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97. Firey, W.J., 'Blaschke sums of convex bodies and mixed bodies' in *Proceedings of the colloquium on convexity, Copenhagen, 1965* (ed. W. Fenchel), Københavns Universitets Matematiske Institut, Copenhagen (1967)
98. Efimov, N.V., *Flächenverbiegung im Grossen,* Akademie Verlag, Berlin (1957); translated from Russian
99. Pogorelov, A.V., *Extrinsic geometry of convex surfaces,* Translations of Mathematical Monographs 35, American Mathematical Society, Providence, R.I. (1973); translated from Russian
100. Pogorelov, A.V., *Die eindeutige Bestimmung allgemeiner konvexer Flächen,* Akademie Verlag, Berlin (1956); translated from Russian
101. Pogorelov, A.V., *Die Verbiegung konvexer Flächen,* Akademie Verlag, Berlin (1957); translated from Russian
102. Matheron, G., *Integral geometry and geometric probability,* Wiley, New York (1975)
103. Larman, D.G., and Rogers, C.A., 'The existence of a centrally symmetric convex body with central sections that are unexpectedly small', *Mathematika,* **22**, 164-175 (1975)
104. Schneider, R., 'Zu einem Problem von Shephard über die Projektionen konvexer Körper', *Mathematische Zeitschrift,* **101**, 71-82 (1967)
105. Kendall, M.G. and Moran, P.A.P., *Geometric probability,* Griffin, London (1963)
106. John, F., 'Extremum problems with inequalities as subsidiary conditions' in *Studies and essays presented to R. Courant* (ed. K.O. Friedrichs, O.E. Neugebauer and J.J. Stoker), Wiley, New York (1948)
107. Danzer, L., Laugwitz, D. and Lenz, H., 'Über das Löwnersche Ellipsoid und sein Analogen unter den einem Eikörper einbeschriebenen Ellipsoiden', *Archiv der Mathematik,* **8**, 214-219 (1957)
108. Grünbaum, B., 'Borsuk's problem and related questions', *American Mathematical Society. Proceedings of Symposia in Pure Mathematics,* **7**, 271-284 (1963)
109. Knast, R., 'An approximate theorem for Borsuk's conjecture', *Cambridge Philosophical Society. Proceedings,* **75**, 75-76 (1974)
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13

Topology

*J.F. Adams and A.R. Pears**

13.1 INTRODUCTION

The object of this section is to give guidance on, and help with, the use of the literature in two fields, which can broadly be described as (1) classical, homotopy-invariant algebraic topology, and (2) its application to the study of manifolds. The former is a subject which enjoyed a period of rapid growth starting in the early 1950s; it now presents a well-developed body of theory, and seems to be moving more slowly. By contrast there is now (1975) more research activity in the application of algebraic topology to problems arising elsewhere, and here one must include almost all problems about manifolds. Of course, on the one hand, problems about manifolds do not exhaust the applications of algebraic topology, and, on the other hand, their solution may require other techniques in addition to those of algebraic topology. Still, it would seem hard to do serious work on manifolds without a knowledge of the classical methods of algebraic topology.

The primary sources in these areas are of course papers in the learned journals; and these can be located in the usual way via *Mathematical Reviews* and similar bibliographical works (cf. Section 3.4). Of particular use here are the *Reviews of papers in algebraic and differential topology, topological groups and homological*

* J.F. Adams contributed the sections on algebraic topology and manifolds; A.R. Pears, the section on general topology [editor].

algebra, edited by N.E. Steenrod, and published in two volumes by the American Mathematical Society, Providence, R.I. (1968). Unfortunately it only goes up to 1967, and this in the sense that if a paper appeared in 1967 and its review appeared in 1968, then it will not be included. However, there are now (1977) plans to bring it up to date under the editorship of A. Clark.

The expert will require no further guidance; and the layman should be directed to the articles in the *Encyclopaedia Britannica.* Editions before 1974 carried an article on algebraic topology by Prof. W.S. Massey which is basically excellent, but now shows its age a little. A new article was commissioned for the 1974 edition; this does cover more up-to-date material, but the coverage attempted is perhaps a little ambitious for complete understanding by a lay reader. With these two classes of reader dismissed, I may address the main body of readers, whom I suppose to have at least a first degree in mathematics, but not to be expert in this particular area.

