Recent Advances in Topological Manifolds

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Introduction

A *topological n-manifold* is a Hausdorff space which is locally n-Euclidean (like $\mathbb{R}^n$).

No progress was made in their study (unlike that in PL and differentiable manifolds) until 1968 when Kirby, Siebenmann and Wall solved most questions for high dimensional manifolds (at least as much as for the PL and differentiable cases).

Question: can compact n-manifolds be triangulated? Yes, if $n \leqslant 3$ (Moise 1950's). This is unknown in general.

However, there exist manifolds (of dimension $\geqslant 5$) which don't have PL structures. (They might still have triangulations in which links of simplices aren't PL spheres.) There is machinery for deciding whether manifolds of dimension $\geqslant 5$ have a PL structure.

Not much is known about 4-manifolds in the topological, differentiable, and PL cases.

Question 2: the generalized Schönflies theorem. Let $B^n = \big\{x \in \mathbb{R}^{n+1} \colon \|x\| \leqslant 1\big\}$, $S^n = \big\{x \in \mathbb{R}^{n+1} \colon \|x\| = 1\big\}$. Given an embedding $f : B^n \to S^n$ (i.e. a 1-1 continuous map), is $\overline{S^n \setminus f(B^n)} \cong B^n$? No–the Alexander horned sphere.

Let $\lambda B^n = \big\{x \in \mathbb{R}^{n+1} \colon \|x\| \leqslant \lambda\big\}$. Question 2′: is $\overline{S^n \setminus f(\lambda B^n)} \cong B^n$ (where $0 < \lambda < 1$)? Yes.

In 1960, Morton Brown, Mazur, and Morse proved the following: if $g : S^{n-1} \times [-1, 1] \to S^n$ is an embedding, then $S^n \backslash g(S^{n-1} \times \{0\})$ has 2 components, D_1, D_2, such that $\overline{D_1} \cong \overline{D_2} \cong B^n$, which implies 2′ as a corollary. (The proof is easier than that of PL topology).

Question 3: the annulus conjecture. Let $f : B^n \to \operatorname{Int} B^n$ be an embedding. Is $\overline{B^n \setminus f(\frac{1}{2}B^n)} \cong \overline{B^n \setminus \frac{1}{2}B^n}$ $(\cong S^{n-1} \times I)$? In 1968, Kirby, Siebenmann, and Wall proved this for $n \geqslant 5$. This was already known for $n \leqslant 3$. The $n = 4$ case is still unknown.

Outline of course:

- Basic facts about topological manifolds
- Morton Brown's theorem – the first "recent" result

- Kirby's trick: $\mathrm{Homeo}(M)$ is a topological group (with the compact-open topology). This is locally contractible: any homeomorphism h near 1 can be joined by a path in $\mathrm{Homeo}(M)$ to 1
- Product structure theorem: if M^n is a topological manifold and $M \times \mathbb{R}^k$ has a PL structure, then M^n has a PL structure ($n \geqslant 5$).
- sketch of proof of the annulus conjecture (complete except for deep PL theorems).

1 Basic Properties of Topological Manifolds

Let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_n \geqslant 0\}$. Identify $\mathbb{R}^{n-1}$ with

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_n = 0\} = \partial \mathbb{R}^n_+.$$

Definition 1.1. A (topological) *n-manifold* (with boundary) is a Hausdorff space M such that each point of M has a neighborhood homeomorphic to $\mathbb{R}^n_+$. The *interior* of M, $\operatorname{Int} M$, is the set of points in M which have neighborhoods homeomorphic to $\mathbb{R}^n$. The *boundary* of M, $\partial M = M \setminus \operatorname{Int} M$.

$\operatorname{Int} M$ is an open set in M, ∂M is closed in M.

M is an *open manifold* if it is non-compact and $\partial M = \emptyset$.

M is a *closed manifold* if it is compact and $\partial M = \emptyset$.

Example. Any open subset of an n-manifold is an n-manifold.

Let M be a connected manifold with $\partial M = \emptyset$. If $x, y \in M$ then there is a homeomorphism $h : M \to M$ with $h(x) = y$.

Theorem 1.2 (Invariance of domain)**.** *Let $U, V \subset \mathbb{R}^n$ be subsets such that $U \cong V$. Then if U is open in $\mathbb{R}^n$, then so is V.*

Corollary 1.3. *If M is an N-manifold, then ∂M is an $(n-1)$-manifold without boundary.*

Proof. Suppose $x \in M$ and $f : \mathbb{R}^n_+ \to M$ be a homeomorphism onto a neighborhood N of x in M. Then

$$x \in \partial M \iff x \in f(\mathbb{R}^{n-1}) \tag{1}$$

If $x \notin f(\mathbb{R}^{n-1})$, then $x \in f(\mathbb{R}^n_+ \setminus \mathbb{R}^{n-1}) \cong \mathbb{R}^n$, so $x \in \operatorname{Int} M$ and $x \notin \partial M$.

If $x \notin \partial M$, then $x \in \operatorname{Int} M$, i.e. there is a neighborhood U of x homeomorphic to $\mathbb{R}^n \subset f(\mathbb{R}^n_+)$. So there is a neighborhood V of x which is open in M such that $V \subset U$, homeomorphic to an open set in $\mathbb{R}^n$. Therefore $f^{-1}(V) \subset \mathbb{R}^n_+ \subset \mathbb{R}^n$. By theorem 1.2, $f^{-1}(V)$ is open in $\mathbb{R}^n$.

Suppose $x \notin f(\mathbb{R}^{n-1})$. Then $f^{-1}(x) \in \mathbb{R}^{n-1}$, but then $f^{-1}(V)$ can't be a neighborhood of $f^{-1}(x)$, so $f^{-1}(V)$ is not open. This is a contradiction, therefore $x \in f(\mathbb{R}^{n-1}) \implies x \in \partial M$.

Now suppose $y \in \partial M$. Let $g : \mathbb{R}^n_+ \to M$ be a homeomorphism onto a neighborhood P of y in M. P contains an open neighborhood W of y in M.

Now $W \cap \partial M = W \cap g(\mathbb{R}^{n-1})$ by (1). Therefore $W \cap g(\mathbb{R}^{n-1})$ is a neighborhood of y in ∂M homeomorphic to an open set N, so y has a neighborhood in ∂M homeomorphic to $\mathbb{R}^{n-1}$, as required. ☐

Corollary 1.4. *If M^m, N^n are manifolds then $M \times N$ is an $(m+n)$-manifold with $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$, i.e.* $\operatorname{Int}(M \times N) = \operatorname{Int} M \times \operatorname{Int} N$.

Proof. If $x \in M \times N$, then x has a neighborhood homeomorphic to $\mathbb{R}^m_+ \times \mathbb{R}^n_+ \cong \mathbb{R}^{m+n}_+$, so $M \times N$ is an $(m+n)$-manifold.

Clearly $\operatorname{Int} M \times \operatorname{Int} N \subset \operatorname{Int}(M \times N)$.

If $x \in (\partial M \times N) \cup (M \times \partial N)$, then x has a neighborhood homeomorphic to $\mathbb{R}^m_+ \times \mathbb{R}^n$, $\mathbb{R}^m \times \mathbb{R}^n_+$, or $\mathbb{R}^m_+ \times \mathbb{R}^n_+$ – all homeomorphic to $\mathbb{R}^{m+n}_+$ by a homeomorphism carrying x to $\mathbb{R}^{m+n-1}$. By (1), $x \in \partial(M \times N)$. Hence the result. ☐

Example. Examples of manifolds:

- $\mathbb{R}^m$ is an m-manifold without boundary, open.
- S^m is a closed m-manifold. (Stereographic projection gives neighborhoods.)
- B^m is a compact manifold with boundary S^{m-1}.
- $\mathbb{R}^m_+$ is an m-manifold with boundary $\mathbb{R}^{m-1}$.
- Products of these,
- $\mathbb{C}P^n$, orthogonal groups $O(n)$ are manifolds.

These are all differentiable manifolds. There exist topological manifolds which do not possess a differentiable structure.

Lemma 1.5. *If $X \subset S^n$ is homeomorphic to B^k, then $\widetilde{H}_r(S^n \setminus X) = 0$ for all $r \in \mathbb{Z}$.*

Proof. By induction on k. The lemma is true if $k = 0$: $S^n \setminus \{\text{pt.}\} \cong \mathbb{R}^n$.

Assume true if $k = l$, we prove it for $k = l + 1$. Choose a homeomorphism $f : B^l \times I \cong B^{l+1} \to X$, suppose $\alpha \in \widetilde{H}_r(S^n \setminus X)$. Take $t \in I$. By induction hypothesis, $\widetilde{H}_r(S^n \setminus f(B^l \times \{t\})) = 0$. Therefore α is represented by the boundary of some singular chain c lying in $S^n \setminus f(B^l \times \{t\})$. There is a neighborhood N_t of $f(B^l \times \{t\})$ in S^n such that c lies in $S^n \setminus N_t$.

Therefore there is an open interval $J_t \subset I$ containing t such that c lies in $S^n \setminus f(B^l \times J_t)$. Since the unit interval is compact, we can cover by finitely many of the J_t's. Therefore there is a dissection $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $[t_{p-1}, t_p] \subset$ some J_t.

Let $\phi_{p,q} : \widetilde{H}_r(S^n \setminus X) \to \widetilde{H}_r(S^n \setminus f(B^l \times [t_p, t_q]))$ where $p < q$ and the map is induced by inclusion. Now $\phi_{p-1,p}(\alpha) = 0$ for all p.

Suppose inductively that $\phi_{0,i}(\alpha) = 0$ starts with $i = 1$. By the main inductive hypothesis, $\widetilde{H}_s(S^n \setminus f(B^l \times \{t_i\})) = 0$ for $s = r, r+1$. The sets $S^n \setminus f(B^l \times [t_p, t_q])$ are open. We have the lattice

$$\begin{tikzcd}
S^n \setminus f(B^l \times [0,t_i]) \arrow[r] & S^n \setminus f(B^l \times \{t_i\}) \\
S^n \setminus f(B^l \times [0,t_{i+1}]) \arrow[r] \arrow[u] & S^n \setminus f(B^l \times [t_i,t_{i+1}]) \arrow[u]
\end{tikzcd}$$

and the corresponding Mayer-Vietoris sequence:

$$\begin{aligned}0 \longrightarrow &\widetilde{H}_r(S^n \setminus f(B^l \times [0,t_{i+1}])) \\ &\longrightarrow \widetilde{H}_r(S^n \setminus f(B^l \times [0,t_i])) \oplus \widetilde{H}_r(S^n \setminus f(B^l \times [t_i,t_{i+1}])) \longrightarrow 0,\end{aligned}$$

with the maps induced by inclusion.

Since $\phi_{0,i}(\alpha) = 0$ and $\phi_{i,i+1}(\alpha) = 0$, we have $\phi_{0,i+1}(\alpha) = 0$. Therefore, $\phi_{0,k}(\alpha) = 0$, i.e. $\alpha = 0$ and $\widetilde{H}_r(S^n \setminus X) = 0$ as required. ☐

Lemma 1.6. *If $X \subset S^n$ is homeomorphic to S^k, then*

$$\widetilde{H}_r(S^n \setminus X) \cong \widetilde{H}_r(S^{n-k-1}) = \begin{cases} \mathbb{Z} & \text{if } r = n-k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By induction on k. The result is true if $k = 0$, for $S^k \setminus$ pair of points $\cong S^{n-1}$. (???) Now assume the result holds for $k = l-1$ and try to prove it for $k = l$.

Choose a homeomorphism $f : S^l \to X$. Let D_1, D_2 be northern and southern hemispheres of S^l so that $D_1 \cup D_2 = S^l$ and $D_1 \cap D_2 \cong S^{l-1}$. The sets $S^n \setminus X$, $S^n \setminus f(D_i)$, and $S^n \setminus f(D_1 \cap D_2)$ are open. We have the lattice

$$\begin{tikzcd}
S^n \setminus f(D_1) \arrow[r] & S^n \setminus f(D_1 \cap D_2) \\
S^n \setminus X \arrow[r] \arrow[u] & S^n \setminus f(D_2) \arrow[u]
\end{tikzcd}$$

and the Mayer-Vietoris sequence

$$0 \longrightarrow \widetilde{H}_{r+1}(S^n \setminus f(D_1 \cap D_2)) \longrightarrow \widetilde{H}_r(S^n \setminus X) \longrightarrow 0$$

since $\widetilde{H}_{r+1}(S^n \setminus f(D_1)) \cong \widetilde{H}_{r+1}(S^n \setminus f(D_2)) \cong 0$ by the previous lemma. The result follows from the inductive hypothesis. ☐

Corollary 1.7. *If $f : S^{n-1} \to S^n$ is 1-1 and continuous, then $S^n \setminus f(S^{n-1})$ has just two components.*

Proof. By 1.6, $\widetilde{H}_0(S^n \setminus f(S^{n-1})) \cong \widetilde{H}_0(S^0) \cong \mathbb{Z}$. Therefore, $S^n \setminus f(S^{n-1})$ has two components. ☐

Corollary 1.8. *If $f : B^n \to S^n$ is 1-1 and continuous, then $f(\operatorname{Int} B^n)$ is open in S^n.*

Proof. By lemma 1.5, $\widetilde{H}_0(S^n \setminus f(B^n)) = 0$, so $S^n \setminus f(B^n)$ is connected. Now $S^n \setminus f(S^{n-1}) = f(\operatorname{Int} B^n) \cup S^n \setminus f(B^n)$, and $f(\operatorname{Int} B^n)$ and $S^n \setminus f(B^n)$ are connected, while $S^n \setminus f(S^{n-1})$ is not (by corollary 1.7). Thus $f(\operatorname{Int} B^n)$ and $S^n \setminus f(B^n)$ are the components of $S^n \setminus f(S^{n-1})$, and are closed in $S^n \setminus f(S^{n-1})$.

$f(\operatorname{Int} B^n)$ is open in $S^n \setminus f(S^{n-1})$, therefore open in S^n. ☐

Proof of theorem 1.2. We have $U, V \subset \mathbb{R}^n$, a homeomorphism $f : U \to V$, U open in $\mathbb{R}^n$. Choose $x \in U$. Then there exists a closed n-ball $B^n \subset U$ with center x and a map $g : \mathbb{R}^n \to S^n$ which is a homeomorphism onto $g(\mathbb{R}^n)$ (e.g. the inverse of stereographic projection). We have that $gf : U \to S^n$ is 1-1 and continuous, so by 1.7, $gf(\operatorname{Int} B^n)$ is open in S^n and $f(B^n)$ is open in $\mathbb{R}^n$.

Now $f(x) \in f(\operatorname{Int} B^n) \subset f(U) = V$, so V is a neighborhood of $f(x)$. Since $V = f(U)$, V is open in $\mathbb{R}^n$. ☐

2 The Generalized Schönflies Theorem

Definition 2.1. If M, N are manifolds, an *embedding* of M in N is a map $f : M \to N$ which is a homeomorphism onto $f(M)$. (If M is compact then any 1-1 continuous map $f : M \to N$ is an embedding, but this is not true in general.)

