AN EXPLICIT PROJECTION

by Andrew Ranicki

A module P over a ring A is f.g. (finitely generated) projective if it is isomorphic to the image $\mathrm{im}(p : A^n \longrightarrow A^n)$ of a projection $p = p^2 : A^n \longrightarrow A^n$ of a f.g. free module A^n. Projective modules and the projective class groups $K_0(A)$, $\widetilde{K}_0(A)$ entered topology via the work of Swan [8] on finite group actions on homotopy spheres, and more generally via the finiteness obstruction theory of Wall [10]. In various papers (Munkholm and Ranicki [5], Ranicki [7], Lück [1], Pedersen and Weibel [6], Lück and Ranicki [2]) it has actually been found more convenient to work with the projections rather than the modules.

In this note an explicit projection is obtained for the f.g. projective A-module constructed by the standard Mayer-Vietoris procedure (Milnor [4,§2]) from an automorphism of a f.g. free A'-module, with A and A' related by a cartesian square of rings

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ B' & \xrightarrow{\ g'\ } & A' \end{array}$$

with $f' : A \longrightarrow B'$ and $g : B \longrightarrow A'$ onto. In view of the theorem of Swan that the map $\widetilde{K}_0(\mathbb{Z}[G]) \longrightarrow \widetilde{K}_0(\mathbb{Q}[G])$ is trivial this is the generic construction of f.g. projective $\mathbb{Z}[G]$-modules for finite groups G. By way of an example an explicit projection is constructed for a generator of $\widetilde{K}_0(\mathbb{Z}[Q(8)]) = \mathbb{Z}_2$, with $Q(8)$ the quaternion group $Q(8)$ of order 8. This is the simplest example of a group G with non-trivial reduced projective class group $\widetilde{K}_0(\mathbb{Z}[G])$.

A commutative square of rings (as above) is *cartesian* if the sequence of additive groups

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} f \\ f' \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} g & -g' \end{pmatrix}} A' \longrightarrow 0$$

is exact.

Given an automorphism $\alpha' : A'^n \longrightarrow A'^n$ of a f.g. free A'-module define the pullback f.g. projective A-module

$$P(\alpha') \ = \ \{(x, x') \in B^n \oplus B'^n \,|\, \alpha'(g(x)) = g'(x') \in A'^n\}$$

which fits into an exact sequence of additive groups

$$0 \longrightarrow P(\alpha') \longrightarrow B^n \oplus B'^n \xrightarrow{\begin{pmatrix} \alpha' g & -g' \end{pmatrix}} A'^n \longrightarrow 0$$

with A acting by

$$A \times P(\alpha') \longrightarrow P(\alpha')\ ;\ (a,(x,x')) \longmapsto (f(a)x, f'(a)x')\ .$$

The construction is used to define the connecting map ∂ in the Mayer-Vietoris exact sequence (Milnor [4,§4]) of algebraic K-groups

$$K_1(A) \xrightarrow{\begin{pmatrix} f \\ f' \end{pmatrix}} K_1(B) \oplus K_1(B') \xrightarrow{(g \quad -g')} K_1(A') \xrightarrow{\partial}$$
$$K_0(A) \xrightarrow{\begin{pmatrix} f \\ f' \end{pmatrix}} K_0(B) \oplus K_0(B') \xrightarrow{(g \quad -g')} K_0(A')\ ,$$

with

$$\partial\ :\ K_1(A') \longrightarrow K_0(A)\ ;\ \tau(\alpha' : A'^n \longrightarrow A'^n) \longmapsto [P(\alpha')] - [A^n]\ .$$

Given A'-module automorphisms $\alpha' : A'^n \longrightarrow A'^n$, $\alpha'' : A'^m \longrightarrow A'^m$, and also a B-module morphism $\beta : B^n \longrightarrow B^m$ and a B'-module morphism $\beta' : B'^n \longrightarrow B'^m$ such that the square

$$\begin{array}{ccc} A'^n & \xrightarrow{g(\beta)} & A'^m \\ {\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\alpha''} \\ A'^n & \xrightarrow{g'(\beta')} & A'^m \end{array}$$

commutes let

$$(\beta,\beta')\ :\ P(\alpha') \longrightarrow P(\alpha'')\ ;\ (x,x') \longmapsto (\beta(x),\beta'(x'))$$

be the pullback A-module morphism.

