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WHITEHEAD GROUPS OF GENERALIZED FREE PRODUCTS

- preliminary report -

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The generalized free products referred to are of two types. The first one generalizes "taking the free product of A and B", and is the usual free product with amalgamation, $A *_C B$. The other one generalizes in a similar way "taking the free product of A with an infinite cyclic group $\{t\}$, generated by t ", and is defined as follows. Let C_1 and C_2 be subgroups of A, and $\psi: C_1 \to C_2$ an isomorphism. Form the free product $A*\{t\}$, and introduce the relations $t^{-1}ct = \psi(c)$, for $c \in C_1$. The resulting group is denoted $A *_C \{t\}$. There is a pushout definition of $A *_C \{t\}$, using groupoids however.

The construction $A *_C \{t\}$ occurs implicitly in Magnus' analysis of one-relator groups. Subsequently, it has been employed in deriving embedding theorems for groups. It has become quite popular among logicians, and is sometimes called the Higman-Neumann-Neumann-Britton extension.

As an example consider a CW pair X,Y which, for convenience, is a codimension 1 pair in the sense that Y is closed in X, and has a neighborhood U in X so that U,Y is cellularly equivalent to the pair $Y \times [-1,+1]$, $Y \times 0$. If $\pi_1 Y \to \pi_1 X$ is a monomorphism, then $\pi_1 X$ is a generalized free product $A *_C B$ or $A *_C \{t\}$, respectively, according to whether Y does or does not separate X, with $C = \pi_1 Y$.

This is our main result:

Let $\alpha: C \to A$, $\beta: C \to B$ be the inclusions defining $A *_C B$, and $\alpha, \beta: C \to A$ those defining $A *_C \{t\}$, respectively.

In the situation $D = A *_C B$, there is a natural direct sum splitting

$$(*)\quad Wh(D) \approx Wh(A,B;C) \oplus \tilde{K}_0(C;A,B) \oplus \tilde{C}(C;A,B) .$$

Here, by definition,

$$Wh(A,B;C) = \operatorname{coker}(\alpha_* \oplus -\beta_* : Wh(C) \to Wh(A) \oplus Wh(B))$$
$$\tilde{K}_0(C;A,B) = \ker(\alpha_* \oplus -\beta_* : \tilde{K}_0(C) \to \tilde{K}_0(A) \oplus \tilde{K}_0(B))$$

(we omit the "Z" in the notation for projective class groups of (integral) group rings).

In the situation $D = A *_C \{t\}$, there is a similar splitting

$$(*)\quad Wh(D) \approx Wh(A,A;C) \oplus \tilde{K}_0(C;A,A) \oplus \tilde{D}(C;A,A) .$$

Here,

$$Wh(A,A;C) = \operatorname{coker}(\alpha_* - \beta_* : Wh(C) \to Wh(A))$$
$$\tilde{K}_0(C;A,A) = \ker(\alpha_* - \beta_* : \tilde{K}_0(C) \to \tilde{K}_0(A)) .$$

The definitions of $\tilde{C}(C;A,B)$ and $\tilde{D}(C;A,A)$ are long, and are given in section 2.

There is an appealing geometric interpretation of the formulas (*). Let X,Y be a codimension 1 pair of finite CW complexes, as described above, and $\pi_1 Y \to \pi_1 X$ a monomorphism, so $\pi_1 X$ is a generalized free product $A *_C B$ or $A *_C \{t\}$, in a natural way. Let W be a finite CW complex and $f: W \to X$ a homotopy equivalence. We ask for the obstruction to finding a formal deformation from $f:W$ to $f':W'$ so that, for $V' = f'^{-1}(Y)$, V' is a subcomplex of W', and $f'|V'$ is a homotopy equivalence. It turns out that these obstructions are classified by the term $\tilde{K}_0(C;A,B) \oplus \tilde{C}(C;A,B)$ or $\tilde{K}_0(C;A,A) \oplus \tilde{D}(C;A,A)$, respectively, cf. section 6.

This geometric interpretation is closely related to the very way of proving (*). The underlying idea is just, given $f: W \to X$ as above, to attempt to make $f|f^{-1}(Y)$ a homotopy equivalence. Our proof is essentially a translation into algebra of this idea. The relevant machinery, leading eventually to the proof of the sum theorem (*), occupies the bulk of the paper, sections 3 through 5.

The present work was motivated by a comparison of classification results in high-dimensional and low-dimensional manifolds, respectively. Whereas in the former the Whitehead group enters in an essential way (via the s-cobordism theorem), it frequently is totally absent in the latter. On the other hand, 3-dimensional lens spaces show that dimension 3 does have sufficient room for the Whitehead group to enter. This suggested that for the manifolds considered in [11], $Wh(\pi_1)$ should be trivial, and my original attempt was to mimick the techniques of [11] in the world of simple homotopy types. The outcome is however somewhat different.

An application of the sum theorem (*) is in deriving vanishing theorems for the Whitehead group, cf. section 7. Our results in this direction justify the above guess.

For a while, I thought that prop. 1.4 could be proved without the assumption of coherence. This would have verified conjecture 7.3, of which [12] was conceived as an application. [12] certainly makes it clear enough why conjecture 7.3 is important.

Before publishing this material (probably under some title like "Algebraic K-functors of generalized free products"), I will try for some time to get hold of conjecture 7.3 in some other way. The plan is to recast the proof of the sum theorem in such a way that it applies to other K-functors as well. - It would be easy to obtain analogues of the sum theorem for the functors derived from the Whitehead group as "contracted functors" [1], just by working with various coefficient rings, and comparing. The problem is with the exotic nil-functors of sections 1 and 2.

References in the paper are by page numbers.

1. A SPECIAL SPLITTING OBSTRUCTION GROUP

Let R be a ring, and S a R-bimodule which is free (or projective - this doesn't matter) both as a left or right R-module. S need not be finitely generated as a module. We will define a functor, defined on pairs (R,S), using the following category.

An <u>object</u> is a pair (P,p), P a finitely generated projective (left) R-module, and p a R-homomorphism

$$p\colon P \to S \otimes_R P ,$$

a <u>morphism</u> (P,p) → (Q,q) is a commutative square

$$\begin{array}{ccc} P & \xrightarrow{p} & S \otimes_R P \\ m \downarrow & & \downarrow 1\otimes m \\ Q & \xrightarrow{q} & S \otimes_R Q \end{array}$$

for some R-map m.

For an object (P,p), a filtration

$$0 = A_0 \subset A_1 \subset \dots \subset A_j = P$$

is called an <u>assailable filtration</u> if

$$p(A_{i+1}) \subset S \otimes A_i .$$

It is a <u>finitely assailable filtration</u> if, in addition, all the A_i are finitely generated. The object is <u>nilpotent</u> if it has an assailable filtration.

<u>Lemma</u>. Given an assailable filtration $0 \subset B_1 \subset \dots \subset B_j = P$, there exists a finitely assailable filtration $A_1 \subset \dots \subset A_j = P$ so that $A_i \subset B_i$.

<u>Proof</u>. By induction on decreasing i, we define A_i so that $p(A_{i+1}) \subset S\otimes A_i \subset S\otimes B_i$. Suppose A_{i+1} has been defined already. Let a be an element of A_{i+1}. Since $a \in B_{i+1}$, there is an expression $p(a) = \Sigma\, s_\alpha \otimes b_\alpha$ with $b_\alpha \in B_i$. We let A_i be generated by the b_α, for a running through a generating set of A_{i+1}.

A sequence of morphisms $(P_1,p_1) \to (P_2,p_2) \to (P_3,p_3)$ is <u>short exact</u>, if it induces a short exact sequence on the P's. In that case, there is a direct sum splitting $P_2 = P_1 \oplus P_3$, and by definition of a morphism we must have $p_2|P_1 = p_1$, and $p_2|P_3 = p_3 \oplus$ some R-map $P_3 \to S\otimes P_1$. In particular, (P_2,p_2) is nilpotent if and only if both (P_1,p_1) and (P_3,p_3) are.

An object (F,f) is a standard trivial object if there exists a finitely assailable filtration $0 = F_0 \subset F_1 \subset \ldots \subset F_j = F$, so that all the F_i and all the quotients F_i/F_{i-1} are free.

Given a nilpotent object (P,p), there exists a surjective morphism $(F,f) \to (P,p)$, for some standard trivial object (F,f). Namely, if $A_1 \subset \ldots \subset A_j = P$ is a finitely assailable filtration of P, we pick a free module $F_i^!$ with generators corresponding to a generating set of A_i, and a surjection $h_i^!\colon F_i^! \to A_i$, $i = 1,\ldots,j$. And by induction we define $f_i^!\colon F_i^! \to S\otimes F_{i-1}^!$ so that

$$\begin{array}{ccc} F_i^! & \xrightarrow{f_i^!} & S\otimes F_{i-1}^! \\ h_i \downarrow & & \downarrow 1\otimes h_{i-1}^! \\ A_i & \xrightarrow{p|A_i} & S\otimes A_{i-1} \end{array}$$

commutes. We now set $F_i = F_i^! \oplus \ldots \oplus F_1^!$, $f_i = f_i^! \oplus \ldots \oplus f_1^!$, and $h_i = h_i^! + \ldots + h_1^!$. With this we have

$$\begin{array}{ccc} F_i & \xrightarrow{f_i} & S\otimes F_{i-1} \\ h_i \downarrow & & \downarrow 1\otimes h_{i-1} \\ A_i & \xrightarrow{p|A_i} & S\otimes A_{i-1} \end{array}$$

and the F_i give the desired filtration of $(F,f) = (F_j,f_j)$.

Define $(Q,q) = (\ker(F \to P),\ f|\ker(F \to P))$. Then $(Q,q) \to (F,f) \to (P,p)$ is short exact, and the $B_i = F_i \cap Q$ give a (not necessarily finitely) assailable filtration of (Q,q). We refer to this construction as suspension, and call (Q,q) a suspension of (P,p) (there is of course no question of uniqueness).

We now introduce an equivalence relation on the nilpotent objects. The equivalence relation is to be generated by this definition: (Q,q) is equivalent to the inverse of (P,p) if it is a suspension of (P,p). (So far, inverse is just a language, exploiting the convention $(-1)\cdot(-1) = (+1)$.)

It is clear that the equivalence classes form a group under direct sum. On the other hand, we can form the Grothendieck group of the subcategory of nilpotent objects, and can pass to the quotient in which a free module with the zero homomorphism represents zero. From the following lemma it is clear that these two groups are in fact the same.

Lemma. Any suspension can be suspended to a split exact sequence.

Proof. Suppose we have a diagram

$$\begin{array}{ccccc}
(Q_1,q_1) & \to & (Q_1,q_1)\oplus(Q_3,q_3) & \to & (Q_3,q_3) \\
\downarrow & & \downarrow & & \downarrow \\
(F_1,f_1) & \to & (F_2,f_2) & \to & (F_3,f_3) \\
\downarrow & & \downarrow & & \downarrow \\
(P_1,p_1) & \to & (P_2,p_2) & \to & (P_3,p_3)
\end{array}$$

with short exact rows and columns, but with the middle term missing, and we want to fill in the middle term. For the moment replace (P_2,p_2) by $(P_1,p_1)\oplus(P_3,p_3)$. Then we can fill in $(F_1,f_1)\oplus(F_3,f_3)$ as the middle term. And we have $F_2 = F_1 \oplus F_3 = Q_1 \oplus P_1 \oplus Q_3 \oplus P_3$. Now (P_2,p_2) differs from $(P_1,p_1)\oplus(P_3,p_3)$ precisely by some (arbitrary R-) homomorphism $P_3 \to S\otimes P_1$. But this can be taken care of nicely by redefining $f_2|P_3$ in adjoining a component $P_3 \to S\otimes P_1$. The only non-trivial thing about the whole business is to verify that the (F_2,f_2) so obtained is standard trivial. But this is obvious.

Definition. The above group is denoted C(R,S). There are mappings

$$\tilde{K}_0(R) \to C(R,S) \to \tilde{K}_0(R) ,$$

defined by $P \to (P,0)$, and by forgetting the homomorphism, respectively. Hence there is a natural direct sum splitting

$$C(R,S) = \tilde{K}_0(R) \oplus \tilde{C}(R,S) ,$$

defining $\tilde{C}(R,S)$.