Theorem 2.2 (Morton Brown's Schönflies Theorem)**.** *If $f : S^{n-1} \times [-1, 1] \to S^n$ is an embedding, then each component of $S^n \setminus f(X^{n-1} \times \{0\})$ has closure homeomorphic to B^n.*

Definition 2.3. Let M be a manifold and $X \subset \operatorname{Int} M$. X is *cellular* if it is closed and, for any open set U containing X there is a set $Y \subset U$ such that $Y \cong B^n$ and $X \subset \operatorname{Int} Y$.

Example. Any collapsible polyhedron in $\mathbb{R}^n$ is cellular.

If $f : B^n \to S^n$ is any embedding, then $\overline{S^n \setminus f(B^n)}$ is cellular.

Lemma 2.4. *If M is a manifold and $X \subset M$ is cellular, then M/X is homeomorphic to M by a homeomorphism fixed on ∂M.*

Proof. Since X is cellular, there is a $Y_0 \subset \operatorname{Int} M$ such that $Y_0 \cong B^n$ and $X \subset \operatorname{Int} Y_0$. Y_0 has a metric d. Let $U_r = \left\{y \in Y_0 \colon d(X, y) < \frac{1}{r}\right\}$. Define Y_r inductively: assume $Y_{r-1} \subset M$ is constructed with $X \subset \operatorname{Int} Y_{r-1}$. X is cellular implies that there is a $Y_r \subset (\operatorname{Int} Y_{r-1}) \cap U_r$ such that $Y_r \cong B^n$ and $X \subset \operatorname{Int} Y_r$, where $\operatorname{Int} Y_r$ is the interior or Y_r in M. We have

$$Y_0 \supset \operatorname{Int} Y_0 \supset Y_1 \supset \operatorname{Int} Y_1 \supset \cdots \supset X = \bigcap_{r=0}^{\infty} Y_r.$$

We construct homeomorphisms $h_r : M \to M$ such that

i. $h_0 = 1$,

ii. $h_r|_{M\setminus Y_{r-1}} = h_{r-1}|_{M\setminus Y_{r-1}}$, and

iii. $h_r(Y_r)$ has diameter $< \frac{1}{r}$ with respect to the metric d.

Suppose h_{r-1} is defined. Choose a homeomorphism $f : h_{r-1}(Y_{r-1}) \to B^n$. Now, $Y_r \subset \operatorname{Int} Y_{r-1}$, so $f(h_{r-1}(Y_r)) \subset \operatorname{Int} B^n$ and there is a $\lambda < 1$ and $\epsilon > 0$ such that $f(h_{r-1}(Y_r)) \subset \lambda B^n$ and $f^{-1}(\epsilon B^n)$ has diameter $< \frac{1}{r}$. There is a homeomorphism $g : B^n \to B^n$ such that $g|_{\partial B^n} = 1$ and $g(\lambda B^n) \subset \epsilon B^n$. Define $h_r : M \to M$ by

$$h_r(x) = \begin{cases} h_{r-1}(x) & \text{if } x \in M \setminus Y_{r-1}, \\ f^{-1}gfh_{r-1}(x) & \text{if } x \in Y_{r-1}. \end{cases}$$

To verify (3), note that

$$\begin{aligned} h_r(Y_r) &\subseteq f^{-1}gfh_{r-1}(Y_{r-1}) \\ &\subset f^{-1}g(\lambda B^n) \\ &\subset f^{-1}(\epsilon B^n) \end{aligned}$$

has diameter $< \frac{1}{r}$.

Define $h(x) = \lim_{r\to\infty} h_r(x)$ for each $x \in M$. If $x \in M \setminus X$, then $x \in M \setminus Y_r$ for some r, and $h_r(x) = h_{r+1}(x) = \cdots = h(x)$ by (2), so $h(x)$ exists. Since $h_r(Y_r) \supset h_{r+1}(Y_r) \supset \ldots$, with diameter $h_r(Y_r) \to 0$, $\bigcap_{r=1}^{\infty} h_r(Y_r) = \{y\}$ for some $y \in M$. If $x \in X$, $h_r(x) \in h_r(Y_r)$, so $d(h_r(x), y) < \frac{1}{r}$ by (3), so $h_r(x) \to y$ as $r \to \infty$ and $h(x) = y$.

h is continuous at $x \in M \setminus X$ because $h = h_r$ in a neighborhood of x for some r. h is continuous at $x \in X$ because Y_r is a neighborhood of x and $h(Y_r) \subset \frac{1}{r}$ neighborhood of Y. Thus h induces a continuous map $\hat{h} : M/X \to M$ with $\hat{h}|_{\partial M} = 1$.

Since h coincides with some h_r outside X, $h|_{M\setminus X} \to M \setminus \{y\}$ is a homeomorphism. $h(X) = y$, so $\hat{h}$ is bijective. Further, $\hat{h}|_{M\setminus X}$ is open: If U is a neighborhood of X is M, then $U \supset Y_r$ for some r, so $y \in h_{r+1}(Y_{r+1}) \subset \operatorname{Int} h_r(Y_r) \subset h(U)$ and $h(U)$ is a neighborhood of y, so h is open.

Therefore $\hat{h}$ is a homeomorphism. □

Lemma 2.5. *If $X \subset \operatorname{Int} B^n$ is closed and B^n/X is homeomorphic to some subset of S^n, then X is cellular.*

Proof. Let $f : B^n \to S^n$ induce an embedding $\hat{f} : B^n/X \to S^n$. Suppose $f(x) = y$. Then $f(B^n) = \hat{f}(B^n/X) \neq S^n$. (Apply theorem 1.2 to neighborhoods of points of ∂B^n). Let U be any neighborhood of X in B^n; $f(U)$ is a neighborhood of y in S^n. $f(B^n)$ is a proper closed subset of S^n.

There is a homeomorphism $h : S^n \to S^n$ such that $h|_V = 1$ for some neighborhood V of y and $h(f(B^n)) \subset f(U)$: there is a $Y \subset S^n$ such that $Y \cong B^n$ and $f(B^n) \subset \operatorname{Int} Y$. Let Z be a small convex ball with $y \in \operatorname{Int} Z$. The radial map gives the homeomorphism.

Define $g : B^n \to B^n$ by

$$g(x) = \begin{cases} f^{-1}hf(x) & \text{if } x \notin X, \\ x & \text{if } x \in X. \end{cases}$$

Here, $hf(x) \neq y$ implies that $f^{-1}hf(x)$ is well defined. g is continuous since $h = 1$ in a neighborhood of y. Also, g is 1-1. Now $g(B^n) \cong B^n$ and $g(B^n) \subset f^{-1}hf(B^n) \subset f^{-1}f(U) = U$, and $g = 1$ on a neighborhood of X. Therefore, $\operatorname{Int} g(B^n) \supset X$ and X is cellular. □

Proof of Theorem 2.2. $f : S^{n-1} \times [-1,1] \to S^n$ is an embedding, $S^n \setminus f(X^{n-1} \times \{0\})$ has two components, D_+ and D_-. Say $f(S^{n-1} \times \{-1\}) \subset D_-$. Let $X_+ = D_+ \setminus f(X^{n-1} \times (0,1))$ and $X_- = D_- \setminus f(X^{n-1} \times (-1,0))$.

Then X_+ and X_- are both closed, and $X_+ \cup X_- = S^n \setminus f(X^{n-1} \times (-1,1))$. Note that $(S^n/X_+)/X_- \cong (S^{n-1} \times [-1,1]/S^{n-1} \times \{-1\})/S^{n-1} \times \{1\} \cong S^n$. Therefore there is a map $g : S^n \to S^n$ such that $g(X_+) = y_+$, $g(X_-) = y_-$, and $g|_{S^n \setminus (X_+ \cup X_-)}$ is a homeomorphism onto $S^n \setminus \{y_+, y_-\}$ where y_+, y_- are the poles of S^n.

$X_+ \cup X_-$ is a proper closed subset of S^n, so there exists $Y \subset S^n$ with $Y \cong B^n$ and $X_+ \cup X_- \subset \operatorname{Int} Y$. Since $g(Y)$ is a proper closed subset of S^n, there is a homeomorphism $h : S^n \to S^n$ such that $h = 1$ on a neighborhood of y_- and $h(g(Y)) \subset S^n \setminus \{y_+, y_-\}$.

Define $\phi : Y \to S^n$ by

$$\phi(x) = \begin{cases} g^{-1}hg(x) & \text{if } x \notin X_-, \\ x & \text{if } x \in X_-. \end{cases}$$

Since $h = 1$ on a neighborhood of y_-, ϕ is injective on $Y \setminus X_+$ and $\phi(X_+) = g^{-1}h(y_+)$. Therefore ϕ induces an embedding $\hat{\phi} : Y/X_+ \to S^n$, $Y \cong$???????????????. By lemma 2.5, X_+ is cellular.

$\overline{D_+}$ is a manifold with $X_+ \subset D_+ = \operatorname{Int} \overline{D_+}$. By lemma 2.4, $\overline{D_+} \cong \overline{D_+}/X_+ \cong S^{n-1} \times [0,1]/S^{n-1} \times \{1\} \cong B^n$. Similarly for D_-. □

Corollary 2.6. *If $f, g : S^{n-1} \times [-1,1] \to S^n$ are embeddings, then there is a homeomorphism $h : S^n \to S^n$ such that*

$$hf|_{S^{n-1} \times \{0\}} = g|_{S^{n-1} \times \{0\}} .$$

Proof. If $\phi : \partial B^n \to \partial B^n$ is a homeomorphism, then ϕ extends to a homeomorphism $\phi : B^n \to B^n$ in an obvious way along radii: $\phi(rx) = r\phi(x)$ for $0 \leqslant r < 1, x \in \partial B^n$. Therefore, if Y_1, Y_2 are homeomorphic to balls and $\phi : \partial Y_1 \to \partial Y_2$ is a homeomorphism, then ϕ extends to a homeomorphism $\phi : Y_1 \to Y_2$.

Let D_+, D_- be the components of $S^n \setminus f(S^{n-1} \times \{0\})$ and E_+, E_- be the components of $S^n \setminus g(S^{n-1} \times \{0\})$. Define $h|_{f(S^{n-1} \times \{0\})}$ to be gf^{-1}, so $h : \partial\overline{D_+} \to \partial\overline{E_+}$. Since $\overline{D_+} \cong \overline{E_+} \cong B^n$, h can be extended to a homeomorphism $h : \overline{D_+} \to \overline{E_+}$.

Extend $h|_{\partial\overline{D_-}} \to \partial E_-$ (already defined) to a homeomorphism $h|_{\overline{D_-}} \to \overline{E_-}$. We obtain a homeomorphism $h : S^n \to S^n$ with $hf|_{S^{n-1}\times\{0\}} = g|_{S^{n-1}\times\{0\}}$. $\square$

Definition 2.7. A *collar* of ∂M in M is an embedding $f : \partial M \times I \to M$ such that $f(x, 0) = x$ for $x \in \partial M$.

Exercise. $f(\partial M \times I)$ is a neighborhood of ∂M in M.

Remark. From now on, we only consider metrizable manifolds, i.e. ones which are second countable.

Exercise. Compact manifolds are metrizable.

Theorem 2.8 (Morton Brown). *If M is metrizable, then ∂M has a collar in M.*

If U is an open set in ∂M, say that U is *collared* if U has a collar in the manifold $\operatorname{Int} M \cup U$.

Let $V \subset U$ be a smaller open set and $\lambda : U \to I = [0, 1]$ be a continuous map such that $\lambda(x) = 0$ iff $x \notin V$. Define a *spindle neighborhood* of V in $U \times I$ to be

$$S(V, \lambda) = \{(x, t) \in U \times I \colon t < \lambda(x)\}\,.$$

$S(V, \lambda)$ is open, therefore a neighborhood of $V \times \{0\}$.

Lemma 2.9. *Let $f : S(V, \lambda) \to U \times I$ be an embedding with $f|_{V\times\{0\}} = 1$. Then there is a homeomorphism $h : U \times I \to U \times I$ such that:*

i. $hf = 1$ on $S(V, \mu)$ for some μ such that $\mu \le \lambda$, and

ii. $h|_{U\times I\setminus f(S(V,\lambda))}$ is the identity.

Proof. Spindle neighborhoods form a base of neighborhoods of $V \times \{0\}$ in $U \times I$. Suppose $V \times \{0\} \subset W$, W open. Let d be a metric on U and define a metric d on $U \times I$ by

$$d((x, t), (x', t')) = d(x, x') + |t - t'|.$$

Let $\nu(x) = \min\{d(x, U \times I \setminus W), d(x, U \setminus V)\}$. Then $(x, t) \in S(V, \nu)$ implies that $t < \nu(x)$, and so $(x, t) \in W$. Therefore $S(V, \nu) \subset W$.

There exists μ such that $S(V, 2\mu) \subset S(V, \frac{1}{2}\lambda) \cap f(S(V, \frac{1}{2}\lambda))$. There is an embedding $g : U \times I \to U \times I$ defined by

$$(x, t) \longmapsto \begin{cases} (x, t) & \text{if } t \geqslant 2\mu(x), \\ (x, \mu(x) + \frac{1}{2}t) & \text{otherwise.} \end{cases}$$

g has image $U \times I \setminus S(V, \mu)$ and $g|_{U\times I\setminus S(V,2\mu)} = 1$.

Define $h : U \times I \to U \times I$ by

$$h(x) = \begin{cases} f^{-1}(x) & \text{if } x \in f(S(V, \mu)), \\ gfg^{-1}f^{-1}(x) & \text{if } x \in f(S(V, \lambda)) \setminus f(S(V, \mu)), \\ x & \text{otherwise.} \end{cases}$$

Continuity of h is simply verified. In fact, h is a homeomorphism such that $hf = 1$ on $S(V, \mu)$ and $h = 1$ off $f(S(V, \lambda))$. $\square$

Lemma 2.10. *If $U, V \subset \partial M$ are collared, then $U \cup V$ is collared.*

Proof. Let $f : U \times I \to M$, $g : V \times I \to M$ be collars. Choose $\lambda : U \cup V \to I$ so that $S(U \cap V, \lambda) \subset f^{-1}g(V \times I)$. Apply lemma 2.9 to the embedding

$$g^{-1}f : S(U \cap V, \lambda) \to V \times I.$$

There is an $S(U \cap V, \mu) \subset S(U \cap V, \lambda)$ and a homeomorphism $h : V \times I \to V \times I$ such that $hg^{-1}f|_{S(U \cap V, \mu)} = 1$. Then gh^{-1} and f agree on $S(U \cap V, \mu)$.

Define an open set $U_1 \subset U \times I$ by

$$U_1 = \{x \in U \times I \colon d(x, (U \setminus V) \times \{0\}) < d(x, (V \setminus U) \times \{0\})\}.$$

Define $V_1 \subset V \times I$ similarly. Let U_2 be

$$U_2 = \{y \in M \colon d(y, U \setminus V) < d(y, V \setminus U)\}$$

and define V_2 similarly. Then $U_1 \cap V_1 = \emptyset$ and $U_2 \cap V_2 = \emptyset$.

Put $U_3 = U_1 \cap f^{-1}(U_2)$, $V_3 = V_1 \cap hg^{-1}(V_2)$. Then U_3, V_3 are open, $U_3 \cap V_3 = \emptyset$, $f(U_3) \cap gh^{-1}(V_3) = \emptyset$, $(U \setminus V) \times \{0\} \subset U_3$, and $(V \setminus U) \times \{0\} \subset V_3$, so $W = U_3 \cup S(U \cap V, \mu) \cup V_3$ is a neighborhood of $(U \cup V) \times \{0\}$ in $(U \cup V) \times I$.