PROPOSITION *Given an A'-module automorphism $\alpha' : A'^n \longrightarrow A'^n$ and any lifts of α', α'^{-1} to B-module endomorphisms $\beta,\gamma : B^n \longrightarrow B^n$ there is defined an A-module projection*

$$p(\alpha') = \begin{pmatrix} ((2-\beta\gamma)\beta\gamma, 1) & ((2-\beta\gamma)(1-\beta\gamma)\beta, 0) \\ (\gamma(1-\beta\gamma), 0) & ((1-\gamma\beta)^2, 0) \end{pmatrix}\ :\ A^n \oplus A^n \longrightarrow A^n \oplus A^n$$

such that up to isomorphism

$$P(\alpha') = \mathrm{im}(p(\alpha'))\ .$$

PROOF: Lift the Whitehead lemma identity of A'-module automorphisms

$$\begin{pmatrix} \alpha' & 0 \\ 0 & \alpha'^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha'^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$: A'^n \oplus A'^n \longrightarrow A'^n \oplus A'^n$$

to define a B-module automorphism

$$\phi = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (2-\beta\gamma)\beta & \beta\gamma - 1 \\ 1-\gamma\beta & \gamma \end{pmatrix}$$

$$: B^n \oplus B^n \longrightarrow B^n \oplus B^n$$

with inverse

$$\phi^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & 1-\gamma\beta \\ \beta\gamma - 1 & (2-\beta\gamma)\beta \end{pmatrix}$$

$$: B^n \oplus B^n \longrightarrow B^n \oplus B^n .$$

Identifying $A^n \oplus A^n = A^{2n}$ define an A-module isomorphism

$$h = (\phi, 1) : P(\alpha') \oplus P(\alpha'^{-1}) \longrightarrow P(1 : A'^{2n} \longrightarrow A'^{2n}) = A^{2n}$$

with inverse

$$h^{-1} = (\phi^{-1}, 1) : P(1 : A'^{2n} \longrightarrow A'^{2n}) = A^{2n} \longrightarrow P(\alpha') \oplus P(\alpha'^{-1}) .$$

It is now immediate from the identity

$$\begin{aligned} p(\alpha') &= h(1 \oplus 0)h^{-1} : \\ & A^{2n} \xrightarrow{h^{-1}} P(\alpha') \oplus P(\alpha'^{-1}) \xrightarrow{1 \oplus 0} P(\alpha') \oplus P(\alpha'^{-1}) \xrightarrow{h} A^{2n} \end{aligned}$$

that $p(\alpha') : A^{2n} \longrightarrow A^{2n}$ is a projection with image isomorphic to $P(\alpha')$. Explicitly, the restriction of h defines an A-module isomorphism

$$P(\alpha') \longrightarrow \mathrm{im}(p(\alpha')) \ ; \ (x, x') \longmapsto ((2-\beta\gamma)\beta(x), x') \oplus ((1-\gamma\beta)(x), 0) .$$

□

EXAMPLE Given a finite group G consider the Rim cartesian square of rings

$$\begin{array}{ccc} \mathbb{Z}[G] & \longrightarrow & \mathbb{Z}[G]/N \\ \downarrow \epsilon & & \downarrow \epsilon \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/|G| \end{array}$$

in which all the morphisms are onto, with

$$N = \sum_{g\in G} g \in \mathbb{Z}[G] \ , \ \epsilon : \mathbb{Z}[G] \longrightarrow \mathbb{Z} \ ; \ g \longmapsto 1 \ .$$

The canonical isomorphism of rings $\mathbb{Z}[G]\longrightarrow(\mathbb{Z}[G]/N,1,\mathbb{Z})$ has inverse

$$(\mathbb{Z}[G]/N,1,\mathbb{Z}) \longrightarrow \mathbb{Z}[G] \ ; \ (b,b') \longmapsto a + (b'-\epsilon(a))(N/|G|)$$

with $a \in \mathbb{Z}[G]$ any lift of $b \in \mathbb{Z}[G]/N$ (so that $\epsilon(a) \equiv b' (\bmod |G|)$). In this case the boundary map in the Mayer-Vietoris sequence is given by

$$\begin{aligned} \partial : K_1(\mathbb{Z}/|G|) = (\mathbb{Z}/|G|)^\times &\longrightarrow K_0(\mathbb{Z}[G]) \ ; \\ \tau(\alpha') &\longmapsto [\mathrm{im}(p(\alpha'))] - [\mathbb{Z}[G]^2] \end{aligned}$$

for any unit $\alpha' \in (\mathbb{Z}/|G|)^\times$, with $\beta,\gamma \in \mathbb{Z}$ such that $[\beta]=\alpha'$, $[\gamma]=\alpha'^{-1} \in \mathbb{Z}/|G|$, and $p(\alpha')$ the $\mathbb{Z}[G]$-module projection

$$p(\alpha') = \begin{pmatrix} 1-(1-\beta\gamma)^2(N/|G|) & (2-\beta\gamma)(1-\beta\gamma)\beta(N/|G|) \\ \gamma(1-\beta\gamma)(N/|G|) & (1-\gamma\beta)^2(N/|G|) \end{pmatrix}$$
$$: \mathbb{Z}[G]\oplus\mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]\oplus\mathbb{Z}[G] \ .$$

