14th April, 1996

Dear Frank

I have obtained a closed form expression of the group $Q_0(B,\beta)$, which I believe shows that (after all) there is 4-torsion in $\mathrm{Unil}_3(\mathbb{Z})$!

There are two general principles underlying what follows:

1. The relative twisted quadratic Q-groups $Q_*(f,\chi)$ are defined for any map of chain bundles

$$(f,\chi)\ :\ (C,\gamma)\longrightarrow(D,\delta)$$

to fit into a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \widehat{Q}^{n+1}(C) & \longrightarrow & Q_n(C,\gamma) & \longrightarrow & Q^n(C) & \xrightarrow{J_\gamma} & \widehat{Q}^n(C) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \widehat{f}^{\%}} & & \downarrow{\scriptstyle (f,\chi)_{\%}} & & \downarrow{\scriptstyle f^{\%}} & & \downarrow{\scriptstyle \widehat{f}^{\%}} & & \\
\dots & \longrightarrow & \widehat{Q}^{n+1}(D) & \longrightarrow & Q_n(D,\delta) & \longrightarrow & Q^n(D) & \xrightarrow{J_\delta} & \widehat{Q}^n(D) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \widehat{Q}^{n+1}(f) & \longrightarrow & Q_n(f,\chi) & \longrightarrow & Q^n(f) & \longrightarrow & \widehat{Q}^n(f) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \widehat{Q}^{n}(C) & \longrightarrow & Q_{n-1}(C,\gamma) & \longrightarrow & Q^{n-1}(C) & \xrightarrow{J_\gamma} & \widehat{Q}^{n-1}(C) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

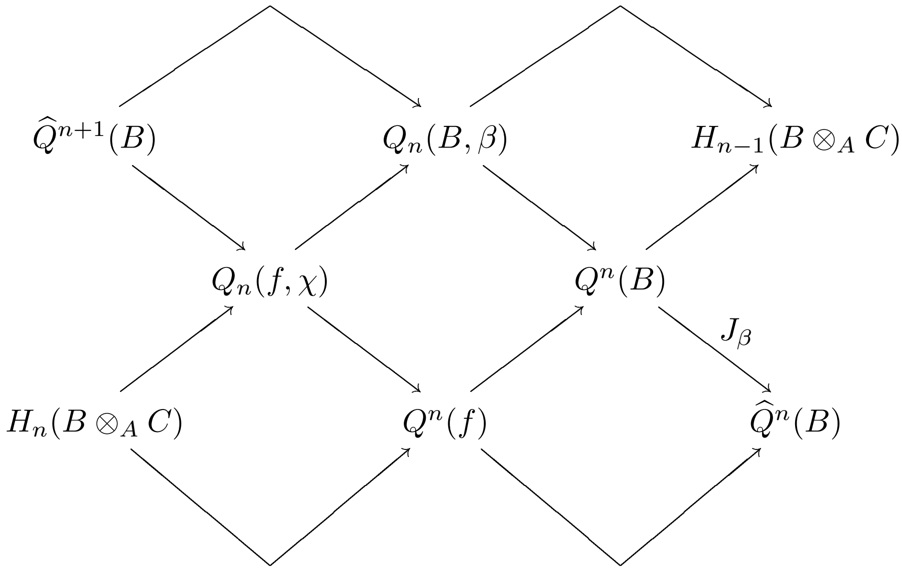
2. In the special case of 1. when $\gamma=0$ there is defined a chain bundle (B,β) on the algebraic mapping cone

$$B\ =\ \mathcal{C}(f)\ ,$$

such that the inclusion $g:D\longrightarrow B$ is covered by a map of chain bundles

$$(g,\eta)\ :\ (D,\delta)\longrightarrow(B,\beta)$$

with $(g,\eta)(f,\chi) \simeq (0,0)$. In this case the relative twisted quadratic Q-groups $Q_*(f,\chi)$ of 1. are related to the (absolute) twisted quadratic Q-groups $Q_*(B,\beta)$ by a commutative braid of exact sequences



involving the exact sequence

$$\dots \longrightarrow H_n(B \otimes_A C) \longrightarrow Q^n(f) \longrightarrow Q^n(B) \longrightarrow H_{n-1}(B \otimes_A C) \longrightarrow \dots$$

in the braid of §5 of the EPSRC report.

