Differentiable Manifolds Which Are Homotopy Spheres

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§1. Introduction

This paper will study the problem of classifying differentiable n-manifolds which are homotopy spheres, under the relation of J-equivalence. (See the "dictionary" below.) It is shown that the equivalence classes form an abelian group which is denoted by Θ^n. The only groups Θ^n which I have been able to determine completely are the following:

$$\Theta^1 = \Theta^2 = 0, \quad \Theta^5 = 0, \quad \Theta^7 = Z_{28}, \quad \Theta^{11} = Z_{992} .$$

However partial information is obtained in many other cases. For example (according to 3.7, 5.8 and 6.9):

<u>Theorem</u>. For $k > 1$ the group Θ^{4k-1} is finite but non-trivial.

Section 2 of this paper will study a sum operation for connected manifolds of the same dimension. Section 3 defines an invariant λ' for certain (4k-1)-manifolds. Section 4 contains examples of homotopy spheres for which the invariant λ' takes on all possible values.

Section 5 describes a construction for simplifying manifolds, which was communicated to the author by R. Thom. Using this construction it is shown that the invariant $\lambda'(M)$ determines the J-equivalence class of M uniquely. A corresponding result for dimensions of the form $4k + 1$ is stated without proof. Section 6 studies the following question: Is every homotopy sphere the boundary of a π-manifold?

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Section 7 contains further discussion and a list of unsolved problems. Operations of "pasting together" manifolds and "straightening angles" are described in an appendix.

Dictionary of terms used. The word manifold will mean a compact, oriented, differentiable manifold, with or without boundaries. (The phrase "topological manifold" will be used in case the differentiable structure has not yet been specified.) The symbol $-M$ will be used for the manifold M with orientation reserved.

Two unbounded manifolds M_1, M_2 of the same dimension are J-equivalent if there exists a manifold W such that

1) the boundary ∂W is the disjoint union of M_1 and $-M_2$, and

2) both M_1 and M_2 are deformation retracts of W.

Thus J-equivalent manifolds belong to the same cobordism class and to the same homotopy type. This concept is due to Thom [3]. It is not known whether J-equivalent manifolds are necessarily diffeomorphic.

By a homotopy sphere we mean a (differentiable) manifold without boundary which has the homotopy type of a sphere. Similarly a homology sphere M must be unbounded and satisfy $H_*(M) \approx H_*(S^n)$. Here H_* denotes homology with integer coefficients, and S^n denotes the unit sphere in Euclidean space R^{n+1}. The notation D^{n+1} will be used for the disk bounded by S^n.

1.1 Lemma. Let $M^n = \partial W^{n+1}$ where M^n is simply connected and W^{n+1} is contractible. Then M^n is J-equivalent to S^n.

Proof. Choose an imbedding of D^{n+1} in the interior of W^{n+1}. Then $(W^{n+1}$-interior $(D^{n+1}))$ has* boundary equal to the disjoint union

*Here the symbol - stands for set theoretic subtraction.

of M^n and S^n. It is not difficult to see that both boundaries are deformation retracts of W^{n+1}-interior (D^{n+1}).

A π-manifold W^n is characterized by the following property. If W^n is imbedded in a high dimensional Euclidean space R^{n+q}, then the normal bundle ν^q is trivial. This concept is due to J. H. C. Whitehead [2]. If W is a π-manifold, then clearly ∂W is also a π-manifold.

W^n will be called almost parallelizable if there exists a finite subset F so that $W^n - F$ is parallelizable.

1.2 Lemma (J.H.C.Whitehead) Every parallelizable manifold is a π-manifold. Every π-manifold is almost parallelizable.

Proof. A field of tangent n-frames on $W^n \subset R^{n+q}$ induces a map f from W^n to the Stiefel manifold $V_{n+q,n}$. Note that f is covered by a bundle map from ν^q to a corresponding SO_q-bundle over $V_{n+q,n}$. But the space $V_{n+q,n}$ is (q-1)-connected. (See Steenrod [1] §25.6.) For $q > n$ this implies that f is homotopic to a constant; hence that ν^q is trivial.

Similarly a field of normal q-frames on W^n induces $f: W^n \to V_{n+q,q}$. Since $V_{n+q,q}$ is (n-1)-connected, the only obstruction to contracting f lies in

$$H^n(W^n; \pi_n(V_{n+q,q})).$$

But this cohomology group can be killed by removing a finite number of points from W^n.

A similar argument shows the following.

1.3 Lemma. If every component of W^n has a non-vacuous boundary, then the three concepts: parallelizable, π-manifold, and almost parallelizable, are equivalent.

The J-homomorphism of H. Hopf and G. Whitehead will be denoted by

$$J_n: \pi_n(SO_q) \longrightarrow \pi_{n+q}(S^q) .$$

(For a definition see Kervaire [4] §1.8. Caution: this homomorphism has nothing to do with J-equivalence.) It will always be assumed that q is large. This homomorphism will play a fundamental role in what follows.

§2. The connected sum of manifolds

Let M_1, M_2 be connected differentiable manifolds of the same dimension n. The sum $M_1 \# M_2$ is obtained by removing an n-cell from each, and then pasting the resulting boundaries together. There are three difficulties with this:

1) The pasting must be done in such a way that $M_1 \# M_2$ has an orientation compatible with that of both M_1 and M_2.

2) Even allowing for orientation, not every diffeomorphism between the boundaries will give rise to the same composite manifold. (According to Milnor [1] it is possible to paste together the boundaries of two 7-cells, obtaining a manifold which is not diffeomorphic to S^7.)

3) It is necessary to show that the result does not depend on which n-cell is chosen.

Definition. Choose an orientation preserving imbedding $h_1: R^n \to M_1$ and an orientation reversing imbedding $h_2: R^n \to M_2$. Let $M_1 \# M_2$ be obtained from the disjoint union of $M_1 - h_1(0)$ and $M_2 - h_2(0)$ by identifying $h_1(x)$ with $h_2(x/\|x\|^2)$ for each $x \neq 0$ in R^n.

Remark. It would be sufficient to specify $h_1(x)$ and $h_2(x)$ for $\|x\| < 1 + \varepsilon$ in order to construct this manifold $M_1 \# M_2$. In fact by removing all $h_i(x)$ with $\|x\| \leq 1/(1 + \varepsilon)$ from each M_i, and then identifying $h_1(x)$ with $h_2(x/\|x\|^2)$ for $1+\varepsilon > x > 1/(1+\varepsilon)$, we obtain the identical manifold $M_1 \# M_2$.

The following will be proved in a paper by J. Cerf.

2.1 Theorem of Cerf. Let M be a connected n-manifold. Given two orientation preserving imbeddings $f,f':D^n \longrightarrow$ (interior M), there exists a diffeomorphism $g:M \longrightarrow M$ which satisfies $gf = f'$.

2.2 Corollary. The sum $M_1 \# M_2$ is well defined up to orientation preserving diffeomorphism.

Proof of the corollary. The only choice which occurred in the definition was the choice of imbeddings h_1,h_2 . Given other imbeddings h_1', h_2', there exist diffeomorphisms g_i of M_i so that

$$g_i h_i(x) = h_i'(x) \quad \text{for} \quad \|x\| \leqq 1 + \varepsilon \ .$$

These g_i give rise to a diffeomorphism $g:M_1 \# M_2 \longrightarrow (M_1 \# M_2)'$; which completes the proof.

2.3 Lemma. Suppose that the unbounded manifolds M_1,M_2 are J-equivalent to M_1' and M_2' respectively. Then the sum $M_1 \# M_2$ is J-equivalent to $M_1' \# M_2'$.

