CW transversality by Andrew Ranicki

§1. Algebraic and geometric K-theory

We start with a brief review of the classical applications of the algebraic K-groups K_0, K_1 to the homotopy theory of chain complexes in algebra and CW complexes in topology. It is assumed that the reader is already familiar with the Wall finiteness obstruction and the Whitehead torsion, so that the account is not self-contained. The object is to present the usual invariants in the way appropriate to the generalizations appearing in the splitting theorems.

Rings are to be associative with 1, and morphisms of rings are to preserve 1. We shall have a preference for left modules over right modules, in keeping with the left action of the group of covering translations on a covering space. Modules will be understood to be left modules, with right module structures expressly specified as such.

The elementary geometric operation of attaching an (m+1)-cell

$$(\text{CW complex } X,\ \text{cellular map } f:S^m \longrightarrow X)$$
$$\longmapsto \text{CW complex } X' = X \cup_f D^{m+1}$$

has an evident algebraic analogue.

Let A be a ring, and let C be an A-module chain complex

$$C : \ldots \longrightarrow C_{r+1} \xrightarrow{\ d\ } C_r \xrightarrow{\ d\ } C_{r-1} \longrightarrow \ldots \qquad (r \in \mathbb{Z}) .$$

Given an m-cycle $x \in \ker(d:C_m \longrightarrow C_{m-1})$ regard it as an A-module chain map

$$x : S^m A \longrightarrow C ,$$

with $(S^m A)_r = 0$ $(r \neq m)$, $= A$ $(r = m)$. The algebraic mapping cone of x is the A-module chain complex obtained from C by attaching an algebraic (m+1)-cell at x

$$C' = C(f) : \ldots \longrightarrow C_{m+2} \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} C_{m+1} \oplus A \xrightarrow{(d\quad (-)^m x)} C_m \xrightarrow{\ d\ } C_{m-1} \longrightarrow \ldots .$$

(We are adopting the sign convention that the algebraic mapping cone $C(f)$ of a chain map $f:B \longrightarrow C$ has boundary maps

$$d_{C(f)} = \begin{pmatrix} d_C & (-)^{r-1}f \\ 0 & d_B \end{pmatrix} : C(f)_r = C_r \oplus B_{r-1} \longrightarrow C(f)_{r-1} = C_{r-1} \oplus B_{r-2}).$$

The effect on homology is to kill the homology class $x \in H_m(C)$, with an exact sequence

$$0 \longrightarrow H_{m+1}(C) \longrightarrow H_{m+1}(C') \longrightarrow A \xrightarrow{x} H_m(C) \longrightarrow H_m(C') \longrightarrow 0 ,$$

and $H_r(C) = H_r(C')$ $(r \neq m, m+1)$.

Given a CW complex X and a regular covering $\tilde{X}$ with group of covering translations π let $C(\tilde{X})$ be the cellular $\mathbb{Z}[\pi]$-module chain complex, so that $C(\tilde{X})_r = H_r(\tilde{X}^{(r)}, \tilde{X}^{(r-1)})$ is the free $\mathbb{Z}[\pi]$-module with one generator for each r-cell $e^r \in X^{(r)}$ (with the actual generator depending on a choice of lift $\tilde{e}^r \in \tilde{X}^{(r)}$ and of orientation ± 1), and with the boundary map

$$d : C(\tilde{X})_r = H_r(\tilde{X}^{(r)}, \tilde{X}^{(r-1)}) \longrightarrow C(\tilde{X})_{r-1} = H_{r-1}(\tilde{X}^{(r-1)}, \tilde{X}^{(r-2)})$$

given by the boundary map in the homology of the triad $(\tilde{X}^{(r)}; \tilde{X}^{(r-1)}, \tilde{X}^{(r-2)})$. If $f : S^m \longrightarrow X$ is a cellular map such that either $m \neq 1$ or if $m = 1$ then $f^*(\tilde{X}) = \pi \times S^m$ (i.e. $f \in \pi_1(X)$ has image $1 \in \pi$) it is possible to extend $\tilde{X}$ to a regular cover $\tilde{X}'$ of $X' = X \cup_f D^{m+1}$, and the $\mathbb{Z}[\pi]$-module chain complex $C(\tilde{X}')$ is obtained from $C(\tilde{X})$ by attaching an algebraic $(m+1)$-cell at the r-cycle $x = f_*[S^m] \in \ker(d:C(\tilde{X})_m \longrightarrow C(\tilde{X})_{m-1})$.

Given a commutative diagram of cellular maps of CW complexes

$$\begin{array}{ccc} S^m & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} \\ D^{m+1} & \xrightarrow{g} & Y \end{array}$$

(with $i:S^m \longrightarrow D^{m+1}$ the inclusion) there is defined an extension of h

$$h' = h \cup g : X' = X \cup_f D^{m+1} \longrightarrow Y .$$

If $\tilde{Y}$ is a regular cover of Y with group of covering translations π

the diagram lifts to a commutative square of π-equivariant maps of covers

$$\begin{array}{ccc} \pi \times S^m & \xrightarrow{\tilde{f}} & \tilde{X} \\ {\scriptstyle \tilde{i} = 1 \times i} \downarrow & & \downarrow {\scriptstyle \tilde{h}} \\ \pi \times D^{m+1} & \xrightarrow{\tilde{g}} & \tilde{Y} \end{array}$$

with $\tilde{X} = h^*\tilde{Y}$ the pullback, and $\tilde{h}$ extends to a π-equivariant map of covers

$$\tilde{h}' = \tilde{h} \cup \tilde{g} : \tilde{X}' = \tilde{X} \cup_{\tilde{f}} \pi \times D^{m+1} \longrightarrow \tilde{Y} .$$

The algebraic mapping cone $C(\tilde{h}')$ is obtained from $C(\tilde{h})$ by attaching an algebraic $(m+2)$-cell at $(g,f)_*[D^{m+1},S^m] \in \ker(d:C(\tilde{h})_{m+1} \longrightarrow C(\tilde{h})_m)$.

An A-module chain complex C is <u>n-dimensional</u> if it is positive, $C_r = 0$ for $r < 0$, and $C_r = 0$ for $r > n$

$$C : \ldots \longrightarrow 0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \ldots \longrightarrow C_0 \longrightarrow 0 \longrightarrow \ldots \quad .$$

The complex C is <u>finite</u> if it is finite-dimensional (i.e. n-dimensional for some $n \geqslant 0$) and each C_r $(0 \leqslant r \leqslant n)$ is a based f.g. free A-module.

A <u>finite domination</u> (D,f,g,h) of an A-module chain complex C consists of a finite A-module chain complex D, chain maps

$$f : C \longrightarrow D \ , \ g : D \longrightarrow C$$

and a chain homotopy

$$h : gf \simeq 1 : C \longrightarrow C \ .$$

It was shown in Ranicki [ ] that an A-module chain complex C is finitely dominated if and only if it is chain equivalent to a finite-dimensional chain complex of f.g. projective A-modules

$$P : \ldots \longrightarrow 0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow 0 \longrightarrow \ldots \ .$$

The <u>projective class</u> of C is defined in the usual way by

$$[C] = [P] = \sum_{r=0}^{n} (-)^r [P_r] \in K_0(A)$$

for any such P. For a finitely dominated chain complex C let $\dim(C)$ denote the dimension of any chain equivalent P.

A <u>finite domination</u> (Y,f,g,h) of a CW complex X consists of a finite CW complex Y, maps

$$f : X \longrightarrow Y \ , \ g : Y \longrightarrow X$$

and a homotopy

$$h : gf \simeq 1 : X \longrightarrow X \ .$$

Assuming that X is connected let $\widetilde{X}$ be the universal cover, and let $\widetilde{Y} = g^*\widetilde{X}$ be the pullback cover of Y. Then $C(\widetilde{Y})$ is a finite $\mathbb{Z}[\pi_1(X)]$-module chain complex and $(C(\widetilde{Y}),f,g,h)$ is a finite domination of $C(\widetilde{X})$.

The finiteness obstruction theory of Wall [ ],[ ] associates to a finitely dominated CW complex X the reduced projective class of $C(\widetilde{X})$

$$[X] = [C(\widetilde{X})] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) \ ,$$

proving that X is homotopy equivalent to a finite CW complex if and only if $[X] = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$. The essential properties of a finitely dominated CW complex X used in the proof were that if $\phi : K \longrightarrow X$ is a map from a finite m-dimensional CW complex K such that $\pi_r(\phi) = 0$ for $r \leqslant m$ then the $\mathbb{Z}[\pi_1(X)]$-module

$$\pi_{m+1}(\phi) = H_{m+1}(\widetilde{\phi} : C(\widetilde{K}) \longrightarrow C(\widetilde{X}))$$

is finitely generated (f.g.), and if $m \geqslant n = \dim(C(\widetilde{X}))$ then it is also projective with reduced projective class

$$[\pi_{m+1}(\phi)] = (-)^{m+1}[C(\widetilde{X})] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) \ .$$

Let $\{(g_i,f_i) \in \pi_{m+1}(\phi) \,|\, 1 \leqslant i \leqslant j\}$ be a finite set of $\mathbb{Z}[\pi_1(X)]$-module generators of $\pi_{m+1}(\phi)$. Then attaching j geometric $(m+1)$-cells to ϕ results in a map from a finite $(m+1)$-dimensional CW complex K'

$$\phi' = \phi \cup \bigcup_{i=1}^{j} g_i : K' = K \cup_{\bigcup_{i=1}^{j} f_i} \bigcup_{i=1}^{j} e_i^{m+1} \longrightarrow X$$

such that $\pi_r(\phi') = 0$ for $r \leqslant m+1$. If $m \geqslant n = \dim(X)$ and $\pi_{m+1}(\phi)$ is a f.g. free $\mathbb{Z}[\pi_1(X)]$-module with base $\{(g_i,f_i) \,|\, 1 \leqslant i \leqslant j\}$ then $\pi_*(\phi') = 0$ and $\phi' : K' \longrightarrow X$ is a homotopy equivalence.

The finiteness obstruction for chain complexes can be developed in exactly the same way as for CW complexes. Given a finitely dominated A-module chain complex C and a chain map $\phi:F \longrightarrow C$ from a finite m-dimensional A-module chain complex F such that $H_r(\phi) = 0$ for $r \leqslant m$ it is the case that the A-module $H_{m+1}(\phi)$ is f.g., and that if $m \geqslant n = \dim(C)$ it is also projective with

$$[H_{m+1}(\phi)] = (-)^{m+1}[C] \in \tilde{K}_0(A) .$$

Given a finite set of A-module generators $\{(g_i,f_i) \in H_{m+1}(\phi) \mid 1 \leqslant i \leqslant j\}$ it is possible to attach j algebraic (m+1)-cells to $C(\phi)$, so as to obtain the algebraic mapping cone $C(\phi')$ of a chain map to C from a finite (m+1)-dimensional A-module chain complex F'

$$\phi' = \phi \oplus \sum_{i=1}^{j} g_i : F' = C(\sum_{i=1}^{j} f_i : \sum_{i=1}^{j} S^m A \longrightarrow F) \longrightarrow C$$

such that $H_r(\phi') = 0$ for $r \leqslant m+1$. If $m \geqslant n$ and $H_{m+1}(\phi)$ is a f.g. free A-module with base $\{(g_i,f_i) \mid 1 \leqslant i \leqslant j\}$ then $H_*(\phi') = 0$ and $\phi':F' \longrightarrow C$ is a chain equivalence. Thus a finitely dominated A-module chain complex C is chain equivalent to a finite chain complex if and only if $[C] = 0 \in \tilde{K}_0(A)$, and the reduced projective class $[C] \in \tilde{K}_0(A)$ deserves to be called the <u>finiteness obstruction</u> of C.

Simple homotopy theory studies the uniqueness properties of finite CW complex structures with respect to the <u>elementary expansion operation</u>

$$(\text{finite CW complex } X,\ \text{cellular map } (g,f):(D^{m+1},S^m) \longrightarrow X)$$

$$\longmapsto \text{finite CW complex } X' = (X \cup_f D^{m+1}) \cup_{g\cup 1} D^{m+2} ,$$

where

$$g\cup 1 : S^{m+1} = D^{m+1}_+ \cup_{S^m} D^{m+1}_- \longrightarrow X \cup_f D^{m+1} .$$

Cohen [ ] is a general reference for simple homotopy theory. The <u>Whitehead torsion</u> of a homotopy equivalence $h:X \longrightarrow Y$ of finite CW complexes

$$\tau(h) \in Wh(\pi_1(X))$$

is defined as usual, with $\tau(h) = 0$ if and only if h is a simple homotopy equivalence, that is if h is homotopic to the composite of a finite sequence of elementary expansions or their inverses. A <u>finiteness structure</u> on a CW complex X is an equivalence class of pairs

$$(\text{finite CW complex } K,\ \text{homotopy equivalence } \phi:K \longrightarrow X)$$

under the equivalence relation

$$(K,\phi) \sim (K',\phi') \text{ if } \tau(\phi'^{-1}\phi : K \longrightarrow K') = 0 \in Wh(\pi_1(X)) .$$

An <u>algebraic elementary expansion</u> of a finite A-module chain complex C is the finite A-module chain complex C' determined by any chain $g \in C_{m+1}$

$$C' : \dots \longrightarrow C_{m+3} \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} C_{m+2}\oplus A \xrightarrow{\begin{pmatrix} d & (-)^m g \\ 0 & 1 \end{pmatrix}} C_{m+1}\oplus A \xrightarrow{(d \ \ (-)^{m-1}f)} C_m \longrightarrow \dots$$

with $f = dg \in C_m$. The <u>torsion</u> of a chain equivalence $h:C \longrightarrow D$ of finite A-module chain complexes

$$\tau(h) \in K_1(A)$$

is defined as usual, with $\tau(h) = 0$ if and only if h is a simple chain homotopy equivalence, that is if h is chain homotopic to the composite of a finite sequence of elementary algebraic expansions and their inverses. A finiteness structure on an A-module chain complex C is an equivalence class of pairs

$$(\text{finite A-module chain complex } F, \text{ chain equivalence } \phi : F \longrightarrow C)$$

under the equivalence relation

$$(F,\phi) \sim (F',\phi') \text{ if } \tau(\phi'^{-1}\phi : F \longrightarrow F') = 0 \in K_1(A) \ .$$

In the applications to topology $A = \mathbb{Z}[\pi]$ is a group ring and torsion is measured in the Whitehead group

$$Wh(\pi) = K_1(\mathbb{Z}[\pi])/\{\pm\pi\}$$

rather than $K_1(\mathbb{Z}[\pi])$. In particular, the Whitehead torsion of a homotopy equivalence $h:X \longrightarrow Y$ of finite CW complexes X,Y is just the reduction of the torsion $\tau(\tilde{h}:C(\tilde{X}) \longrightarrow C(\tilde{Y})) \in K_1(\mathbb{Z}[\pi_1(X)])$ to $Wh(\pi_1(X))$, with $\tilde{h}:C(\tilde{X}) \longrightarrow C(\tilde{Y})$ any induced $\mathbb{Z}[\pi_1(X)]$-module chain equivalence. The effect of a geometric elementary expansion $(X,(f,g)) \longmapsto X'$ on the chain level is that of an algebraic elementary expansion $(C(\tilde{X}),\tilde{g} \in C(\tilde{X})_{m+1}) \longmapsto C(\tilde{X}') = C(\tilde{X})'$. The finiteness structures on a CW complex X are in one-one correspondence with the finiteness structures on the $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\tilde{X})$.

Finally, we recall the realization theorems. If π is a finitely presented group every element $x \in \tilde{K}_0(\mathbb{Z}[\pi])$ is the finiteness obstruction $x = [X]$ of a finitely dominated CW complex X with $\pi_1(X) = \pi$, and every element $\tau \in Wh(\pi)$ is the Whitehead torsion $\tau = \tau(h)$ of a homotopy equivalence $h:X \longrightarrow Y$ of finite CW complexes with $\pi_1(X) = \pi$.

