ON MICHEL KERVAIRE'S WORK IN SURGERY AND KNOT THEORY

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http://www.maths.ed.ac.uk/~aar/slides/kervaire.pdf

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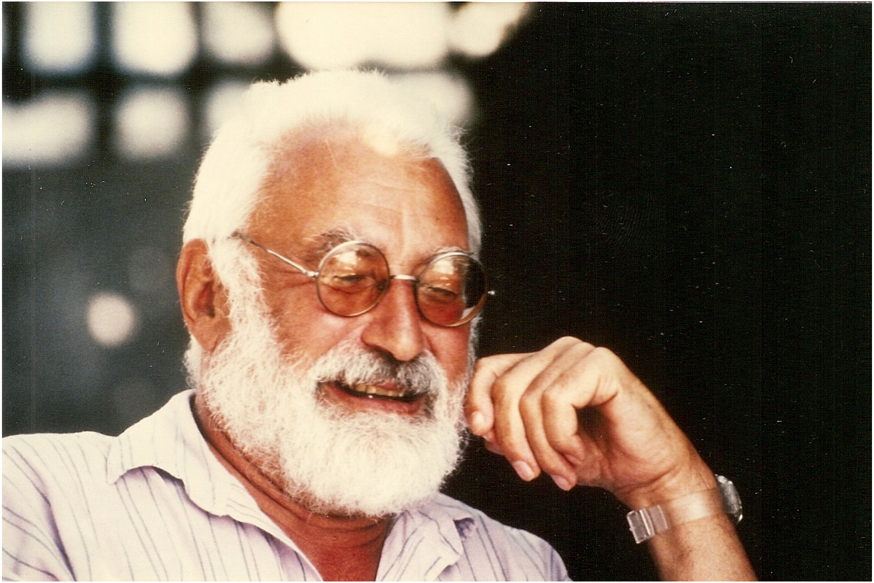

1927 – 2007

Highlights

- Major contributions to the topology of manifolds of dimension $\geqslant 5$.
- Main theme: connection between stable trivializations of vector bundles and quadratic refinements of symmetric forms. 'Division by 2'.
- 1956 Curvatura integra of an m-dimensional framed manifold
  $= $ Kervaire semicharacteristic $+$ Hopf invariant.
- 1960 The Kervaire invariant of a $(4k+2)$-dimensional framed manifold.
- 1960 The 10-dimensional Kervaire manifold without differentiable structure.
- 1963 The Kervaire-Milnor classification of exotic spheres in dimensions $\geqslant 4$: the birth of surgery theory.
- 1965 The foundation of high dimensional knot theory, for $S^n \subset S^{n+2}$ with $n \geqslant 2$.

MATHEMATICAL REVIEWS + 1

Kervaire was the author of 66 papers listed (1954 – 2007)

+1 unlisted : **Non-parallelizability of the n-sphere for $n > 7$,** Proc. Nat. Acad. Sci. 44, 280–283 (1958)

619 matches for "Kervaire" anywhere, of which 84 in title.

18,600 Google hits for "Kervaire".

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MR0058972 (15,458c) Kervaire, Michel Extension d'un théorème de G. de Rham et expression de l'invariant de Hopf par une intégrale. (French) *C. R. Acad. Sci. Paris* **237,** (1953). 1486--1488. (Reviewer: S. Chern) 56.0X

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The *curvatura integra*

- **Convention** Unless specified otherwise

  manifold = compact connected oriented closed differentiable manifold.

- **Definition** The **curvatura integra** of a codimension 1 immersion of an m-dimensional manifold $M^m \looparrowright \mathbb{R}^{m+1}$ is the degree of the Gauss map

$$c \;:\; M \to V_{m+1,1} = S^m \;;\; x \mapsto \text{unit normal to } x \text{ in } S^m \subset \mathbb{R}^{m+1}\;.$$

- **Theorem** (Gauss-Bonnet, 19th century) For $m = 2$

$$\text{degree}(c) \;=\; \frac{1}{4\pi}\int_M \kappa \;=\; \chi(M)/2 \in \mathbb{Z}$$

  with κ the Gaussian curvature and $\chi(M) \in \mathbb{Z}$ the Euler characteristic. The original division by 2.

- **Theorem** (Hopf, 1925) For any even $m \geqslant 2$

$$\text{degree}(c) \;=\; \chi(M)/2 \in \mathbb{Z}\;.$$

- Is there an analogue of the curvatura integra theorem for more general m-dimensional manifolds?

The Stiefel spaces $V_{m+k,k}$ I.

- **Definition** The **Stiefel space** is

$$\begin{aligned} V_{m+k,k} &= \{\text{orthonormal } k\text{-frames in } \mathbb{R}^{m+k}\} \\ &= \{\text{isometries } u : \mathbb{R}^k \hookrightarrow \mathbb{R}^{m+k}\} \ = \ O(m+k)/O(m) \ . \end{aligned}$$

- **Theorem** (Stiefel, 1935) $V_{m+k,k}$ is $(m-1)$-connected, with

$$H_m(V_{m+k,k}) \ = \ \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 (\text{mod } 2) \text{ or if } k = 1 \\ \mathbb{Z}_2 & \text{if } m \equiv 1 (\text{mod } 2) \text{ and } k > 1 \ . \end{cases}$$

- **Definition** The **Grassmann space** is

$$\begin{aligned} G_{m+k,k} &= V_{m+k,k}/O(k) \\ &= \{k\text{-dimensional subspaces } U = u(\mathbb{R}^k) \subseteq \mathbb{R}^{m+k}\} \ . \end{aligned}$$

  The **canonical** m**-plane bundle** η over $G_{m+k,k}$ is

$$E(\eta) \ = \ \{(U \subseteq \mathbb{R}^{m+k}, v) \,|\, v \in U^{\perp} \subseteq \mathbb{R}^{m+k}\} \ .$$

The Stiefel spaces $V_{m+k,k}$ II.

- **Theorem** (Steenrod, 1950) The classifying space for m-plane bundles is
$$BO(m) = \varinjlim_k G_{m+k,k} .$$
The map $p : V_{m+k,k} \to G_{m+k,k} \subset BO(m)$ fits into a fibration
$$V_{m+k,k} = O(m+k)/O(m) \xrightarrow{p} BO(m) \xrightarrow{\oplus \epsilon^k} BO(m+k) .$$
Also stable classifying space $BO = \varinjlim_m BO(m)$.
- **Definition** The **canonical m-plane bundle** $\theta = p^*\eta$ over $V_{m+k,k}$
$$E(\theta) = \{(u,v) \,|\, (u : \mathbb{R}^k \hookrightarrow \mathbb{R}^{m+k}) \in V_{m+k,k} \, , \, v \in u(\mathbb{R}^k)^{\perp}\}$$
has the **canonical k-stable trivialization** $\delta\theta : \theta \oplus \epsilon^k \cong \epsilon^{m+k}$ given by
$$u(\mathbb{R}^k)^{\perp} \oplus u(\mathbb{R}^k) = \mathbb{R}^{m+k} .$$
- For a CW complex X
$$[X, BO(m)] = \{m\text{-plane bundles } \xi \text{ over } X\} ,$$
$$[X, V_{m+k,k}] = \{\xi \text{ with a } k\text{-stable trivialization } \delta\xi : \xi \oplus \epsilon^k \cong \epsilon^{m+k}\} .$$

Framed manifolds

- **Definition** A manifold M^m is **framed** if there is given an embedding $f : M^m \hookrightarrow \mathbb{R}^{m+k}$ with trivialized normal bundle $\nu_f \cong \epsilon^k$, so that

$$M \times \mathbb{R}^k \subset \mathbb{R}^{m+k} .$$

  The tangent bundle τ_M is k-stably trivialized, $\tau_M \oplus \epsilon^k \cong \epsilon^{m+k}$.
- **Definition** The **Pontryagin-Thom** map of framed M

$$F \ : \ S^{m+k} \to S^{m+k}/(S^{m+k} \backslash (M \times \mathbb{R}^k)) \ = \ \Sigma^k M^+ \to S^k$$

  is such that $F^{-1}(*) = M \subset S^{m+k}$, with $M^+ = M \cup \{\text{pt.}\}$.
- **Theorem** (P-T, 1954) The framed cobordism groups Ω_*^{fr} are related to the stable homotopy groups of spheres π_*^S by the isomorphism

$$\Omega_m^{fr} \xrightarrow{\cong} \pi_m^S \ = \ \varinjlim_k \pi_{m+k}(S^k) \ ; \ M^m \mapsto F \ .$$

- **Theorem** (Serre, 1951) The groups π_*^S are finite.