Proof. If the dimension n is $\leqq 2$, then the assertion is clear. Hence we may assume that $n \geqq 3$. Choose manifolds W_i so that ∂W_i is the disjoint union of the deformation retracts M_i and $-M_i'$. Choose a differentiable arc a_i from $p_i \varepsilon M_i$ to $p_i' \varepsilon M_i'$ in W_i , so that the interior of a_i lies in the interior of W_i . We will see that the inclusion map

$$j: M_i - p_i \longrightarrow W_i - a_i$$

is a homotopy equivalence.

Since the codimension n of p_i in M_i is $\geqq 3$, the homomorphisms $\pi_1(M_i - p_i) \longrightarrow \pi_1(M_i)$, $\pi_1(W_i - a_i) \longrightarrow \pi_1(W_i)$ are isomorphisms. Hence

$$j_*: \pi_1(M_i - p_i) \longrightarrow \pi_1(W_i - a_i)$$

is an isomorphism.

Let $\hat{M}_i \subset \hat{W}_i$ denote the universal covering spaces, and let $\hat{p}_i \subset \hat{a}_i$ denote the inverse images of p_i, a_i. The inclusion

$$(\hat{M}_i, \hat{M}_i - \hat{p}_i) \longrightarrow (\hat{W}_i, \hat{W}_i - \hat{a}_i)$$

gives rise to a homomorphism between exact sequences of homology groups. Using the Five Lemma it follows that

$$\hat{j}_*: H_k(\hat{M}_i - \hat{p}_i) \longrightarrow H_k(\hat{W}_i - \hat{a}_i)$$

is an isomorphism for all k. Therefore j is a homotopy equivalence. (Compare J.H.C.Whitehead [3].)

Choose tubular neighborhoods N_i of a_i, and let W be a manifold obtained from $W_1 - N_1$ and $W_2 - N_2$ by pasting together the boundaries in such a way that ∂W is the disjoint union of $M_1 \# M_2$ and $-(M_1' \# M_2')$. Since the inclusions

$$M_i - (M_i \cap N_i) \longrightarrow W_i - N_i$$

are homotopy equivalences, it follows easily that the inclusion

$$M_1 \# M_2 \longrightarrow W$$

is a homotopy equivalence. A corresponding argument takes care of the inclusion $(M_1' \# M_2') \longrightarrow W$. This completes the proof of 2.3.

It is clear that the operation # is associative and commutative, providing that we do not distinguish between diffeomorphic manifolds. Furthermore the sphere acts as a zero element: $M \# S^n \approx M$.

2.4 Lemma. Suppose that M is a homotopy n-sphere. Then $M \# (-M)$ is J-equivalent to S^n .

Proof. Let U denote the interior of a disk $D^n \subset M$. Consider the topological manifold $(M-U) \times [0,1]$. This is differentiable, except along the "angles" $\partial U \times [0]$ and $\partial U \times [1]$. Let W be a differentiable manifold obtained from $(M-U) \times [0,1]$ by straightening these angles. (See the Appendix.) Then W is a contractible manifold with boundary $M \# (-M)$. Together with 1.1 this completes the proof.

Now combining 2.3 and 2.4 this proves:

2.5 Theorem. The set of all J-equivalence classes of homotopy n-spheres forms an abelian group under the operation $\#$.

This group will be denoted by Θ^n. It is clear that $\Theta^1 = 0$. Since Munkres [1] has shown that a 2-manifold has an essentially unique differentiable structure, it follows that $\Theta^2 = 0$.

[Two subgroups of Θ^n will also be studied. $\Theta^n(\pi)$ will denote the subgroup formed by all π-manifolds in Θ^n, and $\Theta^n(\partial\pi)$ will denote the subgroup formed by all boundaries of π-manifolds.]

§3. The invariant $\lambda'(M^{4k-1})$

Let M be a (4k-1)-manifold which is (1) a homology sphere, and (2) the boundary of some π-manifold W . The intersection number of two homology class α, β of W will be denoted by $\langle\alpha,\beta\rangle$. Let $I(W)$ denote the index of the quadratic form

$$\alpha \longrightarrow \langle \alpha,\alpha \rangle ,$$

where α varies over the Betti group $H_{2k}(W)/(\text{torsion})$. Integer coefficients are to be understood.

Define I_k as the greatest common divisor of $I(M)$ where M ranges over all almost parallelizable manifolds of dimension 4k which have no boundary. This number has been studied by Kervaire and Milnor [1]. (See 3.7.)

3.1 Lemma. The residue class of $I(W)$ modulo I_k is an invariant of the boundary M .

Proof. If M is the boundary of two parallelizable manifolds W_1 and W_2, let N be the unbounded 4k-manifold obtained from W_1 and $-W_2$ by pasting together the common boundary. Clearly

$$I(N) = I(W_1) - I(W_2) .$$

Let p be a point of M . Then the complement N-p is parallelizable. In fact N-p is the union of parallelizable manifolds W_1-p and W_2-p , having an intersection M-p which is acyclic. Given a field of 4k-frames on W_1-p and on W_2-p , it is possible to deform one of the two so that they coincide along M-p . Therefore N is almost parallelizable; and

$$I(N) \equiv 0 \quad (\mathrm{mod}\ I_k) .$$

This completes the proof.

Not every residue class can occur:

3.2 Lemma. The index $I(W)$ of an almost parallelizable manifold is always divisible by 8; providing that ∂W is a homology sphere.

Proof. First observe that the intersection number $< \alpha, \alpha >$ is always an even integer. This is the homology translation of the statement that

$$Sq^{2k} : H^{2k}(W, \partial W ; Z) \longrightarrow H^{4k}(W, \partial W ; Z_2)$$

is zero. If Sq^{2k} were not zero then the formulae of Wu (see Wu [1], Kervaire [2]) would imply that W had a non-trivial Stiefel-Whitney class in dimension $\leqq 2k$.

Since ∂W is a homology sphere it follows by Poincare duality that the matrix of intersection numbers has determinant ± 1. But a quadratic form with determinant ± 1 which takes on only even values must have index divisible by 8. (Compare Milnor [4].) This completes the proof.

Definition. The residue class of $\frac{1}{8} I(W)$ modulo $\frac{1}{8} I_k$ will be denoted by $\lambda'(M)$.

3.3 Lemma. The properties of being (1) a homotopy n-sphere, and (2) the boundary of a π-manifold, are invariant under J-equivalence; and are preserved by the sum operation #.

Hence the manifolds which have these properties give rise to a subgroup of Θ^n.

Definition. This subgroup will be denoted by $\Theta^n(\partial \pi)$.

3.4 Lemma. The invariant $\lambda'(M)$ depends only on the J-equivalence class of M. Furthermore

$$\lambda'(M_1 \# M_2) = \lambda'(M_1) + \lambda'(M_2).$$

The proofs of 3.3 and 3.4 are straightforward. Hence λ' gives rise to a homomorphism

$$\Lambda': \Theta^{4k-1}(\partial \pi) \longrightarrow Z_{\frac{1}{8} I_k}$$

It will be proved in Sections 4, 5 that Λ' is an isomorphism, at least for $k > 1$.

The principal difficulty with the invariant λ' is that it is extremely difficult to compute. For example it would be very interesting to evaluate λ' for the topological spheres which are constructed in Milnor [1, 5] and Shimada [1]. The invariant λ which is defined in these papers is somewhat weaker, but much easier to compute.

The numbers $\frac{1}{8} I_k$ can be described as follows. Let B_k denote the k-th Bernoulli number:

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \ldots, \quad B_6 = 691/2730, \ldots .$$

Define j_k as the order of the image

$$J_{4k-1}(SO_q) \subset \pi_{q+4k-1}(S^q) \qquad \text{for large } q .$$

Define a_k to be 2 if k is odd and 1 if k is even. Then according to Kervaire and Milnor [1]:

3.5 Lemma. I_k is equal to

$$2^{2k-1}(2^{2k-1}-1)\ B_k\ j_k\ a_k / k .$$

The only unknown quantity here is the integer j_k.