13.2 ALGEBRAIC TOPOLOGY

So far as algebraic topology goes an attempt has been made to give the necessary guidance and help in J.F. Adams, *Algebraic topology; a student's guide,* Cambridge University Press (1972; London Mathematical Society Lecture Note Series 4). In that book I begin with a 31-page survey of the material facing the student of algebraic topology, and commenting on the sources from which it can most conveniently be studied; the book continues as a reprint collection. The introductory survey, of course, is more full than I can manage in the present space, so I refer the reader to it. It seems to wear reasonably well. However, because of publication difficulties, the references were not up to date when the book appeared; and, in particular, the list of books on pp. 1-4 contains nothing later than 1967. Today it would be advisable to add a number of books to this list. The following may all be considered as 'first textbooks', but have different slants:

E. Artin and H. Braun, *Introduction to algebraic topology,* Charles E. Merrill Publishing Co, Columbus, Ohio (1969)
C. Godbillon, *Éléments de topologie algébrique,* Hermann, Paris (1971)
A. Gramain, *Topologie des surfaces,* Presses Universitaires de France, Paris (1971)
C.R.F. Maunder, *Algebraic topology,* Van Nostrand, Reinhold, London (1970)
J. Mayer, *Algebraic topology,* Prentice-Hall, Englewood Cliffs, N.J. (1972)

C.T.C. Wall, *A geometric introduction to topology,* Addison-Wesley, Reading, Mass. (1972)
A.H. Wallace, *Algebraic topology: homology and cohomology,* Benjamin, New York (1970)

Wall's book can be recommended for its individual approach. Otherwise, the book by Maunder is the most comprehensive, and should probably be recommended. In addition, the old book on *Algebraic topology* by W. Franz has been translated (Frederick Ungar, New York, 1968); but there seems no particular reason to recommend it in view of later texts.

Of the books recently published, the following two seem the most useful as 'second textbooks': A. Dold, *Lectures on algebraic topology,* Grundlehren der Mathematischen Wissenschaften 200, Springer-Verlag, Berlin (1972); R.M. Switzer, *Algebraic topology – homotopy and homology,* Grundlehren der Mathematischen Wissenschaften 212, Springer-Verlag, Berlin (1975).

Dold has an excellent style, but the subject matter of his book is a little restricted, centering around classical homology. Switzer's book covers a wider range of material than Dold's, and indeed is very substantial; apart from its price, which is excessive, it is probably the most recommendable choice as a second textbook.

The following books combine expository intent and specialist content in various proportions:

J.F. Adams, *Stable homotopy and generalised cohomology,* University of Chicago Press, Chicago (1974)
R. F. Brown, *The Lefschetz fixed point theorem,* Scott, Foresman and Co, Glenview, Ill. (1971)
G.E. Cook and R.L. Finney, *Homology of cell complexes,* Princeton University Press, Princeton (1967)
E. Dyer, *Cohomology theories,* W.A. Benjamin, New York (1969)
F. Harary, *Graph theory,* Addison-Wesley, Reading, Mass. (1969) (Harary's is the standard book on this topic.)
P.J. Hilton (ed.), *Studies in modern topology,* MAA Studies in Mathematics 5, Mathematical Association of American, Buffalo, N.Y. (1968) (This contains interesting expository surveys.)
P.J. Hilton, G. Mislin and J. Roitberg, *Localisation of nilpotent groups and spaces,* Mathematics Studies 15, North-Holland, Amsterdam (1975) (This is essentially a specialist monograph; but localisation is important.)
S.T. Hu, *Homology theory,* Holden-Day, San Francisco (1966)
S.T. Hu, *Cohomology theory,* Markham, Chicago (1968) (In general, books by Hu have no advantage over the sources from which they derive.)