As in the report let A be a commutative ring with the identity involution, with no additive 2-torsion. Given an integer $r \geq 1$ let

$$\begin{aligned}
M_r(A) &= \text{additive group of } r \times r \text{ matrices } (a_{ij}) \text{ with } a_{ij} \in A \ , \\
T \ : \ M_r(A) &\longrightarrow M_r(A) \ ; \ M \ = \ (a_{ij}) \longrightarrow M^t \ = \ (a_{ji}) \ , \\
\mathrm{Sym}_r(A) &= \ker(1-T) \ = \ \{(a_{ij}) \in M_r(A) \,|\, a_{ij} = a_{ji}\} \ , \\
\mathrm{Quad}_r(A) &= \mathrm{im}(1+T) \ = \ \{(a_{ij}) \in \mathrm{Sym}_r(A) \,|\, a_{ii} \in 2A\} \ .
\end{aligned}$$

Given $x_1, x_2, \dots, x_r \in A$ let

$$X \ = \ \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_r \end{pmatrix} \in \mathrm{Sym}_r(A) \ .$$

(In the application $\{x_1, x_2, \dots, x_r\}$ represents a basis of the $A/2A$-module $\widehat{H}^0(\mathbb{Z}_2; A)$, but the general theory works for any $\{x_1, x_2, \dots, x_r\}$). Define a map of chain bundles over A

$$(f,\chi) \ : \ (C,0) \longrightarrow (D,\delta)$$

by

$$
\begin{aligned}
f \ &= \ 2 \ : \ C_0 \ = \ A^r \longrightarrow D_0 \ = \ A^r \ , \\
C_r \ &= \ D_r \ = \ 0 \ \ (r \neq 0) \ , \\
\delta_0 \ &= \ X \ : \ D_0 \ = \ A^r \longrightarrow D^0 \ = \ A^r \ , \\
\chi_{-1} \ &= \ 2X \ : \ C_0 \ = \ A^r \longrightarrow C^0 \ = \ A^r \ .
\end{aligned}
$$

In order to compute $Q_0(B,\beta)$ with $B = \mathcal{C}(f)$ and β as in 2. above consider first $Q_0(f,\chi)$. The part of the commutative diagram in 1. with exact rows and columns

$$
\begin{array}{ccccccccc}
Q^1(C) & \xrightarrow{J} & \widehat{Q}^1(C) & \longrightarrow & Q_0(C,0) & \longrightarrow & Q^0(C) & \xrightarrow{J} & \widehat{Q}^0(C) \\
\big\downarrow{\scriptstyle f^{\%}} & & \big\downarrow{\scriptstyle \widehat{f}^{\%}} & & \big\downarrow{\scriptstyle (f,\chi)_{\%}} & & \big\downarrow{\scriptstyle f^{\%}} & & \big\downarrow{\scriptstyle \widehat{f}^{\%}} \\
Q^1(D) & \xrightarrow{J_\delta} & \widehat{Q}^1(D) & \longrightarrow & Q_0(D,\delta) & \longrightarrow & Q^0(D) & \xrightarrow{J_\delta} & \widehat{Q}^0(D) \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
Q^1(f) & \longrightarrow & \widehat{Q}^1(f) & \longrightarrow & Q_0(f,\chi) & \longrightarrow & Q^0(f) & \longrightarrow & \widehat{Q}^0(f) \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
Q^0(C) & \xrightarrow{J} & \widehat{Q}^0(C) & \longrightarrow & Q_{-1}(C,0) & \longrightarrow & Q^{-1}(C) & \xrightarrow{J} & \widehat{Q}^{-1}(C)
\end{array}
$$