3.6 Lemma. j_k is a multiple of the denominator of $B_k/4k$.

Proof. For k even, this is proved in Kervaire and Milnor [1]. For k odd this follows from the arguments of that paper, together with the following:

Theorem of Hirzebruch (Not yet published.) If the unbounded manifold M^{4k} has Stiefel-Whitney class w_2 equal to zero, and if k is odd, then the $\hat{A}$-genus $\hat{A}[M^{4k}]$ is an even integer.

On the other hand an upper bound for j_k is given by the order of the largest cyclic subgroup of $\pi_{q+4k-1}(S^q)$. The p-primary component of $\pi_{q+4k-1}(S^q)$ is known for $k < p^2(p-1)/2$ for any prime p. (See Toda [1,2].) The full group is known (to me) only for $k = 1, 2, 3$. It turns out that the upper bound for the p-primary factor of j_k is exactly equal to the lower bound in each known case.

Combining the preceding information, we have:

3.7 Lemma. The number $\frac{1}{8} I_k$ is equal to $a_k 2^{2k-2}(2^{2k-1}-1)$ (numerator B_k/k), multiplied by an integer whose prime factors p satisfy $p^2(p-1) \leq 2k$. In particular

$$\frac{1}{8} I_1 = 2 , \quad \frac{1}{8} I_2 = 28 , \quad \frac{1}{8} I_3 = 992, \quad \frac{1}{8} I_4 \text{ equals 8128 times}$$

a power of 2.

§4. Construction of (4k-1)-manifolds

The following is perhaps the simplest example of a symmetric matrix with determinant ± 1, with only even elements on the diagonal, and with index different from zero. (Compare Milnor [4].)

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

We will construct a manifold W^{4k} which has the above intersection matrix.

Let T be a tubular neighborhood of the diagonal in $S^{2k} \times S^{2k}$: say the set of all pairs (x,y) with distance $d(x,y) \leq \varepsilon$. Thus T is a 4k-manifold having the homotopy tupe of S^{2k}. The intersection number of the fundamental 2k-cycle with itself is $+2$.

Let $\alpha : S^{2k} \longrightarrow S^{2k}$ be the "twelve hour rotation" which leaves the north pole p fixed, and satisfies $\alpha(x) = -x$ for x on the equator. Let $T' = (1 \times \alpha)\, T$ be the set of pairs (x,y) with $d(x,\alpha\, y) \leq \varepsilon$. Then T and T' intersect only in a small neighborhood of the pair (p,p), and a small neighborhood of the pair $(-p,-p)$.

The universal covering space of $T \cup T'$ consists of infinitely many disjoint copies of T and infinitely many disjoint copies of T'. Numbering these copies T_i and T_i', we may assume that each T_i' intersects only T_i and T_{i+1}.

Define W_1 as the subset

$$T_1 \cup T_1' \cup T_2 \cup T_2' \cup T_3 \cup T_3' \cup T_4 \cup T_4'$$

of this universal covering space. Thus W_1 is a topological $4k$-manifold, having the homotopy type of a union of eight $2k$-spheres with a single point in common. Choosing an appropriate basis for $H_{2k}(W_1)$, the intersection matrix is as follows:

$$\begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & 1 \\ & & & & & & 1 & 2 \end{pmatrix}$$

To correct this intersection matrix it is necessary to introduce an intersection between T_1' and T_3, so as to obtain an intersection number -1. Choose a rotation of $S^{2k} \times S^{2k}$ which carries a region of T' near the "equator" onto a region of T near the "equator", so as to obtain an intersection number of

-1 . Matching the corresponding regions of T_1' and T_3 , we obtain a topological manifold W_2, with the required intersection matrix.

This manifold W_2 is differentiable except along eight "angles" which have been introduced in the boundary. Let W_3 be a differentiable manifold obtained by straightening these angles. (See the appendix.)

Unfortunately the transition from W_1 to W_2 changed the homotopy type. In fact the fundamental group $\pi_1(W_2) = \pi_1(W_3)$ is infinite cyclic. Next we will kill this fundamental group. A generator can be represented by a simple closed differentiable curve C lying on the boundary of W_3 .

Choose an imbedding $h: S^1 \times D^{4k-2} \longrightarrow \partial W_3$ which carries $S^1 \times 0$ onto the given curve C . Let W_4 be the space obtained from the disjoint union

$$W_3 \cup D^2 \times D^{4k-2}$$

by identifying $S^1 \times D^{4k-2}$ with its image under h . Then W_4 is simply connected. In fact W_4 has the same homotopy type as W_1; but the same intersection matrix as W_2 or W_3 .

This space W_4 is a differentiable manifold, except along the "angle" corresponding to $S^1 \times S^{4k-3}$. Let W_0 be a differentiable manifold obtained by "straightening" this angle.

4.1 <u>Theorem.</u> W_0 is a parallelizable $4k$-manifold with boundary M_0 which is a homology $(4k-1)$-sphere. In fact for $k > 1$, M_0 is a homotopy sphere. The index $I(W_0)$ equals $+8$.

Thus the invariant $\lambda'(M_0)$ is defined and equal to $+1$.

4.2 <u>Corollary.</u> The homomorphism Λ' from $\Theta^{4k-1}(\partial\pi)$ to the cyclic group of order $\frac{1}{8} I_k$ is onto, providing that $k > 1$.

4.3 <u>Corollary</u>. The group Θ^{4k-1} is non-trivial, providing that $k > 1$.

<u>Proof</u> that W_o is parallelizable. The only obstruction to parallelizability lies in the group

$$H^{2k}(W_o; \pi_{2k-1}(SO_{4k})).$$

But $H_{2k}(W_o)$ is generated by eight cycles, each of which is contained in a sub-manifold diffeomorphic to $T \subset S^{2k} \times S^{2k}$. Since S^{2k} is a π-manifold, it follows that T is a π-manifold, hence parallelizable. Therefore W_o is parallelizable.

<u>Computation</u> of $H_*(M_o)$. Since the groups $H_i(W_o)$ have no torsion, it follows by Poincaré-Lefschetz duality that $H_i(W_o, M_o) \approx H^{4k-i}(W_o)$ is isomorphic to Hom $(H_{4k-i}(W_o), Z)$. The natural homomorphism

$$H_i(W_o) \longrightarrow H_i(W_o, M_o) \approx \text{Hom}\,(H_{4k-i}(W_o), Z)$$

is determined by the matrix of intersection numbers.

Now recall that $H_o(W_o) \approx Z$; that $H_i(W_o) = 0$ for $i \neq 0, 2k$; and that the matrix of intersection numbers in dimension $2k$ has determinant $+1$. Plugging this information into the exact sequence of the pair (W_o, M_o), it follows that M_o has the homology of a $(4k-1)$-sphere.

<u>Proof</u> that M_o is simply connected, providing that $k > 1$. (For k equal to 1 the group $\pi_1(M_o)$ depends on the choice of the curve C. If C could be chosen so that $\pi_1(M_o) = 0$, then M_o would provide a counter-example to the Poincaré hypothesis.)

Let $K \subset W_3$ denote the union of 8 copies of S^{2k}, one in the center of each T_i and T_i'. Since the codimension $2k$ of K is greater than 2, it follows that

$$\pi_2(W_3, W_3 - K) = 0 .$$

But it is clear that ∂W_3 is a deformation retract of $W_3 - K$. Hence $\pi_2(W_3, \partial W_3) = 0$. From the exact sequence of this pair it follows that $\pi_1(\partial W_3)$ is infinite cyclic, generated by the closed curve C .

The manifold ∂W_4 can be obtained from ∂W_3 in two steps, as follows. (Compare Lemma 5.3.)