S.Y. Husseini, *The topology of the classical groups and related topics,* Gordon and Breach, New York (1969)
K. Lamotke, *Semisimpliziale algebraische Topologie,* Grundlehren der Mathematischen Wissenschaften 147, Springer-Verlag, Berlin (1968)
A.T. Lundell and S. Weingram, *The topology of CW-complexes,* Van Nostrand Reinhold, London (1969)
J.W. Milnor and J.D. Stasheff, *Characteristic classes,* Annals of Mathematics Studies 76, Princeton University Press, Princeton (1974) (Milnor's notes on the subject date from 1957; their appearance in print has been long awaited, and is most welcome.)
R.E. Mosher and M.C. Tangora, *Cohomology operations and applications in homotopy theory,* Harper and Row, New York (1968) (For getting a certain sort of know-how this is definitely readable. The trouble is that it may encourage students to do unintelligent calculations in an area where unintelligent calculations are the last thing we need.)
J. Stallings, *Group theory and three-dimensional manifolds,* Yale Mathematical Monographs 4, Yale University Press, New Haven, Conn. (1971)
R. Stong, *Notes on cobordism theory,* Princeton University Press, Princeton (1968) (This is the most useful standard reference work on cobordism.)

13.3 MANIFOLDS

I now turn to the subject of manifolds. Here one might remark that many subjects seem to pass rather rapidly from a state in which there are too few books to one in which there are too many; and large areas of manifold theory (if not the whole of it) are still in the former state. So, in a sense, there are fewer 'resources' to be surveyed than in the last section.

It is convenient to subdivide our topic, according to the nature of the assumptions made on the manifolds, into the study of topological manifolds, piecewise-linear manifolds and differentiable (or smooth) manifolds. Of course, there are large bodies of work which seek to relate manifolds, as defined under one set of rules, to manifolds defined under another set of rules; for example, one may take a given piecewise-linear manifold and enquire how many smooth structures (if any) may be put on it. Indeed, it is only recently that it has been *proved* that these differing assumptions lead to different theories, in the sense that there are manifolds which satisfy one assumption but not another; it seems that some of the pioneers hoped for the contrary state of affairs, so that the three theories would have been equivalent; this

would no doubt have been simpler, but it turns out that the state of affairs is not so simple.

It is perhaps natural to begin with differentiable (smooth) manifolds, for those which arise in the rest of mathematics are most often of this type. The assumptions made on the manifolds are such that one can do differential geometry. Therefore the basic justification for the geometric constructions which are made comes from analysis; but one can perform 'obvious' geometric constructions quite freely, provided that one avoids non-differentiable means which might introduce kinks and corners. The theory of fibre bundles provides the most important single tool, and often allows one to reduce geometrical questions to problems which can be solved by algebraic topology.

The following are among the books which might serve as a starting point:

S. Lang, *Differential manifolds,* Addison-Wesley, Reading, Mass. (1972) (This is a revised and greatly expanded version of the author's earlier book, *Introduction to differential manifolds,* John Wiley, New York, 1962. It presents the way an expert would like to see the foundations set up – but parts of it may be a bit abstract for the beginner.)
J.W. Milnor, *Topology from the differentiable viewpoint,* University Press of Virginia, Charlottesville (1965) (Recommended.)
J.R. Munkres, *Elementary differential topology,* Annals of Mathematics Studies 54, Princeton University Press, Princeton (1963)

One more advanced book has been found useful by many students:

J.W. Milnor, *Lectures on the h-cobordism theorem,* Princeton University Press, Princeton, N.J. (1965)

Then we have the following survey articles:

J.W. Milnor, 'Differential topology' (pp. 165-183 of T.L. Saaty (ed.), *Lectures on modern mathematics,* Vol. 2, Wiley, New York, 1964)
S. Smale, 'A survey of some recent developments in differential topology', *American Mathematical Society. Bulletin,* **69**, 131-145 (1963)
C.T.C. Wall, 'Topology of smooth manifolds', *London Mathematical Society. Journal,* **40**, 1-20 (1965)

These are strongly recommended; and of course they have references. However, I will cite two papers which I feel mark the birth of modern differential topology:

J.W. Milnor, 'On manifolds homeomorphic to the 7-sphere', *Annals of Mathematics,* **64**, 399-405 (1956)
S. Smale, 'Generalised Poincaré conjecture in dimensions greater than four', *Annals of Mathematics,* **74**, 391-406(1961)