<u>Proposition</u>. Suppose R is coherent, and its global homological dimension is finite. Then $\tilde{C}(R,S) = 0$.

<u>Proof</u>. Let $p: P \to S \otimes P$ be an object, and $0 \subset A_1 \subset \ldots \subset P$ a finitely assailable filtration of (P,p). Pick a suspension

$$\begin{array}{ccccc} F_1 & \subset & F_2 & \subset \; . \; . \; . \subset & F \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \subset & A_2 & \subset \; . \; . \; . \subset & P \end{array}$$

for some standard trivial (F,f). As we obseved earlier, the $B_i = \ker(F_i \to A_i)$ form an assailable filtration of $(Q,q) = (\ker(F \to P), f|\ker(F \to P))$.

Now R is coherent, so all the B_i are finitely generated, cf. 4.4, so they form a finitely assailable filtration of (Q,q). Use this filtration to suspend again.

Going on this way, it is clear that in particular we are building up a projective resolution of A_1. But A_1 has finite homological dimension. So after finitely many steps, we will have replaced (P,p) by an object (P',p'), equivalent to (P,p) up to sign, which has a finitely assailable filtration starting with A_1', and A_1' is projective. Let P'' be an inverse for A_1', i.e., $A_1' \oplus P''$ is free. The object $(P',p') \oplus (P'',0)$ is equivalent to (P',p') in $\tilde{C}(R,S)$, and it has a finitely assailable filtration with first term free. So, performing one more suspension, we can reduce the length of the assailable filtration. By induction on this length it follows then that (P,p) represents zero in $\tilde{C}(R,S)$.

2. THE SPLITTING OBSTRUCTION GROUPS

Case $D = A *_C B$.

As part of the structure of $A *_C B$, we have inclusions $C \to A$ and $C \to B$, inducing inclusions $ZC \to ZA$ and $ZC \to ZB$. By pullback along the inclusion, ZA is both a left and right free ZC-module. And there is a natural splitting of bimodules,

$$ZA = ZC \oplus \widehat{ZA},$$

where $\widehat{ZA}$ is generated (either as a left or as a right module) by the non-trivial cosets of C in A. Similarly, $ZB = ZC \oplus \widehat{ZB}$.

We define a category as follows. An <u>object</u> is a quadruple $(P,Q;p,q)$, with P and Q finitely generated (left) projective ZC-modules, and p and q ZC-homomorphisms

$$p\colon P \to \widehat{ZB} \otimes_{ZC} Q$$
$$q\colon Q \to \widehat{ZA} \otimes_{ZC} P,$$

a <u>morphism</u> $(P_1,Q_1;p_1,q_1) \to (P_2,Q_2;p_2,q_2)$ is a pair of ZC-homomorphisms $m\colon P_1 \to P_2$ and $n\colon Q_1 \to Q_2$ so that

$$\begin{array}{ccc} P_1 & \xrightarrow{p_1} & ZB \otimes Q_1 \\ m \downarrow & & \downarrow 1\otimes n \\ P_2 & \xrightarrow{p_2} & ZB \otimes Q_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} Q_1 & \xrightarrow{q_1} & ZA \otimes P_1 \\ n \downarrow & & \downarrow 1\otimes m \\ Q_2 & \xrightarrow{q_2} & ZA \otimes P_2 \end{array}$$

commute. A <u>short exact sequence</u> is the obvious thing.

An <u>assailable filtration</u> is a pair of converging filtrations, by submodules,

$$0 = K_0 \subset K_1 \subset \cdots \subset K_j = P$$
$$0 = L_0 \subset L_1 \subset \cdots \subset L_j = Q$$

so that $p(K_i) \subset \widehat{ZB} \otimes L_{i-1}$ and $q(L_i) \subset \widehat{ZA} \otimes K_{i-1}$, for all i; it is <u>finitely assailable</u> if all the K_i and L_i are finitely generated.

An object is <u>nilpotent</u> if it has an assailable (and hence also a finitely assailable) filtration. It is <u>standard trivial</u> if it has a finitely assailable filtration so that all the K_i and L_i, and all the K_i/K_{i-1} and L_i/L_{i-1} are free.

We now have by arguments analogous to those in the preceeding section: Given a nilpotent object $(P,Q;p,q)$, there exists a surjective morphism $(F,G;f,g) \to (P,Q;p,q)$, for some standard trivial object $(F,G;f,g)$. We define the kernel of this morphism, $(P',Q';p',q')$ (with, of course, $P' = \ker(F \to P)$, $Q' = \ker(G \to Q)$, $p' = f|P'$, $q' = g|Q'$), also called a <u>suspension</u> of $(P,Q;p,q)$, to be <u>equivalent to the inverse</u> of $(P,Q;p,q)$. Then the equivalence classes of nilpotent objects form a group under direct sum, denoted $C(C;A,B)$. This is just the quotient of the Grothendieck group of the subcategory of nilpotent objects, in which $(F,G;0,0)$, with F,G free, represents zero.

There are natural maps

$$\tilde{K}_0(ZC) \oplus \tilde{K}_0(ZC) \to C(C;A,B) \to \tilde{K}_0(ZC) \oplus \tilde{K}_0(ZC) ,$$

defined by $P \oplus Q \to (P,Q;0,0)$, and by forgetting the homomorphisms, respectively. Therefore

$$C(C;A,B) = \tilde{K}_0(ZC) \oplus \tilde{K}_0(ZC) \oplus \tilde{C}(C;A,B) ,$$

defining $\tilde{C}(C;A,B)$. And we have the vanishing theorem that $\tilde{C}(C;A,B) = 0$, provided ZC is coherent and its global homological dimension is finite.

<u>Case</u> $D = A *_C \{t\}$.

As part of the structure of $A *_C \{t\}$, we have inclusions

$$\alpha: C \to A \quad \text{and} \quad \beta: C \to A ,$$

inducing inclusions $ZC \to ZA$ also denoted by α and β.

With respect to either inclusion α or β, ZA may be considered as a ZC-bimodule which is free both as a left or right ZC-module. And there are splittings of bimodules

$$ZA = ZC \oplus \widehat{ZA}^{\alpha} , \quad \text{and}$$

$$ZA = ZC \oplus \widehat{ZA}^{\beta} ,$$

where $\widehat{ZA}^{\alpha}$ (resp. $\widehat{ZA}^{\beta}$) is generated (either as a left or as a right module) by the non-trivial cosets of the inclusion $\alpha: C \to A$ (resp. $\beta: C \to A$).

Our interest is in the category the <u>objects</u> of which are quadruples $(P,Q;p,q)$ where P, Q are finitely generated (left) projective ZC-modules and p, q are ZC-homomorphisms

$$p\colon P \to_\beta ZA \otimes_\alpha P \oplus \widehat{ZA}^\beta \otimes_\beta Q$$

$$q\colon Q \to_\alpha ZA \otimes_\beta Q \oplus \widehat{ZA}^\alpha \otimes_\alpha P$$

(the tensor products are taken over ZC, the subscript α or β indicates that the action comes from the inclusion $\alpha\colon C \to A$ or $\beta\colon C \to A$, respectively; similarly for the homomorphisms).

With <u>morphisms</u> in the obvious way, and <u>nilpotency</u> similarly as above, we have a Grothendieck group of nilpotent objects. Its maximal quotient in which $(F,G;0,0)$, with F, G free, represents zero, is denoted $D(C;A,A)$. There is a further quotient, namely the cokernel of the natural (split) inclusion $\tilde{K}_0(ZC) \oplus \tilde{K}_0(ZC) \to D(C;A,A)$; it is denoted $\tilde{D}(C;A,A)$.

We may now formulate propositions, analogous to the above, concerning $D(C;A,A)$ and $\tilde{D}(C;A,A)$. In particular, there is the vanishing theorem saying that $\tilde{D}(C;A,A) = 0$, provided ZC is coherent and its global homological dimension is finite.

<u>Remark</u>. If both $\alpha, \beta\colon C \to A$ are isomorphisms, $A *_C \{t\}$ is an extension $A \times_\gamma Z$, with γ given by $\alpha^{-1}\beta$ (or its inverse). Just as it ought to do, $\tilde{D}(C;A,A)$ in this case reduces to twice Farrell's group (or Bass' group in case $\gamma = \mathrm{id.}$), $\tilde{C}(ZC,\gamma) \oplus \tilde{C}(ZC,\gamma^{-1})$ in the customary notation.

3. MAYER VIETORIS SPLITTINGS

A based free ZE-module, E a group, has a basis which is well defined up to order, sign, and multiplication by elements of E. A based isomorphism is the obvious thing. A short exact sequence is based, if the middle term splits as a direct sum, the summands being generated by complementary subsets of the basis, so that there result two based isomorphisms; similarly for a based monomorphism.

A based ZE-complex is a (positive) chain complex of based (free) ZE-modules. A short exact sequence of based ZE-complexes is based if each short exact sequence of chain modules, one for each dimension, is based. If $\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3$ is a based short exact sequence of ZE-complexes, the bases give a splitting (of chain modules only) $\mathcal{C}_2 = \mathcal{C}_1 \oplus \mathcal{C}_3$. And $\mathcal{C}_2$ may be canonically identified with the algebraic mapping cone of a certain map $\mathcal{C}_3 \to \mathcal{C}_1$; this map is that part of the boundary operator in $\mathcal{C}_2$ which goes from $\mathcal{C}_3$ to $\mathcal{C}_1$.

In what follows $\mathcal{C}_D$ denotes a based chain complex over ZD. We will investigate structures on $\mathcal{C}_D$, called Mayer Vietoris presentations (or MV presentations, for short), which in some sense reflect the generalized free product structure on D.

Case $D = A *_C B$.

The data of a MV presentation of $\mathcal{C}_D$ consist of ba~~/~~sed chain complexes $\mathcal{C}_C, \mathcal{C}_A, \mathcal{C}_B$ over ZC, ZA, ZB, respectively, and monomorphisms

$$\iota_\alpha : \mathcal{C}_C \to \mathcal{C}_A, \quad \iota_\beta : \mathcal{C}_C \to \mathcal{C}_B \quad \text{over ZC, and}$$
$$\iota^*_\alpha : \mathcal{C}_A \to \mathcal{C}_D, \quad \iota^*_\beta : \mathcal{C}_B \to \mathcal{C}_D \quad \text{over ZA, resp. ZB,}$$

inducing based monomorphisms

$$ZA \otimes_C \mathcal{C}_C \to \mathcal{C}_A\ , \quad ZB \otimes_C \mathcal{C}_C \to \mathcal{C}_B \quad \text{over ZA resp. ZB,}$$

so that

$$0 \to ZD \otimes_C \mathcal{C}_C \xrightarrow{1\otimes\iota_\alpha \,\oplus\, (-1)\otimes\iota_\beta} \left(ZD \otimes_A \mathcal{C}_A\right) \oplus \left(ZD \otimes_B \mathcal{C}_B\right)$$
$$\xrightarrow{1\otimes\iota^*_\alpha \,+\, 1\otimes\iota^*_\beta} \mathcal{C}_D \to 0$$

is exact, and the given basis of $\mathcal{C}_D$ coincides with the induced one.

Case $D = A *_C \{t\}$.

The data of a MV presentation consist of based chain complexes $\mathcal{C}_C$, $\mathcal{C}_A$ over ZC, ZA, respectively, a monomorphism $\iota^* : \mathcal{C}_A \to \mathcal{C}_D$ over ZA and monomorphisms

$$\iota_\alpha : \mathcal{C}_C \to {}_\alpha\mathcal{C}_A, \quad \iota_\beta : \mathcal{C}_C \to {}_\beta\mathcal{C}_A$$

over ZC, inducing based ZA-monomorphisms

$$ZA_\alpha \otimes_C \mathcal{C}_C \to \mathcal{C}_A, \quad ZA_\beta \otimes_C \mathcal{C}_C \to \mathcal{C}_A$$

(here the subscript α, resp. β, on a ZA- module indicates that. we are using the ZC-structure induced from the inclusion $\alpha: C \to A$, resp. $\beta: C \to A$) so that

$$ZD \otimes_C \mathcal{C}_C \xrightarrow{1\otimes\iota_\alpha - t\otimes\iota_\beta} ZD \otimes_A \mathcal{C}_A \xrightarrow{1\otimes\iota^*} \mathcal{C}_D$$

is short exact, and the given basis of $\mathcal{C}_D$ coincides with the induced one (notice $t\otimes\iota_\beta$, in contrast to $1\otimes\iota_\alpha$).