Kervaire's first thesis I.

- (1955) ETH Zürich, under supervision of Hopf.
- The **generalized Gauss map** of an m-dimensional framed manifold M^m with $f : M^m \hookrightarrow \mathbb{R}^{m+k}$, $\nu_f \cong \epsilon^k$

$$c \ : \ M \to V_{m+k,k} \ ; \ x \mapsto ((\nu_f)_x = \mathbb{R}^k \hookrightarrow (\tau_{\mathbb{R}^{m+k}})_{f(x)} = \mathbb{R}^{m+k})$$

classifies the tangent m-plane bundle $\tau_M : M \to BO(m)$ with the k-stable trivialization

$$\delta\tau_M \ : \ \tau_M \oplus \epsilon^k \ \cong \ \tau_M \oplus \nu_f \ = \ \tau_{\mathbb{R}^{m+k}}|_M \ = \ \epsilon^{m+k} \ .$$

- **Problem** Compute the **generalized curvatura integra**

$$c_*[M] \in H_m(V_{m+k,k}) \ = \ \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 (\mathrm{mod}\ 2) \\ \mathbb{Z}_2 & \text{if } m \equiv 1 (\mathrm{mod}\ 2) \end{cases} \ (k > 1) \ .$$

The Hopf invariant

- (Hopf, 1931 in formulation of Steenrod, 1949) The **Hopf invariant**

$$H\ :\ \pi_{2k-1}(S^k) \to \mathbb{Z}\ ;\ (F : S^{2k-1} \to S^k) \mapsto H(F)$$

is determined by the cup product structure in the mapping cone

$$X\ =\ S^k \cup_F D^{2k}\ .$$

If $a = 1 \in H^k(X) = \mathbb{Z}$, $b = 1 \in H^{2k}(X) = \mathbb{Z}$ then $H(F) \in \mathbb{Z}$ given by

$$a \cup a\ =\ H(F)b \in H^{2k}(X)\ .$$

- $H : \pi_3(S^2) \to \mathbb{Z}$ is an isomorphism: for a map $F : S^3 \to S^2$

$$H(F)\ =\ L(M, N) \in \mathbb{Z}$$

with $L(M, N)$ the linking number of generic disjoint inverse images

$$M^1\ =\ F^{-1}(x)\ ,\ N^1\ =\ F^{-1}(y) \subset S^3\ (x \neq y \in S^2)\ .$$

- If k is odd then $H(F) = 0$.
- (Adams, 1960) $H : \pi_{2k-1}(S^k) \to \mathbb{Z}$ is onto if and only if $k = 2, 4, 8$.

The mod 2 Hopf invariant

- (Steenrod, 1949) The **mod 2 Hopf invariant** is the morphism

$$H \ : \ \pi_{m+k}(S^k) \to \mathbb{Z}_2 \ ; \ (F : S^{m+k} \to S^k) \mapsto H_2(F) \ (m \geqslant 1)$$

determined by the Steenrod square in the mapping cone

$$X \ = \ S^k \cup_F D^{m+k+1} \ .$$

- If $a = 1 \in H^k(X;\mathbb{Z}_2) = \mathbb{Z}_2$, $b = 1 \in H^{m+k+1}(X;\mathbb{Z}_2) = \mathbb{Z}_2$ then

$$Sq^{m+1}(a) \ = \ H_2(F)b \in H^{m+k+1}(X;\mathbb{Z}_2) \ .$$

- $H_2 : \pi_m^S \to \mathbb{Z}_2$ is an isomorphism for $m = 1$, onto for $m = 3, 7$.
- (Adams, 1960) $H_2 = 0$ for $m \neq 1, 3, 7$.
- **Definition** For an m-dimensional framed manifold M^m let

$$\text{Hopf}(M) \ = \ \begin{cases} 0 \in \mathbb{Z} & \text{if } m \equiv 0 (\text{mod } 2) \\ H_2(F) \in \mathbb{Z}_2 & \text{if } m \equiv 1 (\text{mod } 2) \end{cases}$$

with $F : S^{m+k} \to S^k$ the Pontryagin-Thom map.

Kervaire's first thesis II.

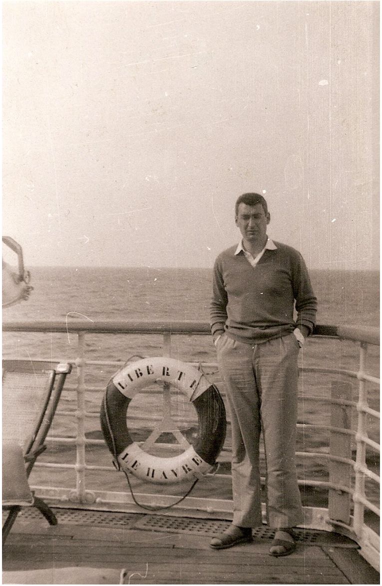
- 1. **Courbure integrale généralisée et homotopie**, Math. Annalen 131, 219–252 (1956)
- 2. **Relative characteristic classes**, Amer.J.Math. 79, 517–558 (1957)
- **Theorem** The *curvatura integra* of an m-dimensional framed manifold M^m is

$$c_*[M] \;=\; \text{Hopf}(M) + \begin{cases} \chi(M)/2 \\ \chi_{1/2}(M) \end{cases} \in \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 (\text{mod } 2) \\ \mathbb{Z}_2 & \text{if } m \equiv 1 (\text{mod } 2) \end{cases}$$

involving the **Kervaire semicharacteristic**

$$\chi_{1/2}(M) \;=\; \sum_{j=0}^{(m-1)/2} \dim_{\mathbb{Z}_2} H_j(M;\mathbb{Z}_2) \in \mathbb{Z}_2$$

dividing the Euler characteristic by 2.
- $\chi_{1/2}(M)$ and its $\mathbb{R}$-coefficient version have taken on a life of their own (E.Thomas, Atiyah-Dupont on the index of a vector field, with analytic interpretation, Davis-Ranicki in surgery theory, Gibbons-Hawking on the topology of the universe).



En route to the US and exotic spheres, 1956

Non-parallelizability of spheres

- **Definition** An m-dimensional manifold M is **parallelizable** if the tangent m-plane bundle $\tau_M : M \to BO(m)$ is trivial, $\tau_M \cong \epsilon^m$.
- A parallelizable manifold is framed.
- The n-sphere S^n is framed, with $S^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$, $\tau_{S^n} \oplus \epsilon \cong \epsilon^{n+1}$, but not necessarily parallelizable.
- 3. **Non-parallelizability of the n-sphere for $n > 7$**, Proc. Nat. Acad. Sci. 44, 280–283 (1958)
- **Theorem** (Bott-Milnor, Kervaire) The n-sphere S^n is parallelizable if and only if $n = 1, 3, 7$.
- **Corollary** The morphisms

$$\varinjlim_k H_n(V_{n+k,k}) = \pi_{n+1}(BO, BO(n)) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 (\text{mod } 2) \\ \mathbb{Z}_2 & \text{if } n \equiv 1 (\text{mod } 2) \end{cases}$$

$$\to \ker(\pi_n(BO(n)) \to \pi_n(BO)) \ ; \ 1 \mapsto \tau_{S^n}$$

are 0 for $n = 1, 3, 7$ and isomorphisms for $n \neq 1, 3, 7$.

The generalized Hopf invariant I.

- 4. **An interpretation of G. Whitehead's generalization of H. Hopf's invariant**, Ann. of Maths. 69, 345–365 (1959)
- Framed cobordism interpretation.
- The generic inverse images of a map $F : S^{d+n+1} \to S^{n+1}$ are disjoint d-dimensional framed submanifolds

$$M^d = F^{-1}(x) \ , \ N^d = F^{-1}(y) \subset S^{d+n+1} \ (x \neq y \in S^{n+1}) \ .$$

Regard M, N as subsets of $\mathbb{R}^{d+n+1} \subset S^{d+n+1}$ and let

$$G \ : \ M \times N \to S^{d+n} \ ; \ (a,b) \mapsto \frac{a-b}{\|a-b\|} \ .$$

- **Definition** The **linking manifold** of f is the $(d-n)$-dimensional framed submanifold

$$L(M,N)^{d-n} \ = \ G^{-1}(z) \subset M \times N \subset \mathbb{R}^{2d+2n+2}$$

for a generic $z \in S^{d+n}$, with a Pontryagin-Thom map

$$h(F) \ : \ S^{2d+2n+2} \to \Sigma^{d+3n+2} L(M,N)^+ \to S^{d+3n+2} \ .$$

The generalized Hopf invariant II.