1) Remove a tubular neighborhood of $C \subset \partial W_3$. Since the co-dimension $4k-2$ of C in ∂W_3 is greater than 2, it follows that $\pi_1(\partial W_3 - C)$ is also infinite cyclic.

2) Fill in the resulting hole with a copy of $D^2 \times S^{4k-3}$. The effect of this addition on the fundamental group is to kill the generator. Hence $\pi_1(\partial W_4) = 0$.

Since the differentiable manifold $M_0 = \partial W_0$ is homeomorphic to ∂W_4, this completes the proof that M_0 is a homotopy sphere. Since the index $I(W_0)$ is easily shown to be $+8$, this proves Theorem 4.1.

§5. Simplifying manifolds by surgery

This section will describe an operation, suggested to the author by Thom, which can be used to kill off the lower homotopy groups of a manifold. To illustrate the method, the following will first be proved.

5.1 Theorem. Let M be an unbounded $4k$-manifold which is almost parallelizable. (That is there exists a finite subset F, so that $M-F$ is parallelizable.) Then there exists an unbounded $4k$-manifold M' which satisfies:

1) the index $I(M')$ equals $I(M)$,
2) M' is also almost parallelizable, and
3) M' is $(2k-1)$-connected.

Remark 1. It is not possible to kill off any further homotopy groups: If M' were 2k-connected, then the index $I(M')$ would have to be zero.

Remark 2. The hypothes.s that M is almost parallelizable is essential here. As an example, for n=12, the complex projective space $P_6(C)$ has index 1. But for any 5-connected 12-manifold, the index must be divisible by $I_3 = 7936$. (This follows sincd the only obstruction to almost parallelizability lies in $H^6(M; \pi_5(SO_{12})) = 0$.)

Proof of 5.1. If M has several components, let M_1 denote the connected sum of these components. It is not hard to show that M_1 is almost parallelizable, and that $I(M_1) = I(M)$.

Suppose by induction that M is (q-1)-connected, where $0 < q < 2k$. Any given element

$$\alpha \in \pi_q(M)$$

can be represented by an imbedding

$$f : S^q \longrightarrow M .$$

(This presents no difficulty since the dimension 4k is greater than 2q. Compare Whitney [1].)

5.2 Lemma. Let $f: S^q \longrightarrow M^n$ be an imbedding, with $q < \frac{1}{2}n$; and suppose that the bundle $f^*(\tau^n)$ induced from the tangent bundle of M^n is trivial. Then the normal bundle ν^{n-q} is trivial.

Proof. Let o^k denote the trivial SO_k-bundle over S^q , and let τ^q denote the tangent bundle. It is well known that the Whitney sum $\tau^q \oplus o^1$ is trivial. We are assuming that the bundle $\nu^{n-q} \oplus \tau^q \sim f^*(\tau^n)$ is trivial. Therefore

$$\nu^{n-q} \oplus o^{q+1} \sim \nu^{n-q} \oplus \tau^q \oplus o^1 \sim o^n \oplus o^1 \sim o^{n+1}$$

is trivial. That is: the inclusion $SO_{n-q} \longrightarrow SO_{n+1}$ carries the

SO_{n-q}- bundle ν^{n-q} into the trivial bundle. But the homomorphism

$$\pi_{q-1}(SO_{n-q}) \longrightarrow \pi_{q-1}(SO_{n+1})$$

is an isomorphism in the stable range $n-q > q$. This completes the proof.

Let T be a tubular neighborhood of $f(S^q)$. Then T can be identified with the total space of the D^{4k-q}-bundle which is associated with ν^{4k-q}. Choosing a specific product structure for ν^{4k-q}, it follows that T is homeomorphic to $S^q \times D^{4k-q}$. Let M_1 denote a differentiable manifold obtained from M by

1) removing the interior of T, and

2) pasting a copy of $D^{q+1} \times S^{4k-q-1}$ in its place, matching the common boundary $S^q \times S^{4k-q-1}$.

5.3 <u>Lemma</u>. The manifold M_1 is also (q-1)-connected. Furthermore $\pi_q(M_1)$ is isomorphic to $\pi_q(M)/(\alpha)$, where (α) denotes the normal subgroup generated by α.

<u>Proof</u>. Since $f(S^q)$ has codimension $4k-q$ in M, it follows that $\pi_i(M-f(S^q))$ is isomorphic to $\pi_i(M)$ for $i < 4k-q-1$. In particular

$$\pi_i(M-f(S^q)) = 0 \quad \text{for} \quad i < q \text{ , and}$$

$$\pi_q(M-f(S^q)) \approx \pi_q(M) \text{ .}$$

The manifold $M - f(S^q)$ can be obtained from M_1 by removing the sphere $0 \times S^{4k-q-1}$ of codimension $q + 1$.

Therefore

$$\pi_i(M_1) \approx \pi_i\,(M - f(S^q)) = 0 \quad \text{for} \quad i < q \text{ .}$$

<u>Case</u> 1. $q = 1$. Then $\pi_1(M_1)$ can be computed as follows. The manifold M_1 can be obtained from $M-T$ by first adjoining a 2-cell

$D^2 \times$ (constant) and then adjoining a 4k-cell. The first operation introduces the relation $\alpha = 0$ into the fundamental group ; while the second operation leaves the group unchanged.

Case 2. $q > 1$. Then the group $\pi_{q+1}(M_1, M-f(S^q))$ is isomorphic to $H_{q+1}(M_1, M-f(S^q)) \approx Z$. In the exact sequence

$$Z \longrightarrow \pi_q\ (M-f(S^q)) \longrightarrow \pi_q\ (M_1) \longrightarrow 0\ ,$$

it is clear that $\partial(1) = \alpha$, so that $\pi_q(M_1) \approx \pi_q(M)/(\alpha)$, as required. This completes the proof of 5.3.

5.4 Lemma. If the product structure for the normal bundle ν^{4k-q} is correctly chosen, then the manifold M_1 will also be almost parallelizable.

Before giving the proof, here is a description of some vector fields on $S^q \subset D^{q+1}$. Let $\varepsilon_1, \ldots, \varepsilon_{q+1}$ be the standard basis for the tangent vector space of D^{q+1}. The outward normal vector at a point $(t_1, \ldots, t_{q+1}) \in S^q$ is $\zeta = t_1\, \varepsilon_1 + \ldots + t_{q+1}\, \varepsilon_{q+1}$. Let ε_i' denote the projection of ε_i into the tangent bundle of S^q . Thus $\varepsilon_i' = \varepsilon_i - t_i\zeta$, so that $\varepsilon_i = \varepsilon_i' + t_i\zeta$.

Proof of 5.4. Choose some field φ_1 of vectors normal to $f(S^q)$. The "endpoints" of the vectors φ_1 sweep out a subset of $\partial\, T$ which will be denoted by $S^q \times (1, 0, \ldots, 0)$. The outward normal vector to $\partial\, T$ at a point of $S^q \times (1, 0, \ldots, 0)$ will also be denoted by φ_1 (since it can be considered as a translate of the vector φ_1 at the corresponding point of $f(S^q)$). Now consider the vector fields

$$\varepsilon_1' + t_1\, \varphi_1,\quad \varepsilon_2' + t_2\, \varphi_1,\ \ldots,\quad \varepsilon_{q+1}' + t_{q+1}\, \varphi_1\ ,$$

along $S^q \times (1, 0, \ldots, 0)$. These are orthogonal unit vectors in the tangent bundle of M.

Since M is almost parallelizable, there exists a field $(\psi_1, \ldots, \psi_{4k})$ of 4k-frames which is defined over $M-F$. Here F denotes some finite set which we may assume is disjoint from T.