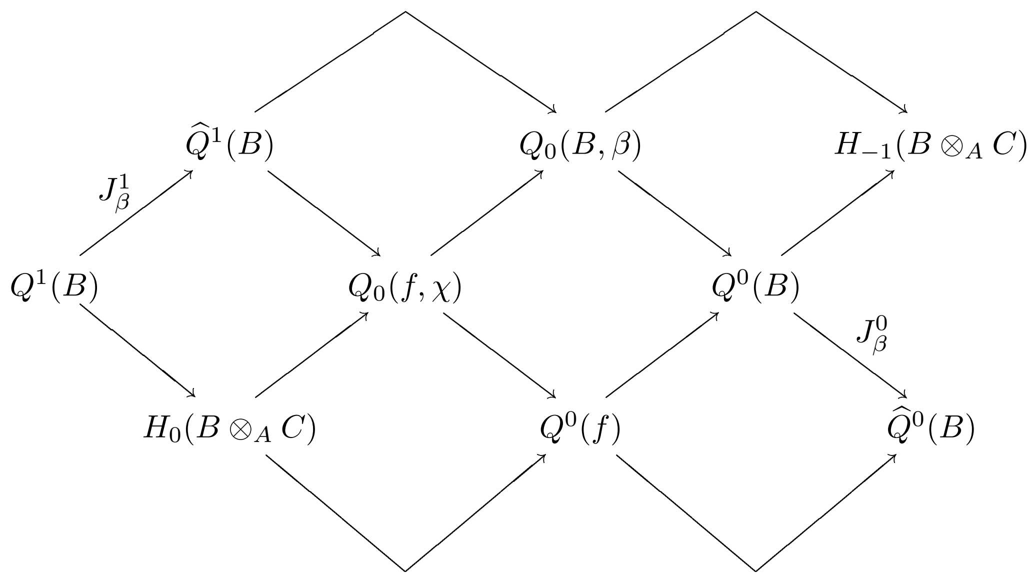
is given by

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Quad}_r(A) & \longrightarrow & \mathrm{Sym}_r(A) & \longrightarrow & \mathrm{Sym}_r(A)/\mathrm{Quad}_r(A) \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle 4} & & \big\downarrow{\scriptstyle 4} & & \big\downarrow{\scriptstyle 0} \\
0 & \longrightarrow & 0 & \longrightarrow & Q_0(D,\delta) & \longrightarrow & \mathrm{Sym}_r(A) & \xrightarrow{J_X} & \mathrm{Sym}_r(A)/\mathrm{Quad}_r(A) \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle 1} \\
0 & \longrightarrow & \mathrm{Sym}_r(A)/\mathrm{Quad}_r(A) & \xrightarrow{4} & Q_0(f,\chi) & \longrightarrow & \mathrm{Sym}_r(A)/4\mathrm{Sym}_r(A) & \rightarrow & \mathrm{Sym}_r(A)/\mathrm{Quad}_r(A) \\
\big\downarrow & & \big\downarrow{\scriptstyle 1} & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Sym}_r(A) & \rightarrow & \mathrm{Sym}_r(A)/\mathrm{Quad}_r(A) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

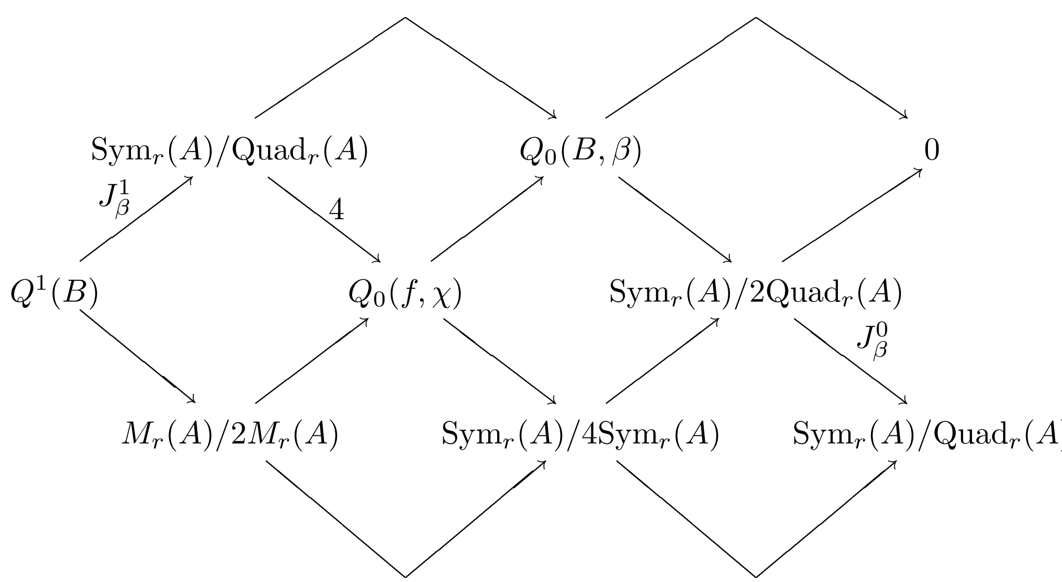
with

$$J_X \ : \ \mathrm{Sym}_r(A) \longrightarrow \mathrm{Sym}_r(A)/\mathrm{Quad}_r(A) \ ; \ M \longrightarrow M - M^t X M \ ,$$
$$Q_0(D,\delta) \ = \ \ker(J_X) \ ,$$
$$Q_0(f,\chi) \ = \ \ker(J_X)/4\mathrm{Quad}_r(A) \ .$$