- By the Freudenthal suspension theorem $h(F) = \Sigma^{d+n+1} H(F)$ is the $(d+n+1)$-fold suspension of a map
$$H(F) \ : \ S^{d+n+1} \to S^{2n+1} \ .$$
- **Theorem** (K) The linking manifold construction induces the generalized Hopf invariant map
$$H \ : \ \pi_{d+n+1}(S^{n+1}) \to \pi_{d+n+1}(S^{2n+1}) \ ; \ F \mapsto H(F) \ .$$
- **Example** For $n = d$, $F : S^{2n+1} \to S^{n+1}$, $L(M^n, N^n)^0 = G^{-1}(z)$ is 0-dimensional, and H is the original Hopf invariant
$$\begin{aligned} H(F) \ &= \ \text{degree}(G : M \times N \to S^{2n}) \\ &= \ \text{number of points in } L(M,N)^0 \in \pi_{2n+1}(S^{2n+1}) \ = \ \mathbb{Z} \ . \end{aligned}$$
- The linking manifold is a foundation of a geometric understanding of the generalized Hopf invariant and the Wall surgery obstruction (Boardman-Steer 1967, Koschorke-Sanderson 1977, Crabb-Ranicki 1998–).

The J-homomorphism

- (G. Whitehead, 1942) The J-**homomorphism**

$$J \ : \ \pi_m(O(k)) \to \pi_{m+k}(S^k)$$

sends $\omega : S^m \to O(k)$ to the Pontryagin-Thom map $J(\omega) : S^{m+k} \to S^k$ of $S^m \subset S^{m+k}$ with the framing

$$b_\omega \ : \ S^m \times D^k \subset S^{m+k} \ = \ S^m \times D^k \cup D^{m+1} \times S^{k-1} \ ;$$
$$(x, y) \mapsto (x, \omega(x)(y)) \ .$$

- There is also a stable version

$$J \ : \ \varinjlim_k \pi_m(O(k)) \ = \ \pi_m(O) \to \varinjlim_k \pi_{m+k}(S^k) \ = \ \pi_m^S \ = \ \Omega_m^{fr} \ ;$$
$$\omega \mapsto (S^m, b_\omega) \ .$$

Symmetric and quadratic forms

- Let A be a commutative ring, $\epsilon = +1$ or -1.
- **Definition** An ϵ-**symmetric form** over A (H, λ) is a f.g. free A-module H with a bilinear pairing $\lambda : H \times H \to A$ such that

$$\lambda(x,y) \;=\; \epsilon\lambda(y,x) \in A \;\; (x,y \in H)\ .$$

- The form (H, λ) is **nonsingular** if the A-module morphism

$$H \to \mathrm{Hom}_A(H,A)\ ;\ x \mapsto (y \mapsto \lambda(x,y))$$

an isomorphism.
- **Definition** An ϵ-**quadratic form** over A (H, λ, μ) is an ϵ-symmetric form (H, λ) together with a function

$$\mu \;:\; H \to Q_\epsilon(A) \;=\; \mathrm{coker}(1-\epsilon : A \to A)$$

such that for all $x, y \in H$, $a \in A$

$$\lambda(x,y) \;=\; \mu(x+y) - \mu(x) - \mu(y)\ ,\ \mu(ax) \;=\; a^2\mu(x) \in Q_\epsilon(A)\ ,$$
$$\lambda(x,x) \;=\; (1+\epsilon)\mu(x) \in \mathrm{im}(1+\epsilon : A \to A) \subseteq \ker(1-\epsilon : A \to A)\ .$$

The signature of forms over $\mathbb{Z}$

- **Definition** The **signature** of a 1-symmetric form (H, λ) over $\mathbb{Z}$ is

$$\text{signature}(H, \lambda) \;=\; p - q \in \mathbb{Z}$$

  if $\mathbb{R} \otimes (H, \lambda)$ has p positive eigenvalues and q negative eigenvalues.
- $\text{signature}(\mathbb{Z}, 1) = 1$.
- A 1-symmetric form (H, λ) over $\mathbb{Z}$ has a 1-quadratic function

$$\mu \;:\; H \to Q_+(\mathbb{Z}) \;=\; \mathbb{Z}$$

  if and only if it has even diagonal entries

$$\lambda(x, x) \;\equiv\; 0 (\bmod 2) \;\; (x \in H)$$

  with $\mu(x) = \lambda(x, x)/2$ (division by 2).
- For a nonsingular 1-quadratic form (H, λ, μ) over $\mathbb{Z}$

$$\text{signature}(H, \lambda) \;\equiv\; 0 (\bmod 8)$$

The E_8-form

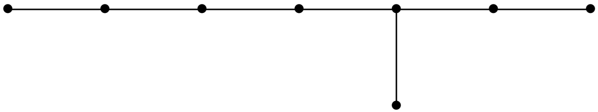
- The nonsingular 1-quadratic form $(\mathbb{Z}^8, \lambda, \mu)$ over $\mathbb{Z}$ with

$$\lambda = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}$$

  has

$$\text{signature}(\mathbb{Z}^8, \lambda) = 8 .$$

- The Dynkin diagram of E_8



The signature of manifolds

- The **intersection form** of a $2n$-dimensional topological manifold with boundary $(M^{2n}, \partial M)$ is the $(-)^n$-symmetric form (H, λ) over $\mathbb{Z}$

$$H \;=\; H^n(M, \partial M)/\text{torsion} \;,\; \lambda(x, y) \;=\; \langle x \cup y, [M] \rangle \in \mathbb{Z} \;.$$

  If $\partial M = \emptyset$ or $H_*(\partial M) = H_*(S^{2n-1})$ then (H, λ) is nonsingular.
- **Definition** The **signature** of a $4k$-dimensional $(M^{4k}, \partial M)$ is

$$\text{signature}(M) \;=\; \text{signature}(H_{2k}(M)/\text{torsion}, \lambda) \in \mathbb{Z} \;.$$

- **Theorem** (Hirzebruch, 1952) If $\partial M = \emptyset$ then

$$\text{signature}(M^{4k}) \;=\; \langle \mathcal{L}(p(M)), [M] \rangle \in \mathbb{Z} \;.$$

- **Theorem** (Milnor, 1956 for $k = 1$) The intersection form of a $4k$-dimensional manifold M^{4k} has a 1-quadratic function if and only if it has $2k^{\text{th}}$ Wu class $v_{2k}(M) = 0 \in H^{2k}(M; \mathbb{Z}_2)$, in which case signature$(M) \equiv 0 (\text{mod } 8)$.
- **Theorem** (Rohlin, 1952) The signature of a 4-dimensional manifold M^4 with $v_2(M) = 0 \in H^2(M; \mathbb{Z}_2)$ has signature$(M) \equiv 0 (\text{mod } 16)$.

Almost framed manifolds I.

- 5. **On the Pontryagin classes of certain $O(n)$-bundles over manifolds**, Amer. J. Math. 80, 632–638 (1958)
- 6. (with Milnor) **Bernoulli numbers, homotopy groups, and a theorem of Rohlin**, Proc. 1958 Edinburgh ICM
- **Definition** An m-dimensional manifold with boundary $(M^m, \partial M)$ is **almost framed** if the open manifold $M \backslash \{\text{pt.}\}$ is framed

$$(M \backslash \{\text{pt.}\}) \times \mathbb{R}^k \subset \mathbb{R}^{m+k} \ (k \text{ large}) \ .$$

- If $\partial M \neq \emptyset$ almost framed = framed = parallelizable.
- An almost framed closed M^m has a **framing obstruction**

$$\mathfrak{o}(M) \in \ker(J : \pi_{m-1}(O) \to \pi^S_{m-1})$$

  such that M is framed if and only if $\mathfrak{o}(M) = 0$.
- **Theorem** (K+M) The cobordism group A_m of m-dimensional almost framed manifolds fits into the exact sequence

$$A_m \xrightarrow{\mathfrak{o}} \pi_{m-1}(O) \xrightarrow{J} \pi^S_{m-1} \ .$$

Almost framed manifolds II.