<u>Assertion.</u> It is possible to deform this field $(\psi_1, \ldots, \psi_{4k})$ so as to obtain a field $(\psi_1', \ldots, \psi_{4k}')$ of 4k-frames such that along $S^q \times (1, 0, \ldots, 0)$:

$$\psi_1' = \varepsilon_1' + t_1 \varphi_1, \ldots, \psi'_{q+1} = \varepsilon'_{q+1} + t_{q+1} \varphi_1 .$$

This is proved as follows. Define a matrix $a_{ij}(t_1, \ldots, t_{q+1})$ by the formulae

$$\varepsilon_i' + t_i \varphi_1 = \sum_{j=1}^{4k} a_{ij} \psi_j, \qquad i = 1, 2, \ldots, q+1 .$$

This defines a map from S^q to the Stiefel manifold $V_{4k\ q+1}$. This map is null-homotopic since $\pi_q(V_{4k,\ q+1}) = 0$. Hence it can be lifted to a null-homotopic map of S^q into $V_{4k,\ 4k}$

$$(t_1, \ldots, t_{q+1}) \longrightarrow \| a_{ij} \| , \quad i = 1, \ldots, 4k .$$

Let $\psi_i' = \psi_i$ outside of a neighborhood of $S^q \times (1, 0, \ldots, 0)$ but let

$$\psi_i' = \Sigma\, a_{ij} \psi_j$$

for points in $S^q \times (1, 0, \ldots, 0)$, and for all i. The null-homotopy can now be used to define ψ_i' throughout the neighborhood.

We may assume that the vectors ψ_i' along $f(S^q)$ are translates of those along $S^q \times (1, 0, \ldots, 0)$.

Now choose the product structure for the normal bundle of $f(S^q)$ which is determined by the field

$$\varphi_1,\ \psi'_{q+2},\ \ldots,\ \psi'_{4k}$$

of normal (4k-q)-frames. In terms of this product structure, construct the manifold

$$M_1 = M\text{-(interior T)} \quad \cup \quad D^{q+1} \times S^{4k-q-1} .$$

The 4k-frame $(\psi'_1, \ldots, \psi'_{4k})$ in M-(interior T) can be extended throughout $D^{q+1} \times (1,0,\ldots,0)$ as follows. Note that the vector φ_1 along $S^q \times (1,0,\ldots,0)$ can be identified with the normal vector to S^q in D^{q+1}. Hence the vectors $\psi'_i = \varepsilon'_i + t_i\varphi_1$ $(1 \leq i \leq q+1)$ along $S^q \times (1,0,\ldots,0)$ can be identified with the standard basis for the tangent bundle of $D^{q+1} \times (1,0,\ldots,0)$. Hence these vectors $\psi'_1,\ldots,\psi'_{q+1}$ can be extended.

The remaining vectors $\psi'_{q+2}, \ldots, \psi'_{4k}$ are normal to D^{q+1}. The projection $S^q \times S^{4k-q-1} \longrightarrow S^{4k-q-1}$ carries these vectors onto a fixed (4k-q-1)-frame at the point $(1,0,\ldots,0) \in S^{4k-q-1}$. Hence it is certainly possible to extend $(\psi'_{q+2},\ldots, \psi'_{4k})$ over $D^{q+1} \times (1,0,\ldots,0)$ as a field of normal (4k-q-1)-frames.

Thus a field of 4k-frames has been defined over the subset

$$(M\text{-(interior T)-F}) \cup (D^{q+1} \times (1,0,\ldots,0))$$

of $M_1 - F$. The complement of this set in $M_1 - F$ consists of a single 4k-cell: (interior D^{q+1}) $\times$ (S^{4k-q-1} -point). Let F' consist of F together with a single point in this cell. Then it is clearly possible to extend the 4k-frame field throughout $M_1\text{-}F'$. This completes the proof that M_1 is almost parallelizable.

<u>Remark</u>. It cannot be proved that M_1 is parallelizable, even assuming that M is parallelizable. As an example take $M = S^1 \times S^3$, $M_1 = S^4$

5.5 <u>Lemma</u> (Thom). The manifold M_1 belongs to the same cobordism class as M.

<u>Proof</u>. Let W be the space obtained from the disjoint union of $M \times [0,1]$ and $D^{q+1} \times D^{4k-q}$ by pasting together $T \times [1]$ and $S^q \times D^{4k-q}$, using the product structure for T constructed above. A differentiable manifold W_1 is obtained from W by "straightening" the angle $\partial T \times [1] = S^q \times S^{4k-q-1}$. It is clear that ∂W_1 is the disjoint union of M_1 and -M, which completes the proof.

<u>Proof</u> of 5.1. Suppose that M is (q-1)-connected, almost parallelizable, and that $\pi_q(M)$ has r generators. The above construction yields a manifold M_1 which is (q-1)-connected, almost parallelizable, and such that $\pi_q(M_1)$ has r-1 generators. Iterating the construction r times, this yields a manifold M_r which is q-connected. Continue by induction on q until we obtain a manifold M′ which is (2k-1)-connected.

According to 5.5 the manifold M′ has the same cobordism class as M. Therefore the index I(M′) is equal to I(M). (Compare Thom [1].) This completes the proof of 5.1.

5.6 <u>Theorem</u>. Let M be a homology sphere of dimension 4k-1, $k > 1$, which bounds a π - manifold. If $\lambda'(M) = 0$ then M bounds a contractible manifold.

5.7 <u>Corollary</u>. If M is a homotopy (4k-1)-sphere with $\lambda'(M) = 0$, then M is J-equivalent to S^{4k-1}; providing that $k > 1$.

5.8 <u>Corollary</u>. For $k > 1$ the group $\Theta^{4k-1}(\partial \pi)$ is cyclic of order $\frac{1}{8} I_k$.

The proof of 5.6 is similar to that of 5.1, but also uses the following three results.

5.9. <u>Theorem</u>. Let W be a simply connected manifold of dimension $2n$, $n > 2$. Then every element of $\pi_n(W)$ is represented by an imbedding $f: S^n \longrightarrow W$.

The proof is a modification of Whitney's proof that every n-manifold can be imbedded in 2n-space. (See Whitney [2].) Details will not be given. I do not know whether this theorem is true for $n = 2$.

5.10. <u>Theorem</u>. Suppose that a quadratic form over the integers has determinant ± 1, index 0, and takes on only even values. Then it is equivalent to a quadratic form with matrix $\mathrm{diag}(U,U,\ldots,U)$, where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

<u>Proof</u>. This follows from theorems 1, 2 of Milnor [4] (making use of theorems of Eichler, etc).

[The following remark is due to H. Sah. In order to prove 5.10 it is sufficient to prove that there exists an isotropic vector: that is an $\alpha \neq 0$ such that the value $< \alpha,\alpha >$ of the quadratic form is zero. The existence of an isotropic vector is not difficult to prove; using the Hasse-Minkowski theorem that such a vector exists if and only if (1) the form is indefinite and (2) for each prime p the corresponding form over the p-adic numbers has an isotropic vector. Given such an α let $\alpha_1 = \alpha/r$ be indivisible. Since the determinant is ± 1, there exists β with $< \alpha_1, \beta > = 1$. Define

$$\beta_1 = \beta - \frac{1}{2} < \beta, \beta > \alpha_1 .$$

Then

$$< \alpha_1, \alpha_1 > = < \beta_1, \beta_1 > = 0 \ , \ < \alpha_1, \beta_1 > = 1.$$

Now consider the set of all γ which satisfy $< \alpha_1, \gamma > = < \beta_1, \gamma > = 0$. By induction on the rank we can choose a basis for this set so that the matrix has the required form.]

5.11. Lemma. Let $f: S^{2k} \longrightarrow W^{4k}$ be an imbedding, and suppose that

1) the homology class β of $f(S^{2k})$ has self intersection number $< \beta, \beta > = 0$ and

2) the induced bundle $f^*(\tau^{4k})$ over S^{2k} is trivial.

Then the normal bundle ν^{2k} is trivial.