A <u>finite</u> MV presentation has all chain complexes finitely generated.

Morphisms, and (possibly based) exact sequences, and mapping cones, of MV presentations are the obvious things. E.g., in case $D = A *_C B$, a morphism of MV presentations consists of chain mappings

$$\mathcal{C}_C \to \mathcal{C}'_C \ , \ \mathcal{C}_A \to \mathcal{C}'_A \ , \ \mathcal{C}_B \to \mathcal{C}'_B \ , \ \mathcal{C}_D \to \mathcal{C}'_D \ ,$$

all over the appropriate rings, so that the obvious diagram commutes.

Again it is a convenient fact that for a based short exact sequence of MV presentations, $M_1 \to M_2 \to M_3$, M_2 may be canonically identified with the algebraic mapping cone of a certain morphism $M_3 \to M_1$.

We now describe our basic operations on MV presentations. The most important one is <u>surgery</u>; it is given by a MV presentation, M^*, which admits the following description. There are precisely six basis elements in M^*, four in dimension n, denoted a, and two in dimension n+1, denoted b, subject to mappings and boundary maps as follows

$$\begin{tikzcd}
& b_A \arrow[r] \arrow[d, "\partial"'] & b_D \arrow[d, "\partial"] \\
a_C \arrow[r, "\iota_\alpha"] \arrow[dr, "\iota_\beta"'] & a_A \arrow[r] & a_D \\
& a_B \arrow[ur] &
\end{tikzcd}$$

or similarly with A and B interchanged (in case $D = A *_C \{t\}$, both a_A and a_B are of course elements of a ZA-module). And a <u>surgery from</u> M_1 <u>to</u> M_2 is a based short exact sequence of MV presentations $M_1 \rightarrow M_2 \rightarrow M^*$.

There are also various sorts of simple expansions.An <u>elementary A-expansion</u> is given by a MV presentation $\widetilde{M}$ like this. There are four basis elements, two in dimension n, denoted a, and two in dimension n+1, denoted b, subject to these maps

$$\begin{tikzcd}
b_A \arrow[r] \arrow[d, "\partial"'] & b_D \arrow[d, "\partial"] \\
a_A \arrow[r] & a_D
\end{tikzcd}$$

an <u>elementary</u> B-<u>expansion</u> is similar, with A and B interchanged. An <u>elementary</u> C-<u>expansion</u> is this

$$\begin{tikzcd}
& & b_A \arrow[d, "\partial"'] \arrow[ddrr] & & \\
& & a_A \arrow[dr] & & \\
b_C \arrow[r, "\partial"] \arrow[uurr] \arrow[ddrr] & a_C \arrow[ur] \arrow[dr] & & a_D & b_D \arrow[l, "\partial"'] \\
& & a_B \arrow[ur] & & \\
& & b_B \arrow[u, "\partial"] \arrow[uurr] & &
\end{tikzcd}$$

And an <u>elementary</u> A- (resp. B- or C-) <u>expansion</u> <u>from</u> M_1 <u>to</u> M_2 is a based short exact sequence of MV presentations $M_1 \rightarrow M_2 \rightarrow \widetilde{M}$, where $\widetilde{M}$ is of the appropriate type.

Two finite MV presentations M and M' are <u>equivalent</u> if there is a MV presentation M" which can be reached from both M and M' by finitely many surgeries and elementary expansions.

<u>Proposition</u>. Up to simple homotopy type of ZD-complexes, any finitely generated ZD-complex has a finite MV presentation which is unique up to equivalence of MV presentations. (wo $C \subset A, B$)

<u>Proof</u>. The proof is a translation of geometric arguments. We will always have in mind the geometry, and not be too explicit about the algebra.

If X is a connected CW complex, Y a closed connected subcomplex with a trivial normal bundle of dimension 1 in X, so that $\ker(\pi_1 Y \to \pi_1 X) = 0$, then, by the van Kampen theorem, we have $\pi_1 X = A *_C B$ or $= A *_C \{t\}$, with $C = \pi_1 Y$, according to whether Y does or does not separate X. In the universal cover of X, the subspace covering Y gives a certain decomposition pattern which may be described by a graph, Γ, a sort of dual to the decomposition. Γ is a tree and may abstractly be described thus.

<u>Case</u> $D = A *_C B$. <u>Vertices</u> of Γ are the A-cosets and B-cosets of D (occasionally we will speak of A-<u>vertices</u> or B-<u>vertices</u>), and <u>segments</u> are the C-cosets of D. (We are having $\pi_1 X$ act from the left, so our cosets are right cosets). Representatives for all the cosets can be fixed by merely choosing representatives for the right C-cosets in A and B. Whenever a choice has to be made, we will make this one.) For each C-coset there are precisely one A-coset and one B-coset which contain it. Hence with inclusion of cosets as <u>incidence</u>, we obtain an honest graph which is a tree. - Algebraically, these statements are less obvious than geometrically. They contain, for example, the proposition that any element of D can be written in the usual normal form in an essentially unique way.

<u>Case</u> $D = A *_C \{t\}$. <u>Vertices</u> of Γ are the A-cosets, and <u>segments</u> are the C-cosets. For each C-coset there is precisely one A-coset which contains it via the inclusion $\alpha \colon C \to A$ or $\beta \colon C \to A$, respectively. So again we get an honest graph which again is a tree. (Similarly as in the other case, representatives of all the cosets can be specified by merely fixing representatives for the right cosets of both the inclusions $\alpha, \beta \colon C \to A$.)

We now specialize to the case $D = A *_C B$, the other case being similar.

We describe a MV presentation which we call a <u>subdivided n-cell</u>, or, more explicitly, a n-cell subdivided according to Δ, where Δ is a finite tree in Γ. Let a_i, b_i, c_i be basis elements indexed respectively by the A-vertices, the B-vertices, and the segments of Δ, and let x_i, y_i, z_i be representatives of the respective cosets. We let the (n-1)-chain module of each of $\mathcal{C}_C$, $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_D$ be generated by the c_i, and the n-chain module of $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_D$ by the a_i, the b_i, and the a_i and b_i, respectively. All other modules are trivial, and the chain maps are the obvious ones, except for a change of sign in $\mathcal{C}_C \to \mathcal{C}_B$. The non-trivial boundary maps are given by

$\partial x_j a_j$ (resp. $-\partial y_j b_j$) = Σ all those $z_i c_i$ for which there is incidence of the corresponding segment to the given verte

(notice that $\partial a_j = \sum x_j^{-1} z_i c_i$ does not lead out of $\mathcal{C}_A$).

Let $\mathcal{C}'_D$ be the ZD-complex which is trivial, except for dimension n where it is free of rank one, generated by e, say. There is a chain map $\mathcal{C}'_D \to \mathcal{C}_D$ over ZD, defined by

$$e \to \Sigma a_i + \Sigma b_i$$

which both is an inclusion and induces a simple homotopy equivalence. Let there be given a map $f\colon \mathcal{C}'_D \to \mathcal{C}''_D$, some $\mathcal{C}''_D$, and a MV presentation of $\mathcal{C}''_D$. Then, we claim, if Δ has been chosen suitably, there is a morphism of MV presentations covering an extension $\mathcal{C}_D \to \mathcal{C}''_D$ of f. In fact, write f(e) in terms of the basis elements of $\mathcal{C}''_D$, and then spell out each coefficient in terms of cosets. In some more detail, first normalize the basis elements of $\mathcal{C}''_D$ so that each comes from a basis element of either $\mathcal{C}''_A$ or $\mathcal{C}''_B$. Let $\mathcal{C}''_{DA}$ (resp. $\mathcal{C}''_{DB}$) be the submodule of n-chains of $\mathcal{C}''_D$ generated by all those basis elements which come from $\mathcal{C}''_A$ (resp. $\mathcal{C}''_B$). Consider it as a sum of ZA-modules (resp. ZB-modules) by taking as generators all the basis elements multiplied by all the right A-cosets (resp. B-cosets). And now assume Δ chosen so large that it contains all the vertices corresponding to cosets occuring in the expression for f(e). The required morphism of MV presentations can then be defined: There is a canonical way of defining the maps on n-chains. And one verifies that this automatically, and consistently, gives the map on (n-1)-chains.

Now any finitely generated ZD-complex can be built up by repeatedly "attaching a (n+1)-cell at a n-cycle" which by definition is just taking the mapping cone of some map $\mathcal{C}'_D \to \mathcal{C}''_D$ as above. This verifies the existence part of our proposition.

We now turn to uniqueness. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are finitely generated based ZD-complexes, then an <u>elementary expansion</u> from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a based short exact sequence $\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3$, where $\mathcal{C}_3$ has precisely two non-vanishing chain modules, generated respectively by a and b, with boundary relation $\partial b = a$. We have the <u>lemma</u>. If $\mathcal{C}$ and $\mathcal{C}'$ have the same simple homotopy type, then there is $\mathcal{C}''$ which can be reached from both $\mathcal{C}$ and $\mathcal{C}'$ by elementary expansions. (More precisely, it should be stated that the identity on $\mathcal{C}''$ is in the given homotopy class $\mathcal{C} \to \mathcal{C}'$). This is just the algebraic analogue of a well-known geometric lemma, due to Whitehead.

So if we have MV presentations of $\mathcal{C}_1$ and $\mathcal{C}_2$, and a simple homotopy equivalence $\mathcal{C}_1 \to \mathcal{C}_2$, we can perform elementary expansions to make this map an identity, on $\mathcal{C}$, say. Covering the expansions by operations on MV presentations, namely by the process of attaching subdivided cells, we then obtain complexes $\mathcal{C}'$ and $\mathcal{C}''$ which are both obtained from $\mathcal{C}$ by subdividing cells.

Let I be the interval, i.e., the Z-complex with two base elements, x and y, in dimension 0, and one base element, z, in dimension 1, and boundary relation $\partial z = x + y$. We may consider $\mathcal{C}' \oplus \mathcal{C}''$ as a subdivision of $\mathcal{C} \otimes \partial z$, and we already have a MV presentation of $\mathcal{C}' \oplus \mathcal{C}''$. So, attaching one at a time the cells of $\mathcal{C} \otimes z$, and subdividing, we get a MV presentation of a complex $\mathcal{C}^*$ which is a subdivision of $\mathcal{C} \otimes I$. And this MV presentation has the property it can be obtained from both the MV presentations of $\mathcal{C}'$ and $\mathcal{C}''$, and therefore also of those of $\mathcal{C}_1$ and $\mathcal{C}_2$, by repeatedly performing an elementary expansion on the ZD-complex, and subdividing the two cells involved according to the receipt given.

Our uniqueness assertion will thus follow when we show that if $M_1 \to M_2 \to M_3$ is a short exact sequence of MV presentations, and M_3 is obtained by subdividing the two cells in an elementary expansion, then we can obtain isomorphic MV presentations by surgery on M_2, and by elementary (A-, B-, C-) expansions on M_1.

We first treat the special case $M_1 = 0$. Let the subdivided n-cell of M_3 be described by the tree Δ, and the subdivided (n+1)-cell by the tree ∇. Clearly, ∇ contains Δ. If $\nabla \neq \Delta$, then there is an extreme vertex v in ∇ which is incident to a single segment, s, both not contained in Δ. The segment s corresponds to an n-cycle in $\mathcal{C}_C$ which bounds in either $\mathcal{C}_A$ or $\mathcal{C}_B$, according to whether v is an A- or B-vertex, respectively. So we can perform a surgery to make this n-cycle bound in $\mathcal{C}'_C$ (prime denoting the MV presentation resulting after the surgery has been performed). Let Δ' be the collection of segments and vertices used in the construction (there are s and v to begin with). If now s', v' is any extreme pair in $\nabla - \Delta - \Delta'$, then the n-cycle corresponding to s' again bounds in either $\mathcal{C}'_A$ or $\mathcal{C}'_B$ because the still more extreme stuff already has been killed by surgery. So we can continue to perform surgery until all of $\nabla - \Delta$ has been used.