- **Theorem** (K+M) The framing obstruction of a $4k$-dimensional almost framed manifold M^{4k} is

$$\begin{aligned}\mathfrak{o}(M) &= p_k(M)/(a_k(2k-1)!) \\ &\in \ker(J:\pi_{4k-1}(O)\to\pi^S_{4k-1}) \;=\; j_k\mathbb{Z}\subset\pi_{4k-1}(O) \;=\; \mathbb{Z}\end{aligned}$$

with $p_k(M)\in H^{4k}(M)=\mathbb{Z}$ the Pontryagin class, and

$$a_k = \begin{cases} 1 & \text{for } k\equiv 0(\text{mod } 2) \\ 2 & \text{for } k\equiv 1(\text{mod } 2)\end{cases}, \; j_k = \text{den}(B_k/4k)$$

(using Adams, Bott).

- **Important special case**. For $k=1$ have $j_1=24$. The identity

$$\begin{aligned}\mathfrak{o}(M^4) &= p_1(M)/2 \\ &\in \ker(J:\pi_3(O)\to\pi^S_3) \;=\; \ker(\mathbb{Z}\to\mathbb{Z}_{24}) \;=\; 24\mathbb{Z}\subset\pi_3(O) \;=\; \mathbb{Z}\end{aligned}$$

is required for the proof of Rohlin's theorem, with

$$\text{signature}(M^4) \;=\; \langle p_1(M)/3,[M]\rangle \;\equiv\; 0(\text{mod } 16) \; .$$

On 2-spheres in 4-manifolds

- (Whitney, 1944) For a simply-connected $2n$-dimensional manifold M with $n \geqslant 3$ every element $x \in \pi_n(M)$ is represented by an embedding $S^n \hookrightarrow M^{2n}$. Proved by the Whitney trick for removing double points.
- 7. (with Milnor) **On 2-spheres in 4-manifolds**, Proc. Nat. Acad. U.S.A. 47, 1651–1657 (1961)
- **Theorem** (K+M) Let M^4 be a 4-dimensional manifold with intersection form $(H_2(M)/\text{torsion}, \lambda)$. If $v_2(M) \in H^2(M;\mathbb{Z}_2) = H_2(M;\mathbb{Z}_2)$ is represented by an embedding $x : S^2 \hookrightarrow M$ then

$$\text{signature}(M) \equiv \lambda(x,x) \pmod{16} .$$

(For $v_2(M) = 0$ this is Rohlin's theorem, with $x = 0 \in H_2(M)$).
- **Corollary** (Failure of Whitney trick in dimension 4.) For $M^4 = S^2 \times S^2$

$$x = (2,2) \in H_2(M) = \mathbb{Z} \oplus \mathbb{Z}$$

is not represented by $x : S^2 \hookrightarrow M^4$, since

$$\text{signature}(M) = 0 , \ \lambda(x,x) = 8 \in \mathbb{Z} , \ v_2(M) = 0 \in H^2(M;\mathbb{Z}_2) .$$

Exotic spheres

- **Definition** A **homotopy sphere** Σ^m is an m-dimensional manifold which is homotopy equivalent to S^m.
- **Theorem** (Milnor, 1956) There exist **exotic spheres**, homotopy spheres Σ^m which are homeomorphic but not diffeomorphic to S^m.
- Initial construction for $m = 7$ as the (D^4, S^3)-bundles of certain 4-plane bundles $\eta : S^4 \to BO(4)$ over S^4

$$(D^4, S^3) \to (D(\eta)^8, S(\eta)) \to S^4$$

  with signature$(D(\eta)) = 1$. The 7-manifold $\Sigma^7 = S(\eta)$ is homeomorphic to S^7 by Morse theory. If Σ^7 were diffeomorphic to S^7 then $M^8 = D(\eta) \cup_{\Sigma^7} D^8$ would be an 8-dimensional manifold with signature$(M) = 1$ and Pontryagin classes $p_1(M), p_2(M)$ contradicting the Hirzebruch signature theorem. Failure of signature theorem for manifolds with boundary.
- 8. Kervaire and Milnor, **Groups of homotopy spheres I.**, Annals of Maths. 77, 504–537 (1963)

The Arf invariant

- Every nonsingular (-1)-symmetric form (H, λ) over $\mathbb{Z}$ has a basis $\{b_1, b_2, \ldots, b_p, c_1, c_2, \ldots, c_p\}$ for H with

  $\lambda(b_i, b_{i'}) = 0$, $\lambda(c_j, c_{j'}) = 0$, $\lambda(b_i, c_j) = 0$ for $i \neq j$, $\lambda(b_i, c_i) = 1$.
- (H, λ) has (-1)-quadratic functions $\mu : H \to Q_-(\mathbb{Z}) = \mathbb{Z}_2$, but these are not determined by λ.
- **Definition** The **Arf invariant** of a nonsingular (-1)-quadratic form (H, λ, μ) over $\mathbb{Z}$ is

$$\text{Arf}(H, \lambda, \mu) = \sum_{i=1}^{p} \mu(b_i)\mu(c_i) \in \mathbb{Z}_2 = \{0, 1\} .$$

- The form $(\mathbb{Z} \oplus \mathbb{Z}, \lambda, \mu)$ with

$$\lambda((v, w), (x, y)) = wx - vy \ , \ \mu(x, y) = x^2 + xy + y^2$$

  has

$$\text{Arf}(\mathbb{Z} \oplus \mathbb{Z}, \lambda, \mu) = 1 \in \mathbb{Z}_2 .$$

The $(-)^n$-quadratic form of a $2n$-dimensional framed manifold for $n = 1, 2$

- Pontryagin (1950) The intersection (-1)-symmetric form $(H^1(M), \lambda)$ over $\mathbb{Z}$ of a framed surface $M^2 \times \mathbb{R}^k \subset \mathbb{R}^{k+2}$ has a geometrically defined (-1)-quadratic function

$$\mu \ : \ H^1(M) \ = \ H_1(M) \to Q_{-1}(\mathbb{Z}) \ = \ \Omega_1^{fr} \ = \ \mathbb{Z}_2 \ ; \ x \mapsto (x : S^1 \hookrightarrow M) \ .$$

Represent each $x \in H^1(M)$ by an embedding $x : S^1 \hookrightarrow M$ with a corresponding framing $S^1 \times \mathbb{R}^{k+1} \subset \mathbb{R}^{k+2}$, $\delta\nu_x : \nu_x \oplus \epsilon^k \cong \epsilon^{k+1}$.

- Isomorphism

$$\text{Arf} \ : \ \Omega_2^{fr} \ = \ \pi_2^S \ \xrightarrow{\cong} \ \mathbb{Z}_2 \ ; \ M^2 \mapsto \text{Arf}(H^1(M), \lambda, \mu) \ .$$

- Milnor (1956) The intersection symmetric form $(H^2(M), \lambda)$ of a simply-connected 4-dimensional manifold M^4 admits a quadratic function $\mu : H^2(M) \to Q_+(\mathbb{Z}) = \mathbb{Z}$ if and only if M is almost framed, if and only if $v_2(M) = 0 \in H^2(M; \mathbb{Z}_2)$, in which case $\mu(x) = \lambda(x, x)/2$.

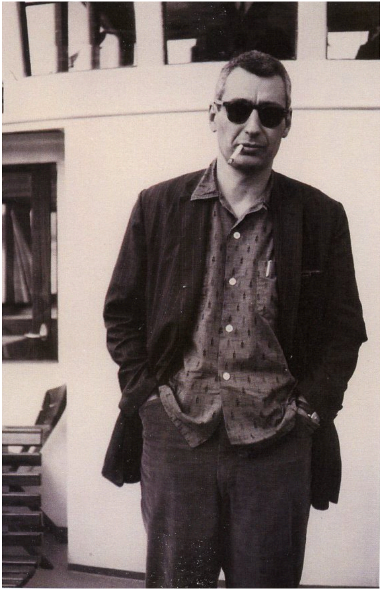
The $(-)^n$-quadratic form of a $2n$-dimensional almost framed manifold $(M^{2n}, \partial M)$ for $n \geqslant 1$

- **Theorem** (Kervaire-Milnor 1959/1963) If M is $(n-1)$-connected the $(-)^n$-symmetric intersection form $(H_n(M), \lambda)$ over $\mathbb{Z}$ has a $(-)^n$-quadratic function $\mu : H_n(M) \to Q_{(-)^n}(\mathbb{Z})$, using Steenrod squares.
- The almost framing $(M\backslash\{\text{pt.}\}) \times \mathbb{R}^k \subset \mathbb{R}^{2n+k}$ (k large) determines a k-stable trivialization $\delta\nu_x : \nu_x \oplus \epsilon^k \cong \epsilon^{n+k}$ of $\nu_x : S^n \to BO(n)$ for any $x : S^n \hookrightarrow M$. For $n \geqslant 3$ can define μ geometrically by

$$\mu \ : \ H_n(M) \to H_n(V_{n+k,k}) = \pi_{n+1}(BO, BO(n)) = Q_{(-)^n}(\mathbb{Z}) \ ;$$
$$(x : S^n \hookrightarrow M) \mapsto (\delta\nu_x, \nu_x) \ .$$