Proof. Just as in 5.2 it is seen that ν^{2k} corresponds to an element $\alpha \in \pi_{2k-1}(SO_{2k})$ which is annihilated by the homomorphism

$$\pi_{2k-1}(SO_{2k}) \longrightarrow \pi_{2k-1}(SO_{4k+1}).$$

Since the group $\pi_{2k-1}(SO_{2k+1})$ is already stable, it follows from the exact sequence

$$\pi_{2k}(S^{2k}) \approx Z \xrightarrow{\partial} \pi_{2k-1}(SO_{2k}) \longrightarrow \pi_{2k-1}(SO_{2k+1})$$

that $\alpha = \partial n$ for some $n \in Z$.

The element $\partial 1 \in \pi_{2k-1}(SO_{2k})$ corresponds to the tangent bundle of S^{2k}, with Euler class equal to twice the generator of $H^{2k}(S^{2k})$. Therefore the Euler class of ν^{2k} is equal to $2n$ times a generator of $H^{2k}(S^{2k})$. But this Euler class can be interpreted as the self-intersection number $< \beta, \beta >$ times a generator. Therefore $2n = 0$, hence $\alpha = 0$. This completes the proof.

Proof of 5.6. Let M be a $(4k-1)$-manifold with $\lambda'(M) = 0$. An argument similar to the proof of 5.1 shows that M bounds a manifold W which is almost parallelizible (hence parallelizable) and $(2k-1)$-connected. The index $I(W)$ is congruent to zero modulo I_k. Hence there exists an almost parallelizable $4k$-manifold N, without boundary, which satisfies $I(N) = -I(W)$. By 5.1 we may assume that N is also $(2k-1)$-connected.

Now consider the sum $W_1 = W \# N$. This is a parallelizable $4k$-manifold with index zero, and with boundary M. The self-intersection matrix of W_1 has determinant ± 1 by the Poincaré duality theorem, and has only even elements on the diagonal. (Compare the proof of 3.2.) Therefore, according to 5.10, it is possible to choose a basis $\{\alpha_1,\ldots,\alpha_r, \beta_1,\ldots,\beta_r\}$ for $H_{2k}(W_1)$ so that the intersection matrix is given by:

$$< \alpha_i, \alpha_j > = 0, < \beta_i, \beta_j > = 0, < \alpha_i, \beta_j > = \delta_{ij}.$$

(Here δ_{ij} is a Kronecker delta.)

According to 5.9 there exists an imbedding $f: S^{2k} \longrightarrow W_1$ which represents the homology class α_1. According to 5.11 the normal bundle of $f(S^{2k})$ is trivial. Hence we can remove a tubular neighborhood and replace it by $D^{2k+1} \times S^{2k-1}$, yielding a new manifold W_2.

From the pair $(W_1, W_1-f(S^{2k}))$ we obtain an exact sequence

$$\ldots \longrightarrow 0 \xrightarrow{\partial} H_{2k}(W_1-f(S^{2k})) \longrightarrow H_{2k}(W_1) \xrightarrow{j} Z \longrightarrow \ldots$$

where $j(\gamma)$ is the intersection number of γ with the homology class α_1 of $f(S^{2k})$. Therefore $H_{2k}(W_1 - f(S^{2k}))$ is the subgroup of $H_{2k}(W_1)$ generated by $\{\alpha_1, \ldots, \alpha_r, \beta_2, \ldots, \beta_r\}$.

From the pair $(W_2, W_1 - f(S^{2k}))$ we obtain an exact sequence

$$\longrightarrow Z \xrightarrow{\partial} H_{2k}(W_1 - f(S^{2k})) \longrightarrow H_{2k}(W_2) \xrightarrow{j} 0 \longrightarrow \cdots ,$$

where $\partial 1$ is the class α_1. Hence $H_{2k}(W_2)$ is freely generated by the classes $\{\alpha_2, \ldots, \alpha_r, \beta_2, \ldots, \beta_r\}$. Note that the intersection numbers of these classes in W_2 is the same as that in W_1. In fact any 2k-cycle in W_2 can be deformed so that it does not intersect the submanifold $0 \times S^{2k-1}$ which has codimension $2k + 1$.

Now choose an imbedding $f_2 \colon S^{2k} \longrightarrow W_2$ which represents the class α_2. We may assume that $f_2(S^{2k})$ is contained in the parallelizable manifold

$$W_2 - (0 \times S^{2k-1}) = W_1 - f(S^{2k}),$$

hence the normal bundle is trivial. Iterating this procedure r times, we obtain a manifold W_{r+1} which is 2k-connected, and therefore contractible. This completes the proof of 5.6.

This argument can be modified slightly to prove the following.

5.12. <u>Theorem</u>. The groups $\Theta^5(\partial\pi)$ and $\Theta^{13}(\partial\pi)$ are zero.

<u>Proof</u>. Let $M^5 = \partial W^6$ where W^6 is parallelizable. Just as above, we may assume that W^6 is 2-connected. The self intersection matrix of $H_3(W^6)$ is skew symmetric with determinant ± 1. Hence it is equivalent to a matrix of the form $\mathrm{diag}(U', U', \ldots, U')$ where

$U' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (See for example Veblen [1] pg. 183.) The normal bundle of any 3-sphere in W^6 is trivial since $\pi_2(SO_3) = 0$. Hence the argument above shows that we can kill $H_3(W^6)$.

The argument in dimension 13 is similar, using the fact that $\pi_6(SO_7) = 0$. This completes the proof of 5.12.

Remark. The following assertion will be proved in a later paper. For any n of the form $4k + 1$, the group $\Theta^n(\partial\pi)$ is either zero or cyclic of order two. The proof will make use of the Arf invariant of a certain quadratic form over the field Z_2.

5.13. Theorem. The groups $\Theta^6(\partial\pi)$ and $\Theta^{14}(\partial\pi)$ are zero.

Outline of proof. Let $M^{2k} = \partial W^{2k+1}$, where W^{2k+1} is parallelizable. Just as above we may assume that W^{2k+1} is $(k-1)$-connected. Furthermore the group $H_k(W^{2k+1}; Q)$ with rational coefficients is not difficult to kill. Thus we may assume that $H_k(W^{2k+1})$ is a finite group. Any element of this group is represented by an imbedded k-sphere with trivial normal bundle. Hence one can form W_1^{2k+1} as before. However the homology group $H_k(W_1^{2k+1})$ depends on the particular product structure which is chosen for the normal bundle. The following question arises: Given an arbitrary normal vector field φ_1, does there exist a field of normal $(k+1)$-frames $(\varphi_1, \ldots, \varphi_{k+1})$? For k equal to 1,3 or 7 this is possible, since the homomorphism

$$\pi_k(SO_{k+1}) \longrightarrow \pi_k(S^k)$$

is onto. Hence it is possible to choose $(\varphi_1, \ldots, \varphi_{k+1})$ so that

$H_k(W_1^{2k+1})$ is smaller than $H_k(W^{2k+1})$. However for other values of k this homomorphism is not onto, so that the proof does not go through.

§6. The group $\Theta^n/\Theta^n(\partial\pi)$

The main result of this section will be that the factor group $\Theta^n/\Theta^n(\partial\pi)$ is always finite. [This is the group of all homotopy n-spheres modulo those which bound π-manifolds.] Upper bounds for this group are given, but no lower bounds. It is possible that every homotopy sphere is the boundary of a π-manifold.

Let $M^n \subset R^{n+q}$ be a homology sphere, with $q > n$. The only obstruction to triviality of the normal bundle is an element

$$\mathcal{O} \in H^n(M^n; \pi_{n-1}(SO_q)) \approx \pi_{n-1}(SO_q) .$$

This coefficient group has been computed as follows by R. Bott [1]:

n modulo 8	0	1	2	3	4	5	6	7
$\pi_{n-1}(SO_q)$	Z	Z_2	Z_2	0	Z	0	0	0

(This table is valid for $q > n \geq 2$.)

