Homological Algebra 18.715 D.G. Quillen. ①
Algebraic K-Theory

(Notes by Howard Hiller)

A ring.

$K_0 A$ = Grothendieck group of fin. gen. projective A-modules.
$K_1 A = GL_\infty(A)/[GL_\infty(A), GL_\infty(A)]$

Look for higher K-functors $K_n A$, $n \in \mathbb{Z}$.

$K_n A = \pi_n(FA)$ FA = some space associated with A.
so part of homotopy theory, so algebraic topology.

Acyclic spaces and maps:

Def: Let X be a space. X is called <u>acyclic</u> if $\tilde{H}_*(X) = 0$.
(integral homology). <u>aspherical</u> if $\pi_*(X, x_0) = 0$.

If X has homotopy type of CW complex $\Longrightarrow$ $X \sim *$ by Whitehead's Theorem.

Assumption: All spaces have homotopy type of CW-complex, with basepoint, connected.

Poincaré: $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$.

Def: A group G is <u>perfect</u> if $G = [G, G]$ $G \longrightarrow A$ (abelian) is trivial.

X acyclic $\Rightarrow$ $\pi_1(X)$ is perfect (by Poincaré).

X acyclic and simply connected $\Rightarrow$ $X \sim *$ (by Whitehead.

All simple non-abelian groups are perfect; e.g. A_n $n \geq 5$.

Question: Does $\exists$ a finite acyclic polyhedron?

op & Def: Let $f: X \longrightarrow Y$ be a map. TFAE:

(i) The homotopy fibre of f is acyclic.
(Replace $f: X \to Y$ by a Serre fibration, and all fibers ha
same homotopy type).

$$\begin{tikzcd} X \arrow[dr, "f"'] & X \times_Y Y^I \arrow[l, "p_1"'] \arrow[d, dashed] \arrow[dr, "p_2"] & \\ & Y & Y^I \arrow[l, "\bar{\lambda}"] \end{tikzcd} \quad X \times_Y Y^I = \{(x,\lambda) : f(x) = \lambda(0)\}$$

homotopy-fibre of f = $X \times_Y Y^I \times_Y \{y_0\}$

(ii) For any local coefficient system of abelian groups L on Y

$$f_* : H_q(X, f^*L) \xrightarrow{\approx} H_q(Y, L)$$

(iii) Let $\tilde{Y}$ = universal cover of Y. Then:

$$\begin{tikzcd} X \times_Y \tilde{Y} \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

$$(p_2)_* : H_*(X \times \tilde{Y}) \xrightarrow{\approx} H_*(\tilde{Y})$$

Proof: Assume $f: X \longrightarrow Y$ is fibration and put $F = f^{-1}(\{y_0\})$.
(i) $\Rightarrow$ (ii) Assume F is acyclic.

Consider Serre spectral sequence:

$$\begin{tikzcd} F \arrow[r] & X^{\urcorner L'} \arrow[d, twoheadrightarrow] \\ & Y \end{tikzcd} \qquad E^2_{p,q} = H_p(Y, H_q(F, i^*L')) \Rightarrow H_{p+q}(X, L')$$

universal coefficient theorem:

$$H_q(F,A) \simeq H_q(F) \otimes A \oplus \operatorname{Tor}_1(H_{q-1}(F), A) = 0 \quad , \ F \text{ acyclic.}$$

Take $L' = f^*L$. Then i^*L' is constant.

Thus $H_q(F, i^*L') = \begin{cases} L & q=0 \\ 0 & q>0 \end{cases}$

So spectral sequence degenerates, $E^2_{p,0} \xleftarrow{\simeq} H_p$, yielding (ii).