Let M''_3 be the resulting MV presentation. Corresponding to each segment or vertex of ∇, there are precisely none or two base elements in each of $\mathcal{C}''_C$, $\mathcal{C}''_A$, $\mathcal{C}''_B$, $\mathcal{C}''_D$, namely a (n-1, n) pair for segments of Δ, a (n+1, n+2) pair for vertices of $\nabla - \Delta$, and a (n, n+1) pair otherwise. And it is obvious that M''_3 can be built up from the trivial MV presentation by expansions: first C-expansions to accomodate the situation at the segments, and then A- and B-expansions.

Actually, we have done a bit more than just treating that very special case. For, M_2 is the mapping cone of a certain morphism $M_3 \to M_1$. So our assertion will be established generally if we can show that this morphism extends to $M''_3 \to M_1$, where M''_3 is the MV presentation constructed above. But doing surgery is itself taking a mapping cone. So there is a unique way of obtaining $M''_3 \to M_1$, namely, at each step we take the composition of the attaching map defining the surgery, and of the attaching map $M' \to M$ already constructed.

4. HOMOLOGY OF MAYER VIETORIS PRESENTATIONS

Homology groups will have universal coefficients. So, for example, $H_n(\mathcal{C}_D)$ is a ZD-module in a natural way. Since ZD is a free ZC-module, the functor ZD $\otimes_{ZC}$ is exact; similarly for other such functors. So we have a natural isomorphism $H_n(ZD \otimes_{ZC} \mathcal{C}_C) \approx ZD \otimes_{ZC} H_n(\mathcal{C}_C)$, say, and a natural equivalence $H(1 \otimes \iota_\alpha) = 1 \otimes H(\iota_\alpha)$, say, where $1 \otimes \iota_\alpha$: $ZD \otimes_{ZC} \mathcal{C}_C \to ZD \otimes_{ZA} \mathcal{C}_A$. These facts will henceforth be used without further mentioning.

We will be considering $\mathcal{C}_D$ as a Z-complex for the purpose of presenting it in various ways as a sum of subcomplexes, and comparing homology. Whenever there is extra structure on such a decomposition, this will be mentioned.

It will be convenient to make extensive use of the tree Γ introduced in the previous section. Recall that the vertices of Γ correspond bijectively to the right A-cosets and B-cosets of D (resp. A-cosets in the $A *_C \{t\}$ case), and the segments to the C-cosets. We may then consider $ZD \otimes_{ZA} \mathcal{C}_A$ as a direct sum of Z-complexes, each isomorphic to $\mathcal{C}_A$, and indexed by the A-vertices of Γ (more precisely, we are considering the symbol $d\otimes$ as an index, and the action occurs to the right of the symbol $d\otimes$); and similarly for $ZD \otimes_{ZB} \mathcal{C}_B$ and $ZD \otimes_{ZC} \mathcal{C}_C$. We refer to the summands as <u>vertex-complexes</u> and <u>segment-complexes</u>, respectively. Under the projection $(ZD \otimes_{ZA} \mathcal{C}_A) \oplus (ZD \otimes_{ZB} \mathcal{C}_B) \to \mathcal{C}_D$ (resp. $ZD \otimes_{ZA} \mathcal{C}_A \to \mathcal{C}_D$), and the composite projection $ZD \otimes_{ZC} \mathcal{C}_C \to \mathcal{C}_D$, any vertex-complex or segment-complex is embedded in $\mathcal{C}_D$. Any segment-complex is the intersection of the two vertex-complexes corresponding to the vertices incident to the segment. And there is no other intersection.

If V is any collection of vertices of Γ, we define $\mathcal{C}(V)$ to be the subcomplex of $\mathcal{C}_D$ which is the union of all the vertex-complexes corresponding to vertices in V; similarly $\mathcal{C}(S)$, for S a collection of segments.

<u>Lemma</u>. Let x be any vertex v or segment s of Γ, and α any element of $\ker(H_n(\mathcal{C}(x)) \to H_n(\mathcal{C}_D))$. Then α can be killed by a finite number of (n-dimensional) surgeries.

Proof. $\mathcal{C}_D$ is the direct limit, via inclusion, of the $\mathcal{C}(V)$, V ranging over the collections of vertices of finite trees in Γ. Therefore there is such a tree that $\alpha \in \ker(H_n(\mathcal{C}(x)) \to H_n(\mathcal{C}(V)))$. We now proceed by induction on the number of vertices in V. If V has but one vertex, and v is this vertex, the assertion is trivial. If s is incident to this vertex, our task is precisely to kill an element of $\ker(\iota_{\alpha*}: H_n(\mathcal{C}_C) \to H_n(\mathcal{C}_A))$, say. But this is exactly what n-dimensional surgery can do.

In the general case, let v be x or a vertex incident to x, respectively, and S the collection of segments incident to both x and $V' = V - x$. There is a short exact sequence $\mathcal{C}(S) \to \mathcal{C}(x) \oplus \mathcal{C}(V') \to \mathcal{C}(V)$. Moreover $\mathcal{C}(S) = \oplus_{s\in S}\mathcal{C}(s)$, and $\mathcal{C}(V') = \oplus_{s\in S}\mathcal{C}(V_s)$, where the $\mathcal{C}(V_s)$ have been indexed by their intersection with $\mathcal{C}(x)$. From the above short exact sequence, we have an exact sequence

$$\oplus_{s\in S}H_n(\mathcal{C}(s)) \to H_n(\mathcal{C}(x)) \oplus \oplus_{s\in S}H_n(\mathcal{C}(V_s)) \to H_n(\mathcal{C}(V))$$

where the first mapping is a diagonal of inclusion induced mappings (with the second component going componentwise). Hence, looking at elements $\beta\oplus 0\oplus\ldots\oplus 0$, we see that any element of $\ker(H_n(\mathcal{C}(x)) \to H_n(\mathcal{C}(V)))$ is a sum of elements of the $\ker(H_n(\mathcal{C}(s)) \to H_n(\mathcal{C}(V_s)))$, or rather of their images in $H_n(\mathcal{C}(x))$. The induction step now follows on observing that a n-dimensional surgery does not create any new homology below dimension n+1, so that no unwanted things resulted from previous steps.

Lemma. Suppose the inclusion induced homomorphism $H_n(\mathcal{C}_C) \to H_n(\mathcal{C}_D)$ is a monomorphism. Then so is

$$ZD \otimes_C H_n(\mathcal{C}_C) \xrightarrow{(1\otimes\iota_{\alpha*}\oplus(-1)\otimes\iota_{\beta*})} \left(ZD \otimes_A H_n(\mathcal{C}_A)\right) \oplus \left(ZD \otimes_B H_n(\mathcal{C}_B)\right),$$

resp. $ZD \otimes_C H_n(\mathcal{C}_C) \xrightarrow{(1\otimes\iota_{\alpha*} - t\otimes\iota_{\beta*})} ZD \otimes_A H_n(\mathcal{C}_A)$.

Note the assertion is not about an inclusion induced homomorphism, but about a diagonal of two such. There is no corresponding result for, say, $ZD \otimes_C H_n(\mathcal{C}_C) \to ZD \otimes_A H_n(\mathcal{C}_A)$.

Proof. Let α be an element of the kernel. Then

$$\alpha = \oplus_{s \in S} \alpha_s \in \oplus_{s \in S} d_s \otimes H_n(\mathcal{C}_C) ,$$

for some finite collection of segments, S. Let S be chosen as small as possible; suppose S is non-empty, and let V be the smallest tree in Γ containing S. Pick a segment s' which is incident to an extreme vertex v of V. Then $\alpha_{s'}$ is the only component of α which is mapped into $H_n(\mathcal{C}(v))$. But then it must be an element of $\ker(H_n(\mathcal{C}(s')) \to H_n(\mathcal{C}(v)))$. Therefore $\alpha_{s'} \in \ker(H_n(\mathcal{C}(s')) \to H_n(\mathcal{C}_D))$, $\alpha_{s'} = 0$, and we have a contradiction to our choice of S.

We now turn to a first application of the results derived so far, an estimate for the global homological dimension of group rings of generalized free products.

Proposition. Let $D = A *_C B$ or $= A *_C \{t\}$, respectively. Then gl.dim.(ZD) $\leq$ max(gl.dim.(ZA), gl.dim.(ZB), gl.dim.(ZC) + 1).

Proof. We wish to prove that any ZD-module has a projective resolution of length at most the number given above. It suffices to prove this for countable ZD-modules, in fact: for quotients of ZD. Let $F_1 \to F_0$ be a map of free ZD-modules, with cokernel the given module. We consider $F_1 \to F_0$ as a ZD-complex. By prop. (3.4) (and direct limit, if necessary) there is a MV presentation of a ZD-complex homotopy equivalent to our original complex, giving an exact sequence

$$ZD \otimes_C H_0(\mathcal{C}_C) \to ZD \otimes_A H_0(\mathcal{C}_A) \oplus ZD \otimes_B H_0(\mathcal{C}_B) \to H_0(\mathcal{C}_D) \to 0$$

or $\quad ZD \otimes_C H_0(\mathcal{C}_C) \to ZD \otimes_A H_0(\mathcal{C}_A) \to H_0(\mathcal{C}_D) \to 0$,

respectively. Then, by lemmas (4.1) and (4.2) (and direct limit, if necessary) we can perform surgery on the MV presentation to make the sequence short exact. Now by assumption the first two terms have projective resolutions of lengths at most gl.dim.(ZC) and max(gl.dim.(ZA), gl.dim.(ZB)) (resp. gl.dim.(ZA)), respectively. There is a map of resolutions, covering the monomorphism in our short exact sequence. Taking its mapping cone, we obtain a resolution of $H_0(\mathcal{C}_D)$, which is the given module. This resolution has the required length.

There are some variants of the proposition, like this

<u>Complement to proposition</u>. If every ZC-, ZA-, ZB-module has a finite projective resolution (over the appropriate ring), then so has every ZD-module. - Same proof.

<u>Complement to complement</u>. Before each of ZA-, ZB-, ZD-module insert <u>finitely presented</u>, provided ZC is noetherian. - Cf. below.

<u>Definition</u>. A ring R is <u>coherent</u>, if it has any of the properties (a), (b), (c) below.

(a) For any finitely presented R-module $\operatorname{coker}(F_1 \to F_0)$, there exists a partial projective resolution

$$P_2 \to P_1 \to P_0 \to \operatorname{coker}(F_1 \to F_0) \to 0$$

with all the P_i finitely generated.

(b) Any finitely generated submodule of a free R-module is finitely presented.

(c) For any R-map $f: M_1 \to M_2$, M_1 finitely generated, and M_2 finitely presented, $\ker(f)$ is finitely generated.

((a) $\Rightarrow$ (b) follows by comparison of partial resolution; (b) $\Rightarrow$ (c) may be proved using a presentation of M_2, and an obvious pullback diagram; the other implications are trivial)

<u>Proposition</u>. Let $D = A *_C B$ or $= A *_C \{t\}$, respectively. Suppose ZA and ZB are coherent, and ZC is noetherian. Then ZD is coherent.

<u>Proof</u>. This is a corollary of the proof of prop. (4.3). If the given ZD-module is finitely presented, we can choose a finite MV presentation to start with. Then the kernel to be killed is finitely generated because ZC is noetherian. So we need only perform finitely many surgeries to kill the kernel, ending up with a finite MV presentation. By assumption there exist then finitely generated partial projective resolutions of arbitrary length. Combining these into a mapping cone gives the required thing.

5. THE SUM THEOREM

Case $D = A *_C B$.

$\alpha: C \to A$, $\beta: C \to B$ are, as usual, the inclusions which are part of the structure of $A *_C B$. An asterisk denotes an induced functor.

Definition. $Wh(A,B;C)$ and $\tilde{K}_0(C;A,B)$ are defined by exact sequences (as a pushout or copushout, respectively)

$$Wh(C) \xrightarrow{\alpha_* \oplus -\beta_*} Wh(A) \oplus Wh(B) \to Wh(A,B;C) \to 0$$

$$0 \to \tilde{K}_0(C;A,B) \to \tilde{K}_0(C) \xrightarrow{\alpha_* \oplus -\beta_*} \tilde{K}_0(A) \oplus \tilde{K}_0(B)$$

$\tilde{C}(C;A,B)$ is the group introduced in section 2.