§2. Geometric transversality for CW complexes

The algebraic methods used to prove the splitting theorems in the algebraic K- and L-groups of generalized free products are best understood as abstract versions of the geometric transversality construction of codimension 1 framed submanifolds $Y^{n-1} \subset X^n$ of a connected compact n-manifold X, and the corresponding construction of the universal cover $\widetilde{X}$ by cutting X along Y. However, manifold transversality is too geometric in nature to translate directly into algebra. We shall now interpret the combinatorial methods of Waldhausen [ ] as a transversality theory for CW complexes, which will serve as a model for the algebraic transversality of chain complexes and algebraic Poincaré complexes in §§3,5 below. In the first instance we recall the connections between generalized free products of groups, groups acting on trees and covering spaces. Apart from [ ] the main references for these are Cappell [ ] and Serre [ ].

The <u>free product with amalgamation</u> determined by two injections of a group H into distinct groups G_1, G_2

$$i_1 : H \longrightarrow G_1 \ , \ i_2 : H \longrightarrow G_2$$

is the quotient group of the free product $G_1 * G_2$ defined by

$$G_1 *_H G_2 = G_1 * G_2 / \langle i_1(h) i_2(h)^{-1} \mid h \in H \rangle \ .$$

There are defined injections

$$j_1 : G_1 \longrightarrow G_1 *_H G_2 \ , \ j_2 : G_2 \longrightarrow G_1 *_H G_2$$

$$k = j_1 i_1 = j_2 i_2 : H \longrightarrow G_1 *_H G_2 \ ,$$

such that

$$H = G_1 \cap G_2 \subset G_1 *_H G_2 \ .$$

The commutative square

$$\begin{array}{ccc} H & \xrightarrow{i_1} & G_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ G_2 & \xrightarrow{j_1} & G_1 *_H G_2 \end{array}$$

is a pushout in the category of groups, and the commmutative square

$$\begin{array}{ccc} H \cup H & \xrightarrow{i_1 \cup i_2} & G_1 \cup G_2 \\ \downarrow & & \downarrow{\scriptstyle j_1 \cup j_2} \\ H \times I & \longrightarrow & G_1 *_H G_2 \end{array}$$

is a pushout in the category of groupoids. The group $G_1 *_H G_2$ is infinite and the subgroups G_1, G_2, H are of infinite index, except possibly in the case when one of i_1, i_2 (say i_2) is an isomorphism and $G_1 *_H G_2 \cong G_1$. We exclude this case.

The <u>HNN extension</u> determined by two injections of a group H into the same group G

$$i_1\ ,\ i_2\ :\ H \longrightarrow G$$

is the quotient group of the free product $G*\{t\}$ of G and the infinite cyclic group $\{t\}$ generated by t

$$G*_H\{t\}\ =\ G*\{t\}/\langle i_1(h)ti_2(h)^{-1}t^{-1}\,|\,h \in H\rangle\ .$$

There are defined injections

$$j\ :\ G \longrightarrow G*_H\{t\}\ ,\ k_1 = ji_1,\ k_2 = ji_2\ :\ H \longrightarrow G*_H\{t\}\ .$$

Using i_1 to identify H with $i_1(H) \subseteq G$ there are identities

$$i_2(H)\ =\ t^{-1}Ht \subseteq G\quad ,\quad H\ =\ G \cap tGt^{-1} \subset G*_H\{t\}.$$

The commutative square

$$\begin{array}{ccc} H \cup H & \xrightarrow{\ i_1 \cup i_2\ } & G \\ \big\downarrow & & \big\downarrow j \\ H \times I & \longrightarrow & G*_H\{t\} \end{array}$$

is a pushout in the category of groupoids. The group $G*_H\{t\}$ is infinite, and the subgroups $G,H,t^{-1}Ht$ are of infinite index. The normalizer N of G in $G*_H\{t\}$ is the subgroup generated by t^ngt^{-n} $(g \in G, n \in \mathbb{Z})$, a normal subgroup with infinite cyclic quotient

$$G*_H\{t\}/N\ =\ \{t\}\ =\ \mathbb{Z}\ .$$

Conjugation by t restricts to an automorphism

$$\alpha\ :\ N \longrightarrow N\ ;\ x \longmapsto t^{-1}xt$$

such that $G*_H\{t\}$ may be identified with the α-twisted extension $N \times_\alpha \mathbb{Z}$ $(t\alpha(x) = xt)$ of N by $\mathbb{Z}$

$$G*_H\{t\}\ =\ N \times_\alpha \mathbb{Z}\ .$$

In particular, given an automorphism $\alpha : H \longrightarrow H$ of a group H the HNN extension $G*_H\{t\}$ determined by

$$i_1\ =\ 1\ ,\ i_2\ =\ \alpha\ :\ H \longrightarrow G\ =\ H$$

is such that N = H and

$$G*_H\{t\}\ =\ H \times_\alpha \mathbb{Z}\ .$$

In the case $\alpha = 1$ this is just the product $H \times \mathbb{Z}$.

A group π is a <u>generalized free product</u> if

either A) $\pi = G_1 *_H G_2$ is a free product with amalgamation

or B) $\pi = G *_H \{t\}$ is an HNN extension.

Given a group G and a subgroup $H \subset G$ denote the sets of left and right H-cosets in G by

$$[G:H] = \{Hg \mid g \in G\} \ , \ \overline{[G:H]} = \{gH \mid g \in G\} ,$$

which are related by a natural duality bijection

$$[G:H] \xrightarrow{\sim} \overline{[G:H]} \ ; \ Hg \longmapsto \overline{Hg} = g^{-1}H \ .$$

The right G-action on [G:H]

$$[G:H] \times G \longrightarrow [G:H] \ ; \ (Hg,x) \longmapsto Hgx$$

is then related to the left G-action on $\overline{[G:H]}$

$$G \times \overline{[G:H]} \longrightarrow \overline{[G:H]} \ ; \ (x,gH) \longmapsto xgH$$

by the involution $^{-}: G \longrightarrow G ; x \longmapsto \bar{x} = x^{-1}$, with

$$\overline{(Hg)x} = \bar{x}.(\overline{Hg}) \ .$$

Given a generalized free product $\pi = \begin{cases} G_1 *_H G_2 \\ G *_H \{t\} \end{cases}$ let T be the infinite tree defined by left cosets in π, with

$$T^{(0)} = \begin{cases} [\pi:G_1] \cup [\pi:G_2] \\ [\pi:G] \end{cases} , \ T^{(1)} = [\pi:H] \ .$$

The vertices $\begin{cases} G_1 r_1 , G_2 r_2 \\ Gr_1 , Gr_2 \end{cases} \in T^{(0)}$ $(r_1, r_2 \in \pi)$ are connected by the segment $Hs \in T^{(1)}$ $(s \in \pi)$ if and only if

$$\begin{cases} Hs = G_1 r_1 \cap G_2 r_2 \\ Hs = Gr_1 \cap tGr_2 \ . \end{cases}$$

Use the subscripts of r_1, r_2 to orient each segment $Hs \in T^{(1)}$. The right π-action on the left cosets defines a right π-action on T

$$T \times \pi \longrightarrow T \ ; \ (\rho , x) \longmapsto \rho x$$

which preserves the orientation of each segment. The quotient graph T/π has $\begin{cases} \text{two vertices} \\ \text{one vertex} \end{cases}$ and one segment. Conversely, if

T is an oriented tree and a group π has an orientation-preserving action on T such that the quotient graph T/π has $\begin{cases}\text{two vertices}\\ \text{one vertex}\end{cases}$ and one segment (resp. is finite) then π is a type $\begin{cases}A)\\ B)\end{cases}$ generalized free product (resp. π can be built up out of $\{1\}$ by a finite sequence of generalized free products).

A generalized free product $\pi = \begin{cases} G_1 *_H G_2 \\ G*_H\{t\}\end{cases}$ also determines an infinite tree $\overline{T}$ defined by means of right cosets in π, with

$$\overline{T}^{(0)} = \begin{cases}\overline{[\pi:G_1]} \cup \overline{[\pi:G_2]} \\ \overline{[\pi:G]}\end{cases} \quad , \quad \overline{T}^{(1)} = \overline{[\pi:H]} \ .$$

The vertices $\begin{cases}\overline{r}_1G_1, \overline{r}_2G_2 \\ \overline{r}_1G, \overline{r}_2G\end{cases} \in \overline{T}^{(0)}$ $(\overline{r}_1, \overline{r}_2 \in \pi)$ are connected by the segment $\overline{s}H \in \overline{T}^{(1)}$ $(\overline{s} \in \pi)$ if and only if

$$\begin{cases}\overline{s}H = \overline{r}_1G_1 \cap \overline{r}_2G_2 \\ \overline{s}H = \overline{r}_1G \cap \overline{r}_2Gt^{-1} .\end{cases}$$

As for T the segment $\overline{s}H \in \overline{T}^{(1)}$ is oriented by the order of subscripts in $\overline{r}_1, \overline{r}_2$. The left action of π on the right cosets defines an orientation-preserving left π-action on $\overline{T}$

$$\pi \times \overline{T} \longrightarrow \overline{T} \ ; \ (x, \overline{\rho}) \longmapsto \overline{x}\,\overline{\rho} \ .$$

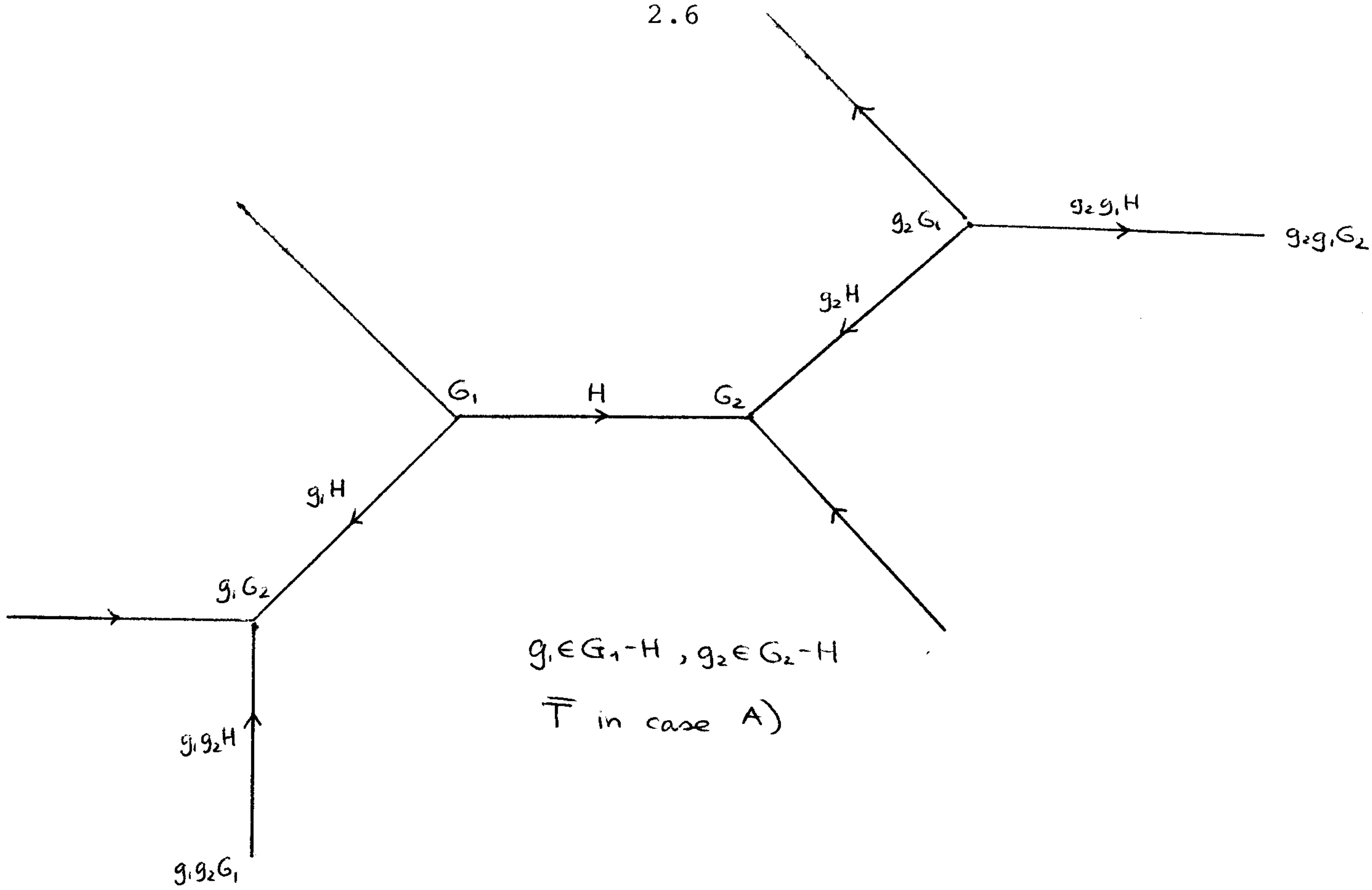
The duality between left and right cosets defines an isomorphism of oriented trees

$$T \xrightarrow{\sim} \overline{T} \ ; \ \rho \longmapsto \overline{\rho}$$

such that

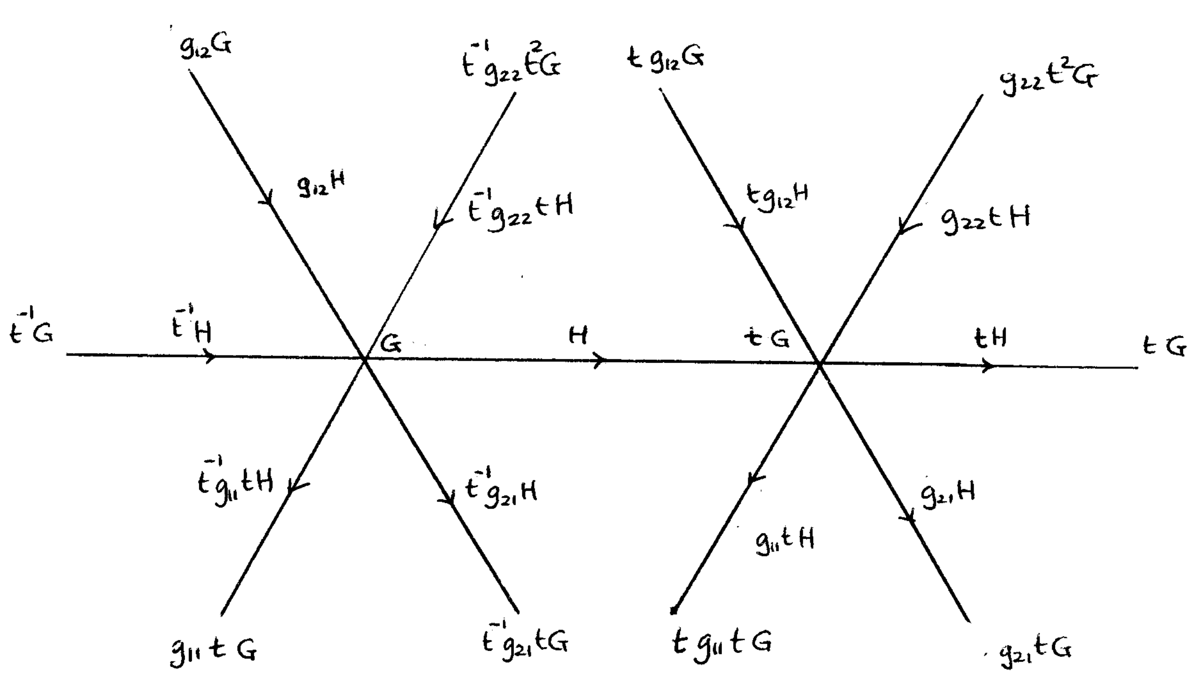
$$(\overline{x\rho}) = (\overline{\rho})(\overline{x}) \quad (x \in \pi, \rho \in T) \ .$$

We shall find both T and $\overline{T}$ useful.