(ii)$\Rightarrow$(iii)

$$(*) \quad \begin{tikzcd} X \times_Y \tilde{Y} \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \tilde{Y} \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

vertical maps are covering spaces.

$$\pi = \pi_1 Y \quad \begin{tikzcd} H_q(X \times_Y \tilde{Y}) \arrow[r, "(p_2)_*"] \arrow[d, "(p_1)_*"', "\simeq"] & H_q(\tilde{Y}) \arrow[d, "\simeq"] \\ H_q(X, \mathbb{Z}[\pi]) \arrow[r, "f_*"'] & H_q(Y, \mathbb{Z}[\pi]). \end{tikzcd}$$

canonical isomorphism from Serre spectral sequenc for p.

(ii) $\Rightarrow$ f_* is isomorphism, so $(p_2)_*$ is.

(iii)$\Rightarrow$(i):

$$\begin{tikzcd} F \arrow[d] \arrow[r, phantom, "="] & F \arrow[d] & \\ X \times_{X'} (X' \times_Y \tilde{Y}) \arrow[r] \arrow[d] & X' \times_Y \tilde{Y} \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] \\ X \arrow[r, "\sim"] & X' \arrow[r] & Y \end{tikzcd}$$

Look at homotopy exact sequence.

So assume $f: X \to Y$ is fibration. But then pr_2 has same fiber as f by pull-back square $(*)$. Call it F.

Replace $f: X \longrightarrow Y$ by $pr_2: X \times_Y \tilde{Y} \longrightarrow \tilde{Y}$. In which case we are trying to show if $f_*: H_*(X) \xrightarrow[\approx]{} H_*(Y)$ and Y simply connec[ted] $\Rightarrow F$ acyclic. This is a special case of (ii) $\to$ (i)

(ii) $\Rightarrow$ (i)

$$\begin{tikzcd} \Omega Y \arrow[r] \arrow[d, equal] & PY \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Omega Y \arrow[r] & PY \arrow[r] & Y \end{tikzcd}$$

Claim: $PY \times_Y X = F$, i.e. homotopy fibre of f.

Show this is acyclic; Look at spectral sequence.

$$E^2_{p,q} = H_p(X, H_q(\Omega Y)) \Longrightarrow H_{p+q}(PY \times_Y X).$$

$$E^2_{p,q} = H_p(Y, H_q(\Omega Y)) \Rightarrow H_{p+q}(PY) \overset{!}{=} 0.$$

Def: f is <u>acyclic</u> if it satisfies conditions of Proposition.

Ex. $X \longrightarrow *$ is acyclic $\Leftrightarrow$ X is acyclic.
trivially fibration.
(General Grothendieck method of extending properties of obj to maps)

Remarks:

(1) $X \xrightarrow{f} Y \xrightarrow{g} Z$ f, g acyclic $\to$ $g \circ f$ acyclic.
f, $g \circ f$ acyclic $\Rightarrow$ g acyclic.

(2)
$$\begin{tikzcd} Y' \times_Y X \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

pullback square, f, g are fibrations.
then f acyclic $\Rightarrow$ f' acyclic.

③
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ Z \arrow[r, "f'"'] & Z \coprod_X Y)_L \end{tikzcd}$$

Assume f, g are cofibrations. then f acyclic $\Rightarrow$ f' acyclic.

Pf: Assume g cofibration.

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ H(X; fg'L) \arrow[r, "f_*"] \arrow[d] & H(Y; g'L) \arrow[d] \\ H(Z; f'L) \arrow[r, "f'_*"] \arrow[d] & H(Z \coprod_X Y; L) \arrow[d] \\ H(Z, X; f'L) \arrow[r, "f'_*"'] \arrow[d] & H(Z \coprod_X Y, Y; L) \arrow[d] \\ \vdots & \vdots \end{tikzcd}$$

f_* iso $\overset{\text{5 Lemma}}{\Longrightarrow}$ f'_* is iso.

Local coefficient Whitehead Theorem:

④ $f: X \to Y$ is acyclic, $\pi_1 f: \pi_1 X \xrightarrow{\cong} \pi_1 Y$, then f is h. eq.

Proof: Hypotheses $\to$ $X \times_Y \tilde{Y}$ = universal cover of $X = \tilde{X}$ and that $\tilde{X} \to \tilde{Y}$ induces isomorphisms of H_*.