Theorem. There is a natural isomorphism

$$Wh(D) \approx Wh(A,B;C) \oplus \tilde{K}_0(C;A,B) \oplus \tilde{C}(C;A,B) .$$

For $\tau \in Wh(D)$, we will write $\tau = \sigma \oplus \kappa \oplus \rho$.

Case $D = A *_C \{t\}$.

$\alpha: C \to A$ and $\beta: C \to A$ are, as usual, the inclusions which are part of the structure of $A *_C \{t\}$.

Definition. $Wh(A,A;C)$ and $\tilde{K}_0(C;A,A)$ are defined by exact sequences (as an equalizer or coequalizer, respectively)

$$Wh(C) \xrightarrow{\alpha_* - \beta_*} Wh(A) \to Wh(A,A;C) \to 0$$

$$0 \to \tilde{K}_0(C;A,A) \to \tilde{K}_0(C) \xrightarrow{\alpha_* - \beta_*} \tilde{K}_0(A)$$

$\tilde{D}(C;A,A)$ is the group introduced in section 2.

Theorem. There is a natural isomorphism

$$Wh(D) \approx Wh(A,A;C) \oplus \tilde{K}_0(C;A\ A) \oplus \tilde{D}(C;A,A)$$

For $\tau \in Wh(D)$, we will write $\tau = \sigma \oplus \kappa \oplus \rho$.

In both cases, we have this interpretation of the exotic terms.

Addendum. The component $\kappa \oplus \rho$ of τ is zero, if and only if there is a finite MV presentation of an acyclic ZD-complex with torsion τ such that $\mathcal{C}_C$ is acyclic.

Similarly, the component ρ is zero, if and only if there is such a MV presentation that the maps $\iota_{\alpha *}: H_*(\mathcal{C}_C) \to H_*(\mathcal{C}_A)$ and $\iota_{\beta *}: H_*(\mathcal{C}_C) \to H_*(\mathcal{C}_B)$ (resp. $H_*(\mathcal{C}_A)$ in case $A *_C \{t\}$) represent $H_*(\mathcal{C}_C)$ as a direct sum of the kernels.

The proof of the sum theorem and its addendum will occupy the present section. We treat both cases simultaneously, unless otherwise stated.

By the realization lemma, prop. (3.4), there is a finite MV presentation of an acyclic ZD-complex $\mathcal{C}_D$ so that the torsion of $\mathcal{C}_D$ is a prescribed element of Wh(D).

Let j be the smallest index such that $H_j(\mathcal{C}_C)$ is non-trivial. Denoting X_i and Z_i the i-th chain- and cycle-module of $\mathcal{C}_C$, respectively, we have, for $i \leq j$, a short exact sequence $Z_i \to X_i \to Z_{i-1}$. So, by induction, Z_i is stably free, and finitely generated, for $i \leq j$.

Any element of $H_j(\mathcal{C}_C)$ maps to zero under the inclusion induced homomorphism $H_j(\mathcal{C}_C) \to H_j(\mathcal{C}_D) = 0$. So, by lemma (4.1), it can be killed by finitely many surgeries in dimension j. The effect of this on $\mathcal{C}_C$ is to attach (j+1)-cells; in particular, $\mathcal{C}_C$ does not acquire any new homology, except possibly in dimension j+1. Working up dimensions, we can thus achieve that $\mathcal{C}_C$ is both (n-1)-connected and n-dimensional, for some n. By an observation above, this implies $H_n(\mathcal{C}_C)$ is stably free and finitely generated.

It is slightly inconvenient to require n to be the top dimension. So from now on we only ask that the homology of $\mathcal{C}_C$ be concentrated in dimension n, some n, and stably free.

We will have to decompose $\mathcal{C}_D$ in various ways. This is most conveniently expressed in terms of the tree Γ (cf. the two previous sections). Let the segment $s_0 \in \Gamma$ correspond to the inclusion $\mathcal{C}_C \to \mathcal{C}_D$ which is the composite $\mathcal{C}_C \to \mathcal{C}_A \to \mathcal{C}_D$. Let

v_l and v_r ("left" and "right") be the vertices incident to s_0, numbered so that the incidence $v_l \subset \partial s_0$ corresponds to the inclusion $\iota_\alpha: \mathcal{C}_C \to \mathcal{C}_A$ (hence $v_r \subset \partial s_0$ corresponds to $\iota_\beta: \mathcal{C}_C \to \mathcal{C}_B$, resp. $t\otimes\iota_\beta: \mathcal{C}_C \to t\otimes\mathcal{C}_A$). Denote by Γ_l and Γ_r the components of $\Gamma - s_0$ (Γ_l containing v_l). There is a Mayer Vietoris sequence

$$0 \to \mathcal{C}(s_0) \to \mathcal{C}(\Gamma_l) \oplus \mathcal{C}(\Gamma_r) \to \mathcal{C}_D \to 0 ,$$

and ZC acts naturally on this decomposition. Hence from the exact homology sequence we obtain an isomorphism

(!) $$0 \to H_n(\mathcal{C}_C) \to H_n(\mathcal{C}(\Gamma_l)) \oplus H_n(\mathcal{C}(\Gamma_r)) \to 0$$

of ZC-modules. We abbreviate

$$\boxed{P = H_n(\mathcal{C}(\Gamma_l)) \quad \text{and} \quad Q = H_n(\mathcal{C}(\Gamma_r))}.$$

Similarly, let Λ be any tree in Γ, and $\tilde\Lambda$ the complementary graph, i.e., $S = \Gamma - (\Lambda \cup \tilde\Lambda)$ consists of segments only, each incident to both Λ and $\tilde\Lambda$. Each component of $\tilde\Lambda$ is a tree, and isomorphic to either Γ_l or Γ_r above. Via incidence, these components are in one-one correspondence to the components of S. We denote them $\tilde\Gamma_s$, $s \in S$. For each s, let Γ_s be the tree complementary to $\tilde\Gamma_s$. There is a Mayer Vietoris sequence (of Z-complexes, in general)

$$0 \to \mathcal{C}(S) \to \mathcal{C}(\Lambda) \oplus \mathcal{C}(\tilde\Lambda) \to \mathcal{C}_D \to 0 ,$$

whence

$$0 \to H_n(\mathcal{C}(S)) \to H_n(\mathcal{C}(\Lambda)) \oplus H_n(\mathcal{C}(\tilde\Lambda)) \to 0 .$$

Evaluating terms using sum splittings, and isomorphisms of type (!), we obtain an isomorphism

$$\oplus_{s\in S} (H_n(\mathcal{C}(\Gamma_s)) \oplus H_n(\mathcal{C}(\tilde\Gamma_s))) \to H_n(\mathcal{C}(\Lambda)) \oplus \oplus_{s\in S} H_n(\mathcal{C}(\tilde\Gamma_s)) .$$

Now the map into the second component is just the projection. Hence the map into the first component is the sum of an isomorphism

(!!) $$\boxed{\oplus_{s\in S} H_n(\mathcal{C}(\Gamma_s)) \xrightarrow{\approx} H_n(\mathcal{C}(\Lambda))}$$

and a map

(!!!) $$\oplus_{s\in S} H_n(\mathcal{C}(\tilde\Gamma_s)) \to H_n(\mathcal{C}(\Lambda)) .$$

It is easy to identify the maps, namely the components of the inverse of (!!) are given by the inclusions $\mathcal{C}(\Lambda) \to \mathcal{C}(\Gamma_s)$, and the components of (!!!) are given by the inclusion homomorphisms

$$\ker(H_n(\mathcal{C}(s)) \to H_n(\mathcal{C}(\Gamma_s))) \to H_n(\mathcal{C}(s)) \to H_n(\mathcal{C}(\Lambda)) .$$

In special cases, we can be even more specific about the isomorphism (!!). Let $\Lambda = v_1$, and $S = \delta v_1$, the collection of segments incident to v_1.

Then, in case $D = A *_C B$, the A-action on Γ, fixing v_1, is transitive on δv_1 (with isotropy group $\alpha(C)$), and (!!) is an isomorphism of ZA-modules

$$\boxed{\begin{array}{l} H_n(\mathcal{C}_A) \approx ZA \otimes_C P . \\ H_n(\mathcal{C}_B) \approx ZB \otimes_C Q . \end{array}}$$

Similarly, (second line above).

In case $A *_C \{t\}$, the A-action on Γ, fixing v_1, has two orbits in δv_1 (with isotropy groups $\alpha(C)$ and $\beta(C)$, respectively), and (!!) is an isomorphism of ZA-modules

$$H_n(\mathcal{C}_A) \approx ZA_\alpha \otimes P \oplus ZA_\beta \otimes Q$$

(tensor products over ZC; the subscript α or β on ZA indicates which ZC-structure is used).

We are now through the preliminaries to proving

<u>Proposition</u>. (1) The pair of inclusion induced homomorphisms $\iota_\alpha : H_n(\mathcal{C}_C) \to H_n(\mathcal{C}_A)$, $\iota_\beta : H_n(\mathcal{C}_C) \to H_n(\mathcal{C}_B)$ (resp. $H_n(\mathcal{C}_A)$ in case $A *_C \{t\}$) determines a nilpotent object in the sense of section 2.

(2) The equivalence class of this object depends only on the torsion of $\mathcal{C}_D$.

(3) The map $Wh(D) \to C(C;A,B)$ (resp. $D(C;A,A)$) so obtained is a split surjection onto $\tilde{K}_0(C;A,B) \oplus \tilde{C}(C;A,B)$ (resp. $\tilde{K}_0(C;A,A) \oplus \tilde{D}(C;A,A)$).

(4) $\rho = 0$ if and only if there is a MV presentation (of some $\mathcal{C}_D$ representing τ) so that $\ker(\iota_{\alpha *})$ and $\ker(\iota_{\beta *})$ present $H_n(\mathcal{C}_C)$ as a direct sum. And $\rho \oplus \kappa = 0$ if and only if $\mathcal{C}_C$ can be made acyclic.

<u>Addendum</u>. If $\rho \oplus \kappa = 0$, and if $H_i(\mathcal{C}_C) = 0$, for $i < j$, then the normalization of $\mathcal{C}_C$ under (4) can be achieved by surgeries in dimensions exceeding $j-1$.

<u>Proof</u>. <u>Ad</u> (1). We treat the case $D = A *_C B$, the other case being similar. Above we obtained a natural splitting $H_n(\mathcal{C}_C) = P \oplus Q$, and a natural isomorphism $H_n(\mathcal{C}_A) = ZA \otimes_C P$. The latter splits as a ZC-module,

$$ZA \otimes_C P = \left(ZC \otimes_C P\right) \oplus \left(\widehat{ZA} \otimes_C P\right)$$

(cf. section 2). Now the discussion preceeding the proposition shows us what the map $P \oplus Q \to P \oplus \left(\widehat{ZA} \otimes_C P\right)$ looks like: it is a direct sum of an identity $P \to P$, and a map

$$Q \to \widehat{ZA} \otimes_C P ,$$

which is just

$$\ker(H_n(\mathcal{C}(s_0)) \to H_n(\mathcal{C}(\Gamma_1))) \to H_n(\mathcal{C}(s_0)) \to H_n(\mathcal{C}_A) .$$

Similarly, $H_n(\mathcal{C}(s_0)) \to H_n(\mathcal{C}(v_r))$ splits into an isomorphism $Q \to Q$, and a map

$$P \to \widehat{ZB} \otimes_C Q$$

which is just

$$P \to H_n(\mathcal{C}(s_0)) \to H_n(\mathcal{C}(v_r)).$$

So we indeed have an object in the sense of section 2. This object is nilpotent: We must show that iteration eventually gives a zero map. Consider, e.g., the composite

$$Q \to \widehat{ZA} \otimes P \to \widehat{ZA} \otimes \widehat{ZB} \otimes Q .$$

This can be identified to $Q \subset H_n(\mathcal{C}(s_0)) \to H_n(\mathcal{C}(\Lambda))$ where Λ is the subtree of Γ_1, any vertex of which has distance at most 1 from v_1. And similarly, the n-th iterate describes the map induced on homology by the inclusion of s_0 into the tree any vertex of which has distance at most n from v_1 or v_r. But the direct limit of this is the zero map, induced from $s_0 \subset \Gamma$.