The paper by Milnor is the one which first showed that the category of smooth manifolds is different from the other two categories. The paper by Smale is important not only for its result, but also for its method, which I may summarise as follows. In algebraic topology one usually begins by studying finite simplicial complexes; these are spaces which can be subdivided into points, line-segments, triangles and their higher-dimensional analogues, which are called simplexes. However, it turns out to be more efficient to study CW-complexes, which are made up out of 'cells' instead of simplexes; one needs only a small number of cells compared with a large number of simplexes, and it turns out that (under suitable assumptions) a decomposition into cells can be made to follow the topological invariants of the space very closely. The analogue for a manifold of a cell-decomposition is a decomposition into 'handles'. It is important to study such decompositions for piecewise-linear and topological manifolds also, but they were originally introduced for smooth manifolds; the idea probably arose from modern interpretations, owing much to R. Bott, of the work of M. Morse. A decomposition into handles can be rearranged and manipulated; and as for cells, it turns out that (under suitable assumptions) a decomposition into handles can be made to follow the topological invariants of a manifold very closely. This is Smale's method.

Somewhat related to handle-decompositions is the method of 'surgery'. In this we take a manifold and cut out entirely a suitable part of it (which for present purposes we may think of as being a little more than one 'handle'). We then glue in a new part, so obtaining a new manifold, different from but related to the one we started with. R. Thom's concept of 'cobordism' may be interpreted in this way; and the method is very useful. The original reference is as follows: J.W. Milnor, 'A procedure for killing homotopy groups of differentiable manifolds', *American Mathematical Society. Proceedings of Symposia in Pure Mathematics,* **3**, 39-55 (1961).

There are two books: W. Browder, *Surgery on simply-connected manifolds,* Ergebnisse der Mathematik und ihrer Grenzgebiete 65, Springer-Verlag, Berlin (1972); and C.T.C. Wall, *Surgery on compact manifolds,* London Mathematical Society Monographs 1, Academic Press, London (1970). The student should start on Browler's: Wall's is harder.

I turn now to the topology of piecewise-linear manifolds. These are ones which can be made up from points, line-segments, triangles, and more generally simplexes, assembled in a prescribed way. The basic justification for the geometric constructions which are made now comes from elementary linear algebra. Kinks and corners, which were taboo in the smooth theory, are now the order of the day. One can

perform 'obvious' geometric constructions quite freely. Contrasting this theory with that of topological manifolds (where one has to construct some fairly pathological homeomorphisms), one is tempted to say that in the piecewise-linear case the 'obvious' constructions are the only ones; but the reader who pauses to consider (for example) knots in 4, 5 and 6 dimensions will see that genuine geometric insight is required, and in that sense the constructions may be far from obvious. Once one gets past the elementary results which can be proved by 'general-position' arguments, proofs tend to proceed by the inductive repetition of elementary steps or moves; this method is very appropriate because of the finitistic nature of the material. This finitistic or combinatorial character of the subject probably helped to attract and encourage the early workers, and the subject has a history going back to the 1930s. The reader may like the following survey article by one who helped to shape that history: M.H.A. Newman, 'Geometrical topology', *International Congress of Mathematicians. Proceedings,* **9**, 139-146 (1962).

At the beginning of the 'modern' period the following notes had a considerable influence: E.C. Zeeman, *Seminar on combinatorial topology,* mimeographed notes, Institut des Hautes Études Scientifiques, Paris (1963; Chapter 7 revised 1965, Chapter 8 revised 1966).

Unfortunately these are not easily available. However, the material is mostly available in book form:

L.C. Glaeser, *Geometrical combinatorial topology,* Vol. 1, Mathematical Studies 27, Van Nostrand Reinhold, London (1970)

J.F.P. Hudson, *Piecewise-linear topology,* Benjamin, New York (1969)

C.P. Rourke and B.J. Sanderson, *Introduction to piecewise-linear topology,* Ergebnisse der Mathematik und ihrer Grenzgebiete 69, Springer-Verlag, Berlin (1972)

J.R. Stallings, *Lectures on polyhedral topology,* Tata Institute of Fundamental Research, Bombay (1968)

The first is shorter than the other three, and from a slightly different tradition; the other three seem more influenced by Zeeman. The fourth is not quite so easily available as the others; it has some nice material near the end, but there are too many pages to read before you get to it. Presumably one recommends the second and third.

