{\Delta^n \to X} \mathrm{Sd}(\Delta^n)$$

(this definition agrees with our earlier definition $\mathrm{Sd}(X) = \mathrm{N}(\Sigma(X))$ in the case where X is nonsingular). For any X, there is a canonical map $\rho_X : \mathrm{Sd}(X) \to X$ which is given by the colimit of the maps $\mathrm{Sd}(\Delta^n) \to \Delta^n$

which assign to each facet of Δ^n its final vertex. We saw in the proof of Proposition 7 that this map is a weak homotopy equivalence (in fact, even cell-like) when X is nonsingular. It follows formally (working simplex-by-simplex) that ρ_X is always a weak homotopy equivalence.

We now define a map $\delta : \mathrm{Sd}(\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})) \to \mathrm{N}(\mathcal{C}_{\mathrm{cell}}^{\mathrm{op}})$. To give such a map, we must associate to every n-simplex $e : \Delta^n \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$ a map $\mathrm{Sd}(\Delta^n) \to \mathrm{N}(\mathcal{C}_{\mathrm{cell}}^{\mathrm{op}})$, which we can identify with a functor $v : \Sigma(\Delta^n)^{\mathrm{op}} \to \mathcal{C}_{\mathrm{cell}}$. The simplex e is given by a diagram

$$P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \leftarrow P_n$$

of finite partially ordered sets and left cofinal maps. Let $P = \bigcup P_i$ be equipped with the partial ordering described in Definition 3. Given a pair of chains $\sigma, \sigma' \in \mathrm{Chain}(P)$, we will write $\sigma \leq \sigma'$ if $p \leq p'$ for every $p \in \sigma$ and $p' \in \sigma'$ (of course, it suffices to check this when p is the largest element of σ and p' is the least element of σ').

Let τ be an m-dimensional facet of Δ^n, corresponding to a set

$$\{i_0 < \cdots < i_m\} \subseteq \{0 < 1 < \cdots < n\}.$$

We define $v : \Sigma(\Delta^n)^{\mathrm{op}} \to \mathcal{C}_{\mathrm{cell}}$ by the formula

$$v(\tau) = \{(\sigma_0, \ldots, \sigma_m) \in \mathrm{Chain}(P_{i_0}) \times \cdots \times \mathrm{Chain}(P_{i_m}) | \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_m.\}$$

It is easy to see that every inclusion of facets $\tau' \subseteq \tau$ induces a Cartesian fibration $v(\tau) \to v(\tau')$, and we saw in Proposition 1 that such maps have weakly contractible fibers. This completes the construction of the map δ.

We claim that δ supplies a homotopy inverse to β. More precisely, we claim that the composite maps

$$\mathrm{Sd}(\mathrm{N}(\mathcal{C}_{\mathrm{cell}}^{\mathrm{op}})) \stackrel{\mathrm{Sd}(\beta)}{\to} \mathrm{Sd}(\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})) \stackrel{\delta}{\to} \mathrm{N}(\mathcal{C}_{\mathrm{cell}}^{\mathrm{op}})$$

and

$$\mathrm{Sd}(\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})) \stackrel{\delta}{\to} \mathrm{N}(\mathcal{C}_{\mathrm{cell}}^{\mathrm{op}}) \stackrel{\beta}{\to} \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$$

are homotopic to the natural maps

$$\rho_{\mathrm{N}(\mathcal{C}_{\mathrm{cell}}^{\mathrm{op}})} : \mathrm{Sd}(\mathrm{N}(\mathcal{C}_{\mathrm{cell}}^{\mathrm{op}})) \to \mathrm{N}(\mathcal{C}_{\mathrm{cell}}^{\mathrm{op}})$$

$$\rho_{\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})} : \mathrm{Sd}(\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})) \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}}).$$

To prove the second of these statements, we will construct a homotopy

$$h : \mathrm{Sd}(\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}}) \times \Delta^1) \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}}).$$

To define h, we need to supply for each n-simplex $e : \Delta^n \to \mathrm{Sd}(\mathrm{N}(\mathcal{C}_{\mathrm{cell}}^{\mathrm{op}})$ a functor $v : \Sigma(\Delta^n \times \Delta^1)^{\mathrm{op}} \to \mathcal{C}_{\mathrm{cell}}$. Let us regard e as given by a diagram

$$P_0 \leftarrow \cdots \leftarrow P_n$$

of left cofinal maps, and let $P = \bigcup P_i$ be defined as above. For $p \in P$ and $\sigma \in \mathrm{Chain}(p)$, we will write $p \leq \sigma$ if $p \leq q$ for each $q \in \sigma$. For $p \in P_i$ and $q \in P_j$, we will write $p \leftarrow q$ if $i \leq j$ and p is the image of q under the map $P_j \to P_i$ (note that this implies that $p \leq q$).

Let τ be a simplex of $\Delta^n \times \Delta^1$, which we can identify with a chain

$$\{(i_1, 0) < \cdots < (i_m, 0) < (j_1, 1) < \cdots < (j_{m'}, 1)\} \subseteq [n] \times [1].$$

We then define

$$v(\tau) = \{(p_1, \ldots, p_m, \sigma_1, \ldots, \sigma'_m) \in P_{i_1} \times \cdots \times P_{i_m} \times \mathrm{Chain}(P_{j_1}) \times \cdots \times \mathrm{Chain}(P_{j_{m'}}) | p_1 \leftarrow p_2 \leftarrow \cdots \leftarrow p_m \leq \sigma_1 \leq \cdots \leq \sigma_{m'}\}.$$

Every inclusion $\tau' \subseteq \tau$ induces a map of partially ordered sets $\theta : v(\tau) \rightarrow v(\tau')$. We claim that each of these maps is left cofinal. To prove this, we may assume without loss of generality that τ' is obtained from τ by omitting a single vertex. If this vertex has the form $(i_a, 0)$ for $a < m$, then θ is an isomorphism and there is nothing to prove. If the vertex has the form $(j_b, 1)$, then θ is a Cartesian fibration whose fibers are of the type analyzed in the proof of Proposition 1, and therefore weakly contractible. it therefore suffices to treat the case where τ' is obtained from τ by omitting the vertex $(i_m, 0)$ (in which case we must have $m > 0$). If $m = 1$, then θ is a Cartesian fibration whose fibers have the form $\{p \in P_{i_m} : p \leq p'\}$ for some $p' \in P_{i_m}$, and is therefore a weak homotopy equivalence. Let us therefore assume that $m > 1$. Fix an element of $v(\tau')$ given by a sequence $(p_1, \ldots, p_{m-1}, \sigma_1, \ldots, \sigma_m)$; we wish to show that the partially ordered set

$$S = \{(p'_1, \ldots, p'_m, \sigma'_1, \ldots, \sigma'_m) \in v(\tau) | p_{m-1} \leq p'_{m-1} \text{ and } \sigma_j \subseteq \sigma'_j\}$$

Let S' be the subset of S consisting of those tuples where $\sigma'_j = \sigma_j$ for all j. The inclusion $S' \hookrightarrow S$ admits a right adjoint and is therefore a weak homotopy equivalence. It will therefore suffice to show that S' is weakly contractible. Unwinding the definitions, we see that S' either has the form

$$\{p'_m \in P_{i_m} : p_{m-1} \leq p'_m \leq \sigma'_1\} \text{ or } \{p'_m \in P_{i_m} : p_{m-1} \leq p'_m\}$$

depending on whether or not $m' = 0$. In the former case, S' has a largest element; in the latter, it is weakly contractible by virtue of the left cofinality of $P_{i_m} \rightarrow P_{i_{m-1}}$.

The above analysis shows that the homotopy h is a well-defined map of simplicial sets. It follows from unwinding the definitions that the restriction of h to $\mathrm{Sd}(\mathrm{N}(\mathcal{C}^{op}_{cof}) \times \{0\})$ is given by $\rho_{\mathrm{N}(\mathcal{C}^{op}_{cof})}$ and the restriction of h to $\mathrm{Sd}(\mathrm{N}(\mathcal{C}^{op}_{cof}) \times \{0\})$ agrees with $\beta \circ \delta$.

To construct the other homotopy, it suffices to observe that if the maps

$$P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_n$$

are Cartesian fibrations with contractible fibers, then each of the maps $v(\tau) \rightarrow v(\tau')$ has the same property (arguing as above, this reduces easily to the case where τ' is obtained from τ by omitting the vertex $(i_m, 0)$ and where $m > 1$, in which case the desired result can be deduced from the assumption that the map $P_{i_m} \rightarrow P_{i_{m-1}}$ is a Cartesian fibration). Consequently, the restriction of h to $\mathrm{Sd}(\mathrm{N}(\mathcal{C}^{op}_{cell}) \times \Delta^1)$ factors through $\mathrm{N}(\mathcal{C}^{op}_{cell})$, and determines a homotopy from $\rho_{\mathrm{N}(\mathcal{C}^{op}_{cell})}$ to $\delta \circ \mathrm{Sd}(\beta)$.

Equivalence of the Combinatorial Definition (Lecture 11)

September 26, 2014

Our goal in this lecture is to complete the proof of our first main theorem by proving the following:

Theorem 1. *The map of simplicial sets*

$$\gamma : \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}}) \to \mathcal{M}$$

constructed in the previous lecture is a weak homotopy equivalence.

Here $\mathcal{C}_{\mathrm{cof}}$ denotes the category of finite partially ordered sets and left cofinal maps.

Recall that the simplicial set $\mathcal{M}$ is characterized by the following property: for any finite nonsingular simplicial set X, maps from X into $\mathcal{M}$ can be identified with finite polyhedra $E \subseteq |X| \times \mathbb{R}^\infty$ for which the projection map $E \to |X|$ is a fibration. In what follows, we will abuse notation by ignoring the embeddings into $\mathbb{R}^\infty$ and simply identifying maps $X \to \mathcal{M}$ with the corresponding PL fibration $E \to |X|$.

The first observation is that γ is surjective on connected components. To prove this, it suffices to observe that every finite polyhedron X is PL homeomorphic to the nerve of a finite partially ordered set P. In fact, we can take P to be the set $\Sigma(X)$ of simplices taken with respect to some chosen triangulation τ of X (so that the canonical triangulation of $\mathrm{N}(P)$ can be identified with the barycentric subdivision of the triangulation τ).

The preceding observation can be carried out in families. Let us first introduce a bit of terminology.

Definition 2. Let us say that a Cartesian fibration of partially ordered sets $f : P \to Q$ is *good* if, for every inequality $q \leq q'$ in Q, the induced map $P_{q'} \to P_q$ is left cofinal.

Remark 3. Let $f : P \to Q$ be a Cartesian fibration finite partially ordered sets. The following conditions are equivalent:

- (i) The map f is good.
- (ii) For every $p \in P$ and every $q \geq f(p)$ in Q, the poset $\{a \in P : f(a) = q \text{ and } a \geq p\}$ is weakly contractible.
- (iii) The induced map of topological spaces $|\mathrm{N}(P)| \to |\mathrm{N}(Q)|$ is a fibration.

Remark 4. Let $f : P \to Q$ be a good Cartesian fibration of finite partially ordered sets. Then the construction $(q \in Q) \mapsto P_q$ determines a functor $Q \to \mathcal{C}_{\mathrm{cof}}^{\mathrm{op}}$, hence a map of simplicial sets $\chi_f : \mathrm{N}(Q) \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$. The composite map

$$\mathrm{N}(Q) \xrightarrow{\chi_f} \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}}) \xrightarrow{\gamma} \mathcal{M}$$

classifies the PL fibration $|\mathrm{N}(P)| \to |\mathrm{N}(Q)|$.

Construction 5. Let $q : E \to B$ be a PL fibration of polyhedra. Suppose that we have chosen compatible triangulations of E and B, and let $\Sigma(E)$ and $\Sigma(B)$ denote the posets of simplices of E and B, respectively. Since q is a fibration, the map $\Sigma(E) \to \Sigma(B)$ is good; we therefore obtain a map $\chi_q : \mathrm{N}(\Sigma(B)) \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$, whose composition with γ classifies the PL fibration $|\mathrm{N}(\Sigma(E))| \to |\mathrm{N}(\Sigma(B))|$ which is isomorphic (in the category of polyhedra) to our original fibration q.

Note that the map χ_q of Construction 5 depends not only on q, but also on our chosen triangulations of E and B. In fact, even the domain of the map χ_q depends on the choice of triangulation. However, we would like to argue that this dependence is not essential. For this, we introduce the following definition:

Definition 6. Let X be a simplicial set and suppose that we are given a pair of maps $f : A \to X$, $f' : A' \to X$. We will say that f and f' are *homotopic* if there exists another map $\overline{f} : \overline{A} \to X$ and trivial cofibrations equivalences $i : A \to \overline{A}$ and $i' : A' \to \overline{A}$ such that $f = \overline{f} \circ i$ and $f' = \overline{f} \circ i'$. (In this case, we will refer to f' as a *homotopy* from f to f').

The proof of Theorem 1 rests on the following three propositions. First, we claim that Construction 5 is not very sensitive to our chosen triangulations:

Proposition 7. *Given diagram of triangulated polyhedra*

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "q'"] & E \arrow[d, "q"] \\ B' \arrow[r] & B \end{tikzcd}$$

where the vertical maps are fibrations which are compatible with our triangulations and the horizontal maps are PL homeomorphisms, the maps $\chi_q : \mathrm{N}(\Sigma(B)) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ and $\chi_{q'} : \mathrm{N}(\Sigma(B')) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ are homotopic to one another.

We next claim that Construction 5 is homotopy inverse to γ:

Proposition 8. *Let $P \to Q$ be a good Cartesian fibration of finite partially ordered sets, classified by a map $e : \mathrm{N}(Q) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ so that $\gamma(e)$ classifies the PL fibration $q : |\,\mathrm{N}(P)| \to |\,\mathrm{N}(Q)|$. Applying Construction 5 to the standard triangulations of $\mathrm{N}(P)$ and $\mathrm{N}(Q)$, we obtain a map $\chi_q : \mathrm{N}(\Sigma(\mathrm{N}(Q))) = \mathrm{N}(\mathrm{Chain}(Q)) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$. Then e and χ_q are homotopic.*

To state our next result, we need a bit of notation. For every finite polyhedron X, we let $[X]$ denote the associated vertex of $\mathcal{M}$. We will be a bit sloppy and identify $[X]$ with $[Y]$ when we are given a PL homeomorphism from X to Y (recall that to fully specify a vertex of $\mathcal{M}$, we need to supply an embedding of the corresponding polyhedron into $\mathbb{R}^\infty$, but we are suppressing this). Similarly, if P is a finite partially ordered set, we let $[P]$ denote the corresponding vertex of $\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$, so that $\gamma([P]) = [\mathrm{N}(P)]$.

If E is a finite polyhedron equipped with a triangulation, then Construction **??** (applied in the case $B = *$) determines a map $* \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ which is given by the vertex $[\Sigma(E)]$. If we are given another triangulation of the same polyhedron (which we will denote by E' to avoid confusion), then Proposition 7 determines a path p from $[\Sigma(E)]$ to $[\Sigma(E')]$ in the simplicial set $\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$. The image of this path under γ is a loop based at $[E] = [E']$.

Proposition 9. *In the situation above, the homotopy of Proposition 7 can be chosen so that $\gamma(p)$ is homotopic to the constant loop in $\mathcal{M}$.*

Let us now show that Propositions 7, 8, and 9 imply Theorem 1.

Proof of Theorem 1. Fix a finite polyhedron X and a triangulation of X. We will prove the following for each $n \geq 0$:

(*a*) If $\eta \in \pi_n(|\,\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})|, [\Sigma(X)])$ has the property that $\gamma(\eta)$ vanishes in $\pi_n(|\,\mathcal{M}\,|, [X])$, then e vanishes.

(*b*) Every element of $\pi_n(|\,\mathcal{M}\,|, [X])$ has the form $\gamma(\eta)$ for some $\eta \in \pi_n(|\,\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})|, [\Sigma(X)])$.

Note that when $n = 0$, condition (a) asserts that the connected component of $[\Sigma(X)]$ in $|\operatorname{N}(\mathcal{C}^{op}_{cof})|$ is the unique connected component which lies over the connected component of $[X]$ in $|\mathcal{M}|$. Allowing X to vary, it follows that the map $\pi_0|\operatorname{N}(\mathcal{C}^{op}_{cof})| \to \pi_0|\mathcal{M}|$ is bijective. For $n > 0$, condition (a) implies that the map of homotopy groups

$$\pi_n(|\operatorname{N}(\mathcal{C}^{op}_{cof})|, [\Sigma(X)]) \to \pi_n(|\mathcal{M}|, [X])$$

is injective and condition (b) implies that it is surjective. Theorem 1 will then follow from Whitehead's theorem.

Let us begin with the proof of (a). Suppose we are given a homotopy class $\eta \in \pi_n(|\operatorname{N}(\mathcal{C}^{op}_{cof})|, [\Sigma(X)])$ such that $\gamma(\eta)$ vanishes. The simplicial set $\operatorname{N}(\mathcal{C}^{op}_{cof})$ is not a Kan complex, so the class η cannot necessarily be represented by a map of simplicial sets from $\partial \Delta^{n+1}$ into $\operatorname{N}(\mathcal{C}^{op}_{cof})$. However, it can always be represented by a map of simplicial sets after subdividing $\partial \Delta^{n+1}$ sufficiently many times. We may therefore assume that η is represented by a map

$$f : \operatorname{N}(Q) \to \operatorname{N}(\mathcal{C}^{op}_{cof}),$$

where Q is some finite partially ordered set such that $|\operatorname{N}(Q)|$ is homeomorphic to S^n. The data of the map f is equivalent to the data of a functor $Q \to \mathcal{C}^{op}_{cof}$, which determines a PL fibration $q : |\operatorname{N}(P)| \to |\operatorname{N}(Q)|$.

Since $\gamma(\eta)$ vanishes, we can extend q to a PL fibration $\overline{q} : E \to B$ where $B \simeq D^{n+1}$ is contractible. Let us choose compatible triangulations of E and B, so that we have partially ordered sets of simplices $\Sigma(E)$ and $\Sigma(B)$. Refining these triangulations if necessary, we may assume that the images of $|\operatorname{N}(P)|$ and $|\operatorname{N}(Q)|$ are subcomplexes $E' \subseteq E$ and $B' \subseteq B$, and that the triangulations on E' and B' refine the triangulations of $\operatorname{N}(P)$ and $\operatorname{N}(Q)$, respectively. Construction 5 then yields a map of simplicial sets $\chi_{\overline{q}} : \operatorname{N}(\Sigma(B)) \to \operatorname{N}(\mathcal{C}^{op}_{cof})$. It follows from Propositions 7 and 8 that the restriction of $\chi_{\overline{q}}$ to $\operatorname{N}(\Sigma(B_0))$ is homotopic to our original map f. Consequently, $\chi_{\overline{q}}$ determines a homotopy from f to a constant map; this proves (a).

Let us now prove (b). Suppose we are given an element $\eta \in \pi_n(|\mathcal{M}|, [X])$, represented by a map $f : B = \partial \Delta^{n+1} \to \mathcal{M}$. This map classifies a PL fibration $q : E \to |B|$, whose fiber over some fixed base point $b \in |B|$ coincides with X. Choosing compatible triangulations of E and $|B|$ (and arranging that our triangulation of $|B|$ contains b as a vertex) and applying Construction 5, we obtain a map $\chi_q : \operatorname{N}(\Sigma(|B|)) \to \operatorname{N}(\mathcal{C}^{op}_{cof})$. It is not hard to see that $\gamma(\chi_q)$ is homotopic to our original map f. This *almost* proves (b): it shows that every map from an n-sphere into $|\mathcal{M}|$ is *freely* homotopic to a map which factors through $|\operatorname{N}(\mathcal{C}^{op}_{cof})|$. However, this only tells us that η is conjugate under the action of $\pi_1(|\mathcal{M}|, [X])$ to an element which lies in the image of the map

$$\pi_n(|\operatorname{N}(\mathcal{C}^{op}_{cof})|, [\Sigma(X)]) \to \pi_n(|\mathcal{M}|, [X]).$$

The problem is that the map χ_q need not be pointed: it carries the base point b to the partially ordered set $\Sigma(E_b)$ of simplices for the triangulation of E_b determined by our chosen triangulation of E, which might well differ from our original triangulation of X. However, Proposition 9 implies that there is a path from $[\Sigma(E_b)]$ to $[\Sigma(X)]$ whose image in $\mathcal{M}$ is homotopic to the identity, so that we can our free homotopy can be replaced by a based homotopy. $\square$

We now turn to the proofs of Propositions 7, 8, and 9. We will need a device for producing some homotopies.

Definition 10. Let $f : P \to Q$ be a Cartesian fibration of partially ordered sets. We will say that a pair $(p, p') \in P \times P$ is *Cartesian* if p is a largest element of $\{a \in P : a \leq p' \text{ and } f(a) \leq f(p)\}$.

Construction 11. Let $f : P \to P'$ be a map of partially ordered sets. We let $P' \amalg_f P$ denote the disjoint union of P and P', partially ordered so that there is a Cartesian fibration $g : P' \amalg_f P \to \{0 < 1\}$ with $P' = g^{-1}\{0\}$, $P = g^{-1}\{0\}$, and f the induced map from P to P'. In other words, the ordering on $P' \amalg_f P$ is defined so that $p' \leq p$ if and only if $p' \leq f(p)$ for $p \in P$ and $p' \in P'$.

Exercise 12. Suppose we are given a commutative diagram of partially ordered sets

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & P' \arrow[d] \\ Q \arrow[r, "g"] & Q'. \end{tikzcd}$$

Show that the induced map $P' \amalg_f P \to Q' \amalg_g Q$ is a Cartesian fibration if and only if the following conditions are satisfied:

(*i*) The map f is a Cartesian fibration.

(*ii*) The map g is a Cartesian fibration.

(*iii*) The map f carries Cartesian pairs in P to Cartesian pairs in P'.

Exercise 13. In the situation of Exercise 12, show that the map $P' \amalg_f P \to Q' \amalg_g Q$ is good if and only if the following conditions are satisfied:

(*i*) The map f is good.

(*ii*) The map g is good.

(*iii*) For each $q \in Q$, the induced map $P_q \to P'_{g(q)}$ is left cofinal.

Remark 14. Let $u : P \to Q$ be a good Cartesian fibration of finite partially ordered sets. Then the construction $q \mapsto P_q$ determines a functor $Q \to \mathcal{C}_{\text{cof}}^{\text{op}}$, hence a map of simplicial sets $\chi_u : \mathrm{N}(Q) \to \mathrm{N}(\mathcal{C}_{\text{cof}}^{\text{op}})$.

Suppose we are given a commutative diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "u"] & P' \arrow[d, "u'"] \\ Q \arrow[r, "g"] & Q'. \end{tikzcd}$$

which satisfies the hypotheses described in Example 12 and 13. We note that $\mathrm{N}(Q')$ is a deformation retract of $\mathrm{N}(Q' \amalg_g Q)$. Moreover, the inclusion $\mathrm{N}(Q) \hookrightarrow \mathrm{N}(Q' \amalg_g Q)$ is a weak homotopy equivalence if and only if g is a weak homotopy equivalence. In this case, the Cartesian fibration $P' \amalg_f P \to Q' \amalg_g Q$ is classified by a map $\mathrm{N}(Q' \amalg_g Q) \to \mathrm{N}(\mathcal{C}_{\text{cof}}^{\text{op}})$ which determines a homotopy from χ_u to $\chi_{u'}$.

Proof of Proposition 8. For every partially ordered set P, let $T(P) = \{(p, \sigma) \in P \times \mathrm{Chain}(P) | p \leq \min(\sigma)\}$. Note that if $u : P \to Q$ is a Cartesian fibration, then the induced map $T(P) \to T(Q)$ is also a Cartesian fibration, where a pair $((p, \sigma), (p', \sigma'))$ is Cartesian if and only if the pairs (p, p') and (σ, σ') are individually Cartesian. Moreover, if u is good then the map $T(P) \to T(Q)$ is good. For each pair $(q, \sigma) \in T(Q)$, the projection maps

$$\mathrm{Chain}(P)_\sigma \leftarrow T(P)_{(q,\sigma)} \to P_q$$

are Cartesian fibrations with weakly contractible fibers, hence left cofinal. It follows that the diagrams

$$\begin{tikzcd} \mathrm{Chain}(P) \arrow[d] & T(P) \arrow[l] \arrow[r] \arrow[d] & P \arrow[d] \\ \mathrm{Chain}(Q) & T(Q) \arrow[l] \arrow[r] & Q \end{tikzcd}$$

satisfy the hypotheses of Remark 14, so that the maps $\chi_u : \mathrm{N}(Q) \to \mathrm{N}(\mathcal{C}_{\text{cof}}^{\text{op}})$ and $\chi_{\mathrm{Chain}(u)} : \mathrm{N}(\mathrm{Chain}(Q)) \to \mathrm{N}(\mathcal{C}_{\text{cof}}^{\text{op}})$ are homotopic (since both are homotopic to $\chi_{T(u)}$). $\square$

Proof of Proposition 7. Let $q : E \to B$ be a PL fibration of polyhedra, and choose compatible triangulations of E and B; we will denote the simplices of these triangulations by $\Sigma(E)$ and $\Sigma(B)$, respectively. Suppose we are given PL homeomorphisms $E' \simeq E$ and $B' \simeq B$, where E' and B' are equipped with triangulations which refine the given triangulations on E and B (we can always arrange to be in this case, since any pair of (compatibly chosen) triangulations admits a common refinement). Set

$$P = \{(\sigma', \sigma) \in \Sigma(E') \times \Sigma(E) : \sigma' \subseteq \sigma\}$$

$$Q = \{(\tau', \tau) \in \Sigma(B') \times \Sigma(B) : \tau' \subseteq \tau\}.$$

The evident map $P \to Q$ is a Cartesian fibration. We claim that it is good: that is, for each $(\sigma'_0, \sigma_0) \in P$ and each $(\tau', \tau) \in Q$ containing $(q(\sigma'_0), q(\sigma_0))$, the partially ordered set

$$S = \{(\sigma', \sigma) \in P | q(\sigma') = \tau', q(\sigma) = \tau, \sigma'_0 \subseteq \sigma', \sigma_0 \subseteq \sigma\}$$

is weakly contractible. Note that since the map $\Sigma(E) \to \Sigma(B)$ is good, the partially ordered set $S' = \{\sigma \in \Sigma(E) | q(\sigma) = \tau, \sigma_0 \subseteq \sigma\}$ is weakly contractible. The projection map $S^{\mathrm{op}} \to S'^{\mathrm{op}}$ is Cartesian fibration, so it will suffice to show that it has weakly contractible fibers. The fiber over an element $\sigma \in S$ has the form

$$T = \{\sigma' \in \Sigma(E') | \sigma' \subseteq \sigma, q(\sigma') = \tau', \sigma'_0 \subseteq \sigma'\}.$$

This follows from the criterion of Lecture 9, since the projection map $\sigma \to \tau$ (a surjective map of simplices) is a fibration.

It is easy to see that the diagrams

$$\begin{tikzcd}
\Sigma(E) \arrow[d] & P \arrow[l] \arrow[r] \arrow[d] & \Sigma(E') \arrow[d] \\
\Sigma(B) & Q \arrow[l] \arrow[r] & \Sigma(B')
\end{tikzcd}$$

satisfy the criterion of Exercise 12. We claim that they also satisfy the criterion of Example 13. This amounts to two assertions:

- For every pair $(\tau', \tau) \in Q$, the natural map $P_{(\tau', \tau)} \to \Sigma(E)_\tau$ is left cofinal. For this, we must show that if $\sigma_0 \in \Sigma(E)$ satisfies $q(\sigma_0) = \tau$, then the partially ordered set

  $$S = \{(\sigma', \sigma) \in P : q(\sigma') = \tau', q(\sigma) = \tau, \sigma_0 \subseteq \sigma\}$$

  is weakly contractible. Let $S' = \{\sigma \in \Sigma(E) : q(\sigma) = \tau, \sigma_0 \subseteq \sigma\}$. Then S' is weakly contractible (it has a smallest element) and the map $S^{\mathrm{op}} \to S'^{\mathrm{op}}$ is a Cartesian fibration. It will therefore suffice to show that the fibers of $S \to S'$ are weakly contractible. That is, we must show that if $\sigma \in S'$, then the partially ordered set $\{\sigma' \in \Sigma(E') : \sigma' \subseteq \sigma, f(\sigma') = \tau'\}$ is weakly contractible. Again, this follows by applying our combinatorial criterion for fibrations to the map $\sigma \to \tau$.

- For every pair $(\tau', \tau) \in Q$, the natural map $P_{(\tau', \tau)} \to \Sigma(E')_{\tau'}$ is left cofinal. This map is a Cartesian fibration; it will therefore suffice to show that it has weakly contractible fibers. In other words, we must show that for $\sigma' \in \Sigma(E')$ with $q(\sigma') = \tau'$, the partially ordered set $S = \{\sigma \in \Sigma(E) : q(\sigma) = \tau, \sigma' \subseteq \sigma\}$ is weakly contractible. Let σ_0 be the smallest simplex of E which contains σ', so we can write $S = \{\sigma \in \Sigma(E) : q(\sigma) = \tau, \sigma_0 \subseteq \sigma\}$. The weak contractibility of this partially ordered set follows from the fact that the map $E \to B$ is a fibration.

Applying Remark 14, we deduce that the good Cartesian fibrations

$$\Sigma(E) \to \Sigma(B) \qquad \Sigma(E') \to \Sigma(B')$$

determine homotopic maps $\mathrm{N}(\Sigma(B)) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$, $\mathrm{N}(\Sigma(B')) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$. $\square$

Proof of Proposition 9. We will show that the homotopy constructed in the proof of Proposition 7 has the desired property. Suppose we are given a PL homeomorphism of triangulated polyhedra $E' \to E$; as before, we may assume without loss of generality that the triangulation of E' refines the triangulation of E. Let $P = \{(\sigma', \sigma) \in \Sigma(E') \times \Sigma(E) : \sigma' \subseteq \sigma\}$ be defined as in the proof of Proposition 7 (in the special case $B = B' = *$). Let $\rho_1 : P \to \Sigma(E')$ and $\rho_2 : P \to \Sigma(E)$ be the projections onto the first and second factor, respectively. Then the partially ordered sets $Q = \Sigma(E) \amalg_{\rho_1} P$ and $Q' = \Sigma(E') \amalg_{\rho_2} P$ determine paths from $[\Sigma(E)]$ to $[P]$ and $[\Sigma(E')]$ to $[P]$ in $|\operatorname{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})|$; we wish to prove that the images of these paths in $\mathcal{M}$ (which go from $[E]$ to $[|\operatorname{N}(P)|]$ are homotopic to one another. Let us denote these images by α and α', respectively.

We now construct some auxiliary partially ordered sets $\overline{Q}$ and $\overline{Q}'$ enlarging Q and Q', respectively. Let $\Sigma'(E)$ denote another copy of the partially ordered set $\Sigma(E)$ (given a different name to avoid confusion) and set

$$\overline{Q}' = Q' \amalg \Sigma'(X) = \Sigma(E') \amalg P \amalg \Sigma'(E),$$

equipped with the partial ordering where $q \leq q'$ if only if one of the following conditions holds:

- q and q' belong to $\Sigma(E')$ and $q \subseteq q'$.
- q and q' belong to P and $q \leq q'$ in P.
- q and q' belong to $\Sigma'(E)$ and $q \subseteq q'$.
- $q \in \Sigma(E')$, $q' = (\sigma', \sigma) \in P$, and $q \leq \sigma'$.
- $q \in \Sigma(E')$, $q' \in \Sigma'(X)$, and $q \subseteq q'$.
- $q = (\sigma', \sigma) \in P$, $q' \in \Sigma'(E)$, and $\sigma \subseteq q'$.

Let $\overline{Q} = Q \amalg \Sigma'(E) = \Sigma(E) \amalg P \amalg \Sigma'(E)$, which we regard as a partially ordered set in an analogous way. Note that we have evident maps of partially ordered sets

$$\overline{Q} \to \{0 < 1 < 2\} \leftarrow \overline{Q}'$$

Claim 15. *The PL maps*

$$|\operatorname{N}(\overline{Q})| \xrightarrow{u} \Delta^2 \xleftarrow{v} |\operatorname{N}(\overline{Q}')|$$

are fibrations.

For $i, j \in \{0, 1, 2\}$, let $\Delta^{\{i,j\}}$ denote the edge of Δ^2 containing i and j. It follows from Claim 15 that u and v are classified by maps $\overline{\alpha}, \overline{\alpha}' : \Delta^2 \to \mathcal{M}$ whose restrictions to $\Delta^{\{0,1\}}$ are the paths α and α', respectively. By construction, the inverse images $u^{-1}\Delta^{\{1,2\}}$ and $v^{-1}\Delta^{\{1,2\}}$ are canonically isomorphic and therefore determine the same path in $\mathcal{M}$. Consequently, to show that α and α' are homotopic, it suffices to show that the restrictions of $\overline{\alpha}$ and $\overline{\alpha}'$ to $\Delta^{\{0,2\}}$ are homotopic. Note that the inverse image $u^{-1}\Delta^{\{0,2\}}$ is canonically isomorphic to $E \times [0,1]$, so that the restriction of $\overline{\alpha}$ to $\Delta^{\{0,2\}}$ is constant. To complete the proof, it suffices to show that the restriction of $\overline{\alpha}'$ to $\Delta^{\{0,2\}}$ is a nullhomotopic path from $[E']$ to $[E]$ (which we identify with each other). This can be established by showing that the the PL homeomorphism $E' \to E$ extends to a PL homoemorphism

$$v^{-1}\Delta^{\{0,2\}} = \operatorname{N}(\Sigma(E') \amalg \Sigma(E)) \to E \times [0,1]$$

which restricts to the identity over the point $1 \in [0,1]$ (the relevant homeomorphism can be constructed by defining it on vertices and extending linearly). $\square$

Proof of Claim 15. We will prove that the map v is a fibration; the proof for u is similar (and slightly easier). Let $g : \overline{Q}' \to \{0, 1, 2\}$ be the projection map. For each nonempty subset $I \subseteq \{0, 1, 2\}$, we let $\operatorname{Chain}_I(\overline{Q}')$ denote the partially ordered set of chains in $\overline{Q}'$ whose image in $\{0, 1, 2\}$ is I. Recall that to show that v is a fibration, it will suffice to show that for $I \subseteq J$ the restriction map $\theta : \operatorname{Chain}_J(\overline{Q}') \to \operatorname{Chain}_I(\overline{Q}')$ has weakly contractible fibers. Moreover, we may assume without loss of generality that I is obtained from J by removing a single element. If $J = \{0,1\}$, the desired result was established in Proposition 8. If $J = \{0,2\}$ or $J = \{1,2\}$, then the map $g^{-1}J \to J$ is the opposite of a Cartesian fibration; it will therefore suffice to show (by the opposite of our previous results) that the induced maps $\Sigma(E') \to \Sigma'(E)$ and $P \to \Sigma'(E)$ are right cofinal. The first statement is equivalent to the assertion that for $\sigma \in \Sigma'(E)$, the partially ordered set $\{\sigma' \in \Sigma(E') : \sigma' \subseteq \sigma\}$ is weakly contractible: this is clear, since the geometric realization of this poset is homeomorphic to σ. For the second statement, we observe that $P^{\mathrm{op}} \to \Sigma'(E)^{\mathrm{op}}$ is a Cartesian fibration; we are therefore reduced to proving that it has constructible fibers, and those fibers are again of the form $\{\sigma' \in \Sigma(E) : \sigma' \subseteq \sigma\}$.

It remains to treat the case where $J = \{0,1,2\}$ and $I = J - \{i\}$ for some $i \in \{0,1,2\}$. In the case $i = 0$ each fiber of θ has the form $\mathrm{Chain}(S)$ where S has a largest element, and in the case $i = 2$ each fiber of θ has the form $\mathrm{Chain}(S)$ where S has a smallest element. In the case $i = 1$, each fiber of θ has the form $\mathrm{Chain}(S)$ where S has the form

$$\{(\sigma', \sigma) \in \Sigma(E') \times \Sigma(E) : \sigma'_0 \subseteq \sigma' \subseteq \sigma \subseteq \sigma_0\}$$

for some fixed pair of simplices $\sigma'_0 \in \Sigma(E')$, $\sigma_0 \in \Sigma(E)$ with $\sigma'_0 \subseteq \sigma_0$. Let S' be the subset of S consisting of those pairs (σ', σ) where $\sigma' = \sigma_0$. The inclusion $S' \hookrightarrow S$ admits a right adjoint and is therefore a weak homotopy equivalence. It therefore suffices to show that S' is weakly contractible, which follows from the observation that S' has a largest element. $\square$

Some Loose Ends (Lecture 12)

September 29, 2014

Earlier in this course, we stated the following result:

Theorem 1. *Let X and Y be finite polyhedra and let $f : X \to Y$ be a homotopy equivalence. The following conditions are equivalent:*

(1) *The map f is a simple homotopy equivalence.*

(2) *There exists a concordance $q : E \to [0,1]$ from X to Y such that the induced homotopy equivalence $X \simeq q^{-1}\{0\} \to q^{-1}\{1\} \simeq Y$ is homotopic to f.*

The implication (2) $\Rightarrow$ (1) was proved in Lecture 6. We are now *almost* in a position to prove the converse. Let us begin by introducing a bit of (nonstandard) terminology.

Definition 2. Let X be a polyhedron and let D^n be the standard n-disk (regarded as a finite polyhedron). Given a PL embedding $e : D^n \hookrightarrow X$, the mapping cylinder $M(e) = X \amalg_{D^n} (D^n \times [0,1])$ has the structure of a polyhedron and is equipped with a cell-like PL map $\pi : M(e) \to X$.

We will say that a PL map of finite polyhedra $Y \to X$ is a *polyhedral elementary contraction* if it has the form $\pi : M(e) \to X$ for some PL embedding $e : D^n \hookrightarrow X$, and a *polyhedral elementary expansion* if it is isomorphic to the canonical inclusion $X \hookrightarrow M(e)$ for some PL embedding $e : D^n \hookrightarrow X$. We say that f is a *polyhedral simple homotopy equivalence* if it is homotopic to a composition of polyhedral elementary expansions and polyhedral elementary contractions.

Claim 3. *Let $f : X \to Y$ be a map of finite polyhedra. Then f is a simple homotopy equivalence if and only if it is a polyhedral simple homotopy equivalence.*

To prove Claim 3, it suffices to show that f is a polyhedral simple homotopy equivalence if and only if it has vanishing Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1 X)$ (in the case where X is connected). This can be established by carrying out the argument sketched in Lecture 4 entirely in the setting of polyhedra.

Assuming Claim 3, we can prove the implication (1) $\Rightarrow$ (2) of Theorem 1. For this, it suffices to show that any polyhedral elementary contraction $f : X \to Y$ is a homotopy equivalence which arises from a concordance between X and Y. This is a special case of the following:

Proposition 4. *Let X and Y be finite polyhedra and let $f : X \to Y$ be a cell-like PL map. Then there exists a concordance $q : E \to [0,1]$ from X to Y such that the induced homotopy equivalence $X \simeq q^{-1}\{0\} \to q^{-1}\{1\} \simeq Y$ is homotopic to f.*

Proof. Choose triangulations of X and Y that are compatible with f, so that f induces a Cartesian fibration $q : \Sigma(X) \to \Sigma(Y)$. Since f is cell-like, the fibers of q are weakly contractible. We complete the proof by setting $E = \mathrm{N}(\Sigma(Y) \amalg_q \Sigma(X))$ (which we proved to be a concordance in Lecture 10). $\square$

Remark 5. In the second part of this course, we will give an alternative approach Theorem 1, which does not require revisiting Lecture 4 in the polyhedral setting.

In Lectures 10 and 11, we constructed a weak homotopy equivalence $\rho : \mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}})^{\mathrm{op}} \to \mathcal{M}$, where $\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}}$ is the category whose objects are finite nonsingular simplicial sets and whose morphisms are cell-like maps. We next show that the condition of nonsingularity is not essential.

Definition 6. Let $f : X \to Y$ be a map of finite simplicial sets (not necessarily nonsingular). We will say that f is *cell-like* if the induced map of topological spaces $|X| \to |Y|$ is cell-like (which is equivalent to the assertion that the fibers of $|f|$ are contractible).

We let $\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{cl}}$ denote the category whose objects are finite simplicial sets and whose morphisms are cell-like maps.

Remark 7. Suppose we are given a pullback diagram of finite simplicial sets

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y. \end{tikzcd}$$

Then the associated diagram of geometric realizations is also a pullback diagram. It follows that f' is cell-like if f is; the converse holds if g is surjective.

Definition 8. Let X be a finite simplicial set. A *desingularization* of X is a finite nonsingular simplicial set Y equipped with a cell-like map $\pi : Y \to X$. We let $\mathcal{D}(X)$ denote the category whose objects are desingularizations $\pi : Y \to X$, and whose morphisms are commutative diagrams of cell-like maps

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "\pi"] & & Y' \arrow[dl, "\pi'"'] \\ & X. & \end{tikzcd}$$

We will prove the following:

Proposition 9. *For every finite simplicial set X, the category $\mathcal{D}(X)$ is weakly contractible.*

Corollary 10. *The inclusion functor $\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}} \hookrightarrow \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{cl}}$ induces a homotopy equivalence $|\,\mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}})| \hookrightarrow |\,\mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{cl}})|$.*

Proof. Combine Proposition 8 with Quillen's Theorem A. $\square$

Proposition 8 is an easy consequence of the following special case:

Lemma 11. *Let X be a finite simplicial set. Then there exists a desingularization of X.*

Proof of Proposition 8. Fix a desingularization $\pi : Y_0 \to X$. We define a functor $T : \mathcal{D}(X) \to \mathcal{D}(X)$ by the formula $T(Y) = Y \times_X Y_0$; note that $T(Y)$ is nonsingular since it is contained in the nonsingular simplicial set $Y \times Y_0$, and Remark 6 implies that $T(Y)$ is again a desingularization of X. The evident maps $Y \leftarrow T(Y) \to Y_0$ show that the identity map $\mathrm{id} : |\,\mathrm{N}(\mathcal{D})| \to |\,\mathrm{N}(\mathcal{D})|$ is homotopic to the constant map taking the value Y_0. $\square$

Proof of Lemma 10. We proceed by induction on the number of nondegenerate simplices of X. If X is empty, there is nothing to prove; otherwise, we can write X as a pushout $X' \amalg_{\partial\,\Delta^n} \Delta^n$. The inductive hypothesis the implies that there exists a desingularization $Y' \to X'$. Set $K = Y' \times_{X'} \partial\,\Delta^n$; note that K is nonsingular since it is contained in the product $Y' \times \partial\,\Delta^n$. Choose an embedding $K \hookrightarrow C(K)$, where $C(K)$ is nonsingular and weakly contractible (for example, we can take $C(K)$ to be the join $K \star \Delta^0$). The diagonal map $K \to \Delta^n \times C(K)$ is then an embedding so that the mapping cylinder

$$M = K \times \Delta^1 \amalg_{K \times \{1\}} \Delta^n \times C(K)$$

is nonsingular. We have a commutative diagram of simplicial sets

$$\begin{tikzcd} Y' \times C(K) \arrow[d, "\alpha"] & K \arrow[l] \arrow[r] \arrow[d, "\beta"] & M \arrow[d, "\gamma"] \\ X' & \partial \Delta^n \arrow[l] \arrow[r] & \Delta^n. \end{tikzcd}$$

Note that the upper horizontal maps are injective so that the pushout $Y = (Y' \times C(K)) \amalg_K M$ is a nonsingular simplicial set. We claim that the canonical map $\pi : Y \to X$ is cell-like. To prove this, choose any point $x \in |X|$; we claim that the inverse image $\pi^{-1}\{x\} \subseteq |Y|$ is contractible. If x does not belong to $|X'|$, then we have $\pi^{-1}\{x\} \simeq \gamma^{-1}\{x\}$, which is contractible, since γ is the composition of cell-like maps

$$M \to \Delta^n \times C(K) \to \Delta^n.$$

If x belongs to $|X'|$, let $Z \subseteq |\partial \Delta^n|$ be the inverse image of x; then $\pi^{-1}\{x\}$ is given by the pushout

$$\alpha^{-1}\{x\} \amalg_{\beta^{-1}Z} \gamma^{-1}Z.$$

The map α is a composition of cell-like maps

$$Y' \times C(K) \to Y' \to X',$$

so that $\alpha^{-1}\{x\}$ is contractible. It will therefore suffice to show that the inclusion $\beta^{-1}Z \hookrightarrow \gamma^{-1}Z$ is a homotopy equivalence. This is clear, since $\gamma^{-1}Z$ is homeomorphic to the mapping cylinder of $\beta^{-1}Z \to Z \times C(K)$ which is a homotopy equivalence since β is cell-like. $\square$

Since $\mathcal{M}$ is a Kan complex, it follows from Corollary 9 that the map $\rho : \mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}})^{\mathrm{op}} \to \mathcal{M}$ admits an extension $\overline{\rho} : \mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{cl}})^{\mathrm{op}} \to \mathcal{M}$. It is not obvious how to construct such an extension explicitly because there there is no functorial way to endow the geometric realization $|X|$ with the structure of a polyhedron for an arbitrary finite simplicial set X. However, if we do not insist on working with polyhedra, then this problem goes away:

Definition 12. Let $Q = \prod_{n \geq 0} [0, 1]$ be the Hilbert cube (or any other "sufficiently large" contractible space). We define a simplicial set $\mathcal{M}^+$ as follows: the n-simplices of $\mathcal{M}^+$ are subsets $E \subseteq \Delta^n \times Q$ with the following properties:

- As a topological space, E is a compact absolute neighborhood retract.
- The projection $E \to \Delta^n$ is a fibration.

Any choice of embedding $\mathbb{R}^\infty \hookrightarrow Q$ determines a map of simplicial sets $\mathcal{M} \hookrightarrow \mathcal{M}^+$; roughly speaking, this map is given by "forgetting" PL structures.

Example 13. Suppose we are given maps of topological spaces

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n.$$

The *mapping simplex* of the sequence $\{X_i\}$ is defined to be the iterated pushout

$$X_0 \amalg_{X_1} (X_1 \times \Delta^1) \amalg_{X_2 \times \Delta^1} (X_2 \times \Delta^2) \amalg \cdots \amalg (X_n \times \Delta^n).$$

We will denote this mapping simplex by $M(X_0 \leftarrow \cdots \leftarrow X_n)$. There is an evident map

$$\pi : M(X_0 \leftarrow \cdots \leftarrow X_n) \to \Delta^n.$$

If each X_i is a compact ANR, one can show that $M(X_0 \leftarrow \cdots \leftarrow X_n)$ is a compact ANR. If, in addition, each of the maps $X_i \to X_{i-1}$ is cell-like, then the map π is a fibration: this follows from the main result of Lecture 8.

Construction 14. Suppose we are given an n-simplex σ of $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{cl}})$, given by a sequence of cell-like maps

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n.$$

Choosing an embedding of each $|X_i|$ into Q, we can regard the mapping simplex $M(|X_0| \leftarrow \cdots \leftarrow |X_n|)$ as a subset of $\Delta^n \times Q$ which defines an n-simplex of $\mathcal{M}^+$. This determines for us a map of simplicial sets

$$\rho' : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{cl}})^{\mathrm{op}} \to \mathcal{M}^+ .$$

The maps ρ and ρ' are closely related:

Proposition 15. *The diagram of simplicial sets*

$$\begin{tikzcd} \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}})^{\mathrm{op}} \arrow[r, "\rho"] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{cl}})^{\mathrm{op}} \arrow[r, "\rho'"] & \mathcal{M}^+ \end{tikzcd}$$

commutes up to homotopy.

Remark 16. In the final portion of this course, if we get there, we will show that the inclusion $\mathcal{M} \hookrightarrow \mathcal{M}^+$ is a homotopy equivalence (so that the diagram of Proposition 14 consists of homotopy equivalences).

Let us sketch the proof of Proposition 14. Suppose we are given an n-simplex σ of $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}})^{\mathrm{op}}$, given by a sequence of cell-like maps

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n$$

of *nonsingular* simplicial sets. We will abuse notation by identifying σ with its image in $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{cl}})^{\mathrm{op}}$ and $\rho(\sigma)$ with its image in $\mathcal{M}^+$. We wish to relate $\rho(\sigma)$ to $\rho'(\sigma)$. Both can be identified with spaces which are fibered over the topological n-simplex Δ^n, given by the geometric realizations of certain finite simplicial sets: in the former case, we use the simplicial set

$$Y = \mathrm{N}(\Sigma(X_0) \amalg \Sigma(X_1) \amalg \cdots \amalg \Sigma(X_n)),$$

and in the latter case we use the mapping simplex

$$Z = M(X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n)$$

(given by carrying out the construction of Example 12 in the category of simplicial sets rather than the category of topological spaces). Let W denote the simplicial set

$$M(\mathrm{Sd}(X_0) \leftarrow \mathrm{Sd}(X_1) \leftarrow \mathrm{Sd}(X_2) \leftarrow \cdots \leftarrow \mathrm{Sd}(X_n)).$$

Amalgamating the maps $\mathrm{Sd}(X_i) \to X_i$, we obtain a map $W \to Z$. There is also a canonical map $W \to Y$, given by amalgamating maps

$$\mathrm{Sd}(X_i) \times \Delta^i = \mathrm{N}(\Sigma(X_i) \times \{0, \ldots i\}) \xrightarrow{\theta_i} \mathrm{N}(\Sigma(X_0) \amalg \cdots \Sigma(X_n)),$$

where θ_i is given by natural map of posets carrying $\Sigma(X_i) \times \{j\}$ to $\Sigma(X_j)$ for $0 \leq j \leq i$.