Define $\phi : W \to M$ by

$$\phi(x) = \begin{cases} f(x) & \text{if } x \in U_3 \cup S(U \cap V, \mu), \\ gh^{-1}(x) & \text{if } x \in S(U \cap V, \mu) \cup V_3. \end{cases}$$

Then ϕ is well defined, continuous, and 1-1.

There is a $\nu : U \cup V \to I$ such that $S(U \cup V, \nu) \subset W$. Define $\psi : (U \cup V) \times I \to M$ by $(x, t) \mapsto \phi(x, t\frac{\nu(x)}{2})$. This is continuous and 1-1, and hence an embedding by invariance of domain. □

Proof of theorem 2.8. Collared sets cover ∂M because if $x \in \partial M$, then there is a homeomorphism $f : \mathbb{R}^n \times \mathbb{R}_+ \to M$ onto a neighborhood of x in M. We proved in corollary 1.3 that $f(\mathbb{R}^n \times \{0\})$ contains a neighborhood U of x in ∂M. Then U has a collar given by $g : U \times I \to M$ sending $(y, t) \mapsto f(p_1 f^{-1}(y), t)$.

If ∂M is compact, then ∂M is collared by lemma 2.10. Before proceeding to the general case, we prove:

Lemma 2.11. *Let $U_\alpha, \alpha \in A$ be a disjoint family of open collared sets. Then $\bigcup_{\alpha \in A} U_\alpha$ is collared.*

Proof. Let $V_\alpha = \left\{y \in M \colon d(y, U_\alpha) < d(y, \bigcup_{\beta \neq \alpha} U_\beta)\right\}$. This is an open neighborhood of U_α in M, and $\alpha \neq \beta$ implies that $V_\alpha \cap V_\beta = \emptyset$.

Let $f_\alpha : U_\alpha \times I \to M$ be a collar of U_α. Let $W_\alpha = f_\alpha^{-1}(V_\alpha)$, a neighborhood of $U_\alpha \times \{0\}$ in $U_\alpha \times I$. There are maps $\nu_\alpha : U_\alpha \to I$ such that $S(U_\alpha, \nu_\alpha) \subset W_\alpha$. Define $g_\alpha : U_\alpha \times I \to M$ by

$$g_\alpha(x, t) = f_\alpha(x, \frac{t\nu_\alpha(x)}{2}) \in V_\alpha.$$

Define $g = \bigcup g_\alpha : (\bigcup_\alpha U_\alpha) \times I \to M$. This is a collar of $\bigcup_{\alpha \in A} U_\alpha$ in M. □

We have proved that if $X = \partial M$ then

i. X is covered by collared sets,

ii. a finite union of collared sets is collared,

iii. a disjoint union of collared sets is collared, and

iv. open subsets of collared sets are collared.

Then (i)–(iv) together with X metric imply that X is collared.

Lemma 2.12. *Any countable union of collared sets is collared.*

Proof. It is enough to consider countable nested unions $U = \bigcup_{n=1}^{\infty} U_n$ with $U_1 \subset U_2 \subset \ldots$.

Put $V_n = \{x \in U_n \colon d(x, X \setminus U_n) > 2^{-n}\}$. Then $U = \bigcup_{n=1}^{\infty} V_n$ since $x \in U_k$ means there is an $n > k$ such that $B(x, 2^{-n}) \subset U_k$. Therefore $d(x, X \setminus U_k) > 2^{-n}$, so $d(x, X \setminus U_n) > 2^{-n}$ and $x \in V_n$.

We have that $\overline{V_n} \subset V_{n+1}$. Let $A_k = V_{2k+1} \setminus \overline{V_{2k-1}}$ and $B_k = V_{2k+2} \setminus \overline{V_{2k}}$. Then $A = \bigcup_{k=1}^{\infty} A_k$ is a disjoint union of collared sets, hence collared. Similarly for $B = \bigcup_{k=1}^{\infty} B_k$. Now $U = A \cup B \cup V_2$ is collared. ☐

A family of subsets of X is *discrete* if each $x \in X$ has a neighbourhood which intersects at most one member of the family. Call a family of subsets of X *σ-discrete* if it is a countable union of locally finite discrete subfamilies (Kelley, p. 127).

Lemma 2.13. *Every open cover of a metric space X has a σ-discrete refinement.*

Proof (cf Kelley, p. 129). Let $\mathcal{U}$ be an open cover of a metric space X. If $U \in \mathcal{U}$ let $U_n = \{x \in U \colon d(x, X \setminus U) > 2^{-n}\}$. Then $d(U_n, X \setminus U_{n+1}) \geqslant 2^{-(n+1)}$.

Well order $\mathcal{U}$ by the relation $<$. Let $U_n^* = U_n \setminus \bigcup_{V<U} V_{n+1}$. If $U \neq V$ then $U < V$ or $U > V$. The first implies that $V_n^* \subset X \setminus U_{n+1}$, the second that $U_n^* \subset X \setminus V_{n+1}$, and in either case $d(U_n^*, V_n^*) \geqslant 2^{-(n+1)}$.

Let U_n' be an open $2^{-(n+2)}$ neighborhood of U_n^*, similarly for V_n'. If $U \neq V$, then $U_n' \cap V_n' = \emptyset$.

It is enough to prove that $\bigcup_{n,U} U_n' = X$. If $x \in X$ let U be the first (with respect to $<$) member of $\mathcal{U}$ containing x. Then $x \in U_n$ for some n and so $x \in U_n^* \subset U_n'$. Now $\{U_n'\}$ is a σ-discrete refinement of $\mathcal{U}$. ☐

☐ (theorem 2.8)

References:

Morton Brown: "A proof of the generalized Schönflies conjecture" Bull. Amer. Math. Soc. 66 (1960) 74–76

Morton Brown: "Locally flat embeddings of topological manifolds" Annals of Math. 75 (1962) 331–341

A shortened version of the second reference is included in the book "Topology of 3-manifolds.

Definition 2.14. Let M^m, N^n be manifolds without boundary. An embedding $f : M^m \to N^n$ is *locally flat* if for all $x \in M$, there is a neighborhood U of x and an embedding $F : U \times \mathbb{R}^{n-m} \to N^n$ such that $F(y, 0) = f(y)$ for $y \in U$.

Remark. There needn't be an embedding $G : M \times \mathbb{R}^{n-m} \to N$ such that $G(y, 0) = f(y)$ for all y. For example, $S^1 \to$ Möbius strip along the center line. This is locally flat but there is no embedding $S^1 \times \mathbb{R} \to M$ agreeing with the previous one on $S^1 \times \{0\}$.

Example. If $f : S^{n-1} \to S^n$ is locally flat then each component of $S^n \setminus f(X^{n-1})$ has closure homeomorphic to B^n.

If ∂M is compact and $f, g : \partial M \times I \to M$ are two collars, then there is a homeomorphism $h : M \to M$ such that hf agrees with g on $\partial M \times [0, \frac{1}{2}]$ and $h = 1$ outside $f(\partial M \times I) \cup g(\partial M \times I)$, so "the collaring of ∂M in M is unique." This is not true if ∂M is noncompact, e.g. Milnor's rising sun.

Exercise. Suggest a generalization that does work.

Given two manifolds M^m, N^n let $\mathcal{E}(M, N)$ be the set of embeddings of M in N with the compact-open topology.

A continuous map $f : X \to Y$ is *proper* if $C \subseteq Y$ compact implies $f^{-1}(C) \subseteq X$ is compact.

Let $\mathcal{E}_p(M, N)$ be the set of embeddings which are proper maps. We will be interested in $\mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n)$, which consists of embeddings $f : \mathbb{R}^n \setminus \operatorname{Int} B^n \to \mathbb{R}^n$ *onto* neighborhoods of ∞ (by propriety).

Let $\widehat{\mathbb{R}^n}$ be the one point compactification of $\mathbb{R}^n$. $f : \mathbb{R}^n \setminus \operatorname{Int} B^n \to \mathbb{R}^n$ extends to a continuous map $\widehat{f} : \widehat{\mathbb{R}^n} \setminus \operatorname{Int} B^n \to \widehat{\mathbb{R}^n}$ with $\widehat{f}(\infty) = \infty$ iff f is proper.

(In general, $f : X \to Y$ extends to a continuous map $\widehat{f} : \widehat{X} \to \widehat{Y}$ with $\widehat{f}(\infty) = \infty$ iff f is proper.)

Theorem 2.15. *There is a neighborhood U of 1 in $\mathcal{E}(6B^n \setminus \operatorname{Int} B^n, \mathbb{R}^n)$ and a continuous map $\theta : U \to \mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n)$ such that $\theta(f)|_{S^{n-1}} = f|_{S^{n-1}}$.*

Proof. Take $U = \{f \in \mathcal{E}(6B^n \setminus \operatorname{Int} B^n, \mathbb{R}^n) : d(x, f(x)) < 1, x \in 6B^n \setminus \operatorname{Int} B^n\}$. If $f \in U$, then $f(2B^n \setminus \operatorname{Int} B^n) \subseteq \operatorname{Int} 3B^n \setminus \{0\}$ and $f(6B^n \setminus \operatorname{Int} 5B^n) \subset f(7B^n \setminus \operatorname{Int} 4B^n)$.

Define inductively $f_k : (4k + 6)B^n \setminus \operatorname{Int} B^n \to \mathbb{R}^n$ such that

i. $f_0 = f$,

ii. $f_{k+1}|_{(4k+5)B^n \setminus \operatorname{Int} B^n} = f_k|_{(4k+5)B^n \setminus \operatorname{Int} B^n}$,

iii. $f_k((4r + 6)B^n \setminus \operatorname{Int}(4r + 5)B^n) \subset \operatorname{Int}(4r + 7)B^n \setminus (4r + 4)B^n$ for $r \leqslant k$, and

iv. f_k depends continuously on f.

Suppose f_k is constructed. If $c, d \in (a, b)$ $(a, b, c, d \in \mathbb{R})$, let $\rho(a, b, c, d) : \mathbb{R}^n \to \mathbb{R}^n$ be the radial homeomorphism fixed outside $6B^n \setminus aB^n$ taking cB^n onto dB^n. Let $\rho_k(a, b, c, d) = \rho(4k + a, 4k + b, 4k + c, 4k + d)$.

Define $g_k : (4k+6)B^n \setminus \operatorname{Int} B^n \to \mathbb{R}^n$ by $g_k = \rho_k(3, 11, 4, 8) f_k \rho_k(1, 5\frac{2}{3}, 5\frac{1}{3}, 2)$. Define a homeomorphism $h_k : \mathbb{R}^n \to \mathbb{R}^n$ by

$$h_k(x) = \begin{cases} f_k \rho_k(1, 5\frac{2}{3}, 2, 5\frac{1}{3}) f_k^{-1} & \text{if } x \text{ is in the image of } f_k, \\ x & \text{otherwise.} \end{cases}$$

Let $\sigma_k : (4k + 10)B^n \to (4k + 6)B^n$ be a radial homeomorphism fixed on $(4k + 5)B^n$, sending $(4k + 6)B^n \to (4k + 5\frac{1}{2})B^n$ and $(4k + 9)B^n \to (4k + 5\frac{2}{3})B^n$.

Define $f_{k+1} = h_k g_k \sigma_k : (4k+10)B^n \setminus \operatorname{Int} B^n \to \mathbb{R}^n$. Check (ii): let $x \in (4k+5)B^n$ so $\sigma_k(x) = x$, $\rho_k(1, 5\frac{2}{3}, 5\frac{1}{3}, 2)(x) \in (4k+2)B^n$, $y = f_k \rho_k(1, 5\frac{2}{3}, 5\frac{1}{3}, 2)(x) \in (4k+3)B^n$ (inductive hypothesis), $g_k(x) = \rho_k(3, 11, 4, 8)(y) = y$ etc. XXXXXXXXXX

Similarly we can verify (iii). To prove (iv), that f_k depends continuously on f, it is enough to show that h_k depends continuously on f_k. Let f'_k be near f_k, and let

$$h'_k = \begin{cases} f'_k \rho_k f_k^{-1} & \text{on } \operatorname{Im} f \text{ where } \rho_k = \rho_k(1, 5\frac{2}{3}, 2, 5\frac{1}{2}), \\ 1 & \text{otherwise.} \end{cases}$$

If C is a compact set in $\mathbb{R}^n$, we must prove that $\sup_{x \in C} d(h_k x, h'_k x)$ can be made less than ϵ by requiring $d(f_k y, f'_k y) = \delta$ for all $y \in A_k$, $\epsilon > 0$.

Let $A_k = (4k + 6)B^n \setminus \operatorname{Int} B^n =$ domain of f_k. Given $\epsilon > 0$ there is an $\eta > 0$ such that $y, y' \in A$ and $d(y, y') < \eta$ imply $d(f_k \rho_k(y), f_k \rho_k(y')) < \frac{\epsilon}{2}$. Since δ_k is injective, there is a $\delta > 0$ such that $y, y' \in A$ and $d(y, y') \geq \eta$ imply $d(f_k y, f_k y') \geq \delta$. We suppose $\delta < \frac{\epsilon}{2}$. Suppose $d(f_k y, f'_k y) < \frac{\delta}{2}$ for all $y \in A$. Let $x \in C$. We split into cases:

i. $x \in \operatorname{Im} f_k \cap \operatorname{Im} f'_k$, say $x = f_k y_k = f'_k y'_k$. Then $d(f_k y_k, f_k y'_k) = d(f'_k y'_k, f_k y'_k) < \frac{\delta}{2} < \delta$. Therefore $d(y, y') < \eta$, so

$$\begin{aligned} d(h_k x, h'_k x) &= d(f_k \rho_k y_k, f'_k \rho_k y'_k) \\ &\leqslant d(f_k \rho_k y_k, f_k \rho_k y'_k) + d(f_k \rho_k y'_k, f'_k \rho_k y'_k) \\ &< \frac{\epsilon}{2} + \frac{\delta}{2} \\ &< \epsilon \end{aligned}$$

ii. If $x \in \operatorname{Im} f_k \setminus \operatorname{Im} f'_k$, say $x = f_k(y)$, then $d(x, f'_k y) < \frac{\delta}{2}$, so there is a $z \in \partial A$ such that $d(f'_k z, f'_k y) < XXXXX$ and $d(x, f'_k z) < \frac{\delta}{2}$. But $d(f'_k z, f_k z) < \frac{\delta}{2}$, so $d(x, f_k z) < \delta$, so $d(y, z) < \eta$. Therefore

$$\begin{aligned} d(h_k x, h'_k x) &= d(f_k \rho_k y, x) \\ &\leqslant d(f_k \rho_k y, f_k \rho_k z) + d(f_k z, x) \\ &< \frac{\epsilon}{2} + \delta \\ &< \epsilon. \end{aligned}$$

Here we used the fact that $f_k \rho_k z = z$ since $z \in \partial A$.

iii. If $x \in \operatorname{Im} f_k' \setminus \operatorname{Im} f_k$, the proof is similar.

iv. If $x \notin \operatorname{Im} f_k' \cup \operatorname{Im} f_k$, there is nothing to prove.