One research monograph should be mentioned here: M.W. Hirsch and B. Mazur, *Smoothings of piecewise-linear manifolds,* Annals of Mathematics Studies, 80, Princeton University Press, Princeton, N.J. (1974).

In certain situations, the question of what can be done by a finite number of elementary steps or moves can be reduced to pure algebra. This brings us to the theory of 'torsion' or 'simple-homotopy type'.

Here the word 'simple' is not to be taken in its everyday sense; the theory is no more simple than the theory of 'simple groups'. However, there is a theory, and it is useful both in piecewise-linear and in smooth topology. I would direct the reader firmly to Milnor's excellent exposition: J.W. Milnor, 'Whitehead torsion', *American Mathematical Society. Bulletin,* **72**, 358-426 (1966).

There are also two accounts available in book form: M.M. Cohen, *A course in simple-homotopy theory,* Graduate Texts in Mathematics series 10, Springer-Verlag, Berlin (1973); and G. de Rham, S. Maumary and M.A. Kervaire, *Torsion et type simple d'homotopie,* Lecture Notes in Mathematics 48, Springer-Verlag, Berlin (1967).

In all three categories, studies of the 'position' of a subset in a manifold are fundamental. The problem reaches perhaps its most characteristic form in the study of knots. There is one canonical book on 'ordinary' knots: R.H. Crowell and R.H. Fox, *Introduction to knot theory,* Ginn, Boston (1963).

However some feel that one can get the same ideas by reading fewer words in the following survey article: R.H. Fox, 'A quick trip through knot theory' (pp. 120-167 of M.K. Fort (ed.), *Topology of 3-manifolds and related topics,* Prentice-Hall, Englewood Cliffs, N.J., 1962). This is perhaps the recommended source.

Of course, from the point of view of the rest of mathematics, knots in higher-dimensional space deserve just as much attention as knots in 3-space. On this topic I am reduced to citing, with some misgivings, a selection of the original sources:

A. Haefliger, 'Knotted $(4k-1)$-spheres in $6k$-space', *Annals of Mathematics,* **75**, 452-466 (1962)

E.C. Zeeman, 'Unknotting combinatorial balls', *Annals of Mathematics,* **78**, 501-526 (1963)

M.A. Kervaire, 'Les noeuds de dimension supérieures', *Société Mathématique de France. Bulletin,* **93**, 225-271 (1965)

J. Levine, 'Unknotting spheres in codimension two', *Topology,* **4**, 9-16 (1965)

J. Levine, 'A classification of differentiable knots', *Annals of Mathematics,* **82**, 15-50 (1965)

Experts in the subject tell their students to read Levine.

Finally we come to the subject of topological manifolds. Here the basic justification for the geometric constructions which are made must come from general topology – that is, 'analytic' or 'point-set' topology; and indeed the subject draws much of its flavour and much of its impetus from that source. More precisely, in order to make any use of the difference between the topological category and the smooth or piecewise-linear category, one must construct maps (and, more especially, homeomorphisms) which are not locally 'good' at all, but to smooth or piecewise-linear eyes present the most extreme singularities. Inescapably the methods for handling such things have much in common with those for constructing and handling 'pathological examples' in general topology; but these days they also require extensive awareness of algebraic topology and of developments elsewhere in the theory of manifolds (surgery obstructions, handlebodies).

We may regard the distinctive flavour of the subject as beginning to emerge with the work of Moise and, particularly, Bing; but the many works of these authors are not all easy to read. Another choice for a paper making the birth of the 'modern' period might be the following: M. Brown, 'A proof of the generalised Schoenflies theorem', *American Mathematical Society. Bulletin,* **66**, 74-76 (1960).

There has been much activity in this field recently; in particular, the student should be aware that Kirby and Siebenmann have made important progress, and have (for example) proved the Annulus Conjecture. There is exactly one book: T.B. Rushing, *Topological embeddings,* Academic Press, New York (1973). This therefore becomes the recommended source; it has useful references.