If n is congruent to 3, 5, 6 or 7 modulo 8, this clearly implies that $\mathcal{O}$ is zero.

If n is equal to 4k then the obstruction class $\mathcal{O}$ can be identified with a certain fraction of the Pontrjagin class $p_k(M^n)$. (See Kervaire [3] or Kervaire and Milnor [1].) But Hirzebruch's index formula (Hirzebruch [1] p. 85) implies that the Pontrjagin class of a homology sphere is zero. Again it follows that $\mathcal{O} = 0$.

Finally suppose that n is congruent to 1 or 2 modulo 8, so that $\pi_{n-1}(SO_q) \approx Z_2$. A theorem of Rohlin asserts that the obstruction class $\mathcal{O}$

is annihilated by the homorphism

$$J_{n-1}: \pi_{n-1}(SO_q) \longrightarrow \pi_{n+q-1}(S^q) .$$

(See Rohlin [1] or Kervaire and Milnor [1].) If J_{n-1} is non-trivial, it follows that $\sigma' = 0$. This proves:

6.1 __Theorem.__ Every homology n-sphere is a π-manifold, unless

1) $n \equiv 1$ or 2 modulo 8, and

2) the homomorphism J_{n-1} is zero.

For $n = 2$ it is well known that J_1 is an isomorphism. For $n = 9, 10$ we have:

6.2 __Lemma of Kervaire__ [5]. The homomorphisms J_8 and J_9 are non-trivial.

Therefore:

6.3 __Corollary.__ For $n < 17$ every homology n-sphere is a π-manifold.

I do not know whether conditions (1) and (2) of 6.1 are ever satisfied. However in any case the following is true.

6.4 __Lemma.__ For any n the homotopy n-spheres which are π-manifolds form a subgroup $\Theta^n(\pi) \subset \Theta^n$ which has index at most 2.

__Proof.__ We may assume that $\pi_{n-1}(SO_q) \approx Z_2$. The obstruction correspondence $M^n \longrightarrow \sigma'(M^n) \in \pi_{m-1}(SO_q)$ is easily seen to be additive, and invariant under J-equivalence. This completes the proof.

Now let M^n be any π-manifold without boundary, and consider the question: __Is__ M^n __the boundary of a π-manifold?__ The theory of Thom [2] can be used to give an answer as follows.

Choose an imbedding of M^n in the interior of a cube $[0, 1] \times \ldots \times [0, 1] = I^{n+q}$, and choose a field φ of normal q-frames. Then the

Thom construction yields a map

$$(I^{n+q}, \partial I^{n+q}) \longrightarrow (S^q, \text{base point}),$$

and hence a homotopy class

$$t(\varphi) \in \pi_{n+q}(S^q) .$$

(See Thom [2] p.30, or Kervaire [1] p.223, or Kervaire and Milnor [1], proof of Lemma 1.) This class is zero if and only if there exists a π-manifold $W \subset I^{n+q+1}$ such that

1) $\partial W = M^n \times [0]$, and

2) the field φ of normal q-frames can be extended throughout W.

Now let φ range over all possible fields of normal q-frames. The set of all homotopy classes $t(\varphi)$ will be denoted by

$$t'(M^n) \subset \pi_{n+q}(S^q) .$$

Evidently M^n bounds a π-manifold if and only if

$$0 \in t'(M^n) .$$

6.5 Lemma. If M_1 and M_2 are π-manifolds, then $t'(M_1 \# M_2) \supset t'(M_1) + t'(M_2)$.

(I do not know whether equality holds.) Proof. Let W be a manifold formed from the disjoint union of $M_1 \times [0, 1]$, $M_2 \times [0, 1]$ and $D^n \times D^1$ by matching $D^n \times [-1]$ with a cell in $M_1 \times [1]$; matching $D^n \times [1]$ with a cell in $M_2 \times [1]$; and then straightening corners. If the orientations are correct, then ∂W will be the disjoint union of $M_1 \# M_2$, $-M_1$ and $-M_2$. Furthermore W has the homotopy type of the union of M_1 and M_2 with a single point in common.

Choose an imbedding of W in $R^{n+q} \times [0,1]$ so that $-M_1$ and $-M_2$ go into $R^{n+q} \times [0]$, while $M_1 \# M_2$ goes into $R^{n+q} \times [1]$. Now given fields φ_1, φ_2 of normal q-frames on M_1 and M_2 respectively, there exists an extension ψ which is defined throughout W.

If φ denotes the restriction of ψ to $M_1 \# M_2$, then it is clear that $t(\varphi) = t(\varphi_1) + t(\varphi_2)$. This completes the proof of 6.5.

Now consider the special case $M = S^n$. Every field φ of normal q-frames determines an element

$$\alpha \in \pi_n(SO_q) .$$

Kervaire has shown that $t(\varphi)$ is equal to $\pm J_n(\alpha)$. (See Kervaire [4].) Since any α may occur this proves:

6.6 Lemma. The set $t'(S^n)$ is equal to Image $J_n \subset \pi_{n+q}(S^q)$.

Applying 6.5 to the identity

$$M^n \# S^n = M^n$$

this shows that $t'(M^n) \supset t'(M^n) + (\text{image } J_n)$. In other words $t'(M^n)$ is a union of cosets of (image J_n). This suggests that we define $t(M^n)$ as the subset of

$$\text{cokernel } J_n = \pi_{n+q}(S^q)/(\text{image } J_n)$$

which corresponds to $t'(M^n)$.

6.7 Theorem. The Thom construction yields a correspondence

$$M^n \longrightarrow t(M^n) \subset (\text{cokernel } J_n)$$

with the following properties:

a) $t(M^n)$ is defined and non-vacuous for every unbounded π-manifold.

b) $t(M^n)$ contains 0 if and only if M^n bounds a π-manifold.

c) $t(M_1 \# M_2) \supset t(M_1) + t(M_2)$.

d) $t(S^n) = \{0\}$.

e) If M_1 is J-equivalent to M_2 then $t(M_1) = t(M_2)$.

f) If M^n is a homotopy sphere, then $t(M^n)$ consists of a single element.

Proof. Properties (a) through (d) follow from the discussion above. Property (e) follows immediately from the definition. To prove (f) recall that $M^n \# (-M^n)$ is J-equivalent to S^n. Therefore $\{0\} \supset t(M^n) + t(-M^n)$. But this would be impossible if $t(M^n)$ contained more than one element.

6.8 Corollary. The factor group $\Theta^n(\pi)/\Theta^n(\partial \pi)$ is naturally isomorphic to a subgroup of (cokernel J_n).

6.9 Corollary. This factor group is finite for every n. Hence the subgroup $\Theta^n(\partial \pi) \subset \Theta^n$ has finite index.

To conclude this section, here is a summary of what is known about the group (cokernel J_n). Toda has computed the p-primary component of the stable group $\pi_{n+q}(S^q)$ for the range $n < 2p^2(p-1)-3$. (See Toda [2].) Combining this information with §3.6 the p-primary component of (cokernel J_n) is determined for the same range. As an example (compare Milnor [3]):

Assertion. The p-primary component of (cokernel J_n) is zero for $n < 2p(p-1)-2$, and is Z_p for $n = 2p(p-1)-2$.

The 2-primary component can be determined for $n \leq 13$, making use of Toda [1], together with §6.2 and §3.6. The following is a tabulation of the first thirteen groups.

n =	1	2	3	4	5	6	7	8	9	10	11	12	13	$14 \leq n \leq 19$
coker J_n =	0	Z_2	0	0	0	Z_2	0	Z_2	$Z_2 + Z_2$	Z_6	0	0	Z_3	2-group

Since Θ^2 is known to be zero, the first unsolved case occurs for $n = 6$. Is the group $\Theta^6/\Theta^6(\partial \pi)$ non-trivial?