<u>Ad</u> (2). According to the uniqueness part of the realization lemma, prop. 3.4, if we have MV presentations M and M' of acyclic ZD-complexes $\mathcal{C}_D$ and $\mathcal{C}'_D$ with the same torsion, there is a MV presentation M" which can be reached from both M and M' by surgery and elementary expansions. By executing some more surgery on M", we can get M" into our normal form, so that $H_j(\mathcal{C}''_C) = 0$ unless $j = n''$, for some $n'' \geq n, n'$. So we now assume

M'' has this property. We will investigate what happens in passing from M, say, to M''.

Clearly we can arrange the surgeries in order of increasing dimension. Let $M^0 = M$, and let M^{j+1} be obtained from M^j by performing the surgeries of dimension j. There is a short exact sequence of MV presentations, $M^j \to M^{j+1} \to N^j$, defining N^j. N^j may be described as being obtained from the trivial MV presentation by j-dimensional surgeries. Suppose N^j comprises only one surgery. Then $H_{j+1}(\mathcal{C}_C(N^j))$ is free of rank one, and one of the induced maps, say $\iota_{\alpha *}: H_{j+1}(\mathcal{C}_C(N^j)) \to H_{j+1}(\mathcal{C}_A(N^j))$, is a zero map. In the general case, there is a short exact sequence of MV presentations, $N^{j'} \to N^j \to N^{j''}$, where $N^{j'}$ is given by all j-dimensional surgeries except the last one, and $N^{j''}$ by that last one. By induction, this shows that the nilpotent object determined by N^j is a standard trivial object in the sense of section 2.

For $i \leq n-2$, we have a short exact sequence

$$H_{i+1}(\mathcal{C}_C(M^{i+1})) \to H_{i+1}(\mathcal{C}_C(N^i)) \to H_i(\mathcal{C}_C(M^i)) .$$

By naturality of the maps, we see that if the i-th homology of M^i determines a nilpotent object, then the maps $H_{i+1}(N^i) \to H_i(M^i)$ just describe the operation on objects called suspension in sections 1 and 2. So, by induction, $H_i(M^i)$, for $i \leq n-1$, indeed describes an object, and this object is equivalent to zero.

For $i = n-1$, a somewhat longer sequence is obtained. We may write down instead a short exact sequence

$$H_{i+1}(\mathcal{C}_C(M^i)) \to H_{i+1}(\mathcal{C}_C(M^{i+1})) \to$$
$$\ker(H_{i+1}(\mathcal{C}_C(N^i)) \to H_i(\mathcal{C}_C(M^i))) .$$

By the above argument there is an object (equivalent to zero) corresponding to the third term. So we have objects corresponding to the extreme terms. We must then also have an object corresponding to the middle term, and these three objects form a short exact sequence. So $H_{i+1}(M^{i+1})$ indeed describes an object, and this object is equivalent to the one given by $H_{i+1}(M)$.

For $i \geqslant n$, finally, we again have a short exact sequence

$$H_{i+1}(\mathcal{C}_C(M^{i+1})) \to H_{i+1}(\mathcal{C}_C(N^i)) \to H_i(\mathcal{C}_C(M^i)) ,$$

and other sequences of the same type, and natural maps between them. We conclude that the object determined by $H_{i+1}(M^{i+1})$ is obtained from the one determined by $H_i(M^i)$ by suspension. So the former has the same equivalence class as the latter, multiplied by (-1). And so we have verified that if we normalize a MV presentation of an acyclic $\mathcal{C}_D$ as in (1), determine the equivalence class of the object obtained (which lives in $C(C;A,B)$, resp. $D(C;A,A)$), and multiply it by $(-1)^n$, then the result depends on $\tau(\mathcal{C}_D)$ only.

<u>Ad</u> (3). We assume the normalization of (1). So the homology of each of $\mathcal{C}_C$, $\mathcal{C}_A$, $\mathcal{C}_B$, is concentrated in dimension n, and is a stably free module over the appropriate ring.

In case $D = A *_C B$, $H_n(\mathcal{C}_A) \approx ZA \otimes_C P$. Therefore $[P] \in \ker(\tilde{K}_0(C) \to \tilde{K}_0(A))$. Similarly, $-[P] = [Q] \in \ker(\tilde{K}_0(C) \to \tilde{K}_0(B))$.

In case $D = A *_C \{t\}$, $H_n(\mathcal{C}_A) \approx ZA_\alpha \otimes_C P \oplus ZA_\beta \otimes_C Q$. Therefore (with $\alpha_*, \beta_* : \tilde{K}_0(C) \to \tilde{K}_0(A)$), $\alpha_*[P] + \beta_*(-[P]) = 0$.

To construct a right inverse for the projection, we will again consider cases.

<u>Case</u> $D = A *_C B$. Let $p: P \to \widehat{ZB} \otimes Q$, $q: Q \to \widehat{ZA} \otimes P$ be a nilpotent object so that $P \oplus Q$, $ZA \otimes P$, $ZB \otimes Q$ are stably free over the appropriate rings. On adding to P and Q free modules, with the zero homomorphism, we can assume that these modules are in fact free. We must now endow these modules with bases. We cannot choose any odd base for any of them, as there would result a big ambiguity from this. Fortunately, however, we can manage to do with a choice of just two bases, and then it will turn out the ambiguities cancel.

We choose some basis for $P \oplus Q$, and some <u>stable basis for P</u>, meaning a fixed isomorphism class from P to a reference module $P_0 \subset P_0 \oplus Q_0$. This determines a stable basis for Q, namely a class of isomorphisms $Q \to Q_0$, such that $P \oplus Q \to P_0 \oplus Q_0$ has zero torsion. The choices made so far, determine stable bases for $ZA \otimes P$ and $ZB \otimes Q$. On adding some more free modules to P and Q, we may assume these are actual bases.

We now construct a MV presentation from our data. Let n be even (to accomodate our sign convention). We define the n-th chain module of $\mathcal{C}_C$, $\mathcal{C}_A$, $\mathcal{C}_B$ to be $P \oplus Q$, $ZA \otimes (P \oplus Q)$, $ZB \otimes (P \oplus Q)$, respectively. The (n+1)-th chain module of $\mathcal{C}_A$ (resp. $\mathcal{C}_B$) is $ZA \otimes Q$ (resp. $ZB \otimes P$); the boundary map is induced from

$$Q \xrightarrow{\text{id.} \oplus q} Q \oplus \widehat{ZA} \otimes P \subset ZA \otimes (Q \oplus P) \qquad \text{or}$$

$$P \xrightarrow{\text{id.} \oplus p} P \oplus \widehat{ZB} \otimes Q \subset ZB \otimes (P \oplus Q) \quad ,$$

respectively, by $ZA \otimes_A$, resp. $ZB \otimes_B$. All other chain modules are trivial, and the chain maps are the obvious ones. $\mathcal{C}_D$ is now given by the pushout of

$$\begin{tikzcd}[row sep=tiny] & ZD \otimes_A \mathcal{C}_A \\ ZD \otimes_C \mathcal{C}_C \arrow[ur] \arrow[dr] & \\ & ZD \otimes_B \mathcal{C}_B \end{tikzcd} \quad .$$

$\mathcal{C}_D$ is a free chain complex, and endowed with canonical bases. That it is acyclic may be seen by manipulating with Mayer Vietoris sequences as in the beginning of this section.

It is clear that our construction gives a right inverse for the projection $Wh(D) \to \tilde{K}_0(C;A,B) \oplus \tilde{C}(C;A,B)$. We are left to verify that it gives a homomorphism. Now if we have a short exact sequence of nilpotent objects (and keep it exact on adding all those free modules) then it is also clear that our construction gives a short exact sequence of MV presentations. All we have to worry about, is the torsion of the sequence of ZD-complexes. But for an obvious choice of stable bases in the middle term, this torsion is zero, all right. So if indeed our construction is well defined at all, it will also give a homomorphism. We must now investigate the arbitrariness involved in choosing those stable bases.

Suppose the basis of $P \oplus Q$ is altered by an isomorphism with torsion $\tau \in Wh(C)$. The exact sequence

$$0 \to ZD \otimes_C \mathcal{C}_C \to ZD \otimes_A \mathcal{C}_A \oplus ZD \otimes_B \mathcal{C}_B \to \mathcal{C}_D \to 0$$

shows that this induces an isomorphism of the n-th chain module of $\mathcal{C}_D$ with torsion $\mathrm{Im}(-\tau) + \mathrm{Im}(\tau) + \mathrm{Im}(\tau) = \mathrm{Im}(\tau)$. But by our conventions, this alteration in addition induces a change of the stable basis of Q, also with torsion τ. This in turn contributes another $\mathrm{Im}(\tau)$, this time on the (n+1)-chains of $\mathcal{C}_D$. So eventually the two cancel.

If on the other hand, we alter the stable basis of P by an isomorphism with torsion τ, we must simultaneously alter that of Q, by $-\tau$. This time there is no torsion on the n-chains, and on the (n+1)-chains we get a contribution $\mathrm{Im}(\tau) + \mathrm{Im}(-\tau) = 0$.

Case $D = A *_C \{t\}$. This is entirely similar, and is left to the reader.

Ad (4). This is, in fact, a trivial consequence of the other parts of the proposition. Given a MV presentation of an acyclic ZD-complex with $\kappa \oplus \rho = 0$, normalize it according to (1), to concentrate all homology in dimension n, say, n even. This gives a nilpotent object. Use this to get a MV presentation as under (3) (while mimicking the adding of free modules to P and Q, by performing (n-1)-dimensional surgeries on the old MV presentation). There is an obvious morphism of MV presentations from the new one to the old one. The mapping cone of this morphism has the desired property.

Concerning the other part, observe that if we have an object representing κ, and with zero homomorphisms, then the MV presentation resulting from (3) has the desired property. In the general case, first perform the above trick. This gives the desired result, but kills κ. And then form the direct sum with a MV presentation of the special type.

Proof of addendum. Let M be a MV presentation of an acyclic ZD-complex with $\kappa \oplus \rho = 0$. According to (4) there is a MV presentation M' of a ZD-complex with the same torsion, and with $\mathcal{C}'_C$ acyclic. And by prop. 3.4, there is a MV presentation M" which can be reached from both M and M' surgeries and elementary (A-, B-, C-) expansions. We claim $\mathcal{C}''_C$ can be made acyclic. To see this, observe that sometimes a surgery can be cancelled by another surgery. Specifically, if M^* is a MV presentation with $H_i(\mathcal{C}_C^*) = 0$, then any i-dimensional surgery on M^* will add a direct summand (free of rank one) to $H_{i+1}(\mathcal{C}_C^*)$, and this summand can then be killed by a (i+1)-dimensional surgery. The total effect of these surgeries is then also the resultant of elementary expansions (a C-expansion and an A- (resp. B-) expansion). We now apply this remark to the surgeries leading from M' to M". Arrange these in order of increasing dimension, with lowest dimension i, say. We start by performing some new surgeries in dimension i+1, to cancel the old surgeries in dimension i. This way we get rid of the i-dimensional surgeries at the expense of introducing elementary expansions instead. Afterwards we are in the same position as before, except that now the lowest dimensional surgeries are in dimension i+1, corresponding to the original (i+1)-dimensional surgeries. And so, working up dimensions, we will eventually replace M" by M‴ which can be reached from M' by elementary expansions only, and from M by elementary expansions and surgeries.

Having verified half the assertion, we are left to show that we can do without low-dimensional surgeries on M. But here we can apply the same cancelling trick as before, thus eventually replacing by elementary expansions the surgeries below dimension j. In the end we write down all operations to be performed on M, and from this list omit the elementary expansions. This gives the desired result.

<u>Proposition</u>. There is a natural isomorphism

$$\ker\Big(Wh(D) \to \tilde{K}_0(C;A,B) \oplus \tilde{C}(C;A,B)\Big) \overset{\approx}{\to} Wh(A,B;C) \qquad \text{or}$$

$$\ker\Big(Wh(D) \to \tilde{K}_0(C;A,A) \oplus \tilde{D}(C;A,A)\Big) \overset{\approx}{\to} Wh(A,A;C),$$

respectively.