Finally, it remains to comment on volumes which contain the proceedings of various specialist conferences. Conferences have the drawback that they tend to be ephemeral; who would want to attend last year's conference, if it were available perfectly recorded on videotape? They have the virtue that they give a vivid impression of mathematics as a living enterprise. The proceedings of conferences have both the drawback and the virtue in a diluted form. It often happens that the best contributions to conferences are also published properly in the usual journals; and to this extent conference proceedings present the disadvantage that they duplicate other journals, but appear in an unsystematic way, at irregular intervals, and under a new editor and title each time. One might be excused for consulting them only when a bibliographical search reveals that some relevant paper appeared in one. However, there are a few conferences which, by a fortunate choice of subject at a fortunate time, publish proceedings with a higher ratio than usual of contributions which one wishes to consult; and these may deserve a place on one's bookshelf. Perhaps the best example in this area is:

M.K. Fort (ed.), *Topology of 3-manifolds and related topics,* Prentice-Hall, Englewood Cliffs, N.J. (1962)

For the rest, the most recent reference may be the most useful, as later references allow you to find earlier ones but not vice versa. So as a sample of conference proceedings in general I cite a recent one: L.F. McAuley (ed.), *Algebraic and geometrical methods in topology,* Lecture Notes in Mathematics 428, Springer-Verlag, Berlin (1974).

13.4 GENERAL TOPOLOGY

Topology is a rather diverse subject with many origins. General topology consists in the main of the study of abstract spaces and mappings between them but also includes many other topics which do not belong to the areas of algebraic topology, differential topology and global analysis. Soon after Cantor has initiated the theory of sets, Fréchet began, in 1906, the study of abstract spaces. The concept of topological space soon evolved. Point-set topology is the study of general topological spaces and continuous mappings. Some knowledge of point-set topology is essential for all work in mathematics. The basic notions and general constructions should be known. The most interesting classes of spaces for non-specialists are the compact spaces and the metrisable spaces. The class of paracompact spaces contains both of these classes and has turned out to be the 'right' class of spaces for many purposes. The non-specialist should perhaps also know the Nagata–Smirnov theorem which characterises the topological spaces which are metrisable. Textbooks giving an exposition of this fundamental material are: J. Dugundji, *Topology,* Allyn and Bacon, Boston (1966); R. Engelking, *Outline of general topology,* North-Holland, Amsterdam (1968) and J.L. Kelley, *General topology,* Van Nostrand, Princeton, N.J. (1955).

Very many generalisations of the classes of paracompact and metrisable spaces have been introduced. It is not possible to say which of these classes will be found ultimately to be significant. A unified approach to the problems of classification of spaces and mappings was developed by A.V. Arkhangel'skii, 'Mappings and spaces', *Russian Mathematical Surveys,* **21**, 115-162 (1966). A detailed account of the ring of continuous real-valued functions on a topological space is given by L. Gillman and M. Jerison, *Rings of continuous functions,* Van Nostrand, Princeton, N.J. (1960). The algebraic structure of this ring gives information about topological properties of the space. A closely connected topic is the Stone–Čech compactification, and this is also investigated in detail by Gillman and Jerison.

Peano's example of a space-filling curve forced questioning of the meaning of dimension. The problem of distinguishing topologically between different Euclidean spaces was the starting point of dimension theory. The classical dimension theory of separable metric spaces is elegantly exposed by W. Hurewicz and H. Wallman, *Dimension theory,* Princeton University Press, Princeton, N.J. (1941). There is no satisfactory single extension of the theory of dimension to general topological spaces. For an account of the various dimension theories for non-metrisable spaces, the relations between them, and examples showing the pathological aspects of the theories see: K. Nagami, *Dimension theory,* Academic Press, New York (1970) and A.R. Pears, *Dimension theory of general spaces,* Cambridge University Press, Cambridge (1975).