Claim 17. *In the situation above, the maps $Y \leftarrow W \to Z$ are cell-like.*

Proof. The fact that the map $W \to Z$ is cell-like follows from the fact that each of the maps $\mathrm{Sd}(X_i) \to X_i$ is cell-like. To prove that the map $W \to Y$ is cell-like, we must show that for each point $y \in |Y|$ the fiber $F = |W| \times_{|Y|} \{y\}$ is cell-like. Without loss of generality we may assume that the image of y in Δ^n does not belong to Δ^{n-1} (otherwise, we can simply truncate our sequence of simplicial sets $\{X_i\}$). In this case, the space F is also the fiber of the map

$$\theta_n : \mathrm{N}(\Sigma(X_n) \times [n]) \to \mathrm{N}(\Sigma(X_0) \amalg \cdots \amalg \Sigma(X_n)).$$

Since the underlying map of posets is a Cartesian fibration, it suffices to check that the fibers of θ_n are weakly contractible: this follows from the fact that each of the maps $\Sigma(X_n) \to \Sigma(X_i)$ has weakly contractible fibers. $\square$

Consider now the 2-sided mapping cylinder

$$Y \amalg_{W \times \{0\}} (W \times \Delta^1) \amalg_{W \times \{1\}} Z.$$

This is a simplicial set whose geometric realization is equipped with a canonical map to $\Delta^n \times \Delta^1$; using Claim 16 and the results of Lecture 8, one can show that this map is a fibration. Choosing an embedding into the big contractible space Q, we obtain a map $\Delta^n \times \Delta^1 \to \mathcal{M}^+$ which is a homotopy from $\rho(\sigma)$ to $\rho'(\sigma)$. It is easy to see that these homotopies can be chosen to be compatible as σ varies and therefore supply a proof of Proposition 14.

Homotopy Types and Simple Homotopy Types (Lecture 13)

September 30, 2014

In the first part of this course, we introduced a simplicial set $\mathcal{M}$ whose n-simplices are finite polyhedra $E \subseteq \Delta^n \times \mathbb{R}^\infty$ for which the projection map $E \to \Delta^n$ is a fibration. We can think of $\mathcal{M}$ as a "moduli space of simple homotopy types": we have seen that two polyhedra belong to the same path component of $\mathcal{M}$ if and only if they are simple homotopy equivalent.

Let us now consider a variant of the simplicial set $\mathcal{M}$ where all finiteness conditions are replaced by homotopy-theoretic finiteness conditions.

Definition 1. Let Q be the Hilbert cube (or some other suitably enormous contractible space). We let $\mathcal{M}^h$ be the simplicial set whose n-simplices are subspaces $E \subseteq \Delta^n \times Q$ for which the projection map $E \to \Delta^n$ is a fibration whose fibers are finitely dominated.

Remark 2. The simplicial set $\mathcal{M}^h$ is a Kan complex. For any simplicial set B, the set of homotopy classes of maps from B to $\mathcal{M}^h$ can be identified with the set of homotopy equivalence classes of fibrations $E \to B$ (in the category of simplicial sets) or $E \to |B|$ (in the category of topological spaces) whose fibers are finitely dominated.

Remark 3. The simplicial set $\mathcal{M}^h$ is homotopy equivalent to the disjoint union $\coprod_X \mathrm{BAut}(X)$, where X ranges over all homotopy equivalence classes of finitely dominated spaces and $\mathrm{Aut}(X)$ denotes the simplicial monoid of homotopy equivalences of X with itself.

A choice of inclusion $\mathbb{R}^\infty \hookrightarrow Q$ induces a map of simplicial sets $\mathcal{M} \to \mathcal{M}^h$, which we think of as assigning to each simple homotopy type its underlying homotopy type.

The second part of this course is devoted to the following question:

Question 4. What can one say about the homotopy fibers of the map $\mathcal{M} \to \mathcal{M}^h$?

For every finitely dominated space X, let $\mathfrak{S}(X)$ denote the homotopy fiber product $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$. We can identify the connected components of $\mathfrak{S}(X)$ with equivalence classes of homotopy equivalences $f : X \to P$ where P is a finite polyhedron, where we declare $f : X \to P$ to be equivalent to $f' : X \to P'$ if the induced homotopy equivalence of P with P' is simple. If X is connected with fundamental group G, then the first few lectures yield the following information:

- The space $\mathfrak{S}(X)$ is nonempty if and only if the Wall finiteness obstruction of X vanishes (as an element of $\widetilde{K}_0(\mathbf{Z}[G])$).
- If $\mathfrak{S}(X)$ is nonempty, then it is a torsor for the Whitehead group $\mathrm{Wh}(G) = \widetilde{K}_1(\mathbf{Z}[G])/\operatorname{Im}(G)$.

In particular, if X is a finite polyhedron to begin with (so that $\mathfrak{S}(X)$ has a canonical base point), then we obtain an abelian group structure on the set $\pi_0\,\mathfrak{S}(X)$. We next observe that this is no accident.

Let X be a finite polyhedron. We can identify $\mathfrak{S}(X)$ with the simplicial set whose n-simplices are pairs (E, ϕ), where $E \subseteq \Delta^n \times \mathbb{R}^\infty$ is a finite polyhedron for which the projection $E \to \Delta^n$ is a fibration, and $\phi : X \times \Delta^n \to E$ is a PL homotopy equivalence which commutes with the projection to Δ^n. As usual, we will generally abuse terminology by ignoring the data of the embedding $E \hookrightarrow \Delta^n \times \mathbb{R}^\infty$ when describing our constructions.

Let $\mathfrak{S}'(X)$ denote the simplicial subset of $\mathfrak{S}(X)$ consisting of those pairs (E, ϕ) where ϕ is an embedding. It is not hard to see that $\mathfrak{S}'(X)$ is a deformation retract of $\mathfrak{S}(X)$: we can always replace a map $\phi : X \times \Delta^n \to E$ with the diagonal map $(\phi, j) : X \times \Delta^n \to E \times C(X)$, where $C(X)$ is some contractible polyhedron equipped with an embedding $j : X \to C(X)$.

There is a natural "addition" map on the Kan complex $\mathfrak{S}'(X)$: on n-simplices, it carries a pair of embeddings

$$\phi : X \times \Delta^n \hookrightarrow E \qquad \phi' : X \times \Delta^n \hookrightarrow E'$$

to the induced map $X \times \Delta^n \to E \amalg_{X \times \Delta^n} E'$. This addition is commutative and associative up to coherent homotopy, and therefore endows $\mathfrak{S}'(X) \simeq \mathfrak{S}(X)$ with the structure of an E_∞-space.

At the level of vertices, we note that if we are given embeddings $\phi : X \to E$ and $\phi' : X \to E'$, then the Whitehead torsion of the induced map $X \hookrightarrow E \amalg_X E'$ is the sum $\tau(\phi) + \tau(\phi')$. In other words, the commutative monoid structure on $\pi_0 \mathfrak{S}(X)$ determined by the E_∞-structure on $\mathfrak{S}(X)$ agrees with the abelian group structure arising from the identification $\pi_0 \mathfrak{S}(X) \simeq \mathrm{Wh}(G)$. Consequently, the E_∞-structure on $\mathfrak{S}(X)$ is group-like, and therefore exhibits $\mathfrak{S}(X)$ as the zeroth space of some spectrum. Our goal in the second part of this course will be to give a more explicit description of this spectrum.

Recall that if X is any space for which the homology $\mathrm{H}_*(X; \mathbf{Q})$ is finite-dimensional as a rational vector space (for example, any finitely dominated space), then the *Euler characteristic* of X is given by the alternating sum

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim \mathrm{H}_i(X; \mathbf{Q}).$$

This is manifestly a homotopy invariant quantity. However, it can be described in other ways which are not so obviously invariant:

Example 5. Let X be a finite CW complex. Then we have Euler's formula

$$\chi(X) = \sum_{d \geq 0} (-1)^d s_d,$$

where s_d denotes the number of d-cells of X.

Example 6. If X be a compact Riemannian manifold of even dimension $2n$. Then the Chern-Gauss-Bonnet formula gives

$$\chi(X) = \frac{1}{(2\pi)^n} \int_X \Omega$$

where Ω denotes the Pfaffian of the curvature of X.

In both of these examples, we took advantage of some additional structure on X (a CW structure or a Riemannian structure) to write the homotopy invariant quantity $\chi(X)$ as a "sum" of local contributions which depend on that additional structure. In either case, the additional structure on X that we needed equips X with a preferred simple homotopy type. The next main theorem of this course will be a sort of converse: we will show that if X is a finitely dominated space, then having a "local formula" for the Euler characteristic of X (in a suitably generalized setting) is equivalent to equipping X with a preferred simple homotopy type.

Let us now be a bit more precise. A fundamental property of the Euler characteristic is that it is additive for homotopy pushout squares: that is, given a homotopy pushout square of finitely dominated spaces

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

where the horizontal maps are inclusions of subcomplexes, we have $\chi(Y') + \chi(X) = \chi(X') + \chi(Y)$.

Suppose, more generally, that we are given some "target space" Z. Let $T(Z)$ denote the free abelian group generated by symbols $[X]$, where X is a finitely dominated space with a map to Z, modulo the relations given by

$$[\emptyset] = 0$$

$$[Y'] + [X] = [X'] + [Y]$$

for every homotopy pushout diagram of CW complexes over Z, as above. For every finitely dominated space X with a map $X \to Z$, we can regard $[X]$ as a kind of "generalized Euler characteristic" of X, which is not an integer but an element of the abelian group $T(Z)$.

In the next few lectures, we will see that the abelian group $T(Z)$ can be identified with π_0 of a spectrum $A(Z)$, called the *Waldhausen A-theory spectrum of Z*.

Warning 7. Our definition of $A(Z)$ will differ slightly from the definition more common in the literature, in that we will allow finitely generated projective modules rather than only finitely generated free modules; this has the effect of slightly enlarging the group π_0 and not changing the higher homotopy groups.

The construction $Z \mapsto A(Z)$ is functorial. Consequently, any point $z \in Z$ determines a map $A(*) \to A(Z)$. This map depends continuously on Z, and we therefore obtain an *assembly map*

$$\rho : A(*) \wedge Z_+ \to A(Z).$$

Now suppose that the space Z itself is finitely dominated, so that $[Z]$ can be regarded as element of the abelian group $\pi_0 A(Z)$ (which we will identify with a point of $\Omega^\infty A(Z)$. This point can be regarded as a "universal" version of the Euler characteristic of Z, and the homotopy fiber of the map

$$\Omega^\infty(A(*) \wedge Z_+) \to \Omega^\infty A(Z)$$

can be thought of as the "space of all local formulas for the Euler characteristic of Z." We can now state more precisely the theorem we are after:

Theorem 8. *Let Z be a finitely dominated space. Then there is a canonical homotopy pullback square*

$$\begin{tikzcd}
\mathfrak{S}(Z) \arrow[r] \arrow[d] & \Omega^\infty(A(*) \wedge Z_+) \arrow[d] \\
* \arrow[r, "{[Z]}"] & \Omega^\infty A(Z).
\end{tikzcd}$$

Theorem 8 motivates the following:

Definition 9. Let Z be a space. The *Whitehead spectrum* of Z is the cofiber of the assembly map $A(*) \wedge Z_+ \to A(Z)$. We will denote the Whitehead spectrum of Z by $\mathbf{Wh}(Z)$.

Theorem 8 implies that if Z is a finitely dominated space, then we can associate to Z an obstruction $\eta \in \pi_0 \mathbf{Wh}(Z)$ which vanishes if and only if $\mathfrak{S}(Z)$ is nonempty. If $\eta = 0$, then $\mathfrak{S}(Z)$ is homotopy equivalent to the space $\Omega^{\infty+1}\mathbf{Wh}(Z)$.

We will see that if Z is connected with fundamental group G, then there are canonical isomorphisms

$$\pi_0 \mathbf{Wh}(Z) \simeq \widetilde{K}_0(\mathbf{Z}[G])$$

$$\pi_1 \mathbf{Wh}(Z) \simeq \mathrm{Wh}(G).$$

Consequently, Theorem 8 will subsume the theory of the Wall finiteness obstruction and the theory of the Whitehead torsion.

Warning 10. The groups $\pi_0 \mathbf{Wh}(Z)$ and $\pi_1 \mathbf{Wh}(Z)$ depend only on the fundamental group of Z (if Z is connected), but this is not true in general: the Whitehead spectrum $\mathbf{Wh}(Z)$ is sensitive to the entire homotopy type of the space Z.

(Lower) K-theory of ∞-categories (Lecture 14)

October 2, 2014

We begin by reviewing the definition of an ∞-category.

Notation 1. For every pair of integers $0 \leq i \leq n$, we let Λ^n_i denote the simplicial subset of Δ^n given by the union of all those faces except the one opposite to the ith vertex. We will refer to a simplicial set of the form Λ^n_i as a *horn*. We will say that it is an *inner horn* if $0 < i < n$, and otherwise an *outer horn*.

Definition 2. Let X be a simplicial set. We will say that X is an ∞-*category* if every map $f_0 : \Lambda^n_i \to X$ can be extended to an n-simplex $f : \Delta^n \to X$ provided that $0 < i < n$. (In other words, every *inner horn* in X can be filled.)

Remark 3. Simplicial sets satisfying the requirements of Definition 2 are also referred to as *quasi-categories* or *weak Kan complexes*.

Example 4. Any Kan complex is an ∞-category (recall that a simplicial set X is a Kan complex if *any* horn $f_0 : \Lambda^n_i \to X$ can be extended to an n-simplex of X).

Example 5. For any category $\mathcal{C}$, the nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category.

In fact, one has the following stronger assertion:

Exercise 6. Let X be a simplicial set. Show that X is isomorphic to the nerve of a category if and only if every inner horn $f_0 : \Lambda^n_i \to X$ can be extended *uniquely* to an n-simplex $f : \Delta^n \to X$.

In what follows, we will often abuse notation by identifying a category $\mathcal{C}$ with the ∞-category $\mathrm{N}(\mathcal{C})$. This does not lose any information:

Exercise 7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Show that there is a bijective correspondence between the set of functors $F : \mathcal{C} \to \mathcal{D}$ and the set of maps of simplicial sets $\mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$. In other words, the formation of nerves induces a fully faithful embedding from the category of (small) categories to the category of simplicial sets.

The formation of nerves admits a left adjoint, which sends each simplicial set X to a category which we will denote by hX. Concretely, the category hX admits the following presentation by generators and relations:

- The objects of hX are the vertices of X.
- For each edge e of X joining a vertex x to a vertex y, there is a corresponding morphism $[e]$ from x to y in hX.
- If the edge e is degenerate (so that $x = y$), then $[e] = \mathrm{id}_x$.

- For every 2-simplex of X given pictorially by the diagram

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z, \end{tikzcd}$$

  we have $[h] = [g] \circ [f]$ in hX.

We will refer to hX as the *homotopy category* of X.

In the special case where X is an ∞-category, the homotopy category hX admits a more concrete description: all morphisms in hX have the form $[e]$ for some edge e, and two edges e and e' (with the same initial and final vertices) satisfy $[e] = [e']$ if and only if they are homotopic (meaning that there exists a 2-simplex

$$\begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur, "e"] \arrow[rr, "e'"] & & y \end{tikzcd}$$

whose 0th face (joining y to y) is degenerate).

Most of the basic concepts of category theory (commutative diagrams, limits and colimits, initial and final objects, functors, adjunctions) can be generalized to the setting of ∞-categories. We will henceforth make use of those generalizations, and refer the reader to [1] for more details.

In what follows, we will use the notation $\mathcal{C}$ to denote an ∞-category (emphasizing the idea that $\mathcal{C}$ is some sort of generalized category rather than a simplicial set). We refer to the vertices of $\mathcal{C}$ as its *objects* and to the edges of $\mathcal{C}$ as its *morphisms*.

Definition 8. Let $\mathcal{C}$ be an ∞-category. A *zero object* of $\mathcal{C}$ is an object $*$ which is both initial and final. We will say that $\mathcal{C}$ is *pointed* if it has a zero object. If $\mathcal{C}$ is pointed, then for every pair of objects X and Y there is a canonical morphism from X to Y given by the composition $X \to * \to Y$, which we refer to as the *zero morphism*.

Notation 9. Let $\mathcal{C}$ be a pointed ∞-category with zero object $*$. Suppose that $\mathcal{C}$ admits pushouts. For every morphism $f : X \to Y$ in $\mathcal{C}$, we let $\mathrm{cofib}(f)$ denote the pushout $Y \amalg_X *$. We refer to f as the *cofiber* of f. In the special case where $Y = *$, we refer to $\mathrm{cofib}(f)$ as the *suspension* of X and denote it by ΣX. Note that we have a diagram

$$X \xrightarrow{f} Y \to \mathrm{cofib}(f)$$

where the composition is zero; we refer to such diagrams as *cofiber sequences*.

Definition 10. Let $\mathcal{C}$ be a pointed ∞-category which admits pushouts. We let $K_0(\mathcal{C})$ denote the free abelian group on generators $[X]$, where X is an object of $\mathcal{C}$, modulo the relations given by $[X'] + [X''] = [X]$ whenever there is a cofiber sequence

$$X' \to X \to X''$$

in $\mathcal{C}$.

Remark 11. Using the cofiber sequence

$$* \to * \to *$$

we deduce that $[*] = 0 \in K_0(\mathcal{C})$. Using the cofiber sequence

$$X \to * \to \Sigma(X)$$

we conclude that $[\Sigma(X)] = -[X]$ in $K_0(\mathcal{C})$.

Warning 12. Definition 10 is not interesting for "large" ∞-categories. For example, if $\mathcal{C}$ admits infinite coproducts, then any object X fits into a cofiber sequence

$$\coprod_{n\geq 1} X \to \coprod_{n\geq 0} X \to X$$

where the first two terms are equivalent to one another, so that $[X] = 0 \in K_0(\mathcal{C})$; since X was arbitrary, we have $K_0(\mathcal{C}) \simeq 0$.

Example 13. Let $\mathcal{C}$ be the ∞-category of finite pointed spaces. Then $K_0(\mathcal{C})$ is isomorphic to $\mathbf{Z}$, the isomorphism being given by the "reduced" Euler characteristic

$$[X] \mapsto \chi_{\mathrm{hyp}}(X) = \chi(X) - 1.$$

Example 14. Let R be a ring. A *perfect complex* over R is a bounded chain complex

$$\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

where each P_i is a finitely generated projective R-module. The collection of perfect chain complexes over R can be organized into an ∞-category $\mathrm{Mod}_R^{\mathrm{perf}}$. There is a natural map $K_0(R) \to K_0(\mathrm{Mod}_R^{\mathrm{perf}})$ which carries each finitely generated projective R-module P to the chain complex consisting of P in degree zero. One can show that this map is an isomorphism: it has an inverse which carries a chain complex $[P_*]$ to the alternating sum $\sum_n (-1)^n [P_n]$.

Remark 15. Let $\mathcal{C}$ and $\mathcal{D}$ be pointed ∞-categories which admit pushouts and suppose we are given a functor $F : \mathcal{C} \to \mathcal{D}$ which preserves finite colimits. Then F induces a group homomorphism $K_0(\mathcal{C}) \to K_0(\mathcal{D})$, given by $[X] \mapsto [F(X)]$.

Example 16. Let $\mathcal{C}$ be a pointed ∞-category which admits pushouts. Then the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ satisfies the hypotheses of Remark 15, and induces the map $K_0(\mathcal{C}) \to K_0(\mathcal{C})$ given by multiplication by -1.

Definition 17. We say that an ∞-category $\mathcal{C}$ is *stable* if it is pointed, admits pushouts, and the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence of ∞-categories.

Remark 18. Let $\mathcal{C}$ be a pointed ∞-category which admits pushouts. We define the *Spanier-Whitehead category* $SW(\mathcal{C})$ to be the direct limit

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \to \cdots .$$

Then $SW(\mathcal{C})$ is stable, and is universal among stable ∞-category which receive a functor from $\mathcal{C}$ which preserves finite colimits. Moreover, $K_0(SW(\mathcal{C}))$ can be identified with the direct limit of the sequence

$$K_0(\mathcal{C}) \xrightarrow{-1} K_0(\mathcal{C}) \xrightarrow{-1} K_0(\mathcal{C}) \to \cdots ,$$

and is therefore isomorphic to $K_0(\mathcal{C})$.

In other words, for studying K_0, there is no real loss of generality in assuming that we are working with stable ∞-categories.

In the next lecture, we will need the following result:

Proposition 19. *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full (stable) subcategory. Assume that every object of $\mathcal{C}$ is a direct summand of an object that belongs to $\mathcal{C}_0$. Then:*

(a) *The canonical map $\alpha : K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$ is injective.*

(b) *An object $C \in \mathcal{C}$ belongs to $\mathcal{C}_0$ if and only if $[C]$ belongs to the image of α.*

To prove Proposition 19, it will be convenient to introduce a variant of Definition 10.

Definition 20. Let $\mathcal{C}$ be a stable ∞-category. We let $K_{\mathrm{add}}(\mathcal{C})$ denote the free abelian group generated by symbols $[X]$ where $X \in \mathcal{C}$, modulo the relations

$$[X] = [X'] + [X'']$$

if X is equivalent to a direct sum $X' \oplus X''$.

Remark 21. In the situation of Definition 20, it is easy to see that we have $[X] = [Y]$ in $K_{\mathrm{add}}(\mathcal{C})$ if and only if X and Y are *stably equivalent*: that is, if and only if there exists an object $Z \in \mathcal{C}$ such that $X \oplus Z$ is equivalent to $Y \oplus Z$.

We have an evident surjective map $K_{\mathrm{add}}(\mathcal{C}) \to K_0(\mathcal{C})$; let us denote the kernel of this map by $I(\mathcal{C})$.

Lemma 22. *In the situation of Proposition 19, the canonical map $I(\mathcal{C}_0) \to I(\mathcal{C})$ is surjective.*

Proof. Note that $I(\mathcal{C})$ is generated by expressions of the form $\eta = [X] - [X'] - [X'']$, where

$$X' \to X \to X''$$

is a cofiber sequence in $\mathcal{C}$. For any such cofiber sequence, we can choose objects Y', Y'' in $\mathcal{C}$ such that $X' \oplus Y'$ and $X'' \oplus Y''$ belong to $\mathcal{C}_0$. We then have a cofiber sequence

$$X' \oplus Y' \to X \oplus Y' \oplus Y'' \to X'' \oplus Y''$$

where the outer terms belong to $\mathcal{C}_0$, so that the middle term does as well. It follows that $\eta = [X \oplus Y' \oplus Y''] - [X' \oplus Y'] - [X'' \oplus Y'']$ belongs to the image of the map $I(\mathcal{C}_0) \to I(\mathcal{C})$. $\square$

Proof of Proposition 19. It follows immediately from Remark 21 that the map $K_{\mathrm{add}}(\mathcal{C}_0) \to K_{\mathrm{add}}(\mathcal{C})$ is injective. Assertion (a) now follows by applying the snake lemma to the diagram

$$\begin{tikzcd} 0 \arrow[r] & I(\mathcal{C}_0) \arrow[r] \arrow[d] & K_{\mathrm{add}}(\mathcal{C}_0) \arrow[r] \arrow[d] & K_0(\mathcal{C}_0) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I(\mathcal{C}) \arrow[r] & K_{\mathrm{add}}(\mathcal{C}) \arrow[r] & K_0(\mathcal{C}) \arrow[r] & 0. \end{tikzcd}$$

To prove (b), we note that the snake lemma implies that the natural map

$$K_{\mathrm{add}}(\mathcal{C}) / \operatorname{Im}(K_{\mathrm{add}}(\mathcal{C}_0)) \to K_0(\mathcal{C}) / \operatorname{Im} K_0(\mathcal{C}_0)$$

is an isomorphism of abelian groups. Consequently, if $X \in \mathcal{C}$ has the property that $[X]$ belongs to the image of $K_0(\mathcal{C}_0)$, then $[X]$ belongs to the image of $K_{\mathrm{add}}(\mathcal{C}_0)$. It follows that there exists objects $Y, Y' \in \mathcal{C}_0$ such that $X \oplus Y \simeq Y'$, so that X is equivalent to the cofiber of a map $Y \to Y'$ and therefore belongs to $\mathcal{C}_0$ as desired. $\square$

References

[1] Lurie, J. *Higher Topos Theory.*

The Wall Finiteness Obstruction Revisited (Lecture 15)

October 6, 2014

Let $\mathcal{C}$ be a (small) ∞-category which admits finite colimits. There is a formal procedure for enlarging $\mathcal{C}$ to admit all colimits: namely, one can replace $\mathcal{C}$ by the ∞-category $\mathrm{Ind}(\mathcal{C})$ of Ind-*objects* of $\mathcal{C}$. This enlargement can be characterized as follows:

(a) There is a fully faithful embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ (we will henceforth abuse notation by identifying $\mathcal{C}$ with its image in $\mathrm{Ind}(\mathcal{C})$).

(b) The ∞-category $\mathrm{Ind}(\mathcal{C})$ admits filtered colimits (in fact, it admits all small colimits, and the functor j preserves finite colimits).

(c) Every object of $\mathrm{Ind}(\mathcal{C})$ can be written as a filtered colimit $\varinjlim C_\alpha$, where each C_α belongs to $\mathcal{C}$ (identified with its image in $\mathrm{Ind}(\mathcal{C})$ via j).

(d) Every object $C \in \mathcal{C}$ is *compact* as an object of $\mathrm{Ind}(\mathcal{C})$: that is, the construction $D \mapsto \mathrm{Map}_{\mathcal{C}}(C, D)$ commutes with filtered colimits.

At an informal level, we can see that these properties characterize $\mathrm{Ind}(\mathcal{C})$ as follows. For each filtered diagram $\{C_\alpha\}$ in the ∞-category $\mathcal{C}$, let "$\varinjlim C_\alpha$" denote the colimit of the diagram $\{C_\alpha\}$ in $\mathrm{Ind}(\mathcal{C})$. Assumption (b) ensures that this colimit is well defined and assumption (c) implies that every object of $\mathrm{Ind}(\mathcal{C})$ has this form. Morphism spaces in $\mathrm{Ind}(\mathcal{C})$ are then given by

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\text{“}\varinjlim C_\alpha\text{''}, \text{“}\varinjlim D_\beta\text{''}) &\simeq \varprojlim_\alpha \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(C_\alpha, \text{“}\varinjlim D_\beta\text{''}) \\
&\simeq \varprojlim_\alpha \varinjlim_\beta \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(C_\alpha, D_\beta) \\
&\simeq \varprojlim_\alpha \varinjlim_\beta \mathrm{Map}_{\mathcal{C}}(C_\alpha, D_\beta)
\end{aligned}$$

where the last two equivalences are obtained from (d) and (a), respectively.

Remark 1. Working more formally, one can define $\mathrm{Ind}(\mathcal{C})$ to be the ∞-category of functors from $\mathcal{C}^{\mathrm{op}}$ to spaces which preserve finite limits, from which one can derive properties (a), (b), (c), and (d).

Example 2. Let $\mathcal{C}$ be the category of finitely presented groups. Then $\mathrm{Ind}(\mathcal{C})$ can be identified with the category of all groups. The same conclusion holds if we replace "groups" by any other type of algebraic structure (abelian groups, rings, commutative rings, etcetera).

Example 3. Let $\mathcal{C}$ be the ∞-category of finite CW complexes. Then $\mathrm{Ind}(\mathcal{C})$ can be identified with the ∞-category of spaces.

Example 4. Let R be a ring spectrum (or ordinary ring), and let $\mathcal{C}$ be the ∞-category of perfect (complexes of) R-modules. Then $\mathrm{Ind}(\mathcal{C})$ can be identified with the ∞-category of all (complexes of) R-modules.

It is natural to ask if the converse to (d) is true: does every compact object of $\mathrm{Ind}(\mathcal{C})$ belong to (the essential image of) $\mathcal{C}$? To address, suppose we are given an object $X = $ "$\varinjlim C''_\alpha \in \mathrm{Ind}(\mathcal{C})$ which is compact. Then the identity map $\mathrm{id}_X : X \to$ "$\varinjlim C''_\alpha$ must factor through some C_α. It follows that X is a *retract* of C_α in the ∞-category $\mathrm{Ind}(\mathcal{C})$. Conversely, it is not hard to see that the collection of compact objects of $\mathrm{Ind}(\mathcal{C})$ is closed under retracts, and therefore contains all retracts of objects of $\mathcal{C}$.

Remark 5. Suppose that $X \in \mathrm{Ind}(\mathcal{C})$ is a retract of an object $C \in \mathcal{C}$, so that we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\mathrm{id}_X"] \arrow[dr, "\alpha"'] & & X \\ & C \arrow[ur, "\beta"'] & \end{tikzcd}$$

in $\mathcal{C}$. Let $r = \alpha \circ \beta$; we think of r as the "retraction" from C onto X. Then X can be recovered as the filtered colimit

$$C \xrightarrow{r} C \xrightarrow{r} C \xrightarrow{r} C \to \cdots$$

Conversely, if we start with a map $r : C \to C$ and define X to be the colimit of the above sequence, then one can show that X is a retract of C provided that there exists a homotopy h of r^2 with r for which the diagram

$$\begin{tikzcd} r^3 \arrow[r, "h \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times h"] & r^2 \arrow[d, "h"] \\ r^2 \arrow[r, "h"] & r \end{tikzcd}$$

commutes up to homotopy. Beware that the commutativity of this diagram is a necessary condition: in general, not every idempotent in the homotopy category of $\mathcal{C}$ can be lifted to a "homotopy coherent" idempotent in $\mathcal{C}$.

Example 6. Let $\mathcal{C}$ be an ordinary category which admits finite colimits. If $X \in \mathrm{Ind}(\mathcal{C})$ is a retract of an object $C \in \mathcal{C}$, then X can be recovered as the coequalizer of the maps $r, \mathrm{id} : C \to C$. It follows that every compact object of $\mathrm{Ind}(\mathcal{C})$ belongs to (the essential image of) $\mathcal{C}$.

Example 7. Let $\mathcal{C}$ be the ∞-category of finite CW complexes, so that $\mathrm{Ind}(\mathcal{C})$ can be identified with the ∞-category of spaces. An object $X \in \mathrm{Ind}(\mathcal{C})$ is compact if and only if it is finitely dominated (in the sense of Lecture 2). Consequently, not all compact objects of $\mathrm{Ind}(\mathcal{C})$ come from $\mathcal{C}$.

Remark 8. Let $\mathcal{C}$ be a small ∞-category which admits finite colimits. We let $\overline{\mathcal{C}}$ denote the full subcategory of $\mathrm{Ind}(\mathcal{C})$ spanned by the compact objects. Then $\overline{\mathcal{C}}$ is the *idempotent completion* of $\mathcal{C}$: that is, it can be obtained by formally enlarging $\mathcal{C}$ adding "images" of all coherently idempotent morphisms in $\mathcal{C}$.

Example 9. Let $\mathcal{C}$ be a stable ∞-category. Then the idempotent completion $\overline{\mathcal{C}}$ is also stable. Moreover, every object of $\overline{\mathcal{C}}$ can be obtained as a direct summand of an object of $\mathcal{C}$.

Example 10. Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. The formation of idempotent completions commutes with filtered colimits. Applying this observation to the diagram

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \cdots,$$

we conclude that the Spanier-Whitehead ∞-category $SW(\overline{\mathcal{C}})$ can be identified with the idempotent completion of $SW(\mathcal{C})$. In particular, every object of $SW(\overline{\mathcal{C}})$ can be written as a direct summand of an object of $SW(\mathcal{C})$.

In Lecture 2, we saw that not every finitely dominated space X is homotopy equivalent to a finite CW complex. However, the possible failure of X to be homotopy equivalent to a finite complex was under good control: namely, we could associate to X a certain algebraic invariant w_X (its *Wall finiteness obstruction*) which vanished if and only if X was homotopy equivalent to a finite CW complex. We would now like to do something like this for a general ∞-category $\mathcal{C}$. First, we need to introduce a bit of terminology.

Definition 11. Let $\mathcal{C}$ be an ∞-category which admits finite colimits. We will say that a morphism $f : X \to Y$ in $\mathrm{Ind}(\mathcal{C})$ is $\mathcal{C}$-*finite* if it can be written as a composition

$$X = X_0 \to X_1 \to \cdots \to X_n = Y$$

where each of the maps $X_{i-1} \to X_i$ fits into a pushout diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & D \arrow[d] \\ X_{i-1} \arrow[r] & X_i \end{tikzcd}$$

for $C, D \in \mathcal{C}$.

Example 12. Let $\mathcal{C}$ be the ∞-category of finite CW complexes. Then a map of spaces $f : X \to Y$ is $\mathcal{C}$-finite if and only if Y is homotopy equivalent to a space which is obtained from X by attaching finitely many cells.

Remark 13. The collection of $\mathcal{C}$-finite morphisms contains all equivalences and is closed under composition.

Remark 14. Let $\emptyset$ denote the initial object of $\mathcal{C}$. Then an object $X \in \mathrm{Ind}(\mathcal{C})$ belongs to $\mathcal{C}$ if and only if the map $\emptyset \to X$ is $\mathcal{C}$-finite.

Remark 15. Suppose we are given a pushout diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in $\mathrm{Ind}(\mathcal{C})$. If f' is $\mathcal{C}$-finite, then so is f.

We will also need the following less obvious observation:

Proposition 16. *Suppose we are given morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathrm{Ind}(\mathcal{C})$*. If f and $g \circ f$ are $\mathcal{C}$-finite, then g is $\mathcal{C}$-finite.*

Proof. Factoring f as a composition, we can reduce to the case where Y has the form $X \amalg_C D$ for some $C, D \in \mathcal{C}$. Then g factors as a composition

$$X \amalg_C D \to Z \amalg_C D \xrightarrow{\alpha} Z$$

where the first map is a pushout of $g \circ f$, and is therefore $\mathcal{C}$-finite. It will therefore suffice to show that α is $\mathcal{C}$-finite. This follows from the observation that α is a pushout of the "fold map" $D \amalg_C D \to D$. $\square$

Suppose now that X is an object of $\mathrm{Ind}(\mathcal{C})$. We let $\mathrm{Ind}(\mathcal{C})_{X//X}$ denote the ∞-category of objects of $\mathrm{Ind}(\mathcal{C})$ lying over and under X: that is, the ∞-category whose objects are commutative diagrams

$$\begin{tikzcd} & Y \arrow[dr, "\beta"] & \\ X \arrow[ur, "\alpha"] \arrow[rr, "\mathrm{id}"] & & X. \end{tikzcd}$$

Let $\mathcal{C}[X]$ denote the full subcategory of $\mathrm{Ind}(\mathcal{C})_{X//X}$ spanned by those diagrams where α is $\mathcal{C}$-finite. It is not hard to see that the objects of $\mathcal{C}[X]$ form compact generators for $\mathrm{Ind}(\mathcal{C})_{X//X}$: that is, we have an equivalence of ∞-categories $\mathrm{Ind}(\mathcal{C}[X]) \simeq \mathrm{Ind}(\mathcal{C})_{X//X}$. In particular, we can identify the idempotent completion of $\mathcal{C}[X]$ with the ∞-category of compact objects of $\mathrm{Ind}(\mathcal{C})_{X//X}$ (which can be characterized as those diagrams above for which α exhibits Y as a compact object of $\mathrm{Ind}(\mathcal{C})_{X/}$).

Note that $\mathcal{C}[X]$ and its idempotent completion $\overline{\mathcal{C}[X]}$ are pointed ∞-categories which admit finite colimits, so that we can consider the K-groups

$$K_0(\mathcal{C}[X]) \qquad K_0(\overline{\mathcal{C}[X]})$$

introduced in the previous lecture. Note that if Y is a compact object of $\mathrm{Ind}(\mathcal{C})$ equipped with a map $f : Y \to X$, then the diagram

$$\begin{tikzcd} & X \amalg Y \arrow[dr, "(\mathrm{id}, f)"] & \\ X \arrow[ur] \arrow[rr, "\mathrm{id}"] & & X \end{tikzcd}$$

belongs to $\overline{\mathcal{C}[X]}$ and therefore determines an element of $K_0(\overline{\mathcal{C}[X]})$ which we will denote by $[Y/X]$. If Y belongs to $\mathcal{C}$, then the natural map $X \to X \amalg Y$ is $\mathcal{C}$-finite and we can therefore lift $[Y/X]$ to $K_0(\mathcal{C}[X])$. We are going to prove the following converse:

Theorem 17 (Generalized Wall Finiteness Obstruction)**.** *Let $\mathcal{C}$ be a small ∞-category which admits finite colimits and let $X \in \mathrm{Ind}(\mathcal{C})$. Then:*

(*a*) *The natural map $\alpha : K_0(\mathcal{C}[X]) \to K_0(\overline{\mathcal{C}[X]})$ is a monomorphism of abelian groups.*

(*b*) *Suppose that $X \in \mathrm{Ind}(\mathcal{C})$ is compact. Then X belongs to (the essential image of) $\mathcal{C}$ if and only if the class $[X/X] \in K_0(\overline{\mathcal{C}[X]})$ belongs to the image of α.*

Example 18. Suppose that $\mathcal{C}$ is the ∞-category of finite CW complexes and that X is a connected finitely dominated space. We will later show that the quotient $K_0(\overline{\mathcal{C}[X]})/K_0(\mathcal{C}[X])$ can be identified with the reduced K-group $\widetilde{K}_0(\mathbf{Z}[\pi_1 X])$ and that this identification carries $[X/X]$ to the Wall obstruction w_X. Consequently, assertion (*b*) of Theorem 17 can be regarded as a generalization of the main result of Lecture 2.

Proof of Theorem 17. Using Example 10, we can identify α with the canonical map from $K_0(SW(\mathcal{C}[X]))$ to $K_0(\overline{SW(\mathcal{C}[X])})$, which we proved to be injective in the previous lecture. This proves (*a*). The "only if" direction of (*b*) is obvious. We now prove the converse.

First, without loss of generality we can assume that X is the final object of $\mathrm{Ind}(\mathcal{C})$ (we can achieve this by replacing $\mathcal{C}$ by the ∞-category $\mathcal{C} \times_{\mathrm{Ind}(\mathcal{C})} \mathrm{Ind}(\mathcal{C})_{/X}$). Let us denote this final object by $*$. Let $\overline{\mathcal{C}}$ denote the idempotent completion of $\mathcal{C}$, so that $\overline{\mathcal{C}}$ contains $*$ as a final object. Then $\overline{\mathcal{C}[X]}$ can be identified with the ∞-category $\overline{\mathcal{C}}_*$ of pointed objects of $\overline{\mathcal{C}}$, and $\mathcal{C}[X]$ can be identified with the full subcategory of $\overline{\mathcal{C}}_*$ spanned by those pointed objects $e : * \to C$ where e is a $\mathcal{C}$-finite map.

To prove (*b*), we must show that if $[* \amalg *] \in K_0(\overline{\mathcal{C}}_*)$ belongs to the image of α, then the object $*$ belongs to $\mathcal{C}$. By assumption, we know that $*$ belongs to the idempotent completion of $\mathcal{C}$. In particular, there exists an object $C \in \mathcal{C}$ and a map $* \to C$. We then have a cofiber sequence

$$* \amalg * \to * \amalg C \to C$$

in the ∞-category $\overline{\mathcal{C}}_*$, where the middle term belongs to $\mathcal{C}[X]$. It follows that $[C] \in K_0(\widetilde{\mathcal{C}}_*)$ belongs to the image of α. Invoking the main result from the previous lecture, we conclude that the image of C in the Spanier-Whitehead category $SW(\widetilde{\mathcal{C}}_*)$ belongs to the Spanier-Whitehead category $SW(\mathcal{C}[X])$. In other words, there exists an integer $n \geq 0$ such that $\Sigma^n(C)$ belongs to $\mathcal{C}[X]$, meaning that the base point inclusion $* \to \Sigma^n(C)$ is $\mathcal{C}$-finite. Applying Proposition 16, we conclude that the projection map $\Sigma^n(C) \to *$ is $\mathcal{C}$-finite. If $n = 0$, this tells us that the map $C \to *$ is $\mathcal{C}$-finite so that $* \in \mathcal{C}$, as desired. We will complete the proof by verifying the following:

- $(*)$ If $n \geq 0$ and there exists an object $C \in \mathcal{C}$ with a base point $* \to C$ for which the canonical map $\Sigma^{n+1}C \to *$ is $\mathcal{C}$-finite, then there is another object $D \in \mathcal{C}$ with a base point $* \to D$ for which the canonical map $\Sigma^n D \to *$ is also $\mathcal{C}$-finite.

(In fact, the proof will show that we can take D to be *any* other object of $\mathcal{C}[X]$, for example we can take $D = C$; however, it will be less confusing if we do not identify D with C.) To prove $(*)$, we first note that because the ∞-category $\overline{\mathcal{C}}$ admits finite colimits it is naturally tensored over the ∞-category $\mathcal{S}^{\mathrm{fin}}$ of finite CW complexes. We will denote the action of $\mathcal{S}^{\mathrm{fin}}$ on $\overline{\mathcal{C}}$ by

$$\otimes : \mathcal{S}^{\mathrm{fin}} \times \overline{\mathcal{C}} \to \overline{\mathcal{C}}.$$

Concretely, the tensor product $K \otimes C$ is characterized by the universal property $\mathrm{Map}_{\overline{\mathcal{C}}}(K \otimes C, D) = \mathrm{Map}_{\overline{\mathcal{C}}}(C, D)^K$. We will also abuse notation by identifying each finite CW complex K with the tensor product $K \otimes * \in \overline{\mathcal{C}}$.

The proof of $(*)$ is based on the observation that the suspension $\Sigma^n C$ can be computed in two different ways:

- (a) For any $n \geq 0$, we can identify $\Sigma^n C$ (for a pointed object C) with the pushout $(S^n \otimes C) \amalg_{S^n} *$.
- (b) If $n > 0$, then the definition of $\Sigma^n C$ does not require a base point in C: the "unreduced suspension" can be realized as the pushout $C \amalg_{S^{n-1} \otimes C} S^{n-1}$

Let us now suppose that $C \in \mathcal{C}$ is as in $(*)$ and use (b) to describe the suspension $\Sigma^{n+1}C$. Let D be any object of $\mathcal{C}$ equipped with a base point $* \to D$ (we know that such an object exists; for example we can take $D = C$). We then have a pushout diagram

$$\begin{tikzcd} C \amalg_{S^n \otimes C} S^n \arrow[r] \arrow[d] & C \amalg_{S^n \otimes C} (S^n \otimes D) \arrow[d] \\ * \arrow[r] & * \otimes_{S^n} (S^n \otimes D). \end{tikzcd}$$

Since the left vertical map is $\mathcal{C}$-finite, so is the right vertical map. Since the upper right hand corner belongs to $\mathcal{C}$, we obtain $* \otimes_{S^n} (S^n \otimes D) \in \mathcal{C}$. Using description (a) of the suspension $\Sigma^n D$, we obtain a pushout diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d] & * \amalg_{S^n} (S^n \otimes D) \arrow[d] \\ * \arrow[r] & \Sigma^n D. \end{tikzcd}$$

The upper horizontal map in this diagram is a morphism between objects of $\mathcal{C}$ and is therefore $\mathcal{C}$-finite. It follows that the lower horizontal map is $\mathcal{C}$-finite again. Applying Proposition 16, we deduce that the projection map $\Sigma^n D \to *$ is $\mathcal{C}$-finite as desired. □

References

Higher K-Theory of ∞-Categories (Lecture 16)

October 7, 2014

Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits; we will denote the coproduct on $\mathcal{C}$ by $(C, D) \mapsto C \vee D$ (to emphasize the analogy with pointed spaces). Recall that we defined the K-group $K_0(\mathcal{C})$ to be the free abelian group on symbols $[X]$ for $X \in \mathcal{C}$, modulo the relations $[X] = [X'] + [X'']$ for every cofiber sequence

$$X' \to X \to X''$$

in $\mathcal{C}$. Note that it is not necessary to demand in advance that this group be abelian: for any pair of objects $X, Y \in \mathcal{C}$, we have cofiber sequences

$$X \to X \vee Y \to Y$$

$$Y \to X \vee Y \to X$$

which give the relation $[X] + [Y] = [X \vee Y] = [Y] + [X]$.

Our goal in this lecture is to discuss higher K-groups of an ∞-category $\mathcal{C}$, following ideas of Waldhausen. Our basic plan is to construct a space W whose fundamental group is $K_0(\mathcal{C})$; we will then define the higher K-groups of $\mathcal{C}$ to be the higher homotopy groups of W. Let's have a look at what such a space should look like:

(a) The space W needs to have a base point (so that it makes sense to extract the homotopy groups of W).

(b) Every object $X \in \mathcal{C}$ needs to determine a path p_X in W which begins and ends in the base point, which we take be a representative of the homotopy class $[X] \in K_0(\mathcal{C}) \simeq \pi_1 W$.

(c) For every cofiber sequence

$$X' \to X \to X''$$

in the ∞-category $\mathcal{C}$, we will need a 2-simplex of W with boundary given by

$$\begin{tikzcd} & * \arrow[dr, "p_{X'}"] & \\ * \arrow[ur, "p_{X''}"] \arrow[rr, "p_X"] & & *, \end{tikzcd}$$

which "witnesses" the relation $[X] = [X'] + [X'']$ in the fundamental group of W.

In what follows, it will be convenient to denote the cofiber of a map $f : X \to Y$ by Y/X; our basic relation in $K_0(\mathcal{C})$ can then be rewritten as $[Y] = [X] + [Y/X]$. From this, we can deduce analogous identities for "longer" filtrations. For example, suppose we are given a pair of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Then $[Z] = [X] + [Y/X] + [Z/Y]$. In fact, we can prove this in two different ways: first, we can use the cofiber sequences

$$X \to Z \to Z/X$$

$$Y/X \to Z/X \to Z/Y$$

to deduce $[Z] = [X] + [Z/X] = [X] + ([Y/X] + [Z/Y])$. Alternatively, we could use the cofiber sequences

$$Y \to Z \to Z/Y$$

$$X \to Y \to Y/X$$

to deduce $[Z] = [Y] + [Z/Y] = ([X] + [Y/X]) + [Z/Y]$. If W is a space satisfying (a), (b), and (c), then these two proofs give two *a priori* different homotopies between the path p_Z and the concatenation of the paths p_X, $p_{Y/X}$, and $p_{Z/Y}$, which determines a map from a 2-sphere into W. Concretely, this two sphere is given by a map $\partial \Delta^3 \to W$, whose restriction to each fact of $\partial \Delta^3$ is the 2-simplex associated by assumption (c) to one of the cofiber sequences above. It is natural to demand the following "three-dimensional" analogue of (b) and (c):

(d) For every diagram

$$X \to Y \to Z$$

in $\mathcal{C}$, we should have a 3-simplex in W which we depict as

$$\begin{tikzcd} & * \arrow[r, "p_{Y/X}"] \arrow[drr, "p_Y"] & * \arrow[dr, "p_X"] & \\ * \arrow[ur, "p_{Z/Y}"] \arrow[urr, "p_{Z/X}"] \arrow[rrr, "p_Z"] & & & * \end{tikzcd}$$

whose faces are the 2-simplices given by (c).

Moreover, we want to make an analogous "n-dimensional" demand for every n-step filtration

$$X_1 \to X_2 \to \cdots \to X_n$$

in the ∞-category $\mathcal{C}$. In order to say this properly, we need to get organized.

Definition 1. Let P be a partially ordered set, and let $P^{(2)} = \{(i,j) \in P \times P : i \leq j\}$. Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. We define a *P-gapped object* of $\mathcal{C}$ to be a functor $X : \mathrm{N}(P^{(2)}) \to \mathcal{C}$ with the following properties:

(i) For every $i \in P$, the object $X(i,i)$ is a zero object of $\mathcal{C}$.

(ii) For every $i \leq j \leq k$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} X(i,j) \arrow[r] \arrow[d] & X(i,k) \arrow[d] \\ X(j,j) \arrow[r] & X(j,k) \end{tikzcd}$$

is a pushout square; by virtue of (i), this means that we have a cofiber sequence

$$X(i,j) \to X(i,k) \to X(j,k).$$

Note that if $f : P \to Q$ is a map of partially ordered sets, then f induces a map of simplicial sets $\mathrm{N}(P^{(2)}) \to \mathrm{N}(Q^{(2)})$. Composition with this map carries Q-gapped objects of $\mathcal{C}$ to P-gapped objects of $\mathcal{C}$.

For every integer $n \geq 0$, we let $[n]$ denote the linearly ordered set $\{0 < 1 < \ldots < n\}$. The collection of $[n]$-gapped objects of $\mathcal{C}$ form an ∞-category which we will denote by $\mathrm{Gap}_{[n]}(\mathcal{C})$. Let $S_n(\mathcal{C})$ denote the underlying Kan complex of $\mathrm{Gap}_{[n]}(\mathcal{C})$: that is, the simplicial set obtained from $\mathrm{Gap}_{[n]}(\mathcal{C})$ by removing all those edges which correspond to noninvertible morphisms in $\mathrm{Gap}_{[n]}(\mathcal{C})$ (along with all simplices which contain such edges). Any monotone map $[m] \to [n]$ induces a map of Kan complexes $S_n(\mathcal{C}) \to S_m(\mathcal{C})$; we may therefore regard $S_\bullet(\mathcal{C})$ as a simplicial Kan complex. We will refer to $S_\bullet(\mathcal{C})$ as *the Waldhausen construction* on $\mathcal{C}$.

Remark 2. By definition, an object of $\mathrm{Gap}_{[n]}(\mathcal{C})$ is a diagram

$$\begin{tikzcd}
X(0,0) \arrow[r] & X(0,1) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & X(0,n) \arrow[d] \\
& X(1,1) \arrow[r] & \cdots \arrow[r] \arrow[d] & X(1,n) \arrow[d] \\
& & \cdots \arrow[r] & \cdots \arrow[d] \\
& & & X(n,n)
\end{tikzcd}$$

where every square is a pushout and the objects along the diagonal are zero.