So by simply connected Whitehead Theorem: $\tilde{X} \to \tilde{Y}$ h. eq.

Ex: Suppose X is closed n-manifold, homology n-sphere, $n \geq 2$
$\Rightarrow$ orientable. $H_n(X)$ has fundamental class μ

$$X \xrightarrow{f} X/X-U \cong S^n.$$

$$H_n(X) \to H_n(S^n) \quad \text{maps to canonical genera}$$
μ

f is acyclic map, L is constant since S^n is simply-connec

Poincaré homology 3-sphere: $X = SO(3)/\underset{\substack{\| \\ A_5}}{\text{icosahedral group}} = S^3/\text{binary icosahedral group}$

This is a homology 3-sphere. 3-manifold with no H_1, so no H_2 by Poincaré duality, and orientable.

Classification of acyclic with fixed source X.

Theorem: (i) If $f: X \to Y$ and $f': X \to Y'$ are acyclic and $\ker(\pi_1(f)) = \ker(\pi_1(f'))$, then $\exists$ homotopy equivalence $g: Y \to Y'$ s.t. $g \circ f \simeq f'$

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Y \arrow[rr, "g"'] & & Y' \end{tikzcd}$$

homotopy commutes.

(ii) Given a perfect normal subgroup N of $\pi_1 X$, $\exists$ an acyc map $f: X \to Y$ with $\ker(\pi_1(f)) = N$.

Remark 5: $f: X \to Y$ acyclic $\to$ $\pi_1 f: \pi_1 X \to \pi_1 Y$ is onto and $\ker(\pi_1(f))$ is perfect. ?

$$\pi_1 F \to \pi_1 X \xrightarrow[\pi_1 f]{} \pi_1 Y \to \underset{\substack{\| \\ 0}}{\pi_0 F}$$

stronger form of (i):

Proposition: Given maps:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[dl, dashed, "h"] \\ Z & \end{tikzcd}$$

with f acyclic.

If $\ker(\pi_1(f)) \subseteq \ker(\pi_1(g))$. Then $\exists h: Y \to Z$ st. $h \circ f \sim g$ and any two such are homotopic.

Note: Proposition $\Rightarrow$ (i) by abstract nonsense; f is initial object in maps over Z.

Proof:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Z & \xrightarrow[f']{} & Z \coprod_X Y \end{array}$$

Assume f inclusion of CW complexe

Van Kampen Theorem: $\pi_1(Z \coprod_X Y) = \pi_1(Z) *_{\pi_1(X)} \pi_1(Y)$

$= \pi_1(Z)$ by hypothesis on kernels.

f' is acyclic and $\pi_1(f')$ is isomorphism

By Remark 5, f' is homotopy equivalence.

Any h is obtained from retraction of f', homotopy inverses are unique, so obtain uniqueness.

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classifying acyclic maps $X \to Y$, X fixed

Proof of (ii): Case $N = \pi_1 X$. Choose maps $f_i : S^1 \to X$, $i \in I$ s.t. $[f_i]$ generate $\pi_1 X$.

$$\bigvee_{i\in I} S^1 \xrightarrow{\vee f_i} X \longrightarrow X' = \text{mapping cone of } \vee f_i.$$

where $X' = \bigcup_{i\in I}^{X} X \cup_{f_i} e^2 = X \cup_{(f_i)} \vee_I e^2$

Van Kampen Theorem $\Rightarrow$ $\pi_1 X' = 0$. For homology, have long exact mapping cone sequence.

$$0 \to H_3 X \to H_3 X' \to 0 \to H_2 X \to H_2 X' \xrightarrow{\partial} \bigoplus_{i\in I} \mathbb{Z} \to H_1 X'$$

$H_1 X' = 0$

$H_q X \simeq H_q X'$ $q > 3$. s.e.s. splits.

By Hurewicz theorem: $\pi_2 X' \xrightarrow{\approx} H_2 X'$, since $\pi_1 X' = 0$.