Although topological spaces and continuous mappings are the main concern of point-set topology, other types of 'continuity structure' are studied. The theory of uniform spaces is analogous to the theory of metric spaces but is of wider applicability. This is the setting in which the concept of uniform continuity can be most naturally investigated. There are two approaches to uniformity: by means of uniform covering, the theory being developed from this point of view in J.R. Isbell's book, *Uniform spaces,* Mathematical Surveys 12, American Mathematical Society, Providence, R.I. (1964); and by means of certain relations, called entourages, which are employed by N. Bourbaki in Chapter 2, 'Structures uniformes', of his *Topologie générale,* Hermann, Paris (1961) (translated as *General topology,* 2 vols, Addison-Wesley, Reading, Mass., 1966). In a uniform space there is a notion of 'nearness' of sets and this can be abstracted to provide the definition of a proximity space. An introduction to the theory of proximity spaces and their generalisations is given by S.A. Naimpally and B.D. Warrack, *Proximity spaces,* Cambridge Tracts in Mathematics and Mathematical Physics 59, Cambridge University Press, Cambridge (1973). E. Čech, *Topological spaces,* Publishing House of Czechoslovak Academy of Sciences, Prague; Interscience, London (1966) is an interesting presentation of topologies, uniformities and proximities which is completely self-contained, all necessary mathematical concepts, beginning with class and set, being introduced in the work. A.W. Hager's paper entitled 'Some nearly fine uniform spaces', *London Mathematical Society. Proceedings,* **28**, 517-546 (1974) contains a bibliography of recent work on uniform spaces.

In 1963 P.J. Cohen proved that it is consistent with the usual axioms for set theory that the continuum hypothesis be false. The technique of proof, called forcing, which Cohen introduced, has recently been applied to many questions in general topology of a set-theoretic nature. Many such problems have a translation into questions of cardinal arithmetic; the text by I. Juhász, 'Cardinal functions in topology', *Mathematical Centre Tracts,* **34** (1971), gives much information on this subject. There is at present much interest in the answers to topological questions in special models for set theory – in particular, in the contrasting models (a) satisfying Martin's axiom together with the negation of the continuum hypothesis and (b) Gödel's constructable universe (in which the generalised continuum hypothesis holds). For a survey of the whole area affected by set-theoretic influence and of the methods used see: M.E. Rudin, *Lectures on set theoretic topology,* Regional Conference Series in Mathematics 23, American Mathematical Society, Providence, R.I. (1975).

Finally a selection should be made from the numerous topics in addition to point-set topology which belong to the field of general topology. Many topological results of a general nature can be described and analysed in categorical terms. The notes of H. Herrlich, *Topologische Reflexionen und Coreflexionen,* Lecture Notes in Mathematics 78, Springer-Verlag, Berlin (1968), provide an introduction to categorical topology. The shape of a topological space is a modification of its homotopy type. The articles by S. Mardešić, 'A survey of the shape theory of compacta', *Prague Topological Symposium. Proceedings,* **3**, 291-300 (1971) and 'Shapes for topological spaces', *General Topology and its Applications,* **3**, 265-282 (1973), give the basic definitions of this rapidly developing field. There does not seem to be an expository account of infinite dimensional topology, but the report *Symposium on Infinite Dimensional Topology,* edited by R.D. Anderson (Annals of Mathematics Studies 69, Princeton University Press, Princeton, N.J., 1974), should indicate the nature of this field.

14

Mathematical Programming

J.M. Brown

14.1 INTRODUCTION

Mathematical programming is constrained optimisation. The abstract problem is to find the greatest or least value of a function of many variables with the variables constrained to some subdomain of R_N. Usually, they are non-negative and satisfy other inequalities, and may be further limited to say integer values.

However, this is very much an applied branch of mathematics, recognised as such by the 1975 Nobel awards in economics to Kantorovitch[1] and Koopmans[2] for their early transportation studies. Existence, uniqueness and characterisation of optima are important, but far more emphasis is given to their calculation. The subject developed in parallel with digital computers, and for a technique to become established, reliable and efficient computer program implementations are necessary. This emphasis is important and different from most other chapters of this volume.

The subject is well covered by textbooks, but journal articles and conference proceedings are often very specialised. These are scattered under many headings: operational research; management; economics; military strategy; computing; combinatorics; optimisation; chemical, structural and electrical engineering! This survey is far from comprehensive, but attempts to locate comprehensible treatments, key works in the field and unusually illuminating articles on difficult aspects.