Example 3. When $n = 0$, $\mathrm{Gap}_{[n]}(\mathcal{C})$ is the full subcategory of $\mathcal{C}$ spanned by the zero objects. Since $\mathcal{C}$ is pointed, this is a contractible Kan complex.

Example 4. When $n = 1$, $\mathrm{Gap}_{[n]}(\mathcal{C})$ is the ∞-category whose objects are diagrams

$$\begin{tikzcd}
* \arrow[r] & X \arrow[d] \\
& *'
\end{tikzcd}$$

where $X \in \mathcal{C}$ is arbitrary and $*$ and $*'$ are zero objects of $\mathcal{C}$. This ∞-category is equivalent to $\mathcal{C}$.

Example 5. When $n = 2$, $\mathrm{Gap}_{[n]}(\mathcal{C})$ is the ∞-category of diagrams

$$\begin{tikzcd}
* \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[d] \\
& *' \arrow[r] & X'' \arrow[d] \\
& & *''
\end{tikzcd}$$

where $*$, $*'$, and $*''$ are zero objects and square is a pushout. We can think of the data of such a diagram as determined by a pair of maps

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

together with a nullhomotopy of $g \circ f$, which exhibits X'' as a cofiber of f. More informally: $\mathrm{Gap}_{[2]}(\mathcal{C})$ is the ∞-category whose objects are cofiber sequences in $\mathcal{C}$. This is equivalent to the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C}) = \mathcal{C}^{\Delta^1}$ of morphisms in $\mathcal{C}$, since a cofiber sequence is determined up to equivalence by the morphism $f : X' \to X$.

Remark 6. More generally, for any $n \geq 0$, an $[n]$-gapped object

$$\begin{tikzcd}
X(0,0) \arrow[r] & X(0,1) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & X(0,n) \arrow[d] \\
& X(1,1) \arrow[r] & \cdots \arrow[r] \arrow[d] & X(1,n) \arrow[d] \\
& & \cdots \arrow[r] & \cdots \arrow[d] \\
& & & X(n,n)
\end{tikzcd}$$

is determined by the sequence of maps

$$X(0,1) \to X(0,2) \to \cdots \to X(0,n):$$

the rest of the diagram can be functorially recovered by forming cofibers (note that $X(i,j)$ is the cofiber of the map $X(0,i) \to X(0,j)$). More precisely, evaluation on the pairs $(0,i)$ for $i > 0$ induces an equivalence of ∞-categories

$$\mathrm{Gap}_{[n]}(\mathcal{C}) \to \mathrm{Fun}(\Delta^{n-1}, \mathcal{C}).$$

Remark 7. Using Remark 6, we can describe the Waldhausen construction more informally as follows: for each $n \geq 0$, the Kan complex $S_n(\mathcal{C})$ is a classifying space for diagrams

$$X_1 \to X_2 \to \cdots \to X_n$$

in $\mathcal{C}$. However, the description of $S_\bullet(\mathcal{C})$ as a simplicial Kan complex is not completely apparent from this description. Most of the face maps $S_n(\mathcal{C}) \to S_{n-1}(\mathcal{C})$ correspond to simply "forgetting" one of the objects X_i in the diagram, but the 0th face map instead produces the diagram of cofibers

$$X_2/X_1 \to X_3/X_1 \to \cdots \to X_n/X_1$$

(which are only well-defined up to contractible ambiguity; this ambiguity is resolved in the definition of $\mathrm{Gap}_{[n]}(\mathcal{C})$ by specifying all of the relevant cofibers ahead of time).

Remark 8. From $S_\bullet(\mathcal{C})$ we can produce a new simplicial space $X_\bullet$, given by $X_n = S_{n+1}(\mathcal{C})$; passage from $S_\bullet(\mathcal{C})$ to $X_\bullet$ has the effect of "forgetting" the 0th face map in $S_\bullet(\mathcal{C})$. This simplicial space is equivalent to the one which assigns to each $[n]$ the underlying Kan complex of $\mathrm{Fun}(\Delta^n, \mathcal{C})$. The simplicial space $X_\bullet$ is an example of a *complete Segal space*. One can show that the homotopy theory of ∞-categories is equivalent to the homotopy theory of complete Segal spaces, with the equivalence implemented by the construction $\mathcal{C} \mapsto X_\bullet$. In particular, $\mathcal{C}$ is determined by $X_\bullet$ up to equivalence. Consequently, the Waldhausen construction $\mathcal{C} \mapsto S_\bullet(\mathcal{C})$ does not lose any information about the underlying ∞-category $\mathcal{C}$.

Definition 9 (Waldhausen K-theory of ∞-Categories)**.** Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. We let $|S_\bullet(\mathcal{C})|$ denote the geometric realization of the Waldhausen construction on $\mathcal{C}$ (here we can work in the setting of topological spaces or simplicial sets; in the latter case, the geometric realization is given by the diagonal $[n] \mapsto S_n(\mathcal{C})_n$). The space $|S_\bullet(\mathcal{C})|$ has a canonical base point (up to contractible ambiguity), given by the map $S_0(\mathcal{C}) \to |S_\bullet(\mathcal{C})|$; we let $K(\mathcal{C})$ denote the loop space of $|S_\bullet(\mathcal{C})|$. For each integer $n \geq 0$, we let $K_n(\mathcal{C})$ denote the group $\pi_n K(\mathcal{C}) \simeq \pi_{n+1}|S_\bullet(\mathcal{C})|$.

Remark 10. Since the space $S_0(\mathcal{C})$ is contractible, the geometric realization $|S_\bullet(\mathcal{C})|$ is connected.

Remark 11. The space $|S_\bullet(\mathcal{C})|$ can be written as a direct limit of partial geometric realizations

$$\mathrm{sk}_0 |S_\bullet(\mathcal{C})| \to \mathrm{sk}_1 |S_\bullet(\mathcal{C})| \to \mathrm{sk}_2 |S_\bullet(\mathcal{C})| \to \cdots$$

Here the 0-skeleton $\mathrm{sk}_0 |S_\bullet(\mathcal{C})| = S_0(\mathcal{C})$ is contractible and the 1-skeleton $\mathrm{sk}_1 |S_\bullet(\mathcal{C})|$ is the suspension of $S_1(\mathcal{C})$ (which is equivalent to the underlying Kan complex of $\mathcal{C}$). Since the inclusion $\mathrm{sk}_1 |S_\bullet(\mathcal{C})| \to |S_\bullet(\mathcal{C})|$ is 1-connected, we have a surjection of fundamental groups

$$\pi_1 \Sigma S_1(\mathcal{C}) \to \pi_1 |S_\bullet(\mathcal{C})|,$$

where the left hand side is the free group generated by one symbol $[X]$ for each connected component of $S_1(\mathcal{C})$ (which we can identify with an equivalence class of objects of $\mathcal{C}$) modulo the single relation $[*] = 1$. The inclusion $\mathrm{sk}_2 |S_\bullet(\mathcal{C})| \to |S_\bullet(\mathcal{C})|$ is 2-connected, so all the relations in $\pi_1 |S_\bullet(\mathcal{C})|$ come from connected components of $S_2(\mathcal{C})$ (which we can identify with cofiber sequences in $\mathcal{C}$). These relations say exactly that $[X] = [X'] + [X'']$ whenever we have a cofiber sequence

$$X' \to X \to X''$$

in $\mathcal{C}$. Consequently, the definition of $K_0(\mathcal{C})$ given in Definition 9 agrees with the definition given in Lecture 14.

It follows from the above discussion that the group $\pi_1 |S_\bullet(\mathcal{C})| \simeq K_0(\mathcal{C})$ is abelian. In fact, this is no accident: the space $K(\mathcal{C})$ is an infinite loop space.

Remark 12. Let $\mathcal{C}$ and $\mathcal{D}$ be pointed ∞-categories which admit finite colimits and let $f : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite colimits. Then f induces a map of simplicial spaces $S_\bullet(\mathcal{C}) \to S_\bullet(\mathcal{D})$, hence also a map of K-theory spaces $K(\mathcal{C}) \to K(\mathcal{D})$.

If $\mathcal{C}$ and $\mathcal{D}$ are pointed ∞-categories which admit finite colimits, then the projection functors

$$\mathcal{C} \leftarrow \mathcal{C} \times \mathcal{D} \to \mathcal{D}$$

preserve finite colimits and therefore induce maps

$$K(\mathcal{C}) \leftarrow K(\mathcal{C} \times \mathcal{D}) \to K(\mathcal{D}).$$

These maps induce a homotopy equivalence (even an isomorphism, if we're sufficiently careful with our definitions) $K(\mathcal{C} \times \mathcal{D}) \simeq K(\mathcal{C}) \times K(\mathcal{D})$.

Note that the coproduct functor $\vee : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves finite colimits, and therefore induces a map

$$m : K(\mathcal{C}) \times K(\mathcal{C}) \simeq K(\mathcal{C} \times \mathcal{C}) \to K(\mathcal{C}).$$

Since the coproduct on $\mathcal{C}$ is coherently commutative and associative, the multiplication m is also coherently commutative and associative: that is, it is part of an E_∞-structure on the space $K(\mathcal{C})$.

The E_∞-structure on $K(\mathcal{C})$ determines a commutative monoid structure on the set $K_0(\mathcal{C}) = \pi_0 K(\mathcal{C})$. This coincides with the abelian group structure considered earlier (since $[X \vee Y] = [X] + [Y]$ in $K_0(\mathcal{C})$). It follows that $K(\mathcal{C})$ is a *grouplike* E_∞-space: that is, it is the 0th space of a connective spectrum. We will generally abuse notation and denote this spectrum also by $K(\mathcal{C})$.

Remark 13. One can obtain an explicit delooping of the space $K(\mathcal{C})$ by *iterating* the Waldhausen construction; we refer the reader to [2] for details.

References

[1] Barwick, C. *On the algebraic K-theory of higher categories.*

[2] Waldhausen, F. *Algebraic K-theory of spaces.*

The Additivity Theorem (Lecture 17)

October 10, 2014

Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. In the previous lecture, we introduced an infinite loop space $K(\mathcal{C})$, the *Waldhausen K-theory space* of $\mathcal{C}$. Note that the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$ of arrows in $\mathcal{C}$ satisfies the same hypotheses as $\mathcal{C}$, so we can also consider the K-theory space $K(\mathrm{Fun}(\Delta^1, \mathcal{C}))$. Our goal in this lecture is to prove the following fundamental result:

Theorem 1 (Additivity Theorem)**.** *The functor*

$$F : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$$

$$F(\alpha : C \to D) = (C, \mathrm{cofib}(\alpha))$$

induces a homotopy equivalence

$$K(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to K(\mathcal{C} \times \mathcal{C}) \simeq K(\mathcal{C}) \times K(\mathcal{C}).$$

Before giving the proof of Theorem 1, let us describe some of its consequences. We first note that the functor $F : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ admits a right homotopy inverse, given by the construction

$$(C, D) \mapsto (C \to C \vee D).$$

We therefore obtain:

Corollary 2. *The functor $(C, D) \mapsto (C \to C \vee D)$ induces a homotopy equivalence*

$$K(\mathcal{C} \times \mathcal{C}) \to K(\mathrm{Fun}(\Delta^1, \mathcal{C})).$$

To state the next Corollary, we will need a bit of notation. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are pointed ∞-categories which admit finite colimits, and that $G : \mathcal{C} \to \mathcal{D}$ is a functor which preserves finite colimits. We let G_* denote the associated map (of infinite loop spaces) from $K_0(\mathcal{C})$ to $K_0(\mathcal{D})$.

Corollary 3. *Let $G', G, G'' : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ be the functors given by*

$$G'(C \to D) = C \qquad G(C \to D) = D \qquad G''(C \to D) = D/C.$$

Then we have $G_* = G'_* + G''_*$ (in the abelian group of homotopy classes of maps from $K_0(\mathrm{Fun}(\Delta^1, \mathcal{C}))$ to $K_0(\mathcal{C})$).

Proof. By virtue of Corollary 2, it will suffice to show that the corresponding equality holds in the space of maps from $K(\mathcal{C} \times \mathcal{C})$ to $K(\mathcal{C})$, which follows immediately from the definition of the addition law on $K(\mathcal{C})$. $\square$

Corollary 4. *Let $\mathcal{C}$ and $\mathcal{D}$ be pointed ∞-categories which admit finite colimits, and suppose we are given a cofiber sequence*

$$F' \xrightarrow{\alpha} F \to F''$$

of functors from $\mathcal{D}$ to $\mathcal{C}$ which preserve finite colimits. Then $F_* = F'_* + F''_*$.

Proof. The natural transformation α determines a functor $H : \mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ which preserves finite colimits. Unwinding the definitions, we have

$$F'_* = G'_* H_* \quad F_* = G_* H_* \quad F''_* = G''_* H_*$$

where $G', G, G'' : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ are defined as in Corollary 3. The desired result now follows from the equality $G_* = G'_* + G''_*$. $\square$

Corollary 5. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. Then the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ induces the map $K(\mathcal{C}) \to K(\mathcal{C})$ given by multiplication by -1.*

Proof. Apply Corollary 4 to the cofiber sequence of functors

$$\mathrm{id} \to * \to \Sigma.$$

$\square$

Corollary 6. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. Then the canonical map $\mathcal{C} \to SW(\mathcal{C})$ induces a homotopy equivalence $K(\mathcal{C}) \to K(SW(\mathcal{C}))$*

Proof. The K-theory space of $SW(\mathcal{C})$ can be identified with the direct limit of the sequence

$$K(\mathcal{C}) \xrightarrow{\Sigma_*} K(\mathcal{C}) \xrightarrow{\Sigma_*} K(\mathcal{C}) \to \cdots$$

$\square$

It follows from Corollary 6 that as far as Waldhausen K-theory is concerned, we might as well always be working with stable ∞-categories (which will be the focus of our attention in the next several lectures).

Let us now turn to the proof of Theorem 1. We have an evident evaluation functor

$$e : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$

$$e(C \to D) = C.$$

Fix a zero object $* \in \mathcal{C}$. The fiber of e over $*$ can be identified with the ∞-category $\mathcal{C}_*$ of pointed objects of $\mathcal{C}$ (that is, objects $D \in \mathcal{C}$ equipped with a map $* \to D$). Since $\mathcal{C}$ is pointed, this ∞-category is equivalent to $\mathcal{C}$ itself. We therefore have a fiber sequence of ∞-categories,

$$\mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{e} \mathcal{C}.$$

which gives a diagram of K-theory spaces

$$K(\mathcal{C}) \to K(\mathrm{Fun}(\Delta^1, \mathcal{C})) \xrightarrow{e_*} K(\mathcal{C}).$$

Theorem 1 is equivalent to the assertion that this diagram is again a fiber sequence.

It follows immediately from the definitions that for every integer $n \geq 0$, we have a fiber sequence of ∞-categories

$$\mathrm{Gap}_{[n]}(\mathcal{C}) \to \mathrm{Gap}_{[n]}(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to \mathrm{Gap}_{[n]}(\mathcal{C}).$$

Passing to underlying Kan complexes and allowing n to vary, we obtain a fiber sequence of simplicial spaces

$$S_\bullet(\mathcal{C}) \to S_\bullet(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to S_\bullet(\mathcal{C}).$$

We would like to show that the induced diagram of geometric realizations remains a fiber sequence. This is not obvious: in general, a fiber sequence of simplicial pointed spaces

$$X_\bullet \to Y_\bullet \to Z_\bullet$$

need not yield a fiber sequence $|X_\bullet| \to |Y_\bullet| \to |Z_\bullet|$. One can show that this holds whenever each of the spaces Z_n is connected, but this hypothesis is far too strong for our situation (remember that the connected components of $S_1(\mathcal{C})$ can be identified with the equivalence classes of objects in $\mathcal{C}$; in particular, $S_1(\mathcal{C})$ is never connected unless $\mathcal{C}$ is trivial). We will instead use the following criterion:

Proposition 7. *Let $\mathcal{S}$ denote the ∞-category of spaces and let $\mathcal{J}$ be a small category (or ∞-category). Suppose we are given a natural transformation $X \to Y$ of functors from $\mathcal{J}$ to $\mathcal{S}$ which satisfies the following condition:*

($*$) *For every morphism $I \to J$ in $\mathcal{J}$, the associated map*

$$\varinjlim_{J \to K} X(K) \times_{Y(K)} Y(I) \to \varinjlim_{I \to K} X(K) \times_{Y(K)} Y(I)$$

is a homotopy equivalence.

Suppose we have a pullback diagram of functors

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y. \end{tikzcd}$$

Then:

(a) *The map $X' \to Y'$ also satisfies ($*$).*

(b) *The diagram*

$$\begin{tikzcd} \varinjlim X' \arrow[r] \arrow[d] & \varinjlim X \arrow[d] \\ \varinjlim Y' \arrow[r] & \varinjlim Y \end{tikzcd}$$

is also a pullback square.

Proof. We first prove (a). Fix a morphism $I \to J$ in $\mathcal{J}$. We have a commutative diagram

$$\begin{tikzcd} \varinjlim_{J \to K} X'(K) \times_{Y'(K)} Y'(I) \arrow[r] \arrow[d] & \varinjlim_{I \to K} X'(K) \times_{Y'(K)} Y'(I) \arrow[d] \\ \varinjlim_{J \to K} X'(K) \times_{Y'(K)} Y(I) \arrow[r] \arrow[d] & \varinjlim_{I \to K} X'(K) \times_{Y'(K)} Y(I) \arrow[d] \\ \varinjlim_{J \to K} X(K) \times_{Y(K)} Y(I) \arrow[r] & \varinjlim_{I \to K} X(K) \times_{Y(K)} Y(I) \end{tikzcd}$$

where each square is a pullback. Since the lower horizontal map is a homotopy equivalence, so is the upper horizontal map.

For every morphism $I \to J$ in $\mathcal{J}$, let $F(I \to J)$ denote the colimit $\varinjlim_{J \to K} X(K) \times_{Y(K)} Y(I)$. *A priori*, the functor F is covariant in I and contravariant in J. However, condition ($*$) implies that F is actually independent of J. More precisely, we can write $F(I \to J) = F_0(I)$ for some functor $F_0 : \mathcal{J} \to \mathcal{S}$. Abstractly, we can describe F_0 as the left Kan extension of F along the forgetful functor $(I \to J) \mapsto I$ (from the twisted arrow category of $\mathcal{J}$ to $\mathcal{J}$). Concretely, we can write $F_0(I) = F(I \to I) = \varinjlim_{I \to K} X(K) \times_{Y(K)} Y(I)$. Note that we have an evident projection map $F_0(I) \to Y(I)$, and condition ($*$) implies that for every map $I \to J$, the associated map

$$\begin{tikzcd} F_0(I) \arrow[r] \arrow[d] & F_0(J) \arrow[d] \\ Y(I) \arrow[r] & Y(J) \end{tikzcd}$$

is a pullback square. It follows that for each $I \in \mathcal{I}$, the diagram

$$\begin{tikzcd} F_0(I) \arrow[r] \arrow[d] & \varinjlim F_0 \arrow[d] \\ Y(I) \arrow[r] & \varinjlim Y \end{tikzcd}$$

is a pullback square.

Using (a), we can apply the same reasoning to the natural transformation $X' \to Y'$ to obtain a functor $F'_0 : \mathcal{J} \to \mathcal{S}$. The proof of (a) shows that for each $I \in \mathcal{I}$, the diagram

$$\begin{tikzcd} F'_0(I) \arrow[r] \arrow[d] & F_0(I) \arrow[d] \\ Y'(I) \arrow[r] & Y(I) \end{tikzcd}$$

is a pullback square. It follows that we also have a pullback square

$$\begin{tikzcd} F'_0(I) \arrow[r] \arrow[d] & \varinjlim F_0 \arrow[d] \\ Y'(I) \arrow[r] & \varinjlim Y. \end{tikzcd}$$

Passing to the colimit over I (and using the fact that colimits in $\mathcal{S}$ commute with base change), we obtain a pullback square

$$\begin{tikzcd} \varinjlim F'_0 \arrow[r] \arrow[d] & \varinjlim F_0 \arrow[d] \\ \varinjlim Y' \arrow[r] & \varinjlim Y. \end{tikzcd}$$

We conclude by observing that there are canonical equivalences

$$\begin{aligned} \varinjlim F_0 &\simeq \varinjlim F \\ &\simeq \varinjlim_{I \to J} \varinjlim_{J \to K} X(K) \times_{Y(K)} Y(I) \\ &\simeq \varinjlim_{K} X(K) \times_{Y(K)} \varinjlim_{I \to J \to K} Y(I) \\ &\simeq \varinjlim_{K} X(K) \times_{Y(K)} Y(K) \\ &\simeq \varinjlim X, \end{aligned}$$

and similarly $\varinjlim F'_0 \simeq \varinjlim X'$. □

Exercise 8. Let $X \to Y$ be a natural transformation of functors $\mathcal{J} \to \mathcal{S}$. Let us say that a morphism $f : I \to J$ in $\mathcal{J}$ is *good* if the natural map

$$\varinjlim_{J \to K} X(K) \times_{Y(K)} Y(I) \to \varinjlim_{I \to K} X(K) \times_{Y(K)} Y(I)$$

is a homotopy equivalence. Show that if we are given a pair of morphisms

$$I \xrightarrow{f} J \xrightarrow{g} K$$

in $\mathcal{J}$ such that g is good, then f is good if and only if the composition $g \circ f$ is good.

In order to prove Theorem 1, it will suffice to show that the evaluation map e induces a map of simplicial spaces $S_\bullet(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to S_\bullet(\mathcal{C})$ which satisfies the requirements of Proposition 7 (where we take $\mathcal{J} = \mathbf{\Delta}^{\mathrm{op}}$ to be the opposite of the category of nonempty finite linearly ordered sets). Fix a map of linearly ordered sets $\alpha : [n'] \to [n]$; we wish to show that the induced map

$$\varinjlim_{\beta:[m]\to[n']} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} S_n(\mathcal{C}) \to \varinjlim_{\beta:[m]\to[n]} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} S_n(\mathcal{C})$$

is a homotopy equivalence. Using Exercise 8, we may reduce to the case where $n' = 0$.

Let us fix a point of $S_n(\mathcal{C})$, corresponding to an $[n]$-gapped object $X' \in \mathcal{C}$. Passing to the homotopy fibers over X', we are reduced to proving that the map

$$\theta : \varinjlim_{\beta:[m]\to[0]} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} \{X'\} \to \varinjlim_{\beta:[m]\to[n]} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} \{X'\}$$

is a homotopy equivalence. Let $\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ be the functor given by $(f : C \to D) \mapsto \mathrm{cofib}(f)$, so that cofib induces maps $S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to S_m(\mathcal{C})$. It follows immediately from the definitions that the composite map

$$\varinjlim_{\beta:[m]\to[0]} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} \{X'\} \xrightarrow{\theta} \varinjlim_{\beta:[m]\to[n]} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} \{X'\} \xrightarrow{\theta'} \varinjlim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} S_m(\mathcal{C})$$

is a homotopy equivalence, so that θ is a homotopy equivalence if and only if θ' is a homotopy equivalence. In particular, we see that the condition that θ is a homotopy equivalence is independent of the choice of map $\alpha : [0] \to [n]$. We may therefore assume without loss of generality that $\alpha(0) = n$.

For every map $\beta : [m] \to [n]$, let X'_β denote the image of X' in $\mathrm{Gap}_{[m]}(\mathcal{C})$. Unwinding the definitions, we can identify $S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} \{X'\}$ with the Kan complex

$$Z_\beta = \mathrm{Gap}_{[m]}(\mathcal{C})^{\simeq}_{X'_\beta/},$$

whose vertices are maps $X'_\beta \to X$ in the ∞-category of $[m]$-gapped objects of $\mathcal{C}$. Note that if $\beta \leq \beta'$, then there is a canonical map $\tau_{\beta,\beta'} : Z_\beta \to Z_{\beta'}$ given by $X \mapsto X'_{\beta'} \amalg_{X'_\beta} X$.

Let us regard $[n]$ as fixed, and let $\mathcal{I}$ denote the category whose objects are nonempty finite linearly ordered sets $[m]$ and monotone maps $\beta : [m] \to [n]$. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be the full subcategory consisting of those maps $\beta : [m] \to [n]$ which take the constant value n (so that $\mathcal{I}_0$ is equivalent to the category $\mathbf{\Delta}$). The construction $\beta \mapsto Z_\beta$ determines a functor $\mathcal{I}^{\mathrm{op}} \to \mathcal{S}$, and we wish to show that the canonical map

$$\varinjlim_{\beta\in\mathcal{I}_0^{\mathrm{op}}} Z_\beta \to \varinjlim_{\beta\in\mathcal{I}^{\mathrm{op}}} Z_\beta$$

is a homotopy equivalence.

The key observation is that the construction $\beta \mapsto Z_\beta$ has a little bit of extra functoriality. Let us define an enlargement $\mathcal{I}_+$ of $\mathcal{I}$ as follows:

- The objects of $\mathcal{I}_+$ are nonempty finite linearly ordered sets $[m]$ equipped with monotone maps $\beta : [m] \to [n]$.
- A morphism from $\beta : [m] \to [n]$ to $\beta' : [m'] \to [n]$ in $\mathcal{I}_+$ consists of a monotone map $\gamma : [m] \to [m']$ such that $\beta(i) \geq \beta'(\gamma(i))$ for $0 \leq i \leq m$ (this is an enlargement of the collection of morphisms in $\mathcal{I}$, where we would require the stronger condition $\beta = \beta' \circ \gamma$).

Any morphism γ in $\mathcal{I}_+$ determines a map $Z_{\beta'} \to Z_\beta$, which carries a map of $[m']$-gapped objects $X'_{\beta'} \to X$ to the map of $[m]$-gapped objects $X'_\beta \to Y$ where

$$Y(i,j) = X(\gamma(i), \gamma(j)) \amalg_{X'(\beta'(\gamma(i)), \beta'(\gamma(j)))} X'(\beta(i), \beta(j)).$$

We therefore obtain a commutative diagram

$$\begin{tikzcd}
\varinjlim_{\beta \in \mathcal{I}_0^{op}} Z_\beta \arrow[rr] \arrow[dr] & & \varinjlim_{\beta \in \mathcal{I}^{op}} Z_\beta \arrow[dl] \\
& \varinjlim_{\beta \in \mathcal{I}_+^{op}} Z_\beta . &
\end{tikzcd}$$

To prove that the upper horizontal map is a homotopy equivalence, it will suffice to show that the lower horizontal maps are homotopy equivalences. This follows from the following combinatorial observation:

Lemma 9. *The inclusion maps $\mathcal{I}_0 \hookrightarrow \mathcal{I}_+$ and $\mathcal{I} \hookrightarrow \mathcal{I}_+$ are right cofinal.*

Proof. The right cofinality of $\mathcal{I}_0 \hookrightarrow \mathcal{I}_+$ follows from the fact that it admits a right adjoint (which carries an arbitrary map $\beta : [m] \to [n]$ to the constant map $[m] \to \{n\}$). To prove the right cofinality of the inclusion $\mathcal{I} \hookrightarrow \mathcal{I}_+$, we must work a little bit harder. Fix an object of $\mathcal{I}_+$ given by a map $\beta : [m] \to [n]$. Unwinding the definitions, we see that the overcategory $\mathcal{J} = \mathcal{I} \times_{\mathcal{I}_+} (\mathcal{I}_+)_{/\beta}$ can be identified with the category whose objects are nonempty finite linearly ordered sets $[k]$ equipped with a monotone map $\gamma : [k] \to P$, where $P = \{(i,j) \in [m] \times [n] : \beta(i) \leq j\}$. This category contains as a deformation retract the full subcategory spanned by the injective maps, whose geometric realization is homeomorphic to $|\mathrm{N}(P)|$. It will therefore suffice to show that the partially ordered set P is weakly contractible, which is clear because P has a smallest element. $\square$

References

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Additive K-Theory (Lecture 18)

October 15, 2014

Let $\mathcal{C}$ be a pointed ∞-category ∞-category which admits finite colimits, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under finite colimits, and assume that every object of $\mathcal{C}$ can be obtained as a retract of an object of $\mathcal{C}_0$. In Lectures 14 and 15, we saw that there can be a big difference between $K_0(\mathcal{C})$ and $K_0(\mathcal{C}_0)$: in the stable case, an object $C \in \mathcal{C}$ belongs to (the essential image) of $\mathcal{C}_0$ if and only if the class $[C] \in K_0(\mathcal{C})$ belongs to the image of the map $K_0(\mathcal{C}_0) \hookrightarrow K_0(\mathcal{C})$. Our first goal in this lecture is to show that the difference between $\mathcal{C}$ and $\mathcal{C}_0$ disappears when we look at *higher* K-groups. More precisely, we have the following result:

Proposition 1. *Let $\mathcal{C}$ and $\mathcal{C}_0$ be as above. Then the canonical map $K_n(\mathcal{C}_0) \to K_n(\mathcal{C})$ is an isomorphism for $n > 0$. In other words, the diagram of spaces*

$$\begin{tikzcd} K(\mathcal{C}_0) \arrow[r] \arrow[d] & K(\mathcal{C}) \arrow[d] \\ K_0(\mathcal{C}_0) \arrow[r] & K_0(\mathcal{C}) \end{tikzcd}$$

is a homotopy pullback square.

To prove Proposition 1, we may assume without loss of generality that $\mathcal{C}$ and $\mathcal{C}_0$ are stable (since the construction $\mathcal{C} \mapsto SW(\mathcal{C})$ has no effect on K-theory). For every group G, let $B_\bullet(G)$ denote the simplicial set which models the classifying space of G, so that the set $B_n(G)$ of n-simplices of $B_\bullet(G)$ can be identified with G^n. Let us describe this in a way that makes the simplicial structure of $B_\bullet G$ more apparent. Using additive notation for the group structure on G (we will ultimately be interested in the case where G is abelian), we can identify $B_n G$ with the set of maps

$$f : [n]^{(2)} = \{(i,j) \in [n] \times [n] : i \leq j\} \to G$$

which have the property that $f(i,i) = 0$ and $f(i,j) + f(j,k) = f(i,k)$ for $i \leq j \leq k$.

If $\mathcal{C}$ is a pointed ∞-category which admits finite colimits, then every $[n]$-gapped object $X : [n]^{(2)} \to \mathcal{C}$ determines a map $f : [n]^{(2)} \to K_0(\mathcal{C})$, given by $f(i,j) = [X(i,j)]$, and f will satisfy the condition above. This construction is functorial in $[n]$ and therefore gives rise to a map of simplicial spaces

$$S_\bullet(\mathcal{C}) \to B_\bullet(K_0(\mathcal{C})).$$

The natural map $K(\mathcal{C}) \to K_0(\mathcal{C})$ can then be obtained by passing to classifying spaces and then applying Ω. We may therefore rephrase Proposition 1 as follows:

Proposition 2. *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{C}_0$ be a full stable subcategory such that every object of $\mathcal{C}$ is a direct summand of an object of $\mathcal{C}_0$. Then the diagram*

$$\begin{tikzcd} {|S_\bullet(\mathcal{C}_0)|} \arrow[r] \arrow[d] & {|S_\bullet(\mathcal{C})|} \arrow[d] \\ {|B_\bullet K_0(\mathcal{C}_0)|} \arrow[r] & {|B_\bullet K_0(\mathcal{C})|} \end{tikzcd}$$

is a homotopy pullback square.

Because the map $K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$ is injective and an object $C \in \mathcal{C}$ belongs to $\mathcal{C}_0$ if and only if its K-theory class $[C]$ lifts to $K_0(\mathcal{C}_0)$, the diagram of simplicial spaces

$$\begin{tikzcd} S_\bullet(\mathcal{C}_0) \arrow[r] \arrow[d] & S_\bullet(\mathcal{C}) \arrow[d] \\ B_\bullet K_0(\mathcal{C}_0) \arrow[r] & B_\bullet K_0(\mathcal{C}) \end{tikzcd}$$

is a homotopy pullback square. As in the previous lecture, we need to show that this remains true after geometric realization. Once again, this conclusion is not purely formal, because the spaces $B_n K_0(\mathcal{C})$ are not connected (in fact, they are discrete). Our proof will proceed by taking advantage of some additional structure available in this situation: in this case, the coherently associative addition law on the spaces involved (given by the formation of coproducts in $\mathcal{C}$).

Notation 3. Let $\mathcal{S}$ denote the ∞-category of spaces and let Sp denote the ∞-category of spectra. The formation of 0th spaces determines a functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$. The ∞-category Sp is stable, so that products and coproducts coincide. Consequently, every object $E \in \mathrm{Sp}$ can be regarded as a commutative monoid object of Sp in an essentially unique way. It follows that Ω^∞ determines a map $\mathrm{Sp} \to \mathrm{CAlg}(\mathcal{S})$, where $\mathrm{CAlg}(\mathcal{S})$ denotes the ∞-category of commutative monoid objects of $\mathcal{S}$: that is, the ∞-category of E_∞-spaces.

It follows from abstract nonsense that the functor $\Omega^\infty : \mathrm{Sp} \to \mathrm{CAlg}(\mathcal{S})$ admits a left adjoint, which we will denote by $X \mapsto X^{\mathrm{gp}}$. We will refer to X^{gp} as the *group completion* of X. Tautologically, any E_∞-space X is equipped with an E_∞-map $X \to \Omega^\infty X^{\mathrm{gp}}$. Nontautologically, one can show that this map is a homotopy equivalence if and only if X is grouplike: that is, $\pi_0 X$ is a group.

Let $\mathcal{C}$ be an ∞-category which admits finite coproducts. Then the formation of coproducts endows the underlying Kan complex $\mathcal{C}^{\simeq}$ with the structure of an E_∞-space. We will refer to the group completion $(\mathcal{C}^{\simeq})^{\mathrm{gp}}$ as the *additive K-theory spectrum* of $\mathcal{C}$ and denote it by $K_{\mathrm{add}}(\mathcal{C})$ (note that this conflicts with the notation of Lecture 14, where we used the same notation for the abelian group $\pi_0 K_{\mathrm{add}}(\mathcal{C})$).

If $\mathcal{C}$ is a pointed ∞-category which admits finite colimits, then each $\mathrm{Gap}_{[n]}(\mathcal{C})$ has the same property. It follows that each $S_n(\mathcal{C})$ is an E_∞-space which has a group completion $S_n(\mathcal{C})^{\mathrm{gp}}$. Since the geometric realization $|S_\bullet \mathcal{C}|$ is grouplike (it is connected, we have

$$\begin{aligned} |S_\bullet(\mathcal{C})| &\simeq \Omega^\infty(|S_\bullet(\mathcal{C})^{\mathrm{gp}}) \\ &\simeq \Omega^\infty(|S_\bullet(\mathcal{C})^{\mathrm{gp}}|). \end{aligned}$$

It follows that the diagram of Proposition 2 is obtained by applying Ω^∞ to a diagram of spectra

$$\begin{tikzcd} {|S_\bullet(\mathcal{C}_0)^{\mathrm{gp}}|} \arrow[r] \arrow[d] & {|S_\bullet(\mathcal{C})^{\mathrm{gp}}|} \arrow[d] \\ {|HB_\bullet K_0(\mathcal{C}_0)|} \arrow[r] & {|HB_\bullet K_0(\mathcal{C})|.} \end{tikzcd}$$

The functor Ω^∞ preserves pullback squares, and the formation of geometric realizations of spectra commutes with pullbacks (since the ∞-category Sp is stable). It will therefore suffice to show that each of the diagrams

$$\begin{tikzcd} S_n(\mathcal{C}_0)^{\mathrm{gp}} \arrow[r] \arrow[d] & S_n(\mathcal{C})^{\mathrm{gp}} \arrow[d] \\ HB_n K_0(\mathcal{C}_0) \arrow[r] & HB_n K_0(\mathcal{C}) \end{tikzcd}$$

is a pullback square. Replacing $\mathcal{C}$ by $\mathrm{Gap}_{[n]}(\mathcal{C})$, we can reduce to the case $n = 1$. Proposition 2 is now reduced to the following "additive" version:

Proposition 4. *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{C}_0$ be a full stable subcategory such that every object of $\mathcal{C}$ is a direct summand of an object of $\mathcal{C}_0$. Then the diagram*

$$\begin{tikzcd} K_{\mathrm{add}}(\mathcal{C}_0) \arrow[r] \arrow[d] & K_{\mathrm{add}}(\mathcal{C}) \arrow[d] \\ HK_0(\mathcal{C}_0) \arrow[r] & HK_0(\mathcal{C}) \end{tikzcd}$$

is a homotopy pullback square.

Proof. This is a version of the group completion theorem. Let us indicate a proof. The spectra involved are connective, and the vertical maps are surjective on π_0. Consequently, it will suffice to show that the diagram of 0th spaces

$$\begin{tikzcd} \Omega^\infty K_{\mathrm{add}}(\mathcal{C}_0) \arrow[r] \arrow[d] & \Omega^\infty K_{\mathrm{add}}(\mathcal{C}) \arrow[d] \\ K_0(\mathcal{C}_0) \arrow[r] & K_0(\mathcal{C}) \end{tikzcd}$$

is a pullback square. Let Z denote the inverse image of $K_0(\mathcal{C}_0)$ in $\Omega^\infty K_{\mathrm{add}}(\mathcal{C})$; we wish to show that the canonical map $\theta : \Omega^\infty K_{\mathrm{add}}(\mathcal{C}_0) \to Z$ is a homotopy equivalence. Since $\Omega^\infty K_{\mathrm{add}}(\mathcal{C}_0)$ and X are simple (they are infinite loop spaces), it will suffice to check that θ induces an isomorphism in homology.

Consider the singular chain complexes

$$A_0 = C_*(\Omega^\infty K_{\mathrm{add}}(\mathcal{C}_0); \mathbf{Z}) \qquad A = C_*(\Omega^\infty K_{\mathrm{add}}(\mathcal{C}); \mathbf{Z}).$$

Using the E_∞-structures on the spaces involved, we can regard A_0 and A as E_∞-algebras over $\mathbf{Z}$. Similarly, we have E_∞-algebras

$$B_0 = C_*(\mathcal{C}_0^{\simeq}; \mathbf{Z}) \qquad B = C_*(\mathcal{C}^{\simeq}; \mathbf{Z}).$$

Note that B contains B_0 as a direct summand, and in fact we have a natural grading $B = \bigoplus B_\alpha$ where α ranges over the cosets of $K_0(\mathcal{C}_0)$ in $K_0(\mathcal{C})$.

Using the universal property of the group completion, we see that A_0 can be obtained from B_0 by inverting all elements of the form $[X] \in \mathbf{Z}[K_0(\mathcal{C}_0)] \simeq \mathrm{H}_0(A_0)$ for $X \in \mathcal{C}_0$, and that A can be obtained from B by inverting all elements $[X]$ for $X \in \mathcal{C}$. However, since every object of $\mathcal{C}$ is a direct summand of an object in $\mathcal{C}_0$, we only need to invert the classes $[X]$ for $X \in \mathcal{C}_0$. We therefore have a canonical equivalence $A \simeq A_0 \otimes_{B_0} B$. This equivalence determines a direct sum decomposition

$$A \simeq \bigoplus_\alpha A_0 \otimes_{B_0} B_\alpha,$$

where the chain complex $C_*(X; \mathbf{Z})$ can be identified with the summand corresponding to $\alpha = 0$. From this description, it is clear that $A_0 \simeq C_*(X; \mathbf{Z})$. $\square$

Sometimes there is not much difference between K-theory and additive K-theory. Roughly speaking, we would expect this behavior in a situation where every cofiber sequence

$$X' \to X \to X''$$

splits. However, this hypothesis is unreasonably strong in the context we have been discussing so far: for a cofiber sequence

$$X \to * \to \Sigma(X)$$

to split, we must have $X \simeq *$. It will therefore be useful to consider a slightly more general setup:

Definition 5. An ∞-*category with cofibrations* is a pointed ∞-category $\mathcal{C}$ with a distinguished class of morphisms, which we will call *cofibrations*, which satisfy the following axioms:

- All equivalences are cofibrations and the collection of cofibrations is closed under composition.
- For every object X in $\mathcal{C}$, the canonical map $* \to X$ is a cofibration.
- For a cofibration $f : X \to X'$ and an arbitrary map $X \to Y$, there exists a pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

  and the map g is also a cofibration.

Warning 6. We are using the term "cofibration" in order to follow the language of Waldhausen's paper, but the notion of cofibration considered above does not *a priori* have any relationship to the notion of cofibration in the language of model categories.

Example 7. Let $\mathcal{C}$ be a pointed ∞-category. One way to try to satisfy the axiomatics of Definition 5 is to have as many cofibrations as possible. We can make $\mathcal{C}$ into an ∞-category with cofibrations where *all* morphisms are cofibrations if and only if $\mathcal{C}$ has finite colimits.

Example 8. Let $\mathcal{C}$ be a pointed ∞-category. Another way to try to satisfy the axiomatics of Definition 5 is to have as few cofibrations as possible. Note that if for any pair of objects X and Y, the natural map $* \to X$ is a cofibration and therefore there exists a pushout square

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \end{tikzcd}$$

where the lower horizontal map is a cofibration. Consequently, if $\mathcal{C}$ is an ∞-category with cofibrations, then $\mathcal{C}$ must have coproducts and every map of the form $Y \to X \vee Y$ must be a cofibration.

Conversely, suppose that $\mathcal{C}$ is a pointed ∞-category which admits finite coproducts. Then $\mathcal{C}$ can be made into an ∞-category with cofibrations by declaring that a morphism f is a cofibration if and only if it is equivalent to a morphism of the form $Y \to X \vee Y$; we will refer to such a morphism as a *split cofibration*.

Let $\mathcal{C}$ be an ∞-category with cofibrations. For each integer n, we let $\mathrm{Gap}_{[n]}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}\{(i,j) \in [n] \times [n] : i \leq j\}, \mathcal{C})$ spanned by those functors X satisfying the following three conditions:

- For each $i \leq j \leq k$, the natural map $X(i,j) \to X(i,k)$ is a cofibration.
- For each i, the object $X(i,i)$ is zero.
- For each $i \leq j \leq k$, the diagram

$$\begin{tikzcd} X(i,j) \arrow[r] \arrow[d] & X(i,k) \arrow[d] \\ * \arrow[r] & X(j,k) \end{tikzcd}$$

  is a pushout square.

Arguing as in Lecture 16, we see that an object X of $\mathrm{Gap}_{[n]}(\mathcal{C})$ is determined by the diagram

$$X(0,1) \to X(0,2) \to \cdots \to X(0,n).$$

The only difference is that this time, we consider only those diagrams where each map is a cofibration.

Definition 9. Let $\mathcal{C}$ be an ∞-category with cofibrations. We let $S_\bullet(\mathcal{C})$ denote the simplicial space given by the formula $S_n(\mathcal{C}) = \mathrm{Gap}_{[n]}(\mathcal{C})^{\simeq}$, where $\mathrm{Gap}_{[n]}(\mathcal{C})$ is defined as above. We let $K(\mathcal{C})$ denote the space given by $\Omega|S_\bullet(\mathcal{C})|$.

The simplicial space $S_\bullet(\mathcal{C})$ and $K(\mathcal{C})$ depend not only on the ∞-category $\mathcal{C}$, but also on the class of cofibrations chosen. For example, if $\mathcal{C}$ admits finite colimits and we declare that all morphisms are cofibrations (Example 7), then we recover the definitions of Lecture 16. If $\mathcal{C}$ admits finite coproducts and we use only the split cofibrations (Example 8), then we often recover additive K-theory $K_{\mathrm{add}}(\mathcal{C})$:

Theorem 10. *Let $\mathcal{C}$ be an ∞-category which admits finite products and finite coproducts, and assume that the homotopy category of $\mathcal{C}$ is additive (so that finite products and finite coproducts in $\mathcal{C}$ coincide). For example, we can take any stable ∞-category, or any subcategory of a stable ∞-category which is closed under direct sums. Regard $\mathcal{C}$ as an ∞-category with cofibrations as in Example 8 (allowing only split cofibrations). Then there is a canonical homotopy equivalence $K_{\mathrm{add}}(\mathcal{C}) \to K(\mathcal{C})$ (where we abuse notation by identifying $K_{\mathrm{add}}(\mathcal{C})$ with its 0th space).*

We can identify the 0th space of $K_{\mathrm{add}}(\mathcal{C})$ with the $\Omega|Y_\bullet|$, where $Y_\bullet$ is the simplicial space given by $Y_n = (\mathcal{C}^{\simeq})^n$ (made into a simplicial space using the coproduct on $\mathcal{C}^{\simeq}$). The map $K_{\mathrm{add}}(\mathcal{C}) \to K(\mathcal{C})$ is then obtained from a map of simplicial spaces

$$Y_\bullet \to S_\bullet(\mathcal{C});$$

in degree n this map is given by the construction

$$(C_1, \ldots, C_n) \mapsto (C_1 \to C_1 \oplus C_2 \to \cdots \to C_1 \oplus \cdots \oplus C_n).$$

We wish to show that the induced map of geometric realizations $|Y_\bullet| \to |S_\bullet(\mathcal{C})|$ is a homotopy equivalence. All of the spaces in sight admit E_∞-structures coming from the formation of coproducts in $\mathcal{C}$. Arguing as before, we obtain homotopy equivalences

$$\begin{aligned} |Y_\bullet| &\simeq \Omega^\infty |Y_\bullet|^{\mathrm{gp}} \\ &\simeq \Omega^\infty |Y_\bullet^{\mathrm{gp}}|. \\ |S_\bullet(\mathcal{C})| &\simeq \Omega^\infty |S_\bullet(\mathcal{C})|^{\mathrm{gp}} \\ &\simeq \Omega^\infty |S_\bullet(\mathcal{C})^{\mathrm{gp}}|. \end{aligned}$$

It will therefore suffice to show that for each $n \geq 0$, the map $Y_n \to S_n(\mathcal{C})$ induces a homotopy equivalence of spectra $Y_n^{\mathrm{gp}} \to S_n(\mathcal{C})^{\mathrm{gp}}$.

For simplicity, let us consider the case $n = 2$ (the general case is only notationally more difficult). The space $S_2(\mathcal{C})$ classifies morphisms $f : X \to X'$ which are split cofibrations in $\mathcal{C}$. Let $e : S_2(\mathcal{C}) \to \mathcal{C}^{\simeq}$ be the map given by $e(X \to X') = X$, let $\pi : \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} \to \mathcal{C}^{\simeq}$ be projection onto the first factor, and let $\iota : \mathcal{C}^{\simeq} \to \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$ be given by $X \mapsto (*, X)$. We then have a commutative diagram of E_∞-spaces

$$\begin{tikzcd} \mathcal{C}^{\simeq} \arrow[r, "\iota"] \arrow[d] & \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} \arrow[r] \arrow[d, "\pi"] & S_2(\mathcal{C}) \arrow[d, "e"] \\ * \arrow[r] & \mathcal{C}^{\simeq} \arrow[r, "\mathrm{id}"] & \mathcal{C}^{\simeq}. \end{tikzcd}$$

We wish to show that the upper left horizontal map becomes an equivalence after group completion. In other words, we wish to show that the square on the right becomes a pullback square after group completion. Since the ∞-category of spectra is stable, this is equivalent to the assertion that the square on the right becomes a pushout square after group completion. The left square is clearly a pushout after group completion; it will therefore suffice to show that the outer square is a pushout after group completion. In fact, we claim that the left square is a pushout *before* group completion. In other words, we claim that $\mathcal{C}^{\simeq}$ can be obtained as a one-sided bar construction $S_2(\mathcal{C}) \otimes_{\mathcal{C}^{\simeq}} *$ in the ∞-category of spaces, where the ∞-category $\mathcal{C}^{\simeq}$ acts on $S_2(\mathcal{C})$ via the construction

$$a : \mathcal{C}^{\simeq} \times S_2(\mathcal{C}) \to S_2(\mathcal{C})$$

$$(C, X \to X') \mapsto (X \to X' \oplus C).$$

This is an assertion which can be tested fiberwise over $\mathcal{C}^{\simeq}$. In other words, we are reduced to proving the following:

Proposition 11. *In the situation of Theorem 10, fix an object $X \in \mathcal{C}$, and let $\mathcal{D}$ denote the full subcategory of $\mathcal{C}_{X/}$ spanned by the split cofibrations $X \to X'$. Let $\mathcal{C}^{\simeq}$ act on the space $\mathcal{D}^{\simeq}$ as above. Then the homotopy quotient*

$$\mathcal{D}^{\simeq} / \mathcal{C}^{\simeq} := \mathcal{D}^{\simeq} \otimes_{\mathcal{C}^{\simeq}} *$$

is contractible.

Proof. Note that the ∞-category $\mathcal{D}$ admits finite coproducts (given by pushouts over X), so that $\mathcal{D}^{\simeq}$ is an E_∞-space. We can regard the quotient $\mathcal{D}^{\simeq} / \mathcal{C}^{\simeq}$ as the cofiber of the natural map

$$f : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$$

$$C \mapsto (X \to X \oplus C)$$

in the ∞-category of E_∞-spaces. By construction, the map f is surjective on π_0 so that the quotient $\mathcal{D}^{\simeq} / \mathcal{C}^{\simeq}$ is connected. In particular, $\mathcal{D}^{\simeq} / \mathcal{C}^{\simeq}$ is grouplike, so it can be identified with (the 0th space of) its group completion. It will therefore suffice to show that the map f induces an equivalence of group completions.