We have proved that $f_k \mapsto h_k$ is continuous. $f \mapsto f_{k+1}$ is continuous if $f \mapsto f_k$ is, so the induction is complete.

Define $\theta : U \to \mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n)$ by $\theta(f)(x) = f_k(x)$ for k large and $x \in (4k+5)B^n$. Then $\theta(f)$ is proper (interleaving property (iii)). Also $\theta(f)$ is an embedding, so $\theta(f) \in \mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n)$. $\theta(f)$ depends continuously on f because f_k agrees with f on $(4k+5)B^n$ and f_k depends continuously on f. $\square$

Corollary 2.16. *If* $0 < \lambda < 1$*, there is a neighborhood* V *of* $1 \in \mathcal{E}(B^n \setminus \operatorname{Int} \lambda B^n, \mathbb{R}^n)$ *and a continuous map* $\phi : V \to \mathcal{E}(B^n, \mathbb{R}^n)$ *such that for all* f, $\phi(f)|_{S^{n-1}} = f|_{S^{n-1}}$.

Proof. Let $\widehat{X}$ be the one point compactification of X. $g : X \to Y$ is proper iff g extends to $\widehat{g} : \widehat{X} \to \widehat{Y}$ with $\widehat{g}(\infty) = \infty$.

Example. The map $g \mapsto \widehat{g}$ is not continuous, even if $X = Y = \mathbb{R}^n$.

We first prove that $f \mapsto \widehat{f}$ is continuous (XXXXX seems to contradict the above). Suppose $f \in \mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n)$, $C \subset \mathbb{R}$ is compact, $U \subset \widehat{\mathbb{R}^n}$ is open, and $\widehat{f}(C) = U$. If $\infty \notin C$, C is a compact set in $\mathbb{R}^n \setminus \operatorname{Int} B^n$ so $\{g \in \mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n) \colon g(C) \subset U \cap \mathbb{R}^n\}$ is a neighborhood of f, mapping into a given neighborhood of $\widehat{f}$.

If $\infty \in C$, then $\infty = \widehat{f}(\infty) \in U$ open in $\widehat{\mathbb{R}^n}$ and there is a k such that $\widehat{\mathbb{R}^n} \setminus kB^n \subset U$. Since f is proper, there is an l such that $f^{-1}(2kB^n) \subset lB^n$. Let $N = \big\{g \in \mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n) \colon g(C \cap lB^n) \subset U \cap \mathbb{R}^n, g(lS^{n-1}) \subset \mathbb{R}^n \setminus kB^n\big\}$. This is open in $\mathcal{E}_p$ and contains f.

Now we have to show that $\widehat{g}(C) \subset U$ for all $g \in N$.

$$\begin{aligned} \widehat{g}(C) &= g(C \cap lB^n) \cup \widehat{g}(\widehat{\mathbb{R}^n} \setminus \operatorname{Int} lB^n) \\ &\subset U \cup \text{one of the complementary domains of } g(lS^{n-1}). \end{aligned}$$

In fact $U \cup$outside domain $\subset \widehat{\mathbb{R}^n} \setminus kB^n \subset U$. Hence the map $g \mapsto \widehat{g}$ is continuous.

$$\begin{tikzcd}
\mathcal{E}(6B^n \setminus \operatorname{Int} B^n, \mathbb{R}^n) \supset U \arrow[r] \arrow[dr] & \mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n) \arrow[d] \\
 & \mathcal{E}_p(\widehat{\mathbb{R}^n} \setminus \operatorname{Int} B^n, \widehat{\mathbb{R}^n}).
\end{tikzcd}$$

There exists a homeomorphism $h : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ with

$$h(x) = \begin{cases} \frac{x}{\|x\|^2} & \text{if } x \neq 0, \infty, \\ \infty & \text{if } x = 0, \text{ and} \\ 0 & \text{if } x = \infty, \end{cases}$$

carrying $6B^n \setminus \operatorname{Int} B^n$ onto $B^n \setminus \operatorname{Int} \frac{1}{6}B^n$ taking $\widehat{\mathbb{R}^n} \setminus \operatorname{Int} B^n \to B^n$. Hence the result. (XXXXXX: really?) $\square$

3 Properties of Tori

Definition 3.1. Let $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$. Then $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is the *n-dimensional torus.* Clearly $T^n \cong S^1 \times \cdots \times S^1$, n copies of S^1.

Let $e : \mathbb{R}^n \to T^n$ be the projection map. If $a \in \mathbb{Z}^n$, let $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ send $x \mapsto a + x$.

Proposition 3.2. *$e : \mathbb{R}^n \to T^n$ is a universal covering of T^n. If X is a 1-connected space and $f : X \to T^n$ is any map the there is an $\widetilde{f} : X \to \mathbb{R}$ such that $f = e\widetilde{f}$. ($\widetilde{f}$ is a* lift *of f.) If $\widetilde{f}_1, \widetilde{f}_2$ are lifts of f then $\widetilde{f}_1 = \tau_a \widetilde{f}_2$ for some $a \in \mathbb{Z}^n$.*

If X is simply connected and $f : X \times T^n \to X \times T^n$ is a map, then there exists an $\widetilde{f} : X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ such that $e\widetilde{f} = fe$.

Lemma 3.3. *If f is a homeomorphism, so is $\widetilde{f}$; if f is homotopic to the identity, then $\widetilde{f}$ commutes with the covering translations.*

Proof. Let f be the homeomorphism and g its inverse. We have

$$\begin{aligned} e\widetilde{f}\widetilde{g} &= fe\widetilde{g} \\ &= fge \\ &= e, \end{aligned}$$

so $\widetilde{f}\widetilde{g} = \tau_a$ for some a. Similarly $\widetilde{g}\widetilde{f} = \tau_b$. Therefore $\widetilde{f}$ is a homeomorphism.

Suppose $F : X \times T^n \times I \to X \times T^n$ has $F_0 = f$ and $F_1 = 1$. By 3.2 there is an $\widetilde{F} : X \times \mathbb{R}^n \times I \to X \times \mathbb{R}^n$ with $e\widetilde{F} = Fe$. We have

$$\begin{aligned} e\tau_{-a}\widetilde{F}\tau_a &= e\widetilde{F}\tau_a \\ &= Fe\tau_a \\ &= Fe \\ &= eF. \end{aligned}$$

Therefore there is a $b \in \mathbb{Z}^n$ so that $\tau_{-a}\widetilde{F}\tau_a = \tau_b\widetilde{F}$. We have $e\widetilde{F}_1 = F_1 e = e$, so $\widetilde{F}_1 = \tau_c$ for some c. But $\tau_{-a}\tau_c\tau_a = \tau_b\tau_c$, therefore $b = 0$ and $\tau_b = 1$. Thus $\tau_{-a}\widetilde{F}\tau_a = \widetilde{F}$. Since $F_0 = f, \widetilde{F}_0 = \tau_d\widetilde{f}$ for some d. Therefore $\widetilde{f}$ commutes with τ_d. $\square$

Definition 3.4. Let M, N be manifolds. An *immersion* $f : M \to N$ is a map such that each point $x \in M$ has a neighborhood U_x with $f|_{U_x}$ an embedding. If U_x can be chosen so that $f|_{U_x}$ is locally flat, then f is a *locally flat immersion.*

Theorem 3.5. *There is an immersion of $T^n \setminus$ point in $\mathbb{R}^n$.*

Proof. $T^n \setminus \mathrm{pt}$ is an open parallelizable manifold. Therefore, by Hirsch's theory of immersions there is a C^∞ immersion $T^n \setminus \mathrm{pt} \to \mathbb{R}^n$.

Alternately, regard T^n as the product of n circles, $T = T^1 =$ circle. Let J be a closed interval in T. $T^n \setminus J^n \cong T^n \setminus \mathrm{pt}$. Assume inductively that there is an immersion $f_n : T^n \setminus J^n \to \mathbb{R}^n$ such that $f_n \times 1 : (T^n \setminus J^n) \times [-1,1] \to \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ extends to an immersion $g_n : T^n \times [-1,1] \to \mathbb{R}^{n+1}$.

The induction starts with $n = 1$. Let $\phi_0 : \overline{T \setminus J} \to [-1,1]$. Choose an embedding $\phi : \mathbb{R} \times T \to \mathbb{R} \times \mathbb{R}$ such that if $(x,t) \in [-1,1] \times \overline{T \setminus J}$ then $\phi(x,t) = (x, \phi_0(t))$. Extend $\phi_0^{-1} : [-1,1] \to \overline{T \setminus J}$ to an embedding $\psi : \mathbb{R} \to T$.

Suppose f_n, g_n constructed. We have $T^{n+1} \setminus J^{n+1} = (T^n \setminus J^n) \times T \cup T^n \times (T \setminus J)$. Define $f'_{n+1} : T^{n+1} \setminus J^{n+1} \to \mathbb{R}^{n+1}$ by

$$f'_{n+1} = (1_{\mathbb{R}^{n-1}} \times \phi)[(f_n \times 1_T) \cup (1 \times \psi)g_n(1_{T^n} \times \psi^{-1})].$$

On $(T^n \setminus J^n) \times (T \setminus J)$, $g_n = f_n \times 1$ so $(1 \times \phi)g(1 \times \psi^{-1}) = (1 \times \psi)(f_n \times 1)(1 \times \psi^{-1}) = f_n \times 1$. Let $J' = T \setminus \phi(-\frac{1}{4}, \frac{1}{4})$, so $J \subset \operatorname{Int} J'$.

We shall construct an immersion $g'_{n+1} : T^{n+1} \times [-1,1] \to \mathbb{R}^{n+2}$ which agrees with $f'_{n+1} \times I$ on $T^{n+1} \setminus (J')^{n+1} \times [-\frac{1}{4}, \frac{1}{4}]$. This will be enough, since $T^{n+1} \setminus (J')^{n+1} \cong T^{n+1} \setminus J^{n+1}$.

Define $\theta_t : \mathbb{C} \to \mathbb{C}$ $(= \mathbb{R}^2)$ by

$$\theta_t(z) = \begin{cases} z & \text{if } |z| \leq \frac{1}{2}, \\ z e^{2(|z| - \frac{1}{2})\pi i t} & \text{if } \frac{1}{2} \leq |z| \leq \frac{3}{4}, \text{ and} \\ z e^{\frac{\pi i t}{2}} & \text{if } |z| \geq \frac{3}{4}. \end{cases}$$

Let $J'' = T \setminus \psi(-\frac{3}{4}, \frac{3}{4})$ and $\lambda : T^n \to [0,1]$ be continuous such that $\lambda|_{(J'')^n} = 1$ and $\lambda|_{T^n \setminus (J')^n} = 0$.

Define $g'_{n+1}|_{(T^n \setminus J^n) \times T \times [-1,1]} \to \mathbb{R}^{n+1} \times \mathbb{R}$ by $g'_{n+1}(x,t,u) = (1 \times \theta_{\lambda(x)})(\phi'_{n+1}(x,t), u)$ and define $g'_{n+1}|_{T^n \times (T \setminus J) \times [-1,1]} \to \mathbb{R}^n \times \mathbb{R}$ by $g'_{n+1}(x,t,u) = (f'_{n+1} \times 1)(x, (\psi \times 1)\theta_{\lambda(x)}(\psi^{-1}(t), u))$. Then $g'_{n+1}|_{T^{n+1} \setminus (J')^{n+1}}$ agrees with $f'_{n+1} \times 1$. Define $g'_{n+1}|_{(J'')^{n+1}}$ to be the restriction of

$$\sigma_n(1 \times \phi)\sigma_n(g_n \times 1)\sigma_{n+1}(1 \times \tau) : T^n \times T \times [1,1] \to \mathbb{R}^{n+2}$$

where σ_j swaps the jth and $(j+1)$th factors in the $(n+1)$-fold product and $\tau : [-1,1] \to [-1,1]$ changes sign.

This proof has no end XXXXXXXXXXXXXXX ☐

4 Local Contractibility

Definition 4.1. A space X is *locally contractible* if for each point $x \in X$ and each neighborhood U of x, there is a neighborhood V of x and homotopy $H : V \times I \to U$ such that $H_0 = 1$ and $H_1(V) = x$.

Let X^I be the set of paths in X ending at x. It is enough to find a neighborhood V and map $\phi : V \to X^I$ such that $\phi(y)$ is a path from y to x and $\phi(x)$

is the constant path at x. (Given an open neighborhood U of x, U^I is the open set in X^I so that there is a neighborhood of x in X such that $\phi(V') \subset U^I$.)

If M is a manifold, let $\mathcal{H}(M)$ be the space of homeomorphisms of M together with the compact-open topology.

Definition 4.2. An *isotopy* of M is a path in $\mathcal{H}(M)$. Equivalently, an isotopy is a homeomorphism $H : M \times I \to M \times I$ such that $p_2 H = p_2$. We say that H is an *isotopy from* H_0 *to* H_1, and H_0, H_1 are *isotopic.*

Theorem 4.3 (Černavsky, Kirby)**.** *$\mathcal{H}(\mathbb{R}^n)$ is locally contractible.*

Proof. $\mathcal{H}(\mathbb{R}^n)$ is a group, so it is enough to show that it is locally contractible at 1.

Choose an embedding $i : 4B^n \to T^n$ and choose an immersion $f : T^n \setminus i(0) \to \mathbb{R}^n$. $T^n \setminus i(\operatorname{Int} B^n)$ is compact, so there is a $\delta > 0$ such that for all $x \in T^n \setminus i(\operatorname{Int} B^n)$, $f|_{N_\delta(x)}$ is injective. We may suppose $\delta < d(i(3B^n \setminus \operatorname{Int} 2B^n), i(4S^{n-1} \cup S^{n-1}))$. Since f is open $\epsilon_x = d(f(x), \mathbb{R}^n \setminus N_\delta(f(x))) > 0$ and $\epsilon = \inf\{\epsilon_x \colon x \in T^n \setminus i(\operatorname{Int} B^n)\} > 0$.

If $x \in T^n \setminus i(\operatorname{Int} B^n)$ and $v \in \mathbb{R}^n$ are such that $d(f(x), v) < \epsilon$ then there exists a unique $u \in N_\delta(x)$ such that $f(u) = v$.

Let $h \in \mathcal{H}(\mathbb{R}^n)$. Suppose h is so close to 1 that $d(h(f(x)), f(x)) < \epsilon$ for all $x \in T^n \setminus i(\operatorname{Int} B^n)$. For $x \in T^n \setminus i(\operatorname{Int} 2B^n)$, let $h'(x)$ be the unique point in $N_\delta(x)$ such that $fh'(x) = hf(x)$, $h'(x) \in T^n \setminus i(\operatorname{Int} B^n)$. Since f is an open immersion, h' is an open immersion. If $h'(x) = h'(y)$, then $x, y \in N_\delta(h'(x))$ mean that $f(x) \neq f(y)$ which implies that $h'(x) \neq h'(y)$, a contradiction. Therefore h' is an embedding depending continuously on $h \in \mathcal{H}(\mathbb{R}^n)$.