- If $n \neq 1, 3, 7$ then $Q_{(-)^n}(\mathbb{Z}) \to \pi_n(BO(n)); 1 \mapsto \tau_{S^n}$ is injective, so $\mu(x) = \nu_x \in \pi_n(BO(n))$, and μ is independent of almost framing.
- Browder (1969) Use of functional Steenrod squares to construct form

$$\begin{cases} (H_n(M)/\text{torsion}, \lambda, \mu) \\ (H_n(M; \mathbb{Z}_2), \lambda, \mu) \end{cases} \text{over} \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \end{cases} \text{without } (n-1)\text{-connectivity.}$$



Bonn Arbeitstagung, 1960

The Kervaire invariant

- Let $(M^{4k+2}, \partial M)$ be a $(4k+2)$-dimensional almost framed manifold with boundary such that either $\partial M = \emptyset$ or $H_*(\partial M) = H_*(S^{4k+1})$, so that $(H^{2k+1}(M;\mathbb{Z}_2), \lambda, \mu)$ is a nonsingular quadratic form over $\mathbb{Z}_2$.
- The **Kervaire invariant** of M is

$$\text{Kervaire}(M) \ = \ \text{Arf}(H_{2k+1}(M;\mathbb{Z}_2), \lambda, \mu) \in \mathbb{Z}_2 \ .$$

- If $\partial M = \emptyset$ and $M = \partial N$ is the boundary of a $(4k+3)$-dimensional almost framed manifold N then $\text{Kervaire}(M) = 0 \in \mathbb{Z}_2$.
- **Kervaire invariant 1 problem** In which dimensions $4k+2$ does there exist a framed manifold M^{4k+2} with $\text{Kervaire}(M) = 1$?
- **Theorem** (Browder, 1969) The Kervaire invariant map

$$\text{Kervaire} \ : \ \Omega^{fr}_{4k+2} \ = \ \pi^S_{4k+2} \to \mathbb{Z}_2$$

  is 0 if $k \neq 2^i - 1$. Reduction of problem to homotopy theory.
- There exist $(4k+2)$-dimensional framed manifolds M^{4k+2} with $\text{Kervaire}(M) = 1$ for $k = 0, 1, 3, 7$.
- For $k = 0, 1, 3$ can take $M^{4k+2} = S^{2k+1} \times S^{2k+1}$.

Surgery

- A **surgery** on an m-dimensional manifold M is the procedure of using an embedding $S^r \times D^{m-r} \subset M$ to construct a new m-dimensional manifold M'
$$M \rightsquigarrow M' = (M \backslash S^r \times D^{m-r}) \cup D^{r+1} \times S^{m-r-1} .$$
The **trace** of the surgery is the cobordism $(W^{m+1}; M, M')$ with $W = M \times I \cup D^{r+1} \times D^{m-r}$.
- The surgery **kills** $[S^r] \in \pi_r(M)$, with $\pi_r(W) = \pi_r(M)/\langle [S^r] \rangle$.
- **Cobordism = Surgeries Theorem** (Thom, Milnor, 1950's)
Two m-dimensional manifolds M, M' are cobordant if and only if M' can be obtained from M by a finite sequence of surgeries.
- In a **framed surgery** M is framed and $S^r \times D^{m-r} \subset M$ is chosen sufficiently carefully for M' to be also framed.
- For $m \geqslant 5$ with $m = 2n$ or $2n + 1$ every m-dimensional framed manifold with boundary $(M, \partial M)$ is framed cobordant rel ∂M to an $(n-1)$-connected m-dimensional framed manifold with boundary $(M', \partial M)$.

Kervaire's proof of the *curvatura integra* theorem

- Neither the curvatura integra $c_*[M]$ of a framed manifold M^m nor the Euler characteristic $\chi(M)$ nor the Kervaire semicharacteristic $\chi_{1/2}(M)$ are framed cobordism invariants.
- Kervaire's proof used the Pontrjagin-Thom theorem (framed cobordism = stable homotopy) theorem and a rudimentary form of the Thom-Milnor cobordism=surgeries theorem to prove that the difference

$$c_*[M] - \begin{cases} \chi(M)/2 \\ \chi_{1/2}(M) \end{cases} \in \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 (\bmod 2) \\ \mathbb{Z}_2 & \text{if } m \equiv 1 (\bmod 2) \end{cases}$$

  is a framed cobordism invariant, and then identified this difference with Hopf(M).
- Working outside M.

A generalization of Kervaire's *curvatura integra* theorem

- Let $f : M^m \looparrowright N^{2m}$ be an immersion with normal bundle $\nu_f : M \to BO(m)$, and 0-dimensional double point set

  $$D_2(f) = \{(x,y) \in M \times M \mid x \neq y \in M,\ f(x) = f(y) \in N\}/(x,y) \sim (y,x)$$

- For framed M, N with $M \times \mathbb{R}^{m+k} \subset \mathbb{R}^{2m+k}$, $N \times \mathbb{R}^k \subset \mathbb{R}^{2m+k}$ let $c : M \to V_{m+k,k}$ classify ν_f with the corresponding k-stable trivialization $\delta\nu_f : \nu_f \oplus \epsilon^k \cong \epsilon^{m+k}$.
- **Theorem** (Crabb, 1980) The *curvatura integra* of f is

  $$c_*[M] = \text{Hopf}(M) + \mu([M]^*) - D_2(f) \in H_m(V_{m+k,k}) = Q_{(-)^m}(\mathbb{Z})$$

  with $[M]^* \in H^m(N)$ the Poincaré dual of $f_*[M] \in H_m(N)$.
- **Corollary** For any m-dimensional framed manifold M and $f = \Delta : M \hookrightarrow N = M \times M$ get K's 1956 theorem: $\nu_\Delta = \tau_M$, $D_2(f) = \emptyset$

  $$\mu([M]^*) = \begin{cases} \chi(M)/2 \\ \chi_{1/2}(M) \end{cases} \in Q_{(-)^m}(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 (\text{mod } 2) \\ \mathbb{Z}_2 & \text{if } m \equiv 1 (\text{mod } 2) \end{cases}.$$

  'Division of $\lambda([M]^*, [M]^*) = \chi(M) \in \mathbb{Z}$ by 2.' Working inside M.

Plumbing

- **Theorem** (Milnor, 1959) For $n \geqslant 3$ every $(-)^n$-quadratic form (H, λ, μ) over $\mathbb{Z}$ is realized by an $(n-1)$-connected $2n$-dimensional framed manifold with boundary $(W^{2n}, \partial W)$ with $H_n(W) = H$. The form (H, λ, μ) is nonsingular if and only if $H_*(\partial W) = H_*(S^{2n-1})$.
- If the f.g. free $\mathbb{Z}$-module H has rank r with basis $\{e_1, e_2, \ldots, e_r\}$ let $\mu_1, \mu_2, \ldots, \mu_r \in \mathbb{Z}$ be such that

$$\mu(e_i) \;=\; \mu_i \in Q_{(-)^n}(\mathbb{Z}) \; (1 \leqslant i \leqslant r) \; .$$

The realization is obtained by **plumbing** together the disk bundles

$$(D^n, S^{n-1}) \to (D(\mu_i \tau_{S^n})^{2n}, S(\mu_i \tau_{S^n})) \to S^n$$

of the stably trivialized n-plane bundles over S^n

$$\mu_i(\delta\tau_{S^n}, \tau_{S^n}) \in \pi_{n+1}(BO, BO(n)) \;=\; Q_{(-)^n}(\mathbb{Z}) \;\; (1 \leqslant i \leqslant r)$$

and killing π_1 by framed surgery. $W^{2n} = D^{2n} \cup_{S^{2n-1}} V^{2n}$ with $(V; S^{2n-1}, \partial W)$ the union of the traces of r framed surgeries on $S^{n-1} \times D^n \subset S^{2n-1}$.