Define $q : \mathcal{D}^{\simeq} \to \mathcal{C}^{\simeq}$ by the formula $q(X \to X') = X'/X$. The map q is obviously a left homotopy inverse to f. To complete the proof, it will suffice to show that it is also a right homotopy inverse after group completion. In other words, we wish to show that the composite map

$$(f \circ q) : \mathcal{D}^{\simeq} \to \mathcal{D}^{\simeq}$$

$$(X \to X') \mapsto (X \to X \oplus (X'/X)$$

is homotopic to the identity map after group completion. In fact, we claim that $(f \circ q)$ is homotopic to the identity map id after adding a single copy of the identity map: that is, to any split cofibration $X \to X'$, we can functorially identify the split cofibrations

$$X \to X' \amalg_X X'$$

$$X \to X' \oplus (X'/X).$$

This identification follows from the additivity assumption on $\mathcal{C}$ (the "fold map" $X' \amalg_X X' \to X$ is split by the inclusion of either factor). $\square$

References

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Algebraic K-Theory of Ring Spectra (Lecture 19)

October 17, 2014

Let R be an associative ring spectrum (here by associative we mean A_∞ or associative up to *coherent* homotopy; homotopy associativity is not sufficient for the considerations which follow). Then we can consider the ∞-category Mod_R whose objects are (right) R-module spectra. We say that an R-module M is *perfect* if it is a compact object of Mod_R. We let $\mathrm{Mod}_R^{\mathrm{perf}}$ denote the full subcategory of Mod_R spanned by the perfect R-modules.

Remark 1. The full subcategory $\mathrm{Mod}_R^{\mathrm{perf}}$ can be characterized as the smallest full subcategory of Mod_R which contains R and is closed under finite colimits, desuspensions, and passage to direct summands.

Example 2. Every associative ring R can be regarded as an associative ring spectrum (by identifying R with its Eilenberg-MacLane spectrum HR). In this case, the ∞-category Mod_R has a concrete algebraic description: its objects can be identified with chain complexes of (ordinary) R-modules, and the objects of $\mathrm{Mod}_R^{\mathrm{perf}}$ can be identified with bounded chain complexes of finitely generated projective R-modules.

The ∞-categories Mod_R and $\mathrm{Mod}_R^{\mathrm{perf}}$ are stable. In particular, $\mathrm{Mod}_R^{\mathrm{perf}}$ is a pointed ∞-category which admits finite colimits, so we can consider its K-theory.

Definition 3. Let R be an associative ring spectrum. We set $K(R) = K(\mathrm{Mod}_R^{\mathrm{perf}})$. We will refer to $K(R)$ as the *algebraic K-theory space* of R.

Remark 4. It may appear that Definition 3 is a very special case of the construction described in Lecture 16. However, this is not really the case: the K-theory of an arbitrary pointed ∞-category $\mathcal{C}$ which admits finite colimits can be described in terms of the K-theory of ring spectra. This can be seen as follows:

(a) Since the K-theory of $\mathcal{C}$ is the same as the K-theory of its Spanier-Whitehead ∞-category $SW(\mathcal{C})$, we might as well assume that $\mathcal{C}$ is stable.

(b) In the previous lecture, we showed that replacing $\mathcal{C}$ by its idempotent completion has little effect on the space $K(\mathcal{C})$ (it only changes the set of connected components). We therefore might as well assume that $\mathcal{C}$ is idempotent complete.

(c) We will say that an object $C \in \mathcal{C}$ is a *generator* if the smallest stable subcategory of $\mathcal{C}$ which contains C and is idempotent complete is $\mathcal{C}$ itself. In general, there need not exist a generator for $\mathcal{C}$: however, we can always write $\mathcal{C}$ as a filtered union $\bigcup \mathcal{C}_\alpha$, where each $\mathcal{C}_\alpha$ is a stable subcategory which has a generator. Then $K(\mathcal{C}) \simeq \varinjlim K(\mathcal{C}_\alpha)$. We therefore might as well assume that $\mathcal{C}$ has a generator.

(d) For any object $C \in \mathcal{C}$, the sequence of spaces $\{\mathrm{Map}_{\mathcal{C}}(C, \Sigma^n C)\}_{n \geq 0}$ determine a spectrum which we will denote by $\mathrm{End}(C)$. One can show that $\mathrm{End}(C)$ is a ring spectrum, and that there is a fully faithful embedding

$$\mathrm{Mod}_{\mathrm{End}(C)}^{\mathrm{perf}} \hookrightarrow \mathcal{C}$$
$$M \mapsto M \wedge_{\mathrm{End}(C)} C$$

which is an equivalence if and only if C is a generator. If this condition is satisfied, we then have $K(\mathcal{C}) \simeq K(\mathrm{End}(C))$.

Let R be an associative ring spectrum and let M be an R-module. We will say that M is *finitely generated and projective* if it can be realized as a direct summand of R^n for some n. We let $\mathrm{Mod}_R^{\mathrm{proj}}$ denote the full subcategory of Mod_R spanned by the finitely generated projective R-modules. Note that this subcategory is contained in $\mathrm{Mod}_R^{\mathrm{perf}}$. The ∞-category $\mathrm{Mod}_R^{\mathrm{proj}}$ has finite coproducts (though it does not have finite colimits in general), so we can define its additive K-theory $K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{proj}})$ as in the previous lecture (here we will think of $K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{proj}})$ as a space, rather than a spectrum). Our goal in this lecture is to prove the following result:

Theorem 5. *Let R be a* connective *ring spectrum (meaning that $\pi_n R \simeq 0$ for $n < 0$). Then the inclusion $\mathrm{Mod}_R^{\mathrm{proj}} \hookrightarrow \mathrm{Mod}_R^{\mathrm{perf}}$ induces a homotopy equivalence of K-theory spaces*

$$K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{proj}}) \to K(\mathrm{Mod}_R^{\mathrm{perf}}) = K(R).$$

Theorem 5 will allow us to get a concrete handle on $K(R)$ (and describe some of its homotopy groups) in the case where R is connective; we will return to this point in the next lecture.

To prove Theorem 5, we will need to introduce a family of intermediate objects which interpolate between $\mathrm{Mod}_R^{\mathrm{proj}}$ and $\mathrm{Mod}_R^{\mathrm{perf}}$.

Definition 6. Let M be an R-module and let n be an integer. We say that M is *n-connective* if $\pi_m M \simeq 0$ for $m < n$. We say that M has *projective amplitude* $\leq n$ if, for every $(n+1)$-connective R-module N, every morphism $f : M \to N$ is nullhomotopic.

The hypotheses of n-connectivity and projective amplitude $\leq m$ are of a complementary nature: as m and n grow, the first condition gets stronger and the last condition gets weaker. Note that if M is n-connective and of projective amplitude $< n$, then the identity map $\mathrm{id} : M \to M$ is nullhomotopic and therefore $M \simeq 0$.

Lemma 7. *Let $M \in \mathrm{Mod}_R^{\mathrm{perf}}$. The following conditions are equivalent:*

(1) *The R-module M is finitely generated and projective.*

(2) *The R-module M is 0-connective and of projective amplitude ≤ 0.*

Proof. The implication (1) $\Rightarrow$ (2) is easy. The converse depends on a few basic facts about perfect modules. For every ordinary module N over the ring $\pi_0 R$, we can regard the Eilenberg-MacLane spectrum HN as a module over R. If M is 0-connective, then the space $\mathrm{Map}_{\mathrm{Mod}_R}(M, HN)$ is homotopy equivalent to the discrete set $\mathrm{Hom}_{\pi_0 R}(\pi_0 M, N)$ of module homomorphisms from $\pi_0 M$ into N. If M is perfect, then the construction $N \mapsto \mathrm{Hom}_{\pi_0 R}(\pi_0 M, N)$ commutes with filtered colimits and therefore $\pi_0 M$ is finitely presented as a module over $\pi_0 R$. In particular, we can choose a map $e : R^n \to M$ which is surjective on π_0. Form a cofiber sequence

$$R^n \xrightarrow{e} M \xrightarrow{f} N.$$

The assumption that e is surjective on π_0 ensures that N is 1-connective. If M has projective amplitude ≤ 0, then the map f must be nullhomotopic; a choice of nullhomotopy supplies a section of e which exhibits M as a direct summand of R^n. $\square$

For each integer $n \geq 0$, let $\mathrm{Mod}_R^{(n)}$ denote the full subcategory of $\mathrm{Mod}_R^{\mathrm{perf}}$ spanned by those perfect R-modules M which are 0-connective and of projective amplitude $\leq n$; we let $\mathrm{Mod}_R^{(\infty)} = \bigcup_{n \geq 0} \mathrm{Mod}_R^{(n)}$. We will say that a morphism $f : M' \to M$ in $\mathrm{Mod}_R^{(n)}$ is a *cofibration* if the cofiber of f (formed in the ∞-category Mod_R) also belongs to $\mathrm{Mod}_R^{(n)}$.

Exercise 8. (*a*) Let n be an integer, and suppose we are given a cofiber sequence

$$M' \to M \to M''$$

in Mod_R. Show that if M' and M'' are n-connective (of projective amplitude $\leq n$), then M is also n-connective (of projective amplitude $\leq n$).

(*b*) Show that an R-module M is n-connective (of projective amplitude $\leq n$) if and only if the suspension $\Sigma(M)$ is $(n+1)$-connective (of projective amplitude $\leq n$).

By "rotating" cofiber sequences, we can deduce several formal consequences of (a) and (b):

(*c*) Suppose we are given a cofiber sequence

$$M' \to M \to M''.$$

Show that if M is n-connective and M' is $(n-1)$-connective, then M'' is n-connective. On the other hand, if M is n-connective and M'' is $(n+1)$-connective, then M' is n-connective. Similar statements hold if we replace "n-connective" with "of projective amplitude $\leq n$".

Exercise 9. Show that the notion of cofibration defined above endows $\mathrm{Mod}_R^{(n)}$ with the structure of an ∞-category with cofibrations in the sense of Lecture 18.

It follows from Exercise 8 that if $f : M' \to M$ is a morphism in $\mathrm{Mod}_R^{(n)}$, then $\mathrm{cofib}(f)$ is automatically connective and has projective amplitude at most $\leq n+1$.

Exercise 10. It is somewhat easier to think about cofibration *sequences* in $\mathrm{Mod}_R^{(n)}$ rather than individual cofibrations. The data of a cofibration $f : M' \to M$ in $\mathrm{Mod}_R^{(n)}$ is equivalent to the data of a cofibration sequence

$$M' \to M \to M''$$

in Mod_R where $M', M, M'' \in \mathrm{Mod}_R^{(n)}$. Note that if M and M'' belong to $\mathrm{Mod}_R^{(n)}$, then M' automatically has projective amplitude $\leq n$; it is connective if and only if the map $\pi_0 M \to \pi_0 M''$ is a surjection.

Example 11. When $n = 0$, we are considering cofibration sequences

$$M' \to M \to M''$$

where M', M, and M'' are finitely generated projective R-modules (Lemma 7). It follows that $\mathrm{Mod}_R^{(0)}$ has only split cofibrations.

Example 12. When $n = \infty$, every map in $\mathrm{Mod}_R^{(n)}$ is a cofibration.

We now turn to the proof of Theorem 5. We are interested in studying the composite map

$$K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{perf}}) \to K(\mathrm{Mod}_R^{(0)}) \to K(\mathrm{Mod}_R^{(1)}) \to \cdots \to K(\mathrm{Mod}_R^{(\infty)}) \to K(\mathrm{Mod}_R^{\mathrm{perf}}).$$

We make the following observations:

(*a*) The map $K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{perf}}) \to K(\mathrm{Mod}_R^{(0)})$ is a homotopy equivalence. This follows from the result we proved in Lecture 18, since every cofibration sequence in $\mathrm{Mod}_R^{(0)}$ splits.

(*b*) The ∞-category $\mathrm{Mod}_R^{\mathrm{perf}}$ can be identified with the Spanier-Whitehead ∞-category of $\mathrm{Mod}_R^{(\infty)}$ (this follows from the observation that any perfect R-module M is $(-n)$-connective for $n \gg 0$). It follows from our work in Lecture 17 that the map $K(\mathrm{Mod}_R^{(\infty)}) \to K(\mathrm{Mod}_R^{\mathrm{perf}})$ is a homotopy equivalence.

It will therefore suffice to prove the following:

Proposition 13. *For each integer $n > 0$, the canonical map $K(\mathrm{Mod}_R^{(n-1)}) \to K(\mathrm{Mod}_R^{(n)})$ is a homotopy equivalence.*

To prove Proposition 13, we will need to introduce an auxiliary construction. Let $\mathcal{C}$ be the ∞-category whose objects are cofiber sequences

$$M' \to M \to M'',$$

where $M' \in \mathrm{Mod}_R^{(n-1)}$, $M \in \mathrm{Mod}_R^{\mathrm{proj}}$, and $M'' \in \mathrm{Mod}_R^{(n)}$. We regard $\mathcal{C}$ as an ∞-category with cofibrations, where a cofibration in $\mathcal{C}$ is a map of cofiber sequences whose cofiber (formed in the ∞-category of all cofiber sequences in Mod_R) also belongs to $\mathcal{C}$.

There are evident evaluation maps

$$e' : \mathcal{C} \to \mathrm{Mod}_R^{(n-1)} \qquad e : \mathcal{C} \to \mathrm{Mod}_R^{\mathrm{proj}} \qquad e'' : \mathcal{C} \to \mathrm{Mod}_R^{(n)}.$$

These evaluation maps induce maps of K-theory spaces, which we will denote by e'_*, e_*, and e''_*. We first prove the following:

Lemma 14. *The maps e'_* and e_* induce a homotopy equivalence*

$$K(\mathcal{C}) \to K(\mathrm{Mod}_R^{(n-1)}) \times K(\mathrm{Mod}_R^{\mathrm{proj}}).$$

Proof. We define functors $i : \mathrm{Mod}_R^{(n-1)} \to \mathcal{C}$ and $j : \mathrm{Mod}_R^{\mathrm{proj}} \to \mathcal{C}$ by the formulae

$$i(M') = (M' \to 0 \to \Sigma(M'))$$

$$j(M) = (0 \to M \to M).$$

It is clear that $e'_* i_*$ and $e''_* j_*$ are homotopic to the identity maps on $K(\mathrm{Mod}_R^{(n-1)})$ and $K(\mathrm{Mod}_R^{\mathrm{proj}})$, respectively. To complete the proof, it will suffice to show that the sum $i_* e'_* + j_* e''_*$ is homotopic to the identity map on $K(\mathcal{C})$. This follows by applying the additivity theorem to the natural cofiber sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M \arrow[d] \\
M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[d] \\
M' \arrow[r] & 0 \arrow[r] & \Sigma(M')
\end{tikzcd}$$

(actually we need a slightly more general form of the additivity theorem than the one we proved in Lecture 17, which applies to the K-theory ∞-categories with cofibrations; however, this more general statement can be proven by the same argument). $\square$

Let $k : \mathrm{Mod}_R^{\mathrm{proj}} \to \mathcal{C}$ be the functor given by

$$k(M) = (M \to M \to 0).$$

The composite map

$$K(\mathrm{Mod}_R^{(n-1)}) \times K(\mathrm{Mod}_R^{\mathrm{proj}}) \xrightarrow{i_*, k_*} K(\mathcal{C}) \xrightarrow{e'_*, e_*} K(\mathrm{Mod}_R^{(n-1)}) \times K(\mathrm{Mod}_R^{\mathrm{proj}})$$

is upper triangular and therefore a homotopy equivalence. It follows that i_* induces a homotopy equivalence $K(\mathrm{Mod}_R^{(n-1)}) \to \mathrm{cofib}(k_*)$, where the cofiber is formed in the ∞-category of grouplike E_∞-spaces. Since $e'' \circ k$ is nullhomotopic, we obtain a map

$$\theta : \mathrm{cofib}(k_*) \to K(\mathrm{Mod}_R^{(n)}).$$

The composite map

$$K(\mathrm{Mod}_R^{(n-1)}) \simeq \mathrm{cofib}(k_*) \xrightarrow{\theta} K(\mathrm{Mod}_R^{(n)})$$

is given concretely by the construction

$$\mathrm{Mod}_R^{(n-1)} \to \mathrm{Mod}_R^{(n)}$$
$$M' \to \Sigma(M'),$$

and therefore agrees up to a sign with the map appearing in Proposition 13. It will therefore suffice to show that θ is a homotopy equivalence.

Unwinding the definitions, we see that θ is given by a map

$$\Omega(|S_\bullet\, \mathcal{C}\,|/|S_\bullet\, \mathrm{Mod}_R^{\mathrm{proj}}\,|) \to \Omega|S_\bullet\, \mathrm{Mod}_R^{(n)}\,|,$$

where the quotient means we are forming a bar construction. Since the formation of bar constructions commutes with geometric realization, θ is obtained by looping the geometric realization of a map of simplicial spaces

$$\theta_\bullet : S_\bullet\, \mathcal{C}\,/S_\bullet\, \mathrm{Mod}_R^{\mathrm{proj}} \to S_\bullet\, \mathrm{Mod}_R^{(n)}\,.$$

It will therefore suffice to show that this map is a homotopy equivalence of simplicial spaces. We will take this up in the next lecture.

References

Lower K-Groups of Ring Spectra (Lecture 20)

October 19, 2014

We begin by finishing the proof from the previous lecture. Recall that R is a connective associative ring spectrum, that $\mathrm{Mod}_R^{(n)}$ denotes the ∞-category of connective perfect R-modules which have projective amplitude $\leq n$, and that $\mathcal{C}$ is the ∞-category of cofiber sequences

$$M' \to M \to M''$$

where $M' \in \mathrm{Mod}_R^{(n-1)}$, $M \in \mathrm{Mod}_R^{(0)} = \mathrm{Mod}_R^{\mathrm{proj}}$, and $M'' \in \mathrm{Mod}_R^{(n)}$. We regard $S_\bullet \mathrm{Mod}_R^{\mathrm{proj}}$ as a simplicial E_∞-space (with E_∞-structure given by the formation of direct sums), which acts on $S_\bullet \mathcal{C}$ via the construction

$$(P, M' \to M \to M'') \mapsto (M' \oplus P \to M \oplus P \to M'').$$

We wish to prove the following:

Proposition 1. *The canonical map $S_\bullet \mathcal{C} / S_\bullet \mathrm{Mod}_R^{\mathrm{proj}} \to S_\bullet \mathrm{Mod}_R^{(n)}$ is a homotopy equivalence of simplicial spaces.*

Proposition 1 is a levelwise statement: it asserts that for each integer m, the bar construction

$$S_m \mathcal{C} / S_m \mathrm{Mod}_R^{\mathrm{proj}}$$

is homotopy equivalent to $S_m \mathrm{Mod}_R^{(n)}$. To simplify the exposition, we will consider only the case $m = 1$ (the general case follows by essentially the same argument). In this case, we wish to prove that the natural map

$$\theta : \mathcal{C}^{\simeq} / (\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq} \to (\mathrm{Mod}_R^{(n)})^{\simeq}$$

is a homotopy equivalence. Fix an object $M \in \mathrm{Mod}_R^{(n)}$; we wish to show that the homotopy fiber $\theta^{-1}\{M\}$ is contractible. Set $\mathcal{C}_M = \mathcal{C} \times_{\mathrm{Mod}_R^{(n)}} \{M\}$, so that we can identify $\mathcal{C}_M$ with the ∞-category whose objects are finitely projective R-modules P equipped with a map $P \to M$ which is surjective on π_0. Then $\theta^{-1}\{M\}$ can be identified with the bar construction $\mathcal{C}_M^{\simeq} / (\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$, where $\mathrm{Mod}_R^{\mathrm{proj}}$ acts on $\mathcal{C}_M^{\simeq}$ by the construction

$$(Q, \lambda : P \to M) \mapsto ((\lambda \oplus 0) : P \oplus Q \to M).$$

We wish to show that this space is contractible.

Let us first consider the special case where M is projective. In this case, any map $\lambda : P \to M$ which is surjective on π_0 is automatically split, so that we can write P as a direct sum $M \oplus P_0$ where λ is the identity on M and vanishes on P_0. Here P_0 is a direct summand of P and therefore also projective.

If $P \to M$ and $Q \to M$ are both surjective on π_0, we deduce that $P \times_M Q$ has a direct sum decomposition $M \oplus P_0 \oplus Q_0$, where P_0 and Q_0 are direct summands of P and Q. Consequently, if P and Q are both projective, then so is $P \times_M Q$. It follows that the ∞-category $\mathcal{C}_M$ admits finite products, so that $\mathcal{C}_M^{\simeq}$ has the structure of an E_∞-space. We can now proceed as in Lecture 18. The action of $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ on $\mathcal{C}_M^{\simeq}$ is via an E_∞-map

$$P \mapsto (P \oplus M \to M)$$

which is surjective on connected components. It follows that the quotient $(\mathcal{C}_M^{\simeq})/(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ is connected and in particular grouplike. Consequently, to show that it is contractible, it will suffice to show that the group completion $(\mathcal{C}_M^{\simeq})/(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq})^{\mathrm{grp}}$ is contractible. Equivalently, we must show that the map

$$f : (\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq} \to \mathcal{C}_M^{\simeq}$$

induces a homotopy equivalence after group completion. Note that f has a left homotopy inverse q, given by

$$(\lambda : P \to M) \mapsto \mathrm{fib}(\lambda).$$

To complete the proof, it will suffice to show that q is also a right homotopy inverse after group completion. The composition $f \circ q$ is given by

$$(\lambda : P \to M) \to (\mathrm{fib}(\lambda) \oplus M \to M).$$

This is not homotopic to the identity functor from $\mathcal{C}_M^{\simeq}$ to itself. However, we claim that $(f \circ q) + \mathrm{id}$ is homotopic to $\mathrm{id} \oplus \mathrm{id}$: that is, for each object $(\lambda : P \to M)$, there is a canonical equivalence

$$P \times_M P \simeq P \oplus \mathrm{fib}(\lambda)$$

(which is compatible with the projection to M). This follows from the fact that either projection map $P \times_M P \to P$ has a canonical section, given by the diagonal map $\delta : P \to P \times_M P$. This completes the proof in the case where M is projective.

We now consider the general case. Let $\mathcal{D} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C}_M)$ be the subcategory whose objects are diagrams

$$P \xrightarrow{\alpha} Q \xrightarrow{\beta} M$$

where Q is projective and the maps α and β are surjective on π_0, and whose morphisms are commutative diagrams

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\gamma"] & Q \arrow[r] \arrow[d, "\delta"] & M \arrow[d, "\mathrm{id}"] \\
P' \arrow[r] & Q' \arrow[r] & M
\end{tikzcd}$$

where γ is an equivalence and δ is surjective on π_0. There is an obvious forgetful functor $\mathcal{D} \to \mathcal{C}_M^{\simeq}$ given by "forgetting" the middle term. This functor is a fibration whose fibers are weakly contractible (they have initial objects, given by taking $Q = P$), and therefore a weak homotopy equivalence. Note that the action of $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ on $\mathcal{C}_M^{\simeq}$ extends to an action of $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ on $\mathcal{D}$, given by

$$(P', P \to Q \to M) \mapsto (P \oplus P' \to Q \to M).$$

It will therefore suffice to show that the quotient $\mathcal{D}/(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ is weakly contractible.

Let $\mathcal{C}_M^{\mathrm{surj}}$ denote the subcategory of $\mathcal{C}_M$ containing all objects, whose morphisms are given by commutative diagrams

$$\begin{tikzcd}
Q \arrow[rr] \arrow[dr] & & Q' \arrow[dl] \\
& M &
\end{tikzcd}$$

where the morphisms are surjective on π_0. There is an evident forgetful functor $\mathcal{D} \to \mathcal{C}_M^{\mathrm{surj}}$ (given by forgetting P) which is equivariant with respect to the action of $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ (where we regard $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ as acting trivially on $\mathcal{C}_M^{\mathrm{surj}}$), and therefore induces a map

$$\psi : \mathcal{D}/(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq} \to \mathcal{C}_M^{\mathrm{surj}}.$$

This map is a left fibration whose fiber over an object $(Q \to M)$ can be identified with the bar construction $\mathcal{C}_Q^{\simeq} / \mathrm{Mod}_R^{\mathrm{proj}}$, which we have already proved to be contractible. It follows that ψ is an equivalence of ∞-categories, and therefore a weak homotopy equivalence. We are now reduced to proving the following:

Lemma 2. *The ∞-category $\mathcal{C}_M^{\mathrm{surj}}$ is weakly contractible.*

Proof. Note that $\mathcal{C}_M^{\mathrm{surj}}$ is equipped with a multiplication, given by

$$((\lambda : Q \to M), (\lambda' : Q' \to M)) \to (\lambda \oplus \lambda' : Q \oplus Q' \to M).$$

Let X denote the topological space $|\,\mathcal{C}_M^{\mathrm{surj}}\,|$, so that we obtain a multiplication map $m : X \times X \to X$. The composite map

$$X \to X \times X \overset{m}{\to} X$$

is canonically homotopic to the identity; this follows from the existence of a canonical commutative diagram

$$\begin{tikzcd} Q \oplus Q \arrow[r, "\lambda\oplus\lambda"] \arrow[d] & M \arrow[d, "\mathrm{id}"] \\ Q \arrow[r, "\lambda"] & M \end{tikzcd}$$

where the left vertical map is given by the fold. In particular, for any $x \in X$ we obtain a canonical path h_x from $m(x,x)$ to x. Let ψ denote the induced map

$$\pi_n(X,x) \times \pi_n(X,x) \overset{m}{\to} \pi_n(X, m(x,x)) \overset{h_x}{\to} \pi_n(X,x).$$

Then ψ is a group homomorphism, hence given by

$$\psi(a,b) = \phi_1(a)\phi_2(b)$$

for some commuting group homomorphisms $\phi_1, \phi_2 : \pi_n(X,x) \to \pi_n(X,x)$. By symmetry we must have $\phi_1 = \phi_2$; let us denote them both by ϕ. Using the fact that h extends to a homotopy of the composite map

$$X \to X \times X \overset{m}{\to} X$$

to the identity, we see that $a = \psi(a,a) = \phi(a)^2 = \phi(a^2)$ for all $a \in \pi_n(X,x)$. In particular, the map ϕ is surjective. Using the associativity of the direct sum (and its compatibility with the construction of the homotopy h), we also deduce the identity

$$\psi(a, \psi(b,c)) = \psi(\psi(a,b), c)$$

$$\phi(a)\phi(\phi(b))\phi(\phi(c)) = \phi(\phi(a))\phi(\phi(b))\phi(c)$$

Taking c to be the identity and cancelling, we obtain $\phi(a) = \phi(\phi(a))$: that is, ϕ is the identity when restricted to $\mathrm{Im}(\phi)$. Since ϕ is surjective, we conclude that $\phi = \mathrm{id}$. Thus $a = a^2$ for all $a \in \pi_n(X,x)$, which shows that the connected component of x in X is contractible.

To complete the proof, it will suffice to show that X is connected. This follows from the observation that for any maps $\lambda : Q \to X$ and $\lambda' : Q' \to X$ which are surjective on π_0, we can find a commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d, "\lambda"] \\ Q' \arrow[r, "\lambda"] & X \end{tikzcd}$$

where P is finitely generated and projective and all maps are surjective on π_0. To see this, we note that the fiber product $Q \times_X Q'$ is connective and perfect and therefore there exists map $R^n \to Q \times_X Q'$ which is surjective on π_0. $\square$

This completes the proof that for a connective associative ring spectrum R, we have a homotopy equivalence $K(R) \simeq K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{proj}})$.

Corollary 3. *Let R be a connective associative ring spectrum. Then there is a canonical isomorphism $\pi_0 K(R) \simeq K_0(\pi_0 R)$, where K_0 is defined as in Lecture 2 (so that $K_0(A)$ denotes the Grothendieck group of finitely generated projective A-modules).*

Proof. We have $\pi_0 K(R) = \pi_0 K_{\mathrm{add}}(R) = \pi_0(\mathrm{Mod}_R^{\mathrm{proj}})^{\mathrm{grp}}$, so that $\pi_0 K(R)$ can be identified with the Grothendieck group of the commutative monoid of equivalence classes of finitely generated projective R-modules. To complete the proof, it will suffice to establish the following:

($*$) Let R be a connective associative ring spectrum. Then the construction $P \mapsto \pi_0 P$ induces an equivalence from the homotopy category of (finitely generated) projective R-modules to the ordinary category of (finitely generated) projective $\pi_0 R$-modules.

Note that if P is projective, then we have a canonical isomorphism

$$\pi_0 \mathrm{Map}_{\mathrm{Mod}_R}(P, M) \simeq \mathrm{Hom}_{\pi_0 R}(\pi_0 P, \pi_0 M)$$

for *every* R-module spectrum M (observe that that the collection of all R-modules P for which this map is bijective is closed under retracts, direct sums, and contains R). This proves that the functor described in ($*$) is fully faithful. To prove essential surjectivity, let P_0 be a finitely generated projective module over $\pi_0 R$, so that P_0 is the image of some idempotent map $e_0 : (\pi_0 R)^n \to (\pi_0 R)^n$. Using the full faithfulness, we can lift e_0 to a map $e : R^n \to R^n$. Let P be the direct limit

$$R^n \xrightarrow{e} R^n \xrightarrow{e} \cdots$$

and let Q be the direct limit

$$R^n \xrightarrow{1-e} R^n \xrightarrow{1-e} \cdots .$$

By computing homotopy groups, we see that the natural maps $R^n \to P$ and $R^n \to Q$ exhibit R^n as a product of P with Q. In particular, P is a finitely generated projective R-module, and we clearly have $\pi_0 P = P_0$. $\square$

Corollary 4. *Let R be a connective associative ring spectrum. Then there is a canonical isomorphism $\pi_1 K(R) \simeq K_1(\pi_0 R)$, where K_1 is defined as in Lecture 3 (that is, we have $\pi_1 K(R) \simeq \mathrm{GL}_\infty(\pi_0 R)^{\mathrm{ab}}$).*

Proof. Let $X = (\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$, so that $K(R)$ can be identified with (the 0th space of) the group completion of X. In other words, $K(R)$ is universal among E_∞-spaces which receive a map $X \to K(R)$ having the property that every element of $\pi_0 X$ becomes invertible in $\pi_0 K(R)$. We can identify the elements of $\pi_0 X$ with the equivalence classes of finitely generated projective R-modules. Each of these appears as a direct summand of R^n; consequently, $K(R)$ is universal among E_∞-spaces which receive a map from X for which the image of the point $\{R\}$ becomes invertible in $\pi_0 K(R)$. At the level of singular chain complexes, it follows that $C_*(K(R); \mathbf{Z})$ is universal among E_∞-algebras over $\mathbf{Z}$ which receive an E_∞-map $C_*(X; \mathbf{Z})$ for which the point $[R] \in \mathrm{H}_0(X; \mathbf{Z})$ becomes invertible in $\mathrm{H}_0(K(R); \mathbf{Z})$: that is, it can be identified with the E_∞-algebra obtained from $C_*(X; \mathbf{Z})$ by inverting $[R]$. In particular, we can identify $\mathrm{H}_1(K(R); \mathbf{Z})$ with the direct limit of the sequence

$$\mathrm{H}_1(X; \mathbf{Z}) \xrightarrow{[R]} \mathrm{H}_1(X; \mathbf{Z}) \xrightarrow{[R]} \mathrm{H}_1(X; \mathbf{Z}) \to \cdots .$$

Restricting our attention to the identity component $K(R)^\circ \subseteq K(R)$, we can identify $\mathrm{H}_1(K(R)^\circ; \mathbf{Z})$ with the direct limit

$$\mathrm{H}_1(X(0); \mathbf{Z}) \to \mathrm{H}_1(X(1); \mathbf{Z}) \to \mathrm{H}_1(X(2); \mathbf{Z}) \to \cdots ,$$

where each $X(n)$ is the connected component of X which classifies R-modules which are equivalent to R^n (and the transition maps are given by forming the direct sum with a copy of R). In particular, each $X(n)$

is a classifying space for the group $\mathrm{Aut}(R^n)$ of automorphisms of R^n (in the ∞-category Mod_R). We then have $\pi_1 X(n) = \pi_0 \mathrm{Aut}(R^n) \simeq \mathrm{GL}_n(\pi_0 R)$, so that

$$\pi_1 K(R) \simeq \mathrm{H}_1(K(R)^\circ; \mathbf{Z}) \simeq \varinjlim \mathrm{H}_1(X(n); \mathbf{Z}) \simeq \varinjlim (\pi_1 X(n))^{\mathrm{ab}} \simeq \varinjlim \mathrm{GL}_n(\pi_0 R)^{\mathrm{ab}} \simeq \mathrm{GL}_\infty(\pi_0 R)^{\mathrm{ab}}.$$

□

Warning 5. Corollaries 3 and 4 show that the groups $K_0(R)$ and $K_1(R)$ of a connective ring spectrum R depend only on the associative ring $\pi_0 R$. This is not true for the higher K-groups: in general, $K_{n+1}(R)$ is sensitive to the first n homotopy groups of R. The difference between the higher K-theory of ring spectra (in other words, Waldhausen K-theory) and the higher K-theory of ordinary rings (introduced by Quillen) will become important when studying the "higher" versions of the Whitehead torsion, which is our next objective.

For each $n \geq 0$, let $\mathrm{GL}_n(R)$ denote the space $\mathrm{Aut}(R^n)$ of automorphisms of R^n as an object of the ∞-category Mod_R, let $\mathrm{BGL}_n(R)$ denote its classifying space, and let $\mathrm{BGL}_\infty(R)$ be the direct limit $\varinjlim \mathrm{BGL}_n(R)$. The proof of Corollary 4 shows that there is a canonical map

$$\mathrm{BGL}_\infty(R) \to K(R)^\circ,$$

and that this map is an isomorphism on the first homology group. In fact, something stronger is true: we can identify $K(R)^\circ$ with the space obtained from $\mathrm{BGL}_\infty(R)$ by performing the "plus construction" with respect to the commutator subgroup of $\pi_1 \mathrm{BGL}_\infty(R) = \mathrm{GL}_\infty(\pi_0 R)$. To prove this, it suffices to establish the following:

Proposition 6. *Let M be an abelian group with an action of $\pi_1 K(R)^\circ$, which we view as a local system on both $K(R)^\circ$ and on $\mathrm{BGL}_\infty(R)$. Then the canonical map*

$$\mathrm{H}_*(\mathrm{BGL}_\infty(R); M) \to \mathrm{H}_*(K(R)^\circ; M)$$

is an isomorphism.

Remark 7. Proposition 6 shows that, in the case of a discrete ring R, Waldhausen K-theory (defined via the procedure we have been discussing) agrees with Quillen K-theory (defined via the plus construction).

Proof of Proposition 6. The element $1 = [R] \in K_0(R)$ determines a map of infinite loop spaces $QS^0 \to K(R)$, which induces a map

$$\mathbf{Z}/2\mathbf{Z} \simeq \pi_1 QS^0 \to \pi_1 K(R) = K_1(R).$$

Let $\epsilon \in K_1(R)$ denote the image under this map of the nontrivial element of $\mathbf{Z}/2\mathbf{Z}$ (under the isomorphism $K_1(R) \simeq \mathrm{GL}_\infty(\pi_0 R)^{\mathrm{ab}}$ of Corollary 4, this corresponds to the class represented by an odd permutation of coordinates; equivalently, it is the class represented by $-1 \in \mathrm{GL}_1(R)$, by an exercise from Lecture 3). Then ϵ is an element of $K_1(R)$ having order 2, and therefore induces an involution of the abelian group M.

Let us say that an abelian group M with an action of $K(R)^\circ$ is *good* if the conclusion of Proposition 6 holds for M. It follows immediately that if we are given an exact sequence of representations

$$0 \to M' \to M \to M'' \to 0$$

where two of the terms are good, then so is the third. We may therefore assume without loss of generality that the involution of M determined by ϵ is either id_M or $-\mathrm{id}_M$.

Let us first treat the case where the action of ϵ on M is by the identity, since this is a bit easier. Set $G = K_1(R)/\epsilon$, and regard M as a module over the group ring $\mathbf{Z}[G]$. Then we can regard the direct sum $\mathbf{Z}[G] \oplus M$ as a square-zero extension of $\mathbf{Z}[G]$. We have an exact sequence of multiplicative groups

$$0 \to M \to (\mathbf{Z}[G] \oplus M)^\times \to \mathbf{Z}[G]^\times \to 0.$$

Consequently, to prove that M is good, it will suffice to show that $(\mathbf{Z}[G] \oplus M)^\times$ and $\mathbf{Z}[G]^\times$ are good. Both of these are special cases of the following assertion:

($*$) Let A be a commutative ring and suppose we are given a map $\psi : K_1(R) \to A^\times$ which annihilates ϵ. Then A is good (with respect to the action of $K_1(R)$ via multiplication).

Let $K'(R)$ denote the fiber product $\mathbf{Z} \times_{K_0(R)} K(R)$, so that $K'(R)$ is an infinite loop space with $\pi_0 K'(R) = \mathbf{Z}$ whose connected components are homotopy equivalent to $K(R)^\circ$. Note that ψ can be identified with a map of infinite loop spaces $K(R)^\circ \to K(A^\times, 1)$, and the condition that ψ annihilates ϵ is equivalent to the statement that this map extends to an infinite loop map

$$K'(R) \to K(A^\times, 1).$$

We can regard this infinite loop map as giving a local system $\mathcal{L}$ of A-modules on $K'(R)$ which is *multiplicative* with respect to the E_∞-structure on $K'(R)$, so that $C_*(K'(R); \mathcal{L})$ can be identified with an E_∞-algebra over $\mathbf{Z}$. Note that we have a canonical map of E_∞-spaces

$$\theta : \amalg_{n \geq 0} \mathrm{BGL}_n(R) \to K'(R)$$

which induces a map of E_∞-algebras

$$C_*(\amalg_{n \geq 0} \mathrm{BGL}_n R; \theta^* \mathcal{L}) \to C_*(K'(R); \mathcal{L}).$$

Arguing as in the proof of Corollary 4, we see that this map exhibits the right hand side as obtained from the left hand side by inverting a single element $[X] \in \mathrm{H}_0(\mathrm{BGL}_1(R); \theta^* \mathcal{L})$, from which it follows that $\mathrm{H}_*(K(R)^\circ; A)$ is given by the direct limit of the sequence

$$\mathrm{H}_*(\mathrm{BGL}_0(R); A) \to \mathrm{H}_*(\mathrm{BGL}_1(R); A) \to \cdots$$

This completes the proof in the case where ϵ acts by the identity on M.

In the case where ϵ acts by $-\mathrm{id}_M$, we can use essentially the same strategy, but we need a few modifications. First, we should replace the group algebra $\mathbf{Z}[G]$ by the "twisted group algebra" obtained from $\mathbf{Z}[K_1(R)]$ by setting ϵ equal to -1. We then reduce to proving the following analogue of ($*$):

($*'$) Let A be a commutative ring and suppose we are given a map $\psi : K_1(R) \to A^\times$ which carries ϵ to -1. Then A is good (with respect to the action of $K_1(R)$ via multiplication).

Let Y be the classifying space for invertible *graded* A-modules, so that $Y \simeq \mathbf{Z} \times K(A^\times, 1)$. We regard Y as an infinite loop space using the usual commutativity constraint for tensor products of graded modules. Then multiplication by $\eta \in \pi_1 QS^0$ carries the generator $1 \in \mathbf{Z}$ to $-1 \in A^\times$. It follows that the infinite loop map $K(R)^\circ \times K(A^\times, 1)$ determined by ψ fits into a commutative diagram of E_∞-maps

$$\begin{tikzcd}
K(R)^\circ \arrow[r] \arrow[d, "\psi"] & K'(R) \arrow[r] \arrow[d, "\overline{\psi}"] & \mathbf{Z} \arrow[d] \\
K(A^\times, 1) \arrow[r] & Y \arrow[r] & \mathbf{Z}.
\end{tikzcd}$$

As before, we can identify the map $\overline{\psi}$ with a mutiplicative local system $\mathcal{L}$ on $K'(R)$: the only difference is that $\mathcal{L}$ is now a *graded* local system (on the connected component $\{n\} \times K(R)^\circ$, we regard $\mathcal{L}$ as lying in degree n). We again obtain a map of E_∞-algebras

$$C_*(\amalg_{n \geq 0} \mathrm{BGL}_n R; \theta^* \mathcal{L}) \to C_*(K'(R); \mathcal{L}).$$

which exhibits the right hand side as a localization of the left hand side, this time by inverting a class $[X] \in \mathrm{H}_1(\mathrm{BGL}_1(R); \theta^* \mathcal{L})$. Identifying the restriction of $\mathcal{L}$ to each component with a shift of the local system A, we again obtain the desired isomorphism

$$\mathrm{H}_*(K(R)^\circ; A) \simeq \varinjlim \mathrm{H}_*(\mathrm{BGL}_n(R); A).$$

$\square$

The Algebraic K-Theory of Spaces (Lecture 21)

October 22, 2014

Let X be a topological space. Then the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is a Kan complex, and in particular an ∞-category. If $\mathcal{C}$ is another ∞-category, we define a *local system on X with values in* $\mathcal{C}$ to be a map of simplicial sets

$$\mathrm{Sing}_\bullet(X) \to \mathcal{C}.$$

The collection of all local systems on X with values in $\mathcal{C}$ can be organized into an ∞-category $\mathrm{Fun}(\mathrm{Sing}_\bullet(X), \mathcal{C})$, which we will denote by $\mathcal{C}^X$.

Example 1. If $\mathcal{C}$ is an ordinary category, then every local system on X with values in $\mathcal{C}$ factors through the homotopy category of $\mathrm{Sing}_\bullet(X)$, which is the *fundamental groupoid* of X. If X is connected and we choose a base point $x \in X$, then we can identify $\mathcal{C}^X$ with the category consisting of objects $C \in \mathcal{C}$ with an action of the fundamental group $\pi_1(X, x)$.

Variant 2. We will generally use the term "space" to refer either to a topological space or to a Kan complex (or to an object of some other type which could be used as a model for homotopy theory). In the latter case, the notion of local system takes a simpler form: it is just a map from X into $\mathcal{C}$.

In what follows, we will confine our attention to the case where $\mathcal{C}$ is the ∞-category Sp of spectra. In this case, we will refer to objects of Sp^X as *local systems of spectra on X* or *spectra parametrized by X*. However, many of the notions we introduce make sense for more general ∞-categories $\mathcal{C}$.

Notation 3. Let X be a space and let $\mathcal{L}$ be a local system of spectra on X. Then each point $x \in X$ determines a spectrum $\mathcal{L}_x$, which we will refer to as the *value of $\mathcal{L}$ at x*.

Remark 4. The ∞-category Sp admits small limits and colimits. Consequently, given a local system $\mathcal{L}$ of spectra on a space X, we can take its limit or colimit to obtain a spectrum. We will denote the limit by $C^*(X; \mathcal{L})$ and the colimit by $C_*(X; \mathcal{L})$. In the special case where the local system $\mathcal{L}$ is constant with value E, these can be identified with the function spectrum E^X and the smash product $E \wedge X_+$, respectively.

Remark 5 (Functoriality)**.** Let $f : X \to Y$ be a map of spaces. Then composition with f determines a pullback functor $f^* : \mathrm{Sp}^Y \to \mathrm{Sp}^X$. We will sometimes denote the pullback of a local system $\mathcal{L} \in \mathrm{Sp}^X$ by $\mathcal{L}\,|_Y$.

It follows from abstract nonsense that the functor f^* admits both a left adjoint $f_!$ and a right adjoint f_* (given by left and right Kan extension). If f is a fibration (which we can always arrange), then these functors are given by the formula

$$(f_!\,\mathcal{F})_y = C_*(X_y; \mathcal{L}\,|_{X_y}) \qquad (f_*\,\mathcal{F})_y = C^*(X_y; \mathcal{L}\,|_{X_y})$$

where X_y denotes the fiber of f over the point y.

Proposition 6. *Let X be a space. Then the ∞-category Sp^X is compactly generated. That is, it is equivalent to $\mathrm{Ind}(\mathcal{C})$, where $\mathcal{C} \subseteq \mathrm{Sp}^X$ is the full subcategory spanned by the compact objects.*

Proof. It follows from general nonsense that the inclusion $\mathcal{C} \hookrightarrow \mathrm{Sp}^X$ extends to a fully faithful embedding $F : \mathrm{Ind}(\mathcal{C}) \hookrightarrow \mathrm{Sp}^X$, and that F admits a right adjoint G. To show that F is an equivalence of ∞-categories, it suffices to show that G is conservative. In other words, it will suffice to show that if $\alpha : \mathcal{L} \to \mathcal{L}'$ is a morphism of local systems and that $G(\alpha)$ is an equivalence, then α is an equivalence. Pick a point $x \in X$, and let $i : \{x\} \hookrightarrow X$ denote the inclusion map. The functor i^* preserves filtered colimits, so its left adjoint $i_!$ preserves compact objects. If $G(\alpha)$ is an equivalence, we conclude that it induces a homotopy equivalence

$$\mathrm{Map}(i_! E, \mathcal{L}) \to \mathrm{Map}(i_! E, \mathcal{L}')$$

for every finite spectrum E (viewed as a local system on $\{x\}$). It follows that the map $\mathrm{Map}(E, \mathcal{L}_x) \to \mathrm{Map}(E, \mathcal{L}'_x)$ is a homotopy equivalence for every finite spectrum E, from which we conclude that $\mathcal{L}_x \simeq \mathcal{L}'_x$. Since x is arbitrary, it follows that α is an equivalence. $\square$

The proof of Proposition 6 shows something a bit stronger: the ∞-category Sp^X is generated (under colimits and desuspensions) by compact objects of the form $i_! S$, where S is the sphere spectrum and i ranges over the inclusions of all points$x \in X$. It follows that the collection of compact objects of Sp^X is generated (under finite colimits, desuspensions, and retracts) by objects of the form $i_! S$. Moreover, it suffices to consider one point x lying in each connected component of X. Consequently, if X is connected, then Sp^X is generated by a single compact object $i_! S$, and is therefore equivalent to the ∞-category Mod_R where $R = \mathrm{End}(i_! S)$ is the ring spectrum of endomorphisms of $i_! S$. Note that we can identify R with the spectrum of maps from S to $i^* i_! S$: that is, with the value of $i_! S$ at the point x. Converting i into a fibration and using Remark 5, we see that R can be identified with the spectrum

$$C_*(\Omega(X); S) \simeq \Sigma^\infty_+ \Omega(X).$$

Note that R is connective ring spectrum and that $\pi_0 R$ is isomorphic to the group algebra $\mathbf{Z}[\pi_1 X]$ (specializing to the case of *discrete* R-modules, we recover a more familiar fact: the category of local systems of abelian groups on X is equivalent to the category of $\mathbf{Z}[\pi_1 X]$-modules).

Definition 7. Let X be a space and let $\mathcal{C} \subseteq \mathrm{Sp}^X$ be the full subcategory spanned by the compact objects. Then $K(\mathcal{C})$ is a grouplike E_∞-space, and is therefore the 0th space of a connective spectrum. We will denote this spectrum by $A(X)$ and refer to it as the *A-theory spectrum of* X.

In what follows, we will generally abuse notation and not distinguish between grouplike E_∞-spaces and the corresponding spectra.

Example 8. Let X be a connected space with base point $x \in X$. Then we have $A(X) \simeq K(R)$, where $R = \Sigma^\infty_+ \Omega X$ is the ring spectrum described above. Since R is connective, we can identify $A(X)$ with the group completion of the E_∞-space $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ of finitely generated projective R-modules.

Warning 9. Our definition of $A(X)$ is not standard. The usual convention in the literature is to use $A(X)$ to refer to the group completion of the E_∞-space of finitely generated *free* R-modules. However, we have seen that it does not make a very big difference: the only thing that changes is the group $\pi_0 A(X)$.

Remark 10. Let X be a connected space and let G be its fundamental group. Applying the results of the previous lecture, we obtain isomorphisms

$$\pi_0 A(X) = K_0(\mathbf{Z}[G]) \qquad \pi_1 A(X) = K_1(\mathbf{Z}[G]) = \mathrm{GL}_\infty(\mathbf{Z}[G])^{\mathrm{ab}}.$$

However, the higher homotopy groups of $A(X)$ do not have "classical" names: they depend on the entire homotopy type of X (rather than just its fundamental group) and on the fact that we are working over the sphere spectrum (rather than the ring $\mathbf{Z}$ of integers).

Example 11. Let X be a simply connected space. Then we have

$$\pi_0 A(X) \simeq \mathbf{Z} \qquad \pi_1 A(X) \simeq \mathbf{Z}/2\mathbf{Z}.$$

Let $f : X \to Y$ be a map of spaces. Since the pullback functor f^* preserves filtered colimits, its left adjoint $f_!$ preserves compact objects and therefore induces a map of K-theory spectra $A(X) \to A(Y)$. Consequently, we can view the construction $X \mapsto A(X)$ as a covariant functor from the ∞-category of spaces to the ∞-category of spectra.

Let us now consider some other types of local system:

Example 12. Let $\mathcal{S}$ denote the ∞-category of spaces. For every space X, we can identify the ∞-category $\mathcal{S}^X$ of local systems on X with the ∞-category $\mathcal{S}_{/X}$ of spaces Y with a map $Y \to X$; the identification associates to each map $f : Y \to X$ the local system $x \mapsto Y_x$ where Y_x denotes the homotopy fiber of f over the point $x \in X$. The proof of Proposition 6 shows that $\mathcal{S}^X$ is compactly generated. Under the equivalence $\mathcal{S}^X \simeq \mathcal{S}_{/X}$, the compact objects correspond to those maps $Y \to X$ where Y is a finitely dominated space (note that the finiteness condition here is placed on the space Y itself, not on the homotopy fibers of the map $Y \to X$).