The part of the commutative exact braid in 2.



is given by



with

$$\begin{aligned}
&\mathrm{Sym}_r(A)/2\mathrm{Quad}_r(A) \xrightarrow{\simeq} Q^0(B)\ ;\ M \longrightarrow \phi \text{ (where } \phi_0 = M : B^0 \longrightarrow B_0)\ ,\\
&J^0_\beta\ :\ \mathrm{Sym}_r(A)/2\mathrm{Quad}_r(A) \longrightarrow \mathrm{Sym}_r(A)/\mathrm{Quad}_r(A)\ ;\ M \longrightarrow M - M^tXM\ ,\\
&M_r(A)/2M_r(A) \longrightarrow \mathrm{Sym}_r(A)/4\mathrm{Sym}_r(A)\ ;\ N \longrightarrow 2(N+N^t)\ ,\\
&Q^1(B)\ =\ \ker(M_r(A)/2M_r(A) \longrightarrow \mathrm{Sym}_r(A)/4\mathrm{Sym}_r(A))\\
&\qquad =\ \{N \in M_r(A) \mid N + N^t \in 2\mathrm{Sym}_r(A)\}/2M_r(A)\\
&\qquad\quad \text{(where } \phi \in Q^1(B) \text{ corresponds to } N = \phi_0 \in M_r(A))\ ,\\
&J^1_\beta\ :\ Q^1(B) \longrightarrow \widehat{Q}^1(B)\ =\ \mathrm{Sym}_r(A)/\mathrm{Quad}_r(A)\ ;\ N \longrightarrow \frac{1}{2}(N+N^t) - N^tXN\ ,\\
&M_r(A)/2M_r(A) \longrightarrow Q_0(f,\chi)\ ;\ N \longrightarrow 2(N+N^t) - 4N^tXN\ .
\end{aligned}$$

It follows that

$$\begin{aligned}
Q_0(B,\beta)\ &=\ \mathrm{coker}(M_r(A)/2M_r(A) \longrightarrow Q_0(f,\chi))\\
&=\ \frac{\{M \in \mathrm{Sym}_r(A) \mid M - M^tXM \in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A) + \{2(N+N^t) - 4N^tXN \mid N \in M_r(A)\}}\ ,\\
Q_{-1}(B,\beta)\ &=\ Q_{-1}(f,\chi)\\
&=\ \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A) + \{M - M^tXM \mid M \in \mathrm{Sym}_r(A)\}}\ .
\end{aligned}$$

For any $A, x_1, x_2, \ldots, x_r$ write (B,β) defined above as $(B(x_1,\ldots,x_r), \beta(x_1,\ldots,x_r))$.

The analysis of $Q_0(B,\beta)$ in the special case

$$A\ =\ \mathbb{Z}[x]\quad ,\quad r\ =\ 2\quad ,\quad x_1\ =\ 1\quad ,\quad x_2\ =\ x$$

involves properties of 2×2 matrices over $\mathbb{Z}[x]$ similar to the ones you were looking at last week (but with the crucial difference that there is now 4-torsion). Rather than work directly with the 2×2 matrices it seems easier to first work out the twisted quadratic Q-groups for the two direct summands in

$$(B,\beta)\ =\ (B(1,x),\beta(1,x))\ =\ (B(1),\beta(1)) \oplus (B(x),\beta(x))$$

which only involve 1×1 matrices, and then to combine them.