□

Example For the quaternion group of order 8

$$G = Q(8) = \{\pm 1, \pm i, \pm j, \pm k\}$$

and the unit $\alpha' = 3 \in (\mathbb{Z}/8)^\times$ take $\beta=\gamma=3 \in \mathbb{Z}$ in the previous Example. By the Proposition the corresponding projection

$$p(\alpha') = \begin{pmatrix} 1-8N & 21N \\ -3N & 8N \end{pmatrix} : \mathbb{Z}[Q(8)]\oplus\mathbb{Z}[Q(8)] \longrightarrow \mathbb{Z}[Q(8)]\oplus\mathbb{Z}[Q(8)]$$

is such that $P(\alpha') \cong \mathrm{im}(p(\alpha'))$ is a f.g. projective $\mathbb{Z}[Q(8)]$-module isomorphic to the two-sided ideal

$$\langle 3,N\rangle = \mathrm{im}((3\ N) : \mathbb{Z}[Q(8)]\oplus\mathbb{Z}[Q(8)]\longrightarrow\mathbb{Z}[Q(8)]) \subset \mathbb{Z}[Q(8)]$$

of the type considered by Swan [8,§6], with an isomorphism

$$\langle 3,N\rangle \longrightarrow P(\alpha') \ ; \ 3x+Ny \longmapsto x(1,3)+y(0,8) \ \ (x,y \in \mathbb{Z}[Q(8)]) \ .$$

The reduced projective class

$$\partial\tau(3) \ = \ [P(\alpha')] \in \widetilde{K}_0(\mathbb{Z}[Q(8)]) = \mathbb{Z}/2$$

represents the generator (Martinet [3]). As noted in [3] $P(\alpha')$ is isomorphic to the f.g. projective $\mathbb{Z}[Q(8)]$-module P_3 defined by the f.g. free $\mathbb{Z}$-module $\mathbb{Z}^8$ on 8 generators $\{e_0\} \cup \{e_s | s \in Q(8), s \neq 1\}$, with $Q(8)$ acting by

$$se_0 \ = \ e_0 \quad , \quad se_{s^{-1}} \ = \ 3e_0 - \sum_{t \neq 1} e_t \quad (s \in Q(8)) \ ,$$

$$se_t \ = \ e_{st} \quad (t \neq 1, s^{-1}) \ .$$

The element defined by

$$e_1 \ = \ 3e_0 - \sum_{t \neq 1} e_t \in P_3$$

is such that

$$se_1 \ = \ e_s \in P_3 \quad (s \neq 1) \ .$$

Thus

$$Ne_1 \ = \ e_1 + \sum_{t \neq 1} e_t \ = \ 3e_0 \in P_3 \ ,$$

and there is defined a $\mathbb{Z}[Q(8)]$-module isomorphism

$$\langle 3, N \rangle \longrightarrow P_3 \ ; \ 3x + Ny \longmapsto xe_1 + ye_0 \ .$$

$\square$

Given a ring A and a multiplicative subset $S \subset A$ of central non-zero divisors there is defined a cartesian square of rings

$$\begin{array}{ccc} A & \longrightarrow & S^{-1}A \\ \big\downarrow & & \big\downarrow \\ \widehat{A} & \longrightarrow & S^{-1}\widehat{A} \end{array}$$

with $S^{-1}A$ the localization of A inverting S, and

$$\widehat{A} \ = \ \varprojlim_{s \in S} A/sA$$

the S-adic completion of A. The algebraic K-theory Mayer-Vietoris exact sequence determined by such a square

$$K_1(A) \longrightarrow K_1(S^{-1}A) \oplus K_1(\widehat{A}) \longrightarrow K_1(S^{-1}\widehat{A}) \xrightarrow{\partial}$$
$$K_0(A) \longrightarrow K_0(S^{-1}A) \oplus K_0(\widehat{A}) \longrightarrow K_0(S^{-1}\widehat{A})$$

is widely used in the computations of the K-groups of the group rings $A = \mathbb{Z}[G]$ of finite groups G, with $S = \mathbb{Z}\text{-}\{0\}$, $S^{-1}A = \mathbb{Q}[G]$. Again, the connecting map ∂ is defined by the pullback construction: if $\alpha : S^{-1}\widehat{A}^n \longrightarrow S^{-1}\widehat{A}^n$ is an automorphism of a f.g. free $S^{-1}\widehat{A}$-module then the pullback

$$P(\alpha) \;=\; \{(x,y) \in S^{-1}A^n \oplus \widehat{A}^n \,|\, \alpha(x) = y \in S^{-1}\widehat{A}^n\}$$

is a f.g. projective A-module, and

$$\partial \;:\; K_1(S^{-1}\widehat{A}) \longrightarrow K_0(A) \;;\; \tau(\alpha : S^{-1}\widehat{A}^n \longrightarrow S^{-1}\widehat{A}^n) \longmapsto [P(\alpha)] - [A^n] \;.$$

It is possible to obtain an explicit projection for $P(\alpha)$ from the material in Appendix A of Swan [9], but the actual formula is much more complicated than in the cartesian case with onto maps. (I am grateful to Jim Davis for the reference to [9]).

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