$g_1 \in G_1 - H$, $g_2 \in G_2 - H$

$\overline{T}$ in case A)



$g_{11} \in G - H$ $\quad g_{22} \in t(G-H)t^{-1}$

$g_{12} \in (G-H)t^{-1}$ $\quad g_{21} \in t(G-H)$

$\overline{T}$ in case B)

A CW pair $(X, Y \subset X)$ is <u>bipolar of type</u> $\begin{cases} A) \\ B) \end{cases}$ if there are given CW complexes $\begin{cases} Z_1, Z_2 \\ Z \end{cases}$ with subcomplexes $\begin{cases} Y_1 \subset Z_1,\ Y_2 \subset Z_2 \\ Y_1, Y_2 \subset Z \text{ (disjoint)} \end{cases}$

and an isomorphism of CW complexes

$$\begin{cases} h : X \xrightarrow{\sim} (Z_1 \sqcup Z_2)/\{h_1(y) = h_2(y) \mid y \in Y\} \\ h : X \xrightarrow{\sim} Z/\{h_1(y) = h_2(y) \mid y \in Y\} \end{cases}$$

such that

$$h(y) = [h_1(y)] = [h_2(y)] \quad (y \in Y)$$

with h_1, h_2 given isomorphisms

$$h_1 : Y \xrightarrow{\sim} Y_1 \ , \ h_2 : Y \xrightarrow{\sim} Y_2 \ .$$

The CW complex $\begin{cases} Z_1 \sqcup Z_2 \\ Z \end{cases}$ is the <u>complement</u> of the bipolar pair (X,Y).

The various inclusions are denoted by

$$\begin{cases} i_1 : Y \xrightarrow{h_1} Y_1 \longrightarrow Z_1 \ , \ i_2 : Y \xrightarrow{h_2} Y_2 \longrightarrow Z_2 \\ i_1 : Y \xrightarrow{h_1} Y_1 \longrightarrow Z \ , \ i_2 : Y \xrightarrow{h_2} Y_2 \longrightarrow Z \end{cases}$$

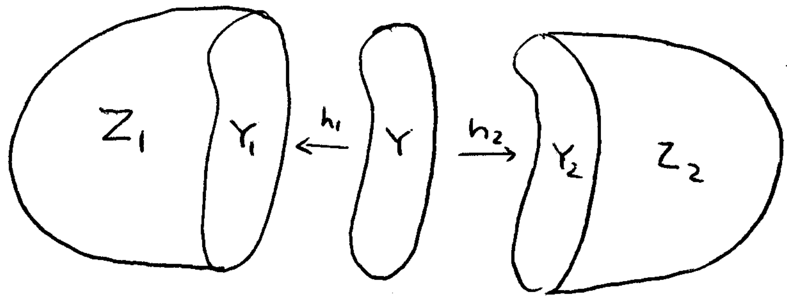
and there are defined natural maps

$$\begin{cases} j_1 : Z_1 \longrightarrow X \ , \ j_2 : Z_2 \longrightarrow X \\ j : Z \longrightarrow X \end{cases}$$

such that the composites

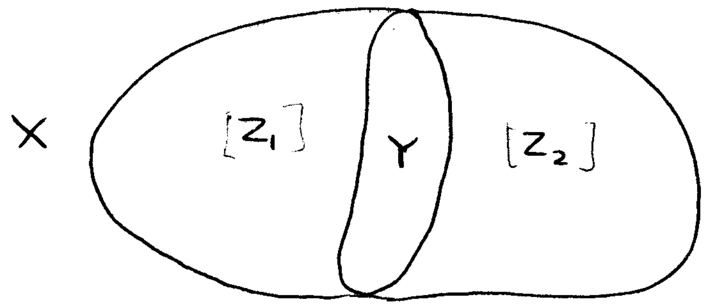
$$\begin{cases} k = j_1 i_1 = j_2 i_2 : Y \longrightarrow X \\ k_1 = j i_1 ,\ k_2 = j i_2 : Y \longrightarrow X \end{cases}$$

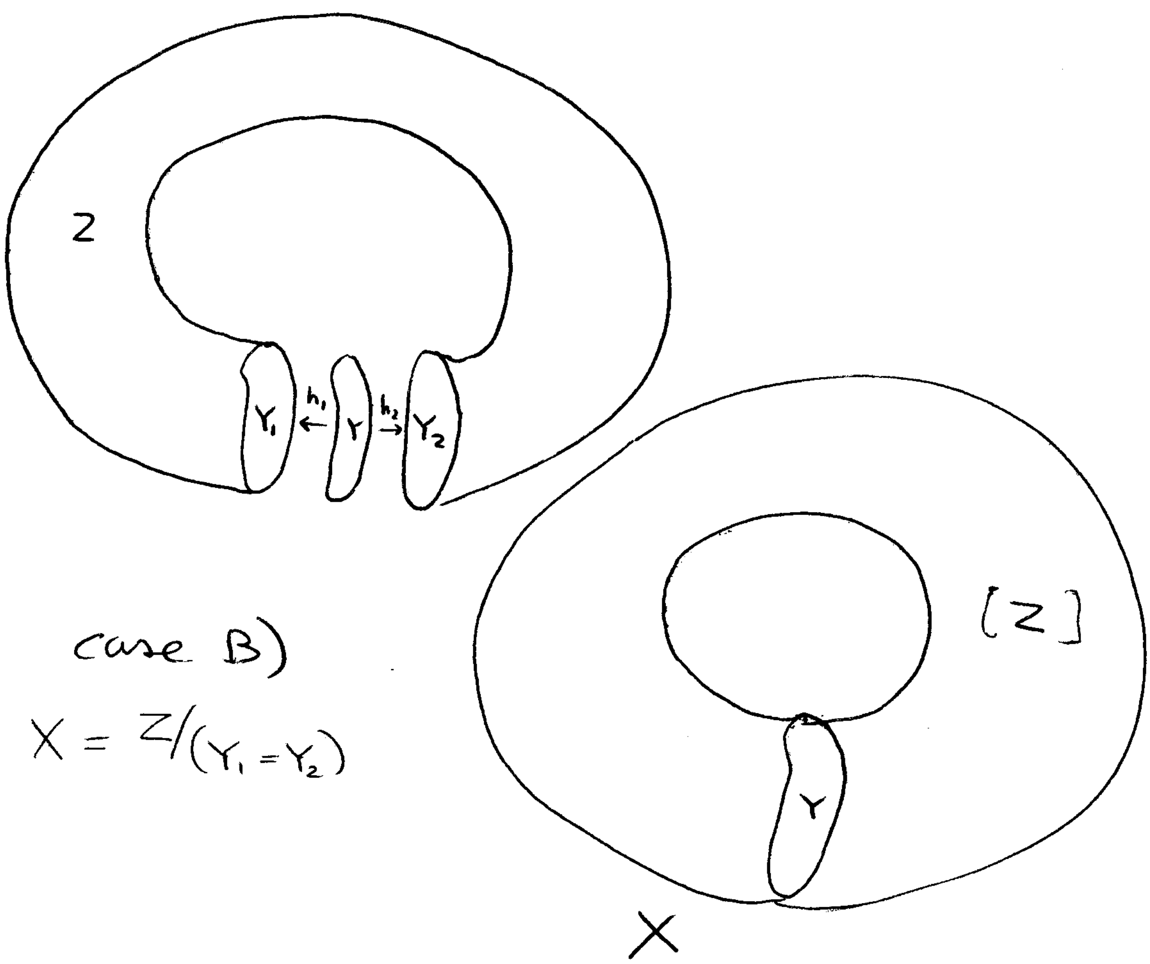
are inclusions.



case A)

$X = Z_1 \cup_Y Z_2$





case B)

$X = Z/(Y_1 = Y_2)$

In particular, if (X,Y) is a pair of connected CW complexes such that Y is bicollared in X, that is the inclusion $Y = Y \times \{0\} \longrightarrow X$ extends to an open embedding $Y \times \mathbb{R} \longrightarrow X$, then (X,Y) has the structure of a bipolar pair with the complement homeomorphic to the actual complement $X - Y \times (-1,1)$ of an open product neighbourhood $Y \times (-1,1) \subset X$, as follows:

A) if $X - Y$ is disconnected it has exactly two components and (X,Y) has type A) bipolar structure, with Z_1 and Z_2 connected

B) if $X - Y$ is connected then (X,Y) has a type B) bipolar structure, with Z connected.

Furthermore, a compact connected n-manifold X^n and a connected codimension 1 framed submanifold $Y^{n-1} \subset X^n$ define such a bicollared finite CW pair (X,Y), and in case $\begin{cases} \text{A) } Z_\lambda \ (\lambda = 1,2) \\ \text{B) } Z \end{cases}$ is a connected n-manifold with boundary $\begin{cases} \partial Z_\lambda = Y_\lambda \\ \partial Z = Y_1 \cup Y_2 \end{cases}$.

For a bipolar CW pair (X,Y) of type $\begin{cases} \text{A)} \\ \text{B)} \end{cases}$ with X,Y $\begin{cases} Z_1, Z_2 \\ Z \end{cases}$ connected and $\pi_1(Y) \longrightarrow \pi_1(X)$ injective the Van Kampen theorem expresses $\pi_1(X)$ as the $\begin{cases} \text{amalgamated free product} \\ \text{HNN extension} \end{cases}$

$$\pi_1(X) = \begin{cases} G_1 *_H G_2 \\ G*_H\{t\} \end{cases}$$

determined by the injections of fundamental groups

$$\begin{cases} i_\lambda : \pi_1(Y) = H \longrightarrow \pi_1(Z_\lambda) = G_\lambda \quad (\lambda = 1,2) \\ i_1, i_2 : \pi_1(Y) = H \longrightarrow \pi_1(Z) = G \ . \end{cases}$$

Bipolar pairs of this type will be called <u>biconnective</u>.

Let X be a CW complex, and let $\tilde{X}$ be a regular covering of X with group of covering translations $\pi = \begin{cases} G_1 *_H G_2 \\ G*_H\{t\} \end{cases}$ a generalized free product, acting on the left

$$\pi \times \tilde{X} \longrightarrow \tilde{X} \ ; \ (g,x) \longmapsto gx \ .$$

A <u>fundamental domain</u> $\begin{cases} (W_1,W_2) \\ W \end{cases}$ for $\tilde{X}$ is defined by subcomplexes $\begin{cases} W_1,W_2 \subset \tilde{X} \\ W \subset \tilde{X} \end{cases}$ such that

$$\begin{cases} G_1W_1 = W_1,\ G_2W_2 = W_2,\ \pi W_1 \cup \pi W_2 = \tilde{X}, \\ \bar{r}_1W_1 \cap \bar{r}_2W_2 = \begin{cases} \bar{s}(W_1 \cap W_2) & \text{if } \bar{r}_1 \in \bar{T}_1^{(0)}, \bar{r}_2 \in \bar{T}_2^{(0)} \text{ are connected by } \bar{s}\in\bar{T}^{(1)} \\ \emptyset & \text{otherwise} \end{cases} \\ GW = W,\ \pi W = \tilde{X} \\ \bar{r}_1W \cap \bar{r}_2W = \begin{cases} \bar{s}(W \cap tW) & \text{if } \bar{r}_1,\bar{r}_2 \in \bar{T}^{(0)} \text{ are connected by } \bar{s} \in \bar{T}^{(1)} \\ \emptyset & \text{otherwise .} \end{cases} \end{cases}$$

The projection $\tilde{X} \longrightarrow \tilde{X}/\pi = X$ sends $\begin{cases} W_1 \cap W_2 \subset \tilde{X} \\ W \cap tW \subset \tilde{X} \end{cases}$ to a subcomplex

$$Y = \begin{cases} (W_1 \cap W_2)/H \subset X \\ (W \cap tW)/H \subset X \end{cases}$$

such that (X,Y) is a bipolar pair of type $\begin{cases} A) \\ B) \end{cases}$ with complement

$$\begin{cases} Z_1 \sqcup Z_2 = W_1/G_1 \cup W_2/G_2 \\ Z = W/G \end{cases} .$$

If $\tilde{X}, \begin{cases} W_1 \cap W_2, W_1, W_2 \\ W \cap tW, W \end{cases}$ are simply-connected then (X,Y) is a biconnective pair, and $\tilde{X}, \begin{cases} \tilde{Y} = W_1 \cap W_2\ ,\ \tilde{Z}_1 = W_1\ ,\ \tilde{Z}_2 = W_2 \\ \tilde{Y} = W \cap tW\ ,\ \tilde{Z} = W \end{cases}$ is the universal cover of $X, Y, \begin{cases} Z_1, Z_2 \\ Z \end{cases}$ respectively. Conversely, if (X,Y) is a biconnective pair the universal cover $\tilde{X}$ of X admits the fundamental domain $\begin{cases} (W_1,W_2) = (\tilde{Z}_1,\tilde{Z}_2) \\ W = \tilde{Z} \end{cases}$ with $\begin{cases} W_1 \cap W_2 = \tilde{Y} \\ W \cap tW = \tilde{Y} \end{cases}$, that is $\tilde{X}$ may be obtained from the universal covers

$\tilde{Y}$, $\begin{cases}\tilde{Z}_1,\tilde{Z}_2\\ \tilde{Z}\end{cases}$ of Y, $\begin{cases}Z_1,Z_2\\ Z\end{cases}$ by 'cutting X along Y': the embeddings

$$\begin{cases} i_1 : Y \xrightarrow{h_1} Y_1 \longrightarrow Z_1 \ , \ i_2 : Y \xrightarrow{h_2} Y_2 \longrightarrow Z_2 \\ i_1 : Y \xrightarrow{h_1} Y_1 \longrightarrow Z \ \ , \ i_2 : Y \xrightarrow{h_2} Y_2 \longrightarrow Z \end{cases}$$

lift to embeddings of the covers

$$\begin{cases} \tilde{i}_1 : \tilde{Y} \longrightarrow \tilde{Z}_1 \ , \ \tilde{i}_2 : \tilde{Y} \longrightarrow \tilde{Z}_2 \\ \tilde{i}_1 \ , \ \tilde{i}_2 : \tilde{Y} \longrightarrow \tilde{Z} \end{cases}$$

such that

$$\tilde{i}_1(hy) = h\tilde{i}_1(y) \ , \begin{cases} \tilde{i}_2(hy) = h\tilde{i}_2(y) \\ \tilde{i}_2(hy) = (t^{-1}ht)\tilde{i}_2(y) \end{cases} \qquad (h \in H, \ y \in \tilde{Y})$$

and $\tilde{X}$ is the quotient

$$\tilde{X} = \begin{cases} (\pi \times (\tilde{Z}_1 \sqcup \tilde{Z}_2))/\sim \\ (\pi \times \tilde{Z})/\sim \end{cases}$$