Example 13. Let $\mathcal{S}_*$ denote the ∞-category of pointed spaces. Then the identification $\mathcal{S}^X \simeq \mathcal{S}_{/X}$ of Example 12 induces an identification $\mathcal{S}_*^X \simeq \mathcal{S}_{X//X}$, where $\mathcal{S}_{X//X}$ denotes the ∞-category of diagrams

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "\mathrm{id}"] & & X \end{tikzcd}$$

which exhibit X as a retract of Y. The proof of Proposition 6 shows that this ∞-category is generated by compact objects. Examples of compact objects include any diagram as above where Y can be obtained from X by attaching finitely many cells. Conversely, any compact object is a *retract* (in the homotopy category) of such a relative cell complex.

There are evident maps

$$\mathcal{S}^X \to \mathcal{S}_*^X \to \mathrm{Sp}^X,$$

given pointwise by "adding a disjoint basepoint" and "taking the suspension spectrum." These constructions preserve compact objects (since they are left adjoint to functors which preserve filtered colimits). In particular, if Y is a finitely dominated space over X, then the construction $x \mapsto \Sigma^\infty_+(Y_x)$ determines a compact object of Sp^X, which determines a point of the space $\Omega^\infty A(X)$ which we will denote by $[Y]$.

Example 14 (Wall Finiteness Obstruction)**.** Suppose that the space X itself is finitely dominated. Then the above construction determines a point $[X] \in \Omega^\infty A(X)$, which is represented by the constant local system $\underline{S}$ which takes each point of x to the sphere spectrum S. We let $\overline{w}_X \in \pi_0 A(X)$ denote the class represented by this point.

Suppose that X is connected with fundamental group G. We claim that under the isomorphism $\pi_0 A(X) \simeq K_0(\mathbf{Z}[G])$ of Remark 10, the class $\overline{w}_X$ is a *lifting* of the Wall finiteness obstruction $w_X \in \widetilde{K}_0(\mathbf{Z}[G])$ introduced in Lecture 2. Recall that to define w_X, we chose a finite complex X' with a map $X' \to X$ such that the relative homology $\mathrm{H}_*(X, X'; \mathbf{Z}[G])$ was a projective module P concentrated in a single degree n, and defined $w_X = (-1)^n[P]$. Choose a base point $x \in X$ and set $R = \Sigma^\infty_+ \Omega(X)$, so that every map of spaces $Y \to X$ determines an R-module spectrum $\Sigma^\infty_+ Y_x$. We then have a cofiber sequence of R-modules

$$\Sigma^\infty_+ X'_x \to \Sigma^\infty_+ X_x \to \Sigma^n \overline{P},$$

where $\overline{P}$ is a projective R-module with $\pi_0 \overline{P} = P$. Since X' admits a finite cell decomposition, the R-module spectrum $\Sigma^\infty_+ X'_x$ admits a finite filtration whose successive quotients are suspensions of R and therefore represents a class in $K(R)$ given by some integer m. We then have

$$\overline{w}_X = m + (-1)^n[P]$$

in $\pi_0 A(X) \simeq K_0(R) \simeq K_0(\mathbf{Z}[G])$.

The abstract version of the Wall finiteness criterion given in Lecture 15 asserts that a finitely dominated space X is homotopy equivalent to a finite cell complex if and only if $\overline{w}_X$ belongs to the image of the canonical map $K_0(\mathcal{S}^{\mathrm{fin}}_{X//X}) \to \pi_0 A(X)$, where $\mathcal{S}^{\mathrm{fin}}_{X//X}$ is the full subcategory of $\mathcal{S}_{X//X}$ spanned by the finite relative cell complexes. It is not hard to see (and we have already invoked above) that the the image of this map is precisely the subgroup $\mathbf{Z} \subseteq K_0(\mathbf{Z}[G])$ corresponding to projective $\mathbf{Z}[G]$-modules which are *free*. We therefore obtain an alternative proof of the main result of lecture 2: the space X is finitely dominated if and only if w_X vanishes in $K_0(\mathbf{Z}[G])$.

Remark 15 (Assembly Maps)**.** Let $\mathcal{S}$ denote the ∞-category of spaces and let $\mathcal{C} \subseteq \mathcal{S}$ be the full subcategory consisting only of the 1-point space $*$. For any functor $F : \mathcal{S} \to \mathrm{Sp}$, we can identify the restriction $F|_{\mathcal{C}}$ with a single spectrum $F(*)$. Let F_+ be the left Kan extension of $F|_{\mathcal{C}}$ along the inclusion $\mathcal{C} \hookrightarrow \mathcal{S}$: this is the functor given by

$$F_+(X) = \varinjlim_{C \to X} F(C)$$

where C ranges over all objects of $\mathcal{C}$ equipped with a $f : C \to X$. By definition, we must have $C = *$ and we can identify f with a point $x \in X$, so that $F_+(X)$ can be identified with the spectrum $C_*(X; F(*)) = X_+ \wedge F(*)$.

The universal property of the left Kan extension F_+ guarantees that there is a natural transformation of functors $F_+ \to F$, determined uniquely (up to homotopy) by the requirement that it is the identity map when evaluated at a point. In other words, for any space X we have a canonical map

$$C_*(X; F(*)) \to F(X).$$

We will refer to this map as *the assembly map* associated to F. It is an equivalence if and only if the functor F commutes with small colimits (in which case F is determined by the spectrum $F(*)$).

Specializing Remark 15 to the case where F is the A-theory functor $X \mapsto A(X)$, we obtain the A-theory assembly map

$$C_*(X; A(*)) \to A(X).$$

This map is *not* an equivalence in general, and we will see that its failure to be an equivalence measures the difference between simple homotopy theory and ordinary homotopy theory.

Definition 16. For every space X, we let $\mathrm{Wh}(X)$ denote the cofiber of the assembly map $C_*(X; A(*)) \to A(X)$. We will refer to $\mathrm{Wh}(X)$ as the *(piecewise linear) Whitehead spectrum* of X.

Remark 17. Let X be a connected space with fundamental group G. Using the isomorphisms $\pi_0 A(*) \simeq \mathbf{Z}$ and $\pi_1 A(*) \simeq \mathbf{Z}/2\mathbf{Z}$, the Atiyah-Hirzebruch spectral sequence supplies an isomorphism

$$\mathrm{H}_0(X; A(*)) \simeq \mathrm{H}_0(X; \mathbf{Z}) \simeq \mathbf{Z}$$

and an exact sequence of low-degree terms

$$\mathrm{H}_0(X; \pi_1 A(*)) \to \mathrm{H}_1(X; A(*)) \to \mathrm{H}_1(X; \mathbf{Z}) \to 0.$$

This sequence is exact on the left and canonically split (we can see this by considering the projection map from X to a point), so we obtain an isomorphism

$$\mathrm{H}_1(X; A(*)) \simeq (\mathbf{Z}/2\mathbf{Z}) \oplus G^{\mathrm{ab}}.$$

The cofiber sequence of spectra

$$C_*(X; A(*)) \to A(X) \to \mathrm{Wh}(X)$$

now supplies a long exact sequence of abelian groups

$$(\mathbf{Z}/2\mathbf{Z}) \oplus G^{\mathrm{ab}} \xrightarrow{\beta} K_1(\mathbf{Z}[G]) \to \pi_1 \mathrm{Wh}(X) \to \mathbf{Z} \xrightarrow{\alpha} K_0(\mathbf{Z}[G]) \to \pi_0 \mathrm{Wh}(X) \to 0.$$

The map α is split injective (via the ring homomorphism $\mathbf{Z}[G] \to \mathbf{Z}$ which annihilates G, say). We can therefore identify $\pi_0 \mathrm{Wh}(X)$ with the reduced K-group $\widetilde{K}_0(\mathbf{Z}[G])$ and $\pi_1 \mathrm{Wh}(X)$ with the cokernel of β, which is the Whitehead group of X as defined in Lecture 4.

For our applications, it will be convenient to have a *geometric* understanding of the assembly map: that is, we would like to understand it not as arising from the general categorical construction of Remark 15, but instead have an interpretation of the domain $C_*(X; A(*))$ as related to some sorts of kind of sheaf theory on X, just as $A(X)$ is related to local systems on X. We will take this up in the next lecture.

Constructible Sheaves (Lecture 22)

October 29, 2014

For any topological space, one can consider the A-theory assembly map

$$X_+ \wedge A(*) \to A(X)$$

defined in the previous lecture. Our goal over the next few lectures is to provide a "geometric" description of the left hand side, analogous to our description of $A(X)$ as the K-theory of an ∞-category of local systems. In what follows, we will confine our attention to the case where X is a finite polyhedron.

Suppose we are given a PL triangulation of X, which we will identify with a finite partially ordered set $\Sigma(X)$ of simplices in X. In this case, X is homeomorphic to the nerve of the poset $\Sigma(X)$. Consequently, the singular complex $\mathrm{Sing}_\bullet X$ is weakly homotopy equivalent to $\mathrm{N}(\Sigma(X))$. Thinking of both $\mathrm{Sing}_\bullet X$ and $\mathrm{N}(\Sigma(X))$ as ∞-categories, this means that the Kan complex $\mathrm{Sing}_\bullet X$ is obtained from the ∞-category $\mathrm{N}(\Sigma(X))$ by formally inverting all morphisms. In other words, for any ∞-category $\mathcal{C}$, there is a fully faithful embedding

$$\mathcal{C}^X = \mathrm{Fun}(\mathrm{Sing}_\bullet X, \mathcal{C}) \to \mathrm{Fun}(\Sigma(X), \mathcal{C}),$$

whose essential image consists of those functors $\mathcal{F} : \Sigma(X) \to \mathcal{C}$ which carry each inclusion of simplices $\sigma_0 \subseteq \sigma$ to an equivalence $\mathcal{F}(\sigma_0) \to \mathcal{F}(\sigma)$. This motivates the following definition:

Definition 1. Let X be a polyhedron equipped with a triangulation $\Sigma(X)$, and let $\mathcal{C}$ be an ∞-category. A *$\mathcal{C}$-valued sheaf on X which is constructible with respect to* $\Sigma(X)$ is a functor $\mathrm{N}(\Sigma(X)) \to \mathcal{C}$. We let $\mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ denote the ∞-category $\mathrm{Fun}(\mathrm{N}(\Sigma(X)), \mathcal{C})$ of $\mathcal{C}$-valued sheaves which are constructible with respect to $\Sigma(X)$.

Example 2. The above analysis shows that we can identify $\mathcal{C}^X$ with a full subcategory of $\mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$.

Remark 3. There is another notion of sheaf which is less combinatorial in flavor: namely, one can define a *$\mathcal{C}$-valued sheaf on* X (for X any topological space) to be a contravariant functor $\mathcal{F}$ from open subsets of X to $\mathcal{C}$, which satisfies the following descent condition: for any collection of open sets $\{U_\alpha\}$, if we set $U = \bigcup U_\alpha$, then $\mathcal{F}(U) \simeq \varprojlim_V \mathcal{F}(V)$, where the limit is taken over all open subsets $V \subseteq X$ which are contained in some U_α.

If $\Sigma(X)$ is a triangulation of X, one can say that such a sheaf $\mathcal{F}$ is *constructible with respect to* $\Sigma(X)$ if its restriction to the interior of each simplex of $\Sigma(X)$ is a constant sheaf. One can show that if $\mathcal{C}$ admits small limits, then this definition is equivalent to Definition 1. However, since we will only be interested in constructible sheaves, we will be content to work with Definition 1.

Exercise 4. Let Sp denote the ∞-category of spectra. Show that an object $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathrm{Sp})$ is compact if and only if each $\mathcal{F}(\sigma)$ is a finite spectrum.

Suppose that we are given another triangulation $\Sigma'(X)$ which refines a triangulation $\Sigma(X)$ (meaning that each simplex of $\Sigma'(X)$ maps linearly into a simplex of $\Sigma(X)$). Then for each simplex $\sigma' \in \Sigma'(X)$, there is a smallest simplex $\sigma \in \Sigma(X)$ which contains it. The construction $\sigma' \mapsto \sigma$ defines a map of partially ordered sets $f : \Sigma'(X) \to \Sigma(X)$. Composition with f induces a map $\iota : \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C}) \to \mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C})$.

Proposition 5. *In the above situation, the functor* $\iota : \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C}) \to \mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C})$ *is fully faithful.*

Remark 6. Proposition 5 is an immediate consequence of the topological picture described in Remark 3.

Proof. We may assume without loss of generality that $\mathcal{C}$ admits finite limits. The functor ι has a left adjoint $\iota_+ : \mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C}) \to \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$, given by left Kan extension along f. Concretely, this functor can be described by the formula

$$(\iota_+ \mathcal{F})(\sigma) = \varinjlim_{\sigma' \in \Sigma'(X), \sigma' \subseteq \sigma} \mathcal{F}(\sigma'),$$

where $\mathcal{F} \in \mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C})$. To prove Proposition 5, we must show that for every $\mathcal{G} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$, the counit map $\iota_+ \iota^* \mathcal{G} \to \mathcal{G}$ is an equivalence. Evaluating at a simplex $\sigma \in \Sigma(X)$, we are required to prove that $\mathcal{G}(\sigma)$ is given by the colimit $\varinjlim_{\sigma' \in \Sigma'(X), \sigma' \subseteq \sigma} \mathcal{G}(f(\sigma'))$. In other words, we wish to show that the canonical map

$$\theta : \varinjlim_{\sigma' \in \Sigma'(X), \sigma' \subseteq \sigma} \mathcal{G}(f(\sigma)) \to \varinjlim_{\sigma' \in \Sigma'(X), \sigma' \subseteq \sigma} \mathcal{G}(\sigma)$$

is an equivalence (the right hand side is given by $\mathcal{G}(\sigma)$, since the diagram is indexed by a contractible partially ordered set: in fact, the geometric realization of this partially ordered set is homeomorphic to σ). Let $P_0 = \{\sigma' \in \Sigma'(X) : \sigma' \subseteq \sigma\}$ and let $P_1 = \{\sigma' \in P : f(\sigma') = \sigma\}$. The map θ is determined by a natural transformation between diagrams $S_0 \to \mathcal{C}$, and this natural transformation is invertible when restricted to S_1. To prove that θ is invertible, it suffices to show that S_1 is left cofinal in S_0. This is a special case of the following more general assertion (applied in the case $M = \sigma$):

Lemma 7. *Let M be a piecewise linear n-manifold with boundary, equipped with a triangulation $\Sigma(M)$. Let Q be the collection of simplices of S which are not contained in ∂M. Then the inclusion $Q \hookrightarrow \Sigma(M)$ is left cofinal.*

Remark 8. Lemma 7 can be regarded as an analogue of the assertion that a manifold with boundary is always homotopy equivalent to its interior.

Proof. To prove Lemma 7, we work by induction on n. Fix a simplex $\sigma \in \Sigma(X)$; we wish to show that the set $Q = \{\sigma' \in P : \sigma \subseteq \sigma'\}$ has weakly contractible nerve. If $\sigma \in P$ this is obvious (since the subset above contains σ as a smallest element). Let us therefore assume that σ is a simplex of the boundary ∂M. Let $V = \{\tau \in \Sigma(X) : \sigma \subsetneq \tau\}$. Then V can be identified with the partially ordered set of simplices of $\mathrm{lk}(\sigma)$, which (since M is a PL manifold with boundary) is PL isomorphic to a disk D^m for $m < n$. We can identify Q with the subset of V consisting of simplices which are not contained in ∂D^m. Using the inductive hypothesis, we deduce that the inclusion $Q \hookrightarrow V$ is left cofinal. Since V has weakly contractible nerve, so does Q. $\square$

$\square$

Emboldened by Proposition 5, let us abuse notation by identifying $\mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ with its essential image in $\mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C})$ whenever $\Sigma'(X)$ is a refinement of $\Sigma(X)$. This motivates the following:

Definition 9. Let $\mathrm{Shv}_{PL}(X; \mathcal{C})$ denote the filtered direct limit $\mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$, where $\Sigma(X)$ ranges over all PL triangulations of X. We will refer to $\mathrm{Shv}_{PL}(X; \mathcal{C})$ as the *∞-category of constructible $\mathcal{C}$-valued sheaves on X.*

Remark 10 (Functoriality)**.** Let $f : X \to Y$ be a PL map of finite polyhedra, and suppose we are given compatible triangulations $\Sigma(X)$ and $\Sigma(Y)$. Then f induces a map of posets $r : \Sigma(X) \to \Sigma(Y)$, which determines a pullback functor

$$f^* : \mathrm{Shv}_{\Sigma(Y)}(Y; \mathcal{C}) \to \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C}).$$

If $\mathcal{C}$ admits finite limits, then f^* admits a right adjoint f_*, given by right Kan extension along r. Since r is a Cartesian fibration, this right Kan extension can be described concretely by the formula

$$(f_* \mathcal{F})(\tau) = \varprojlim_{f(\sigma) = \tau} \mathcal{F}(\sigma).$$

Suppose that we are given refinements $\Sigma'(X)$ and $\Sigma'(Y)$ of the triangulations $\Sigma(X)$ and $\Sigma(Y)$, which are compatible with the map f. It is easy to see that the definition of f^* is compatible with the fully faithful embeddings of Proposition 5. We claim that the same is true of f_*. In other words, we claim that if $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ and $\tau' \in \Sigma'(Y)$, then the canonical map

$$\varprojlim_{f(\sigma)=\tau'^+} \mathcal{F}(\sigma) \to \varprojlim_{f(\sigma')=\tau'} \mathcal{F}((\sigma')^+)$$

is an equivalence, where τ'^+ Is the smallest simplex of $\Sigma(Y)$ containing τ' and the notation $(\sigma')^+$ is defined similarly. To prove this, it suffices to show that construction $\sigma' \mapsto (\sigma')^+$ defines a right cofinal map of posets

$$\{\sigma' \in \Sigma'(X) : f(\sigma') = \tau'\} \to \{\sigma \in \Sigma(X) : f(\sigma) = (\tau')^+\}.$$

Fix a simplex $\sigma \in \Sigma(X)$ with $f(\sigma) = (\tau'^+)$; we wish to prove that the poset $\{\sigma' \in \Sigma'(X) : \sigma' \subseteq \sigma, f(\sigma') = \tau'\}$ is weakly contractible. This follows by applying our criterion for cell-like maps to the map of simplices $\sigma \to (\tau')^+$ (equipped with the triangulations induced by $\Sigma'(X)$ and $\Sigma'(Y)$, respectively).

Passing to the limit over all triangulations, we obtain a pair of adjoint functors

$$\mathrm{Shv}_{PL}(Y; \mathcal{C}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Shv}_{PL}(X; \mathcal{C}).$$

If $\mathcal{C}$ admits finite colimits, then the functor f^* also admits a left adjoint

$$\mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C}) \to \mathrm{Shv}_{\Sigma(Y)}(Y; \mathcal{C}).$$

This functor is *not* compatible with refinement of triangulation, and will therefore be of little use to us.

Universal Local Acyclicity (Lecture 23)

October 27, 2014

Let $\mathcal{C}$ be an ∞-category, which will remain fixed for most of this lecture In the previous lecture, we introduced the notion of a constructible $\mathcal{C}$-valued sheaf on a finite polyhedron X. Suppose we are given a map of finite polyhedra $f : X \to S$. In this Then a constructible sheaf $\mathcal{F}$ on X determines a family of constructible sheaves $\mathcal{F}_s$ on X_s, as s ranges over the points of S. In this lecture, we will study a condition which guarantees that the sheaves $\mathcal{F}_s$ behave nicely as s varies.

Let us henceforth assume that $\mathcal{C}$ admits finite limits.

Definition 1. Let $f : X \to S$ be a PL map of polyhedra. Choose triangulations $\Sigma(X)$ and $\Sigma(S)$ of X and S which are compatible with the map f. We will say that a sheaf $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ is *universally locally acyclic with respect to* f (ULA) at a simplex $\sigma_0 \in \Sigma(X)$ if the following condition is satisfied:

$(*)$ For any simplex $\tau \in \Sigma(S)$ which contains $\tau_0 = f(\sigma_0)$, the canonical map

$$\mathcal{F}(\sigma_0) \to \varprojlim_{\sigma_0 \subseteq \sigma, f(\sigma) = \tau} \mathcal{F}(\sigma)$$

is an equivalence in $\mathcal{C}$.

We will say that $\mathcal{F}$ is *ULA at a point* $x \in X$ if it is ULA at every simplex which contains x. We will say that $\mathcal{F}$ is ULA if it is ULA at every simplex.

Warning 2. The condition that $\mathcal{F}$ be ULA depends not only on the sheaf $\mathcal{F}$ and the polyhedron X, but also on the map f. When it is important to emphasize this dependence, we will instead use the phrase "$\mathcal{F}$ is ULA over S."

Remark 3. It follows tautologically from the definition that the set of points $x \in X$ at which $\mathcal{F}$ is ULA is open: its complement is a subcomplex of X.

Remark 4. In condition $(*)$, we can replace the poset $\{\sigma_0 \subseteq \sigma, f(\sigma) = \tau\}$ with $\{\sigma_0 = \sigma \cap f^{-1}\tau_0, f(\sigma) = \tau\}$: the latter is right cofinal in the former.

Example 5. Let $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$. Then $\mathcal{F}$ is ULA over X if and only if $\mathcal{F}$ is locally constant.

Example 6. Let $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ and $f : X \to S$ be as in Definition 1. If $\sigma \in \Sigma(X)$ has the property that $f(\sigma)$ is a maximal simplex of S (not contained as a facet of any larger simplex), then $\mathcal{F}$ is automatically ULA at σ. In particular, $\mathcal{F}$ is automatically ULA on the open subset of X given by the inverse images of the interiors of the maximal simplices of S.

Proposition 7. *The notion of universal local acyclicity is independent of the choice of triangulation. That is, if $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ and we choose finer triangulations $\Sigma'(X)$ and $\Sigma'(S)$ (still compatible with the map f) and we let $\mathcal{F}' \in \mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C})$ denote the image of $\mathcal{F}$, then $\mathcal{F}$ is ULA at a point $x \in X$ if and only if $\mathcal{F}'$ is ULA at x.*

Proof. Let $g : \Sigma'(X) \to \Sigma(X)$ be the map that takes each simplex of $\Sigma'(X)$ to the smallest simplex of $\Sigma(X)$ which contains it. Then it will suffice to show that $\mathcal{F}'$ is ULA at a simplex σ'_0 if and only if $\mathcal{F}$ is ULA at $\sigma_0 = g(\sigma'_0)$. Suppose first that $\mathcal{F}$ is ULA at σ_0, and consider a simplex $\tau' \in \Sigma'(S)$ which contains $\tau'_0 = f(\sigma'_0)$.

Let τ_0 and τ be the smallest simplices of $\Sigma(S)$ which contain τ'_0 and τ', respectively. We have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(\sigma_0) \arrow[r] \arrow[d] & \varprojlim_{\sigma_0 \subseteq \sigma, f(\sigma)=\tau} \mathcal{F}(\sigma) \arrow[d] \\ \mathcal{F}'(\sigma'_0) \arrow[r] & \varprojlim_{\sigma'_0 \subseteq \sigma', f(\sigma')=\tau'} \mathcal{F}'(\sigma'). \end{tikzcd}$$

By hypothesis, the upper horizontal map is an equivalence, and we wish to prove that the same is true of the lower horizontal map. Since the left vertical map is an equivalence, we are reduced to proving the same is true of the lower vertical map. Invoking the definition of $\mathcal{F}'$, we are reduced to proving that g induces a right cofinal map of posets

$$\{\sigma' \in \Sigma'(X) : \sigma'_0 \subseteq \sigma', f(\sigma') = \tau'\} \to \{\sigma \in \Sigma(X) : \sigma_0 \subseteq \sigma, f(\sigma) = \tau\}.$$

Fix a simplex σ in the latter poset; we wish to prove that the poset

$$\{\sigma' \in \Sigma'(X) : \sigma'_0 \subseteq \sigma' \subseteq \sigma, f(\sigma') = \tau'\}$$

is weakly contractible. This follows from the criterion of Lecture 9, applied to the map $\sigma \to \tau$ (with the triangulations induced by $\Sigma'(X)$ and $\Sigma'(S)$); note that this is a linear map of simplices and therefore automatically a fibration.

Now suppose that $\mathcal{F}'$ is ULA at σ'_0; we wish to show that $\mathcal{F}$ is ULA at σ_0. Let τ be any simplex of $\Sigma(S)$ which contains $f(\sigma_0)$. Then we can choose a simplex τ' of $\Sigma'(S)$ which contains $f(\sigma'_0)$ whose interior is contained in the interior of τ. The desired result now follows by applying the above reasoning to the simplex τ'. □

It follows from Proposition 7 that universal local acyclicity makes sense for objects of $\mathrm{Shv}_{PL}(X; \mathcal{C})$. Moreover, it is a condition which is generically satisfied; Example 6 immediately yields the following:

Proposition 8 ("Sard's Theorem")**.** *Let $f : X \to S$ be a map of finite polyhedra. For each $\mathcal{F} \in \mathrm{Shv}_{PL}(X; \mathcal{C})$, there is a dense open subset $U \subseteq S$ (complementary to a subcomplex of S) such that $\mathcal{F}$ is ULA at every point of $f^{-1}(U)$.*

Example 9. Every sheaf $\mathcal{F} \in \mathrm{Shv}_{PL}(X; \mathcal{C})$ is ULA with respect to the map $X \to *$.

Proposition 10. *The condition of universal local acyclicity is stable under base change. That is, suppose we are given a pullback square of finite polyhedra*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "g'"] & X \arrow[d, "g"] \\ S' \arrow[r, "h"] & S \end{tikzcd}$$

and a constructible sheaf $\mathcal{F}$ on X. For each point $x' \in X'$, if $\mathcal{F}$ is ULA at $f(x')$, then $f^* \mathcal{F}$ is ULA at x'.

Proof. We repeat the proof of Proposition 7. Choose compatible triangulations $\Sigma(X)$ and $\Sigma(S)$ of X and S so that $\mathcal{F}$ is constructible with respect to $\Sigma(X)$. Similarly, we can choose compatible triangulations of $\Sigma(X')$ and $\Sigma(S')$ such that each simplex of X' maps into a simplex of X (not necessarily by a map which preserves vertices) and similarly for $S' \to S$.

Let σ'_0 be a simplex of $\Sigma(X')$ which contains x', let τ'_0 be its image in S', and let τ' be a simplex of $\Sigma(S')$ which contains τ'_0. We wish to show that the canonical map

$$(f^* \mathcal{F})(\sigma'_0) \to \varprojlim_{\sigma'_0 \subseteq \sigma', g'(\sigma') = \tau'} \mathcal{F}(\sigma')$$

is an equivalence. Let ${\sigma'_0}^+$ denote the smallest element of $\Sigma(X)$ which contains the image of σ'_0, and define ${\tau'_0}^+$ and ${\tau'}^+$ similarly. Then ${\sigma'_0}^+$ contains $f(x')$, so our hypothesis gives an equivalence

$$(f^* \mathcal{F})(\sigma'_0) = \mathcal{F}({\sigma'_0}^+) = \varprojlim_{f(\sigma'_0) \subseteq \sigma, g(\sigma) = {\tau'}^+} \mathcal{F}(\sigma).$$

It will therefore suffice to show that the construction $\sigma' \mapsto {\sigma'}^+$ determines a right cofinal map of posets

$$P = \{\sigma' \in \Sigma(X') : \sigma'_0 \subseteq \sigma', g'(\sigma') = \tau'\} \to Q = \{\sigma \in \Sigma(X) : f(\sigma'_0) \subseteq \sigma, g(\sigma) = {\tau'}^+\}.$$

Fix a simplex $\sigma \in Q$, we wish to show that the poset

$$P_{/\sigma} = \{\sigma' \in \Sigma(X') : \sigma'_0 \subseteq \sigma' \subseteq f^{-1}(\sigma), g'(\sigma') = \tau'\}$$

is weakly contractible. Using the criterion from Lecture 9, we are reduced to showing that the projection map $f^{-1}\sigma \to h^{-1}\tau'^+$ is a fibration. Since the diagram in question is a pullback, this reduces to the statement that $\sigma \to {\tau'}^+$ is a fibration, which is clear since it is a linear map of simplices. $\square$

Example 11. Let $f : X \to S$ be a map of finite polyhedra. Then the following conditions are equivalent:

(1) The map f is a fibration.

(2) For every ∞-category $\mathcal{C}$ which admits finite limits and every locally constant sheaf $\mathcal{F}$ with values in $\mathcal{C}$, $\mathcal{F}$ is ULA with respect to f.

To prove that (1) $\Rightarrow$ (2), choose compatible triangulations $\Sigma(X)$ and $\Sigma(S)$; we wish to verify that a locally constant sheaf $\mathcal{F}$ satisfies condition $(*)$ of Definition 1. That is, we wish to show that certain maps of the form

$$\mathcal{F}(\sigma_0) \to \varprojlim_{\sigma_0 \subseteq \sigma, f(\sigma) = \tau} \mathcal{F}(\sigma)$$

are equivalences in $\mathcal{C}$. Since $\mathcal{F}$ is locally constant, we can identify each $\mathcal{F}(\sigma)$ with a fixed object $C = \mathcal{F}(\sigma_0) \in \mathcal{C}$. The result then follows from the contractibility of the poset $\{\sigma | \sigma_0 \subseteq \sigma, f(\sigma) = \tau\}$ (which follows from the combinatorial criterion for f to be a fibration; see Lecture 9).

Conversely, suppose that (2) is satisfied. Taking $\mathcal{C} = \mathcal{S}^{\mathrm{op}}$ and $\mathcal{F}$ to be the constant sheaf taking the value $*$, condition $(*)$ of Definition 1 asserts that posets of the form $\{\sigma | \sigma_0 \subseteq \sigma, f(\sigma) = \tau\}$ are weakly contractible, which is equivalent to the requirement that f is a fibration (again by the result of Lecture 9).

Warning 12. We will be primarily interested in the case where $\mathcal{C} = \mathrm{Sp}^{\mathrm{fin}}$ is the ∞-category of finite spectra. In this case, the condition that every locally constant sheaf $\mathcal{F}$ on X be ULA with respect to a map $f : X \to S$ is slightly weaker than the condition that f is a fibration: it implies only that the partially ordered sets $\{\sigma | \sigma_0 \subseteq \sigma, f(\sigma) = \tau\}$ appearing in our fibration criterion are acyclic, not that they are weakly contractible.

Remark 13 (Vanishing Cycles)**.** Let $\mathcal{C}$ be an ∞-category which admits small limits, let $f : X \to S$ be an arbitrary map of topological spaces, and let us consider $\mathcal{C}$-valued sheaves in the sense of Remark **??**.

Suppose we are given a continuous path $p : [0, 1] \to S$ with $p(0) = s \in S$. Let $\mathcal{F}$ be a $\mathcal{C}$-valued sheaf on X, $\mathcal{F}_p$ denote the restriction of $\mathcal{F}$ to the fiber product $X \times_S [0, 1]$, and let $\mathcal{F}^\circ_p$ denote the restriction of $\mathcal{F}_p$ to $X \times_S (0, 1]$. Let

$$j : X \times_S (0, 1] \to X \times_S [0, 1]$$

be the inclusion map; we can then push $\mathcal{F}_p^\circ$ forward to obtain a sheaf $j_* \mathcal{F}_p^\circ$ on $X \times_S [0,1]$. Let $\mathcal{G}$ denote the restriction of $j_* \mathcal{F}_p^\circ$ to the fiber $X \times_S \{0\} = X_s$. We will refer to $\mathcal{G}$ as the *nearby cycles sheaf of $\mathcal{F}$ at the point s in the direction p.* Note that there is a canonical map $\theta : \mathcal{F}|_{X_s} \to \mathcal{G}$. If $\mathcal{C}$ is stable, we will refer the fiber of θ as the *vanishing cycles sheaf of $\mathcal{F}$ at the point s in the direction p.*

In the special case where X and S are finite polyhedra, f is piecewise linear, $\mathcal{F}$ is constructible, then we can choose compatible triangulations $\Sigma(X)$ and $\Sigma(S)$ such that $\mathcal{F}$ is constructible with respect to $\Sigma(X)$. If $p : [0,1] \to S$ is piecewise linear, then there exist simplices $\tau_0 \subseteq \tau$ of $\Sigma(S)$ such that $s = p(0)$ lies in the interior of τ_0 and $p(t)$ belongs to the interior of τ for sufficiently small nonzero values of t. In this case, there is an identification of the fiber X_s with the nerve of the poset $\{\sigma_0 \in \Sigma(S) : f(\sigma_0) = \tau_0\}$ which exhibits $\mathcal{F}|_{X_s}$ and $\mathcal{G}$ as constructible sheaves on X_s, which correspond to the functors

$$\sigma_0 \mapsto \mathcal{F}(\sigma_0) \qquad \sigma_0 \mapsto \varprojlim_{\sigma_0 \subseteq \sigma, f(\sigma) = \tau} \mathcal{F}(\sigma),$$

respectively. We may therefore reformulate Definition 1 heuristically as follows: a constructible sheaf $\mathcal{F}$ on X is ULA with respect to S if it has no vanishing cycles with respect to any point $s \in S$ and any direction $p : [0,1] \to S$ with $p(0) = s$.

Example 11 shows that if $f : X \to S$ is a fibration, then any locally constant sheaf is universally locally acyclic. However, if we are willing to depart from the world of locally constant sheaves, then the condition of universal local acyclicity has nothing at all to do with the ambient space X:

Proposition 14. *Suppose we are given a commutative diagram of finite polyhedra*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Let $\mathcal{F} \in \mathrm{Shv}_{PL}(X; \mathcal{C})$ be ULA over S. Then $f_* \mathcal{F} \in \mathrm{Shv}_{PL}(Y)$ is ULA over S. The converse holds if f is a closed embedding.

Proof. Choose compatible triangulations $\Sigma(X)$, $\Sigma(Y)$, and $\Sigma(S)$ such that $\mathcal{F}$ is constructible with respect to $\Sigma(X)$. Fix a simplex $\sigma_0 \in \Sigma(Y)$, let $\tau_0 = g(\sigma_0)$, and let $\tau \in \Sigma(S)$ be a simplex containing τ_0. We wish to show that the canonical map

$$(f_* \mathcal{F})(\sigma_0) \to \varprojlim_{\sigma_0 \subseteq \sigma, g(\sigma) = \tau} (f_* \mathcal{F})(\sigma)$$

is an equivalence. Invoking the definition of f_*, we can write this map as

$$\varprojlim_{f(\theta_0) = \sigma_0} \mathcal{F}(\theta_0) \to \varprojlim_{\sigma_0 \subseteq f(\theta), h(\theta) = \tau} \mathcal{F}(\theta).$$

Let $P = \{\theta_0 \in \Sigma(X) : f(\theta_0) = \sigma_0\}$ and let $Q = \{\theta \in \Sigma(X) : h(\theta) = \tau \text{ and } f(\theta) \supseteq \sigma_0\}$. Intersection with $f^{-1}\sigma_0$ induces a map of posets from Q to P; to complete the proof, it will suffice to show that $\mathcal{F}|_P$ is a right Kan extension of $\mathcal{F}|_Q$ along this map. In other words, it will suffice to show that for each $\theta_0 \in P$, the canonical map

$$\mathcal{F}(\theta_0) \to \varprojlim_{\theta \in Q, \theta_0 \subseteq \theta} \mathcal{F}(\theta)$$

is an equivalence. This follows immediately from our assumption that $\mathcal{F}$ is ULA over S.

Now suppose that f is a closed embedding and that $f_* \mathcal{F}$ is ULA over S. To prove that $\mathcal{F}$ is ULA over S, we must show that for each $\theta_0 \in \Sigma(X)$, if $\tau_0 = h(\theta_0)$ and $\tau \in \Sigma(S)$ contains τ_0, then the canonical map

$$\mathcal{F}(\theta_0) \to \varprojlim_{\theta_0 \subseteq \theta, h(\theta) = \tau} \mathcal{F}(\theta)$$

is an equivalence. Let us identify X with its image in Y via f, and let us identify $\mathcal{F}$ with the restriction of $f_* \mathcal{F}$ to simplices of X. Then we can write our map as a compsition

$$(g_* \mathcal{F})(\theta_0) \to \varprojlim_{\theta_0 \subseteq \sigma \in \Sigma(Y), g(\sigma) = \tau} (g_* \mathcal{F})(\sigma) \to \varprojlim_{\theta_0 \subseteq \sigma \in \Sigma(X), g(\sigma) = \tau} (g_* \mathcal{F})(\sigma).$$

The first map is an equivalence by virtue of our assumption that $g_* \mathcal{F}$ is ULA over S, and the second map is an equivalence because the restriction of $g_* \mathcal{F}$ to $\{\sigma \in \Sigma(Y) : \theta_0 \subseteq \sigma, g(\sigma) = \tau\}$ is a right Kan extension of the restriction of $\mathcal{F}$ to $\{\sigma \in \Sigma(X) : \theta_0 \subseteq \sigma, g(\sigma) = \tau\}$. □

Remark 15. The proof of Proposition 14 shows something a bit stronger: to verify that $f_* \mathcal{F}$ is ULA over S at a point $y \in Y$, it suffices to know that $\mathcal{F}$ is ULA over S at every point $x \in X$ such that $f(x) = y$.

The Assembly Map (Lecture 24)

October 29, 2014

We begin this lecture by proving one more basic property of universal local acyclicity:

Proposition 1. *Suppose we are given a commutative diagram of finite polyhedra*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Suppose that g is a fibration and let $\mathcal{F} \in \mathrm{Shv}_{PL}(X; \mathcal{C})$. Then $\mathcal{F}$ is ULA over Y if and only if the following conditions are satisfied:

(1) *The sheaf $\mathcal{F}$ is ULA over S.*

(2) *For each point $s \in S$, the restriction $\mathcal{F}_s = \mathcal{F}|_{X_s}$ is ULA over Y_s.*

Example 2. When $X = Y$, Proposition 1 asserts that $\mathcal{F}$ is locally constant if and only if it is ULA over S and locally constant along each fiber X_s. In particular, if $\mathcal{F}$ is locally constant then it is ULA over S, as we saw in the previous lecture.

Proof of Proposition 1. Since the property of being ULA is stable under base change, condition (2) is clearly necessary. Let us therefore assume that $\mathcal{F}$ satisfies (2) and show that $\mathcal{F}$ is ULA over Y if and only if it is ULA over Z.

Fix compatible triangulations $\Sigma(X)$, $\Sigma(Y)$, and $\Sigma(S)$ of X, Y, and S such that $\mathcal{F}$ is constructible with respect to $\Sigma(X)$. Consider a simplex $\theta_0 \in \Sigma(X)$, and let $\sigma_0 \in \Sigma(Y)$ and $\tau_0 \in \Sigma(S)$ be its images in Y and S, respectively. Let $\tau \in \Sigma(S)$ be a simplex containing τ_0, let $P = \{\theta \in \Sigma(X) : \theta_0 \subseteq \theta, h(\theta) = \tau\}$ and let $Q = \{\sigma \in \Sigma(Y) : \sigma_0 \subseteq \sigma, g(\sigma) = \tau\}$. The map f determines a map of posets $P \to Q$. For each $\sigma \in Q$, let P_σ denote the inverse image of σ in P. We claim the following:

($*$) The right Kan extension of $\mathcal{F}|_P$ along the map $P \to Q$ is a locally constant functor (that is, it sends inequalities in Q to equivalences in $\mathcal{C}$).

Assuming ($*$), we can complete the proof as follows. Let $\mathcal{G}$ be the right Kan extension of $\mathcal{F}|_P$ along the map $P \to Q$. Since g is a fibration, the partially ordered set Q is weakly contractible. It follows from ($*$) that for any element $\sigma \in Q$, the canonical map $\varprojlim \mathcal{G} \to \mathcal{G}(\sigma)$ is an equivalence in $\mathcal{C}$. Rewriting this in terms of $\mathcal{F}$, we conclude that the canonical map

$$\varprojlim_{\theta \in P} \mathcal{F}(\theta) \to \varprojlim_{\theta \in P_\sigma} \mathcal{F}(\theta)$$

is an equivalence. It follows that the natural maps

$$\alpha : \mathcal{F}(\theta_0) \to \varprojlim_{\theta \in P} \mathcal{F}(\theta)$$

$$\beta : \mathcal{F}(\theta_0) \to \varprojlim_{\theta \in P_\sigma} \mathcal{F}(\theta)$$

can be identified with one another. But the condition that $\mathcal{F}$ is ULA over S is equivalent to the requirement that α is an equivalence (for any θ_0 and any τ), while the condition that $\mathcal{F}$ is ULA over X is equivalent to the requirement that β is an equivalence (for any θ_0 and any σ).

To prove $(*)$, suppose we are given an inclusion of simplices $\sigma \subseteq \sigma'$ which belong to Q; we wish to show that the canonical map $\mathcal{G}(\sigma) \to \mathcal{G}(\sigma')$ is an equivalence. Note that the construction $\theta \mapsto \theta \cap f^{-1}\sigma$ induces a map of posets $\lambda : P_{\sigma'} \to P_\sigma$. Condition $(*)$ will follow if we can show that $\mathcal{F}|_{P_\sigma}$ is a right Kan extension of $\mathcal{F}|_{P_{\sigma'}}$ along λ. Unwinding the definitions, this amounts to the assertion that for $\theta \in P_\sigma$, the canonical map

$$\mathcal{F}(\theta) \to \varprojlim_{\theta' \supseteq \theta, f(\theta') = \sigma'} \mathcal{F}(\theta')$$

is an equivalence. This follows from condition (2), applied to any point $s \in S$ which belongs to the interior of the simplex τ. □

Exercise 3. Let $f : X \to S$ be a map of finite polyhedra. Let $\mathcal{F} \in \mathrm{Shv}_{PL}(X; \mathrm{Sp})$ and let $\mathcal{G}$ be a local system of spectra on X. Show that if $\mathcal{F}$ is ULA over S, then so is the constructible sheaf $\mathcal{F} \wedge \mathcal{G}$ given by

$$(\mathcal{F} \wedge \mathcal{G})(\sigma) = \mathcal{F}(\sigma) \wedge \mathcal{G}(\sigma).$$

We now use the theory of ULA sheaves to build a variant of the construction $X \mapsto A(X)$.

Definition 4. Let X be a finite polyhedron. For every finite polyhedron S, we let $\mathrm{Shv}^S_{PL}(X \times S)$ denote the full subcategory of $\mathrm{Shv}_{PL}(X \times S, \mathrm{Sp}^{\mathrm{fin}})$ spanned by those sheaves which are ULA over S. Here $\mathrm{Sp}^{\mathrm{fin}}$ denotes the ∞-category of finite spectra.

Note that the construction $S \mapsto \mathrm{Shv}^S_{PL}(X \times S)$ is contravariant in S (since the condition of being ULA is stable under base change in S). In particular, the construction

$$[n] \mapsto \mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n)$$

determines a simplicial object in the ∞-category of stable ∞-categories. Applying the Waldhausen construction pointwise, we obtain a simplicial space

$$K(\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n)).$$

We will denote the geometric realization of this simplicial space by $K_\Delta(X)$.

Remark 5. Recall that each $K(\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n))$ is the 0th space of a connective spectrum. In the above definition, it does not matter if we take the geometric realization at the level of spaces or at the level of spectra (if we do the latter, we obtain a connective spectrum having $K_\Delta(X)$ as its 0th space).

Our next goal is to relate $K_\Delta(X)$ to the A-theory of X. First, we need a bit of a digression.

Definition 6. Let $\mathcal{C}$ be a stable ∞-category which admits small colimits. We let $\mathcal{C}^\vee$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp})$ spanned by those functors which preserve small colimits. We will refer to $\mathcal{C}^\vee$ as the *dual* of $\mathcal{C}$.

Example 7. Let R be an associative ring spectrum and let $\mathcal{C}$ be the ∞-category of *left* R-modules. Then $\mathcal{C}^\vee$ can be identified with the ∞-category of *right* R-modules; every right R-module M can be identified with the colimit-preserving functor

$$\mathcal{C} \to \mathrm{Sp}$$

$$N \mapsto M \otimes_R N.$$

Example 8. More generally, let $\mathcal{C}$ be a compactly generated stable ∞-category, so that $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ where $\mathcal{C}_0 \subseteq \mathcal{C}$ is the full subcategory spanned by the compact objects. For every object $C \in \mathcal{C}_0$, the functor

$$D \mapsto \mathrm{Map}(C, D)$$

is a colimit-preserving functor from $\mathcal{C}$ to spectra (if we let $\mathrm{Map}(C, D)$ denote the *spectrum* of maps from C to D), which we can identify with an element $h^C \in \mathcal{C}^\vee$. The construction $C \mapsto h^C$ determines a contravariant functor from $\mathcal{C}_0$ to $\mathcal{C}^\vee$, and one can show that this functor induces an equivalence $\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}) \simeq \mathcal{C}^\vee$. In particular, the ∞-category $\mathcal{C}^\vee$ is also compactly generated and its ∞-category of compact objects $\mathcal{C}_0^\vee$ is opposite to the ∞-category $\mathcal{C}_0$. We therefore have a canonical homotopy equivalence of K-theory spaces

$$K(\mathcal{C}_0^\vee) \simeq K(\mathcal{C}_0).$$

Construction 9. Let X be a finite polyhedron and let Sp^X denote the ∞-category of local systems of spectra on X. Let S be another finite polyhedron and let $\phi : X \times S \to X \qquad \psi : X \times S \to S$ be the projection maps. Let $\mathcal{F} \in \mathrm{Shv}^S_{PL}(X \times S)$ and $\mathcal{G} \in \mathrm{Sp}^X$. Then $\mathcal{F} \wedge \phi^* \mathcal{G}$ is a constructible sheaf of spectra on $X \times S$ which is ULA over S (Exercise 3), and therefore $\psi_*(\mathcal{F} \wedge \phi^* \mathcal{G})$ is a local system of spectra on S. This construction yields a map

$$\theta_S : S \to \mathrm{Fun}(\mathrm{Shv}^S_{PL}(X \times S) \times \mathrm{Sp}^X, \mathrm{Sp}).$$

Note that the construction $\mathcal{G} \mapsto \psi_*(\mathcal{F} \wedge \phi^* \mathcal{G})$ preserves colimits in $\mathcal{G}$; we may therefore identify θ_S with a map

$$S \to \mathrm{Fun}(\mathrm{Shv}^S_{PL}(X \times S), (\mathrm{Sp}^X)^\vee).$$

Note that θ_S depends functorially on S.

Let $(\mathrm{Sp}^X)^\vee_0$ denote the full subcategory of $(\mathrm{Sp}^X)^\vee$ spanned by the compact objects (by virtue of Example 8 this is the *opposite* of the ∞-category of compact objects of Sp^X). We claim that θ_S factors through a map

$$S \to \mathrm{Fun}(\mathrm{Shv}^S_{PL}(X \times S), (\mathrm{Sp}^X)^\vee_0).$$

To prove this, it suffices (by functoriality) to treat the case where S is a point; in this case, we are considering a single functor

$$\lambda : \mathrm{Shv}_{PL}(X) \to (\mathrm{Sp}^X)^\vee$$

which is characterized by the formula

$$\lambda(\mathcal{F})(\mathcal{G}) = \psi_*(\mathcal{F} \wedge \mathcal{G}).$$

The collection of those objects $\mathcal{F}$ for which $\lambda(\mathcal{F}) \in (\mathrm{Sp}^X)^\vee_0$ is a stable subcategory of $\mathrm{Shv}_{PL}(X)$. To show that it is all of $\mathrm{Shv}_{PL}(X)$, it will suffice to show that it contains the direct image of any constant sheaf $\underline{S}$ (with value the sphere spectrum) along an inclusion $i : \Delta^k \to X$. If $\mathcal{F} = i_* \underline{S}$, we have

$$\lambda(\mathcal{F})(\mathcal{G}) = \psi_*(i_* \underline{S} \wedge \mathcal{G}) \simeq \mathcal{G}_x$$

for any point $x \in X$ which belongs to the image of i; it follows that the functor $\lambda(\mathcal{F})$ is corepresentable by the (compact) local system $i_{x!}S$, where $i_x : \{x\} \to X$ is the inclusion map.

Passing to K-theory, we see that θ_S induces a map

$$\begin{array}{rcl} K(\theta_S) & \to & \mathrm{Fun}(K(\mathrm{Shv}^S_{PL}(X \times S)), K((\mathrm{Sp}^X)^\vee_0)) \\ & \simeq & \mathrm{Fun}(K(\mathrm{Shv}^S_{PL}(X \times S)), K((\mathrm{Sp}^X_0)^{\mathrm{op}})) \\ & \simeq & \mathrm{Fun}(K(\mathrm{Shv}^S_{PL}(X \times S)), K(\mathrm{Sp}^X_0)) \\ & \simeq & \mathrm{Fun}(K(\mathrm{Shv}^S_{PL}(X \times S)), \Omega^\infty A(X)). \end{array}$$

These maps depend functorially on S. Specializing to the case where S is a simplex, we obtain a map from the simplicial space $K(\mathrm{Shv}^{\Delta^\bullet}_{PL}(X \times \Delta^\bullet))$ to the constant simplicial space with the value $\Omega^\infty A(X)$. Passing to geometric realizations, we obtain a map of infinite loop spaces

$$K_\Delta(X) \to \Omega^\infty A(X).$$

Our goal over the next several lectures is to prove the following:

Theorem 10. *The map $K_\Delta(X) \to \Omega^\infty A(X)$ is a model for the assembly map in A-theory. In particular, there is a canonical homotopy equivalence $K_\Delta(X) \simeq \Omega^\infty(X_+ \wedge A(*))$.*

The Assembly Map, Part II (Lecture 25)

November 3, 2014

Let X be a finite polyhedron. In the previous lecture, we introduced an infinite loop space $K_{\Delta}(X)$, which is given by the geometric realization of the simplicial space

$$[n] \mapsto K(\mathrm{Shv}_{PL}^{\Delta^n}(X \times \Delta^n)).$$

Moreover, we constructed a map of connective spectra

$$\Omega^{-\infty} K_{\Delta}(X) \to A(X).$$

Our goal in this lecture and the next is to prove the following:

Theorem 1. *The map $K_{\Delta}(X) \to \Omega^{\infty} A(X)$ is a model for the assembly map in A-theory. In particular, there is a canonical homotopy equivalence $K_{\Delta}(X) \simeq \Omega^{\infty}(X_+ \wedge A(*))$.*

As a first step, we consider functoriality in X. Recall that for any map of finite polyhedra $f : X \to Y$, the pushforward map

$$(f \times \mathrm{id})_* : \mathrm{Shv}(X \times S) \to \mathrm{Shv}(Y \times S)$$

preserves the property of being ULA over S. It follows that we obtain a map of simplicial ∞-categories

$$\mathrm{Shv}_{PL}^{\Delta^\bullet}(X \times \Delta^\bullet) \to \mathrm{Shv}_{PL}^{\Delta^\bullet}(Y \times \Delta^\bullet).$$

Taking K-theory and passing to geometric realizations, we obtain a map of infinite loop spaces $K_{\Delta}(X) \to K_{\Delta}(Y)$. In other words, we can regard K_{Δ} as a functor from the ordinary category Poly of finite polyhedra (with morphisms given by PL maps) to the ∞-category of spaces (or even of E_{∞}-spaces).