The Kervaire manifold

- 9. **A manifold which does not admit any differentiable structure**, Comm. Math. Helv. 34, 257–270 (1960)
- A 4-connected 10-dimensional manifold M has a homotopy invariant (-1)-quadratic form $(H^5(M), \lambda, \mu)$ over $\mathbb{Z}$, allowing the definition

$$\text{Kervaire}(M) \;=\; \text{Arf}(H^5(M), \lambda, \mu) \in \mathbb{Z}_2 \ .$$

- If M is differentiable then it is framed and μ is the quadratic function given by the normal bundles ν_x of $x : S^5 \subset M^{10}$. It is possible to kill $\pi_5(M)$ by framed surgery, so that M is framed cobordant to a homotopy 10-sphere and $\text{Kervaire}(M) = 0 \in \mathbb{Z}_2$.
- The plumbing $(W^{10}, \partial W)$ of 2 copies of τ_{S^5} realizing the Arf invariant 1 form $(\mathbb{Z} \oplus \mathbb{Z}, \lambda, \mu)$ has $\partial W = \Sigma^9$ homeomorphic to S^9
- The **Kervaire manifold** $M^{10} = W^{10} \cup_{\Sigma^9} D^{10}$ is a 4-connected 10-dimensional PL manifold with $\text{Kervaire}(M) = 1 \in \mathbb{Z}_2$.
- **Theorem** (K) M^{10} is a topological manifold without a differentiable structure. In fact, M is not even homotopy equivalent to a differentiable manifold.

April 1963 Symposium in honor of Marston Morse,
Institute for Advanced Study, Princeton, New Jersey.

First row, seated, from left: S. S. Chern, R. J. Pohrer, A. Selberg, M. Morse, W. Leighton, M. Hirsch, S. S. Cairns, H. Whitney.
Second row, standing, from left: R. Bott, B. Mazur, G. A. Hedlund, T. Frankel, S. Smale, N. Kuiper, J. F. Adams, W. Browder, J. W. Milnor, M. Kervaire.

The classification of exotic spheres I.

- **Definition** An h-**cobordism** of m-dimensional manifolds is a cobordism $(W^{m+1}; M^m, M'^m)$ such that the inclusions $M^m \hookrightarrow W$, $M'^m \hookrightarrow W$ are homotopy equivalences.
- **Theorem** (Smale 1962) For $m \geqslant 5$
  (i) h-cobordant homotopy spheres Σ^m, Σ'^m are diffeomorphic.
  (ii) Every homotopy sphere Σ^m is PL homeomorphic to S^m.
- **Definition** (K+M, 1963) Let Θ_m be the abelian group of h-cobordism classes of m-dimensional homotopy spheres, with addition by connected sum.
- **Theorem** (K+M, 1963) For $m \geqslant 4$ Θ_m is finite, with a short exact sequence

$$0 \to \mathrm{coker}(a : A_{m+1} \to P_{m+1}) \xrightarrow{b} \Theta_m \xrightarrow{c} \ker(a : A_m \to P_m) \to 0\,.$$

  Every homotopy sphere is framed. P_m is the cobordism group of m-dimensional framed manifolds with homotopy sphere boundary, and

$$\ker(a) \subseteq \mathrm{coker}(J : \pi_m(O) \to \pi_m^S) \;=\; \ker(\mathfrak{o} : A_m \to \pi_{m-1}(O))\,.$$

The classification of exotic spheres II.

- **Computation of the simply-connected surgery obstruction groups**

$$P_m = \begin{cases} \mathbb{Z} \text{ (signature/8)} & \text{if } m \equiv 0 (\bmod 4) \\ 0 & \text{if } m \equiv 1 (\bmod 4) \\ \mathbb{Z}_2 \text{ (Kervaire-Arf)} & \text{if } m \equiv 2 (\bmod 4) \\ 0 & \text{if } m \equiv 3 (\bmod 4) \end{cases} \text{ for } m \geqslant 5 \ .$$

- $P_{2n+1} = 0$ proved geometrically by framed surgery and the Kervaire semicharacteristic $\chi_{1/2}(M)$ for a $(2n+1)$-dimensional framed manifold M with $\partial M = \Sigma^{2n}$: for $n \geqslant 2$ M is framed cobordant rel ∂M to D^{2n+1}.
- The simply-connected surgery groups P_m are now seen as the special case $\pi = \{1\}$ of the Wall (1970) obstruction groups $L_m(\mathbb{Z}[\pi])$ for surgery on m-dimensional manifolds with fundamental group π.

The classification of exotic spheres III.

- **Definition** $a : A_{2n} \to P_{2n}$ sends a $2n$-dimensional almost framed manifold M^{2n} to

$$a(M) = \begin{cases} \text{signature}(M)/8 \\ \text{Kervaire}(M) \end{cases} \in P_{2n} = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 (\text{mod } 2) \\ \mathbb{Z}_2 & \text{if } n \equiv 1 (\text{mod } 2) \end{cases}$$

using the nonsingular $(-)^n$-quadratic form $\begin{cases} (H^n(M)/\text{torsion}, \lambda, \mu) \\ (H^n(M; \mathbb{Z}_2), \lambda, \mu) \end{cases}$ over $\begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \end{cases}$ constructed using Steenrod squares.

- **Definition** $b : P_{2n} \to \Theta_{2n-1}$ is given by the plumbing construction of $(2n-1)$-dimensional exotic spheres as boundaries of $(n-1)$-connected $2n$-dimensional framed manifolds.
- **Definition** $c : \Theta_m \to A_m$ sends a homotopy sphere Σ^m to its almost framed cobordism class in A_m.

The classification of exotic spheres IV.

- Extract from **Groups of homotopy spheres I.**

The main result of the present Part I will be:

THEOREM 1.2. *For $n \neq 3$ the group Θ_n is finite.*

(Our methods break down for the case $n = 3$. However, if one assumes the Poincaré hypothesis, then it can be shown that $\Theta_3 = 0$.)

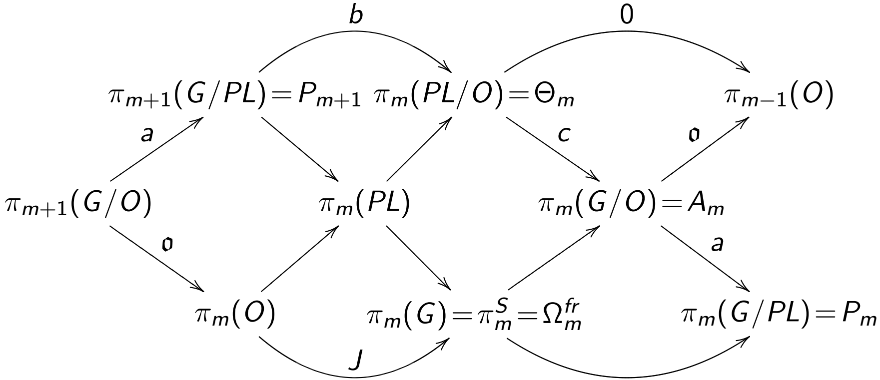
More detailed information about these groups will be given in Part II. For example, for $n = 1, 2, 3, \cdots, 18$, it will be shown that the order of the group Θ_n is respectively:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[\Theta_n]$	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16.

- **Example** $\Theta_7 = bP_8 = \mathbb{Z}_{28}$, generated by the boundary $\Sigma^7 = \partial W$ of the E_8-plumbing W^8 of 8 copies of τ_{S^4} used to construct the Milnor PL manifold $M^8 = W \cup_{\Sigma^7} D^8$.
- **Example** $bP_{10} = \mathbb{Z}_2 \subset \Theta_9$, generated by the boundary $\Sigma^9 = \partial W$ of the Arf plumbing W^{10} of 2 copies of τ_{S^5} used to construct the Kervaire PL manifold $M^{10} = W \cup_{\Sigma^9} D^{10}$.

The classification of exotic spheres V.

- J. Levine's **Lectures on groups of homotopy spheres** (1969/1983) is generally regarded as Part II of **Groups of homotopy spheres I.**.
- Commutative braid of 4 interlocking exact sequences for $m \geqslant 5$



with

$$\mathfrak{o} \;:\; \pi_m(G/O) \;=\; A_m \to \pi_{m-1}(O) \;;\; M^m \mapsto \mathfrak{o}(M) \;.$$

Kervaire's second thesis I. High dimensional knot theory

- (1964) Thése Sc. math., Paris. Examined by Cartan and Serre.
- 10. **Les noeuds de dimensions superiéures**,
  Bull. Soc. Math. France 93, 225–271 (1965)
- 11. **On higher dimensional knots**,
  Proc. Morse Symposium, Princeton, 105–119 (1965)
- **Definition** An n-**knot** is an embedding $f : S^n \hookrightarrow S^{n+2}$.
- *Of course, from the point of view of the rest of mathematics, knots in higher-dimensional space deserve just as much attention as knots in 3-space.* (Frank Adams)
- The **complement** $X = S^{n+2} \backslash f(S^n)$ is an open $(n+2)$-dimensional manifold with $H_*(X) = H_*(S^1)$, $\pi_1(X) \twoheadrightarrow \mathbb{Z}$.
- The **exterior** $(n+2)$-dimensional manifold with boundary

  $$(M, \partial M) \ = \ (\text{cl.}(S^{n+2} \backslash (f(S^n) \times D^2)), S^n \times S^1)$$

  is such that $M \subset X$ is a deformation retract.