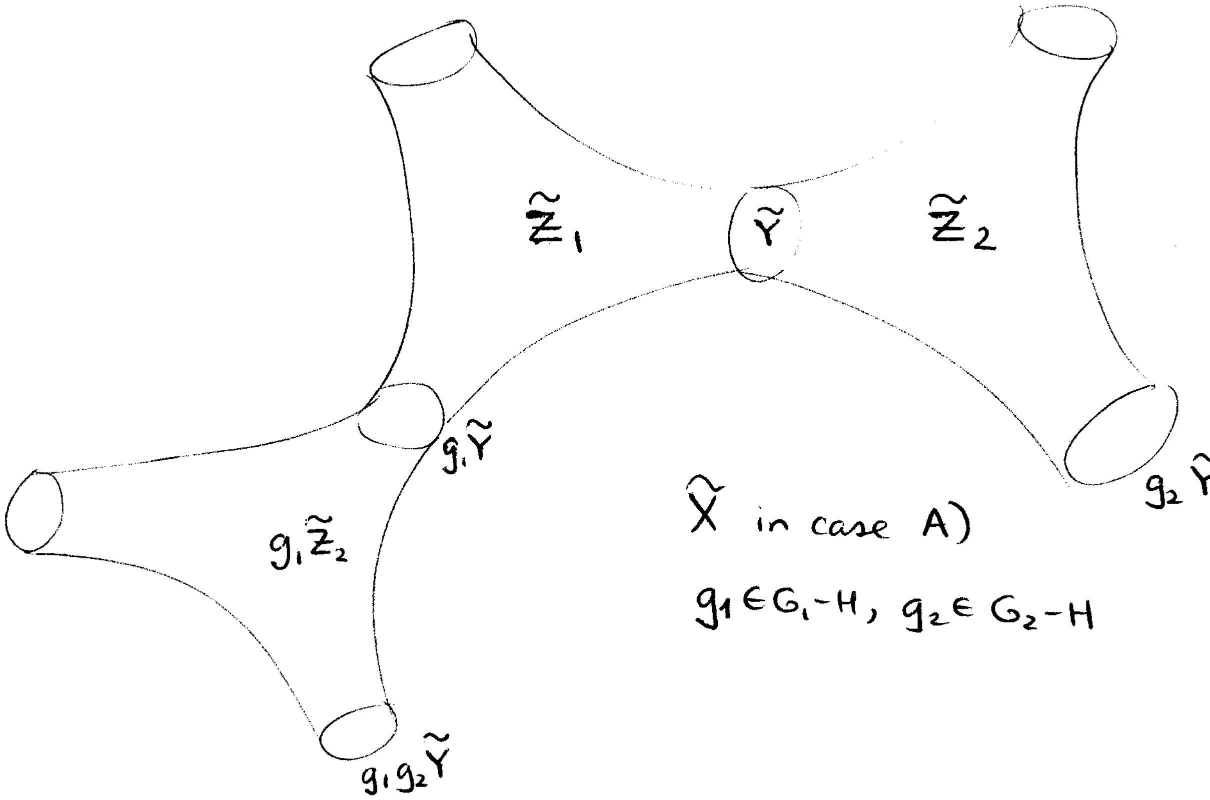
of $\begin{cases}\pi \times (\tilde{Z}_1 \sqcup \tilde{Z}_2)\\ \pi \times \tilde{Z}\end{cases}$ by the equivalence relation generated by

$$\begin{cases} (x,z_\lambda) \sim (xg_\lambda, g_\lambda^{-1}z_\lambda) \\ (x,z) \sim (xg, g^{-1}z) \end{cases} \text{ for all } x \in \pi \ , \begin{cases} g_\lambda \in G_\lambda, \ z_\lambda \in \tilde{Z}_\lambda \quad (\lambda = 1,2) \\ g \in G, \ z \in \tilde{Z} \end{cases}$$

$$\begin{cases} (x,\tilde{i}_1(y)) \sim (x,\tilde{i}_2(y)) \\ (x,\tilde{i}_1(y)) \sim (xt,\tilde{i}_2(y)) \end{cases} \text{ for all } x \in \pi, \ y \in \tilde{Y} \ ,$$

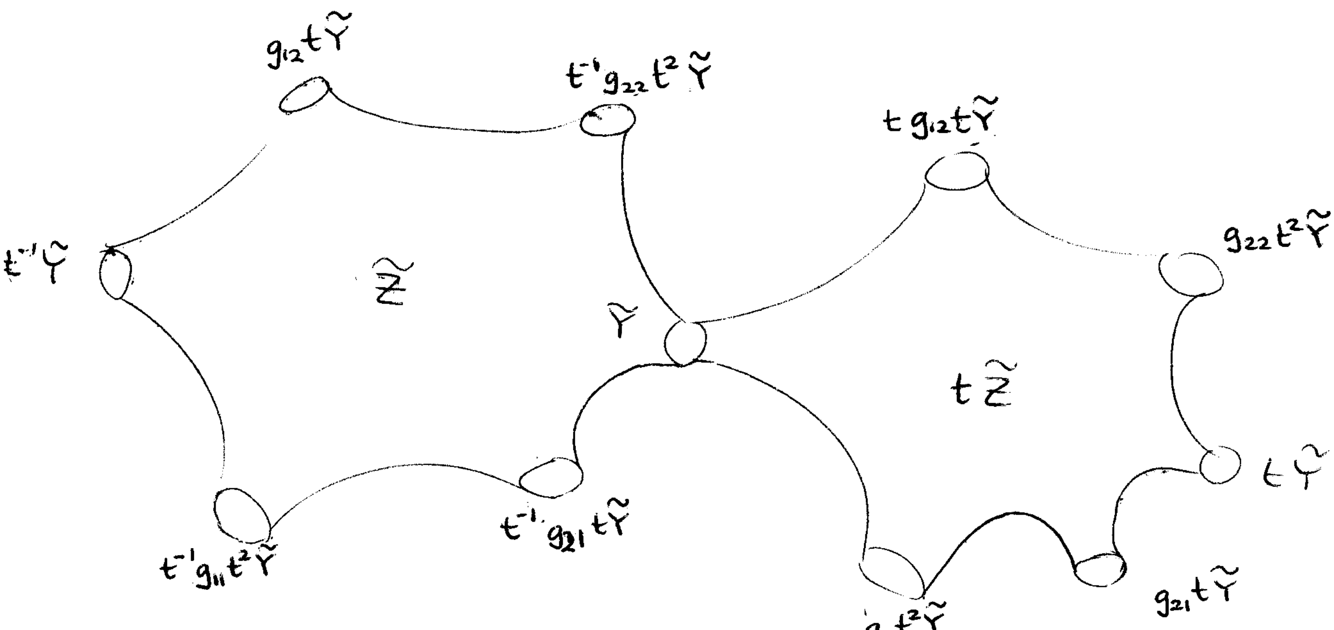
with the action of π given by

$$\pi \times \tilde{X} \longrightarrow \tilde{X} \ ; \ (w,[x,z]) \longmapsto [wx,z] \ .$$



$\tilde{X}$ in case A)

$g_1 \in G_1 - H$, $g_2 \in G_2 - H$



$g_{11} \in G - H \quad g_{22} \in t(G-H)t^{-1}$

$g_{12} \in (G-H)t^{-1} \quad g_{21} \in t(G-H)$

$\tilde{X}$ in case B)

with i_1, i_2, j_1, j_2 the algebraic inclusions induced by the geometric inclusions

$$i_1 : X_1 \cap X_2 \longrightarrow X_1 \quad , \quad i_2 : X_1 \cap X_2 \longrightarrow X_2$$
$$j_1 : X_1 \longrightarrow X \quad , \quad j_2 : X_2 \longrightarrow X \ ,$$

such that $j_1 i_1 = j_2 i_2 = k : X_1 \cap X_2 \longrightarrow X$ is also an inclusion.

Let now $\tilde{X}$ be a regular covering of a CW complex X with group $\pi = \begin{cases} G_1 *_H G_2 \\ G*_H\{t\} \end{cases}$, and fundamental domain $\begin{cases} (W_1, W_2) \\ W \end{cases}$, as before.

The <u>geometric MV presentation</u> of $C(\tilde{X})$ is the Mayer-Vietoris exact sequence of free $\mathbb{Z}[\pi]$-modules

$$\begin{cases} 0 \longrightarrow k_! C(W_1 \cap W_2) \xrightarrow{q} j_{1!} C(W_1) \oplus j_{2!} C(W_2) \xrightarrow{p} C(\tilde{X}) \longrightarrow 0 \\ 0 \longrightarrow k_{1!} C(W \cap tW) \xrightarrow{q} j_! C(W) \xrightarrow{p} C(\tilde{X}) \longrightarrow 0 \end{cases}$$

with $\begin{cases} j_1 : G_1 \longrightarrow \pi \ , \ j_2 : G_2 \longrightarrow \pi \ , \ k = j_1 i_1 = j_2 i_2 : H \longrightarrow \pi \\ j : G \longrightarrow \pi \ , \ k_1 = j i_1 : H \longrightarrow \pi \end{cases}$

the usual inclusions, and p,q defined by

$$\begin{cases} p : j_{1!} C(W_1) \oplus j_{2!} C(W_2) \longrightarrow C(\tilde{X}) \ ; \ (r_1 \otimes z_1, r_2 \otimes z_2) \longmapsto r_1 z_1 + r_2 z_2 \\ p : j_! C(W) \longrightarrow C(\tilde{X}) \ ; \ r \otimes z \longmapsto rz \end{cases}$$

$$(r, r_1, r_2 \in \mathbb{Z}[\pi] , \ z \in C(W) , \ z_1 \in C(W_1) , \ z_2 \in C(W_2))$$

$$\begin{cases} q : k_! C(W_1 \cap W_2) \longrightarrow j_{1!} C(W_1) \oplus j_{2!} C(W_2) \ ; \ s \otimes y \longmapsto (s \otimes y, -s \otimes y) \\ q : k_{1!} C(W \cap tW) \longrightarrow j_! C(W) \ ; \ s \otimes y \longmapsto s \otimes y - st \otimes t^{-1} y \end{cases}$$

$$(s \in \mathbb{Z}[\pi] \ , \ y \in \begin{cases} C(W_1 \cap W_2) \\ C(W \cap tW) \end{cases}) \ .$$

In particular, if (X,Y) is a biconnective pair with complement $\begin{cases} Z_1 \sqcup Z_2 \\ Z \end{cases}$ the $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\tilde{X})$ of the universal cover $\tilde{X}$ has geometric MV presentation

$$\begin{cases} 0 \longrightarrow k_! C(\tilde{Y}) \xrightarrow{q} j_{1!} C(\tilde{Z}_1) \oplus j_{2!} C(\tilde{Z}_2) \xrightarrow{p} C(\tilde{X}) \longrightarrow 0 \\ 0 \longrightarrow k_{1!} C(\tilde{Y}) \xrightarrow{q} j_! C(\tilde{Z}) \xrightarrow{p} C(\tilde{X}) \longrightarrow 0 \end{cases}$$

with $\tilde{Y}, \begin{cases} \tilde{Z}_1, \tilde{Z}_2 \\ \tilde{Z} \end{cases}$ the universal cover of $Y, \begin{cases} Z_1, Z_2 \\ Z \end{cases}$.

The regular covers $\widetilde{X}$ of a CW complex X with a given group of covering translations π are classified by the homotopy classes of maps $\phi : X \longrightarrow B\pi$. The classifying space $B\pi$ is a connected CW complex with fundamental group π and contractible universal cover $E\pi$. The map $\phi : X \longrightarrow B\pi$ classifies the pullback cover of X

$$\widetilde{X} = \phi^*(E\pi) = \{(x,y) \in X \times E\pi \mid \phi(x) = [y] \in B\pi\}$$

with π acting by

$$\pi \times \widetilde{X} \longrightarrow \widetilde{X} \ ; \ (g,(x,y)) \longmapsto (x,gy) \ .$$

If $\pi = \begin{cases} G_1 *_H G_2 \\ G*_H\{t\} \end{cases}$ is a generalized free product of type $\begin{cases} A) \\ B) \end{cases}$ then $(B\pi, BH)$ is a biconnective pair of type $\begin{cases} A) \\ B) \end{cases}$ with complement $\begin{cases} BG_1 \sqcup BG_2 \\ BG \end{cases}$

$$B\pi = \begin{cases} BG_1 \cup_{BH} BG_2 \\ BG/((BH)_1 = (BH)_2) \end{cases} ,$$

and $E\pi$ has fundamental domain $\begin{cases} (EG_1, EG_2) \\ EG \end{cases}$. For any CW complex X and any map $\phi : X \longrightarrow B\pi$ it is possible to subdivide X in such a way that ϕ is homotopic to a map (also denoted by $\phi : X \longrightarrow B\pi$) which is cellular and such that $Y = \phi^{-1}(BH) \subset X$ is a subcomplex, in which case (X,Y) is a bipolar pair of type $\begin{cases} A) \\ B) \end{cases}$ and the pullback cover $\widetilde{X} = \phi^*(E\pi)$ has fundamental domain

$$\begin{cases} (W_1, W_2) = (\widetilde{\phi}^{-1}(EG_1), \widetilde{\phi}^{-1}(EG_2)) \\ W = \widetilde{\phi}^{-1}(EG) \end{cases}$$

such that $Y = \begin{cases} (W_1 \cap W_2)/H \\ (W \cap tW)/H \end{cases}$.

(Moreover, if X is a compact n-manifold the classifying map $\phi : X \longrightarrow B\pi$ can be chosen to be transverse regular at $BH \subset B\pi$, so that $Y^{n-1} = \phi^{-1}(BH) \subset X^n$ is a framed codimension 1 submanifold).

The transversality construction of fundamental domains as inverse images of universal examples is too implicit in nature to translate directly into algebraic K- and L-theory. The combinatorial methods of Waldhausen [ ] inspire a more explicit construction, which uses the mapping cylinders of covering projections to obtain for any CW complex X equipped with a regular covering $\widetilde{X}$ with group $\pi = \begin{cases} G_1 *_H G_2 \\ G *_H \{t\} \end{cases}$ a bipolar pair (X', Y') such that X' is homotopy equivalent to X, and such that the corresponding cover $\widetilde{X}'$ of X' has a fundamental domain $\begin{cases} (W_1', W_2') \\ W' \end{cases}$.
An algebraic version of this construction will be used in §3 below.

Following Cohen [ ] define a map of CW complexes $f : X \longrightarrow Y$ to be <u>contractible</u> if the inverse image of each $y \in Y$ is a non-empty contractible subspace $f^{-1}(y) \subseteq Y$, in which case f is a homotopy equivalence. A contractible map of finite CW complexes is a simple homotopy equivalence ([ ]).

The <u>mapping cylinder</u> of a cellular map of CW complexes $f : X \longrightarrow Y$ is the CW complex

$$M(f) = (X \times [0,1] \cup Y) / \{(x,1) = f(x) \mid x \in X\},$$

with cells $e_X \times \{0\}$, $e_X \times [0,1]$, e_Y ($e_X \subset X$, $e_Y \subset Y$). Then $X = X \times \{0\}$ is a subcomplex of $M(f)$, and the projection

$$M(f) \longrightarrow Y \; ; \; (x,s) \longmapsto f(x) \; , \; y \longmapsto y \quad (x \in X, \; y \in Y, \; s \in [0,1])$$

is a contractible map.

Let then X be a CW complex with a regular covering $\widetilde{X}$ such that the group of covering translations π is a generalized free product $\pi = \begin{cases} G_1 *_H G_2 \\ G*_H\{t\} \end{cases}$. Let T be the associated tree, and let T' be the tree obtained from T by barycentric subdivision, so that

$$\begin{cases} T'^{(0)} = T^{(0)} \sqcup T^{(1)} \\ T'^{(1)} = \{(r,s) \in T^{(0)} \times T^{(1)} \,|\, r \text{ is a vertex of the segment } s\} \end{cases}$$

with $r,s \in T'^{(0)}$ the vertices of the segment $(r,s) \in T'^{(1)}$.
The right action of π on T determines a right action of π on T'.
The <u>universal bipolar pair</u> (X',Y') of X associated to $\widetilde{X}$ is the bipolar pair defined by

$$X' = T' \times_\pi \widetilde{X} = T' \times \widetilde{X} / \{(\rho,x) = (\rho g^{-1}, gx) \,|\, \rho \in T', x \in \widetilde{X}, g \in \pi\},$$

with $Y' = \widetilde{X}/H$ included in X' by

$$Y' \longrightarrow X' \ ; \ [y] \longmapsto [H,y] \quad (y \in \widetilde{X},\ H \in T'^{(0)}) \ .$$

The complement Z' of (X',Y') is defined by the mapping cylinders

$$Z' = \begin{cases} M(i_1 : \widetilde{X}/H \longrightarrow \widetilde{X}/G_1) \sqcup M(i_2 : \widetilde{X}/H \longrightarrow \widetilde{X}/G_2) \\ M(i_1 \sqcup i_2 : \widetilde{X}/H \sqcup \widetilde{X}/H \longrightarrow \widetilde{X}/G) \quad (i_2 = i_1 t^{-1}) \end{cases}$$

with $\begin{cases} i_1, i_2 \\ i_1 \end{cases}$ the covering projection, and $\widetilde{X}'$ has fundamental domain

$$\begin{cases} (\widetilde{M(i_1)}, \widetilde{M(i_2)}) = (M(\widetilde{i}_1 : \overline{[G_1:H]} \times \widetilde{X} \longrightarrow \widetilde{X}), M(\widetilde{i}_2 : \overline{[G_2:H]} \times \widetilde{X} \longrightarrow \widetilde{X})) \\ \widetilde{M(i_1 \sqcup i_2)} = M(\widetilde{i}_1 \sqcup \widetilde{i}_2 : \overline{[G:H]} \times \widetilde{X} \sqcup \overline{[G:t^{-1}Ht]} \times \widetilde{X} \longrightarrow \widetilde{X}) \end{cases}$$

There is a natural identification of CW pairs

$$(X',Y') = \begin{cases} (\widetilde{X}/G_1 \cup_{i_1 \times \{0\}} \widetilde{X}/H \times [0,1]' \cup_{i_2 \times \{1\}} \widetilde{X}/G_2 \ , \ \widetilde{X}/H \times \{\tfrac{1}{2}\}) \\ (\widetilde{X}/G \cup_{i_1 \times \{0\} \sqcup i_2 \times \{1\}} \widetilde{X}/H \times [0,1]' \ , \ \widetilde{X}/H \times \{\tfrac{1}{2}\}) \end{cases}$$

where $[0,1]'$ denotes the barycentric subdivision of $[0,1]$ with CW complex structure

$$[0,1]'^{(0)} = \{0,\tfrac{1}{2},1\} \ , \ [0,1]'^{(1)} = \{[0,\tfrac{1}{2}],[\tfrac{1}{2},1]\} \ .$$

The projection $X' \longrightarrow \widetilde{X}/\pi = X$ is a contractible map, in fact a fibration with contractible fibre T', with the inverse image of each $x \in X$ a copy of T'. In particular, $X' \longrightarrow X$ is a homotopy equivalence.