The category Poly has a canonical simplicial enrichment: to every pair of finite polyhedra X and Y, we can associate a Kan complex $\mathrm{Map}(X, Y)$ whose n-simplices are given by piecewise linear maps from $X \times \Delta^n$ into Y. As a simplicially enriched category, Poly is weakly equivalent to the simplicially enriched category of finite CW complexes. This follows from two observations:

- Every finite CW complex is homotopy equivalent to a finite polyhedron.
- For every pair of finite polyhedra X and Y, the Kan complex $\mathrm{Map}(X, Y)$ is homotopy equivalent to the singular simplicial set of the topological space Y^X of all continuous maps from X into Y (more informally: there are no obstructions to approximating arbitrary continuous maps between polyhedra by piecewise-linear maps).

Using this fact, it is not difficult to show that the ∞-category $\mathcal{S}^{\mathrm{fin}}$ of finite spaces can be obtained from the ordinary category Poly by formally inverting all maps of the form $X \times \Delta^n \to X$. In fact, it suffices to consider the case $n = 1$ (since the n-simplex Δ^n is a retract of a product of copies of Δ^1). In other words, we have the following:

Claim 2. *Let $\mathcal{C}$ be an ∞-category. Then composition with the canonical map* $\mathrm{Poly} \to \mathcal{S}^{\mathrm{fin}}$ *induces a fully faithful embedding*

$$\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Poly}, \mathcal{C}),$$

whose essential image is spanned by the collection of those functors $F : \mathrm{Poly} \to \mathcal{C}$ *with the property that for any finite polyhedron X, the induced map $F(X \times \Delta^1) \to F(X)$ is an equivalence in $\mathcal{C}$.*

We would like to apply Claim 2 to the functor $X \mapsto K_\Delta(X)$.

Proposition 3. *For any finite polyhedron X, the canonical map $K_\Delta(X \times \Delta^1) \to K_\Delta(X)$ is a homotopy equivalence.*

The map of Proposition 3 has a right homotopy inverse, induced by the inclusion $X \times \{0\} \hookrightarrow X \times \Delta^1$. To check that this map is a left homotopy inverse, it will suffice to establish the following:

Lemma 4. *Let $f, g : X \to Y$ be homotopic maps of finite polyhedra. Then f and g induce homotopic maps $f_* g_* : K_\Delta(X) \to K_\Delta(Y)$.*

Proof. It suffices to prove Lemma 4 in the "universal" case where $Y = X \times \Delta^1$ and f and g are the two inclusions $X \times \{i\} \hookrightarrow X \times \Delta^1$. Let $\mathcal{J}$ denote the slice category $\mathbf{\Delta}_{/[1]}$ of nonempty finite linearly ordered sets $[n]$ equipped with a map $[n] \to [1]$. Then $\mathcal{J}$ contains full subcategories $\mathcal{J}_0, \mathcal{J}_1 \subseteq \mathcal{J}$, spanned by those objects of the form $[n] \to \{0\} \subseteq [1]$ and $[n] \to \{1\} \subseteq [1]$, respectively. Each of these subcategories is equivalent to $\mathbf{\Delta}$. To each object $[n] \to [1]$ in $\mathcal{J}$, we can associate a map of finite polyhedra

$$X \times \Delta^n \to X \times \Delta^1 \times \Delta^n,$$

which induces a pushforward functor

$$\mathrm{Shv}_{PL}^{\Delta^n}(X \times \Delta^n) \to \mathrm{Shv}_{PL}^{\Delta^n}(X \times \Delta^1 \times \Delta^n).$$

Taking K-theory and passing to the colimit, we obtain a map

$$\varinjlim_{[n] \in \mathcal{J}} K(\mathrm{Shv}_{PL}^{\Delta^n}(X \times \Delta^n)) \to K_\Delta(X \times \Delta^1).$$

Note that the composition of this map with the natural maps

$$\varinjlim_{[n] \in \mathcal{J}_0} K(\mathrm{Shv}_{PL}^{\Delta^n}(X \times \Delta^n)) \to \varinjlim_{[n] \in \mathcal{J}} K(\mathrm{Shv}_{PL}^{\Delta^n}(X \times \Delta^n))$$

$$\varinjlim_{[n] \in \mathcal{J}_1} K(\mathrm{Shv}_{PL}^{\Delta^n}(X \times \Delta^n)) \to \varinjlim_{[n] \in \mathcal{J}} K(\mathrm{Shv}_{PL}^{\Delta^n}(X \times \Delta^n))$$

coincide with f_* and g_*, respectively. It will therefore suffice to show that these latter maps are homotopy equivalences. Both have left homotopy inverses induced by the map

$$\varinjlim_{[n] \in \mathcal{J}} K(\mathrm{Shv}_{PL}^{\Delta^n}(X \times \Delta^n)) \to K_\Delta(X)$$

determined by the forgetful functor $\pi : \mathcal{J} \to \mathbf{\Delta}$. To show that this map is a homotopy equivalence, it will suffice to show that π is right cofinal. In other words, it will suffice to show that for each object $[n] \in \mathbf{\Delta}$, the category

$$\mathcal{J} \times_{\mathbf{\Delta}} \mathbf{\Delta}_{/[n]} = \mathbf{\Delta}_{/[1]} \times_{\mathbf{\Delta}} \mathbf{\Delta}_{/[n]}$$

is weakly contractible. This is clear, since it is the category of simplices of the weakly contractible simplicial set $\Delta^1 \times \Delta^n$. $\square$

It follows from Proposition 3 and Claim 2 that we can regard the construction $X \mapsto K_\Delta(X)$ as a functor from the ∞-category $\mathcal{S}^{\mathrm{fin}}$ of finite spaces to the ∞-category of E_∞ spaces. Moreover, since the map

$$\Omega^{-\infty} K_\Delta(X) \to A(X)$$

constructed in the previous lecture was functorial for maps of finite polyhedra, it can be regarded as a natural transformation between functors from $\mathcal{S}^{\mathrm{fin}}$ to spectra. We next make a simple observation:

Proposition 5. *The map $\Omega^{-\infty} K_\Delta(X) \to A(X)$ is an equivalence when X is a point.*

Proof. When X is a point, a constructible sheaf $\mathcal{F}$ on $X \times \Delta^n$ is ULA over Δ^n if and only if it is locally constant. Consequently, we can identify $\mathrm{Shv}_{PL}^{\Delta^\bullet}(X \times \Delta^\bullet)$ with the constant simplicial ∞-category taking the value $\mathrm{Sp}^{\mathrm{fin}}$. It follows that $K_\Delta(X)$ can be identified with $K(\mathrm{Sp}^{\mathrm{fin}}) \simeq \Omega^\infty A(X)$. □

It follows from Proposition 5 that the colimit-preserving approximations to the functors $\Omega^{-\infty} K_\Delta(X)$ and $A(X)$ are the same. In other words, for any finite space X we have a commutative diagram

$$\begin{tikzcd} & X_+ \wedge A(*) \arrow[dl, "\theta_X"'] \arrow[dr] & \\ \Omega^{-\infty} K_\Delta(X) \arrow[rr] & & A(X). \end{tikzcd}$$

We can now formulate a more precise version of Theorem 1: the map θ_X is a homotopy equivalence of spectra. To prove this, we note that the collection of those spaces X for which θ_X is a homotopy equivalence contains the one-point space (by Proposition 5) and the empty space (since the domain and codomain of θ_X both vanish in this case). Consequently, to show that it contains all finite spaces, it will suffice to show that it is closed under (homotopy) pushouts. Since the functor $X \mapsto X_+ \wedge A(*)$ preserves pushout squares, we are reduced to the following:

Proposition 6. *Suppose we are given a pushout diagram*

$$\begin{tikzcd} X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\ X_1 \arrow[r] & X \end{tikzcd}$$

in the ∞-category of finite spaces. Then the diagram of spectra

$$\begin{tikzcd} \Omega^{-\infty} K_\Delta(X_{01}) \arrow[r] \arrow[d] & \Omega^{-\infty} K_\Delta(X_0) \arrow[d] \\ \Omega^{-\infty} K_\Delta(X_1) \arrow[r] & \Omega^{-\infty} K(X) \end{tikzcd}$$

is also a homotopy pushout square.

In the statement of Proposition 6, we may assume without loss of generality that X_{01}, X_0, and X_1 are represented by finite polyhedra, and that the maps $X_{01} \to X_0$ and $X_{01} \to X_1$ are given by embeddings of finite polyhedra. We will assume that X is given by the homotopy pushout

$$X_0 \amalg_{X_{01} \times \{0\}} (X_{01} \times [0,1]) \amalg_{X_{01} \times \{1\}} X_1.$$

For each real number t with $0 < t < 1$, let X_t denote the subcomplex of X given by $X_{01} \times \{t\}$. We will say that a sheaf $\mathcal{F} \in \mathrm{Shv}_{PL}^S(X \times S)$ is *transverse* to X_t if the restriction $\mathcal{F}|_{X_t \times S}$ is ULA over S. Let $\mathrm{Shv}_{PL}^{S,t}(X \times S)$ denote the full subcategory of $\mathrm{Shv}_{PL}^S(X \times S)$ spanned by those sheaves which are transverse to t. As S ranges over all simplices, we obtain a simplicial ∞-category $\mathrm{Shv}_{PL}^{\Delta^\bullet}(X \times \Delta^\bullet)$. Let $K_\Delta^t(X)$ denote the geometric realization of the simplicial space

$$K(\mathrm{Shv}_{PL}^{\Delta^\bullet,t}(X \times \Delta^\bullet)).$$

Let us compute $K_\Delta^t(X)$.

Let S be a finite polyhedron, and let $\mathcal{C}_-^S$ and $\mathcal{C}_+^S$ denote the full subcategories of $\mathrm{Shv}_{PL}(X_{\leq t} \times S)$ and $\mathrm{Shv}_{PL}(X_{\geq t} \times S)$ spanned by those sheaves which are ULA over S and which vanish when restricted to X_t. Note that $X_{\leq t}$ and $X_{\geq t}$ contain X_0 and X_1 as deformation retracts. Using a slight variant of the proof of Lemma 4, one can show that the canonical maps

$$K_\Delta(X_0) \to |\, \mathcal{C}_-^{\Delta^\bullet} \,|$$

$$K_\Delta(X_1) \to |\, \mathcal{C}_+^{\Delta^\bullet} \,|$$

are homotopy equivalences.

Note that if $\mathcal{F} \in \mathrm{Shv}_{PL}^{S,T}(X \times S)$, then we have a canonical fiber sequence

$$\mathcal{F}' \to \mathcal{F} \to i_* i^* \mathcal{F}.$$

Since $\mathcal{F}$ and $i^* \mathcal{F}$ are ULA over S, it follows that $i_* i^* \mathcal{F}$ and $\mathcal{F}'$ are ULA over S. Because $\mathcal{F}' |_{X_t}$ vanishes, we can write $\mathcal{F}'$ as a direct sum $\mathcal{F}'_- \oplus \mathcal{F}'_+$, where $\mathcal{F}'_- \in \mathcal{C}_-^S$ and $\mathcal{F}'_+ \in \mathcal{C}_+^S$. The construction $\mathcal{F} \mapsto (\mathcal{F}'_-, \mathcal{F}'_+, i^* \mathcal{F})$ determines an exact functor

$$\mathrm{Shv}_{PL}^{S,T}(X \times S) \to \mathcal{C}_-^S \times \mathcal{C}_+^S \times \mathrm{Shv}_{PL}^S(X_t \times S).$$

This map has a right homotopy inverse (given by pushing forward to $X \times S$ and forming the direct sum). It follows from the additivity theorem that this right homotopy inverse is actually a two-sided homotopy inverse after passing to K-theory. In particular, we obtain a homotopy equivalence

$$K(\mathrm{Shv}_{PL}^{S,t}(X \times S)) \simeq K(\mathcal{C}_-^S) \times K(\mathcal{C}_+^S) \times K(\mathrm{Shv}_{PL}^S(X_t \times S)).$$

Taking S to range over simplices and passing to geometric realizations, we obtain an equivalence

$$K_\Delta^t(X) \simeq K_\Delta(X_0) \times K_\Delta(X_1) \times K_\Delta(X_t).$$

We will elaborate more on this in the next lecture.

The Assembly Map, Part III (Lecture 26)

November 4, 2014

Our goal in this lecture is to complete the proof of the following result:

Proposition 1. *Suppose we are given a pushout diagram*

$$\begin{tikzcd} X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\ X_1 \arrow[r] & X \end{tikzcd}$$

in the ∞-category of finite spaces. Then the diagram

$$\begin{tikzcd} K_\Delta(X_{01}) \arrow[r] \arrow[d] & K_\Delta(X_0) \arrow[d] \\ K_\Delta(X_1) \arrow[r] & K_\Delta(X) \end{tikzcd}$$

is also a pushout square of E_∞-spaces.

To prove Proposition 1, it will be convenient to consider instead the diagram of connected deloopings

$$\begin{tikzcd} \Omega^{-1}K_\Delta(X_{01}) \arrow[r] \arrow[d] & \Omega^{-1}K_\Delta(X_0) \arrow[d] \\ \Omega^{-1}K_\Delta(X_1) \arrow[r] & \Omega^{-1}K_\Delta(X). \end{tikzcd}$$

We wish to compare $\Omega^{-1}K_\Delta(X)$ with the homotopy pushout of the rest of the diagram, which is computed as the geometric realization of a simplicial space

$$\mathrm{Bar}_\bullet(\Omega^{-1}K_\Delta(X_0), \Omega^{-1}K_\Delta(X_{01}), \Omega^{-1}K_\Delta(X_1))$$

given by

$$\mathrm{Bar}_n(\Omega^{-1}K_\Delta(X_0), \Omega^{-1}K_\Delta(X_{01}), \Omega^{-1}K_\Delta(X_1)) = (\Omega^{-1}K_\Delta(X_0)) \times (\Omega^{-1}K_\Delta(X_{01})^n) \times (\Omega^{-1}K_\Delta(X_1)).$$

As in the previous lecture, let us assume that X_0 and X_1 are finite polyhedra, that X_{01} is a subpolyhedron of each, and that X is given by the two-sided mapping cylinder

$$X_0 \amalg_{\{0\}\times X_{01}} ([0,1] \times X_{01}) \amalg_{\{1\}\times X_{01}} X_1.$$

For each $t \in (0,1)$, let X_t denote the subpolyhedron $\{t\} \times X_{01} \subseteq X$. Recall that a sheaf $\mathcal{F} \in \mathrm{Shv}^S_{PL}(X \times S)$ is *transverse to t* if $\mathcal{F}|_{X_t \times S}$ is ULA over S. More generally, if T is a finite subset of the open interval $(0,1)$, we will say that $\mathcal{F} \in \mathrm{Shv}^{S}_{PL}(X \times S)$ is *transverse* to T if it is transverse to t for each $t \in T$. Let $\mathrm{Shv}^{S,T}_{PL}(X \times S)$ denote the full subcategory of $\mathrm{Shv}^{S}_{PL}(X \times S)$ spanned by those sheaves which are transverse to T, and let $K^{T}_{\Delta}(X)$ denote the geometric realization of the simplicial space given by

$$K(\mathrm{Shv}^{\Delta^\bullet,T}_{PL}(X \times \Delta^\bullet).$$

We proved in the last lecture that if T has a single element, then $K^{T}_{\Delta}(X)$ can be identified with the product

$$K_\Delta(X_0) \times K_\Delta(X_{01}) \times K_\Delta(X_1).$$

One can carry out a similar argument for larger finite sets $T = \{t_1, \ldots, t_n\}$: every object of $\mathrm{Shv}^{S,T}_{PL}(X \times S)$ has a canonical filtration whose subquotients are sheaves which are supported either on $X_{t_i} \times S$ for $1 \leq i \leq n$, on $X_{<t_1} \times S$ (which contains $X_0 \times S$ as a deformation retract), on $X_{>t_n} \times S$ (which contains $X_1 \times S$ as a deformation retract), or on $X_{01} \times (t_i, t_{i+1}) \times S$ for $1 \leq i < n$. This analysis supplies a homotopy equivalence

$$K^{T}_{\Delta}(X) \simeq K_\Delta(X_0) \times K_\Delta(X_{01})^{2n-1} \times K_\Delta(X_1).$$

We can be more precise: the construction $T \mapsto K^{T}_{\Delta}(X)$ is contravariantly functorial in T, and there is a functorial identification

$$\Omega^{-1} K^{T}_{\Delta}(X) \simeq \mathrm{Bar}_{\alpha(T)}(\Omega^{-1} K_\Delta(X_0), \Omega^{-1} K_\Delta(X_{01}), \Omega^{-1} K_\Delta(X_1))$$

where $\alpha(T)$ is the finite linearly ordered set given by $T \times \{0, 1\}$, equipped with the lexicographical ordering.

Lemma 2. *Let P be the poset of nonempty finite subsets of $(0,1)$ and let Δ be the category of finite nonempty linearly ordered sets. Then the construction*

$$T \mapsto T \times \{0, 1\}$$

determines a right cofinal functor

$$\alpha : P \to \Delta.$$

Proof. Fix an object $[n] = \{0 < 1 < \cdots < n\} \in \Delta$; we wish to show that the category (in fact, poset) $Q = P \times_\Delta \Delta_{/[n]}$ is weakly contractible. For each $\epsilon > 0$, let $P_\epsilon \subseteq P$ denote the collection of all nonempty finite subsets of $(\epsilon, 1)$, and let Q_ϵ denote the inverse image in Q of P_ϵ. Then Q can be written as a directed union of the posets Q_ϵ. Consequently, to show that Q is weakly contractible, it will suffice to show that each of the inclusion maps $Q_\epsilon \hookrightarrow Q$ is nullhomotopic. We can identify the elements of Q with pairs (T, f) where $T \subseteq (0,1)$ is a nonempty finite set and $f : T \times \{0,1\} \to [n]$ is a monotone map. If $(T, f) \in Q_\epsilon$, then we have natural maps

$$(T, f) \hookrightarrow (T \cup \{\epsilon\}, f_+) \hookleftarrow (\{\epsilon\}, g),$$

where $g : \{\epsilon\} \times \{0, 1\} \to [n]$ is the constant map taking the value 0 and f_+ is the amalgamation of the maps f and g. These maps determine a homotopy from the inclusion $Q_\epsilon \hookrightarrow Q$ to the constant map $Q_\epsilon \to \{(\{\epsilon\}, g)\} \subseteq Q$. $\square$

It follows from Lemma 2 that the pushout

$$\Omega^{-1} K_\Delta(X_0) \amalg_{\Omega^{-1} K_\Delta(X_{01})} \Omega^{-1} K_\Delta(X_1)$$

(formed in the ∞-category of E_∞-spaces) can also be written as

$$\varinjlim_{T \subseteq (0,1)} \Omega^{-1} K^{T}_{\Delta}(X)$$

(formed in the ∞-category of spaces). We may therefore rewrite Proposition 1 as follows:

Proposition 3. *The canonical map*

$$\varinjlim_{T \subseteq (0,1)} \Omega^{-1} K^T_\Delta(X) \to \Omega^{-1} K_\Delta(X)$$

is a homotopy equivalence of spaces.

Note that the map of Proposition 3 can be obtained as the geometric realization of a map of bisimplicial spaces which is given (in bidegree (m,n)) by

$$\theta_m : \varinjlim_{T \subseteq (0,1)} S_n \operatorname{Shv}_{PL}^{\Delta^m, T}(X \times \Delta^m) \to S_n \operatorname{Shv}_{PL}^{\Delta^m}(X \times \Delta^m).$$

Let us regard n as fixed for the remainder of this lecture. We will prove Proposition 3 by showing that the map of geometric realizations

$$| \varinjlim_{T \subseteq (0,1)} S_n \operatorname{Shv}_{PL}^{\Delta^\bullet, T}(X \times \Delta^\bullet)| \to |S_n \operatorname{Shv}_{PL}^{\Delta^\bullet}(X \times \Delta^\bullet)|.$$

is a homotopy equivalence. Note that for fixed m and $T \subseteq (0,1)$, the map

$$S_n \operatorname{Shv}_{PL}^{\Delta^m, T}(X \times \Delta^m) \to S_n \operatorname{Shv}_{PL}^{\Delta^m}(X \times \Delta^m)$$

is the inclusion of a summand: the target space classifies diagrams

$$\mathcal{F}_1 \to \cdots \to \mathcal{F}_n$$

in $\operatorname{Shv}_{PL}^{\Delta^m}(X \times \Delta^m)$, and the domain classifies diagrams which have the additional property that each $\mathcal{F}_i$ is transverse to each $t \in T$. If the diagram $\mathcal{F}_1 \to \cdots \to \mathcal{F}_n$ is fixed, then the collection of those $t \in (0,1)$ such that each $\mathcal{F}_i$ is transverse to T comprise a subset $U \subseteq (0,1)$, and the homotopy fiber of the map θ_m over the point corresponding to $\mathcal{F}_1 \to \cdots \to \mathcal{F}_n$ can be identified with the nerve of the category of nonempty finite subsets of U. This category is either empty (if $U = \emptyset$) or contractible (if $U \neq \emptyset$). It follows that each of the maps θ_m exhibits $\varinjlim_{T \subseteq (0,1)} S_n \operatorname{Shv}_{PL}^{\Delta^m, T}(X \times \Delta^m)$ as a summand of $S_n \operatorname{Shv}_{PL}^{\Delta^m}(X \times \Delta^m)$: namely, the summand consisting of those diagrams $\mathcal{F}_1 \to \cdots \to \mathcal{F}_n$ which are all transverse to *some* $t \in (0,1)$.

Construction 4. Let $Z_\bullet$ be a simplicial space. For every simplicial set K, we let $Z_\bullet(K)$ denote the homotopy inverse limit $\varprojlim_{\Delta^p \to K} Z_p$.

We define a new simplicial space $\operatorname{Ex}(Z_\bullet)$ by the formula

$$\operatorname{Ex}(Z_\bullet)_q = Z_\bullet(\operatorname{Sd} \Delta^q).$$

Note that the "last vertex maps" $\operatorname{Sd} \Delta^q \to \Delta^q$ are compatible as q varies and give rise to a map of simplicial spaces $\rho_{Z_\bullet} : Z_\bullet \to \operatorname{Ex}(Z_\bullet)$.

Lemma 5. *For any simplicial space $Z_\bullet$, the map $\rho_{Z_\bullet} : Z_\bullet \to \operatorname{Ex}(Z_\bullet)$ induces a homotopy equivalence of geometric realizations*

$$|Z_\bullet| \to |\operatorname{Ex}(Z_\bullet)|.$$

Sketch. If $Z_\bullet$ is a simplicial set (meaning that each Z_n is discrete), then this is classical. The general case can be reduced to this: the ∞-category of simplicial spaces is the underlying ∞-category of the model category of *bisimplicial sets*, equipped with the Reedy model structure. Moreover, if $Z_{\bullet\bullet}$ is a bisimplicial set which is Reedy fibrant, then the Ex of Construction 4 can be computed by levelwise application of the usual Ex functor on simplicial sets. □

Let us now specialize to the case where $Z_\bullet$ is the simplicial space $S_n \operatorname{Shv}^{\Delta^\bullet}_{PL}(X \times \Delta^\bullet)$. In this case, we can identify $\operatorname{Ex}(Z_\bullet)$ with the simplicial space whose m-simplices are given by $S_n \operatorname{Shv}^{|\operatorname{Sd}\Delta^m|}_{PL}(X \times |\operatorname{Sd}\Delta^m|)$. Recall that there is a canonical piecewise linear homeomorphism of $|\operatorname{Sd}\Delta^m|$ with $|\Delta^m|$, which is functorial for *injective* maps between simplices. These homeomorphisms determine an equivalence $\theta_{Z_\bullet}$ between the underlying semisimplicial spaces of $Z_\bullet$ and $\operatorname{Ex}(Z_\bullet)$, which we will denote by $Z^s_\bullet$ and $\operatorname{Ex}(Z_\bullet)^s$.

Let $Y_\bullet$ denote the simplicial subspace of $Z_\bullet$ given by $\varinjlim_{T \subseteq (0,1)} S_n \operatorname{Shv}^{\Delta^\bullet, T}_{PL}(X \times \Delta^\bullet)$. The map $\theta_{Z_\bullet}$ restricts to a morphism of underlying semisimplicial spaces

$$\theta_{Y_\bullet} : Y^s_\bullet \to \operatorname{Ex}(Y_\bullet)^s$$

(which is now *not* a levelwise homotopy equivalence). This is not identical to the map of semisimplicial spaces $\rho_{Y_\bullet}$ appearing in Lemma 5. However, they differ by a simplicial homotopy and therefore induce the same map after passing to geometric realizations. It follows from Lemma 5 that $\theta_{Y_\bullet}$ induces a homotopy equivalence

$$|Y^s_\bullet| \to |\operatorname{Ex}(Y_\bullet)^s|.$$

We wish to show that the inclusion $i_\bullet : Y_\bullet \hookrightarrow Z_\bullet$ induces a homotopy equivalence of geometric realizations. For each integer $p \geq 0$, let $\operatorname{Ex}^p(Y_\bullet)$ denote the result of p-fold application of the functor Ex to the simplicial space $Y_\bullet$, and define $\operatorname{Ex}^p(Z_\bullet)$ similarly. Using the above arguments we can identify $|i_\bullet|$ with the induced of geometric realizations

$$|\varinjlim \operatorname{Ex}^p(Y_\bullet)^s| \to |\varinjlim \operatorname{Ex}^p(Z_\bullet)^s|.$$

It will therefore suffice to prove the following:

Proposition 6. *For each integer m, the canonical map $\varinjlim \operatorname{Ex}^p(Y_\bullet)_m \to \varinjlim \operatorname{Ex}^p(Z_\bullet)_m$ is a homotopy equivalence.*

Unwinding the definitions, Proposition 6 asserts that for every diagram $\mathcal{F}_1 \to \cdots \to \mathcal{F}_n$ in $\operatorname{Shv}^{\Delta^m}_{PL}(X \times \Delta^m)$, there exists an integer $p \geq 0$ such that on each simplex σ of the p-fold barycentric subdivision of Δ^m, there exists a real number t_σ which is transverse to each $\mathcal{F}_i|_{X \times \sigma}$.

Choose compatible triangulations Σ of $X_{01} \times [0,1] \times \Delta^m$, Σ' of $[0,1] \times \Delta^m$, and Σ'' of Δ^m, such that each $\mathcal{G}_i$ is constructible with respect to Σ. Let $V \subseteq [0,1] \times \Delta^m$ be the union of the interiors of those simplices $\sigma \in \Sigma'$ for which the map $\sigma \to \Delta^m$ is not injective. It is easy to see that $V \subseteq (0,1) \times \Delta^m$ and that the projection map $V \to \Delta^m$ is surjective. Since V is open, there exists an open cover U_α of Δ^m and a collection of real numbers $t_\alpha \in (0,1)$ such that each product $\{t_\alpha\} \times U_\alpha$ is contained in V. Choose $p \gg 0$ so that each simplex σ of the p-fold barycentric subdivision of Δ^m is contained in some U_α. We claim that p satisfies our requirements: more precisely, each restriction $\mathcal{F}_i|_{X \times \sigma}$ is transverse to t_α. To prove this, it will suffice to show that $\mathcal{F}_i|_{X_{01} \times [0,1] \times \Delta^m}$ is ULA over $[0,1] \times \Delta^m$ over the open set V.

Let

$$f : X_{01} \times [0,1] \times \Delta^m \to [0,1] \times \Delta^m$$

and

$$g : [0,1] \times \Delta^m \to \Delta^m$$

be the projection maps, let $\theta_0 \in \Sigma$, let $\sigma_0 = f(\theta_0) \in \Sigma'$, and let $\sigma \in \Sigma'$ be a simplex containing σ_0. Set $\tau_0 = g(\sigma_0) \in \Sigma''$ and $\tau = g(\sigma) \in \Sigma''$. We wish to show that if the interior of σ_0 is contained in V, then the canonical map

$$\mathcal{F}_i(\theta_0) \to \varprojlim_{\theta_0 \subseteq \theta, f(\theta) = \sigma} \mathcal{F}_i(\theta)$$

is an equivalence. Our assumption that the interior of σ_0 is contained in V guarantees that σ is the *unique* simplex of Σ' which contains σ_0 and whose image is τ, so we can rewrite our map as

$$\mathcal{F}_i(\theta_0) \to \varprojlim_{\theta_0 \subseteq \theta, (g \circ f)(\theta) = \tau} \mathcal{F}_i(\theta).$$

This map is an equivalence by virtue of our assumption that $\mathcal{F}_i$ is ULA over Δ^m.

Higher Torsion (Lecture 27)

November 5, 2014

Let Poly denote the ordinary category of finite polyhedra, and let $\mathcal{S}$ denote the ∞-category of spaces. Over the last few lectures, we have studied the functor

$$K_{\Delta} : \mathrm{Poly} \to \mathcal{S}\,.$$

given by

$$K_{\Delta}(X) = |K(\mathrm{Shv}_{PL}^{\Delta^{\bullet}}(X \times \Delta^{\bullet}))|.$$

Since every finite polyhedron has an underlying topological space, there is a forgetful functor $\iota : \mathrm{Poly} \to \mathcal{S}$. Let us (temporarily) use the notation $\iota_! K_{\Delta}$ to denote the left Kan extension of K_{Δ} along ι. This left Kan extension can be computed in two steps:

- First, we can form the left Kan extension of ι along the forgetful functor $\mathrm{Poly} \to \mathcal{S}^{\mathrm{fin}}$, where $\mathcal{S}^{\mathrm{fin}}$ is the ∞-category of finite spaces. Since K_{Δ} is homotopy invariant, this is equivalent to lifting K_{Δ} along the fully faithful embedding
$$\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{S}) \to \mathrm{Fun}(\mathrm{Poly}, \mathcal{S}).$$
- We then form the left Kan extension along the fully faithful embedding $\mathcal{S}^{\mathrm{fin}} \to \mathcal{S}$. This is the process of formally extending a functor $\mathcal{S}^{\mathrm{fin}} \to \mathcal{S}$ to a functor $\mathcal{S} \to \mathcal{S}$ so that it commutes with filtered colimits.

It follows from this analysis that the restriction of $i_! K_{\Delta}$ to Poly agrees with the original functor K_{Δ}. We will henceforth abuse notation by denoting the functor $i_! K_{\Delta}$ also by K_{Δ}, so that we view K_{Δ} as a functor from spaces to spaces. The main theorem of the previous lectures gives us an explicit description of this functor: it is the domain of the assembly map in Waldhausen A-theory. That is, we have

$$K_{\Delta}(X) \simeq \Omega^{\infty}(X_{+} \wedge A(*)).$$

We can use this identification to produce some $A(*)$-homology classes. Let X be a space, and suppose we are given a finite polyhedron Y, a map $f : Y \to X$, and a constructible sheaf $\mathcal{F}$ on Y (with values in the ∞-category of finite spectra). Then $\mathcal{F}$ is an object of $\mathrm{Shv}_{PL}(Y)$ and therefore determines a point of $K(\mathrm{Shv}_{PL}(Y))$, and therefore also of $K_{\Delta}(Y)$. Using the map f, we obtain a point of $K_{\Delta}(X)$ which we will denote by $\langle Y, \mathcal{F} \rangle$. In the special case where $\mathcal{F}$ is the constant sheaf on Y (with value the sphere spectrum), we will denote this point simply by $\langle Y \rangle$.

We have an assembly map $K_{\Delta}(X) \to \Omega^{\infty} A(X)$. Unwinding the definitions, we see that this assembly map carries $\langle Y, \mathcal{F} \rangle$ to $[\mathcal{F}']$, where $\mathcal{F}'$ is the local system of spectra on X which corepresents the functor

$$\mathrm{Sp}^{X} \to \mathrm{Sp}$$

$$\mathcal{G} \mapsto \Gamma(Y, \mathcal{F} \wedge f^{*}\,\mathcal{G})$$

(here Γ denotes the global sections functor). In the special case where $\mathcal{F}$ is the constant sheaf, this functor is given by

$$\Gamma(Y, f^{*}\,\mathcal{G}) = \mathrm{Map}_{\mathrm{Sp}^{Y}}(\underline{S}_{Y}, f^{*}\,\mathcal{G}) = \mathrm{Map}_{\mathrm{Sp}^{X}}(f_{!}\underline{S}_{Y}, f^{*}\,\mathcal{G}).$$

It follows that $\mathcal{F}' \simeq f_{!}\underline{S}_{Y}$ (where $f_{!}$ denotes the left adjoint to pullback on local systems), so that $[\mathcal{F}']$ can be identified with the point $[Y] \in \Omega^{\infty} A(X)$ studied in Lecture 21. This analysis proves the following:

Proposition 1. *Let X be any space. For any finite polyhedron Y and any map $f : Y \to X$, the assembly map $K_\Delta(X) \to \Omega^\infty A(X)$ carries $\langle Y \rangle \in K_\Delta(X)$ to $[Y] \in \Omega^\infty A(X)$.*

All of the preceding considerations can be generalized to "allow parameters". Let us be more precise. Fix a topological space X. We define Kan complexes $\mathcal{M}_X$ and $\mathcal{M}_X^h$ as follows:

- The n-simplices of $\mathcal{M}_X$ are finite polyhedra $Y \subseteq \Delta^n \times \mathbb{R}^\infty$ equipped with a map $f : Y \to X$, for which the projection $Y \to \Delta^n$ is a PL fibration.
- The n-simplices of $\mathcal{M}_X^h$ are subspaces $Y \subseteq \Delta^n \times \mathbb{R}^\infty$ equipped with a map $f : Y \to X$ for which the projection $Y \to \Delta^n$ is a fibration with finitely dominated fibers.

The construction $(Y \to X) \mapsto [Y]$ can be naturally refined to a map of Kan complexes $\mathcal{M}_X^h \to \Omega^\infty A(X)$, and the construction $(Y \to X) \mapsto \langle Y \rangle$ can be naturally refined to a map of Kan complexes $\mathcal{M}_X \to K_\Delta(X)$. Repeating the analysis that preceded Proposition 1, we obtain the following refinement:

Proposition 2. *Let X be any space. Then the diagram*

$$\begin{tikzcd}
\mathcal{M}_X \arrow[r] \arrow[d] & K_\Delta(X) \arrow[d] \\
\mathcal{M}_X^h \arrow[r] & \Omega^\infty A(X)
\end{tikzcd}$$

commutes (up to canonical homotopy).

Let us now suppose that the space X itself is finitely dominated. In this case, the Kan complex $\mathcal{M}_X^h$ contains a contractible path component whose vertices are *homotopy equivalences* $Y \to X$. Let us denote this path component by $\mathcal{M}_X^{h\circ}$. We have a diagram of homotopy pullback squares

$$\begin{tikzcd}
\mathcal{M}_X \times_{\mathcal{M}_X^h} \mathcal{M}_X^{h\circ} \arrow[r] \arrow[d] & \mathcal{M}_X \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\
\mathcal{M}_X^{h\circ} \arrow[r] & \mathcal{M}_X^h \arrow[r] & \mathcal{M}^h .
\end{tikzcd}$$

In other words, the homotopy fiber of the map $\mathcal{M} \to \mathcal{M}^h$ over X can be identified with $\mathcal{M}_X \times_{\mathcal{M}_X^h} \mathcal{M}_X^{h\circ}$. Applying Proposition 2, we obtain a map

$$\begin{aligned}
\mathcal{M} \times_{\mathcal{M}^h} \{X\} &\simeq \mathcal{M}_X \times_{\mathcal{M}_X^h} \mathcal{M}_X^{h\circ} \\
&\to K_\Delta(X) \times_{\Omega^\infty A(X)} \mathcal{M}_X^{h\circ} \\
&\to K_\Delta(X) \times_{\Omega^\infty A(X)} \{[X]\}.
\end{aligned}$$

We can now give a more precise formulation of the main result of the second part of this course:

Theorem 3. *Let X be a finitely dominated space. Then the map*

$$\mathcal{M} \times_{\mathcal{M}^h} \{X\} \to K_\Delta(X) \times_{\Omega^\infty A(X)} \{[X]\}$$

is a homotopy equivalence.

Example 4. Theorem 3 implies that the homotopy fiber $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$ is either empty (in case X has non-vanishing Wall obstruction) or a torsor for the infinite loop space

$$\mathrm{fib}(K_\Delta(X) \to \Omega^\infty A(X)) \simeq \Omega^{\infty+1} \mathrm{Wh}(X),$$

where $\mathrm{Wh}(X)$ denotes the (piecewise-linear) Whitehead spectrum of X.

If X itself is given as a finite polyhedron, then the space $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$ has a canonical base point. In this case, we obtain a *canonical* homotopy equivalence

$$\tau : \mathcal{M} \times_{\mathcal{M}^h} \{X\} \simeq \Omega^{\infty+1} \mathrm{Wh}(X).$$

Note that the points of $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$ can be identified with pairs (Y, f), where Y is a finite polyhedron and $f : Y \to X$ is a homotopy equivalence. If X itself is a finite polyhedron, then the "identity component" of $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$ consists of those pairs (Y, f) where f is a *simple* homotopy equivalence. It follows from Theorem 3 that f is a simple homotopy equivalence if and only if a certain element $\tau(Y, f) \in \pi_1 \mathrm{Wh}(X)$ vanishes. If X is connected with fundamental group G, we have seen that there is a canonical isomorphism of $\pi_1 \mathrm{Wh}(X)$ with the Whitehead group $\mathrm{Wh}(G)$ of G, so we can identify $\tau(Y, f)$ with an element of $\mathrm{Wh}(G)$.

Proposition 5. *In the situation above, the element $\tau(Y, f) \in \mathrm{Wh}(G)$ coincides with the Whitehead torsion of the homotopy equivalence f (as defined in Lectures 3 and 4).*

Combining Proposition 5 with Theorem 3, we obtain another proof of the main result from Lecture 4: the homotopy equivalence $f : Y \to X$ is simple if and only if its Whitehead torsion vanishes. In other words, Proposition 5 allows us to regard Theorem 3 as a *generalization* of the main result of Lecture 4 (and, as we have already noted, Theorem 3 also generalizes the theory of the Wall obstruction).

Let us informally sketch a proof of Proposition 5. Without loss of generality, we may assume that Y and X have been equipped with triangulations that are compatible with the map f. Assume that X is connected with fundamental group G. We have a pair of points

$$\langle X \rangle, \langle Y \rangle \in K_{\Delta}(X),$$

having images $[X], [Y] \in \Omega^{\infty} A(X)$. Our assumption that f is a homotopy equivalence supplies an equivalence of local systems $f_! \underline{S}_Y \simeq \underline{S}_X$, which gives a path p joining $[X]$ and $[Y]$ in $\Omega^{\infty} A(X)$. This path gives a lift of $\langle X \rangle - \langle Y \rangle$ to the homotopy fiber

$$\Omega^{\infty+1} \mathrm{Wh}(X) \simeq \mathrm{fib}(K_{\Delta}(X) \to \Omega^{\infty} A(X),$$

and $\tau(Y, f)$ is the path component of this lift. Note that the map $\pi_0 K_{\Delta}(X) \to \pi_0 A(X)$ is injective, so that $\langle X \rangle$ and $\langle Y \rangle$ belong to the same path component of $\Omega^{\infty} A(X)$. If we choose a path q from $\langle X \rangle$ to $\langle Y \rangle$, then we can combine p with the image of q to form a closed loop in the space $\Omega^{\infty} A(X)$. This loop determines an element $\eta \in \pi_1 A(X) \simeq K_1(\mathbf{Z}[G])$, which is preimage of $\tau(Y, f)$ under the connecting homomorphism

$$\pi_1 A(X) \to \pi_0(\mathrm{fib}\, K_{\Delta}(X) \to \Omega^{\infty} A(X)).$$

Note that the element η depends on the choice of path q.

Let $\Sigma(X)$ and $\Sigma(Y)$ denote the set of simplices of X and Y, respectively. Let $\underline{S}_X$ and $\underline{S}_Y$ denote the constant sheaves (with value the sphere spectrum) on X and Y, respectively. For each simplex σ of X (or Y), let $\underline{S}_{\sigma}$ denote the constructible sheaf on X (or Y) taking the value S on σ and 0 elsewhere (in other words, the sheaf which is "extended by zero" from the interior of σ) and let $\underline{S}^n_{\sigma}$ denote the nth suspension of $\underline{S}_{\sigma}$. Note that

$$f_* \underline{S}_{\sigma} \simeq \underline{S}^{\dim f(\sigma) - \dim(\sigma)}_{f(\sigma)}.$$

For each $\sigma \in \Sigma(X) \cup \Sigma(Y)$, consider the point $e_{\sigma} \in K_{\Delta}(X)$ given by

$$e_{\sigma} = \begin{cases} [\underline{S}_{\sigma}] & \text{if } \Sigma \in \Sigma(X) \\ -[\underline{S}^{\dim f(\sigma) - \dim(\sigma)}_{f(\sigma)}] & \text{if } \Sigma \in \Sigma(Y) \end{cases}$$

Using the additivity theorem, we can choose a path from the difference $\langle X \rangle - \langle Y \rangle$ to the sum $\sum_{\sigma \in \Sigma(X) \cup \Sigma(Y)} e_{\sigma}$. Let E denote the union of the set of even-dimensional simplices of X and odd-dimensional simplices of Y, and let E' denote the union of the set of odd-dimensional simplices of X and even-dimensional simplices of Y. Note that $\pi_0 K_\Delta(X) \simeq \mathbf{Z}$, and e_σ belongs to the path component 1 if $\sigma \in E$ and the path component -1 if $\sigma \in E'$. Since f is a homotopy equivalence, X and Y have the same Euler characteristic and therefore E and E' have the same size. We may therefore choose a bijection $\beta : E \simeq E'$. For each $\sigma \in E$, we can choose a path q_σ in $K_\Delta(X)$ from $e_\sigma + e_{\beta(\sigma)}$ to the base point; note that these paths are ambiguous up to an element of $\pi_1 K_\Delta(X) \simeq G \oplus \mathbf{Z}/2\mathbf{Z}$. The sum of these paths determines a path q from $\langle X \rangle - \langle Y \rangle$ to the base point.

Unwinding the definitions, we see that the image of $[X] - [Y]$ in $K(\mathbf{Z}[G])$ can be represented by the relative cellular chain complex $C_*(X, Y; \mathbf{Z}[G])$. The given triangulations of X and Y determine a basis for $C_*(X, Y; \mathbf{Z}[G])$ as a $\mathbf{Z}[G]$-module, where the basis elements are ambiguous up to $\pm G$. We have two paths from $[C_*(X, Y; \mathbf{Z}[G])]$ to the base point of $K(\mathbf{Z}[G])$, given as follows:

- (a) The image of p determines a path from $[C_*(X, Y; \mathbf{Z}[G])]$ to the base point of $K(\mathbf{Z}[G])$ which arises from the observation that $C_*(X, Y; \mathbf{Z}[G])$ is an acyclic complex (because f is a homotopy equivalence), and therefore represents a zero object of the ∞-category $\mathrm{Rep}_{\mathbf{Z}[G]}$.

- (b) The image of the path q determines a path from $[C_*(X, Y; \mathbf{Z}[G])]$ to the base point of $K(\mathbf{Z}[G])$. After possibly modifying our choice of basis, we can arrange that this path is obtained by first invoking the additivity theorem to construct a path from $[C_*(X, Y; \mathbf{Z}[G])]$ to the point represented by the sum

$$\bigoplus_{\sigma \in \Sigma(X)} [\Sigma^{\dim(\sigma)} \mathbf{Z}[G]] + \bigoplus_{\sigma \in \Sigma(Y)} [\Sigma^{\dim(\sigma)+1} \mathbf{Z}[G]],$$

  and then connecting this latter sum to the base point by matching factors using the bijection β.

We are therefore reduced to the following statement, which we leave as a (tedious) exercise:

Exercise 6. Let R be a ring and let F_* be a bounded acyclic chain complex of free R-modules, where $\chi(F_*) = 0$ (the latter condition is automatic if R has the form $\mathbf{Z}[G]$). Suppose we have chosen a basis $\{e_i, e'_i\}$ for F_*, where each e_i is homogeneous of even degree d_i, and each e'_i is homogeneous of odd degree d'_i. Then the torsion $\tau(F_*) \in K_1(R) \simeq \pi_1 K(R)$ (as defined in Lecture 3) can be represented as the "difference" between two paths from $[F_*]$ to the base point of the space $K(R)$:

- (a) The path obtained from the observation that the chain complex F_* represents the zero object of Mod_R (since F_* is acyclic).

- (b) The path obtained by first using the additivity theorem to construct a path from $[F_*]$ to the sum $\sum_i [\Sigma^{d_i} R] \oplus [\Sigma^{d'_i} R]$, then connecting each $[\Sigma^{d_i} R] + [\Sigma^{d'_i} R]$ to the base point using the fact that d_i and d'_i have different parities.

References

[1] Waldhausen, F. *The Algebraic K-Theory of Spaces.*

[2] Dywer, W., Weiss, M., and B. Williams. *A Parametrized Index Theorem for the Algebraic K-Theory Euler Class.*

Another Model of the Assembly Map (Lecture 28)

November 7, 2014

Our goal over the next few lectures is to prove the following result:

Theorem 1. *Let X be a finitely dominated space. Then the map*

$$\mathcal{M} \times_{\mathcal{M}^h} \{X\} \to K_\Delta(X) \times_{\Omega^\infty A(X)} \{[X]\}$$

is a homotopy equivalence.

The proof will follow [1]. Our first goal is to construct another model for the domain of the assembly map. This will require us to consider a slightly more general version of Waldhausen's construction.

Definition 2. Let $\mathcal{C}$ be an ∞-category with cofibrations. A collection of morphisms w in $\mathcal{C}$ is a *class of weak equivalences* if it satisfies the following conditions:

(1) The collection of morphisms w contains all equivalences and is closed under composition.

(2) If we are given a diagram

$$\begin{tikzcd} X \arrow[d] & Y \arrow[l] \arrow[r, "f"] \arrow[d] & Z \arrow[d] \\ X' & Y' \arrow[l] \arrow[r, "f'"] & Z' \end{tikzcd}$$

where the vertical maps belong to w and both f and f' are cofibrations, then the induced map $X \amalg_Y Z \to X' \amalg_{Y'} Z'$ also belongs to w.

For each $n \geq 0$, we let $wS_n\,\mathcal{C}$ denote the subcategory of $\mathrm{Gap}([n], \mathcal{C})$ containing all objects, whose morphisms are maps of $[n]$-gapped objects $X \to Y$ with the property that each of the maps $X(i,j) \to Y(i,j)$ belongs to w. Then $wS_\bullet\,\mathcal{C}$ is a simplicial ∞-category. We let $K(\mathcal{C}, w)$ denote the loop space $\Omega|wS_\bullet\,\mathcal{C}\,|$.

When w is the collection of all equivalences in $\mathcal{C}$, Definition 2 reduces to our previous notion of K-theory for an ∞-category with cofibrations. We will be primarily interested in the case where $\mathcal{C}$ is an ordinary category.

Example 3. Let X be a simplicial set. Let Set_Δ denote the ordinary category of simplicial sets, and let $(\mathrm{Set}_\Delta)_{X//X}$ denote the category of simplicial sets over and under X (so that the objects of $(\mathrm{Set}_\Delta)_{X//X}$ are triples (Y, i, r) where Y is a simplicial set, $i : X \to Y$ is a morphism of simplicial sets, and $r : Y \to X$ is a morphism satisfying $r \circ i = \mathrm{id}_X$). We let $\mathcal{C}_X$ denote the full subcategory of $(\mathrm{Set}_\Delta)_{X//X}$ spanned by those objects where Y is obtained from X by adding finitely many simplices.

We will regard $\mathcal{C}_X$ as an ∞-category with cofibrations, where the cofibrations are given by monomorphisms of simplicial sets. We will say that a morphism $f : Y \to Y'$ in $\mathcal{C}_X$ is *cell-like* if the induced map $|Y| \to |Y'|$ has contractible fibers (this is a slight variant on our earlier definition of cell-like, since the simplicial sets Y and Y' may not be finite; note however that the inverse image of any simplex in Y' is a finite simpicial set). It is not difficult to see that this collection of maps satisfies the requirements of Definition 2. We will denote this collection of morphisms by s.

For every simplicial set X, let $F(X)$ denote the connective spectrum $\Omega^{-\infty}K(\mathcal{C}_X, s)$.