Consider $i^{-1}h'i : 3B^n \setminus \operatorname{Int} 2B^n \to \operatorname{Int} 4B^n$. By corollary 2.16 there is a neighborhood W of 1 in $\mathcal{E}(3B^n \setminus \operatorname{Int} 2B^n, \operatorname{Int} 4B^n)$ and continuous map $\phi : W \to \mathcal{E}(3B^n, \operatorname{Int} 4B^n)$ such that $\phi(g)|_{3S^{n-1}} = g|_{3S^{n-1}}$. Define $h'' : T^n \to T^n$ by

$$h''(x) = \begin{cases} h'(x) & \text{if } x \notin i(3B^n), \\ i\phi(i^{-1}h'i)i^{-1}(x) & \text{if } x \in i(3B^n). \end{cases}$$

Then h'' is a homeomorphism, depending continuously on $h \in V$ where $V = \big\{h \in \mathcal{H}(\mathbb{R}^n) \colon h' \text{ is defined and } i^{-1}h'i \in W\big\}$. If V is sufficiently small, then $h \in V$ implies that h'' is homotopic to 1.

Let $e : \mathbb{R}^n \to T^n$ be the (universal) covering map. By 3.3 there exists a homeomorphism $\widetilde{h} : \mathbb{R}^n \to \mathbb{R}^n$ such that $e\widetilde{h} = h''e$. If V is sufficiently small, there is a unique choice of $\widetilde{h}$ such that $d(\widetilde{h}(0), 0) < \frac{1}{2}$. Then $\widetilde{h}$ depends continuously on h. By 3.3, $\widetilde{h}$ commutes with covering translations. Let $I = [0, 1]$: every point of $\mathbb{R}^n$ can be moved into I^n by covering translations.

If $A = \sup_{x \in I^n} d(\widetilde{h}(x), x) < \infty$, we have $d(\widetilde{h}(x), x) \leqslant A$ for all $x \in \mathbb{R}^n$, that is, $\widetilde{h}$ is a bounded homeomorphism of $\mathbb{R}^n$.

Suppose without loss of generality that $e(0) \notin i(4B^n)$. Choose once and for all $r > 0$ such that $f|_{e(rB^n)}$ is injective and $r < 1$ and $e(rB^n) \cap i(4B^n) = \emptyset$.

Define a homeomorphism $\rho : \operatorname{Int} B^n \to \mathbb{R}^n$ fixed on rB^n by

$$\rho(x) = \begin{cases} x & \text{if } x \in rB^n, \\ \frac{r-1}{|x|-1}x & \text{if } x \notin rB^n. \end{cases}$$

Then $\rho^{-1}\widetilde{h}\rho$ is a homeomorphism from $\operatorname{Int} B^n \to \operatorname{Int} B^n$ fixed on rB^n. Suppose $|x| < 1$ is close to 1. Then $d(x, \rho^{-1}\widetilde{h}\rho(x)) \leq \frac{2A(|x|-1)}{r-1} \to 0$ as $|x| \to 1$. So $\rho^{-1}\widetilde{h}\rho$ extends to a homeomorphism of B^n, fixed on ∂B^n.

Define an isotopy R_t of B^n by

$$R_t(x) = \begin{cases} x & \text{if } |x| \geqslant t, \\ t\rho^{-1}\widetilde{h}\rho(\frac{x}{t}) & \text{if } |x| < t. \end{cases}$$

Extend $fe : rB^n \to \mathbb{R}^n$ to a homeomorphism $\sigma : \operatorname{Int} B^n \to \mathbb{R}^n$ (e.g. by Schönflies theorem). Choose s, $0 < s < r$. If V is small enough, $h \in V$ implies that $\widetilde{h}(sB^n) \subset \operatorname{Int} tB^n$.

Define an isotopy S_t of $\mathbb{R}^n$ by $S_t(x) = \sigma R_t \sigma^{-1}(x)$. This depends continuously on h. $S_0 = 1$, and $S_1|_{fe(sB^n)} = h|_{fe(sB^n)}$. Without loss of generality, $0 \in \operatorname{Int} fe(sB^n)$. $S_1^{-1}h$ is 1 on a neighborhood of 0.

Define $F_t : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F_t(x) = \begin{cases} t^{-1}S^{-1}h(tx) & \text{if } t \neq 0, \\ x & \text{if } t = 0. \end{cases}$$

Define $H_t = S_t F_t$, i.e. $H_t(x) = S_t(F_t(x))$. This is an isotopy from 1 to h. H_t depends continuously on $h \in V$, and $h = 1$ implies that $H_t = 1$. So $\mathcal{H}(\mathbb{R}^n)$ is locally contractible. □

What about $\mathcal{H}(M)$ for (say) M compact? (Use a handle decomposition.)

Let $\mathcal{E}$(k-handle) be the space of embeddings of $B^k \times B^n \to B^k \times \mathbb{R}^n$ leaving $(\partial B^k) \times B^n$ fixed.

Theorem 4.4. *There is a neighborhood V of 1 in $\mathcal{E}$(k-handle) and a homotopy $H : V \times I \to \mathcal{E}$(k-handle) such that*

i. $H_t(1) = 1$ *for all* t,

ii. $H_0(h) = h$ *for all* $h \in V$,

iii. $H_t(h)|_{B^k \times \frac{1}{2}B^n} = 1$, *and*

iv. $H_t(h)|_{\partial B^k \times B^n} = h|_{\partial B^k \times B^n}$ *for all* t, h.

Proof. Let $i : 4B^n \to T^n$ be a fixed embedding, $f : T^n \setminus \{0\} \to \operatorname{Int} B^n$ a fixed immersion. Choose $0 < r < 1$ such that $f|_{e(rB^n)}$ is injective and $e(rB^n) \cap i(4B^n) = \emptyset$. Modify f so that $f(e(\operatorname{Int} rB^n)) \supset \frac{1}{2}B^n$.

XXX: this junk is all messed up: Let $h \in \mathcal{E}(\text{k-handle})$ be close to 1. Define a preliminary isotopy G from h to $g \in \mathcal{E}(\text{k-handle})$ such that XXXXXXX:

$$G_t(x,y) = \begin{cases} (x,y) & \text{if } |x| \geq 1 - \frac{t}{2}, \\ ((1-\frac{t}{2})h_1((1-\frac{t}{2})^{-1}x, y), h_2(x,y)) & \text{if } |x| \leqslant 1 - \frac{t}{2} \end{cases}$$

where $h(x,y) = XXXX$. $G_0 = h$, $G_1 = g$ is an embedding fixed on $\overline{B^n \setminus \frac{1}{2}B^k} \times \frac{3}{4}B^n$. G depends continuously on h and

$$G_t|_{B^k \times \partial B^n} = h|_{B^k \times \partial B^n}.$$

As in 4.3 construct an embedding $g' : B^k \times (T^n \setminus i(\operatorname{Int} 2B^n)) \to B^k \times (T^n \setminus i(\operatorname{Int} B^n))$ such that $(1 \times f)g' = g(1 \times f)$ and $g'|_{\overline{B^k \setminus \frac{1}{2}B^k} \times (T^n \setminus -)} = 1$.

Put $g'|_{\overline{B^k \setminus \frac{1}{2}B^k} \times T^n} = 1$. This extends the g' defined above. Use 2.16 to extend $g'|_{\frac{3}{4}B^k \times i(3B^n) \setminus \operatorname{Int}(\frac{1}{2}B^k \times i(2B^n))}$ to an embedding $g'' : \frac{3}{4}B^k \times i(3B^n) \to B^k \times i(4B^n)$ such that $g'' = g'$ on $\partial(\frac{3}{4}B^k \times i(3B^n))$.

Let $\widetilde{g} : B^k \times \mathbb{R}^n \to B^k \times \mathbb{R}^n$ be such that $(1 \times e)\widetilde{g} = g''(1 \times e)$ and $\widetilde{g}|_{\partial B^k \times B^n} = 1$. $\widetilde{g}$ is bounded, i.e. $d(x, \widetilde{g}(x)) \leqslant A$ for $x \in B^k \times \mathbb{R}^n$. Extend $\widetilde{g}$ to a homeomorphism of $\mathbb{R}^k \times \mathbb{R}^n$ by $\widetilde{g}|_{(\mathbb{R}^k \setminus B^k) \times \mathbb{R}^n} = 1$.

Define $\rho :\ \operatorname{Int}(2B^k \times 2B^n) \to \mathbb{R}^k \times \mathbb{R}^n$, a homeomorphism fixing $B^k \times B^n$, by

$$\rho(x,y) = \begin{cases} (x,y) & \text{if } (x,y) \in B^k \times B^n, \\ (2 - \max\{|x|,|y|\})^{-1}(x,y) & \text{otherwise.} \end{cases}$$

Then $\rho^{-1}\widetilde{g}\rho :\ \operatorname{Int}(2B^k \times 2B^n) \to \operatorname{Int}(2B^k \times 2B^n)$ extends to a homeomorphism of $2B^k \times 2B^n$ fixed on $\partial(2B^k \times 2B^n)$. In fact, $\rho^{-1}\widetilde{g}\rho$ fixes $(2B^k \setminus \operatorname{Int} B^k) \times 2B^n$. Thus $\rho^{-1}\widetilde{g}\rho$ defines a homeomorphism of $B^k \times 2B^n$ fixed on $\partial(B^k \times 2B^n)$. Define an isotopy R_t of $B^k \times 2B^n$ by

$$R_t(x,y) = \begin{cases} (x,y) & \text{if } \max\left\{|x|, \frac{1}{2}|y|\right\} \geqslant t, \\ t\rho^{-1}\widetilde{g}\rho(t^{-1}(x,y)) & \text{otherwise.} \end{cases}$$

Let $\sigma : B^k \times 2B^n \to B^k \times \operatorname{Int} B^n$ be an embedding with $\sigma|_{B^k \times rB^n} = fe$. Now define an isotopy S_t of $B^k \times B^n$ by

$$S_t(x) = \begin{cases} \sigma R_t \sigma^{-1} & \text{if } x \in \operatorname{Im}\sigma, \\ x & \text{otherwise.} \end{cases}$$

Then $S_0 = 1$ and S_t fixes $\partial(B^k \times B^n)$.

Suppose V is so small that $h \in V$ implies that $\widetilde{g}$ is defined and $g(\frac{1}{2}B^n) \subset fe(\operatorname{Int} rB^n)$. Then $S_1|_{B^k \times \frac{1}{2}B^n} = g$.

Define $H_t : B^k \times B^n \to B^k \times \mathbb{R}^n$ by

$$H_t(x) = \begin{cases} G_{2t}(x) & \text{if } 0 \leqslant t \leq \frac{1}{2}, \\ gS_{2t-1}^{-1}(x) & \text{otherwise.} \end{cases}$$

This does what is required. $\square$

Lemma 4.5. *If $C \subset \mathbb{R}^n$ is compact and $\epsilon > 0$, then C lies in the interior of a handlebody with handles of diameter $< \epsilon$. Explicitly, there exist finitely many embeddings $h_i : B_i^k \times B^{n-k_i} \to \mathbb{R}^n$, $i = 1, 2 \ldots, l$, such that if $W_j = \bigcup_{i \leqslant j} h_i(B^{k_i} \times \frac{1}{2}B^{n-k_i})$ then*

i. $h_i(B^{k_i} \times B^{n-k_i}) \cap W_{i-1} = h_i(\partial B^{k_i} \times B^{n-k_i})$,

ii. W_l *is a neighborhood of* C, *and*

iii. $h_i(B^{k_i} \times B^{n-k_i})$ *has diameter* $< \epsilon$ *and* $h_i(B^{k_i} \times B^{n-k_i}) \subset N_\epsilon$.

Proof. Cover C by a lattice of cubes of side $\frac{1}{2}\epsilon$. Since C is compact, C only needs a finite number of these cubes. Let $\gamma_1, \ldots, \gamma_i$ be all the faces of all the cubes meeting C.

Let $k_i = \dim \gamma_i$ and order γ_i so that $k_0 \leqslant k_1 \leq \cdots \leqslant k_l$. Define a metric on $\mathbb{R}^n$ by $d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{1 \leqslant i \leqslant n} |x_i - y_i|$. Let

$$H_i = \overline{N_{\epsilon 2^{-i-3}}(\gamma_i) \setminus \bigcup_{j<i} N_{\epsilon 2^{-j-4}}(\gamma_j)}$$

and $\frac{1}{2}H_i = H_i \cap N_{\epsilon 2^{-i-4}}(\gamma_i)$.

Then $H_i \cap \gamma_i \cong \gamma_i$ (radial projection) $\cong B^k$ and clearly $H_i \cong (H_i \cap \gamma_i) \times B^{n-k_i}$.

There exist homeomorphisms $h_i : B^{k_i} \times B^{n-k_i} \to H_i$ carrying $B^{k_i} \times \frac{1}{2}B^{n-k_i}$ onto $\frac{1}{2}H_i$ and $(\partial B^{k_i}) \times B^{n-k_i}$ onto $H_i \cap \bigcup_{j<i} \frac{1}{2}H_j$. Then $h_1, h_2, \ldots, h_i$ do what is required. □

Addendum 4.6. *If $D \subset C$ is compact, then we can select $h_{i_1}, \ldots h_{i_m}$ so that (i) is still satisfied, and (ii) and (iii) are satisfied by $h_{i_1}, \ldots, h_{i_m}$ with respect to D instead of C. That is, $\bigcup h_{i_r}(B^{k_{i_r}} \times \frac{1}{2}B^{n-k_{i_r}})$ is a neighborhood of D, $h_{i_r}(B^{k_{i_r}} \times B^{n-k_{i_r}})$ has diameter less than ϵ and is contained in $N_\epsilon(0)$.*

Proof. Select h_i iff γ_i is a face of a cube which meets D. □

Theorem 4.7 (Kirby-Edwards)**.** *Let C, D be compact in $\mathbb{R}^n$ and U, V be neighborhoods of C, D. Let $\mathcal{E}$ be the space of embeddings of U in $\mathbb{R}^n$ which restrict to 1 on V. There is a neighborhood N of 1 in $\mathcal{E}$ and a homotopy $H : N \times I \to \mathcal{H}(\mathbb{R}^n)$ such that*

i. $H_t(1) = 1$ *for all* t,

ii. $H_0(g) = 1$ *for all* $g \in N$,

iii. $H_1(g)|_C = g|_C$, *and*

iv. $H_t(g)|_{D \cup (\mathbb{R}^n \setminus U)} = 1$ *for all* t, g.

Proof. Let $\epsilon = \min \{d(C, \mathbb{R}^n \setminus U), d(D, \mathbb{R}^n \setminus V)\}$. Cover $C \cup D$ by a handlebody in $U \cup V$ with handles of diameter $< \epsilon$. Let $h_1, \ldots, h_l$ be the handles, with W_i as in 4.5. Select $h_{i_1}, \ldots, h_{i_m}$ to form a sub-handlebody covering D, contained in V. Let $X = \bigcup_r h_{i_r}(B^{k_{i_r}} \times B^{n-k_{i_r}})$.

Suppose inductively that we have constructed a neighborhood N_{i-1} of 1 in $\mathcal{E}$ and a homotopy $H^{(i-1)} : N_{i-1} \times I \to \mathcal{H}(\mathbb{R}^n)$ such that (i) and (ii) are satisfied, $H_1^{(i-1)}(g)|_{W_{i-1}} = g$, $H_t^{(i-1)}(1) = 1$, $H_0^{(i-1)}(g) = 1$, and $H_r^{(i-1)}(g)|_{X\cup(\mathbb{R}^n\setminus U)} = 1$.