For $r=1$, $x_1 = x \in A$ (arbitrary x, A)

$$\begin{aligned}
Q_0(B(x),\beta(x))\ &=\ \frac{\{a \in A \mid a - a^2x \in 2A\}}{8A + \{4(b - b^2x) \mid b \in A\}}\ ,\\
Q_{-1}(B(x),\beta(x))\ &=\ \frac{A}{2A + \{c - c^2x \mid c \in A\}}
\end{aligned}$$

and the maps in the exact sequence

$$Q^1(B(x)) \xrightarrow{J_{\beta(x)}} \widehat{Q}^1(B(x)) \longrightarrow Q_0(B(x),\beta(x)) \longrightarrow Q^0(B(x)) \xrightarrow{J_{\beta(x)}} \widehat{Q}^0(B(x)) \longrightarrow Q_{-1}(B(x),\beta(x))$$

are given by

$$\begin{aligned}
&J_{\beta(x)} \;:\; Q^1(B(x)) \;=\; A/2A \longrightarrow \widehat{Q}^1(B(x)) \;=\; A/2A \;;\; a \longrightarrow a - a^2x \;,\\
&\widehat{Q}^1(B(x)) \;=\; A/2A \longrightarrow Q_0(B(x),\beta(x)) \;;\; a \longrightarrow 4a \;,\\
&Q_0(B(x),\beta(x)) \longrightarrow Q^0(B(x)) \;=\; A/4A \;;\; a \longrightarrow a \;,\\
&J_{\beta(x)} \;:\; Q^0(B(x)) \;=\; A/4A \longrightarrow \widehat{Q}^0(B(x)) \;=\; A/2A \;;\; a \longrightarrow a - a^2x \;,\\
&\widehat{Q}^0(B(x)) \;=\; A/2A \longrightarrow Q_{-1}(B(x),\beta(x)) \;;\; a \longrightarrow a \;.
\end{aligned}$$

For $r = 2$ (arbitrary x_1, x_2, A)

$$(B(x_1,x_2),\beta(x_1,x_2)) \;=\; (B(x_1),\beta(x_1)) \oplus (B(x_2),\beta(x_2))$$

and

$$\begin{aligned}
&Q_0(B(x_1,x_2),\beta(x_1,x_2))\\
&\quad= \frac{\{\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathrm{Sym}_2(A) \,|\, a - a^2x_1 - b^2x_2, c - b^2x_1 - c^2x_2 \in 2A\}}{4\mathrm{Quad}_2(A) + \{2(N+N^t) - 4N^tXN \,|\, N \in M_2(A)\}} \;,\\
&Q_{-1}(B(x_1,x_2),\beta(x_1,x_2)) \;=\; \frac{\mathrm{Sym}_2(A)}{\mathrm{Quad}_2(A) + \{M - M^tXM \,|\, M \in \mathrm{Sym}_2(A)\}}
\end{aligned}$$

with $X = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$. The twisted quadratic Q-groups of the sum are related the sum of the twisted quadratic Q-groups by the exact sequence on page 27 of *Algebraic Poincaré complexes*

$$\begin{aligned}
\dots \longrightarrow\; & H_1(B(x_1) \otimes_A B(x_2))\\
\longrightarrow\; & Q_0(B(x_1),\beta(x_1)) \oplus Q_0(B(x_2),\beta(x_2))\\
\longrightarrow\; & Q_0(B(x_1,x_2),\beta(x_1,x_2)) \longrightarrow H_0(B(x_1) \otimes_A B(x_2))\\
\longrightarrow\; & Q_{-1}(B(x_1),\beta(x_1)) \oplus Q_{-1}(B(x_2),\beta(x_2)) \longrightarrow \dots
\end{aligned} \qquad (*)$$

with

$$\begin{aligned}
Q_0(B(x_i),\beta(x_i)) \;&=\; \frac{\{a \in A \,|\, a - a^2x_i \in 2A\}}{8A + \{4(b - b^2x_i) \,|\, b \in A\}} \;,\\
Q_{-1}(B(x_i),\beta(x_i)) \;&=\; \frac{A}{2A + \{a - a^2x_i \,|\, a \in A\}} \quad (i = 1,2) \;,
\end{aligned}$$

$$H_1(B(x_1)\otimes_A B(x_2)) = A/2A \longrightarrow Q_0(B(x_1),\beta(x_1))\oplus Q_0(B(x_2),\beta(x_2)) ;$$
$$a \longrightarrow (4a^2x_2, 4a^2x_1) ,$$
$$Q_0(B(x_1,x_2),\beta(x_1,x_2)) \longrightarrow H_0(B(x_1)\otimes_A B(x_2)) = A/2A ; \begin{pmatrix} a & b \\ b & c \end{pmatrix} \longrightarrow b ,$$
$$H_0(B(x_1)\otimes_A B(x_2)) = A/2A \longrightarrow Q_{-1}(B(x_1),\beta(x_1))\oplus Q_{-1}(B(x_2),\beta(x_2)) ;$$
$$a \longrightarrow (a^2x_2, a^2x_1) .$$