Remark 4 (Functoriality)**.** If $f : X \to X'$ is a map of simplicial sets, then the construction $Y \mapsto Y \amalg_X X'$ induces a functor from $\mathcal{C}_X$ to $\mathcal{C}_{X'}$ which preserves cofibrations and cell-like maps. It follows that f induces a map of spectra $F(X) \to F(X')$. In other words, we can regard F as a functor from the ordinary category of simplicial sets to the ∞-category Sp of spectra.

Exercise 5. Let $f : X \to X'$ be a map of finite simplicial sets. Show that the construction $Y' \mapsto X \times_{X'} Y'$ determines a functor $\mathcal{C}_{X'} \to \mathcal{C}_X$ which preserves cofibrations and weak equivalences, and therefore induces a map $f^* : F(X') \to F(X)$. If f is cell-like, show that the map f^* is homotopy inverse to the map $F(X) \to F(X')$ of Remark 4. (Argue that for $Y \in \mathcal{C}_X$ and $Y' \in \mathcal{C}_{X'}$, the unit and counit maps

$$Y \to X \times_{X'} (Y \amalg_X X')$$

$$(X \times_{X'} Y') \amalg_X X' \to Y'$$

are cell-like).

Corollary 6. *The functor* $F : \mathrm{Set}_\Delta \to \mathrm{Sp}$ *is a left Kan extension of its restriction to finite nonsingular simplicial sets.*

Proof. It is easy to see that F commutes with filtered colimits and is therefore a left Kan extension of its restriction to finite simplicial sets. To complete the proof, it will suffice to show that for every finite simplicial set X, the canonical map $\varinjlim F(X') \to F(X)$ is a homotopy equivalence, where the colimit is taken over the category $\mathcal{E}$ of all nonsingular finite simplicial sets X' with a map $f : X' \to X$. Choose a cell-like map $\widetilde{X} \to X$, where X is nonsingular. Using Exercise 5, we can reduce to showing that the map

$$\theta : \varinjlim_{X' \in \mathcal{E}} F(X' \times_X \widetilde{X}) \to F(\widetilde{X})$$

is an equivalence. Let $\mathcal{E}'$ denote the category of finite nonsingular simplicial sets with a map to $\widetilde{X}$, so that the construction $X' \mapsto X' \times_X \widetilde{X}$ induces a functor $\mathcal{E} \to \mathcal{E}'$. This functor has a left adjoint and is therefore left cofinal. We may therefore identify θ with the natural map $\varinjlim_{\widetilde{X}' \in \mathcal{E}'} F(\widetilde{X}') \to F(\widetilde{X})$, which is clearly an equivalence because $\mathcal{E}'$ contains $\widetilde{X}$ as a final object. □

Remark 7. For every finite nonsingular simplicial set Y, let $\underline{S}_Y$ denote the constant sheaf on the polyhedron $|Y|$ taking the value S.

Let X be a finite nonsingular simplicial set. For each object $Y \in \mathcal{C}_X$, let $\lambda(Y)$ denote the fiber of the canonical map

$$\underline{S}_X \to \varprojlim_{i:\Delta^n \to Y} (r \circ i)_* \underline{S}_{\Delta^n},$$

where $r : Y \to X$ denotes the structural retraction. Then λ determines a functor from the ordinary category $\mathcal{C}_X$ to the ∞-category $\mathrm{Shv}_{PL}(|X|)^{\mathrm{op}}$, which preserves pushouts by cofibrations and carries cell-like maps to equivalences. Passing to K-theory, we obtain a map

$$F(X) \to \Omega^{-\infty} K(\mathrm{Shv}_{PL}(|X|)^{\mathrm{op}}).$$

Let us abuse notation by regarding $\Omega^{-\infty} K_\Delta$ as a functor from the *ordinary* category of simplicial sets to the ∞-category of spectra, so that the above construction gives a map

$$F(X) \to \Omega^{-\infty} K_\Delta(X).$$

Using Corollary 6, we see that this construction formally extends (in an essentially unique way) to the case where the simplicial set X is arbitrary.

The map $F(X) \to \Omega^{-\infty} K_\Delta(X)$ is generally not an equivalence: the left hand side is not homotopy invariant, but the right hand side is. However, we can attempt to correct that as follows:

Definition 8. Let G be a functor from the category of simplicial sets to the ∞-category of spectra. For every simplicial set X, the construction $[n] \mapsto X^{\Delta^n}$ determines a simplicial object of Set_Δ. We define $\widehat{G} : \mathrm{Set}_\Delta \to \mathrm{Sp}$ by the formula

$$\widehat{G}(X) = |G(X^{\Delta^\bullet})|.$$

Note that we have an evident natural transformation $G \to \widehat{G}$.

Exercise 9. Let G be as in Definition **??**. Show that if a map of simplicial sets $X \to X'$ is a simplicial homotopy equivalence, then the induced map $\widehat{G}(X) \to \widehat{G}(X')$ is an equivalence of spectra.

Warning 10. It is not generally true that the functor $\widehat{G}$ carries weak homotopy equivalences of simplicial sets to equivalences in Sp. However, it will be true for the functors we are interested in.

Remark 11. In the situation of Definition 8, suppose that G carries simplicial homotopy equivalences to homotopy equivalences of spectra. Then the simplicial object $G(X^{\Delta^\bullet})$ is essentially constant, so the natural transformation $G \to \widehat{G}$ is an equivalence.

The natural transformation $F(X) \to \Omega^{-\infty} K_\Delta(X)$ constructed above yields a natural map

$$\widehat{F}(X) \to \widehat{\Omega^{-\infty} K_\Delta}(X) \simeq \Omega^{-\infty} K_\Delta(X),$$

where the second equivalence follows from Remark 11. We will soon show that this map is an equivalence. The main step will be the verify the following:

Proposition 12. *The functor $\widehat{F}$ is a homology theory. In other words:*

- (a) *The functor $\widehat{F}$ carries weak homotopy equivalences of simplicial sets to equivalences in $\mathcal{S}$.*
- (b) *The functor $\widehat{F}$ commutes with filtered colimits.*
- (c) *The spectrum $\widehat{F}(\emptyset)$ vanishes*
- (d) *Given a pushout diagram of simplicial sets*

$$\begin{tikzcd} X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\ X_1 \arrow[r] & X, \end{tikzcd}$$

where the horizontal maps are monomorphisms, the diagram of spectra

$$\begin{tikzcd} \widehat{F}(X_{01}) \arrow[r] \arrow[d] & \widehat{F}(X_0) \arrow[d] \\ \widehat{F}(X_1) \arrow[r] & \widehat{F}(X) \end{tikzcd}$$

is a pushout square.

Remark 13. Part (a) of Proposition 12 implies that $\widehat{F}$ factors through a functor $\mathcal{S} \to \mathrm{Sp}$, and parts (b), (c), and (d) assert that this functor preserves colimits.

Remark 14. Assuming Proposition 12, the assertion that the natural map $\widehat{F}(X) \to \Omega^{-\infty} K_\Delta(X)$ is an equivalence can be reduced to the case where X has a single point.

The proof of Proposition 12 can be broken into two parts, one of which is entirely formal and the other of which depends on the details of our situation:

Lemma 15. *The functor F satisfies conditions (b), (c), and (d) of Proposition 12.*

Lemma 16. *Let $G : \mathrm{Set}_\Delta \to \mathrm{Sp}$ be a functor which satisfies conditions (b), (c), and (d) of Proposition 12. Then $\widehat{G}$ satisfies conditions (a), (b), (c), and (d) of Proposition 12.*

Proof of Lemma 15. Assertions (b) and (c) are trivial. Let us focus on (d). Suppose we are given a pushout diagram of simplicial sets σ :

$$\begin{tikzcd} X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\ X_1 \arrow[r] & X, \end{tikzcd}$$

where the horizontal maps are monomorphisms. Let $\mathcal{C}_{X_0,X_{01}}$ denote the full subcategory of $\mathcal{C}_{X_0}$ spanned by those objects Y for which the projection map $Y \times_{X_0} X_{01}$ is an isomorphism. Then any object $Y \in \mathcal{C}_{X_0}$ sits in a canonical cofiber sequence

$$X_{01} \times_{X_0} Y \to Y \to Y'$$

where $Y' \in \mathcal{C}_{X_0,X_{01}}$. Using a variant of the additivity theorem proved in Lecture 17, we obtain a homotopy equivalence

$$K(\mathcal{C}_{X_0}, s) \simeq K(\mathcal{C}_{X_{01}}, s) \times K(\mathcal{C}_{X_0,X_{01}}, s).$$

Similarly, we have a homotopy equivalence

$$K(\mathcal{C}_X, s) \simeq K(\mathcal{C}_{X_1}, s) \times K(\mathcal{C}_{X,X_1}, s).$$

We conclude by observing that because σ is a pushout square, the map $X_0 \to X$ induces an equivalence of categories $\mathcal{C}_{X_0,X_{01}} \to \mathcal{C}_{X,X_1}$. □

Proof of Lemma 16. Let $\mathbf{\Delta}$ denote the category of simplices, which we regard as a full subcategory of the category Set_Δ of simplicial sets and also the larger ∞-category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$ of simplicial spaces. Let $G_0 = G|_{\mathbf{\Delta}}$ and let $\overline{G}_0 : \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Sp}$ be a left Kan extension of G_0. Then $\overline{G}_0$ preserves small colimits. Consequently, the functor $\overline{G}_0|_{\mathrm{Set}_\Delta}$ satisfies conditions (b), (c), and (d) of Proposition 12. The universal property of $\overline{G}_0$ supplies a natural transformation

$$\alpha : \overline{G}_0|_{\mathrm{Set}_\Delta} \to G.$$

By construction, α is an equivalence when evaluated on simplices. Since the domain and codomain of α both satisfy (c) and (d), it follows by a simple induction that α is an equivalence when evaluated on any finite simplicial set. Since the domain and codomain of α both satisfy (b), we can conclude that α is an equivalence when evaluated on *any* simplicial set: that is, the functor G is a left Kan extension of its restriction to $\mathbf{\Delta}$. We now compute

$$\begin{aligned} \widehat{G}(X) &\simeq \varinjlim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} G(X^{\Delta^m}) \\ &\simeq \varinjlim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \varinjlim_{\Delta^n \to X^{\Delta^m}} G(\Delta^n) \\ &\simeq \varinjlim_{[n]\in\mathbf{\Delta}} \varinjlim_{\Delta^m \to X^{\Delta^n}} G(\Delta^n). \end{aligned}$$

Since the diagonal map $X \to X^{\Delta^n}$ is a simplicial homotopy equivalence, we can rewrite the latter colimit as

$$\begin{aligned} \varinjlim_{[n]\in\mathbf{\Delta}} \varinjlim_{\Delta^m \to X} G(\Delta^n) &\simeq \varinjlim_{\Delta^m \to X} \varinjlim_{[n]\in\mathbf{\Delta}} G(\Delta^n) \\ &\simeq \varinjlim_{\Delta^m \to X} G(\Delta^0) \\ &\simeq G(\Delta^0) \wedge X_+. \end{aligned}$$

□

References

[1] Waldhausen, F. *The Algebraic K-Theory of Spaces.*

Another Model of the Assembly Map II (Lecture 29)

November 12, 2014

Let X be a simplicial set. In the previous lecture, we introduced the category $\mathcal{C}_X$ whose objects are simplicial sets Y over and under X, which are obtained from X by adding finitely many simplices. We can regard $\mathcal{C}_X$ as a category with cofibrations and weak equivalences (where the latter is given by the collection s of cell-like maps), and we proved that the construction

$$F(X) = \Omega^{-\infty} K(\mathcal{C}_X, s)$$

has the property that $\widehat{F}(X) = |F(X^{\Delta^{\bullet}})$ is a homology theory (that is, it induces a colimit-preserving functor from spaces to spectra). There is a natural map $\widehat{F}(X) \to A(X)$; when X is finite and nonsingular, this comes from a composite map

$$F(X) \to \Omega^{-\infty} K(\mathrm{Shv}_{PL}(X)^{\mathrm{op}}) \to \Omega^{-\infty} K_{\Delta}(X) \to A(X)$$

where the first map is obtained from a functor

$$\lambda : \mathcal{C}_X \to \mathrm{Shv}_{PL}(|X|)^{\mathrm{op}}$$

which assigns to each retraction diagram

$$\begin{tikzcd} & Y \arrow[dr, "r"] & \\ X \arrow[ur] \arrow[rr, "\mathrm{id}"] & & X \end{tikzcd}$$

the constructible sheaf given by the cofiber of the unit map $\underline{S}_X \to r_* \underline{S}_Y$. To verify that this construction yields a well-defined map on K-theory, we observe that if $Y, Y' \in \mathcal{C}_X$ are related by a cell-like map $Y \to Y'$, then the constant sheaf $\underline{S}_{Y'}$ can be identified with the direct image of the constant sheaf $\underline{S}_Y$, so that the induced map $\lambda(Y') \to \lambda(Y)$ is an equivalence in $\mathrm{Shv}_{PL}(|X|)$.

Consider the functor $\mu : \mathrm{Shv}_{PL}(|X|)^{\mathrm{op}} \to (\mathrm{Sp}^X)^c$ (here $(\mathrm{Sp}^X)^c$ denotes the ∞-category of compact objects of Sp^X) characterized by the formula

$$\mathrm{Map}_{\mathrm{Sp}^X}(\mu(\mathcal{F}), \mathcal{G}) = \Gamma(|X|, \mathcal{F} \wedge \mathcal{G}).$$

Unwinding the definitions, we see that $\mu \circ \lambda : \mathcal{C}_X \to (\mathrm{Sp}^X)^c$ is given by the formula $Y \mapsto r_! \underline{S}_Y$, where $r_! : \mathrm{Sp}^Y \to \mathrm{Sp}^X$ is the homological pushforward on local systems. Here we have a bit more flexibility: in order to ensure that a map $Y \to Y'$ in $\mathcal{C}_X$ induces an equivalence $(\mu \circ \lambda)(Y) \to (\mu \circ \lambda)(Y')$ in Sp^X, it is sufficient to assume that $Y \to Y'$ is a weak homotopy equivalence.

Exercise 1. Let h be the collection of all weak homotopy equivalences in $\mathcal{C}_X$. Show that $(\mathcal{C}_X, h)$ satisfies the axioms for a category with cofibrations and weak equivalences (where the cofibrations, as before, are given by the monomorphisms).

The above analysis supplies a diagram of infinite loop spaces

$$\begin{tikzcd} K(\mathcal{C}_X, s) \arrow[r] \arrow[d] & K(\mathcal{C}_X, h) \arrow[d, "\theta"] \\ K_\Delta(X) \arrow[r] & \Omega^\infty A(X) \end{tikzcd}$$

which commutes up to canonical homotopy and depends functorially on X. Our goal in this lecture is to show that the right vertical map is close to being a homotopy equivalence. To this end, recall that $(\mathrm{Sp}^X)^c$ can be identified with the Spanier-Whitehead ∞-category of the ∞-category $(\mathcal{S}_*^X)^c \simeq \mathcal{S}^c_{X//X}$, where $\mathcal{S}_{X//X}$ denotes the ∞-category of spaces over and under X and $\mathcal{S}^c_{X//X}$ is the full subcategory of $\mathcal{S}_{X//X}$ spanned by the compact objects. Let $\mathcal{S}^{\mathrm{fin}}_{X//X} \subseteq \mathcal{S}^c_{X//X}$ denote the full subcategory spanned by those objects Y which can be obtained from X by attaching finitely many cells. Then θ is the map on K-theory induces by the composition

$$\mathcal{C}_X \to \mathcal{S}^{\mathrm{fin}}_{X//X} \subseteq \mathcal{S}^c_{X//X} \to (\mathrm{Sp}^X)^c.$$

We have seen that the map $K(\mathcal{S}^c_{X//X}) \to K((\mathrm{Sp}^X)^c) \simeq \Omega^\infty A(X)$ is a homotopy equivalence, and that the map $K(\mathcal{S}^{\mathrm{fin}}_{X//X}) \to K(\mathcal{S}^c_{X//X})$ exhibits the domain as a union of connected components of the target.

Notation 2. Let X be a space. We let $A^{\mathrm{free}}(X)$ denote the spectrum given by $\Omega^{-\infty} K(\mathcal{S}^{\mathrm{fin}}_{X//X})$. Then $A^{\mathrm{free}}(X)$ is a connective spectrum whose homotopy groups are given by

$$\pi_i A^{\mathrm{free}}(X) = \begin{cases} \pi_i A(X) & \text{if } i > 0 \\ \mathrm{H}_0(X; \mathbf{Z}) & \text{if } i = 0. \end{cases}$$

(note: this is the definition of $A(X)$ that appears in Waldhausen's paper).

Our next goal is to prove:

Proposition 3. *Let X be a simplicial set. Then the natural map $\mathcal{C}_X \to \mathcal{S}^{\mathrm{fin}}_{X//X}$ induces homotopy equivalences*

$$K(\mathcal{C}_X, h) \to K(\mathcal{S}^{\mathrm{fin}}_{X//X})$$

$$\Omega^{-\infty} K(\mathcal{C}_X, h) \to A^{\mathrm{free}}(X).$$

Note that the domain and codomain of the map appearing in the statement of Proposition 3 can be identified with the geometric realization of simplicial spaces obtained from Waldhausen's construction. It will therefore suffice to show that that the map $\mathcal{C}_X \to \mathcal{S}^{\mathrm{fin}}_{X//X}$ induces an equivalence in each simplicial degree. In other words, Proposition 3 is a consequence of the following more precise assertion:

Proposition 4. *Let X be a simplicial set and let $n \geq 0$ be an integer. Then the natural map*

$$hS_n\,\mathcal{C}_X \to S_n\,\mathcal{S}^{\mathrm{fin}}_{X//X}$$

is a weak homotopy equivalence (here we regard the left hand side as the nerve of a category and the right hand side as a Kan complex).

The proof of Proposition 4 will proceed by induction on n. The case $n = 0$ is trivial (since both sides are contractible), so let us assume $n > 0$. Let us identify the objects of $hS_n\,\mathcal{C}_X$ with chains of cofibrations

$$Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_n$$

of simplicial sets over and under X. There is a canonical map $\tau : hS_n\,\mathcal{C}_X \to hS_{n-1}\,\mathcal{C}_X$ obtained by "forgetting" Y_n (one of the face maps appearing in the simplicial category $hS_\bullet\,\mathcal{C}_X$). Similarly, we have a forgetful map $S_n\,\mathcal{S}^{\mathrm{fin}}_{X//X} \to S_{n-1}\,\mathcal{S}^{\mathrm{fin}}_{X//X}$ which is a Kan fibration. These maps fit into a commutative diagram

$$\begin{tikzcd} hS_n\,\mathcal{C}_X \arrow[r] \arrow[d, "\tau"] & S_n\,\mathcal{S}^{\mathrm{fin}}_{X//X} \arrow[d] \\ hS_{n-1}\,\mathcal{C}_X \arrow[r] & S_{n-1}\,\mathcal{S}^{\mathrm{fin}}_{X//X}. \end{tikzcd}$$

The inductive hypothesis implies that the lower horizontal map is a weak homotopy equivalence. Consequently, to prove Proposition 4, it will suffice to show that the diagram induces a weak homotopy equivalence after taking homotopy fibers in the vertical direction. Fix an object $\vec{Y} = (Y_1 \to \cdots \to Y_{n-1})$ in $hS_{n-1}\,\mathcal{C}_X$. We will show that the category

$$(hS_n\,\mathcal{C}_X) \times_{(hS_{n-1}\,\mathcal{C}_X)} \{\vec{Y}\}$$

is weakly homotopy equivalent to the Kan complex

$$S_n\,\mathcal{S}^{\mathrm{fin}}_{X//X} \times_{S_{n-1}\,\mathcal{S}^{\mathrm{fin}}_{X//X}} \{\vec{Y}\}.$$

It will then follow that every map $\vec{Y} \to \vec{Y}'$ in $hS_{n-1}\,\mathcal{C}_X$ induces a weak homotopy equivalence

$$(hS_n\,\mathcal{C}_X) \times_{(hS_{n-1}\,\mathcal{C}_X)} \{\vec{Y}\} \to (hS_n\,\mathcal{C}_X) \times_{(hS_{n-1}\,\mathcal{C}_X)} \{\vec{Y}'\}.$$

Applying Quillen's Theorem B (and the observation that τ is a coCartesian fibration), it follows that $(hS_n\,\mathcal{C}_X) \times_{(hS_{n-1}\,\mathcal{C}_X)} \{\vec{Y}\}$ can be identified with the homotopy fiber of τ over $\vec{Y}$, thereby completing the proof of the inductive step. We are therefore reduced to proving the following lemma (applied in the case $Z = Y_{n-1}$):

Lemma 5. *Let $f : Z \to X$ be a map of simplicial sets. Let $\mathcal{C}_f$ denote the category whose objects are diagrams of simplicial sets*

$$\begin{tikzcd} & Y \arrow[dr] & \\ Z \arrow[ur, "j"] \arrow[rr, "f"] & & X \end{tikzcd}$$

where j is a cofibration and Y is obtained from Z by adding only finitely many simplices, and whose morphisms are weak homotopy equivalences. Let $\mathcal{S}^{\mathrm{fin}}_{Z//X}$ denote the ∞-category given by the full subcategory of $\mathcal{S}_{Z//X}$ spanned by those objects Y which can be obtained from Z by attaching finitely many cells. Then the canonical map

$$\nu : \mathcal{C}_f \to (\mathcal{S}^{\mathrm{fin}}_{Z//X})^{\simeq}$$

is a weak homotopy equivalence of simplicial sets.

Proof. Let us compute the homotopy fiber of ν over a point $\eta \in (\mathcal{S}^{\mathrm{fin}}_{Z//X})^{\simeq}$. Let us represent η by a diagram of simplicial sets

$$\begin{tikzcd} & W \arrow[dr, "q"] & \\ Z \arrow[ur, "j"] \arrow[rr, "f"] & & X \end{tikzcd}$$

where j is a cofibration and q is a Kan fibration. Then the homotopy fiber $\nu^{-1}\{\eta\}$ can be identified with the homotopy colimit

$$\varinjlim_{Y \in \mathcal{C}_f} \underline{\mathrm{Hom}}(Y, W),$$

where $\underline{\mathrm{Hom}}(Y, W)$ denotes the Kan complex parametrizing maps from Y to W in $(\mathrm{Set}_{\Delta})_{Z//X}$ which are weak homotopy equivalences. It follows that $\nu^{-1}\{\eta\}$ can be identified with the geometric realization of a simplicial space which is given in degree m by the homotopy colimit

$$\varinjlim_{Y \in \mathcal{C}_f} \underline{\mathrm{Hom}}(Y, W)_m.$$

It will therefore suffice to show that this homotopy colimit is contractible for each m. Replacing W by $W^{\Delta^m} \times_{X^{\Delta^m}} X$, we can reduce to the case where $m = 0$. In this case, the homotopy colimit can be identified with the nerve of the category $\mathcal{D}$ whose objects are commutative diagrams

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ Z \arrow[ur] \arrow[rr] & & W \end{tikzcd}$$

where Y is obtained from Z by adjoining finitely many simplices and the map f is a homotopy equivalence. It will therefore suffice to show that $\mathcal{D}$ is weakly contractible. In fact, we claim that $\mathcal{D}$ is filtered. Note that $\mathcal{D}$ is a full subcategory of a filtered category $\mathcal{D}^+$, where we drop the requirement that the map f be a weak homotopy equivalence. To prove that $\mathcal{D}$ is also filtered, it will suffice to verify that for every object $Y \in \mathcal{D}^+$ there exists a morphism $Y \to Y'$ where $Y' \in \mathcal{D}$. We are therefore reduced to proving the following general assertion about simplicial sets:

Lemma 6. *Let $g : Y \to W$ be a map of simplicial sets. Suppose that $|W|$ is homotopy equivalent to a space obtained from $|Y|$ by attaching finitely many cells. Then g factors as a composition*

$$Y \to Y' \xrightarrow{f} W$$

where f is a weak homotopy equivalence and Y' is obtained from Y by adding finitely many simplices.

Proof. For simplicity, let us assume that $|W|$ is homotopy equivalent to a space obtained from $|Y|$ by attaching a single n-cell (the proof in the general case is similar). This n-cell is attached via a map $S^{n-1} \to |Y|$, which can be obtained as the geometric realization of a map of simplicial sets $A \to Y$ where A is some subdivision $\partial\,\Delta^n$. The map $|Y| \amalg_{S^{n-1}} D^n \to |Z|$ determines a nullhomotopy h of the composite map

$$A \to Y \to Z$$

after geometric realization. Replacing A by a subdivision if necessary, we may assume that the nullhomotopy h arises from a nullhomotopy in the category of simplicial sets. For $n \gg 0$, we may assume that h arises from a simplicial nullhomotopy of the composite map

$$A \to Y \to Z \to \mathrm{Ex}^n\, Z.$$

We can then take

$$Y' = Y \amalg_{\mathrm{Sd}^n A} \mathrm{Sd}^n(A \times \Delta^1) \amalg_{\mathrm{Sd}^n A} \Delta^0.$$

□

□

References

[1] Waldhausen, F. *The Algebraic K-Theory of Spaces.*

The Whitehead Space (Lecture 30)

November 19, 2014

Recall that our goal is to prove the following result:

Theorem 1. *Let X be a finitely dominated space. Then the map*

$$\mathcal{M} \times_{\mathcal{M}^h} \{X\} \to K_\Delta(X) \times_{\Omega^\infty A(X)} \{[X]\}$$

is a homotopy equivalence.

In the last two lectures, we introduced explicit combinatorial models for the spaces $K_\Delta(X)$ and $\Omega^\infty A(X)$. Our goal in this lecture is to do the same for $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$.

Definition 2. Let X be a simplicial set. We let $\mathcal{D}_X$ denote the subcategory of $(\mathrm{Set}_\Delta)_{X/}$ whose objects are monomorphisms $i : X \hookrightarrow Y$ such that Y is obtained from X by adding finitely many simplices and i is a weak homotopy equivalence; the morphisms in $\mathcal{D}_X$ are diagrams

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Y \arrow[rr, "f"] & & Y' \end{tikzcd}$$

where f is cell-like. We let $W(X)$ denote the nerve of the category $\mathcal{D}_X$. We will soon see that $W(X)$ is equivalent to the 1st space of the Whitehead spectrum $\mathrm{Wh}(X)$. We will generally abuse notation by not distinguishing between $W(X)$ and its image in the ∞-category of spaces (obtained by choosing a fibrant replacement for $W(X)$).

Recall that $\mathcal{M}$ can be identified with the nerve of the category of finite simplicial sets and cell-like maps. Consequently, the next result should not be so surprising:

Theorem 3. *Let X be a finite simplicial set. Then there is a homotopy equivalence*

$$\mathcal{M} \times_{\mathcal{M}^h} \{|X|\} \simeq W(X),$$

which is natural with cell-like maps.

Corollary 4. *The construction $X \mapsto W(X)$ preserves weak homotopy equivalences.*

We will discuss Theorem 3 and Corollary 4 in the next lecture.

Note that $W(X)$ is a covariant functor of X, and that this functor commutes with filtered colimits. Consequently, $W(X)$ is formally determined by its restriction to finite simplicial sets.

The formation of pushouts over X determines a symmetric monoidal structure on the category $\mathcal{D}_X$, so that $W(X)$ inherits the structure of an E_∞-space. When X is finite, this E_∞-structure is compatible with the homotopy equialence of Theorem 3 (where we regard $\mathcal{M} \times_{\mathcal{M}^h} \{|X|\}$ as an E_∞-space as in Lecture 13). It follows that E_∞-structure on $W(X)$ is grouplike (when X is connected, we have $\pi_0 W(X) = \mathrm{Wh}(\pi_1 X)$). Since the functor $X \mapsto W(X)$ commutes with filtered colimits, this result extends formally to arbitrary simplicial sets:

Corollary 5. *For every simplicial set X, the E_∞-structure on $W(X)$ is grouplike.*

Let $\mathcal{C}_X$ denote the category studied in the previous two lectures: the objects of $\mathcal{C}_X$ are diagrams

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur, "i"] \arrow[rr, "\mathrm{id}"] & & X \end{tikzcd}$$

where Y is obtained from X by adding finitely many simplices. We let $\mathcal{C}_X^h$ denote the full subcategory of $\mathcal{C}_X$ spanned by those objects where the map i is a weak homotopy equivalence.

Exercise 6. Show that we can regard $\mathcal{C}_X^h$ as a category with cofibrations and weak equivalences in two ways:

- We can take cofibrations in $\mathcal{C}_X^h$ to be monomorphisms and weak equivalences to be cell-like maps (we will denote this class of weak equivalences by s).
- We can take cofibrations in $\mathcal{C}_X^h$ to be monomorphisms and weak equivalences to be weak homotopy equivalences (we will denote this class of weak equivalences by h; note that it is the collection of *all* morphisms in $\mathcal{C}_X^h$).

We can apply Waldhausen's construction in either of these cases to obtain K-theory spaces $K(\mathcal{C}_X^h, h)$ and $K(\mathcal{C}_X^h, s)$. Note that each level of the simplicial category $hS_\bullet \mathcal{C}_X^h$ has an initial object, so that $K(\mathcal{C}_X^h, h)$ is contractible. The inclusion $\mathcal{C}_X^h \hookrightarrow \mathcal{C}_X$ induces maps on K-theory, giving us a commutative diagram

$$\begin{tikzcd} K(\mathcal{C}_X^h, s) \arrow[r] \arrow[d] & K(\mathcal{C}_X, s) \arrow[d] \\ K(\mathcal{C}_X^h, h) \arrow[r] & K(\mathcal{C}_X, h). \end{tikzcd}$$

We will deduce Theorem **??** from the following result, which we defer to a future lecture:

Theorem 7. *For every simplicial set X, the preceding diagram is a homotopy pullback square. In other words, we have a fiber sequence of spaces*

$$K(\mathcal{C}_X^h, s) \to K(\mathcal{C}_X, s) \to K(\mathcal{C}_X, h).$$

Corollary 8. *Let $X = \Delta^0$. Then the canonical map $K(\mathcal{C}_X, s) \to K(\mathcal{C}_X, h)$ is a homotopy equivalence.*

Proof. By virtue of Theorem 7 (and the evident surjectivity on π_0), it will suffice to show that $K(\mathcal{C}_X^h, s)$ is contractible. In fact, every level of the Waldhausen construction $sS_\bullet \mathcal{C}_X^h$ is weakly contractible: each of the categories $sS_n \mathcal{C}_X^h$ has a final object. $\square$

We have seen that there is a canonical homotopy equivalence $K(\mathcal{C}_X, h) \simeq \Omega^\infty A^{\mathrm{free}}(X)$, and that the functor $X \mapsto |K(\mathcal{C}_{X^{\Delta^\bullet}}, s)|$ is a homology theory. Corollary 8 implies that this is the homology theory associated to $A^{\mathrm{free}}(*) \simeq A(*)$:

Corollary 9. *For every simplicial set X, there is a canonical homotopy equivalence*

$$|K(\mathcal{C}_{X^{\Delta^\bullet}}, s)| \simeq \Omega^\infty(A(*) \wedge X_+) \simeq K_\Delta(X).$$

Note that since the assembly map

$$(A(*) \wedge X_+) \to A^{\mathrm{free}}(X)$$

is surjective on π_0, Theorem 7 implies that we have a fiber sequence of spectra

$$\Omega^{-\infty} K(\mathcal{C}^h_X, s) \to \Omega^{-\infty} K(\mathcal{C}_X, s) \to \Omega^{-\infty} K(\mathcal{C}_X, h).$$

Replacing X by $X^{\Delta^\bullet}$ and passing to the geometric realization, we obtain the following:

Corollary 10. *For every simplicial set X, there is a canonical fiber sequence of spectra*

$$|\Omega^{-\infty} K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)| \to A(*) \wedge X_+ \to A^{\mathrm{free}}(X).$$

Passing to 0th spaces (and noting that $\Omega^\infty A^{\mathrm{free}}(X)$ is a union of path components of $\Omega^\infty A(X)$), we obtain:

Corollary 11. *For every simplicial set X, there is a canonical fiber sequence of spaces*

$$|K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)| \to \Omega^\infty(A(*) \wedge X_+) \to \Omega^\infty A(X).$$

To understand the relationship between Corollary 11 and Theorem **??**, let us investigate the relationship between $K(\mathcal{C}^h_X, s)$ and the space $W(X)$. Let $s\,\mathcal{C}^h_X$ denote the subcategory of $\mathcal{C}^h_X$ whose morphisms are cell-like maps (that is, the first stage of the Waldhausen construction on $(\mathcal{C}^h_X, s)$). We have an evident forgetful functor $s\,\mathcal{C}^h_X \to \mathcal{D}_x$, which induces a diagram

$$K(\mathcal{C}^h_X, s) \leftarrow \mathrm{N}(s\,\mathcal{C}^h_X) \to W(X).$$

Replacing X by $X^{\Delta^\bullet}$ and passing to geometric realizations, we obtain a diagram

$$|K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)| \leftarrow |s\,\mathcal{C}^h_{X^{\Delta^\bullet}}| \to W(X).$$

Proposition 12. *If X is a Kan complex, then the above maps are homotopy equivalences. Consequently, we obtain a preferred homotopy equivalence*

$$W(X) \simeq |K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)|,$$

so that Corollary 11 yields a fiber sequence

$$W(X) \to \Omega^\infty(A(*) \wedge X_+) \to \Omega^\infty A(X).$$

Remark 13. It follows from Corollary 11 that the construction

$$X \mapsto |K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)|$$

preserves weak homotopy equivalences in X. Consequently, the identification $W(X) \simeq |K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)|$ of Proposition 12 persists even if we do not assume that X is a Kan complex.

Proposition 12 is really two separate statements (Proposition 14 and Proposition 15 below):

Proposition 14. *Let X be a Kan complex. Then the canonical map of simplicial spaces*

$$\theta : \mathrm{N}(s\,\mathcal{C}^h_{X^{\Delta^\bullet}}) \to S(X^{\Delta^\bullet})$$

induces a homotopy equivalence after geometric realization.

Proof. For every simplicial set X, let $\mathcal{E}_{X,\bullet}$ denote the simplicial category whose objects (in simplicial degree m) are diagrams

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur, "i"] \arrow[rr, "\delta"] & & X^{\Delta^m} \end{tikzcd}$$

where i is a trivial cofibration of simplicial sets which exhibits Y as obtained from X by adding finitely many simplices, and whose morphisms are cell-like maps. Then $\mathcal{E}_{X,0} \simeq s\,\mathcal{C}^h_X$, and we have a canonical map $\mathcal{E}_{X,\bullet} \to \mathcal{D}_X$ (where we identify $\mathcal{D}_X$ with a constant simplicial category). It follows that θ factors as a composition

$$\mathrm{N}(s\,\mathcal{C}^h_X) \xrightarrow{\theta'} |\,\mathrm{N}(\mathcal{E}_{X,\bullet})| \xrightarrow{\theta''} \mathrm{N}(\mathcal{D}_X).$$

We first claim that θ'' is a homotopy equivalence: this follows from the observation that for each object $Y \in \mathcal{D}_X$, the Kan complex of retractions from Y to X (which are the identity on X) is contractible, since X is a Kan complex and Y is weakly equivalent to X. It will therefore suffice to show that θ' induces a homotopy equivalence

$$|\,\mathrm{N}(s\,\mathcal{C}^h_{X^{\Delta^\bullet}})| \to ||\,\mathrm{N}(\mathcal{E}^h_{X^{\Delta^\bullet},\bullet})||.$$

Let $\mathcal{E}_\bullet$ denote the simplicial category given by $[n] \mapsto \mathcal{E}_{X^{\Delta^n},n}$: the objects of $\mathcal{E}_n$ are given by commutative diagrams

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ X^{\Delta^n} \arrow[ur, "i"] \arrow[rr] & & X^{\Delta^n \times \Delta^n} \end{tikzcd}$$

where i is a trivial cofibration which exhibits Y as obtained from X^{Δ^n} by adding finitely many simplices and f classifies a map of simplicial sets $F : Y \times \Delta^n \times \Delta^n \to X$ whose restriction to $X^{\Delta^n} \subseteq Y$ is obtained by ignoring the second factor of Δ^n and using the evaluation map on the first. Let us abuse notation by identifying the objects of $\mathcal{E}_n$ with the pairs (Y, F).

Let $\mathcal{E}'_\bullet$ be the full simplicial subcategory of $\mathcal{E}_\bullet$ given in simplicial degree n by those objects (Y, F) where the map $F : Y \times \Delta^n \times \Delta^n \to X$ does not depend on the second factor of Δ^n. Unwinding the definitions, we we wish to show that the inclusion $\mathcal{E}'_\bullet \hookrightarrow \mathcal{E}_\bullet$ induces a weak homotopy equivalence $|\,\mathrm{N}\,\mathcal{E}'_\bullet\,| \hookrightarrow |\,\mathrm{N}\,\mathcal{E}_\bullet\,|$.

For each $n \geq 0$, let $r^+_n : \Delta^n \times \Delta^n \to \Delta^n \times \Delta^n$ denote the map given on vertices by

$$r^+_n(i,j) = \begin{cases} (i,j) & \text{if } i \leq j \\ (i,i) & \text{otherwise,} \end{cases}$$

and let r^-_n denote the map given by

$$r^-_n(i,j) = \begin{cases} (i,j) & \text{if } j \leq i \\ (i,i) & \text{otherwise,} \end{cases}$$

Then r^-_n and r^+_n induce idempotent maps R^-_n and R^+_n from $Y \times \Delta^n \times \Delta^n$ to itself. We let $\mathcal{E}^+_n$ and $\mathcal{E}^-_n$ denote the full subcategories of $\mathcal{E}_n$ spanned by those pairs (Y, F) where F factors through R^-_n and R^+_n, respectively. Then $\mathcal{E}^-_\bullet, \mathcal{E}^+_\bullet$ are simplicial subcategories of $\mathcal{E}_\bullet$ whose intersection is $\mathcal{E}'_\bullet$. It will therefore suffice to show that the inclusions

$$|\,\mathrm{N}\,\mathcal{E}'_\bullet\,| \overset{\phi}{\hookrightarrow} |\,\mathrm{N}\,\mathcal{E}^+_\bullet\,| \overset{\psi}{\hookrightarrow} |\,\mathrm{N}\,\mathcal{E}_\bullet\,|$$

are weak homotopy equivalences. We will show that ψ is a weak homotopy equivalence by demonstrating that $R^+_\bullet$ is a deformation retraction: that is, there is a simplicial homotopy the identity map to $R^+_\bullet$ which is fixed on $\mathcal{E}^+_\bullet$. A dual construction will show that $R^-_\bullet$ is a deformation retraction from $\mathcal{E}_\bullet$ to $\mathcal{E}^-_\bullet$; we will leave as an exercise to the reader to verify that this deformation retraction carries $\mathcal{E}_\bullet^+$ to itself and therefore exhibits $\mathcal{E}'_\bullet = \mathcal{E}_\bullet^+ \cap \mathcal{E}_\bullet^-$ as a deformation retract of $\mathcal{E}_\bullet^+$ (which will prove that ϕ is a weak homotopy equivalence).

To construct the desired simplicial homotopy, we must exhibit for each map of finite linearly ordered sets $\alpha : [n] \to [1]$ a functor

$$\rho_\alpha : \mathcal{E}_n \to \mathcal{E}_n$$

which is the identity when α has the constant value 0 and agrees with R_n^+ when α has the constant value 1. We define ρ_α by the formula

$$\rho_\alpha(Y, F) = (Y, F \circ (\mathrm{id}_Y \times r_\alpha)),$$

where $r_\alpha : \Delta^n \times \Delta^n \to \Delta^n \times \Delta^n$ is the map given on vertices by the formula

$$r_\alpha(i,j) = \begin{cases} (i,j) & \text{if } i \leq j \text{ or } \alpha(j) = 0. \\ (i,i) & \text{otherwise.} \end{cases}$$

□

The other half of Proposition 12 is contained in the following:

Proposition 15. *Let X be a Kan complex. Then the canonical map*

$$\rho : |s\,\mathcal{C}^h_{X^{\Delta^\bullet}}| \to |K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)|$$

is a homotopy equivalence.

Proof. Let $\mathrm{Bar}_\bullet(s\,\mathcal{C}^h_X)$ denote the simplicial space obtained by applying the bar construction (with respect to the E_∞-structure given by pushouts over X). It follows from Proposition 14 that $|s\,\mathcal{C}^h_{X^{\Delta^\bullet}}|$ is equivalent to $W(X)$ and is therefore equivalent to its own group completion (with respect to the natural E_∞-structure), so we can identify $|s\,\mathcal{C}^h_{X^{\Delta^\bullet}}|$ with the loop space of $|\mathrm{Bar}_\bullet(s\,\mathcal{C}^h_{X^{\Delta^\bullet}})|$. Under this identification, ρ arises from a map of bisimplicial categories

$$\mathrm{Bar}_\bullet(s\,\mathcal{C}^h_{X^{\Delta^\bullet}}) \to sS_\bullet\,\mathcal{C}^h_{X^{\Delta^\bullet}}.$$

To show that this map is a homotopy equivalence, it will suffice to show that for each $n \geq 0$, the map of simplicial categories

$$\rho_n : \mathrm{Bar}_n(s\,\mathcal{C}^h_{X^{\Delta^\bullet}}) \to sS_n(\mathcal{C}^h_{X^{\Delta^\bullet}})$$

induces a homotopy equivalence (after passing to nerves and taking geometric realizations). Let us identify the objects of $sS_n(\mathcal{C}^h_{X^{\Delta^m}})$ with finite sequences of cofibrations

$$Y_1 \hookrightarrow \cdots \hookrightarrow Y_n$$

of simplicial sets over and under X^{Δ^m}, which are weak homotopy equivalent to X^{Δ^m} and obtained from X^{Δ^m} by adding finitely many simplices. The map ρ_n has a left homotopy inverse, given by a functor

$$\psi_n : sS_n(\mathcal{C}^h_{X^{\Delta^\bullet}}) \to (s\,\mathcal{C}^h_{X^{\Delta^\bullet}})^n$$

$$(Y_1 \hookrightarrow \cdots \hookrightarrow Y_n) \mapsto (Y_1, Y_2/Y_1, \ldots, Y_n/Y_{n-1}).$$

It will therefore suffice to show that ψ_n induces a homotopy equivalence (after taking nerves and passing to the geometric realizaiton). The proof proceeds by induction on n, the case $n \leq 1$ being trivial. To carry out the inductive step, it will suffice (by virtue of the inductive hypothesis and Propoistion 14) to show that the functor

$$\phi : sS_n(\mathcal{C}^h_{X^{\Delta^\bullet}}) \to sS_{n-1}(\mathcal{C}^h_{X^{\Delta^\bullet}}) \times \mathcal{D}_{X^{\Delta^\bullet}}$$

$$(Y_1 \hookrightarrow \cdots \hookrightarrow Y_n) \mapsto ((Y_1 \hookrightarrow \cdots Y_{n-1}), Y_n/Y_{n-1})$$

induces a homotopy equivalence (after taking nerves and passing to the geometric realization).

Let $\mathcal{E}_\bullet$ denote the simplicial category whose objects in degree m are diagrams

$$X^{\Delta^m} \to Y_1 \to \cdots \to Y_{n-1} \to Y_n \times X^{\Delta^m}$$

where each Y_i is obtained from Y_{i-1} by adding finitely many simplices and is weakly homotopy equivalent to Y_{i-1} (with the convention that $Y_0 = X^{\Delta^m}$), and the composite map induces the identity from X^{Δ^m} to itself, and whose morphisms are degreewise cell-like maps. More informally: $\mathcal{E}_\bullet$ is defined just like the simplicial category $sS_n(\mathcal{C}^h_{X^\bullet})$, except that we do not require that the simplicial set Y_n is equipped with a retraction back onto $X^{\Delta^\bullet}$. The map ϕ factors as a composition

$$sS_n(\mathcal{C}^h_{X^{\Delta^\bullet}}) \xrightarrow{\phi'} \mathcal{E}_\bullet \xrightarrow{\phi''} sS_{n-1}(\mathcal{C}^h_{X^{\Delta^\bullet}}) \times \mathcal{D}_{X^{\Delta^\bullet}}$$

Note that when $n = 1$, the map ϕ' coincides with the map studied in Proposition 14, and is therefore a homotopy equivalence (after taking nerves and passing to the geometric realization). Moreover, the proof of Proposition 14 can be adapted (with minor changes of notation) to prove that this assertion holds for all n. We are therefore reduced to proving that the map ϕ'' induces a homotopy equivalence. In fact, we claim that this holds even before passage to geometric realization: that is, for each m, the functor

$$\phi''_m : \mathcal{E}_m \to sS_{n-1}(\mathcal{C}^h_{X^{\Delta^m}}) \times \mathcal{D}_{X^{\Delta^m}}$$

induces a weak homotopy equivalence of nerves. Replacing X by X^{Δ^m}, we may assume that $m = 0$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^0 \arrow[rr] \arrow[dr] & & sS_{n-1}(\mathcal{C}^h_X) \times \mathcal{D}_X \arrow[dl] \\
& sS_{n-1}(\mathcal{C}^h_X) &
\end{tikzcd}$$

where the vertical maps are coCartesian fibrations. Consequently, to prove that the horizontal map is a weak homotopy equivalence, it will suffice to show that it induces a weak homotopy equivalence after taking the fiber over any object

$$(Y_1 \hookrightarrow \cdots \hookrightarrow Y_{n-1}) \in sS_{n-1}(\mathcal{C}^h_X).$$

Unwinding the definitions, this amounts to the assertion that the retraction $Y_{n-1} \to X$ induces a weak homotopy equivalence

$$\mathcal{D}_{Y_{n-1}} \to \mathcal{D}_X,$$

which follows from Corollary 4. □

References

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The Whitehead Space II (Lecture 31)

November 19, 2014

Let X be a simplicial set. As in the previous lecture, we let $\mathcal{D}_X$ denote the subcategory of $(\mathrm{Set}_\Delta)_{X/}$ spanned by those objects $i : X \hookrightarrow Y$ which are trivial cofibrations of simplicial sets obtained by adding finitely many simplices to X, and whose morphisms are cell-like maps. We let $W(X)$ denote the nerve of $\mathcal{D}_X$. Our goal in this lecture is to show that if X is finite, then $W(X)$ can be identified with the homotopy fiber product $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$. Our first step is to establish the following result (already used without proof in the previous lecture):

Proposition 1. *The functor $X \mapsto W(X)$ preserves weak homotopy equivalences.*

We will deduce Proposition 1 from two special cases:

Lemma 2. *Let X be a finite simplicial set. Then the "last vertex" map $\mathrm{Sd}(X) \to X$ induces a weak homotopy equivalence $W(\mathrm{Sd}(X)) \to W(X)$.*

Lemma 3. *Let X be a finite simplicial set. Then the projection map $X \times \Delta^1 \to X$ induces a weak homotopy equivalence $W(X \times \Delta^1) \to W(X)$.*

Proof of Proposition 1. We first show that if $f : X \to Y$ is a weak homotopy equivalence of finite simplicial sets, then the induced map $W(X) \to W(Y)$ is a weak homotopy equivalence. Since X is not a Kan complex, the map f need not be a homotopy equivalence. However, there exists a homotopy inverse to f after fibrant replacement: that is, a map $g : Y \to \mathrm{Ex}^\infty X$ such that the unit map $X \to \mathrm{Ex}^\infty X$ is homotopic to $g \circ f$, and $\mathrm{Ex}^\infty(f) \circ g$ is homotopic to the unit map $Y \to \mathrm{Ex}^\infty Y$. Since X and Y are finite, we can replace Ex^∞ by Ex^n for $n \gg 0$. In this case, we can identify g with a map $G : \mathrm{Sd}^n(Y) \to X$, and we have homotopies

$$h : \mathrm{Sd}^n(X \times \Delta^1) \to X$$

$$h' : \mathrm{Sd}^n(Y \times \Delta^1) \to Y.$$

To show that these maps yield homotopies after applying W, it suffices to show that the maps

$$W(\mathrm{Sd}^n(X \times \Delta^1)) \to W(X)$$

$$W(\mathrm{Sd}^n(X)) \to W(X)$$

are weak homotopy equivalences, and similarly for Y; these assertions are immediate consequences of Lemmas 2 and 3.