Kervaire's second thesis II. Realizing groups by manifolds

- **Lemma** (K) For $m \geqslant 4$ every group π with a finite presentation
$$\Phi \;=\; \{g_1, g_2, \dots, g_p; r_1, r_2, \dots, r_q\}$$
is the fundamental group $\pi_1(M) = \pi$ of an m-dimensional framed manifold M.
- **Proof** Realize the generators $g_1, g_2, \dots, g_p$ by the m-dimensional framed manifold
$$M_0 \;=\; \mathop{\#}_{p}(S^1 \times S^{m-1})$$
with free fundamental group $\pi_1(M_0) = \mathop{*}_{p}\mathbb{Z}$.
- Realize the relations $r_1, r_2, \dots, r_q$ by surgeries on $S^1 \times D^{m-1} \subset M_0$, to get an m-dimensional framed manifold
$$M \;=\; (M_0 \backslash \bigcup_q (S^1 \times D^{m-1})) \cup \bigcup_q (D^2 \times S^{m-2})$$
with $\pi_1(M) = \pi$.

Kervaire's second thesis III. Knot groups

- **Theorem** (K) For $n \geqslant 3$ a group π is the fundamental group $\pi_1(M)$ of an n-knot exterior M if and only if π has a finite presentation Φ, $H_1(\pi) = \mathbb{Z}$, $H_2(\pi) = 0$ and there is an element $x \in \pi$ normally generating π.
- **Proof** Lemma gives an $(n+2)$-dimensional framed manifold M_1 with $\pi_1(M_1) = \pi$. Do further surgeries on $S^2 \times D^n \subset M_1$ to get M_2 with $H_i(M_2) = 0$ for $2 \leqslant i \leqslant n$.
- Realize $x \in \pi_1(M_2) = \pi$ by $D^{n+1} \times S^1 \subset M_2$. The $(n+2)$-dimensional manifold with boundary

$$(M^{n+2}, \partial M) \;=\; (\mathrm{cl.}(M_2 \backslash (D^{n+1} \times S^1)), S^n \times S^1)$$

is such that there is a homotopy sphere $\Sigma^{n+2} = M \cup_{\partial M} S^n \times D^2$ with an embedding $e : S^n \hookrightarrow \Sigma^{n+2}$ such that $\pi_1(\Sigma^{n+2} \backslash e(S^n)) = \pi$.
- Connected sum with $-\Sigma \in \Theta_{n+2}$ gives an n-knot

$$f \;=\; e\#0 \;:\; S^n \hookrightarrow \Sigma \# -\Sigma \;=\; S^{n+2}$$

with $\pi_1(S^{n+2} \backslash f(S^n)) = \pi$.

Kervaire's second thesis IV. Knot modules

- The complement $X = S^{n+2}\backslash f(S^n)$ of an n-knot $f : S^n \hookrightarrow S^{n+2}$ has a canonical infinite cyclic cover $\overline{X}$ with $\pi_q(X) = \pi_q(\overline{X})$ for $q \geqslant 2$.
- **Definition** An n-knot $f : S^n \hookrightarrow S^{n+2}$ is q-**simple** if $\pi_i(X) = \pi_i(S^1)$ for $1 \leqslant i < q$, in which case $\pi_q(X) = H_q(\overline{X})$.
- **Theorem** (K) Let A be a $\mathbb{Z}[t, t^{-1}]$-module and q, n integers such that $1 < q < n/2$. There exists a q-simple n-knot $f : S^n \hookrightarrow S^{n+2}$ with $A = \pi_q(X)$ if and only if A is finitely presented and $1 - t : A \to A$ is an isomorphism.
- More complicated characterization of the $\mathbb{Z}[t, t^{-1}]$-modules $\pi_q(X)$ of q-simple n-knots $f : S^n \hookrightarrow S^{n+2}$ with $n = 2q - 1$ or $2q$ for $q \geqslant 3$.
- **Theorem** (K) For every n-knot $f : S^n \hookrightarrow S^{n+2}$ there exists a **Seifert surface** $F^{n+1} \subset S^{n+2}$ with $\partial F = f(S^n) \subset S^{n+2}$.
- **Proof** Replace X by the exterior M, and represent $1 \in H^1(X) = H^1(M) = \mathbb{Z}$ by a map $g : M \to S^1$ transverse at $* \in S^1$, with $F = g^{-1}(*)$.
- Generalization of the original Seifert surfaces for 1-knots $f : S^1 \hookrightarrow S^3$.

Kervaire's second thesis V. Knot cobordism

- **Definition** Two n-knots $f_0, f_1 : S^n \hookrightarrow S^{n+2}$ are **cobordant** if there exists an embedding $g : S^n \times I \hookrightarrow S^{n+2} \times I$ such that

$$g(x,i) \ = \ f_i(x) \ \ (x \in S^n, \ i = 0,1) \ .$$

- The **knot cobordism group** C_n is the group of cobordism classes of n-knots.
- Generalization of the Fox-Milnor (1957) cobordism group C_1 of 1-knots.
- **Theorem** (K) (i) $C_{2k} = 0$ by ambient surgery on odd-dimensional Seifert surface $F^{2k+1} \subset S^{2k+2}$ rel $\partial F = S^{2k} \subset S^{2k+2}$ to $D^{2k+1} \subset S^{2k+2}$, generalizing $P_{2k+1} = 0$.
- (ii) C_{2k-1} is infinitely generated, detected by Alexander polynomials.
- Kervaire initiated the systematic study of high dimensional knot theory. Carried forward by J. Levine – computation of C_{2k-1} for $k \geqslant 2$ using Seifert forms and signatures. Subsequent non-simply-connected generalization due to Cappell and Shaneson.

The Seifert form

- **Definition** The **Seifert form** of a Seifert surface $F^{2k} \subset S^{2k+1}$ for $S^{2k-1} \hookrightarrow S^{2k+1}$ is the bilinear pairing defined by linking numbers

$$\psi \ : \ H_k(F)/\text{torsion} \times H_k(F)/\text{torsion} \to \mathbb{Z} \ ; \ (x,y) \mapsto L(x, i_+y)$$

  with $i_+ : F \to S^{2k+1} \backslash F$ the map pushing F off itself.
- The expression of the $(-)^k$-symmetric intersection form as $\lambda(x,y) = \psi(x,y) + (-)^k \psi(y,x) \in \mathbb{Z}$ is an extreme 'division by 2'.
- For odd k the function

$$\widehat{\mu} \ : \ H_k(F)/\text{torsion} \to \mathbb{Z} \ ; \ x \mapsto \psi(x,x)$$

  sends $x : S^k \hookrightarrow F$ to

$$\psi(x,x) \ = \ (\delta\nu_x, \nu_x) \in \pi_k(V_{k+1,1}) \ = \ \pi_k(S^k) \ = \ \mathbb{Z}$$

  with $\delta\nu_x : \nu_x \oplus \epsilon \cong \epsilon^{k+1}$ the 1-stable framing determined by

$$F \times \mathbb{R} \subset S^{2k+1} \backslash \{\text{pt.}\} \ = \ \mathbb{R}^{2k+1} \ .$$

- $\widehat{\mu}$ is an integral refinement of $\mu : H_k(F)/\text{torsion} \to Q_-(\mathbb{Z}) = \mathbb{Z}_2$.