Now take $A = \mathbb{Z}[x]$, $x_1 = 1$, $x_2 = x$, so that $(B(1,x),\beta(1,x))$ is the 1-skeleton of the universal chain bundle over $\mathbb{Z}[x]$. Will first compute $Q_0(B(1),\beta(1))$ and $Q_0(B(x),\beta(x))$, and then use $(*)$ to obtain $Q_0(B(1,x),\beta(1,x))$.

For $x_1 = 1 \in A = \mathbb{Z}[x]$ the expressions
$$Q_0(B(1),\beta(1)) = \frac{\{a\in\mathbb{Z}[x]\,|\,a-a^2\in 2\mathbb{Z}[x]\}}{8\mathbb{Z}[x]+\{4(b-b^2)\,|\,b\in\mathbb{Z}[x]\}} ,$$
$$Q_{-1}(B(1),\beta(1)) = \frac{\mathbb{Z}[x]}{2\mathbb{Z}[x]+\{c-c^2\,|\,c\in\mathbb{Z}[x]\}}$$
will now be analyzed in detail.
First $Q_0(B(1),\beta(1))$. For any polynomial $a = \sum\limits_{i=0}^{\infty} a_ix^i \in \mathbb{Z}[x]$
$$a - a^2 = \sum_{i=0}^{\infty} a_ix^i - \sum_{i=0}^{\infty} a_ix^{2i} \in \mathbb{Z}[x]/2\mathbb{Z}[x] .$$
Now $a-a^2 \in 2\mathbb{Z}[x]$ if and only if the coefficients $a_0, a_1, \ldots \in \mathbb{Z}$ are such that
$$a_1 \equiv a_2 - a_1 \equiv a_3 \equiv a_4 - a_2 \equiv \ldots \equiv 0 (\bmod 2) ,$$
if and only if
$$a_1 \equiv a_2 \equiv a_3 \equiv a_4 \equiv \ldots \equiv 0 (\bmod 2) .$$
For such a polynomial a
$$a \in 8\mathbb{Z}[x] + \{4(b-b^2)\,|\,b\in\mathbb{Z}[x]\}$$
if and only if there exist $b_0, b_1, \ldots \in \mathbb{Z}$ such that
$$a_0 \equiv 0 , \; a_1 \equiv 4b_1 , \; a_2 \equiv 4(b_2-b_1) , \; a_3 \equiv 4b_3 , \; a_4 \equiv 4(b_4-b_2) , \; \ldots \; (\bmod 8) ,$$
if and only if
$$a_1 \equiv a_2 \equiv a_3 \equiv a_4 \equiv \ldots \equiv 0 (\bmod 4) ,$$
$$a_0 \equiv a_1 + a_2 + a_3 + \ldots \equiv 0 (\bmod 8) .$$
Thus there is defined an isomorphism
$$Q_0(B(1),\beta(1)) \xrightarrow{\simeq} \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \sum_2^{\infty} \mathbb{Z}_2 ; \; \sum_{i=0}^{\infty} a_ix^i \longrightarrow (a_0, (\sum_{i=1}^{\infty} a_i)/2, a_2/2, a_3/2, \ldots) .$$

The map $\widehat{Q}^1(B(1)) \longrightarrow Q_0(B(1),\beta(1))$ is given by

$$\widehat{Q}^1(B(1)) \;=\; \mathbb{Z}[x]/2\mathbb{Z}[x] \longrightarrow Q_0(B(1),\beta(1)) \;=\; \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \sum_2^\infty \mathbb{Z}_2 \;;$$

$$\sum_{i=0}^\infty c_i x^i \longrightarrow (4c_0, \sum_{i=1}^\infty 2c_i, 0, 0, \dots) \;.$$