It follows from the above argument that when restricted to finite simplicial sets, the functor $W : \mathrm{Set}_\Delta \to \mathcal{S}$ preserves weak homotopy equivalences, and therefore induces a functor of ∞-categories $u : \mathcal{S}^{\mathrm{fin}} \to \mathcal{S}$. The functor u admits an essentially extension $U : \mathcal{S} \to \mathcal{S}$ which commutes with filtered colimits. Since W commutes with filtered colimits, it follows that it is given by the composition

$$\mathrm{Set}_\Delta \to \mathcal{S} \xrightarrow{U} \mathcal{S}.$$

$\square$

Proof of Lemma 3. Pushout along the projection map $X \times \Delta^1 \to X$ induces a functor $f : \mathcal{D}_{X\times\Delta^1} \to \mathcal{D}_X$. Consider the functor $g : \mathcal{D}_X \to \mathcal{D}_{X\times\Delta^1}$ given by $Y \mapsto Y \times \Delta^1$. We claim that, after passing to nerves, these maps are mutually inverse homotopy equivalences relating $W(X\times\Delta^1)$ and $W(X)$. Note that $f\circ g : \mathcal{D}_X \to \mathcal{D}_X$ is the functor given by

$$Y \mapsto X \amalg_{X\times\Delta^1} (Y \times \Delta^1).$$

At the level of nerves, this is homotopic to the identity map, since the projection $Y \times \Delta^1 \to Y$ induces a cell-like map

$$X \amalg_{X\times\Delta^1} (Y \times \Delta^1) \to Y.$$

The functor $g \circ f : \mathcal{D}_{X\times\Delta^1} \to \mathcal{D}_{X\times\Delta^1}$ is given by

$$Y \mapsto (Y \amalg_{X\times\Delta^1} X) \times \Delta^1.$$

In this case, we have a two-step homotopy to the identity, given by the diagram

$$(Y \amalg_{X\times\Delta^1} X) \times \Delta^1 \leftarrow Y \times \Delta^1 \to Y.$$

□

Proof of Lemma 2. We wish to show that the functor

$$f : \mathcal{D}_{\mathrm{Sd}(X)} \to \mathcal{D}_X$$

$$Y \mapsto Y \amalg_{\mathrm{Sd}(X)} X$$

induces a weak homotopy equivalence on nerves. We will show that the construction

$$g : \mathcal{D}_X \to \mathcal{D}_{\mathrm{Sd}(X)}$$

$$Y \mapsto \mathrm{Sd}(Y)$$

provides a homotopy inverse. Note that the composite map $f \circ g : \mathcal{D}_X \to \mathcal{D}_X$ is related to the identity functor by a cell-like natural transformation

$$\mathrm{Sd}(Y) \amalg_{\mathrm{Sd}(X)} X \to Y.$$

The other direction is a bit trickier: the composite functor $g \circ f : \mathcal{D}_{\mathrm{Sd}(X)} \to \mathcal{D}_{\mathrm{Sd}(X)}$ carries an object $Y \in \mathcal{D}_{\mathrm{Sd}(X)}$ to the object $\mathrm{Sd}(Y \amalg_{\mathrm{Sd}(X)} X) = \mathrm{Sd}(Y) \amalg_{\mathrm{Sd}^2(X)} \mathrm{Sd}(X)$. In other words, we can identify $(g \circ f)(Y)$ with the image of $\mathrm{Sd}(Y) \in \mathcal{D}_{\mathrm{Sd}^2(X)}$ under the functor $\mathcal{D}_{\mathrm{Sd}^2(X)} \to \mathcal{D}_{\mathrm{Sd}(X)}$ which is obtained from the map $\mathrm{Sd}(e) : \mathrm{Sd}^2(X) \to \mathrm{Sd}(X)$, where $e : \mathrm{Sd}(X) \to X$ is the "last vertex map". Note that e does not coincide with the "last vertex" map $\mathrm{Sd}^2(X) \to \mathrm{Sd}(X)$, but it is simplicially homotopic to it, and therefore (by virtue of Lemma 3) induces a homotopic map from $W(\mathrm{Sd}^2(X))$ to $W(\mathrm{Sd}(X))$. We are therefore reduced to proving that the functor $Y \mapsto \mathrm{Sd}(Y) \amalg_{\mathrm{Sd}^2(X)} \mathrm{Sd}(X)$ is homotopic to the identity, where $\mathrm{Sd}^2(X)$ maps to $\mathrm{Sd}(X)$ via the "last vertex" map. This follows from the first part of the proof (applied to $\mathrm{Sd}(X)$ rather than X). □

It will be useful for us to consider a slight variant of the category $\mathcal{D}_X$. From this point forward, let us assume that the simplicial set X is finite. Let $\mathcal{D}_X^+$ denote the subcategory of $(\mathrm{Set}_\Delta)_{X/}$ whose objects are weak homotopy equivalences $X \to Y$ of finite simplicial sets, and whose morphisms are cell-like maps. Then $\mathcal{D}_X^+$ contains $\mathcal{D}_X$ as a full subcategory: the only difference is that we no longer require the structure map $X \to Y$ to be a cofibration.

Proposition 4. *For every finite simplicial set X, the inclusion $\mathcal{D}_X \hookrightarrow \mathcal{D}_X^+$ induces a weak homotopy equivalence of nerves.*

Proof. For each morphism $f : X \to Y$, let $M(f) = (X \times \Delta^1) \amalg_{X \times \{1\}} Y$ denote the mapping cylinder of f. Then the construction

$$(f : X \to Y) \mapsto (X \times \{0\} \hookrightarrow M(f))$$

determines a functor from $\mathcal{D}_X^+$ into $\mathcal{D}_X$. Using the natural cell-like map $M(f) \to Y$, we see that this functor determines a deformation retraction of $\mathrm{N}(\mathcal{D}_X^+)$ into $\mathrm{N}(\mathcal{D}_X)$. □

Note that the enlargement $\mathcal{D}_X \mapsto \mathcal{D}_X^+$ comes at a price: if $X \to X'$ is a map of finite simplicial sets, the construction

$$Y \mapsto Y \amalg_X X'$$

generally does not preserve cell-like maps (or weak homotopy equivalences), and therefore does not induce a functor from $\mathcal{D}_X^+$ to $\mathcal{D}_{X'}^+$. However, we get a different sort of functoriality as compensation: if $f : X \to X'$ is a weak homotopy equivalence, then composition with f induces a map $\mathcal{D}_{X'}^+ \to \mathcal{D}_X^+$. We will need the following variant of Proposition 1:

Proposition 5. *Let $f : X \to X'$ be a weak homotopy equivalence of finite simplicial sets. Then composition with f induces a weak homotopy equivalence $\mathcal{D}_{X'}^+ \to \mathcal{D}_X^+$.*

Proof. Arguing as in the proof of Proposition 1, it suffices to treat the case of the maps

$$\mathrm{Sd}(X) \to X \qquad X \times \Delta^1 \to X.$$

Using Propositions 1 and 4, we are reduced to proving the the composite functor

$$\mathcal{D}_X \to \mathcal{D}_{X'} \hookrightarrow \mathcal{D}_{X'}^+ \to \mathcal{D}_X^+$$

$$Y \mapsto Y \amalg_X X'$$

is a weak homotopy equivalence. Since f is cell-like, this functor is related to the inclusion $\mathcal{D}_X \hookrightarrow \mathcal{D}_X^+$ by a natural transformation $Y \to Y \amalg_X X'$; the desired result now follows from Proposition 4. □

Fix a finite simplicial set X. For each $n \geq 0$, the construction $[m] \mapsto \mathrm{Sd}^n(X \times \Delta^m)$ determines a cosimplicial object of $\mathcal{C}$. We therefore obtain a simplicial category

$$\mathcal{D}^+_{\mathrm{Sd}^n(X \times \Delta^\bullet)}.$$

After taking nerves, we obtain a simplicial space which is equivalent to the constant simplicial space with the value

$$\mathrm{N}(\mathcal{D}^+_{\mathrm{Sd}^n(X)}) \simeq \mathrm{N}(\mathcal{D}_X^+) \simeq \mathrm{N}(\mathcal{D}_X) \simeq W(X).$$

We may therefore identify $W(X)$ with the geometric realization

$$|\varinjlim_n \mathrm{N}\, \mathcal{D}^+_{\mathrm{Sd}^n(X \times \Delta^\bullet)}|.$$

Let $\mathcal{E}$ denote the category whose objects are finite simplicial sets Y and whose morphisms are cell-like maps. Then each of the categories $\mathcal{D}^+_{\mathrm{Sd}^n(X \times \Delta^m)}$ is cofibered in sets over $\mathcal{E}$, and can therefore be identified with the Grothendieck construction on the functor

$$f_{n,m} : \mathcal{E} \to \mathcal{S}\mathrm{et}$$

which assigns to each object $Y \in \mathcal{E}$ the set of all weak homotopy equivalences

$$\mathrm{Sd}^n(X \times \Delta^m) \to Y.$$

It follows that the nerve of $\mathcal{D}^+_{\mathrm{Sd}^n(X\times\Delta^m)}$ can be identified with the homotopy colimit of the diagram $f_{n,m}$. It follows that

$$\varinjlim_n \mathrm{N}\,\mathcal{D}^+_{\mathrm{Sd}^n(X\times\Delta^\bullet)}$$

can be identified with the homotopy colimit of the functor

$$f_m : \mathcal{E} \to \mathcal{S}\mathrm{et}$$

$$f_m(Y) = \mathrm{Hom}'(X \times \Delta^m, \mathrm{Ex}^\infty Y).$$

where $\mathrm{Hom}'(X \times \Delta^m, \mathrm{Ex}^\infty Y)$ is the subset of $\mathrm{Hom}(X \times \Delta^m, \mathrm{Ex}^\infty Y)$ consisting of weak homotopy equivalences. Passing to the geometric realization, we can identify $W(X)$ with the homotopy colimit of the diagram

$$\mathcal{E} \to \mathrm{Set}_\Delta$$

$$Y \mapsto H(X, \mathrm{Ex}^\infty Y)$$

where $H(X, \mathrm{Ex}^\infty Y)$ is the simplicial set parametrizing homotopy equivalences from X to $\mathrm{Ex}^\infty Y$. We saw in Lecture 12 that we can identify $\mathcal{M}$ with the nerve of $\mathcal{E}$; this identification induces an equivalence

$$W(X) \simeq \varinjlim_{Y\in\mathcal{E}} H(X, \mathrm{Ex}^\infty Y) \simeq \mathrm{N}(\mathcal{E}) \times_{\mathcal{M}^h} \{X\} \simeq \mathcal{M} \times_{\mathcal{M}^h} \{X\}.$$

We conclude by discussing the extent to which the homotopy equivalence $W(X) \simeq \mathcal{M} \times_{\mathcal{M}^h} \{X\}$ can be made functorial in X. By virtue of the above discussion, this amounts to the question of how functorially we can identify the spaces $\mathrm{N}(\mathcal{D}_X)$ with $\mathrm{N}(\mathcal{D}^+_X)$. It follows from Propositions 1 and 5 that the constructions

$$X \mapsto \mathrm{N}(\mathcal{D}_X) \qquad X \mapsto \mathrm{N}(\mathcal{D}^+_X)$$

define functors

$$u : \mathcal{E} \to \mathcal{S}^\simeq \qquad v : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}^\simeq .$$

We will prove the following:

Proposition 6. *The functor u and v are homotopic to one another (after identifying $\mathcal{S}^\simeq$ with its opposite).*

Warning 7. We can extend u and v to functors defined on the larger category whose objects are finite simplicial sets and whose morphisms are weak homotopy equivalences. However, these enlargements are *not* equivalent to one another (note that if they were, then the fibration $\mathcal{M} \to \mathcal{M}^h$ would be classified by the functor $X \mapsto W(X)$, and would therefore admit a section).

To prove Proposition 6, we begin by applying the Grothendieck construction to the assignments

$$X \mapsto \mathcal{D}_X \qquad X \mapsto \mathcal{D}^+_X$$

to produce coCartesian fibrations

$$\mathcal{D} \to \mathcal{E} \qquad \mathcal{D}^+ \to \mathcal{E}^{\mathrm{op}} :$$

the objects of $\mathcal{D}$ are trivial cofibrations $i : X \to Y$ of finite simplicial sets, and the objects of $\mathcal{D}^+$ are weak homotopy equivalences $i : X \to Y$ of finite simplicial sets. Morphisms in $\mathcal{D}$ are given by commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

where the vertical maps are cell-like and the horizontal maps are trivial cofibrations, and morphisms in $\mathcal{D}^+$ are given by commutative diagrams

$$\begin{tikzcd} X \arrow[r] & Y \arrow[d] \\ X' \arrow[u] \arrow[r] & Y' \end{tikzcd}$$

where the vertical maps are cell-like and the horizontal maps are weak homotopy equivalences.

Remark 8. There is a bit of work hidden in this description of $\mathcal{D}$. *A priori*, the morphisms in the relevant Grothendieck construction are given by commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[d, "g"] \\ X' \arrow[r] & Y' \end{tikzcd}$$

where the horizontal maps are trivial cofibrations, f is cell-like, and the induced map $g' : Y \amalg_X X' \to Y'$ is cell-like. But the assumption that f is cell-like guarantees that the map $Y \to Y \amalg_X X'$ is cell-like, from which it follows that g is cell-like if and only if g' is cell-like.

These coCartesian fibrations induce maps of spaces

$$U : \mathrm{N}(\mathcal{D}) \to \mathrm{N}(\mathcal{E}) \qquad V : \mathrm{N}(\mathcal{D}^+) \to \mathrm{N}(\mathcal{E}^{\mathrm{op}}).$$

Using Propositions 1 and 5 and Quillen's Theorem B, we see that the homotopy fibers of these maps (over an object $X \in \mathcal{E}$) can be identified with $\mathrm{N}(\mathcal{D}_X)$ and $\mathrm{N}(\mathcal{D}_X^+)$, respectively. Consequently, Proposition 6 can be reformulated as follows: the natural homotopy equivalence $\mathrm{N}(\mathcal{E}) \simeq \mathrm{N}(\mathcal{E}^{\mathrm{op}})$ can be lifted to an equivalence between U and V (regarded as objects in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{S})$ of morphisms in the ∞-category of spaces).

To prove this, let $\mathrm{TwArr}(\mathcal{E})$ denote the "twisted arrow category" of $\mathcal{E}$: that is, the category whose objects are weak homotopy equivalences $f : X_0 \to X_1$ of finite simplicial sets, where a morphism from $f : X_0 \to X_1$ to $f' : X_0' \to X_1'$ is a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "f"] & X_1 \arrow[d] \\ X_0' \arrow[u] \arrow[r, "f'"] & X_1'. \end{tikzcd}$$

$(f : X_0 \to X_1) \to (X_0, X_1)$ determines a coCartesian fibration

$$\mathrm{TwArr}(\mathcal{E}) \to \mathcal{E}^{\mathrm{op}} \times \mathcal{E}\,.$$

In particular, we have coCartesian fibrations

$$\mathcal{E}^{\mathrm{op}} \xleftarrow{e_0} \mathrm{TwArr}(\mathcal{E}) \xrightarrow{e_1} \mathcal{E}$$

The fibers of these coCartesian fibrations are weakly contractible (since they have initial objects), so Quillen's Theorem B implies that e_0 and e_1 are weak homotopy equivalences; the diagram of spaces

$$\mathrm{N}(\mathcal{E}^{\mathrm{op}}) \leftarrow \mathrm{N}(\mathrm{TwArr}(\mathcal{E})) \to \mathrm{N}(\mathcal{E})$$

supplies a concrete combinatorial description of the natural equivalence between $\mathrm{N}(\mathcal{E}^{\mathrm{op}})$ and $\mathrm{N}(\mathcal{E})$ (in the ∞-category of spaces). We may therefore reformulate Proposition 6 as follows: the spaces $\mathrm{N}(\mathcal{D}^+ \times_{\mathcal{E}^{\mathrm{op}}} \mathrm{TwArr}(\mathcal{E}))$

and $\mathrm{N}(\mathcal{D} \times_{\mathcal{E}} \mathrm{TwArr}(\mathcal{E}))$ are equivalent (in the ∞-category of spaces over $\mathrm{TwArr}(\mathcal{E})$). Note that we can identify the objects of $\mathcal{D} \times_{\mathcal{E}} \mathrm{TwArr}(\mathcal{E})$ with diagrams of finite simplicial sets $X_0 \xrightarrow{f} X_1 \xhookrightarrow{g} Y$ where f is a weak homotopy equivalence and g is a trivial cofibration, and we can identify the objects of $\mathcal{D}^+ \times_{\mathcal{E}} \mathrm{TwArr}(\mathcal{E})$ with diagrams

$$Y \xleftarrow{h} X_0 \xrightarrow{f} X_1$$

where f and h are weak homotopy equivalences. The construction $(f, g) \mapsto (f, g \circ f)$ determines a functor

$$\mathcal{D} \times_{\mathcal{E}} \mathrm{TwArr}(\mathcal{E}) \to \mathcal{D}^+ \times_{\mathcal{E}^{\mathrm{op}}} \mathrm{TwArr}(\mathcal{E})$$

compatible with the projection to $\mathrm{TwArr}(\mathcal{E})$.

It will therefore suffice to show that this functor is a weak homotopy equivalence. To prove this, it suffices to show that it induces an equivalence on homotopy fibers taken over any point $(f : X_0 \to X_1) \in \mathrm{TwArr}(\mathcal{E})$. Unwinding the definitions, we wish to show that composition with f induces a weak homotopy equivalence

$$\mathcal{D}_{X_1} \to \mathcal{D}^+_{X_0};$$

this follows from Propositions 5 and 4.

A PullBack Square (Lecture 32)

November 24, 2014

Let X be a simplicial set. As before, we let $\mathcal{C}_X$ denote the category whose objects are diagrams

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "\mathrm{id}"] & & X \end{tikzcd}$$

where Y is obtained from X by adding finitely many simplices. Let s denote the collection of cell-like maps in $\mathcal{C}_X$, let h denote the collection of weak homotopy equivalences in $\mathcal{C}_X$, and let $\mathcal{C}_X^h$ denote the full subcategory of $\mathcal{C}_X$ spanned by those objects where the map $X \to Y$ is a weak homotopy equivalence. Our goal in this lecture (and the next) is to complete the second part of this course by establishing the following result:

Proposition 1. *The diagram*

$$\begin{tikzcd} K(\mathcal{C}_X^h, s) \arrow[r] \arrow[d] & K(\mathcal{C}_X^h, h) \arrow[d] \\ K(\mathcal{C}_X, s) \arrow[r] & K(\mathcal{C}_X, h) \end{tikzcd}$$

is a homotopy pullback square.

We will prove Proposition 1 by analyzing the K-theory space $K(\mathcal{C}_X, h)$ (which we know to be homotopy equivalent to $\Omega^\infty A^{\mathrm{free}}(X)$) and eventually showing that it can be identified with the homotopy quotient of $K(\mathcal{C}_X, s)$ by the action of $K(\mathcal{C}_X^h, s)$.

As a first step, it will be convenient to replace $\mathcal{C}_X$ by something slightly closer to $\mathcal{C}_X^h$.

Definition 2. For each integer n, let $\mathcal{C}_X^{(n)}$ denote the full subcategory of $\mathcal{C}_X$ spanned by those objects for which the map $X \to Y$ is n-connected.

Lemma 3. *For each integer n, the inclusion $\mathcal{C}_X^{(n)} \hookrightarrow \mathcal{C}_X$ induces homotopy equivalences*

$$K(\mathcal{C}_X^{(n)}, s) \to K(\mathcal{C}_X, s) \qquad K(\mathcal{C}_X^{(n)}, h) \to K(\mathcal{C}_X, h).$$

Proof. We will give the proof of the second assertion; the proof of the first is similar. When $n = -1$, there is nothing to prove. Proceeding by induction on n, we are reduced to proving that each of the inclusions $\mathcal{C}_X^{(n+1)} \hookrightarrow \mathcal{C}_X^{(n)}$ inducse a homotopy equivalence $K(\mathcal{C}_X^{(n+1)}, h) \to K(\mathcal{C}_X^{(n)}, h)$. Let Y be an object of $\mathcal{C}_X$, given by a diagram

$$\begin{tikzcd} & Y \arrow[dr, "r"] & \\ X \arrow[ur] \arrow[rr, "\mathrm{id}"] & & X. \end{tikzcd}$$

Let $M(r) = (Y \times \Delta^1) \amalg_{Y \times \{1\}} X$ denote the mapping cylinder of r and let $F(Y) = X \amalg_Y M(r)$ denote the two-sided mapping cylinder of r. The construction $Y \mapsto F(Y)$ induces a functor from $\mathcal{C}_X$ to itself which carries $\mathcal{C}_X^{(n)}$ into $\mathcal{C}_X^{(n+1)}$; in particular, it carries both $\mathcal{C}_X^{(n)}$ and $\mathcal{C}_X^{(n+1)}$ to themselves. Note that F preserves cofibrations, pushouts, weak homotopy equivalences, and cell-like maps. It therefore induces maps on K-theory. Applying the two-out-of-six property to the diagram of spaces

$$K(\mathcal{C}_X^{(n+1)}, h) \to K(\mathcal{C}_X^{(n)}, h) \xrightarrow{F} K(\mathcal{C}_X^{(n+1)}, h) \to K(\mathcal{C}_X^{(n)}, h),$$

we are reduced to showing that F induces homotopy equivalences from $K(\mathcal{C}_X^{(n)}, h)$ and $K(\mathcal{C}_X^{(n+1)}, h)$ to themselves. In fact, we claim that on both K-theory spaces F acts by (-1): this follows by applying the additivity theorem to the natural cofiber sequence

$$Y \to M(r) \to F(Y),$$

since the functor $Y \mapsto M(r)$ is related by a cell-like natural transformation to the constrant functor $Y \mapsto X$. $\square$

By virtue of Lemma 3, it will suffice to show that the diagram

$$\begin{tikzcd} K(\mathcal{C}_X^h, s) \arrow[r] \arrow[d] & K(\mathcal{C}_X^h, h) \arrow[d] \\ K(\mathcal{C}_X^{(1)}, s) \arrow[r] & K(\mathcal{C}_X^{(1)}, h) \end{tikzcd}$$

is a homotopy pullback square.

Note that $K(\mathcal{C}_X^{(1)}, h)$ can be obtained as the geometric realization of the simplicial object of Set_Δ given by

$$[n] \mapsto \mathrm{N}(hS_n\, \mathcal{C}_X^{(1)}).$$

Let us fix n for the moment, and consider the category $hS_n\, \mathcal{C}_X^{(1)}$: the objects of this category can be identified with diagrams

$$X \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_n \to X$$

where all but the last map are 1-connected cofibrations (each adding finitely many simplices) and the composition is the identity, and the morphisms are levelwise weak homotopy equivalences. Let us denote such an object simply by $\vec{Y}$. We would like to analyze $hS_n\, \mathcal{C}_X^{(1)}$ in terms of the subcategory where the morphisms are levelwise cell-like maps. To this end, let us consider a bisimplicial set $\mathrm{N}'(hS_n\, \mathcal{C}_X^{(1)})_{\bullet,\bullet}$ whose (p,q)-simplices are diagrams

$$\begin{tikzcd} \vec{Y}_{0,0} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \vec{Y}_{0,q} \arrow[d] \\ \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[d] \\ \vec{Y}_{p,0} \arrow[r] & \cdots \arrow[r] & \vec{Y}_{p,q} \end{tikzcd}$$

where the horizontal maps are levelwise weak homotopy equivalences and the vertical maps are levelwise cell-like.

Lemma 4 (Swallowing Lemma). *In the situation above, the canonical map*

$$\mathrm{N}(hS_n\, \mathcal{C}_X^{(1)})_\bullet \simeq \mathrm{N}'(hS_n\, \mathcal{C}_X^{(1)})_{0,\bullet} \to \mathrm{N}'(hS_n\, \mathcal{C}_X^{(1)})_{\bullet,\bullet}$$

is a homotopy equivalence (after geometric realization).

Proof. It will suffice to show that for each $p \geq 0$, the natural map $\mathrm{N}'(hS_n\,\mathcal{C}_X^{(1)})_{0,\bullet} \to \mathrm{N}'(hS_n\,\mathcal{C}_X^{(1)})_{p,\bullet}$ is a weak homotopy equivalence of simplicial sets. Note that the target can be identified with the nerve of the category $\mathcal{E}$ whose objects are diagrams

$$\vec{Y}_0 \to \vec{Y}_1 \to \cdots \to \vec{Y}_p$$

of (levelwise) cell-like maps in $hS_n\,\mathcal{C}_X^{(1)}$. The diagonal map $hS_n\,\mathcal{C}_X^{(1)} \hookrightarrow \mathcal{E}$ admits a left inverse, given by the construction

$$\vec{Y}_0 \to \vec{Y}_1 \to \cdots \to \vec{Y}_p \mapsto \vec{Y}_0.$$

This left inverse is also a right homotopy inverse by means of the evident natural map

$$\begin{tikzcd}
\vec{Y}_0 \arrow[r, "\mathrm{id}"] \arrow[d] & \vec{Y}_0 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \vec{Y}_0 \arrow[d] \\
\vec{Y}_0 \arrow[r] & \vec{Y}_1 \arrow[r] & \cdots \arrow[r] & \vec{Y}_p.
\end{tikzcd}$$

□

It will be convenient to consider a slightly smaller bisimplicial set. We say that a morphism $\vec{Y} \to \vec{Y}'$ in $hS_n\,\mathcal{C}_X^{(1)}$ is a *cofibration* if the induced map $Y'_i \amalg_{Y_i} Y_{i+1} \to Y'_{i+1}$ is a monomorphism of simplicial sets for each i. Let $\mathrm{N}''(h\,\mathcal{C}_X^{(1)})_{\bullet\bullet}$ denote the bisimplicial set whose objects are diagrams

$$\begin{tikzcd}
\vec{Y}_{0,0} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \vec{Y}_{0,q} \arrow[d] \\
\cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[d] \\
\vec{Y}_{p,0} \arrow[r] & \cdots \arrow[r] & \vec{Y}_{p,q}
\end{tikzcd}$$

where the horizontal maps are cofibrations and levelwise weak homotopy equivalences and the vertical maps are cell-like.

Lemma 5. *The inclusion of bisimplicial sets*

$$\mathrm{N}''(hS_n\,\mathcal{C}_X^{(1)})_{\bullet,\bullet} \hookrightarrow \mathrm{N}'(hS_n\,\mathcal{C}_X^{(1)})_{\bullet,\bullet}$$

is a weak homotopy equivalence (after geometric realization).

Proof. It will suffice to show that for each integer $p \geq 0$, the inclusion

$$\mathrm{N}''(hS_n\,\mathcal{C}_X^{(1)})_{p,\bullet} \hookrightarrow \mathrm{N}'(hS_n\,\mathcal{C}_X^{(1)})_{p,\bullet}$$

is a weak homotopy equivalence. In other words, if we let $\mathcal{E}$ be the category appearing in the proof of Lemma 4 and we let $\mathcal{E}_0 \subseteq \mathcal{E}$ be the subcategory of $\mathcal{E}$ whose morphisms are given by diagrams

$$\begin{tikzcd}
\vec{Y}_0 \arrow[r, "\mathrm{id}"] \arrow[d] & \vec{Y}_1 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \vec{Y}_p \arrow[d] \\
\vec{Y}'_0 \arrow[r] & \vec{Y}'_1 \arrow[r] & \cdots \arrow[r] & \vec{Y}'_p,
\end{tikzcd}$$

where the vertical maps are cofibrations (as well as being weak homotopy equivalences), then we wish to show that the inclusion $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ is a weak homotopy equivalence. Let us assume for simplicity that $p = n = 0$ (the proof in the general case is differs only by notation): then $\mathcal{E}$ is the subcategory of $\mathcal{C}_X^{(1)}$ whose morphisms are weak homotopy equivalences, and $\mathcal{E}_0$ is the subcategory of $\mathcal{C}_X^{(1)}$ whose morphisms are trivial cofibrations. We will prove that the inclusion $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ is a weak homotopy equivalence by showing that it is right cofinal. To this end, fix an object $Y \in \mathcal{E}$; we wish to show that the category $\mathcal{D} = \mathcal{E}_0 \times_{\mathcal{E}} \mathcal{E}_{/Y}$ is weakly contractible. Unwinding the definitions, we can identify the objects of $\mathcal{D}$ with weak homotopy equivalences $f : Y' \to Y$ in $\mathcal{C}_X^{(1)}$. To prove that $\mathcal{D}$ is weakly contractible, it suffices to observe that every such object is connected to the identity map $\mathrm{id} : Y \to Y$ by a canonical zig-zag of trivial cofibrations

$$Y' \hookrightarrow (M(f) \amalg_{X \times \Delta^1} X) \hookleftarrow Y$$

where $M(f) = (Y' \times \Delta^1) \amalg_{Y' \times \{1\}} Y$ is the mapping cylinder of f. $\square$

Let us now reorganize a bit. For each $q \geq 0$, let $F_q(\mathcal{C}_X^{(1)})$ denote the category whose objects are sequences of trivial cofibrations

$$Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_q$$

in $\mathcal{C}_X^{(1)}$. Then we can regard $F_q\, \mathcal{C}_X^{(1)}$ as a category with cofibrations (defined as above, with the roles of n and q switched) and weak equivalences (given by the collection s of levelwise cell-like maps). This category with cofibrations and weak equivalences depends functorially on $[q]$, so we can regard $F_\bullet\, \mathcal{C}_X^{(1)}$ as a *simplicial* category with cofibrations and weak equivalences. Unwinding the definitions, we have

$$K(F_q\, \mathcal{C}_X^{(1)}) \simeq |\, \mathrm{N}(hS_\bullet\, \mathcal{C}_X^{(1)})_{\bullet,q}|.$$

Passing to the geometric realization as $[q]$ varies and invoking Lemmas 4 and 5, we obtain a homotopy equivalence

$$K(\mathcal{C}_X^{(1)}, h) \simeq |K(F_\bullet\, \mathcal{C}_X^{(1)}, s)|.$$

Given a cofibration $Y \hookrightarrow Y'$ in $\mathcal{C}_X$, let Y'/Y denote the pushout $Y' \amalg_Y X$. It is clear that if $Y \hookrightarrow Y'$ is a weak homotopy equivalence, then the quotient Y'/Y is weakly homotopy equivalent to X. If Y' and Y both belong to $\mathcal{C}_X^{(1)}$, then the converse holds: this follows from the observation that for any local system of abelian groups $\mathcal{A}$ on X, we have an isomorphism

$$\mathrm{H}_*(Y', Y; \mathcal{A}\,|_{Y'}) \simeq \mathrm{H}_*(Y'/Y, X; \mathcal{A}\,|_{Y'/Y}).$$

It follows that $F_q(\mathcal{C}_X^{(1)})$ admits an alternative description: it can be identified with the category whose objects are sequences of cofibrations

$$Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_q$$

in $\mathcal{C}_X^{(1)}$ where each quotient Y_i/Y_{i-1} belongs to $\mathcal{C}_X^h$.

There is a natural map

$$\theta_q : \mathcal{C}_X^{(1)} \times (\mathcal{C}_X^h)^q \to F_q(\mathcal{C}_X^{(1)}),$$

given on objects by

$$(Y, (Z_1, \ldots, Z_q)) \mapsto (Y \hookrightarrow Y \amalg_X Z_1 \hookrightarrow \cdots \hookrightarrow Y \amalg_X Z_1 \amalg_X \cdots \amalg_X Z_q)$$

This induces a map on K-theory spaces

$$K(\mathcal{C}_X^{(1)}, s) \times K(\mathcal{C}_X^h, s)^q \to K(F_q(\mathcal{C}_X^{(1)}), s).$$

Passing to the geometric realization as q varies, we obtain a map

$$K(\mathcal{C}_X^{(1)}, s)/K(\mathcal{C}_X^h, s)^q \to |K(F_\bullet\, \mathcal{C}_X^{(1)}, s)| \simeq K(\mathcal{C}_X^{(1)}, h).$$

To prove Proposition 1, it will suffice to show that this map is a homotopy equivalence. In fact, we will prove something stronger: each of the maps θ_q induces a homotopy equivalence at the level of K-theory. Note that θ_q has a left homotopy inverse ρ, given by the construction

$$(Y_0 \hookrightarrow \cdots \hookrightarrow Y_q) \mapsto (Y_0, (Y_1/Y_0, \cdots, Y_q/Y_{q-1})).$$

The composition $\theta_q \circ \rho$ is not homotopic to the identity at the level of categories, but induces the identity map on K-theory spaces (up to homotopy) by virtue of the additivity theorem.

References

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Digression: Microbundles (Lecture 33)

December 1, 2014

Let M be a smooth manifold. To each point $x \in M$, one can associate a real vector space $T_{M,x}$, called the *tangent space* to x in M. The union $\bigcup_{x \in M} T_{M,x}$ can be regarded as a vector bundle over M, which we denote by T_M and refer to as the *tangent bundle* of M.

If M is a topological manifold which does not have a smooth structure, then one generally cannot associate to M a tangent vector bundle. To remedy the situation, Milnor introduced the theory of *microbundles*.

Definition 1 (Milnor)**.** Let B be a topological space. An *topological microbundle* on X (of rank n) is a map $p : E \to B$ equipped with a section $s : B \to E$ satisfying the following condition:

($*$) For every point $x \in B$, there exists a neighborhood of $U \subseteq B$ containing x and an open subset of E homeomorphic to $U \times \mathbb{R}^n$, such that the section s can be identified with the zero section $U \simeq U \times \{0\} \hookrightarrow U \times \mathbb{R}^n$.

An *equivalence* of microbundles E and E' over B is a homeomorphism $h : U \simeq U'$ fitting into a commutative diagram

$$\begin{tikzcd} U \arrow[rr, "h"] \arrow[dr] & & U' \arrow[dl] \\ & X, & \end{tikzcd}$$

where U is an open subset of E containing the image of the section $s : B \to E$, U' is an open subset of E' containing the image of $s' : B \to E'$, and we have $h \circ s = s'$.

Example 2. Let M be a topological manifold. The *tangent microbundle* T_M is defined to be the product $M \times M$, mapping to M via the projection $\pi_1 : M \times M \to M$, with section $s : M \to M \times M$ given by the diagonal map.

Variant 3. In Definition 1, we can require E and B to be (not necessarily finite) polyhedra and all of the relevant homeomorphisms to be PL maps. This gives us the notion of a *PL microbundle* over a polyhedron B.

If M is a PL manifold, then the product $M \times M$ can be regarded as a PL microbundle over M (denoted also by T_M).

Variant 4. In Definition 1, we can require p to be a smooth submersion: that is, we can require E to admit a system of charts homeomorphic to $U \times \mathbb{R}^n$ where U is an open subset of B, where the transition functions between charts are continuous in the first variable and infinitely differentiable in the second. This leads to the notion of a *smooth microbundle* over B.

If M is a smooth manifold, then the product $M \times M$ can be regarded as a smooth microbundle over M (again denoted by T_M).

Example 5. Let $\mathcal{E}$ be a vector bundle over a space B. Then $\mathcal{E}$ can be regarded as a smooth microbundle over B (since linear maps are infinitely differentiable).

Conversely, if $E \to B$ is a smooth microbundle with zero section $s : B \to E$, then the relative tangent bundle $T_{E/B}$ is a vector bundle over E whose pullback $s^* T_{E/B}$ can be regarded as a vector bundle over B. This construction determines a map

$$\{ \text{ smooth microbundles over } B\}/ \text{ equivalence } \to \{ \text{ vector bundles over } B\}/ \text{ isomorphism } .$$

It is easy to see that this construction is left inverse to the construction which regards each vector bundle as a smooth microbundle. If B is paracompact, then it is also a right inverse: in other words, any smooth microbundle $E \to B$ is equivalent (as a smooth microbundle) to the vector bundle $s^* T_{E/B}$. One can produce an equivalence by choosing a Riemannian metric on each fiber of E (depending continuously on B) and using it to define a map $s^* T_{E/B} \to E$ by means of the "exponential spray".

Definition 6. Let B be a topological space. A $\mathrm{Top}(n)$-*bundle over* B is a fiber bundle $p : E \to B$ whose fibers are homeomorphic to $\mathbb{R}^n$.

Remark 7. If B is paracompact (which we should always assume here), then any $\mathrm{Top}(n)$-bundle $E \to B$ admits a section and can therefore be regarded as a topological microbundle over B.

Variant 8. If B is a polyhedron, then we also have the notion of a $\mathrm{PL}(n)$-bundle over B: that is, a map of polyhedra $E \to B$ which is locally PL isomorphic to the product of the base with $\mathbb{R}^n$. Any $\mathrm{PL}(n)$-bundle can be regarded as a PL microbundle.

Theorem 9 (Kister-Mazur)**.** *Let B be paracompact. Then the natural map*

$$\{ \text{ Top}(n)\textit{-bundles over } B\}/ \textit{ isomorphism } \to \{ \textit{ rank } n \textit{ microbundles over } B\}/ \textit{ equivalence}$$

is bijective. In other words, every topological microbundle admits an essentially unique refinement to a $\mathrm{Top}(n)$*-bundle.*

Variant 10 (Kuiper-Lashof)**.** Let B be a polyhedron. Then the natural map

$$\{ \text{ PL}(n)\text{-bundles over } B \ \}/ \text{ isomorphism } \to \{ \text{ rank } n \text{ PL microbundles over } B \ \}/ \text{ equivalence}$$

is bijective. In other words, every PL microbundle admits an essentially unique refinement to a $\mathrm{PL}(n)$-bundle.

Example 11. Let M be a (paracompact) topological manifold. Then every point $x \in M$ admits an open neighborhood U_x which is homeomorphic to a Euclidean space $\mathbb{R}^n$. It follows from Theorem 9 that the open sets U_x can be chosen "uniformly" so that the disjoint union $\amalg_{x \in M} U_x = \{(x, y) \in M \times M : y \in U_x\}$ is an open subset of $M \times M$ (to get a feeling for the content of Theorem 9, try proving this directly).

We can regard $\mathrm{Top}(n)$ and $\mathrm{PL}(n)$ as simplicial groups, whose k-simplices are homeomorphisms (required to be PL in the second case) of $\mathbb{R}^n \times \Delta^k$ with itself which commute with projection onto the second factor. The classifying spaces $\mathrm{BTop}(n)$ and $\mathrm{BPL}(n)$ classify $\mathrm{Top}(n)$-bundles and $\mathrm{PL}(n)$-bundles, respectively. By virtue of the above results, we can think of $\mathrm{BTop}(n)$ and $\mathrm{BPL}(n)$ as classifying spaces for topological and PL microbundles of rank n, respectively. Similarly, $\mathrm{BO}(n)$ is a classifying space for smooth microbundles of rank n.

Overview of Part 3 (Lecture 34)

December 3, 2014

We begin with the following:

Question 1. Let X be a space. Under what circumstances does X have the homotopy type of a compact manifold M?

The answer to this question depends heavily on exactly what sorts of manifolds we allow. If we require M to be a *closed* manifold, then this places strong constraints on the space X: it must satisfy Poincare duality. Let us therefore be a bit more liberal, and allow M to be a manifold with boundary. In this case, there is a simple necessary condition:

Claim 2. *Let M be a compact manifold with boundary. Then M is homotopy equivalent to a finite CW complex.*

Proof. If M is a piecewise-linear manifold, this is clear: any finite polyhedron M is actually *homeomorphic* to a finite CW complex, with a CW structure given by a choice of triangulation of M.

If M is a smooth manifold, then we can choose a Whitehead triangulation of M and thereby reduce to the piecewise-linear case.

If M is merely a topological manifold, then Claim 2 is nontrivial (as we remarked in Lecture 2). It is easy to see that M is a *finitely dominated* space, but it is not easy to show that its Wall finiteness obstruction vanishes. This follows from a theorem of Chapman, which asserts that for any compact ANR M, there exists a finite polyhedron N and a homeomorphism $M \times Q \simeq N \times Q$, where Q is the Hilbert cube. □

This necessary condition turns out to be sufficient. First, we note that any CW complex is homotopy equivalent to a finite polyhedron (since the collection of homotopy types of finite polyhedra contains the empty and one point space and is closed under the formation of homotopy pushouts). If X is a finite polyhedron, then we can choose a PL embedding $X \hookrightarrow \mathbb{R}^n$ for n sufficiently large.

Definition 3. Let $X \subseteq \mathbb{R}^n$ be a finite polyhedron. A *regular neighborhood* of X is a finite polyhedron $N \subseteq \mathbb{R}^n$ satisfying the following conditions:

- The polyhedron X is contained in the interior of N.
- The polyhedron N is a PL manifold with boundary.
- The inclusion of X into N can be written as a composition of (polyhedral) elementary expansions (in other words, N "collapses" onto X).

Theorem 4. *Let $X \subseteq \mathbb{R}^n$ be a finite polyhedron. Then there exists a regular neighborhood of X in $\mathbb{R}^n$. Moreover, if N and N' are two regular neighborhoods of X in $\mathbb{R}^n$, then there exists a PL homeomorphism of N with N' which is the identity on X (in fact, one can be more precise: there is a PL isotopy of $\mathbb{R}^n$ which carries N to N', and is the identity on X).*

Theorem 4 supplies an answer to Question 1:

Corollary 5. *Let X be a finite CW complex. Then there exists a homotopy equivalence $X \simeq N$, where N is a compact manifold with boundary.*

For the existence part of Theorem 4, choose a cube $C \subseteq \mathbb{R}^n$ which contains X in its interior, and choose a triangulation $\Sigma(C)$ of C which restricts to a triangulation $\Sigma(X)$ of X. Let $\Sigma'(C)$ denote the barycentric subdivision of this triangulation and $\Sigma''(C)$ the barycentric subdivision of $\Sigma'(C)$. We can then take N to be the union of those (closed) simplices of $\Sigma''(C)$ which intersect X. For a proof that this construction works (and of the uniqueness asserted in Theorem 4), we refer the reader to [3].

Exercise 6. Let X be the boundary of a 2-simplex σ embedded in $\mathbb{R}^2$. Then we can choose a triangulation of $\mathbb{R}^2$ which includes σ as a simplex. Contemplate this example to appreciate the need to take a *second* barycentric subdivision in the construction sketched above.

The deduction of Corollary 5 from Theorem 4 actually yields more precise information:

(a) The homotopy equivalence $X \simeq N$ can be chosen to be a *simple* homotopy equivalence (after replacing X by a finite polyhedron, we can arrange that the inclusion $X \hookrightarrow N$ is a composition of elementary expansions).

(b) The manifold N can be chosen to be piecewise-linear.

(c) The manifold N can be chosen to have trivial tangent microbundle T_N.

Remark 7. If N is a PL manifold with boundary, then the projection map $N \times N \to N$ is not a PL microbundle in the sense of the previous lecture, because N is not locally homeomorphic to $\mathbb{R}^n$ on its boundary. However, if N° denotes the interior of N, then the projection $N \times N^\circ \to N^\circ$ is a PL microbundle over N°, and the inclusion $N^\circ \hookrightarrow N$ is a homotopy equivalence; this determines a PL microbundle on N, which we denote by T_N and refer to as the *tangent microbundle* to N.

Definition 8. Let N be a PL manifold of dimension n (possibly with boundary). A *parallelization* of N is a microbundle equivalence of T_N with $\mathbb{R}^n \times N$.

More generally, suppose that $p : E \to B$ is a PL fiber bundle whose fibers are PL manifolds of dimension n. Then the projection map $E \times_B E \to E$ can be regarded as a PL microbundle over E (at least away from the boundary), which we denote by $T_{E/B}$. A *parallelization* of $E \to B$ is an equivalence of microbundles $T_{E/B} \simeq E \times R^n$.

Construction 9. For each integer $n \geq 0$, we define a simplicial set $\mathcal{M}^n$ as follows: a k-simplex of $\mathcal{M}^n$ consists of a finite polyhedron $E \subseteq \Delta^k \times \mathbb{R}^\infty$ for which the projection $E \to \Delta^k$ is a PL fiber bundle whose fibers are PL manifolds (with boundary) of dimension n, together with a parallelization of E.

In what follows, we will generally abuse notation and identify k-simplices fo $\mathcal{M}^n$ with the PL fiber bundle $E \to \Delta^k$, regarding the embedding $E \hookrightarrow \Delta^k \times \mathbb{R}^\infty$ and the parallelization of E as implicitly specified.

The construction

$$(E \to \Delta^k) \mapsto (E \times [0,1] \to \Delta^k)$$

determines a *stabilization map* $\sigma_n : \mathcal{M}^n \to \mathcal{M}^{n+1}$. We let $\mathcal{M}^\infty$ denote the direct limit of the sequence

$$\mathcal{M}^0 \to \mathcal{M}^1 \to \mathcal{M}^2 \to \cdots .$$

Note that every fiber bundle $E \to \Delta^k$ as in Construction 9 is, in particular, a fibration. Consequently, we have evident forgetful functors $\theta_n : \mathcal{M}^n \to \mathcal{M}$. Moreover, the diagrams

$$\begin{tikzcd} \mathcal{M}^n \arrow[rr, "\sigma_n"] \arrow[dr, "\theta_n"'] & & \mathcal{M}^{n+1} \arrow[dl, "\theta_{n+1}"] \\ & \mathcal{M} & \end{tikzcd}$$

commute up to *canonical* homotopy, so that the maps θ_n can be amalgamated to a map

$$\theta : \mathcal{M}^{\infty} \to \mathcal{M}.$$

Our objective in the next part of this course is to prove the following result:

Theorem 10. *The map $\theta : \mathcal{M}^{\infty} \to \mathcal{M}$ is a homotopy equivalence.*

Remark 11. The simplest consequence of Theorem 10 is that the map θ is surjective on connected components. This asserts that every point of $\mathcal{M}$ (given by a finite polyhedron X) can be connected by a path in $\mathcal{M}$ (that is, a simple homotopy equivalence) to a point lying in the image of θ (that is, a polyhedron which is a parallelized PL manifold with boundary). This is equivalent to the contents of Corollary 5, together with the strengthenings (a), (b), and (c) indicated above.

Theorem 10 is a stronger result: it asserts that for a finite polyhedron X, not only can we choose a simple homotopy equivalence $X \simeq N$ to a parallelized PL manifold N, but that modulo "stabilization" (given by iterated product with $[0,1]$), the PL manifold N is unique up to a contractible space of choices. In particular, if N and N' are parallelized PL manifolds equipped with simple homotopy equivalences to X (and therefore to each other), then we can find integers $a, b \geq 0$ and a PL homeomorphism $N \times [0,1]^a \simeq N' \times [0,1]^b$. This recovers a form of the uniqueness asserted in Theorem 4.

We can summarize the situation informally by saying that Theorem 10 can be regarded as providing a *parametrized* version of regular neighborhood theory, at least after stabilization.

Combining Theorem 10 with the main result of the second part of this course, we obtain the following:

Corollary 12. *Let X be a finitely dominated space. Then there is a homotopy equivalence*

$$\mathcal{M}^{\infty} \times_{\mathcal{M}^h} \{X\} \simeq \Omega^{\infty}(X_+ \wedge A(*)) \times_{\Omega^{\infty} A(X)} \{[X]\}.$$

One can ask analogous questions in the setting of *smooth* manifolds. For each $n \geq 0$, one can introduce a simplicial set $\mathcal{M}^n_{\mathrm{sm}}$ analogous to $\mathcal{M}^n$, whose k-simplices are given by *smooth* submersions $E \to \Delta^k$ of parallelized manifolds. The direct limit $\mathcal{M}^{\infty}_{\mathrm{sm}} = \varinjlim \mathcal{M}^n_{\mathrm{sm}}$ maps to $\mathcal{M}$ via a map $\theta_{\mathrm{sm}} : \mathcal{M}^{\infty}_{\mathrm{sm}} \to \mathcal{M}$, but the map θ_{sm} is *not* a homotopy equivalence. However, we will show that the relationship between $\mathcal{M}^{\infty}_{\mathrm{sm}}$ and $\mathcal{M}^h$ is also governed by an A-theory assembly map. More precisely, we will prove the following version of Corollary 12:

Variant 13. Let X be a finitely dominated space. Then the homotopy fiber product $\mathcal{M}^{\infty}_{\mathrm{sm}} \times_{\mathcal{M}^h} \{X\}$ can be identified with the homotopy fiber of the map $u : \Omega^{\infty}\Sigma^{\infty}_{+} X \to \Omega^{\infty} A(X)$ over the point $[X] \in \Omega^{\infty} A(X)$. Here u is given by composing the A-theory assembly map with map $\Sigma^{\infty}_{+} X \to X_+ \wedge A(*)$ determined by the unit map $S \to A(*)$.

To understand the relationship between Corollary 12 and Variant 13, let us consider the problem of *smoothing* a PL manifold with boundary. For any PL n-manifold with boundary M, the tangent microbundles of M and ∂M are classified by a map of pairs

$$\chi : (M, \partial M) \to (\mathrm{BPL}(n), \mathrm{BPL}(n-1)).$$

In order to choose a smooth structure on M, we need to factor this classifying map through the pair $(\mathrm{BO}(n), \mathrm{BO}(n-1))$.

If we assume that M is equipped with a parallelization, then the situation simplifies: a parallelization of M is a nullhomotopy of the map $M \to \mathrm{BPL}(n)$, so that we can regard χ as a map from ∂M to the homotopy fiber $\mathrm{fib}(\mathrm{BPL}(n-1) \to \mathrm{BPL}(n)) = \mathrm{PL}(n)/\mathrm{PL}(n-1)$. In order to lift M to a (parallelized) smooth manifold, we need to factor this map through the quotient $\mathrm{fib}(\mathrm{BO}(n-1) \to \mathrm{BO}(n)) \simeq O(n)/O(n-1) \simeq S^{n-1}$.

Note that the sequence of spaces $\{O(n+1)/O(n)\}_{n \geq 0}$ can be regarded as a prespectrum which represents the sphere spectrum S. There is an analogous result for the groups $\mathrm{PL}(n)$:

Theorem 14. *The sequence of spaces $\{\mathrm{PL}(n+1)/\mathrm{PL}(n)\}_{n\geq 0}$ can be regarded as a prespectrum, whose associated spectrum is $A(*)$.*

For any parallelized PL n-manifold M, the classifying map χ above determines a map

$$\partial M \to \mathrm{PL}(n)/\mathrm{PL}(n-1) \to \Omega^{\infty-n+1}A(*),$$

whch we can regard as an element of $\Omega^{\infty-n+1}A(*)^{\partial M}$.

Theorem 15. *The boundary map*

$$\Omega^{\infty-n+1}A(*)^{\partial M} \to \Omega^{\infty-n}A(*)^{M/\partial M}$$

carries the classifying map χ defined above to the image of $\langle M\rangle$ under the Atiyah duality map $\Omega^{-n}A(*)^{M/\partial M} \simeq (M_+ \wedge A(*))$.

We will see that Variant 13 is a formal consequence of Theorem 15, since the natural maps $\mathrm{BO}(n) \to \mathrm{BPL}(n)$ give rise to a map of prespectra

$$S^n \simeq O(n+1)/O(n) \to \mathrm{PL}(n+1)/\mathrm{PL}(n)$$

which represents the unit map $S \to A(*)$.

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