If $h_i(B^{k_i} \times B^{n-k_i}) \subset X$, put $N_i = N_{i-1}$ and $H_i = H_{i-1}$. (This is consistent because if $h_i(B^{k_i} \times B^{n-k_i}) \cap h_j(B^{k_j} \times B^{n-k_j}) \neq \emptyset$ for $j < i$ then $h_j(B^{k_j} \times B^{n-k_j}) \subset X$.)

Now suppose $h_i(B^{k_i} \times b^{n-k_i}) \not\subset X$. Choose N_i so small that

$$g^{-1}H_t^{(i-1)}(g)h_i(B^{k_i} \times \frac{3}{4}B^{n-k_i}) \subset h_i(B^{k_i} \times \operatorname{Int} B^{n-k_i}).$$

Let $f = h_i^{-1}g_i^{-1}H_1^{(i-1)}(g)h_i : B^{k_i} \times \frac{3}{4}B^{n-k_i} \to B^{k_i} \times \operatorname{Int} B^{n-k_i}$. Then f fixes $(\partial B^{k_i}) \times \frac{3}{4}B^{n-k_i}$. Theorem 4.4 gives a continuously varying isotopy $H'_t(y)$ such that

i. $H'_t(1) = 1$,

ii. $H'_0(g) = f$,

iii. $H'_1(g)|_{B^{k_i}\times\frac{1}{2}B^{n-k_i}} = 1$, and

iv. $H'_t(g)|_{\partial(B^{k_i}\times\frac{3}{4}B^{n-k_i})} = 1$.

Define

$$H_t^{(i)}(g)(x) = \begin{cases} \left(H_t^{(i-1)}(g)\right) h_i f^{-1}\left(H'_t(g)\right) h_i^{-1}(x) & \text{if } x \in h_i(B^{k_i} \times \frac{3}{4}B^{n-k_i}), \\ H_t^{(i-1)}(g)(x) & \text{otherwise.} \end{cases}$$

Then $W_i = W_{i-1} \cup h_i(B^{k_i} \times \frac{1}{2}B^{n-k_i})$, $h_i(B_i^k \times \frac{3}{4}B^{n-k_i}) \cap X \subset h_i(\partial(B_i^k \times \frac{3}{4}B^{n-k_i}))$ completes the induction.

$H = H^l$, $N = N^l$ do what is required. □

Theorem 4.8. *If M is a compact manifold then $\mathcal{H}(M)$ is locally contractible.*

Proof. First suppose M is closed, $\partial M = \emptyset$. Cover M by finitely many embeddings $f_i : \mathbb{R}^n \to M$, $i = 1, \ldots, l$. In fact, assume $M = \bigcup F_i(B^n)$.

Let $h : M \to M$ be a homeomorphism near 1. Define inductively an isotopy $H^{(i)}(h)$ of M such that

i. $H_t^{(i)}(h)$ depends continuously on h,

ii. $H_t^{(i)}(1) = 1$,

iii. $H_0^{(i)}(h) = 1$, and

iv. $H_1^{(i)}(h)$ agrees with h on $\bigcup_{j\leqslant i} f_j((1 + 2^{-i})B^n)$.

Suppose $H_t^{(i-1)}$ is defined. Let $C = (1+2^{-i})B^n$, $U = (1+2^{-(i-1)})B^n$, and let $D = f_i^{-1}(\bigcup_{j<i} f_j(C)) \cap 4B^n$, $V = f_i^{-1}(\bigcup_{j<i} f_j(U))$.

Suppose h is so near 1 that $h^{-1}H_t^{(i-1)}(h)f_i(U) \subset f_i(\mathbb{R}^n)$. Apply Theorem 4.7 to $g = f_i^{-1}h^{-1}H_1^{(i-1)}(h)f_i : U \to \mathbb{R}^n$. If h is sufficiently near 1, we get a continuously varying isotopy $H'(h)$ of $\mathbb{R}^n$ such that

i. $H_t'(1) = 1$ for all t,

ii. $H_0'(h) = 1$ for all h,

iii. $H_1'(h)|_C = h|_C$, and

iv. $H_t'(h)|_{D\cup(\mathbb{R}^n\setminus U)} = 1$ for all t, h.

Define $H^{(i)} = H^{(i)}(h)$ by

$$H_t^{(i)}(x) = \begin{cases} H_t^{(i-1)} f_i \left(H_t'(h)\right)^{-1} f_i^{-1}(x) & \text{if } x \in f_i(\mathbb{R}), \\ H_t^{(i-1)}(x) & \text{otherwise.} \end{cases}$$

Then $H^{(i)}$ satisfies (i)–(iii) and completes the induction.

Now suppose $\partial M \neq \emptyset$. Let $\gamma : \partial M \times I \to M$ be a collar of ∂M in M. $\mathcal{H}(\partial M)$ is locally contractible. If $h \in \mathcal{H}(M)$ is near 1 then we have an isotopy $H_t(h)$ of ∂M with $H_0(h) = 1$, $H_1(1) = h|_{\partial M}$.

Define an isotopy $\overline{H}$ of M by

$$\overline{H}_t(\gamma(x,u)) = \gamma(H_{t(1-u)}(x), u)$$

for $x \in \partial M, u \in I$, and

$$\overline{H}_t(y) = y$$

if $y \notin \gamma(\partial M \times I)$. Then $\overline{H}_t$ is an isotopy of M from 1 to $\overline{H}_1$ where $\overline{H}_1$ agrees with h on ∂M.

There exists an isotopy $G_t : M \to M$ from $\overline{H}_1$ to G_1 where G_1 agrees with h on $\gamma(\partial M \times [0, \frac{1}{2}])$. Now the argument goes as for closed manifolds. □

Exercise. If M is compact, then $\mathcal{H}(\operatorname{Int} M)$ is locally contractible.

Theorem 4.9 (Isotopy extension)**.** *Let M, N be n-manifolds with M compact, $\partial N = \emptyset$, and $M \subset N$. Suppose we are given a path $H : I \to \mathcal{E}(M,N)$, $H_0 : M \hookrightarrow N$. If U is a neighborhood of ∂M in M, then there is an isotopy $\overline{H} : I \to \mathcal{H}(N)$ such that $\overline{H}_0 = 1$ and $\overline{H}_t|_{M\setminus U} = H|_{M\setminus U}$.*

Proof. First use the method of 4.8 to generalize 4.7 to deal with compact $C, D \subset N$ (i.e. replace $\mathbb{R}^n$ by N.) Let $f \in \mathcal{E}(M,n)$. Then $f(M) \subset N$ is a neighborhood of $f(M \setminus U)$ (assume that U is open), and there exists an open neighborhood V_f of 1 in $\mathcal{E}(f(M), N)$ and a homotopy $F^{(f)} : V_f \times I \to \mathcal{H}(N)$ such that $F_1^{(f)}(g)|_{M\setminus U} = g|_{M\setminus U}$ for $g \in V_f$.

Let $W_f = \{gf : g \in V_f\}$. Then W_f is an open neighborhood of f in $\mathcal{E}(M,N)$. Now $\{W_f\}_{f\in\mathcal{E}(M,N)}$ is an open cover of $\mathcal{E}(M,N)$. There is a dissection $0 =$

$t_0 < t_1 < \cdots < t_l = 1$ of I such that $H([t_{i-1}, t_i])$ is contained in some W_{f_i}, $f_i \in \mathcal{E}(M, N)$.

Define $\overline{H}_t$ for $t_{i-1} \leqslant t \leqslant t_i$ by

$$\overline{H}_t = F_1^{(f_i)}(H_t \circ f_i^{-1}) \left(F_1^{(f_i)}(H_{t_{i-1}} \circ f_i^{-1})\right)^{-1} \overline{H}_{t_{i-1}}.$$

Then $\overline{H}_t = H_t$ on $M \setminus U$. □

Addendum 4.10. *$\overline{H}_t$ can be chosen to be the identity outside some compact set. (This is because 4.7 also produces isotopies of compact support.)*

Corollary 4.11. *Let $f : B^n \hookrightarrow \operatorname{Int} 2B^n$ be isotopic to 1. Then $2B^n \setminus f(\operatorname{Int} \tfrac{1}{2}B^n) \cong 2B^n \setminus \operatorname{Int} \tfrac{1}{2}B^n$.*

Proof. Let H_t be an isotopy from 1 to f. By 4.10 there is an isotopy $\overline{H}_t$ of $\operatorname{Int} 2B^n$, fixed outside λB^n for some $\lambda < 2$, such that $\overline{H}_1 = f$ on $\tfrac{1}{2}B^n$.

Therefore $\widetilde{H}_1$ defines a homeomorphism $2B^n \setminus \operatorname{Int} \tfrac{1}{2}B^n \to 2B^n \setminus f(\operatorname{Int} \tfrac{1}{2}B^n)$. □

5 Triangulation Theorems

Definition 5.1. An *r-simplex* in $\mathbb{R}^n$ is the convex hull of $r + 1$ linearly independent points.

Let $K \subset \mathbb{R}^n$ be compact. An embedding $f : K \to \mathbb{R}^n$ is *PL* if K is a finite union of simplexes, each mapped linearly by f.

If M is an n-manifold, a *PL structure* on M is a family $\mathcal{F}$ of embeddings $f : \Delta^n \to M$ such that

i. every point of M has a neighborhood of XXXXXXXXXX from $f(\Delta^n)$, $f \in \mathcal{F}$,

ii. if $f, g \in \mathcal{F}$, then $g^{-1}f : f^{-1}g(\Delta^n) \to \mathbb{R}^n$ is PL, and

iii. $\mathcal{F}$ is maximal with respect to (i) and (ii).

If M, N have PL structures $\mathcal{F}, \mathcal{G}$, an embedding $h : M \to N$ is *PL* if $f \in \mathcal{F}$ implies $hf \in \mathcal{G}$.

Example. The composite of 2 PL embeddings is PL, i.e. PL $\cong$ is an equivalence relation.

A PL structure $\mathcal{F}$ on M defines a PL structure $\partial\mathcal{F}$ on ∂M.

A compact n-manifold has PL structure iff it has a triangulation with the link of each vertex PL homeomorphic to $\partial\Delta^n$.

We need 3 deep theorems from PL topology.

Proposition 5.2. i. *Suppose M is a closed PL manifold which is homotopy equivalent to S^n. If $n \geqslant 5$, then M is PL homeomorphic to $S^n = \partial\Delta^{n+1}$.*

ii. Call a non-compact manifold W simply-connected at ∞ *if for every compact set $C \subset W$, there is a compact set $D \subset W$ such that any two loops in $W \setminus D$ are homotopic in $W \setminus C$. (Example: $\mathbb{R}^n$ is simply connected at ∞ iff $n \geqslant 3$.)*

Suppose W^n is an open PL manifold which is simply connected at infinity. If $n \geqslant 6$ then W is PL homeomorphic to $\operatorname{Int} V$ where V is some compact PL manifold. XXXXXXXXXXXX

iii. Let M be a closed PL manifold which is homotopy equivalent to T^n. Then some finite covering of M is PL homeomorphic to $T^n = (\partial\Delta^2)^n$. (Proof in Wall's book.)

Theorem 5.3 (Annulus Conjecture)**.** *If $h : B^n \to \operatorname{Int} B^n$ is an embedding and $n \geqslant 6$, then $B^n \setminus h(\operatorname{Int} \frac{1}{2}B^n) \cong B^n \setminus \operatorname{Int} \frac{1}{2}B^n$.*

Proof. Let $a \in T^n$ and let $f : T^n \setminus \{a\} \to \operatorname{Int} B^n$ be a PL immersion such that $f(T^n \setminus \{a\}) \subset \frac{1}{2}B^n$. Let $h : B^n \to \operatorname{Int} B^n$ be a topological homeomorphism: we shall find a PL Structure $\mathcal{F}'$ on $T^n \setminus \{a\}$ such that hf is PL with respect to $\mathcal{F}'$. Let $\mathcal{F}_0 = \{\phi : \Delta^n \to T^n \setminus \{a\} : (hf)\phi$ is a PL embedding$\}$. Since hf is an open immersion, $\{\phi(\operatorname{Int} \Delta^n) \colon \phi \in \mathcal{F}\}$ covers $T^n \setminus \{a\}$. Extend $\mathcal{F}_0$ to a PL structure $\mathcal{F}'$ on $T^n \setminus \{a\}$. Let XXXXXXX. For $n \geqslant 3$, $(T^n \setminus \{a\})' \cong (T^n \setminus \{a\})$ so $(T^n \setminus \{a\})'$ is simply connected at ∞. Since $n \geqslant 6$, by 5.2 (ii) there is a compact PL manifold w and PL homeomorphism $g : (T^n \setminus \{a\})' \to \operatorname{Int} W$. There exists a PL collar $\gamma : \partial W \times I \to W$. Let $\epsilon > 0$ and A be a neighborhood of a in T^n homeomorphic to B^n and so small that

$$g^{-1}\gamma(\partial W \times I) \supset A \setminus \{a\} \supset g^{-1}\gamma(\partial W \times \{\epsilon\}).$$

The first and last sets are homotopy equivalent, so it follows that $\partial W \cong S^{n-1}$. By 5.2 (i) since $n \geqslant 6$, ∂W is PL homeomorphic to S^{n-1}.

By Schönflies theorem, $\{a\} \cup g^{-1}\gamma(\partial W \times (0, \epsilon]) \cong B^n$. Extend $\mathcal{F}'|_{T^n \setminus (\{a\} \cup g^{-1}\gamma(\partial W \times (0,\epsilon)))}$ to a PL structure $\mathcal{F}''$ on XXX. ($\mathcal{F}'$ induces PL structure on $\partial(\{a\} \cup g^{-1}\gamma(\partial W \times (0, \epsilon]))$; extend this "conewise" to a PL structure on $\{a\} \cup g^{-1}\gamma(\partial W \times (0, \epsilon))$.)

By 5.2 (iii), there is a finite covering of $(T^n)''$ which is PL homeomorphic to T^n. Let $\epsilon'' : T^n \to (T^n)''$ be a finite cover. Let $\epsilon : T^n \to T^n$ be the corresponding cover of T^n. By the theory of covering spaces there exists a homeomorphism $\overline{h} : T^n \to T^n$ (*not* PL) such that $\epsilon = \epsilon''\overline{h}$ ($\overline{h}$ is homotopic to 1). Now let $\widetilde{h} : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism such that $e\widetilde{h} = \overline{h}e$. Then $d(x, \widetilde{h}(x))$ is bounded uniformly for $x \in \mathbb{R}^n$.

Let $\rho\colon \operatorname{Int} B^n \to \mathbb{R}^n$ be a PL "radial" homeomorphism (avoiding the "standard mistake"). Now $\eta = \rho^{-1}\widetilde{h}\rho\colon \operatorname{Int} B^n \to \operatorname{Int} B^n$ extends to a homeomorphism of B^n fixing ∂B^n.