Next, $Q_{-1}(B(1),\beta(1))$. A polynomial $d = \sum\limits_{i=0}^\infty d_i x^i \in \mathbb{Z}[x]$ is such that

$$d \in 2\mathbb{Z}[x] + \{c - c^2 \,|\, c \in \mathbb{Z}[x]\}$$

if and only if there exist $c_0, c_1, \ldots \in \mathbb{Z}$ such that

$$d_0 \;\equiv\; 0 \;\;,\;\; d_1 \;\equiv\; c_1 \;\;,\;\; d_2 \;\equiv\; c_2 - c_1 \;\;,\;\; d_3 \;\equiv\; c_3 \;\;,\;\; d_4 \;\equiv\; c_4 - c_2 \;,\; \ldots \;(\bmod 2) \;,$$

if and only if

$$d_0 \;\equiv\; d_1 + d_2 + d_3 + \ldots \;\equiv\; 0 (\bmod 2) \;.$$

Thus there is defined an isomorphism

$$Q_{-1}(B(1),\beta(1)) \xrightarrow{\;\simeq\;} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \;;\; \sum_{i=0}^\infty d_i x^i \longrightarrow (d_0, d_1 + d_2 + d_3 + \ldots) \;.$$

For $x_2 = x \in A = \mathbb{Z}[x]$ the expressions

$$Q_0(B(x),\beta(x)) \;=\; \frac{\{a \in \mathbb{Z}[x] \,|\, a - a^2 x \in 2\mathbb{Z}[x]\}}{8\mathbb{Z}[x] + \{4(b - b^2 x) \,|\, b \in \mathbb{Z}[x]\}} \;,$$

$$Q_{-1}(B(x),\beta(x)) \;=\; \frac{\mathbb{Z}[x]}{2\mathbb{Z}[x] + \{a - a^2 x \,|\, a \in \mathbb{Z}[x]\}} \;.$$

will now be analyzed in detail.

First $Q_0(B(x),\beta(x))$. For any polynomial $a = \sum\limits_{i=0}^\infty a_i x^i \in \mathbb{Z}[x]$

$$a - a^2 x \;=\; \sum_{i=0}^\infty a_i x^i - \sum_{i=0}^\infty a_i x^{2i+1} \in \mathbb{Z}[x]/2\mathbb{Z}[x] \;.$$

Now $a - a^2 x \in 2\mathbb{Z}[x]$ if and only if the coefficients $a_0, a_1, \ldots \in \mathbb{Z}$ are such that

$$a_0 \;\equiv\; a_1 - a_0 \;\equiv\; a_2 \;\equiv\; a_3 - a_1 \;\equiv\; \ldots \;\equiv\; 0 (\bmod 2) \;,$$

if and only if

$$a_0 \;\equiv\; a_1 \;\equiv\; a_2 \;\equiv\; a_3 \;\equiv\; \ldots \;\equiv\; 0 (\bmod 2) \;.$$

For such a polynomial a

$$a \in 8\mathbb{Z}[x] + \{4(b - b^2 x) \,|\, b \in \mathbb{Z}[x]\}$$

if and only there exist $b_0, b_1, \ldots \in \mathbb{Z}$ such that

$$a_0 \;\equiv\; 4b_0 \;,\; a_1 \;\equiv\; 4(b_1 - b_0) \;,\; a_2 \;\equiv\; 4b_2 \;,\; a_3 \;\equiv\; 4(b_3 - b_1) \;,\; \ldots \;(\bmod 8) \;,$$

if and only if

$$\begin{aligned} a_0 \equiv a_1 \equiv a_2 \equiv a_3 \equiv \ldots &\equiv 0 (\bmod 4) \ , \\ a_0 + a_1 + a_2 + a_3 + \ldots &\equiv 0 (\bmod 8) \ . \end{aligned}$$

Thus there is defined an isomorphism

$$Q_0(B(x), \beta(x)) \xrightarrow{\simeq} \mathbb{Z}_4 \oplus \sum_1^\infty \mathbb{Z}_2 \ ; \ \sum_{i=0}^\infty a_i x^i \longrightarrow ((\sum_{i=0}^\infty a_i)/2, a_1/2, a_2/2, \ldots) \ ,$$

The map $\widehat{Q}^1(B(x)) \longrightarrow Q_0(B(x), \beta(x))$ is given by

$$\widehat{Q}^1(B(x)) \ = \ \mathbb{Z}[x]/2\mathbb{Z}[x] \longrightarrow Q_0(B(x), \beta(x)) \ = \ \mathbb{Z}_4 \oplus \sum_1^\infty \mathbb{Z}_2 \ ; \ \sum_{i=0}^\infty c_i x^i \longrightarrow (\sum_{i=0}^\infty 2c_i, 0) \ .$$