§7. Discussion

Combining the results of the preceding sections, we have the following estimate of Θ^n for small values of n.

$\Theta^1 = \Theta^2 = \Theta^5 = 0$,

Θ^6 is either 0 or Z_2,

Θ^7 is cyclic of order 28,

Θ^9 has order at most 8,

Θ^{11} is cyclic of order 992,

Θ^{13} is either 0 or Z_3,

Θ^{14} is a 2-group,

Θ^{15} has order 127 times a power of 2. This group contains an element of order 8128.

Evidently the biggest hiatus in the results is the following.

Problem 1. Are the groups $\Theta^{2k}(\partial\pi)$ finite for $k \neq 1,3,7$? A solution would probably be based on a detailed study of $(2k+1)$-manifolds which are $(k-1)$-connected. (Compare §5.13.)

Another outstanding problem is the decision as to whether every homotopy sphere bounds a π-manifold. (See §6.)

Problem 2. Is there any theory which related the invariant $t(M^n) \subset$ (cokernel J_n) with the topology of M^n? In particular does this invariant vanish for a homotopy sphere?

Another question would be the relationship between this paper and the Poincare hypothesis.

Problem 3. Does there exist a homotopy 3-sphere M such that $\lambda'(M) \neq 0$?

Such a manifold could not be homeomorphic to S^3. In fact J. Munkres, S. Smale and J. H. C. Whitehead have proved that the differentiable structure of a topological 3-manifold is unique up to diffeomorphism.

Problem 4. Are the homotopy spheres M_0^{4k-1} homeomorphic to S^{4k-1}? (See §4. Note that k must be $\geqq 2$.)

The following seems to be a very deep question.

Problem 5. Are J-equivalent manifolds necessarily diffeomorphic?

An affirmative answer would imply the generalized Poincare hypothesis for differentiable manifolds. For if M is a homotopy n-sphere then $M \# (-M)$ is J-equivalent to S^n. But if $M \# M_1$ is diffeomorphic to S^n then an argument due to Mazur [1] implies that M itself is homeomorphic to S^n.

Most known invariants of differentiable manifolds depend only on the J-equivalence class. For example:

Assertion. If M_1 is J-equivalent to M_2 then some homotopy equivalence $M_1 \longrightarrow M_2$ is covered by a bundle map $\tau_1^n \longrightarrow \tau_2^n$ between the tangent bundles.

Proof. Suppose that the boundaries M_1 and $-M_2$ are deformation retracts of W. Choose a non-singular vector field on W which points out of W along M_1 and into W along M_2. The orthogonal complement of this vector field in τ^{n+1} yields an SO_n-bundle ξ^n

over W. Now the bundle maps $\tau_1^n \longrightarrow \xi^n \longleftarrow \tau_2^n$ can be used to construct the required bundle map.

Problem 6. Is the "simple homotopy type" of M invariant under J-equivalence? (See J. H. C. Whitehead [1], [4].)

Appendix: Pasting and straightening

Let R_+ denote the set of real numbers t with $0 \leq t < \infty$.

Assertion. If W is a differentiable manifold with boundary, then there exists a neighborhood U of ∂W, and a diffeomorphism

$$h: \partial W \times R_+ \longrightarrow U$$

which satisfies the identity $h(x,0) = x$.

(A proof of this assertion is given in Milnor [2]. Alternatively this may be taken as part of the definition of "manifold with boundary".)

Given two manifolds W_1, W_2 and an orientation reversing diffeomorphism

$$f: \partial W_1 \longrightarrow \partial W_2 ,$$

let M denote the space obtained from the disjoint union of W_1 and W_2 by identifying each $x \in \partial W_1$ with $f(x)$.

8.1. Lemma. The topological manifold M can be given a differentiable structure which is compatible with that of W_1 and W_2.

8.2. Lemma. If two such differentiable structures are given, then the resulting differentiable manifolds are diffeomorphic.

Proof of 8.1. Choose neighborhoods U_i of ∂W_i in W_i, and diffeomorphisms

$$h_i : \partial W_i \times R_+ \longrightarrow U_i$$

as above. A homeomorphism

$$h: \partial W_1 \times R \longrightarrow U_1 \cup U_2 \subset M$$

is defined by the formula

$$h(x,t) = \begin{matrix} h_1(x,t) & \text{for} & t \geqq 0 \\ h_2(fx,-t) & \text{for} & t \leqq 0. \end{matrix}$$

Taking h to be a diffeomorphism, this defines the required differentiable structure.

Proof of 8.2. Let M and M' be the two differentiable manifolds. Choose a contravariant vector field φ_1 along the boundary of W_1 which points out of W_1. Considering W_1 and W_2 as submanifolds of M, this yields a vector field φ_2 along the boundary of W_2 which points into W_2. On the other hand, considering W_1 and W_2 as submanifolds of M', the field φ_1 corresponds to some other field φ_2' along the boundary of W_2.

Choose a diffeomorphism $g_2: W_2 \longrightarrow W_2$ which leaves ∂W_2 pointwise fixed, and carries the vector field φ_2 into φ_2'. (The construction is not difficult.) Then a homeomorphism $g: M \longrightarrow M'$ is obtained by combining g_2 with the identity map of W_1. It is easily verified that g and g^{-1} are differentiable of class C^1.

Approximate g by a C^∞-differentiable map g'; where the approximation must be close enough so that the Jacobian of g' has rank n everywhere. (See Whitney [1].) Then $g'\colon M \longrightarrow M'$ is the required diffeomorphism.

Several times in this paper it has been necessary to consider n-manifolds with boundary which are differentiable except along some (n-2)-dimensional submanifold of the boundary. The simplest example of such an object is the quadrant $R_+ \times R_+ \subset R^2$. This example can be "straightened" by introducing new coordinates as follows. Map $R_+ \times R_+$ onto the half-plane $R \times R_+$ by the correspondence

$$(r \cos \theta,\ r \sin \theta) \xrightarrow{f} (r \cos 2\theta,\ r \sin 2\theta)$$

for $0 \leq r$, $0 \leq \theta \leq \frac{\pi}{2}$. Thus f is a diffeomorphism, except at the singular point. Another example is provided by the three-quarter-plane $R_+ \times R \cup R \times R_+$. This can be straightened by the transformation

$$(r \cos \theta,\ r \sin \theta) \longrightarrow (r \cos ((2\theta+\pi)/3),\ r \sin ((2\theta+\pi)/3)),$$

for $0 \leq r$, $-\frac{\pi}{2} \leq \theta \leq \pi$.

A higher dimensional example is given as follows. Let W_1 and W_2 be differentiable manifolds with boundary. Then $W_1 \times W_2$ is differentiable except along $\partial W_1 \times \partial W_2$. Some neighborhood $U_1 \times U_2$ of this singular set is "diffeomorphic" to

$$(\partial W_1 \times \partial W_2) \times (R_+ \times R_+).$$

Form a new differentiable manifold W as follows. Take the disjoint union of $W_1 \times W_2 - \partial W_1 \times \partial W_2$ and $\partial W_1 \times \partial W_2 \times R \times R_+$, and identify

$$h_1(x_1, r \cos \theta) \times h_2(x_2, r \sin \theta) \in U_1 \times U_2$$

with

$$(x_1, x_2, r \cos 2\theta, r \sin 2\theta)$$

for each $x_1 \in \partial W_1$, $x_2 \in \partial W_2$, $0 < r$, $0 \leq \theta \leq \frac{\pi}{2}$. This construction will be referred to as "straightening the angle". Note that the differentiable structure of $\partial W_1 \times W_2$, and of $W_1 \times \partial W_2$, is left fixed, so that Lemma 8.2 applies to their union ∂W.

A similar construction works for each of the examples considered in this paper.

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