Sir Michael Atiyah

Early landmarks

Oxford, Cambridge, Princeton, Harvard, Bonn

Atiyah's subsequent career oscillated between Cambridge, Oxford and Princeton. In 1957 he returned to Cambridge as a University Lecturer. In 1961 he moved to Oxford, first as a Reader and then from 1963-69 as Savilian Professor of Geometry.

After his stay at Princeton from 1969-72, Atiyah returned to Oxford as a Royal Society Research Professor. He remained there until in 1990 he moved back to Cambridge as Master of Trinity College. In 1997 he retired from Cambridge, and moved to the University of Edinburgh, where he is an Honorary Professor.

Bott and Atiyah. Bombay, 1963

If Cambridge, Oxford and Princeton were the universities where Atiyah had permanent positions, in his career an important part was also played by Harvard and Bonn.

Hirzebruch and Atiyah

During his close collaborations with R. Bott and I. Singer, he spent two sabbatical terms at Harvard and for around twenty-five years he used to go to Bonn for the annual Arbeitstagung. Organized by Hirzebruch, they were enormously exciting events with many new results being discussed and with a stellar cast of participants. The opening lecture was always assigned to Atiyah.

K-theory

During the early years in Bonn much attention was paid to Hirzebruch's Riemann-Roch Theorem and its subsequent generalization by A. Grothendieck.

Almost at the same time Bott made his famous discovery of the periodicity theorem in the classical groups. Atiyah soon realised that there existed close links between the work of Bott and Grothendieck. This led to new concrete results in algebraic topology which convinced him that it was worth developing a topological version of Grothendieck's *K*-theory.

The development of this idea, which grew into a signifcant enterprise, was done jointly with Hirzebruch. They wrote many joint papers on various aspects and applications of *K*-theory. About this collaboration Atiyah has expressed that he "learnt much, not least in how to write papers and present lectures", and also that Hirzebruch was "an elder brother" who continued his education.

Some of the remarkable consequences of Hirzebruch's Riemann-Roch Theorem had been the integrality of various expressions in characteristic classes.

A priori, since these formulae had denominators, the answers were rational numbers. But in fact, under appropriate hypotheses, they turned out to be integers. For complex algebraic manifolds this followed from their interpretation as holomorphic Euler characteristics, a consequence of the Riemann-Roch Theorem.

For other manifolds Hirzebruch had been able to deduce integrality by various topological tricks, but this seemed unsatisfactory. Topological K-theory gave a better explanation for these integrality theorems, closer to the analytic proofs derived from sheaf theory in the case of complex manifolds.

Index theory

A particularly striking case was the fact that an expression called by Hirzebruch the $\hat{A}$-genus was an integer for spin-manifolds.

It was the attempt to understand this fact that eventually led Atiyah and Singer to their celebrated theorem.

Because of the comparison with analytic methods on complex manifolds, it was natural to ask if there was any analytical counterpart for spin-manifolds.

A key breakthrough came with the realization that Dirac had, thirty years before, introduced the famous differential operator that bears his name. Next, with Singer's background in physics and differential geometry, they were able to define (on a spin-manifold with a Riemannian metric) a Dirac operator acting naturally on spinor fields. Finally, with Atiyah's familiarity with the character formulae for the spin manifolds (from his apprenticeship with Hirzebruch) they easily saw that the index of the Dirac operator should be equal to the mysterious $\hat{A}$-genus.

This work started while Singer was spending a sabbatical term in Oxford. They also had a brief visit from S Smale, just returned from Moscow, who told them that Gel'fand had proposed the general problem of computing the index of any elliptic differential operator.

Their knowledge of *K*-theory allowed them to see that the Dirac operator was in fact the primordial elliptic operator and that, in a sense, it generated all others. Thus a proof of the conjectured index formula for the Dirac operator would yield a formula for all elliptic operators.

Over the subsequent decades the index theorem in its various forms and generalizations occupied most of the efforts of Atiyah and Singer.

THE INDEX OF ELLIPTIC OPERATORS ON COMPACT MANIFOLDS

BY M. F. ATIYAH AND I. M. SINGER[1]

Communicated by Raoul Bott, February 1, 1963

Introduction. In his paper [16] Gel'fand posed the general problem of investigating the relationship between topological and analytical invariants of elliptic differential operators. In particular he suggested that it should be possible to express the *index* of an elliptic operator (see §1 for the definition) in topological terms. This problem has been taken up by Agranovic [2; 3], Dynin [3; 14; 15], Seeley [20; 21] and Vol'pert [22] who have solved it in special cases. The purpose of this paper is to give a general formula for the index of an elliptic operator on any compact oriented differentiable manifold (Theorem 1). As a special case of this formula we get the Hirzebruch-Riemann-Roch theorem for any compact complex manifold (Theorem 3). This was previously known only for projective algebraic manifolds. Some other special cases, of interest in differential topology, are discussed in §3.

We are greatly indebted to A. P. Calderon, L. Nirenberg, and R. T. Seeley for their generous help.

1. Elliptic operators. Let X be a compact oriented smooth mani-

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A particularly interesting strand was a Lefschetz fixed point formula which Atiyah developed with Bott (now known as the Atiyah-Bott fixed point formula). They also reached a fuller understanding of elliptic boundary value problems. It was during this period that Atiyah spent two sabbatical terms at Harvard, which he recalls "as a particularly stimulating and fruitful time".

178 M. F. ATIYAH AND R. BOTT

$$\text{index }(D,B) = \int_{\partial B(X)} \sigma(D,B)^* ch \wedge \pi^* \mathscr{T}(X).$$

We will give the definition of $\sigma(D, B)$ in rather complete detail in Sections 2 and 3. The proof of Theorem 2 is then sketched in Section 4.

Another important extension of the index theorem which required the collective efforts of Atiyah, Bott, Singer and Patodi was the local form of the index theorem and the contribution of the boundary arising from the η-invariant. This was a spectral invariant, analogous to the L-functions of number theory and originating in fact in a beautiful conjecture of Hirzebruch on the cusps of Hilbert modular surfaces. Most of this work was done while Atiyah was a professor at Princeton.

Graeme Segal, who was one of Atiyah's early research students, collaborated with him on the equivariant version of the index theorem as well as on aspects of *K*-theory.

Sources

The text on this poster is extracted from [1], with some slight changes to fit the narrative style. The pictures with Bott and with Hirzebruch are digital reproductions from parts of pictures in [2]. The second image on this column is taken from the paper *The index problem for manifolds with boundary*, in ***Differential Analysis*** (papers presented at the Bombay Colloquium 1964), Oxford University Press, 175-186.

[1] M. Atiyah: *Autobiography* (text written for the Abel Prize Committee, 2004).
[2] *An Interview With Michael Atiyah*. Conducted by Roberto Minio and published in ***The Mathematical Intelligencer***, vol. 6, Number 1 (1984), 9-19.