Next, $Q_{-1}(B(x), \beta(x))$. A polynomial $d = \sum\limits_{i=0}^\infty d_i x^i \in \mathbb{Z}[x]$ is such that

$$d \in 2\mathbb{Z}[x] + \{c - c^2 x \,|\, c \in \mathbb{Z}[x]\}$$

if and only if there exist $c_0, c_1, \ldots \in \mathbb{Z}$ such that

$$d_0 \equiv c_0 \ , \ d_1 \equiv c_1 - c_0 \ , \ d_2 \equiv c_2 \ , \ d_3 \equiv c_3 - c_1 \ , \ d_4 \equiv c_4 \ , \ \ldots (\bmod 2) \ ,$$

if and only if

$$d_0 + d_1 + d_2 + d_3 + \ldots \equiv 0 (\bmod 2) \ .$$

Thus there is defined an isomorphism

$$Q_{-1}(B(x), \beta(x)) \xrightarrow{\simeq} \mathbb{Z}_2 \ ; \ \sum_{i=0}^\infty d_i x^i \longrightarrow d_0 + d_1 + d_2 + d_3 + \ldots \ .$$

For $A = \mathbb{Z}[x]$, $x_1 = 1$, $x_2 = x$ the maps in the exact sequence given by $(*)$

$$\begin{aligned} \ldots \longrightarrow & H_1(B(1) \otimes_{\mathbb{Z}[x]} B(x)) \\ \longrightarrow & Q_0(B(1), \beta(1)) \oplus Q_0(B(x), \beta(x)) \longrightarrow Q_0(B(1,x), \beta(1,x)) \\ \longrightarrow & H_0(B(1) \otimes_{\mathbb{Z}[x]} B(x)) \longrightarrow Q_{-1}(B(1), \beta(1)) \oplus Q_{-1}(B(x), \beta(x)) \longrightarrow \ldots \end{aligned}$$

are such that

$$H_1(B(1)\otimes_{\mathbb{Z}[x]} B(x)) ~=~ \mathbb{Z}[x]/2\mathbb{Z}[x]$$

$$\longrightarrow Q_0(B(1),\beta(1))\oplus Q_0(B(x),\beta(x)) ~=~ \Big(\mathbb{Z}_8\oplus\mathbb{Z}_4\oplus\sum_1^\infty\mathbb{Z}_2\Big)\oplus\Big(\mathbb{Z}_4\oplus\sum_0^\infty\mathbb{Z}_2\Big)$$

$$\sum_{i=0}^\infty a_ix^i \longrightarrow ((0,\sum_{i=0}^\infty 2a_i,0,0,\dots),(\sum_{i=0}^\infty 2a_i,0,0,\dots))~,$$

$$H_0(B(1)\otimes_{\mathbb{Z}[x]} B(x)) ~=~ \mathbb{Z}[x]/2\mathbb{Z}[x]$$

$$\longrightarrow Q_{-1}(B(1),\beta(1))\oplus Q_{-1}(B(x),\beta(x)) ~=~ (\mathbb{Z}_2\oplus\mathbb{Z}_2)\oplus\mathbb{Z}_2~;$$

$$\sum_{i=0}^\infty a_ix^i \longrightarrow ((0,\sum_{i=0}^\infty a_i),\sum_{i=0}^\infty a_i)~.$$

Thus there is defined an exact sequence

$$0\longrightarrow\mathbb{Z}_2\longrightarrow\Big(\mathbb{Z}_8\oplus\mathbb{Z}_4\oplus\sum_2^\infty\mathbb{Z}_2\Big)\oplus\Big(\mathbb{Z}_4\oplus\sum_1^\infty\mathbb{Z}_2\Big)\longrightarrow Q_0(B(1,x),\beta(1,x))$$

$$\longrightarrow\sum_0^\infty\mathbb{Z}_2\longrightarrow\mathbb{Z}_2\longrightarrow 0$$

(with the generator of the first $\mathbb{Z}_2$ sent to $(2,2)\in\mathbb{Z}_4\oplus\mathbb{Z}_4$) and

$$Q_0(B(1,x),\beta(1,x))/\mathbb{Z}_8 ~=~ \mathrm{Unil}_3(\mathbb{Z})$$

has 4-torsion.

Best wishes, Andrew