For example, if $i_1 = 1$, $i_2 = \alpha : H \longrightarrow G = H$ for some automorphism $\alpha : H \longrightarrow H$ of a group H then $\pi_1(X) = H \times_\alpha \mathbb{Z}$, and $\overline{X} = X/H$ is a regular infinite cyclic cover of X. (Moreover, every connected regular infinite cyclic cover $\overline{X}$ of a connected CW complex X arises in this way). The universal bipolar pair (X',Y') of X in this case is the mapping torus of the generating covering translation $t : \overline{X} \longrightarrow \overline{X}$

$$X' = T(t) = \overline{X} \times [0,1]'/\{(x,1) = (tx,0) \mid x \in \overline{X}\}$$

with

$$Y' = X \times \{\tfrac{1}{2}\} \subset X' \ .$$

The universal bipolar pair (X',Y') contains many bipolar subpairs $(X'',Y'') \subset (X',Y')$ such that X'' is a deformation retract of X', and hence also homotopy equivalent to X. Here is a general construction of such subpairs.

A <u>subfundamental domain</u> $\begin{cases}(W_1,W_2)\\ W\end{cases}$ for a regular cover $\widetilde{X}$ of a CW complex X with group $\pi = \begin{cases} G_1 *_H G_2 \\ G*_H\{t\}\end{cases}$ consists of subcomplexes $\begin{cases} W_1,W_2 \subseteq \widetilde{X} \\ W \subseteq \widetilde{X}\end{cases}$ such that

$$\begin{cases} G_1W_1 = W_1 \ , \ G_2W_2 = W_2 \ , \ \pi W_1 \cup \pi W_2 = \widetilde{X} \\ GW = W \ , \ \pi W = \widetilde{X}\end{cases}$$

and such that the subgraph $\Delta(\tilde{e}) \subseteq T$ defined for each cell $\tilde{e} \subset \widetilde{X}$ by

$$\Delta(\tilde{e})^{(0)} = \begin{cases}\{r_1 \in T_1^{(0)} \mid r_1\tilde{e} \subset W_1\} \cup \{r_2 \in T_2^{(0)} \mid r_2\tilde{e} \subset W_2\} \\ \{r \in T^{(0)} \mid r\tilde{e} \subset W\}\end{cases}$$

$$\Delta(\tilde{e})^{(1)} = \begin{cases}\{s \in T^{(1)} \mid s\tilde{e} \subset (W_1 \cap W_2)\} \\ \{s \in T^{(1)} \mid s\tilde{e} \subset (W \cap tW)\}\end{cases}$$

is a subtree. The subtrees $\Delta(\tilde{e}) \subseteq T$ $(\tilde{e} \subset \widetilde{X})$ are non-empty by virtue of the condition $\begin{cases}\pi W_1 \cup \pi W_2 = \widetilde{X} \\ \pi W = \widetilde{X}\end{cases}$, and are such that

i) $\Delta(g\tilde{e}) = \Delta(\tilde{e})g^{-1}$ for all $\tilde{e} \subset \tilde{X}$, $g \in \pi$

ii) $\Delta(\tilde{e}') \subseteq \Delta(\tilde{e})$ if $\tilde{e}, \tilde{e}' \subset \tilde{X}$ are cells such that $\mathring{\tilde{e}} \subset \tilde{e}'$.

Conversely, any collection $\Delta = \{\Delta(\tilde{e}) \subseteq T \mid \tilde{e} \subset \tilde{X}\}$ of non-empty subtrees satisfying i) and ii) determines a subfundamental domain $\begin{cases} (W_1, W_2) \\ W \end{cases}$ with $\begin{cases} W_\lambda \quad (\lambda = 1,2) \\ W \end{cases}$ consisting of all the cells $\tilde{e} \subset \tilde{X}$ for which $\begin{cases} 1 \in \Delta(\tilde{e})_\lambda^{(0)} \\ 1 \in \Delta(\tilde{e})^{(0)} \end{cases}$. A fundamental domain for $\tilde{X}$ is a subfundamental domain $\begin{cases} (W_1, W_2) \\ W \end{cases}$ such that the subtrees $\Delta(\tilde{e}) \subseteq T$ $(\tilde{e} \subset \tilde{X})$ are either single vertices or single segments. However, there are many subfundamental domains which are not fundamental domains, for example $\begin{cases} (W_1, W_2) = (\tilde{X}, \tilde{X}) \\ W = \tilde{X} \end{cases}$ with $\Delta(\tilde{e}) = T$ $(\tilde{e} \subseteq \tilde{X})$. (Excersize: describe all the subfundamental and fundamental domains W for the universal cover $\tilde{X} = \mathbb{R}$ of $X = S^1$, with $t : \tilde{X} \longrightarrow \tilde{X}$; $x \longmapsto x+1$, and with the CW structure

$$\tilde{X}^{(0)} = \{\{n\} \mid n \in \mathbb{Z}\}\ ,\ \tilde{X}^{(1)} = \{[n,n+1] \mid n \in \mathbb{Z}\}\ ).$$

Given a subfundamental domain $\begin{cases} (W_1, W_2) \\ W \end{cases}$ for $\tilde{X}$ define the <u>subuniversal</u> bipolar pair $\begin{cases} (X(W_1,W_2), Y(W_1,W_2)) \\ (X(W), Y(W)) \end{cases}$ to be the subpair of the universal bipolar pair (X', Y') given by

$$\begin{cases} X(W_1,W_2) = W_1/G_1 \cup_{i_1} (W_1 \cap W_2)/H \times [0,1]' \cup_{i_2} W_2/G_2 \subseteq X' \\ X(W) = W/G \cup_{i \sqcup it^{-1}} (W \cap tW)/H \times [0,1]' \subseteq X' \end{cases}$$

$$\begin{cases} Y(W_1,W_2) = (W_1 \cap W_2)/H \times \{\tfrac{1}{2}\} \subseteq Y' \\ Y(W) = (W \cap tW)/H \times \{\tfrac{1}{2}\} \subseteq Y' \end{cases}$$

with complement

$$\begin{cases} Z_1(W_1,W_2) \sqcup Z_2(W_1,W_2) = M(i_1 : (W_1 \cap W_2)/H \longrightarrow W_1/G_1) \\ \qquad\qquad \sqcup\, M(i_2 : (W_1 \cap W_2)/H \longrightarrow W_2/G_2) \subseteq Z_1' \sqcup Z_2' \\ Z(W_1,W_2) = M(i \sqcup it^{-1} : (W \cap tW)/H \sqcup (W \cap tW)/H \longrightarrow W/G) \subseteq Z' \ . \end{cases}$$

The composite

$$\begin{cases} X(W_1,W_2) \xrightarrow{\text{inclusion}} X' \xrightarrow{\text{projection}} X \\ X(W) \xrightarrow{\text{inclusion}} X' \xrightarrow{\text{projection}} X \end{cases}$$

is a contractible map, with the inverse image of $x \in X$ a copy of the tree $\Delta(\tilde{e})$ associated to any lift $\tilde{e} \subset \tilde{X}$ of the unique cell $e \subset X$ such that $x \in \mathring{e}$. The construction converts a subfundamental domain $\begin{cases} (W_1,W_2) \\ W \end{cases}$ for $\tilde{X}$ into a fundamental domain

$$\begin{cases} (W_1'',W_2'') = (M(\tilde{i}_1 : G_1(W_1 \cap W_2) \longrightarrow W_1)\ ,\ M(\tilde{i}_2 : G_2(W_1 \cap W_2) \longrightarrow W_2)) \\ W'' = M(\tilde{i} \sqcup \tilde{i}t^{-1} : G(W \cap tW) \sqcup G(W \cap tW) \longrightarrow W) \end{cases}$$

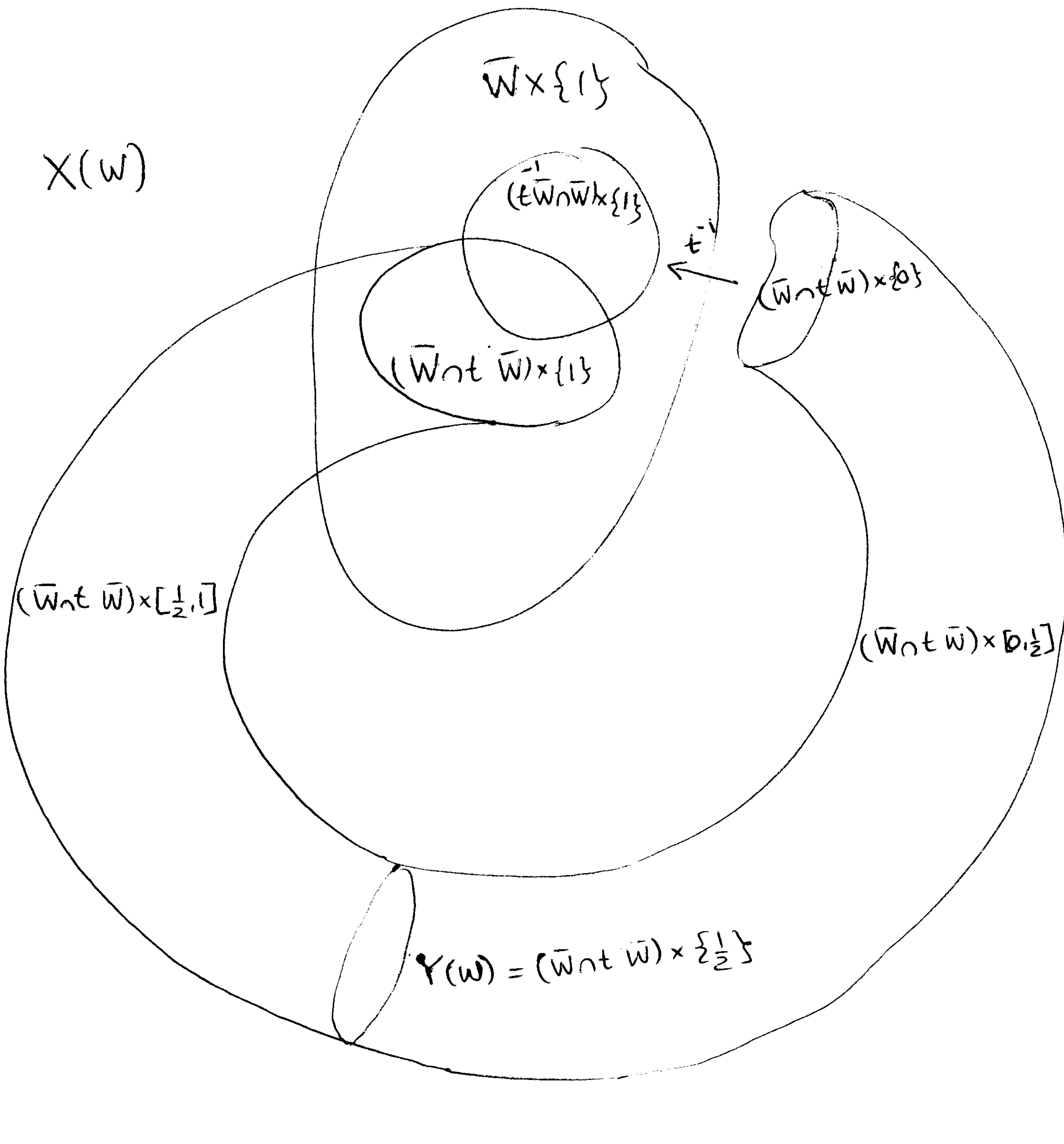
for the the induced cover $\tilde{X}''$ of $X'' = \begin{cases} X(W_1,W_2) \\ X(W) \end{cases}$, with X'', $\begin{cases} W_1'',W_2'' \\ W'' \end{cases}$ homotopy equivalent to X, $\begin{cases} W_1,W_2 \\ W \end{cases}$ respectively and $\begin{cases} W_1'' \cap W_2'' = W_1 \cap W_2 \\ W'' \cap tW'' = W \cap tW \end{cases}$.

For example, if $i_1 = 1$, $i_2 = \alpha : H \longrightarrow G = H$, so that $\overline{X} = \widetilde{X}/H$ is an infinite cyclic cover of X and $(X',Y') = (T(t:\overline{X} \to \overline{X}), \overline{X}\times\{\frac{1}{2}\})$ (as above) then the subuniversal biconnective pair $(X(W),Y(W))$ associated to a subfundamental domain $W \subseteq \widetilde{X}$ is given by

$$Y(W) = (W\cap tW)/H\times\{\tfrac{1}{2}\} = (\overline{W}\cap t\overline{W})\times\{\tfrac{1}{2}\}$$

$$\subset X(W) = (\overline{W}\times\{1\} \cup (\overline{W}\cap tW)\times[0,1]')/\{(x,0) = (t^{-1}x,1)\,|\,x\in\overline{W}\cap t\overline{W}\}$$

with $\overline{W} = W/H \subseteq \overline{X}$ the image of $W \subseteq \widetilde{X}$ under the projection $\widetilde{X} \longrightarrow \overline{X}$.



Let now X be a finite CW complex which is connected and such that $\pi_1(X) = \begin{cases} G_1 *_H G_2 \\ G*_H\{t\} \end{cases}$. As the tree T is infinite the universal biconnective pair (X',Y') is an infinite CW pair, with the complement Z' an infinite CW complex. The following finiteness conditions on a subfundamental domain $\begin{cases} (W_1,W_2) \\ W \end{cases}$ for the universal cover $\tilde{X}$ are equivalent:

i) The CW complexes $\begin{cases} W_1/G_1, W_2/G_2 \\ W/G \end{cases}$ are finite.

ii) The CW pair $\begin{cases} (X(W_1,W_2), Y(W_1,W_2)) \\ (X(W), Y(W)) \end{cases}$ is finite.

iii) For each cell $\tilde{e} \subset \tilde{X}$ the tree $\Delta(\tilde{e})$ is finite.

If these conditions are satisfied $\begin{cases} (W_1,W_2) \\ W \end{cases}$ is a <u>finite</u> subfundamental domain, and $\begin{cases} X(W_1,W_2) \xrightarrow{\sim} X \\ X(W) \xrightarrow{\sim} X \end{cases}$ is a simple homotopy equivalence of finite CW complexes. A fundamental domain is automatically finite.

Translated into the language of CW complexes the algebraic transversality result of Waldhausen [ ] is that a finite CW complex X admits a finite subfundamental domain for $\tilde{X}$.

We shall not actually use the CW complex version of algebraic transversality, but it helps to motivate the development of chain complex transversality in §3 below.