Let U be a nonempty open set in $\operatorname{Int} B^n$ such that $\epsilon e\rho(U) \cap A = \emptyset$ and $\sigma = f\epsilon e\rho|_U$ maps U injectively into $\frac{1}{2}B^n$. Let $\sigma'' = hf\epsilon''e\rho|_{\eta(U)} \to \operatorname{Int} B^n$. Then σ, σ'' are PL embeddings and $\sigma''\eta = h\sigma$. The PL annulus conjecture is true (proof by regular neighborhood theory). There is an n-simplex $\Delta \subset U$

such that $\eta(\Delta)$ is contained in some n-simplex $\Delta'' \subset \eta(U)$. Therefore by the PL annulus theorem, $\overline{\frac{1}{2}B^n \setminus \sigma(\Delta)} \cong$ the standard annulus $\cong \overline{B^n \setminus \frac{1}{2}B^n}$.

We have that $B^n \setminus h(\operatorname{Int} \frac{1}{2}B^n) \cong B^n \setminus h\sigma(\operatorname{Int}\Delta)$ by gluing the standard annulus $h(\frac{1}{2}B^n) \setminus h\sigma(\operatorname{Int}\Delta)$ onto $B^n \setminus h(\operatorname{Int}\frac{1}{2}B^n)$. From there,

$$\begin{aligned} B^n \setminus h\sigma(\operatorname{Int}\Delta) &\cong B^n \setminus \sigma''\eta(\operatorname{Int}\Delta) \\ &\cong \sigma''(\Delta'') \setminus \sigma''\eta(\operatorname{Int}\Delta) \\ &\cong \Delta'' \setminus \eta(\operatorname{Int}\Delta) \\ &\cong B^n \setminus \eta(\operatorname{Int}\Delta) \\ &\cong B^n \setminus \operatorname{Int}\Delta \\ &\cong B^n \setminus \operatorname{Int}\frac{1}{2}B^n. \end{aligned}$$

□

The proof depends only on knowing that given embeddings $f, g : B^n \to T^n$ there exists an $h : T^n \to T^n$ carrying $f(\frac{1}{2}B^n)$ onto $g(\frac{1}{2}B^n)$. If we could do this purely geometrically (i.e. without PL theory) for all dimensions, we would have then proved the annulus conjecture in all dimensions.

New notation: W is any manifold, $]$ is the subset $(\partial W \times I) \cup (W \times \{1\})$ of $W \times I$.

Theorem 5.4. *Let M be a PL manifold and let $h : I \times B^k \times \mathbb{R}^n \to M$ be a homeomorphism which is PL on a neighborhood of $]$. If $k + n \geqslant 6$ then there is an isotopy $H_t : I \times B^k \times \mathbb{R}^n \to M$ such that*

i. $H_0 = h$,

ii. H_1 *is PL on* $I \times B^k \times B^n$, *and*

iii. $H_t = h$ *on* $]$ *and outside* $I \times B^k \times 2B^n$.

Proof. Let $a \in T^n$ and let $f : T^n \setminus \{a\} \to \mathbb{R}^n$ be a PL immersion. As in 5.3, let $\mathcal{F}'$ be a PL structure on $I \times B^k \times (T^n \setminus \{a\})$ such that $h(1 \times f) : (I \times B^k \times (T^n \setminus \{a\}))' \to M$ is PL. Then $\mathcal{F}'$ agrees with $\mathcal{F}$ near $]$.

Let A be a ball neighborhood of a in T^n. First extend $\mathcal{F}'$ over a neighborhood of $]$ in $I \times B^k \times T^n$ (using the standard structure). As in 5.3 extend $\mathcal{F}'$ over $\{0\} \times B^k \times T^n$, obtaining a structure $\mathcal{F}''$. The following argument implies that we can extend $\mathcal{F}'' \cup \mathcal{F}|_{I \times B^k \times (T^n \setminus A)}$ over a neighborhood of $\{0\} \times B^k \times T^n$ in $I \times B^k \times T^n$.

As in 5.3 extend this to a PL structure over $I \times B^k \times T^n$ agreeing with the standard structure near $]$ and with $\mathcal{F}'$ on $I \times B^k \times (T^n \setminus A)$. We can take $\mathcal{F}''$ to be the standard structure near $\{1\} \times B^k \times T^n$. Now $\mathcal{F}''$ is defined near $\partial(I \times B^k \times A)$; we extend over $I \times B^k \times A$ as in 5.3, obtaining a PL manifold $(I \times B^k \times T^n)''$. The inclusion $(I \times B^k \times (T^n \setminus \{a\}))' \hookrightarrow (I \times B^k \times T^n)''$ is PL except on $I \times B^k \times A$, and the identity map $I \times B^k \times T^n \to (I \times B^k \times T^n)''$ is PL near $]$.

Now we need another result from PL topology:

Proposition 5.5. *Let W, V_1, V_2 be compact PL manifolds with $\partial W = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2$. Suppose the inclusions $V_i \to W$ are homotopy equivalent. If $\pi_1(W)$ is free abelian and* $\dim W \geqslant 6$, *then W is PL homeomorphic to $V_1 \times I$.*

Apply this result with $W = (I \times B^k \times T^n)''$, $V_1 =]$ and $V_2 = (\{0\} \times B^k \times T^n)''$. We obtain a PL homeomorphism $(I \times B^k \times T^n)'' \to] \times I$. Since $] \times I \cong I \times B^k \times T^n$ by a PL homeomorphism taking $(x, 0)$ to x, we can find a PL homeomorphism $g : I \times B^k \times T^n \to (I \times B^k \times T^n)''$ which is the identity near $]$.

Let $\widetilde{h} : I \times B^k \times \mathbb{R}^n \to I \times B^k \times \mathbb{R}^n$ be such that $e\widetilde{h} = g^{-1}e$ and $\widetilde{h} = 1$ on $]$. Then $\widetilde{h}$ is a bounded homeomorphism. Extend $\widetilde{h}$ over $[0, \infty) \times \mathbb{R}^k \times \mathbb{R}^n$ by putting $\widetilde{h} = 1$ outside $I \times B^k \times \mathbb{R}^n$. Extend further over $\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^n$ by putting $\widetilde{h}(t, x, y) = (t, p_2\widetilde{h}(0, x, y), p_3\widetilde{h}(0, x, y))$ for $t \leqslant 0$. Note that $d(x, \widetilde{h}(x))$ remains bounded for $x \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^n$.

Suppose $0 < r < 1$, $e(rB^n) \cap A = \emptyset$, and $fe|_{rB^n}$ is injective. We may also suppose $fe(rB^n) \supset sB^n$ for some $s > 0$. There is a PL "radial" homeomorphism $\rho : (-1, 2) \times \operatorname{Int}(2B^k \times B^n) \to \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^n$ fixed near $I \times B^k \times rB^n$. Then $\rho\widetilde{h}\rho^{-1}$ extends to a homeomorphism of $[-1, 2] \times 2B^k \times B^n$ fixing the boundary. Let $\eta = \rho\widetilde{h}\rho^{-1}$.

Note that $I \times B^k \times B^n \times I$ is the join of $(\frac{1}{2}, 0, 0, \frac{1}{2})$ to $(] \times I) \cup (I \times B^k \times B^n \times \partial I)$. Define a PL homeomorphism R of $I \times B^k \times B^n \times i$ by $R(\frac{1}{2}, 0, 0, \frac{1}{2}) = (\frac{1}{2}, 0, 0, \frac{1}{2})$, $R|_{(] \times I) \cup (I \times B^k \times B^n \times \{1\})} = 1$, and $R|_{I \times B^k \times B^n \times \{0\}} = \eta$, extending conewise. Then R defines a PL isotopy R_t of $I \times B^k \times B^n$, fixed near $]$, with $R_0 = \eta$ and $R_1 = 1$.

Let $\sigma : I \times B^k \times B^n \to I \times B^k \times \mathbb{R}^n$ be a PL embedding which agrees with $1 \times fe$ near $I \times B^k \times rB^n$. Then $h\sigma\eta^{-1}$ agrees with $h(1 \times f)ge\rho$ near $\eta(I \times B^k \times rB^n)$, so it is PL there.

$$\begin{tikzcd}
I \times B^k \times B^n \arrow[r, "\eta"] \arrow[d, "\sigma"'] & I \times B^k \times B^n \arrow[d, "\sigma''"] \\
I \times B^k \times \mathbb{R}^n \arrow[r, "h"] & M
\end{tikzcd}$$

$h\sigma\eta^{-1}$ is PL near $\eta(I \times B^k \times rB^n)$. $W = I \times B^k \times B^n \setminus \eta(I \times B^k \times rB^n)$ is a PL manifold (since it is an open subset of a PL manifold). If $n \geqslant 3$, W is simply connected at infinity, so if $n \geqslant 3$ the Browder-Levine-Livesay theorem (5.2B) implies that W is homeomorphic to an open subset of a compact manifold.

If $n \leqslant 2$, the same result, using instead Siebenmann's XXXXXXX. It follows that $\eta(I \times B^k \times rB^n)$ has a neighborhood which is a compact PL manifold such that $\partial N \subset \overline{I \times B^k \times B^n \setminus N}$ is a homotopy equivalence. Now the s-cobordism theorem (5.5) implies that $\overline{I \times B^k \times B^n \setminus N}$ is PL homeomorphic to $\overline{I \times B^k \times B^n \setminus I \times B^k \times rB^n}$.

It follows that there is a $\sigma'' : I \times B^k \times B^n \to M$, a PL embedding such that $\sigma''\eta = h\sigma$ near $I \times B^k \times rB^n$ (regard $\overline{I \times B^k \times B^n \setminus N}$ as a collar of XXXXXX).

Let R_t be an isotopy from η to 1 rel]. Define $S_t : I \times B^k \times \mathbb{R}^n \to M$ by

$$S_t(x) = \begin{cases} \sigma'' R_t \eta^{-1}(\sigma'')^{-1} h(x) & \text{if } h(x) \in \operatorname{Im} \sigma'', \\ h(x) & \text{otherwise.} \end{cases}$$

Then $S_0 = h$ and

$$\begin{aligned} S_1|_{I \times B^k \times sB^n} &= \sigma'' R_1 \eta^{-1} \eta \sigma^{-1} \\ &= \sigma'' \sigma^{-1}|_{I \times B^k \times sB^n} \to M \end{aligned}$$

which is PL. $S_t = h$ on] and also outside h^{-1} (the image of σ'') which is compact. Therefore $S_t = h$ on] and outside $I \times B^k \times RB^n$ for some $R \gg 0$. It is trivial to replace S_t by an isotopy H_t satisfying (i)–(iii). □

Theorem 5.6. *Let C, D be closed subsets of $\mathbb{R}^n$ and let U be an open neighborhood of C. Let $\mathcal{F}$ be a PL structure on $U \times I \subset \mathbb{R}^n \times I$ which agrees with the standard PL structure near $(U \cap D) \times I$ and near $U \times \{0\}$. If $n \geqslant 6$, then there is an isotopy H_t of $\mathbb{R}^n \times I$ such that*

i. $H_0 = 1_{\mathbb{R}^n \times I}$,

ii. $H_1 : (U \times I, \text{standard}) \to (U \times I, \mathcal{F})$ is PL near $C \times I$, and

iii. $H_1 = 1$ near $(D \cup (\mathbb{R}^n \setminus U)) \times I$ and near $\mathbb{R}^n \times \{0\}$.

Proof. If C, D are compact, this is deduced from 5.4 exactly as 4.7 was deduced from 4.4. For the general case, let $C_i = C \cap iB^n$, $U_i = U \cap (i+1) \operatorname{Int} B^n$, $D_i = D \cap (i+1)B^n$. Suppose inductively that $H^{(i)}$ satisfies (i)–(iii) with respect to C_i, D_i, and U_i.

Let $\mathcal{F}_i = (H_1^{(i)})^{-1}(\mathcal{F})$: this is a PL structure on $U \times I$ which agrees with the standard PL structure near $(C_i \times I) \cup (D_i \cup (\mathbb{R}^n \setminus U_i)) \times I$ and near $U \times \{0\}$. Now apply the compact case to get an isotopy H'_t satisfying (i)–(iii) with respect to $\overline{C_{i+1} \setminus C_i}$, $U_{i+1} \setminus \overline{U_{i-2}}$, $C_i \cup D_{i+1}$, $\mathcal{F}_i$. Then $H_t^{(i+1)} = H_t^{(i)} H'_t$ satisfies (i)–(iii) with respect to $C_{i+1}, U_{i+1}, D_{i+1}, \mathcal{F}$. Since $H'_t = 1$ on $(i-1)B^n$, $H_t^{(i+1)} = H_t^{(i)}$ on $(i-1)B^n$.

Now take $H_t = \lim_{i \to \infty} H_t^{(i)}$. This satisfies (i)–(iii) with respect to $C, D, U, \mathcal{F}$. □

Theorem 5.7 (Product Structure Theorem)**.** *Let M^n be a topological manifold, $C \subseteq M$ be a closed subset, and U be an open neighborhood of C. Let $\mathcal{F}_0$ be a PL structure on U, and let $\mathcal{G}$ be a PL structure on $M \times \mathbb{R}^n$ such that $\mathcal{G}$ agrees with $\mathcal{F}_0 \times \mathbb{R}^k$ on $U \times \mathbb{R}^k$. If $n \geqslant 6$ then there is a PL structure $\mathcal{F}$ on M agreeing with $\mathcal{F}_0$ on C and a PL homeomorphism $(M \times \mathbb{R}^k, \mathcal{F} \times \mathbb{R}^k) \to (M \times \mathbb{R}^k, \mathcal{G})$ which is isotopic to 1 by an isotopy fixing a neighborhood of $C \times \mathbb{R}^k$.*

The proof is given below.

Definition 5.8. PL structures $\mathcal{F}_1, \mathcal{F}_2$ on M are *isotopic* if there is a PL homeomorphism $h : (M, \mathcal{F}_1) \to (M, \mathcal{F}_2)$ which is isotopic to 1.

Let PL(M) be the set of isotopy classes of PL structures on M.

Corollary 5.9. *If* $\dim M \geqslant 6$*, the natural map* $\mathrm{PL}(M) \to \mathrm{PL}(M \times \mathbb{R}^k)$ *is a bijection. In particular, if* $M \times \mathbb{R}^k$ *has a PL structure and* $\dim M \geqslant 6$*, then* M *has a PL structure.*

Lemma 5.10. *Any two PL structures on* $\mathbb{R}^n$ *(n XXXXXXX) are isotopic.*

Proof. Let $\mathcal{F}$ be a PL structure on $\mathbb{R}^n$. By 5.2 (ii) $(\mathbb{R}^n, \mathcal{F})$ is PL homeomorphic to $\operatorname{Int} W$ where W is a compact PL manifold with $\partial W \cong S^{n-1}$. By 5.2 (i), ∂W is PL homeomorphic to S^{n-1}. W is contractible, so by 5.2 (i), W is PL homeomorphic to B^n. $W \cup_\partial B^n \cong S^n$, so there exists a PL homeomorphism $h : \mathbb{R}^n \to \operatorname{Int} B^n \to \operatorname{Int} W \to (\mathbb{R}^n, \mathcal{F})$. We may assume h is orientation preserving. We must prove that h is isotopic to 1.

Let $R > r > 0$ be chosen so that $h(rB^n) \subset \operatorname{Int} h(RB^n)$. By the annulus theorem 5.3, there is a homeomorphism $f : RB^n \setminus \operatorname{Int} rB^n \to RB^n \setminus h(\operatorname{Int} rB^n)$ with $f_{\partial(RB^n)} = 1$. Since h is orientation preserving, and using the proof of 5.3, we can choose f so that $f = h$ on $\partial(rB^n)$.