<u>Proof</u>. Let M be a MV presentation of a ZD-complex representing an element of that kernel. Call M <u>split</u>, if $\mathcal{C}_C$ is acyclic. According to prop. 5.4.4, we can assume M is split.

In case $D = A *_C B$, we then have an exact sequence of acyclic ZD-complexes

$$0 \to ZD \otimes_C \mathcal{C}_C \to \big(ZD \otimes_A \mathcal{C}_A\big) \oplus \big(ZD \otimes_B \mathcal{C}_B\big) \to \mathcal{C}_D \to 0$$

Hence $\tau_D = \mathrm{Im}(\tau_A) + \mathrm{Im}(\tau_B) - \mathrm{Im}(\tau_C)$.

Similarly, in case $D = A *_C \{t\}$, there is an exact sequence

$$0 \to ZD \otimes_C \mathcal{C}_C \to ZD \otimes_A \mathcal{C}_A \to \mathcal{C}_D \to 0 ,$$

which gives us $\tau_D = \mathrm{Im}(\tau_A) - \mathrm{Im}(\tau_C)$.

Now consider this construction. Perform surgeries on M in some dimension, n, say, with result M'. Then $H_{n+1}(\mathcal{C}'_C)$ is free (with rank the number of surgeries), and is represented as a direct sum by the kernels of the mappings $\iota_{\alpha *}: H_{n+1}(\mathcal{C}'_C) \to H_{n+1}(\mathcal{C}'_A)$, and $\iota_{\beta *}$. Let $H_{n+1}(\mathcal{C}'_C) = P \oplus Q$ be this splitting ($Q = \ker(\iota_{\alpha *})$); P and Q are both free. Next perform (n+1)-dimensional surgeries (minimal in number) to kill $H_{n+1}(\mathcal{C}'_C)$. The effect of this on $\mathcal{C}'_C$ is fully described by giving based free modules P' and Q', and isomorphisms $P' \to P$, $Q' \to Q$, with torsions, say, τ_P and τ_Q, respectively (τ_P, $\tau_Q \in \mathrm{Wh}(C)$, arbitrary). Call the total operation a <u>twist cancelled surgery</u>.

The effect of the twist cancelled surgery just described on the chain complexes in M is this: To $\mathcal{C}_C$, a summand $\tau_P + \tau_Q$ is added. In case $D = A *_C B$, τ_A inherits a summand $\alpha_*(\tau_P)$, and τ_B inherits a summand $\beta_*(\tau_Q)$ (the change of sign in $(-1) \otimes \iota_\beta$ of course doesn't matter). And in case $D = A *_C \{t\}$, τ_A similarly inherits a summand $\alpha_*(\tau_P) + \beta_*(\tau_Q)$. Of course, τ_D is unaltered in both cases.

We can use twist cancelled surgery to normalize a split MV presentation so that $\tau_C = 0$. This normalization is not unique. But the ambiguity in $\mathrm{Wh}(A) \oplus \mathrm{Wh}(B)$ (resp. $\mathrm{Wh}(A)$) resulting from normalizing by different twist cancelled surgeries, is precisely that which is killed under the projection $\mathrm{Wh}(A) \oplus \mathrm{Wh}(B) \to \mathrm{Wh}(A,B;C)$ (resp. $\mathrm{Wh}(A) \to \mathrm{Wh}(A,A;C)$).

The preceding arguments show that there is a map (obviously a homomorphism) $\mathrm{Wh}(A,B;C)$ (resp. $\mathrm{Wh}(A,A;C)$) $\to \mathrm{Wh}(D)$ which is onto the kernel we are investigating. We must now show that it is injective. Essentially, we will show that all ambiguity there is, comes from something like twist cancelled surgery. As this is no harder, we will in fact show more: There is a natural map $Wh(D) \to Wh(A,B;C)$ (resp. $Wh(A,A;C)$) which is a left inverse to the map above (of course, the existence of such a map follows from injectivity of the above map, and our other results; the interest is in an explicit description).

Let $\tau_D \in Wh(D)$ be any element, let it be the torsion of an acyclic ZD-complex $\mathcal{C}_D$ of which M is a MV presentation. We assume the homology of M is concentrated in dimension n. Then $H_n(\mathcal{C}_C)$ is stably free, and is naturally split into a sum of projectives, $H_n(\mathcal{C}_C) = P \oplus Q$. We now make a choice: we choose some stable basis for each of $H_n(\mathcal{C}_C)$, and P. This determines a stable basis for Q (which is such that $P \oplus Q \to H_n(\mathcal{C}_C)$ has zero torsion), and induces stable bases for $H_n(\mathcal{C}_A) \approx ZA \otimes_C P$ and $H_n(\mathcal{C}_B) \approx ZB \otimes_C Q$ (respectively, $H_n(\mathcal{C}_A) \approx ZA_\alpha \otimes_C P \oplus ZA_\beta \otimes_C Q$).

Once we made this choice, a torsion is defined for each of $\mathcal{C}_C$, $\mathcal{C}_A$, $\mathcal{C}_B$, though these complexes may not be acyclic [ ]. We like to know the ambiguity in these torsions which comes from the choices made. Choose another basis in $H_n(\mathcal{C}_C)$, related to the first one by torsion τ', say. By our conventions, the stable basis for Q must then also be altered by τ'. So we get contributions τ' on τ_C, and $\beta_*(\tau')$ on τ_B (resp. τ_A, in case $A *_C \{t\}$). If, on the other hand, we have some τ'' on P, we must have $-\tau''$ on Q. And so τ_A and τ_B (resp. τ_A) acquire contributions $\alpha_*(\tau'')$ and $\beta_*(-\tau'')$ (respectively, $\alpha_*(\tau'') - \beta_*(\tau'')$).

Hence, if we associate to τ_D the element $(\tau_A - \alpha_*(\tau_C)\ ,\ \tau_B) \in Wh(A) \oplus Wh(B)$, respectively $(\tau_A - \alpha_*(\tau_C)) \in Wh(A)$, then the ambiguity is cancelled when we pass to the quotient $Wh(A,B;C)$ or $Wh(A,A;C)$, respectively.

We are left to investigate the ambiguity resulting from the choice of the MV presentation M. As it turns out, we can do this by arguments we had already: We know that if M' is another such MV presentation, there is M" which can be reached from both M and M' by elementary expansions and surgery. Performing some more surgeries, if necessary, we may assume M" is in normal form, too, i.e., its homology is concentrated in some dimension n", $n'' \geq n, n'$. And then (as we observed in the proof of prop. 5.4.2) to pass from M, say, to M", we can collect the surgeries according to their dimension, and in dimension j the total surgery gives a short exact sequence of MV presentations $M^j \to M^{j+1} \to N^j$. N^j may be considered as the resultant of j-dimensional surgeries on a trivial MV presentation, so has its homology concentrated in dimension j+1. $H_{j+1}(\mathcal{C}_C(N^j))$ splits into free modules P_j and Q_j, corresponding to the surgeries performed at the left or right, respectively. (We ignore extra structure, like nilpotent objects.) And the above sequence induces a surjection $H_{j+1}(\mathcal{C}_C(N^j)) \to H_j(\mathcal{C}_C(M^j))$ which in turn splits into surjections $P_j^* \to P_j$ and $Q_j^* \to Q_j$.

Now consider first the special case that no surgeries below dimension n are needed. Then the homology of M^j and M^{j+1} is concentrated in dimensions j and j+1, respectively. And we have short exact sequences $P_{j+1} \to P_j^* \to P_j$ and $Q_{j+1} \to Q_j^* \to Q_j$. The actual individual surgeries determine a basis for each of P_j^* and Q_j^*. As we do not know which bases are determined we must treat these as an ambiguity. But this ambiguity is of no harm since we can absorb it into the one treated earlier. Namely, we give $P_j^* \oplus Q_j^*$ its natural basis, obtained by composition. There are now unique stable bases for P_{j+1}, Q_{j+1}, and $P_{j+1} \oplus Q_{j+1}$ such that the sequences $P_{j+1} \to P_j^* \to P_j$, $Q_{j+1} \to Q_j^* \to Q_j$, and $P_{j+1} \oplus Q_{j+1} \to P_j^* \oplus Q_j^* \to P_j \oplus Q_j$ all have torsion zero. And if we choose these bases, we find that $\tau_C^{j+1} = \tau_C^j$, $\tau_A^{j+1} = \tau_A^j$, and $\tau_B^{j+1} = \tau_B^j$. Moreover, and this is the point, with these stable bases, the sequence $P_{j+1} \to P_{j+1} \oplus Q_{j+1} \to Q_{j+1}$ has torsion zero. So, whatever the new ambiguity may be, it results in no more than a choice of two out of three of the stable bases of P_{j+1}, Q_{j+1}, and $P_{j+1} \oplus Q_{j+1}$. And this, as we saw earlier, is permissible.

In the general case, we may have to perform surgery in dimensions less than n also. Again we understand that surgery is performed in order of increasing dimension. Then, if $j < n$, it may no longer be true that the homology of M^j is concentrated in one dimension, rather we have homology in dimensions j and n. But in both dimensions, we have free modules $(H(\mathcal{C}_C(M^j)))$, canonically split into projectives. And so we can again define torsions in fixing arbitrarily two out of three stable bases in each of these dimensions. The resulting ambiguity is the same as before.

Finally, if $j < n-1$, the same argument as above shows that the torsions obtained are unaltered in passing from M^j to M^{j+1}. This argument will also work for $j = n-1$, provided there is no interference between these dimensions. And in fact there is no such interference because an exact sequence of projectives, ending in a surjection, splits completely.

The proof of the sum theorem is now complete.

6. A GEOMETRIC INTERPRETATION OF THE SUM THEOREM

We work in the category of finite CW complexes and topological maps. An <u>elementary expansion</u> of a map $f\colon W \to X$ is an elementary expansion applied to W, together with some extension of f. An <u>elementary contraction</u> is the inverse process. A <u>formal deformation</u> of $f{:}W$ is the resultant of a finite sequence of elementary expansions and contractions applied to f. Formal deformations include homotopic deformations.

A pair X,Y is a <u>codimension 1 pair</u> if, in some subdivision of X, there is a cellular neighborhood U of Y, so that the pair U,Y is cellularly equivalent to the pair $Y\times[-1,+1], Y\times 0$ with its natural cell structure.

Given such a pair X,Y, and a map $f\colon W \to X$, this map can be formally deformed into general position, meaning $V = f^{-1}(Y)$ is a codimension 1 subcomplex of W, and $V\times[-1,+1]$ is mapped fibrewise, and homeomorphically on each fibre. Such normalization can be achieved by working up the cells of W, using formal deformations of W, if necessary, to vary each attaching map in its homotopy class.

Typically, a codimension 1 pair is obtained from maps $f_i\colon Y \to X_i$ (X_1 and X_2 different complexes, or one and the same complex) by forming the mapping cylinders, and identifying the common subcomplexes Y (and similarly identifying the common subcomplexes X_1, in case $X_1 = X_2$).

If both X, Y are connected, and $\pi_1 Y \to \pi_1 X$ is a monomorphism, then (by the van Kampen theorem) $\pi_1 X \approx A *_C B$ or $\approx A *_C \{t\}$, respectively, according to whether Y does or does not separate X, with $C \approx \pi_1 Y$ (and $A \approx \pi_1 X_1$, $B \approx \pi_1 X_2$).

<u>Theorem</u>. Let X,Y be a codimension 1 pair with X, Y connected, and $\pi_1 Y \to \pi_1 X$ a monomorphism. And let $f\colon W \to X$ be a homotopy equivalence. Then $f{:}W$ can be formally deformed into a homotopy equivalence of pairs $f'\colon(W',V')\to(X,Y)$, with $V' = f'^{-1}(Y)$, if and only if $\kappa\oplus\rho = 0$.

Proof. We only treat sufficiency. Necessity will be obvious from the proof of sufficiency. Also we assume throughout that Y separates X (into X_1 and X_2, say), leaving it to the reader to modify our arguments in the other case.

We assume $f:W$ is given both in general position and as a cellular map. We now describe the operation we use for altering f. This is a very special kind of elementary expansion of $f:W$ which we call surgery on f.