§3. Algebraic transversality for chain complexes

The algebraic transversality theory of Waldhausen [ ] for the simple homotopy theory of finite chain complexes over generalized free product rings will now be cast in a form suitable for the generalization to L-theory in §5 below. In the first instance we recall the structure theory of generalized free product rings. The main reference for this is Waldhausen [ ], but see also Cohn [ ], Stallings [ ], Casson [ ], Cappell [ ] and Dicks [ ],[ ].

The base of a based free R-module M is denoted by [M:R]. For a based free right R-module M the base is denoted by $\overline{[M:R]}$.

A ring A is a <u>pure extension</u> of a subring $B \subset A$ if there is given a (B,B)-bimodule $A' \subset A$ such that

$$A = B \oplus A'$$

and A' is free and based both as a left and as a right B-module. Then A is free and based both as a left and as a right B-module also, with

$$[A:B] = \{1\} \sqcup [A':B] \quad , \quad \overline{[A:B]} = \{1\} \sqcup \overline{[A':B]} \ .$$

A morphism of rings $i:B \longrightarrow A$ is a <u>pure injection</u> if it is an injection such that A is a pure extension of the subring $i(B) \subseteq A$. For example, if $i : H \longrightarrow G$ is the inclusion of a subgroup H in a group G then $i : \mathbb{Z}[H] \longrightarrow \mathbb{Z}[G]$ is a pure injection of rings, with

$$\mathbb{Z}[G] = \mathbb{Z}[H] \oplus \mathbb{Z}[G-H] \ .$$

A choice of coset representatives for the left cosets $[G:H] = \{Hg \mid g \in G\}$ is a left $\mathbb{Z}[H]$-module base $[\mathbb{Z}[G]:\mathbb{Z}[H]]$ for $\mathbb{Z}[G]$, and similarly for the right cosets $\overline{[G:H]} = \{gH \mid g \in G\}$ and a right $\mathbb{Z}[H]$-module base $\overline{[\mathbb{Z}[G]:\mathbb{Z}[H]]}$.

The <u>free product with amalgamation</u> $A_1 *_B A_2$ determined by two pure injections of a ring B into distinct rings A_1, A_2

$$i_1 : B \longrightarrow A_1 \quad , \quad i_2 : B \longrightarrow A_2$$

is the quotient ring of the free product $A_1 * A_2$

$$A_1 *_B A_2 = A_1 * A_2 / < i_1(b) - i_2(b) \mid b \in B > \ .$$

There are defined pure injections

$$j_1 : A_1 \longrightarrow A_1 *_B A_2 \quad , \quad j_2 : A_2 \longrightarrow A_1 *_B A_2$$

$$k = j_1 i_1 = j_2 i_2 : B \longrightarrow A_1 *_B A_2$$

such that

$$B = A_1 \cap A_2 \subset A_1 *_B A_2 \ .$$

The commutative square

$$\begin{array}{ccc} B & \xrightarrow{i_1} & A_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ A_2 & \xrightarrow{j_2} & A_1 *_B A_2 \end{array}$$

is a pushout in the category of rings.

The HNN extension $A *_B [t,t^{-1}]$ determined by two pure injections of a ring B into the same ring A

$$i_1 \ , \ i_2 \ : \ B \longrightarrow A$$

is the quotient ring of the free product $A * \mathbb{Z}[t,t^{-1}]$

$$A *_B [t,t^{-1}] = A * \mathbb{Z}[t,t^{-1}] / < i_1(b) - t i_2(b) t^{-1} \mid b \in B > .$$

There are defined pure injections

$$j : A \longrightarrow A *_B [t,t^{-1}] \ , \ k_1 = j i_1 , \ k_2 = j i_2 \ : \ B \longrightarrow A *_B [t,t^{-1}]$$

such that using i_1 and k_1 to identify B with $i_1(B) \subset A$ and $k_1(B) \subset A *_B [t,t^{-1}]$, and using j to identify A with $j(A) \subset A *_B [t,t^{-1}]$, there are identities

$$i_2(B) = t^{-1} B t \subset A \quad , \quad B = A \cap t A t^{-1} \subset A *_B [t,t^{-1}] \ .$$

In particular, if $\alpha : A \longrightarrow A$ is an automorphism of a ring A the HNN extension determined by the pure injections

$$i_1 = 1 \ , \ i_2 = \alpha \ : \ B = A \longrightarrow A$$

(with $A_1' = A_2' = 0$) is the α-twisted Laurent polynomial extension of A

$$A *_B [t,t^{-1}] = A_\alpha [t,t^{-1}]$$

consisting of all the polynomials $\sum_{n=-\infty}^{\infty} a_n t^n$ with the coefficients $a_n \in A$ such that $\{n \in \mathbb{Z} \mid a_n \neq 0\}$ is finite, with the multiplication determined by

$$at = t\alpha(a) \quad (a \in A) \ .$$

In the special case $\alpha = 1 : A \longrightarrow A$ the HNN extension is just the usual Laurent polynomial extension

$$A*_B[t,t^{-1}] = A[t,t^{-1}] \ .$$

Any HNN extension is an α-twisted Laurent polynomial extension

$$A*_B[t,t^{-1}] = N_\alpha[t,t^{-1}]$$

with N the subring of $A*_B[t,t^{-1}]$ generated by $\{t^n a t^{-n} \mid a \in A, n \in \mathbb{Z}\}$ and

$$\alpha : N \longrightarrow N \ ; \ x \longmapsto t^{-1}xt \quad .$$

A ring R is a <u>generalized free product</u> if

either A) $R = A_1 *_B A_2$ is a free product with amalgamation

or B) $R = A*_B[t,t^{-1}]$ is an HNN extension .

The group ring of a $\left\{\begin{array}{l}\text{free product with amalgamation}\\ \text{HNN extension}\end{array}\right.$ $\pi = \left\{\begin{array}{l} G_1 *_H G_2 \\ G*_H\{t\}\end{array}\right.$

is a $\left\{\begin{array}{l}\text{free product with amalgamation}\\ \text{HNN extension}\end{array}\right.$

$$\mathbb{Z}[\pi] = \left\{\begin{array}{l} \mathbb{Z}[G_1] *_{\mathbb{Z}[H]} \mathbb{Z}[G_2] \\ \mathbb{Z}[G] *_{\mathbb{Z}[H]} [t,t^{-1}] \end{array}\right. ,$$

so that the group ring of a generalized free product of groups is a generalized free product of rings.

As for groups a generalized free product ring

$$R = \begin{cases} A_1 *_B A_2 \\ A *_B [t,t^{-1}] \end{cases}$$

has an associated oriented tree T, which will now be defined using the given left B-module bases $[A_1':B],[A_2':B]$ of the given (B,B)-bimodules A_1',A_2' such that

$$\begin{cases} A_1 = B \oplus A_1' \ , \ A_2 = B \oplus A_2' \\ A = B \oplus A_1' = t^{-1}Bt \oplus A_2' \end{cases} .$$

In dealing with such bases we shall be considering $M \otimes_B N$ for various (B,B)-bimodules M,N which are free as left B-modules, with given bases $[M:B],[N:B]$. The (B,B)-bimodule $M \otimes_B N$ is also free as a left B-module, and is given the base

$$[M \otimes_B N:B] = \{x \otimes y \in M \otimes_B N \,|\, x \in [M:B] \ , \ y \in [N:B]\} \ .$$

The oriented tree T in the amalgamated free product case $R = A_1 *_B A_2$ is defined by

$$T^{(0)} = T_1^{(0)} \sqcup T_2^{(0)} = [R:A_1] \sqcup [R:A_2] \quad , \quad T^{(1)} = [R:B] \ .$$

The segment $s \in T^{(1)}$ has initial vertex $r_1 \in T_1^{(0)}$ and terminal vertex $r_2 \in T_2^{(0)}$, uniquely characterized by

$$Bs = A_1 r_1 \cap A_2 r_2 \subset R.$$

It remains to describe $[R:A_1],[R:A_2],[R:B]$. Define (B,B)-bimodules $R_{m_1 m_2}$ for $m_1,m_2 \in \{1,2\}$ by

$$R_{m_1 m_2} = \sum_{n=1}^{\infty} R_{m_1 m_2,n}$$

with $R_{m_1 m_2,n}$ the (B,B)-bimodules defined inductively by

$$R_{11,1} = A_1' \ , \ R_{22,1} = A_2' \ , \ R_{12,1} = 0 \ , \ R_{21,1} = 0 \ ,$$

$$R_{11,n+1} = A_1' \otimes_B R_{21,n} \ , \ R_{22,n+1} = A_2' \otimes_B R_{12,n} \ ,$$

$$R_{12,n+1} = A_1' \otimes_B R_{22,n} \ , \ R_{21,n+1} = A_2' \otimes_B R_{11,n} \ .$$

As a (B,B)-bimodule

$$R = B\oplus R_{11}\oplus R_{22}\oplus R_{12}\oplus R_{21} \quad ,$$

and the left B-module base [R:B] is obtained from $[A_1^!:B]$ and $[A_2^!:B]$ by successive application of the above rule for bases of tensor products. As a left A_1-module

$$R = A_1 \otimes_B (B\oplus R_{22}\oplus R_{21})$$

with $[R:A_1] = 1\otimes[B\oplus R_{22}\oplus R_{21}:B]$, and similarly for the left A_2-module structure.

The oriented tree T in the HNN extension case $R = A*_B[t,t^{-1}]$ is defined by

$$T^{(0)} = [R:A],\ T^{(1)} = [R:B] \ .$$

The segment $s\in T^{(1)}$ has the initial and terminal vertices $r_1, r_2 \in T^{(0)}$ uniquely characterized by

$$Bs = Ar_1 \cap tAr_2 \subset R.$$

Again, it remains to describe [R:A],[R:B]. Define (B,B)-bimodules $R_{m_1m_2}$ for $m_1,m_2 \in \{1,2\}$ by

$$R_{m_1m_2} = \sum_{n=1}^{\infty} R_{m_1m_2,n}$$

with $R_{m_1m_2,n}$ the (B,B)-bimodules defined inductively by

$$R_{11,1} = A_1^!,\ R_{22,1} = tA_2^!t^{-1},\ R_{12,1} = At^{-1},\ R_{21,1} = tA,$$

$$R_{11,n+1} = At^{-1}\otimes_B R_{11,n} \oplus A_1^!\otimes_B R_{21,n},\ R_{22,n+1} = tA\otimes_B R_{22,n} \oplus tA_2^!t^{-1}\otimes_B R_{12,n},$$

$$R_{12,n+1} = At^{-1}\otimes_B R_{12,n} \oplus A_1^!\otimes_B R_{22,n},\ R_{21,n+1} = tA\otimes_B R_{21,n} \oplus tA_2^!t^{-1}\otimes_B R_{11,n}.$$

As a (B,B)-bimodule

$$R = B\oplus R_{11}\oplus R_{22}\oplus R_{12}\oplus R_{21},$$

with the corresponding left B-module base [R:B]. As a left A-module

$$R = A\otimes_B(B\oplus R_{22}\oplus R_{21}) \oplus At^{-1}\otimes_B(B\oplus R_{11}\oplus R_{12}),$$

with

$$[R:A] = 1\otimes[B\oplus R_{22}\oplus R_{21}:B] \sqcup t^{-1}\otimes[B\oplus R_{11}\oplus R_{12}:B] \ .$$

It is also possible to construct an oriented tree $\overline{T}$ for a generalized free product ring $R = \begin{cases} A_1 *_B A_2 \\ A*_B[t,t^{-1}] \end{cases}$ using the right B-module bases $\overline{[A_1^!:B]}, \overline{[A_2^!:B]}$ with

$$\overline{T}^{(0)} = \begin{cases} \overline{T}_1^{(0)} \sqcup \overline{T}_2^{(0)} = \overline{[R:A_1]} \sqcup \overline{[R:A_2]} \\ \overline{[R:A]} \end{cases} , \quad \overline{T}^{(1)} = \overline{[R:B]} .$$

The segment $\overline{s} \in \overline{T}^{(1)}$ has the initial and terminal vertices $\begin{cases} \overline{r}_1 \in \overline{T}_1^{(0)}, \ \overline{r}_2 \in \overline{T}_2^{(0)} \\ \overline{r}_1, \overline{r}_2 \in \overline{T}^{(0)} \end{cases}$ uniquely characterized by

$$\begin{cases} \overline{s}B = \overline{r}_1 A_1 \cap \overline{r}_2 A_2 \subset R \\ \overline{s}B = \overline{r}_1 A \cap \overline{r}_2 A t^{-1} \subset R . \end{cases}$$

(In the group ring case $R = \begin{cases} \mathbb{Z}[G_1 *_H G_2] \\ \mathbb{Z}[G*_H\{t\}] \end{cases}$ the group ring involutions $\overline{}: \mathbb{Z}[\pi] \xrightarrow{\sim} \mathbb{Z}[\pi]; g \longmapsto g^{-1}$ $(g \in \pi)$ determine an isomorphism of oriented trees $T \xrightarrow{\sim} \overline{T}$).

A morphism of rings

$$i : B \longrightarrow A$$

determines an (A,B)-bimodule structure on A

$$A \times A \times B \longrightarrow A \ ; \ (a,x,b) \longmapsto axi(b) \ .$$

This is used to define the <u>induction</u> functor

$$i_! : (\text{B-modules}) \longrightarrow (\text{A-modules}) \ ; \ M \longmapsto i_!M = B \otimes_A M \ ,$$

which has already appeared in §1 for injections of group rings. If i is a pure injection define for each $\overline{a} \in \overline{[A:B]}$ the $\mathbb{Z}$-module

$$\overline{a}M = \{\overline{a} \otimes x \in i_!M \mid x \in M\} \ .$$

There is defined an isomorphism of $\mathbb{Z}$-modules

$$M \xrightarrow{\sim} \overline{a}M \ ; \ x \longmapsto ax \ ,$$

and as a $\mathbb{Z}$-module

$$i_!M = \sum_{\overline{a} \in \overline{[A:B]}} \overline{a}M = M \oplus \sum_{\overline{a}' \in \overline{[A':B]}} \overline{a}'M \ .$$

If M is a based free B-module then $i_!M$ is a based free A-module (for any morphism i), with $[i_!M:A] = \{1 \otimes x \in i_!M \mid x \in [M:B]\} = 1 \otimes [M:B]$.

Let $R = \begin{cases} A_1 *_B A_2 \\ A*_B[t,t^{-1}] \end{cases}$ be a generalized free product ring.

An <u>MV presentation</u> of an R-module M is an exact sequence of R-modules

$$\begin{cases} 0 \longrightarrow k_!Q \xrightarrow{q} j_{1!}P_1 \oplus j_{2!}P_2 \xrightarrow{p} M \longrightarrow 0 \\ 0 \longrightarrow k_{1!}Q \xrightarrow{q} j_!P \xrightarrow{p} M \longrightarrow 0 \end{cases}$$

such that $\begin{cases} P_1, P_2 \\ P \end{cases}$, Q is an $\begin{cases} A_1-, A_2- \\ A- \end{cases}$, B-module respectively, with

the R-module morphism q such that

$$\begin{cases} q(Q) \subseteq P_1 \oplus P_2 \\ q(Q) \subseteq P \oplus tP. \end{cases}$$

Thus q is determined by $\begin{cases} q_1 \in \mathrm{Hom}_{A_1}(i_{1!}Q, P_1),\ q_2 \in \mathrm{Hom}_{A_2}(i_{2!}Q, P_2) \\ q_1 \in \mathrm{Hom}_A(i_{1!}Q, P),\ q_2 \in \mathrm{Hom}_A(i_{2!}Q, P) \end{cases}$

such that $\begin{cases} pq_1 = pq_2 \\ pq_1 = ptq_2 \end{cases}$, with

$$\begin{cases} q = \begin{pmatrix} q_1 \\ -q_2 \end{pmatrix} : k_!Q \longrightarrow j_{1!}P_1 \oplus j_{2!}P_2\ ;\ r \otimes x \longmapsto (r \otimes q_1(x), r \otimes q_2(x)) \\ q = q_1 - tq_2 : k_{1!}Q \longrightarrow j_!P\ ;\ r \otimes x \longmapsto r \otimes q_1(x) - rt \otimes q_2(x)\ . \end{cases}$$

The MV presentation is <u>finite</u> if each of the

R-, $\begin{cases} A_1-, A_2- \\ A- \end{cases}$, B-modules M, $\begin{cases} P_1, P_2 \\ P \end{cases}$, Q is based f.g. free, and

the corresponding exact sequence of based f.g. free R-modules

has torsion $\tau = 0 \in K_1(R)$. In the topological context R is a

group ring $R = \mathbb{Z}[\pi] = \begin{cases} \mathbb{Z}[G_1 *_H G_2] \\ \mathbb{Z}[G *_H \{t\}] \end{cases}$, and torsion is measured in

the Whitehead group $Wh(\pi)$, with the corresponding modification

in the definition of finite MV presentation.