Extend f over $\mathbb{R}^n$ by

$$f(x) = \begin{cases} x & \text{if } \|x\| \geqslant R, \\ hx & \text{if } \|x\| \leqslant r. \end{cases}$$

Since $f = 1$ outside RB^n, f is isotopic to 1, so h is isotopic to $f^{-1}h$. Since $f^{-1}h = 1$ in rB^n, $f^{-1}h$ is isotopic to 1. Therefore h is isotopic to 1 as required. □

Proof of Theorem 5.7. Clearly, it is sufficient to prove for the case $k = 1$. Assume first that $M = \mathbb{R}^n$, $\mathcal{G}$ = a PL structure on $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. By 5.10, there exists an isotopy H_t such that $H_1 : \mathbb{R}^{n+1} \to (\mathbb{R}^{n+1}, \mathcal{G})$ is PL and $H_t = 1$ for $t \leq \frac{1}{4}$. H defines a homeomorphism $H : \mathbb{R}^n \times \mathbb{R} \times I \to \mathbb{R}^n \times \mathbb{R} \times I$ (sending $(x, t) \mapsto (H_t(x), t)$) Let $\mathcal{H} = H$(standard PL structure). Then $\mathcal{H}$ agrees with the standard structure near $\mathbb{R}^n \times \mathbb{R} \times \{0\}$ and with $\mathcal{G}$ on $\mathbb{R}^n \times \mathbb{R} \times \{1\}$. Apply theorem 5.6 to $\mathbb{R}^n \times \mathbb{R} \times I$ with $C, U, D, \mathcal{F}$ replaced by $\mathbb{R}^n \times (-\infty, 0]$, $\mathbb{R}^n \times (-\infty, \frac{1}{2})$, $\emptyset$, $\mathcal{H}|_{U \times I}$.

We obtain an isotopy F_t on $\mathbb{R}^n \times \mathbb{R} \times \{1\}$ such that $F_0 = 1$, $F_t = 1$ outside $\mathbb{R}^n \times (-\infty, \frac{1}{2}) \times XXXX$ and $F_1 : (\mathbb{R}^n \times (-\infty, \frac{1}{2}) \times \{1\}, \text{standard}) \to XXXXXX$ is PL near $\mathbb{R}^n \times (-\infty, 0) \times \{1\}$.

Let $\mathcal{G}' = F_1^{-1}(\mathcal{G})$, a PL structure on $\mathbb{R}^n \times \mathbb{R}$. Then $\mathcal{G}'$ agrees with $\mathcal{G}$ near $\mathbb{R}^n \times [1, \infty)$ and $\mathcal{G}'$ agrees with the standard structure near $\mathbb{R}^n \times (-\infty, 0]$. $\mathbb{R}^n \times \{0\}$ is a PL submanifold of $\mathcal{G}'$, $U \times \{1\}$ is a PL submanifold of $\mathcal{G}'$, therefore $\mathcal{G}'$ induces a PL structure on $U \times I$. $\mathcal{G}'$ is equal to the standard structure near $U \times \{0\}$.

Apply Theorem 5.6 to $C, U, \emptyset, \mathcal{G}'|_{U \times I}$ to obtain an isotopy G_t of $\mathbb{R}^n \times \mathbb{R}$ such that $G_1 : (U \times I, \text{standard}) \to (U \times I, \mathcal{G}')$ is PL near $C \times I$. G_t is 1 near $\mathbb{R}^n \times \{0\}$.

Let $g : \mathbb{R}^n \to \mathbb{R}$ be defined by $(g(x), 1) + G(XXXXXX)$. Let $\mathcal{G}'' = (g \times 1)G_1^{-1}(\mathcal{G}')$. Near $C \times I$, $\mathcal{G}''$ agrees with $(g \times 1)(\text{standard structure}) = \mathcal{F}_g \times I$.

Define $\mathcal{F} = \mathcal{G}''|_{\mathbb{R}^n \times \{0\}}$. $\mathcal{F}$ agrees with $\mathcal{F}_0$ near C. XXXXXX Remains constant isotopy (rel $C \times \mathbb{R}$) from $\mathcal{F} \times \mathbb{R}$ to $\mathcal{G}$.

Choose a PL isotopy (of embeddings) $j_t : \mathbb{R} \to \mathbb{R}$ such that $j_t = 1$ when $t \leq \frac{1}{4}$ and $j_1(\mathbb{R}) \subset (1, \infty)$. Let $J : \mathbb{R}^n \times \mathbb{R} \times I \to \mathbb{R}^n \times \mathbb{R} \times I$ be defined by $J(x, y, t) = (x, j_t(y), t)$. Then the PL structure $J^{-1}(\mathcal{G}'' \times I)$ agrees with $\mathcal{G}'' \times \{0\}$ on $\mathbb{R}^n \times \mathbb{R} \times \{0\}$ and agrees with $\mathcal{F}_0 \times \mathbb{R} \times I$ near $C \times \mathbb{R} \times I$.

Apply theorem 5.6 (using the fact that $\mathcal{G}''$ is isotopic to the standard structure by lemma 5.10) to obtain an isotopy from $\mathcal{G}''$ to $J^{-1}(\mathcal{G}'' \times \{1\})$, fixed near $C \times \mathbb{R}$. We have $J^{-1}(\mathcal{G}'' \times \{1\}) = J^{-1}(\mathcal{G} \times \{1\})$ (since $\mathcal{G}'' = \mathcal{G}$ on $\mathbb{R}^n \times (1, \infty)$) and similarly $\mathcal{G}$ is isotopic to $J^{-1}(\mathcal{G} \times \{1\})$ (fixed near $C \times \mathbb{R}$). Therefore $\mathcal{G}, \mathcal{G}''$ are isotopic (relative to a neighborhood of $C \times \mathbb{R}$). Similarly, $\mathcal{G}'', \mathcal{F} \times \mathbb{R}^n$ are isotopic fixing a neighborhood of $C \times \mathbb{R}$. Therefore $\mathcal{G}, \mathcal{F} \times \mathbb{R}$ are isotopic fixing a neighborhood of $C \times \mathbb{R}$.

For general M, with $\partial M = \emptyset$, we may assume WLOG that M is connected. We know that M is metrizable implies that M is second countable. So $M = \bigcup_{i=1}^{\infty} f_i(B^n)$ where $f_i : \mathbb{R}^n \to M$ are embeddings. Let $C_i = C \cup f_1(B^n) \cup \cdots \cup f_i(B^n)$. Suppose inductively we have a PL structure $\mathcal{F}_{i-1}$ on a neighborhood of C_{i-1} in M, extending $\mathcal{F}_0$ and a PL structure $\mathcal{G}_{i-1}$ on $M \times \mathbb{R}$ extending $\mathcal{F}_{i-1} \times \mathbb{R}$ and isotopic to $\mathcal{G}$ by an isotopy fixed near $C \times \mathbb{R}$.

Apply the result for $M = \mathbb{R}^n$ to $\mathcal{F}' = f_i^{-1}(\mathcal{F}_{i-1})$ (near $C' = f_i^{-1}(C_{i-1})$) and $(f_i \times 1)^{-1}(\mathcal{G}_{i-1}) = \mathcal{G}'$. We obtain a PL structure $\mathcal{F}''$ on $\mathbb{R}^n$ ($= \mathcal{F}'$ near C') and isotopy H_t of $\mathbb{R}^n \times \mathbb{R}$ with $H_t = 1$ for $t \leq \frac{1}{4}$ and $H_1^{-1}(\mathcal{G}') = \mathcal{F}'' \times \mathbb{R}$, and H_t fixes a neighborhood of C'.

H defines a homeomorphism on $\mathbb{R}^n \times \mathbb{R} \times I$. Let $\mathcal{H} = H^{-1}(\mathcal{G}' \times I)$. $\mathcal{H}$ agrees with $\mathcal{G}'$ near $\mathbb{R}^n \times \mathbb{R} \times \{0\}$, with $\mathcal{F}'' \times \mathbb{R}$ on $\mathbb{R}^n \times \mathbb{R} \times \{1\}$, and near $C' \times \mathbb{R} \times I$. Apply theorem 5.6 to this: replace $C, U, D, \mathcal{F}$ by $B^n \times \mathbb{R}, \operatorname{Int} 2B^n \times \mathbb{R}, C' \times \mathbb{R}, \mathcal{H}$ to obtain a PL structure $\mathcal{G}''$ on $\mathbb{R}^n \times \mathbb{R}$ which agrees with $\mathcal{F}'' \times \mathbb{R}$ near $(C' \cup B^n) \times \mathbb{R}$ and which is isotopic to $\mathcal{G}'$ rel $(C' \cup (\mathbb{R}^n \setminus \operatorname{Int} 2B^n)) \times \mathbb{R})$.

Define $\mathcal{F}_i = \mathcal{F}_{i-1} \cup f_i(\mathcal{F}'')$ and extend $(f_i \times 1)(\mathcal{G}'')$ to a structure $\mathcal{G}_i$ on $M \times \mathbb{R}$ agreeing with $\mathcal{G}_{i-1}$ off $f_i(\mathbb{R}^n) \times \mathbb{R}$. Then $\mathcal{G}_i$ agrees with $\mathcal{F}_i \times \mathbb{R}$ near $C_i \times \mathbb{R}$ and $\mathcal{G}_i$ is isotopic to $\mathcal{G}_{i-1}$ fixing a neighborhood of $C_{i-1} \times \mathbb{R}$, so $\mathcal{F}_i = \mathcal{F}_{i-1}$ near C_{i-1}.

Since $\mathcal{F}_i = \mathcal{F}_{i-1}$ near C_{i-1} there is a PL structure $\mathcal{F}$ on M agreeing with $\mathcal{F}_i$ near C_i. $\mathcal{G}$ agrees with $\mathcal{F} \times \mathbb{R}$ near $C \times \mathbb{R}$, $\mathcal{F}$ agrees with $\mathcal{F}_0$ near XXXXXXXXXXX. Since $\mathcal{G}_i$ is isotopic to $\mathcal{G}_{i-1}$ (fixing a neighborhood of $C_{i-1} \times \mathbb{R}$). Hence all isotopies can be pieced together to obtain an isotopy of $\mathcal{F} \times \mathbb{R}$ to $\mathcal{G}$, fixing a neighborhood of $C \times \mathbb{R}$. This proves the product theorem when M has no boundary.

If M has nonempty boundary ∂M, then apply the theorem for M unbounded to ∂M, and then to $\operatorname{Int} M$ using a collaring argument. We seem to need $\dim M \geqslant 7$ to ensure $\dim \partial M \geqslant 6$. □

In fact the theorem can be proved for all unbounded 5-manifolds and all 6-manifolds.

As an application, if M is a topological manifold, we can embed M in $\mathbb{R}^N$ with a neighborhood E which fibers over M, i.e. there is a map $\phi : E \to M$ which is locally the projection of product, with fiber $\mathbb{R}^n$ (structural group $\mathcal{H}(\mathbb{R}^n) = \mathrm{Top}_n$).

Let $\nu = \phi$. A *necessary* condition for M to have a PL structure is that ν come from a PL bundle over M. This is also *sufficient* if $\dim M \geqslant 6$.

$E(\nu)$ is an open subset of $\mathbb{R}^N$ so that it inherits a PL structure. Suppose there exists a PL bundle ξ over $E(\nu)$ which is equivalent as a topological bundle to ν. There exists a PL bundle η over $E(\nu)$ such that $\xi \oplus \eta$ is trivial. The the total space $E(\eta)$ is homeomorphic to $M \times \mathbb{R}^k$ and has a PL structure. By the product structure theorem, M has a PL structure.

There exists a classifying space BTop_n classifying such topological bundles by $[M, \mathrm{BTop}_n]$. n is immaterial, so take $\mathrm{BTop} = \bigcup_{n=1}^{\infty} \mathrm{BTop}_n$. Similarly for $\mathrm{BPL}_n, \mathrm{BPL}$. There is a natural map $\mathrm{BPL}_n \to \mathrm{BTop}$ which forgets the extra structure.

Therefore when $\dim M \geqslant 6$, M has a PL structure if the map $\nu : M \to \mathrm{BTop}$ factors (up to homotopy) as

$$\begin{tikzcd}
 & M \arrow[dl, dashed] \arrow[d, "\nu"] \\
\mathrm{BPL} \arrow[r] & \mathrm{BTop}
\end{tikzcd}$$

Therefore M has a PL structure iff the classifying map of the *stable normal bundle* ν *of* M lies in the image of $[M, \mathrm{BPL}] \to [M, \mathrm{BTop}]$.

To show that PL $\neq$ Top: let k be an integer, and $p_k : T^n \to T^n$ be induced by $\mathbb{R}^n \to \mathbb{R}^n$; $x \mapsto kx$. Then p_k is a k^n-fold covering (a fiber bundle with discrete fibers of k^n XXXXXXXXXXXXX). There exists a homeomorphism $h_k : T^n \to T^n$ such that

$$\begin{tikzcd}
T^n \arrow[r, dashed, "h_k"] \arrow[d, "p_k"'] & T^n \arrow[d, "p_k"] \\
T^n \arrow[r, "h"] & T^n
\end{tikzcd}$$

for any given homeomorphism $h : T^n \to T^n$. There are k^n such homeomorphisms. Since all covering translations of $p_k : T^n \to T^n$ are isotopic to 1, any two choices for h_k are isotopic.

Theorem 5.11. *If $h : T^n \to T^n$ is a homeomorphism homotopic to* 1*, then h_k is topologically isotopic to* 1 *for sufficiently large k.*

Proof. First isotope h until $h(0) = 0$ (where $0 = e(0) \in T^n$.) Choose h_k so that $h_k(0) = 0$. Let $\widetilde{h_k} : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism such that $e\widetilde{h_k} = h_k e$ and $\widetilde{h_k}(0) = 0$. Since $h \simeq 1$, $\widetilde{h}_1 = \widetilde{h}$ is bounded. $\widetilde{h_k}(x) = \frac{1}{k}\widetilde{h}(x)$ because $p_k e\widetilde{h_k} = p_k h_k e = h p_k e$, $\widetilde{h_k}(0) = 0$, and these characterize $\widetilde{h_k}$. We have

$$\sup_{x \in \mathbb{R}^n} d(x, \widetilde{h_k}(x)) = \frac{1}{k}\left(\sup_{x \in \mathbb{R}^n} d(x, \widetilde{h}(x))\right) \to 0$$

as $k \to \infty$. So $\sup_{y \in T^n} d(y, h_k(y)) \to 0$ as $k \to \infty$. But $\mathcal{H}(T^n)$ is locally contractible by Theorem 4.8. Therefore if k is large enough, h_k is isotopic to 1. $\square$

But the behavior is different in the PL case:

Proposition 5.12. *Let $n \geqslant 5$. There exists a PL homeomorphism $h : T^n \to T^n$ such that $h \simeq 1$ and h_k is not PL isotopic to 1 for any odd k.*

Exercise. Show that if $h : T^n \to T^n$ is PL and topologically isotopic to 1 but not PL isotopic to 1 then $T^n \times I/(x, 0) \sim (h(x), 1)$ is topologically homeomorphic to T^{n+1} but not PL homeomorphic to T^{n+1}.