Let E be a $(n+1)$-cell with boundary ∂E, and hemispheres E^+ and E^- in ∂E. Suppose we are given a map $g: E^- \to W$ such that $g^{-1}(V) = \partial E^-$. Then, using $g|\partial E^-$ we can attach E^+ to V, and afterwards we can attach E to $W \cup E^+$ by the attaching map $g|E^- \cup \mathrm{id.}|E^+$. The total effect is an elementary expansion of W inducing on V the attaching of E^+. We can modify the construction slightly by attaching instead of E^+ a thickened E^+ (and pulling in a bit E). This gives the same result, and in addition makes $V \cup E^+$ a codimension 1 subcomplex of $W \cup E$.

Now assume the above map g is part of a device representing an element of $\ker(\pi_n(f|V) \to \pi_n(f))$ ($= \pi_n(f|V)$ since f is a homotopy equivalence), where as usual $\pi_n(f)$ means $\pi_n(T(f),W)$, $T(f)$ the mapping cylinder of f. Then this device also gives us an extension of $f \circ g|E^-$ to $h: E \to X$ with $h(E^+) \subset Y$. If indeed our construction is to kill the given element, we have but one choice for extending f, namely $h|E$. Half of this presents no problem. Namely, we use $h|E^+$ to extend $f|V$ over E^+, and mapping the thickened stuff fibrewise we keep f in general position. But now we must insist that $h(\mathrm{Int}(E))$ be disjoint to Y. Otherwise, $f^{-1}(V)$ would inherit new things in a rather uncontrollable fashion, and our construction would not promise to simplify anything. Conversely, under this hypothesis our construction is all right.

So it transpires that $\ker(\pi_n(f|V) \to \pi_n(f))$ is not really relevant to our problem. Rather we must consider $\ker(\pi_n(f|V) \to \pi_n(f|W_i))$, where $W_i = f^{-1}(X_i)$, $i = 1, 2$. And indeed, for any given element of one of the latter sets, we can perform a construction as above to kill this element. It is this construction to which we refer as surgery (in dimension n).

Lemma. Surgery can be performed to make V connected, and $f : \pi_1 V \to \pi_1 Y$ an isomorphism.

The proof uses special arguments which are however fairly well known, so we do not give them here (a description of these arguments is given in [12]).

The lemma enables us to pass to universal covers. Let $\tilde{X}$ and $\tilde{W}$ be the universal covers of X and W. Tilda on top of something else denotes induced covering, e.g., $\tilde{Y}$ is the subspace of $\tilde{X}$ covering Y. And T(f) denotes the mapping cylinder of f. There is a Mayer Vietoris sequence of (cellular) chain complexes

$$\mathcal{C}(\tilde{T}(f|V),\tilde{V}) \to \mathcal{C}(\tilde{T}(f|W_1),\tilde{W}_1) \oplus \mathcal{C}(\tilde{T}(f|W_2),\tilde{W}_2) \to \mathcal{C}(\tilde{T}(f),\tilde{W}) .$$

Lemma. This sequence gives a finite MV presentation of $\mathcal{C}(\tilde{T}(f),\tilde{W})$ in the sense of section 3.

Proof. Exactness and bases are all right. We must verify that certain terms, and maps, are obtained by tensoring. But this is rather obvious: If, say, $\hat{V}$ denotes the universal cover of V, then $\mathcal{C}(\tilde{V}) = ZD \otimes_C \mathcal{C}(\hat{V})$ as a ZD-complex, where $C = \pi_1 Y$, and $D = \pi_1 X$.

Now in pairs like $\tilde{T}(f|V),\tilde{V}$, and $\tilde{T}(f|W_i),\tilde{W}_i$, all spaces are simply connected, and the pairs are 1-connected. Hence the Hurewicz homomorphism is an isomorphism in the lowest dimension n ($\geqslant 2$) where not everything is trivial. And by naturality of the Hurewicz homomorphism, we can conclude that for this lowest n,

$$\begin{array}{l} \ker(\pi_n(\tilde{T}(f|V),\tilde{V}) \to \pi_n(\tilde{T}(f|W_i),\tilde{W}_i)) \\ \downarrow \\ \ker(H_n(\tilde{T}(f|V),\tilde{V}) \to H_n(\tilde{T}(f|W_i),\tilde{W}_i)) \end{array}$$

is an isomorphism. But the first group classifies geometric surgeries (on f), and the second group classifies algebraic surgeries (on the MV presentation). Hence in (the lowest) dimension n, these notions agree.

Finally, we are assuming $\kappa \oplus \rho = 0$. So, by 5.4.4, algebraic surgeries can be performed to make $\mathcal{C}(\tilde{T}(f|V),\tilde{V})$ acyclic. These can be arranged in order of increasing dimension, and by the addendum 5.4, no surgeries below dimension n are needed. Therefore the whole procedure can be performed geometrically, and we are finished.

7. ITERATED GENERALIZED FREE PRODUCTS

Let X,Y be a pair of CW complexes which is a codimension 1 pair in the sense of section 6. We do not ask that X or Y be finite. We do ask that X be connected and countable, and that for any component Y_0 of Y, $\pi_1 Y_0 \to \pi_1 X$ be a monomorphism. For the latter it is sufficient to ask that the homomorphisms induced by inclusion in the adjacent component(s) of $X-Y$ be injective.

In this situation, we call $\pi_1 X$ a <u>very generalized free product</u> of the $\pi_1 X_j$ over the $\pi_1 Y_i$, where the X_j and Y_i are the components of $X-Y$, and Y, respectively. The structure of $\pi_1 X$ may be described somewhat more algebraically using the graph ∇, as follows. To the component Y_i of Y, we associate the edge y_i of ∇, and to y_i in turn, we associate $\pi_1 Y_i$. To the component X_j of $X-Y$, we associate the vertex x_j of ∇, and to x_j in turn, we associate $\pi_1 X_j$. And whenever Y_i is adjacent to X_j, x_j is incident to y_i, and to this incidence there corresponds an inclusion of $\pi_1 Y_i$ in $\pi_1 X_j$. From the graph ∇, and the groups and injections it carries, $\pi_1 X$ may be recovered. ∇ is connected, but otherwise it may have any shape, in particular, it may be infinite.

We now introduce classes of groups, $G_{m,n}$, indexed by pairs of integers in lexicographical ordering. Every class will contain all the preceding ones. We abbreviate $\bigcup_n G_{m,n} = G_m$, and $\bigcup_m G_m = G$.

(1) $G_{0,0}$ contains only the trivial group

(2) $D \in G$ if and only if D is a very generalized free product of groups A_j over groups C_i, all of which are contained in G_m, for some fixed m.

(3) If $D \in G$, then $D \in G_m \Leftrightarrow$ all $A_j \in G_{m,n}$, for some fixed n, and all $C_i \in G_{m-1}$

(4) If $D \in G_m$, then $D \in G_{m,0} \Leftrightarrow$ all $A_j \in G_{m-1}$

$D \in G_{m,n} \Leftrightarrow$ all $A_j \in G_{m,n-1}$

Lemma. If $D \in G_{m,n}$, and E is a subgroup of D, then $E \in G_{m,n}$.

Proof. By induction on (m,n). Represent D by a CW pair X,Y as above. Form the covering $\tilde{X}$ of X with $\pi_1\tilde{X} = E$. The induced covering $\tilde{Y}$ of Y gives the desired decomposition.

Remark. This lemma is the reason for introducing very generalized free products. If instead we had defined $G_{m,n}$ by (non-iterated) generalized free product, then if $D \in G_{m,n}$, there does not seem to be any reason to suppose that the subgroup E is in G_m, even if E be finitely presented.

Examples. (1) G is closed under extensions.

Proof. Let $1 \to \ker(p) \to E \xrightarrow[p]{} D \to 1$ be exact, with $\ker(p), D \in G$. Let $D \in G_{m,n}$. The proof is by induction on (m,n). Let D be the very generalized free product of the groups A_j over the groups C_i. Then E is the very generalized free product of the groups $p^{-1}(A_j)$ over the groups $p^{-1}(C_i)$.

(2) If M is a closed 2-manifold other than a 2-sphere or projective plane, then $\pi_1 M \approx A *_C \{t\}$, with A free (of finite rank), and C cyclic. Hence $\pi_1 M \in G_{2,0}$.

(3) If M is a knot space or, more generally, a compact 3-manifold as considered in [11], then $\pi_1 M$ is in either G_2 or G_3, according to whether M is bounded or not. This is non-trivial, and is essentially due to Haken [4]. The only known upper bound for n is given by an iterated exponential function involving the number of simplices of M [4].

(4) A one-relator group is in G_2 if (and only if) the relator is not a proper power. In fact, an essential element in Magnus' analysis of these groups (cf. [7]) is this. Given a one-relator group A_1, there is a sequence of groups A_1, A_1', A_2, A_2', ..., A_k, so that (a) A_j is a subgroup of finite index in A_j' ; (b) $A_j' = A_{j+1} *_{C_j} \{t\}$, with C_j a free group ; (c) A_k is a finite group with exponent the power of the relator of A_1 .

(5) Ch. Miller III has informed me that there is a finitely presented group in G_3 (or maybe even G_2) with unsolvable word problem.

Proposition. If $D \in G_m$, then gl.dim.(ZD) $\leq$ m+1.

Proof. If D is a very generalized free product, then one can define MV presentations of ZD-complexes, similarly as in section 3. One cannot define finite MV presentations, in general, but otherwise the only difference is in notation. In particular, prop. 4.3 can be proved in this framework. But this is just what we are after.

Digression. It is not true (but for non-trivial reasons) that G contains all groups of finite homological dimension. The counter-examples I know are however finite extensions of groups in $G_{3,0}$. It might thus be worth asking the following question. Is there a group of finite homological dimension which cannot be built up by these processes: forming a generalized free product, taking a direct limit of inclusions, and taking a supergroup of finite index ?

Conjecture. If $E \in G$, then Wh(E) = 0.

The following is a partial result in this direction. It verifies the conjecture for the class G_2, and the examples (2), (3), and (4).

Definition. Let G^1 be the subclass of those $D \in G$ such that ZD is noetherian; and for r = 2, 3, let G^r be the subclass consisting of those $D \in G$ which can be built up so that at any stage, all the amalgamating subgroups C_i are in G^{r-1}.

Lemma. If $D \in G^2$, then ZD is coherent.

Proof. Since a direct limit (via inclusion) of coherent rings is again coherent, this follows from prop. 4.4.

The following observation is quite useful.

(!) If D is a very generalized free product of the groups A_j over the groups C_i, then $D \times Z$ is a very generalized free product of the groups $A_j \times Z$ over the groups $C_i \times Z$.

<u>Corollary</u>. If $D \in G^r$, $r = 1, 2, 3$, then $D \times Z \in G^r$.

<u>Proof</u>. For $r = 1$, this is well known. For $r = 2, 3$, it follows from (!) by induction.

<u>Theorem</u>. If $E \in G^3$, then $Wh(E) = \tilde{K}_0(E) = 0$.

<u>Proof</u>. Write $E = D \times F$, where $D \in G_{m,n}$, and F is a finitely generated free abelian group. The proof is by induction on (m,n). So the induction beginning is the free abelian groups. The induction beginning is all right by [2].

Since D is a very generalized free product, it is a direct limit, via inclusion, of groups D_j, $j = 0, 1, \ldots$, where

$$D_{j+1} \approx D_j *_{C_j} B_j \quad \text{or} \quad \approx D_j *_{C_j} \{t\} \quad ,$$

respectively, and $D_0, B_j, C_j \in G_{m,n-1}$. The functors Wh and $\tilde{K}_0$ commute with direct limit. Therefore it suffices to prove by induction on j that $Wh(D_j \times F) = \tilde{K}_0(D_j \times F) = 0$. By (!) above, we can apply the sum theorem 5.1. This expresses $Wh(D_j \times F)$ as the sum of a Wh term and a $\tilde{K}_0$ term which are both zero by the induction hypothesis, and a $\tilde{C}$ (or $\tilde{D}$) term which is zero since $Z(C_j \times F)$ is coherent, and has finite global homological dimension.

We must now take care of $\tilde{K}_0(D_j \times F)$. But this is a direct summand of $Wh(D_j \times F \times Z)$, so it has been taken care of already.

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