An MV presentation of an R-module chain complex C

$$\begin{cases} 0 \longrightarrow k_!E \xrightarrow{q} j_{1!}D_1 \oplus j_{2!}D_2 \xrightarrow{p} C \longrightarrow 0 \\ 0 \longrightarrow k_{1!}E \xrightarrow{q} j_!D \xrightarrow{p} C \longrightarrow 0 \end{cases}$$

is defined in the same way as for an R-module, but using chain complexes and chain maps, with $\begin{cases} D_1, D_2 \\ D \end{cases}$, E an $\begin{cases} A_1-, A_2- \\ A- \end{cases}$, B-module chain complex. The MV presentation is finite if all the chain complexes are finite and the corresponding exact sequence of finite R-module chain complexes has torsion $\tau = 0 \in K_1(R)$, by definition. In particular, if (X,Y) is a biconnective pair of type $\begin{cases} A) \\ B) \end{cases}$ and $R = \mathbb{Z}[\pi_1(X)]$ the geometric Mayer-Vietoris presentation of the R-module chain complex $C(\tilde{X})$ is an MV presentation

$$\begin{cases} 0 \longrightarrow k_! C(\tilde{Y}) \xrightarrow{\;q\;} j_{1!}C(\tilde{Z}_1) \oplus j_{2!}C(\tilde{Z}_2) \xrightarrow{\;p\;} C(\tilde{X}) \longrightarrow 0 \\ 0 \longrightarrow k_{1!}C(\tilde{Y}) \xrightarrow{\;q\;} j_! C(\tilde{Z}) \xrightarrow{\;p\;} C(\tilde{X}) \longrightarrow 0 \, . \end{cases}$$

For finite (X,Y) this is an exact sequence of finite R-module chain complexes with torsion

$$\begin{cases} \tau(X(Z_1,Z_2) = Z_1 \cup_{i_1} Y \times [0,1]' \cup_{i_2} Z_2 \longrightarrow X = Z_1 \cup_Y Z_2) = 0 \in Wh(\pi_1(X)) \\ \tau(X(Z) = Z \cup_{i_1 \cup i_2} Y \times [0,1]' \longrightarrow X = Z/(i_1(Y) = i_2(Y)) = 0 \in Wh(\pi_1(X)) \, , \end{cases}$$

so that the geometric MV presentation is finite.

Waldhausen [ ] showed that every finite R-module chain complex admits a finite MV presentation, and this was the first step in the splitting theorem for the Whitehead group of a generalized free product. In order to make the analogous first step for the L-groups of a generalized free product it is necessary to consider finite MV presentations with extra structure. We shall now associate to every R-module chain complex a <u>canonical infinite MV presentation</u>: we shall require finite MV presentations to be embedded in this one. We shall show that the methods of [ ] actually supply enough such finite MV presentations for the purposes of L-theory.

Given a morphism of rings $i:B \longrightarrow A$ and an A-module M let $i^!M$ denote the B-module with the same $\mathbb{Z}$-module structure and B acting by

$$B \times i^!M \longrightarrow i^!M \; ; \; (b,x) \longmapsto i(b)x \; .$$

The <u>restriction</u> functor

$$i^! \; : \; (A\text{-modules}) \longrightarrow (B\text{-modules}) \; ; \; M \longmapsto i^!M$$

is adjoint to the induction functor

$$i_! \; : \; (B\text{-modules}) \longrightarrow (A\text{-modules}) \; ; \; M \longmapsto i_!M = A \otimes_B M \; ,$$

meaning that for any A-module M and any B-module N there is defined a natural $\mathbb{Z}$-module isomorphism

$$\mathrm{Hom}_B(N, i^!M) \xrightarrow{\sim} \mathrm{Hom}_A(i_!N, M) \; ; \; f \longmapsto (a \otimes x \longmapsto af(x)) \; .$$

The adjoint of $1 \in \mathrm{Hom}_B(i^!M, i^!M)$ is the A-module morphism

$$p_i \; : \; i_!i^!M \longrightarrow M \; ; \; a \otimes x \longmapsto ax \; .$$

If $i:B \longrightarrow A$ is a pure injection and M is a based free A-module then $i^!M$ is a based free B-module with

$$[i^!M:B] = \{ax \,|\, a \in [A:B], x \in [M:A]\} \; .$$

Given a commutative triangle of rings

$$\begin{array}{ccc} B & \xrightarrow{\;k\;} & R \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle j} & \\ & A & \end{array}$$

and an R-module M there is defined a commutative triangle of R-modules

$$\begin{array}{ccc} k_!k^!M & \xrightarrow{\;p_k\;} & M \\ & {\scriptstyle q_i}\searrow \quad \nearrow{\scriptstyle p_j} & \\ & j_!j^!M & \end{array}$$

with

$$q_i \; : \; k_!k^!M \longrightarrow j_!j^!M \; ; \; r \otimes x \longmapsto r \otimes x \qquad (r \in R, x \in M)$$

or equivalently

$$q_i = j_!p_i \; : \; k_!k^!M = j_!i_!i^!j^!M \longrightarrow j_!j^!M \; .$$

Proposition 3.1 Let $R = \begin{cases} A_1 *_B A_2 \\ A*_B[t,t^{-1}] \end{cases}$ be a generalized free product ring. Every R-module M admits an MV presentation, the universal MV presentation

$$M(T) : \begin{cases} 0 \longrightarrow k_! k^! M \xrightarrow{q = \begin{pmatrix} q_{i_1} \\ -q_{i_2} \end{pmatrix}} j_{1!} j_1^! M \oplus j_{2!} j_2^! M \xrightarrow{p = (p_{j_1} \ p_{j_2})} M \longrightarrow 0 \\ 0 \longrightarrow k_{1!} k_2^! M \xrightarrow{q = q_1 - tq_2} j_! j^! M \xrightarrow{p = p_j} M \longrightarrow 0 \\ \text{with } q_1 = q_{i_1} t : r \otimes x \longmapsto r \otimes tx,\ tq_2 = q_{i_2} t : r \otimes x \longmapsto rt \otimes x \ (r \in R,\ x \in M). \end{cases}$$

Proof: This result was first obtained using general ring-theoretic methods by Dicks $\begin{cases} [\quad] \\ [\quad] \end{cases}$, for $\begin{cases} \text{amalgamated free products} \\ \text{HNN extensions} \end{cases}$ rings determined by arbitrary ring morphisms not just pure injections. A proof specific to the case of pure injections will be given in Proposition 3.2 below.

It is sufficient to verify exactness in the special case M = R, since the universal MV presentation of R consists of (R,R)-bimodules which are free as right R-modules and for any R-module M

(the universal MV presentation of M)

$\qquad$ = (the universal MV presentation of R)$\otimes_R M$.

[]

For a generalized free product of group rings

$$R = \mathbb{Z}[\pi] = \begin{cases} \mathbb{Z}[G_1 *_H G_2] \\ \mathbb{Z}[G*_H\{t\}] \end{cases}$$

it is possible to verify the exactness of the universal MV presentation of an R-module M by identifying it with $S(T)\otimes_{\mathbb{Z}} M$, where $S(T)$ is the augmented chain complex of the oriented tree T

$$S(T) : 0 \longrightarrow S_1(T) \xrightarrow{\partial} S_0(T) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 ,$$

with

$$\partial : S_1(T) = \mathbb{Z}[T^{(1)}] \longrightarrow S_0(T) = \mathbb{Z}[T^{(0)}] \ ; \ s \longmapsto r_1 - r_2$$

$$\varepsilon : S_0(T) = \mathbb{Z}[T^{(0)}] \longrightarrow \mathbb{Z} \ ; \ r \longmapsto 1 .$$

Given a $\mathbb{Z}[\pi]$-module M define a $\mathbb{Z}[\pi]$-action on $S(T)\otimes_{\mathbb{Z}}M$ by

$$\mathbb{Z}[\pi] \times S(T)\otimes_{\mathbb{Z}}M \longrightarrow S(T)\otimes_{\mathbb{Z}}M \ ; \ (g, r\otimes x) \longmapsto rg^{-1}\otimes gx .$$

There is then defined an isomorphism of exact sequences of $\mathbb{Z}[\pi]$-modules

$$f : S(T)\otimes_{\mathbb{Z}}M \xrightarrow{\sim} M(T)$$

by

$$\begin{array}{ccccccccc}
S(T)\otimes_{\mathbb{Z}}M : & 0 & \longrightarrow & S_1(T)\otimes_{\mathbb{Z}}M & \xrightarrow{\partial\otimes 1} & S_0(T)\otimes_{\mathbb{Z}}M & \xrightarrow{\varepsilon\otimes 1} & M & \longrightarrow 0 \\
f \downarrow \wr & & & f_1 \downarrow \wr & & f_0 \downarrow \wr & & \| & \\
M(T) : & 0 & \longrightarrow & \left\{\begin{array}{l} k_! k^! M \\ k_{1!} k_2^! M \end{array}\right. & \xrightarrow{q} & \left\{\begin{array}{l} j_{1!} j_1^! M \oplus j_{2!} j_2^! M \\ j_! j^! M \end{array}\right. & \xrightarrow{p} & M & \longrightarrow 0
\end{array}$$

with the $\mathbb{Z}[\pi]$-module isomorphisms f_0, f_1 given by

$$\left\{\begin{array}{l}
f_0 : S_0(T)\otimes_{\mathbb{Z}}M = \mathbb{Z}[T_1^{(0)}]\otimes_{\mathbb{Z}}M \oplus \mathbb{Z}[T_2^{(0)}]\otimes_{\mathbb{Z}}M \xrightarrow{\sim} j_{1!}j_1^!M \oplus j_{2!}j_2^!M \ ; \\
\qquad\qquad (r_1\otimes x_1, r_2\otimes x_2) \longmapsto (r_1^{-1}\otimes r_1x, r_2^{-1}\otimes r_2x) \\
f_0 : S_0(T)\otimes_{\mathbb{Z}}M \xrightarrow{\sim} j_!j^!M \ ; \ r\otimes x \longmapsto r^{-1}\otimes rx ,
\end{array}\right.$$

$$\left\{\begin{array}{l}
f_1 : S_1(T)\otimes_{\mathbb{Z}}M \xrightarrow{\sim} k_!k^!M \ ; \ s\otimes y \longmapsto s^{-1}\otimes sy \\
f_1 : S_1(T)\otimes_{\mathbb{Z}}M \xrightarrow{\sim} k_{1!}k_2^!M \ ; \ s\otimes y \longmapsto s^{-1}\otimes t^{-1}sy .
\end{array}\right.$$

The universal property of the universal MV presentation is that given any MV presentation of an R-module M

$$\left\{\begin{array}{l} 0 \longrightarrow k_!Q \xrightarrow{q=\binom{q_1}{-q_2}} j_{1!}P_1 \oplus j_{2!}P_2 \xrightarrow{p=(p_1\ p_2)} M \longrightarrow 0 \\ 0 \longrightarrow k_{1!}Q \xrightarrow{q=q_1-tq_2} j_!P \xrightarrow{p} M \longrightarrow 0 \end{array}\right.$$

and any R-module morphism $f \in \mathrm{Hom}_R(M,N)$ there is a unique extension of f to a morphism of MV presentations

$$\left\{\begin{array}{ccccccccc} 0 & \longrightarrow & k_!Q & \xrightarrow{q} & j_{1!}P_1 \oplus j_{2!}P_2 & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle k_!h} & & \downarrow{\scriptstyle j_{1!}g_1 \oplus j_{2!}g_2} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & k_!k^!N & \xrightarrow{q} & j_{1!}j_1^!N \oplus j_{2!}j_2^!N & \xrightarrow{p} & N & \longrightarrow & 0 \\ 0 & \longrightarrow & k_{1!}Q & \xrightarrow{q} & j_!P & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle k_{1!}t^{-1}h} & & \downarrow{\scriptstyle j_!g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & k_{1!}k_2^!N & \xrightarrow{q} & j_!j^!N & \xrightarrow{p} & N & \longrightarrow & 0 \end{array}\right.$$

with $\left\{\begin{array}{l} g_1 \in \mathrm{Hom}_{A_1}(P_1, j_1^!N),\ g_2 \in \mathrm{Hom}_{A_2}(P_2, j_2^!N),\ h \in \mathrm{Hom}_B(Q, k^!N) \\ g \in \mathrm{Hom}_A(P, j^!N),\ h \in \mathrm{Hom}_B(Q, k_1^!N) \end{array}\right.$

the adjoints of $\left\{\begin{array}{l} fp_1 \in \mathrm{Hom}_R(j_{1!}P_1, N),\ fp_2 \in \mathrm{Hom}_R(j_{2!}P_2, N),\ fpq_1 \in \mathrm{Hom}_R(k_!Q, N) \\ fp \in \mathrm{Hom}_R(j_!P, N),\ fpq_1 \in \mathrm{Hom}_R(k_{1!}Q, N) \end{array}\right.$

respectively. In particular, this gives the functoriality of the universal MV presentation: a morphism of R-modules $f : M \longrightarrow N$ induces a morphism of the universal MV presentations

$$\left\{\begin{array}{ccccccccc} 0 & \longrightarrow & k_!k^!M & \xrightarrow{q} & j_{1!}j_1^!M \oplus j_{2!}j_2^!M & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle k_!k^!f} & & \downarrow{\scriptstyle j_{1!}j_1^!f \oplus j_{2!}j_2^!f} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & k_!k^!N & \xrightarrow{q} & j_{1!}j_1^!N \oplus j_{2!}j_2^!N & \xrightarrow{p} & N & \longrightarrow & 0 \\ 0 & \longrightarrow & k_{1!}k_2^!M & \xrightarrow{q} & j_!j^!M & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle k_{1!}k_2^!f} & & \downarrow{\scriptstyle j_!j^!f} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & k_{1!}k_2^!N & \xrightarrow{q} & j_!j^!N & \xrightarrow{p} & N & \longrightarrow & 0 \end{array}\right.$$

.