Non-simply-connected intersection numbers

- 12. **Geometric and algebraic intersection numbers**, Comm. Math. Helv. 39, 271–280 (1965)
- A $2n$-dimensional manifold M^{2n} with universal cover $\widetilde{M}$ has a $(-)^n$-symmetric intersection pairing

$$\lambda \ : \ H_n(\widetilde{M}) \times H_n(\widetilde{M}) \to \mathbb{Z}[\pi_1(M)] \ ; \ (x,y) \mapsto \lambda(x,y) = (-)^n \overline{\lambda(y,x)}$$

  with the involution on $\mathbb{Z}[\pi_1(M)]$ given by $\overline{g} = g^{-1}$ $(g \in \pi_1(M))$.
- **Theorem** If $n \geqslant 3$, no 2-torsion in $\pi_1(M)$ then $x \in \pi_n(M)$ represented by embedding $S^n \hookrightarrow M$ if and only if $\lambda(x,x) \in \mathbb{Z} \subseteq \mathbb{Z}[\pi_1(M)]$.
- **Corollary** For $M^{2n} = S^1 \times S^{2n-1} \# S^n \times S^n$, $\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[t,t^{-1}]$,

$$x \ = \ (1, t + (-)^n t^{-1}) \in \pi_n(M) \ = \ H_n(\widetilde{M}) \ = \ \mathbb{Z}[t,t^{-1}] \oplus \mathbb{Z}[t,t^{-1}]$$

  has

$$\lambda(x,x) \ = \ 2(t + t^{-1}) \notin \mathbb{Z} \subset \mathbb{Z}[t,t^{-1}]$$

  so not represented by $S^n \hookrightarrow M$.

Surgery obstruction theory I.

- The Kervaire-Milnor classification of exotic spheres is the precursor of the Browder-Novikov-Sullivan-Wall surgery theory (1965-1970) for classifying manifolds of dimensions $\geqslant 5$, involving the Wall surgery obstruction groups $L_*(\mathbb{Z}[\pi])$ with π the fundamental group.
- $L_m(A)$ defined for any ring A with an involution $A \to A; a \mapsto \overline{a}$, with $L_m(A) = L_{m+4}(A)$.
- $L_{2n}(A)$ is the Witt group of nonsingular $(-)^n$-quadratic forms (H, λ, μ) over A, with H a f.g. free A-modules and

$$\begin{aligned} \lambda \ &: \ H \times H \to A \ ; \ (x,y) \mapsto \lambda(x,y) \ = \ (-)^n\overline{\lambda(y,x)} \ , \\ \mu \ &: \ H \to Q_{(-)^n}(A) \ = \ A/\{a - (-)^n\overline{a} \,|\, a \in A\} \end{aligned}$$

  such that

$$\lambda(x,x) \ = \ \mu(x) + (-)^n\overline{\mu(x)} \ , \ \lambda(x,y) \ = \ \mu(x+y) - \mu(x) - \mu(y) \ .$$

- $L_{2n+1}(A)$ is the stable group of automorphisms of $(-)^n$-quadratic forms over A.

Surgery obstruction theory II.

- Kervaire's intersection form λ over $\mathbb{Z}[\pi_1(M)]$ was generalized by Wall (1966-70) to a $(-)^n$-quadratic function counting double points

  $$\mu \ : \ \{\text{immersions } S^n \looparrowright M^{2n}\} \to Q_{(-)^n}(\mathbb{Z}[\pi_1(M)])\ ;\ x \mapsto D_2(\widetilde{x} : S^n \looparrowright \widetilde{M})$$

  with $\widetilde{M}$ the universal cover, such that

  $$\lambda(x,x) - (\mu(x) + (-)^n\overline{\mu(x)}) \ = \ \chi(\nu_x) \in \mathbb{Z} \subseteq \mathbb{Z}[\pi_1(M)]\ .$$

  The surgery obstruction of n-connected normal map $(f,b) : M \to X$ is

  $$\sigma_*(f,b) \ = \ (\ker(\widetilde{f}_* : H_n(\widetilde{M}) \to H_n(\widetilde{X})), \lambda, \mu) \in L_{2n}(\mathbb{Z}[\pi_1(X)])\ .$$

- Wall's μ is a regular homotopy invariant such that for $n \geqslant 3$ $x : S^n \looparrowright M^{2n}$ is regular homotopic to an embedding if and only if

  $$\mu(x) \ = \ 0 \in Q_{(-)^n}(\mathbb{Z}[\pi_1(M)])\ .$$

  Generalization of Kervaire's embedding condition to $\pi_1(M)$ with 2-torsion: for $n \geqslant 3$ $x \in \pi_n(M)$ represented by embedding if and only if

  $$\mu(x) \in Q_{(-)^n}(\mathbb{Z}) \subseteq Q_{(-)^n}(\mathbb{Z}[\pi_1(M)])\ .$$

Surgery with $\pi_1 = \mathbb{Z}$

- 13. (with A. Vasquez) **Simple-connectivity and the Browder-Novikov theorem** Trans. A.M.S. 126, 508–513 (1967)
- **Theorem** (Browder, Novikov 1962) For $n \geqslant 2$ a simply-connected $(2n+1)$-dimensional Poincaré duality space X is homotopy equivalent to a manifold if and only if the Spivak normal fibration $\nu_X : X \to BG$ admits a vector bundle reduction $\widetilde{\nu}_X : X \to BO$.
- **Theorem** (K+V) Browder-Novikov result is false if $\pi_1(X) \neq \{1\}$.
- **Proof** Application of the Kervaire invariant and high dimensional knot theory to construct $(8k+3)$-dimensional PL manifolds X with $\pi_1(X) = \mathbb{Z}$ and vector bundle reductions, but which are not homotopy equivalent to differentiable manifolds.
- Now seen as a precursor of the Shaneson-Wall-Novikov-R. splitting

$$L_{4*+3}(\mathbb{Z}[t,t^{-1}]) \ = \ L_{4*+2}(\mathbb{Z}) \ = \ P_{4*+2} \ = \ \mathbb{Z}_2 \ .$$

High dimensional homology spheres

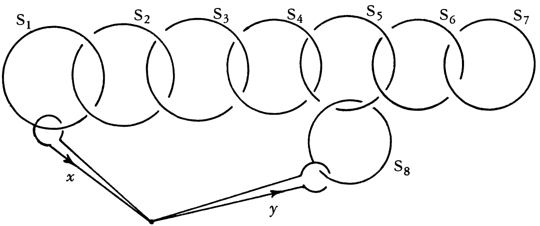
- 14. **Smooth homology spheres and their fundamental groups**, Trans. Amer. Math. Soc. 144, 67–72 (1969)
- **Definition** An H-**cobordism** of m-dimensional manifolds is a cobordism $(W^{m+1}; M^m, M'^m)$ such that $H_*(M) \cong H_*(W) \cong H_(M')$.
- **Definition** An m-dimensional manifold Σ^m is a **homology sphere** if $H_*(\Sigma) = H_*(S^m)$. Let Θ^H_m be the abelian group of H-cobordism classes of m-dimensional homology spheres, with addition by connected sum.
- **Theorem** (K) For $m \geqslant 5$ a group π is the fundamental group $\pi_1(\Sigma)$ of an m-dimensional homology sphere Σ^m if and only if π is finitely presented, $H_1(\pi) = 0$ and $H_2(\pi) = 0$.
- **Theorem** (K) For $m \geqslant 4$ every m-dimensional homology sphere Σ^m is H-cobordant to a homotopy sphere, and the forgetful map $\Theta_m \to \Theta^H_m$ is an isomorphism.

A picture!

- The **Rohlin invariant** map

  $$r \; : \; \Theta_3^H \to \mathbb{Z}_2 \; ; \; \Sigma^3 = \partial W \mapsto \text{signature}(W)/8 \; (W^4 \text{ parallelizable})$$

  is onto. The 4-dimensional E_8-plumbing $(M^4, \partial M)$ of 8 τ_{S^2}'s has ∂M the 3-dimensional Poincaré homology sphere, signature$(M) = 8$, $r(\partial M) = 1$. Picture from Kervaire's 1969 paper:



- Θ_3^H and r play a vital role in the Kirby-Siebenmann (1970) structure theory of high dimensional topological manifolds. After Donaldson (1982) it is known that Θ_3^H is infinitely generated.

The E_8 bread and wine

Influential expositions

- 15. **Le théorème de Barden-Mazur-Stallings**, Comment. Math. Helv. 40, 31–42 (1965)
- 16. (with G. de Rham and S. Maumary) **Torsion et type simple d'homotopie**, Springer Lecture Notes 48 (1964)
- 17. **Lectures on the theorem of Browder and Novikov and Siebenmann's thesis**, Tata, Bombay (1969)
- 18. **Knot cobordism in codimension two**, Manifolds–Amsterdam 1970, Springer Lecture Notes 197, 83–105 (1971)
- 19. (with C. Weber), **A survey of multidimensional knots**, Knot theory (Proc. Sem., Plans-sur-Bex, 1977), Springer Lecture Notes 197, 61–134 (1978)