An MV presentation of an R-module M is <u>subuniversal</u> if the adjoints of the R-module morphisms

$$\begin{cases} p_1 \in \mathrm{Hom}_R(j_{1!}P_1,M),\ p_2 \in \mathrm{Hom}_R(j_{2!}P_2,M),\ p_1q_1 = p_2q_2 \in \mathrm{Hom}_R(k_!Q,M) \\ p \in \mathrm{Hom}_R(j_!P,M),\ pq_1 = ptq_2 \in \mathrm{Hom}_R(k_{1!}Q,M) \end{cases}$$

are injections

$$\begin{cases} g_1 \in \mathrm{Hom}_{A_1}(P_1,j_1^!M),\ g_2 \in \mathrm{Hom}_{A_2}(P_2,j_2^!M),\ h \in \mathrm{Hom}_B(Q,k^!M) \\ g \in \mathrm{Hom}_A(P,j^!M),\ h \in \mathrm{Hom}_B(Q,k_1^!M)\ . \end{cases}$$

Using these injections as identifications note that $\begin{cases} P_1,P_2 \\ P \end{cases},Q$ are additive subgroups of M such that

$$\begin{cases} A_1P_1 = P_1,\ A_2P_2 = P_2,\ BQ = Q,\ RP_1 + RP_2 = M,\ P_1 \cap P_2 = Q \\ AP = P,\ BQ = Q,\ RP = M,\ P \cap tP = Q. \end{cases}$$

A subuniversal MV presentation has a canonical embedding in the universal MV presentation

$$\begin{cases} \begin{array}{ccccccccc} 0 & \longrightarrow & k_!Q & \xrightarrow{q=\begin{pmatrix} q_1 \\ -q_2 \end{pmatrix}} & j_{1!}P_1 \oplus j_{2!}P_2 & \xrightarrow{p=(p_1\ p_2)} & M & \longrightarrow & 0 \\ & & \downarrow k_!h & & \downarrow j_{1!}g_1 \oplus j_{2!}g_2 & & \Vert & & \\ 0 & \longrightarrow & k_!k^!M & \xrightarrow{q} & j_{1!}j_1^!M \oplus j_{2!}j_2^!M & \xrightarrow{p} & M & \longrightarrow & 0 \end{array} \\ \\ \begin{array}{ccccccccc} 0 & \longrightarrow & k_{1!}Q & \xrightarrow{q=q_1-tq_2} & j_!P & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow k_{1!}t^{-1}h & & \downarrow j_!g & & \Vert & & \\ 0 & \longrightarrow & k_{1!}k_2^!M & \xrightarrow{q} & j_!j^!M & \xrightarrow{p} & M & \longrightarrow & 0 \end{array} \end{cases} .$$

Conversely, any MV presentation with an embedding in the universal MV presentation is subuniversal. The subuniversal MV presentations of a fixed R-module M are partially ordered by inclusion, with maximal element the universal MV presentation. In dealing with finite subuniversal MV presentations of a based f.g. free R-module M it is assumed that the bases are such that

$$\begin{cases} g_1[P_1:A_1] \subset [j_1^!M:A_1],\ g_2[P_2:A_2] \subset [j_2^!M:A_2],\ h[Q:B] \subset [k^!M:B] \\ g[P:A] \subset [j^!M:A],\ t^{-1}h[Q:B] \subset [k_2^!M:B] = \{t^{-1}stx \mid s \in [R:B],\ x \in [M:R]\}\ . \end{cases}$$

A subtree $\Delta \subseteq T$ is <u>based</u> if it contains $\begin{cases} \text{either } 1 \in T_1^{(0)} \text{ or } 1 \in T_2^{(0)} \\ 1 \in T^{(0)} \end{cases}$. The based subtrees $\Delta \subseteq T$ are partially ordered by inclusion, with maximal element T.

A <u>fundamental domain</u> Δ for a based free R-module M is a collection of based subtrees of T indexed by the base [M:R]

$$\Delta = \{\Delta(x) \subseteq T \mid x \in [M:R]\} .$$

The set of fundamental domains for M is partially ordered by

$$\Delta \subseteq \Delta' \text{ if } \Delta(x) \subseteq \Delta'(x) \text{ for each } x \in [M:R] .$$

The maximal element $\Delta = \{\Delta(x) = T \mid x \in [M:R]\}$ is denoted by $\Delta = T$. A fundamental domain Δ is <u>finite</u> if [M:R] is finite and each $\Delta(x) \subseteq T$ is a finite based subtree.

Given a based free R-module M (over any ring R) and $x \in M$ let $<x,y> \in R$ be the coefficients of the base elements $y \in [M:R]$ in the expression of x as an R-linear combination

$$x = \sum_{y \in [M:R]} <x,y> y \in M = \sum_{y \in [M:R]} Ry ,$$

so that $\{y \in [M:R] \mid <x,y> \neq 0 \in R\}$ is finite.

<u>Proposition 3.2</u> Let $R = \begin{cases} A_1 *_B A_2 \\ A *_B [t,t^{-1}] \end{cases}$ be a generalized free product ring, and let M be a based free R-module.

i) Given a fundamental domain Δ for M there is defined a subuniversal MV presentation of M

$$M(\Delta) : \begin{cases} 0 \longrightarrow k_! M(\Delta^{(1)}) \xrightarrow{q} j_{1!} M(\Delta_1^{(0)}) \oplus j_{2!} M(\Delta_2^{(0)}) \xrightarrow{p} M \longrightarrow 0 \\ 0 \longrightarrow k_{1!} M(\Delta^{(1)}) \xrightarrow{q} j_! M(\Delta^{(0)}) \xrightarrow{p} M \longrightarrow 0 \end{cases}$$

with $\begin{cases} M(\Delta_1^{(0)}), M(\Delta_2^{(0)}) \\ M(\Delta^{(0)}) \end{cases}$, $M(\Delta^{(1)})$ the based free $\begin{cases} A_1\text{-}, A_2\text{-} \\ A\text{-} \end{cases}$, B-module given by

$$\begin{cases} M(\Delta_\lambda^{(0)}) = \sum_{x\in[M:R]} \sum_{r_\lambda \in \Delta(x)_\lambda^{(0)}} A_\lambda r_\lambda x & (\lambda = 1,2) \\ M(\Delta^{(0)}) = \sum_{x\in[M:R]} \sum_{r\in\Delta(x)^{(0)}} Arx , \end{cases}$$

$$\begin{cases} M(\Delta^{(1)}) = \sum_{y\in[M:R]} \sum_{s\in\Delta(y)^{(1)}} Bsy \\ M(\Delta^{(1)}) = \sum_{y\in[M:R]} \sum_{s\in\Delta(y)^{(1)}} t^{-1}Bsy . \end{cases}$$

If $\Delta\subseteq\Delta'$ then $M(\Delta)\subseteq M(\Delta')$, with $M(T)$ the universal MV presentation of M. If Δ is finite then $M(\Delta)$ is a finite subuniversal presentation of M.

ii) Given another based free R-module N, an R-module morphism $f\in \mathrm{Hom}_R(M,N)$ and a fundamental domain Δ for M there is defined a fundamental domain $f_*\Delta$ for N such that the induced morphism of universal MV presentations

$$f : M(T) \longrightarrow N(T)$$

restricts to a morphism of subuniversal MV presentations

$$f : M(\Delta) \longrightarrow N(\Gamma)$$

if and only if the fundamental domain Γ for N is such that $f_*\Delta\subseteq\Gamma$. If Δ and $[N:R]$ are finite then so is $f_*\Delta$.

Proof: i) Since $M(\Delta) = \sum_{x\in[M:R]} R(\Delta(x))x$ it is sufficient to prove the exactness of $M(\Delta)$ in the special case $M = R$, with $[M:R] = \{1\}$ and $\Delta = \{\Delta(1)\}$ consisting of a single based subtree $\Delta(1)\subseteq T$. Denote $\Delta(1)$ by Δ.

If Δ consists of a single vertex $R(\Delta)$ is the exact sequence

$$R(\Delta) : 0 \longrightarrow 0 \longrightarrow R \xrightarrow{1} R \longrightarrow 0 .$$

If $\Delta\subset T$ is obtained from $\Delta'\subset\Delta$ by adjoining a single segment $s\in\Delta^{(1)}$ with vertex $r\notin\Delta'^{(0)}$ then $R(\Delta')$ embeds in $R(\Delta)$ with quotient the exact sequence

$$R(\Delta)/R(\Delta') : 0 \longrightarrow Rs \longrightarrow Rr \longrightarrow 0 \longrightarrow 0 ,$$

with $s\longmapsto \pm r$. This gives the inductive step in proving that $R(\Delta)$

is a finite subuniversal MV presentation of R for every finite based subtree $\Delta \subset T$. An infinite based subtree $\Delta \subsetneq T$ is the union of all the finite based subtrees $\Delta' \subset \Delta$, so that $R(\Delta)$ is the union of all the $R(\Delta')$, and the exactness of $R(\Delta)$ follows from the finite case.

ii) Define $f_*\Delta(y) \subseteq T$ $(y \in [N:R])$ to be the intersection of all the based subtrees containing the set of vertices

$$\begin{cases} \{v_\lambda \in T_\lambda^{(0)} \mid \langle r_\lambda f(x), v_\lambda y\rangle \neq 0 \in A_\lambda \text{ for some } x \in [M:R], r_\lambda \in \Delta(x)_\lambda^{(0)}, \lambda \in \{1,2\}\} \\ \{v \in T^{(0)} \mid \langle rf(x), vy\rangle \neq 0 \in A \text{ for some } x \in [M:R], r \in \Delta(x)^{(0)}\}. \end{cases}$$

[]

The functoriality of the universal MV presentation of an R-module shows that every R-module chain complex C has a universal MV presentation

$$C(T) : \begin{cases} 0 \longrightarrow k_!k^!C \xrightarrow{q} j_{1!}j_1^!C \oplus j_{2!}j_2^!C \xrightarrow{p} C \longrightarrow 0 \\ 0 \longrightarrow k_{1!}k_2^!C \xrightarrow{q} j_!j^!C \xrightarrow{p} C \longrightarrow 0 \end{cases}$$

which in degree m is just the universal MV presentation of the R-module C_m. Subuniversal MV presentations of R-module chain complexes are defined in the same way as for R-modules, and have the same general properties. In particular, if $\tilde{X}$ is a regular cover of a CW complex X with group $\pi = \begin{cases} G_1 *_H G_2 \\ G*_H\{t\} \end{cases}$ and $\begin{cases} (W_1, W_2) \\ W \end{cases}$ is a fundamental domain then the geometric MV presentation of $C(\tilde{X})$ is subuniversal, with the geometrically induced injections

$$\begin{cases} g_1 : C(W_1) \longrightarrow j_1^!C(\tilde{X}),\ g_2 : C(W_2) \longrightarrow j_2^!C(\tilde{X}),\ h : C(W_1 \cap W_2) \longrightarrow k^!C(\tilde{X}) \\ g : C(W) \longrightarrow j^!C(\tilde{X}),\ h : C(W \cap tW) \longrightarrow k_1^!C(\tilde{X}) \end{cases}$$

defining an embedding in the universal MV presentation of $C(\tilde{X})$

$$\begin{cases} \begin{array}{ccccccccc} 0 & \longrightarrow & k_!C(W_1 \cap W_2) & \xrightarrow{q} & j_{1!}C(W_1) \oplus j_{2!}C(W_2) & \xrightarrow{p} & C(\tilde{X}) & \longrightarrow & 0 \\ & & \downarrow k_!h & & \downarrow j_{1!}g_1 \oplus j_{2!}g_2 & & \| & & \\ 0 & \longrightarrow & k_!k^!C(\tilde{X}) & \xrightarrow{q} & j_{1!}j_1^!C(\tilde{X}) \oplus j_{2!}j_2^!C(\tilde{X}) & \xrightarrow{p} & C(\tilde{X}) & \longrightarrow & 0 \\ 0 & \longrightarrow & k_{1!}C(W \cap tW) & \xrightarrow{q} & j_!C(W) & \xrightarrow{p} & C(\tilde{X}) & \longrightarrow & 0 \\ & & \downarrow k_!t^{-1}h & & \downarrow j_!g & & \| & & \\ 0 & \longrightarrow & k_{1!}k_2^!C(\tilde{X}) & \xrightarrow{q} & j_!j^!C(\tilde{X}) & \xrightarrow{p} & C(\tilde{X}) & \longrightarrow & 0 \end{array} \end{cases}$$

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A <u>fundamental domain</u> Δ for a based free R-module chain complex

$$C : \ldots \longrightarrow C_{m+1} \xrightarrow{d_{m+1}} C_m \xrightarrow{d_m} C_{m-1} \longrightarrow \ldots \qquad (m \in \mathbb{Z})$$

is a set $\{\Delta_m | m \in \mathbb{Z}\}$ of fundamental domains $\Delta_m = \{\Delta_m(x) | x \in [C_m : R]\}$ for the based free R-modules C_m such that

$$(d_m)_*(\Delta_m) \subseteq \Delta_{m-1} \quad (m \in \mathbb{Z}) \ .$$

The fundamental domains for C are partially ordered by

$$\Delta \subseteq \Delta' \text{ if } \Delta_m \subseteq \Delta'_m \text{ for all } m \in \mathbb{Z} \ ,$$

and the maximal element $\Delta = \{\Delta_m = T \mid m \in \mathbb{Z}\}$ is denoted $\Delta = T$. A fundamental domain Δ is <u>finite</u> if C is finite and each Δ_m is finite.

<u>Proposition 3.3</u> Let $R = \begin{cases} A_1 *_B A_2 \\ A *_B [t, t^{-1}] \end{cases}$ be a generalized free product ring, and let C be a based free R-module chain complex.

i) Given a fundamental domain Δ for C there is defined a subuniversal MV presentation $C(\Delta)$ of C, given in degree m by

$$C(\Delta)_m = C_m(\Delta_m) \ .$$

If $\Delta \subseteq \Delta'$ then $C(\Delta) \subseteq C(\Delta')$, and $C(T)$ is the universal MV presentation.

ii) If C is finite there exists a finite fundamental domain Δ, in which case $C(\Delta)$ is a finite subuniversal MV presentation of C.

<u>Proof</u>: i) Immediate from Proposition 3.2.

ii) Let $n \geqslant 0$ be such that $C_m = 0$ for $m > n$. Starting with any finite fundamental domain Δ_n for C_n let Δ_m $(0 \leqslant m \leqslant n-1)$ be the finite fundamental domain for C_m defined inductively by

$$\Delta_m = (d_{m+1})_* \Delta_{m+1} \ .$$

Then $\Delta = \{\Delta_m | 0 \leqslant m \leqslant n\}$ is a finite fundamental domain for C.

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Let X be a CW complex, and let $\widetilde{X}$ be a regular cover of X with group $\pi = \begin{cases} G_1 *_H G_2 \\ G *_H \{t\} \end{cases}$. Given a subfundamental domain $\begin{cases} (W_1, W_2) \\ W \end{cases}$ for $\widetilde{X}$ is possible to choose for each cell $e \subset X$ a lift to a cell $\tilde{e} \subset \widetilde{X}$ such that $\begin{cases} \text{either } \tilde{e} \subset W_1 \text{ or } \tilde{e} \subset W_2 \\ \tilde{e} \subset W \end{cases}$, so that $\Delta(\tilde{e}) \subset T$ is a based subtree. These choices determine a $\mathbb{Z}[\pi]$-module base for $C(\widetilde{X})$

$$[C(\widetilde{X}) : \mathbb{Z}[\pi]] = \{\tilde{e} \mid e \subset X\}$$

and a fundamental domain for $C(\widetilde{X})$

$$\Delta = \{\Delta(\tilde{e}) \mid e \subset X\}$$

such that the corresponding subuniversal MV presentation of $C(\widetilde{X})$ may be expressed as

$$C(\widetilde{X})(\Delta) \ : \begin{cases} 0 \longrightarrow k_! C(W_1 \cap W_2) \xrightarrow{\ q\ } j_{1!} C(W_1) \oplus j_{2!} C(W_2) \xrightarrow{\ p\ } C(\widetilde{X}) \longrightarrow 0 \\ 0 \longrightarrow k_{1!} C(W \cap tW) \xrightarrow{\ q\ } j_! C(W) \xrightarrow{\ p\ } C(\widetilde{X}) \longrightarrow 0 \ . \end{cases}$$

(For a fundamental domain $\begin{cases} (W_1, W_2) \\ W \end{cases}$ this is just the geometric MV presentation of $C(\widetilde{X})$). Let $(X', Y') = (T' \times_\pi \widetilde{X}, \widetilde{X}/H)$ be the universal bipolar pair and let

$$(X'', Y'') = \begin{cases} (X(W_1, W_2), Y(W_1, W_2)) \\ (X(W), Y(W)) \end{cases} \subset (X', Y')$$

be the subuniversal bipolar pair, as constructed at the end of §2. The projection $\widetilde{X}' = T' \times \widetilde{X} \longrightarrow \widetilde{X}$ is a π-equivariant homotopy equivalence inducing a chain equivalence (i.e. a morphism consisting of chain equivalences) between the geometric MV presentation of $C(\widetilde{X}')$ and the universal MV presentation $C(\widetilde{X})(T)$ of $C(\widetilde{X})$. The composite $\widetilde{X}'' \longrightarrow \widetilde{X}' \longrightarrow \widetilde{X}$ is a π-equivariant homotopy equivalence inducing a chain equivalence between the geometric MV presentation of $C(\widetilde{X}'')$ and the subuniversal MV presentation $C(\widetilde{X})(\Delta)$ of $C(\widetilde{X})$.