Let $g_i : S^2 \to X'$ be s.t. it maps onto i^{th} generator under

Then: $H_2 X' \cong H_2 X \oplus \bigoplus_{i\in I} \mathbb{Z}_{g_{i*}(b)}$ b generates $H_2($

$$\bigvee_{i\in I} S^2 \xrightarrow{\vee g_i} X' \longrightarrow Y = X' \cup_{\vee g_i} \bigvee_I e^3$$ Then have:

$$\begin{tikzcd} 0 \arrow[r] & H_3 X' \arrow[r, "\approx"] & H_3 Y \arrow[r] & \bigoplus_I \mathbb{Z} \arrow[r, hook] & H_2 X' \arrow[r] & H_2 Y \arrow[r] & 0 \arrow[r] & \dots \\ & & & & H_2 X \arrow[u, hook] \arrow[ur, "\approx"] & & & \end{tikzcd}$$

Thus $H_q X \simeq H_q Y$ $\forall q$, Y is also simply-connected by van Kampen. Finished, $X \to Y$ is acyclic with $\ker(\pi_1 f) = N$

General case: Let $X_0' =$ covering space of X with $\pi_1 X_0 = N$

By first part:
$$\begin{tikzcd} X_0 \arrow[r, hook, "f_0"] \arrow[d] & Y_0 \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
f_0 acyclic $\pi_1 Y_0 = 0$

Let Y be pushout.

Van Kampen $\Rightarrow$ $\pi_1 Y = \pi_1 X *_{\pi_1 X_0} \pi_1 Y_0 = \pi_1 X *_N 0 = N$.

For fixed target Y harder problem. In particular $Y = *$.

Dror's thesis: "Acyclic spaces" M.I.T.

Remark: Any group G has a largest perfect subgroup N nec normal. (G is generated by all perfect subgroups)

G/N has no non-trivial perfect sub gp.

Notation: $X \to X^+ =$ acyclic map killing the largest perfect subgroup of $\pi_1 X$.

(1) $X \mapsto X^+$ is a homotopy functor

(2) $(X \times Y)^+ \sim X^+ \times Y^+$

(1)
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \\ X^+ \arrow[r, dashed] & Z^+ \end{tikzcd}$$
by universal property on killing π_1.

(2) Convert $X \xrightarrow{f} X^+$, $Y \xrightarrow{g} Y^+$ into fibrations:

$$X \times Y \xrightarrow{f \times g} X^+ \times Y^+ \quad \text{acyclic map.}$$

$\pi_1 X^+ = \pi_1 X / N$ $\qquad$ $\pi_1 (X^+ \times Y^+) = \pi_1 X/N \times \pi_1 Y/N'$

$\qquad\qquad\qquad\qquad\qquad = \pi_1 X \times \pi_1 Y / N \times N'$.

$\pi_1 Y^+ = \pi_1 Y / N'$. $\qquad$ $N \times N'$ is largest perfect subgroup.

So: $\quad X^+ \times Y^+ = (X \times Y)^+$.

Examples: (1) X acyclic $\Leftrightarrow$ X^+ contractible (Hurewicz & Whitehead).

(2) Suppose X is a closed n-manifold with $H_*(X) = H_*(S^n)$; $n \geq 2$ (homology n-sphere). collapsing map $X \to S^n$.

So $\quad X^+ \sim S^n$.

(3) $X =$ Poincaré homology 3-sphere $= SO(3)/G. = S^3/\tilde{G}$
$G =$ symmetries of the icosahedron $\approx A_5$. $\quad |G| = 60$.

$\tilde{G} =$ binary icosahedral group ; $\quad |\tilde{G}| = 120$

$X^+ = S^3$ by (2).

Take space $B\tilde{G} \simeq K(\tilde{G},1)$ for $\tilde{G}$ discrete.

What is $(B\tilde{G})^+$? Look at fibration.

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[d, equal] & S^3/\tilde{G} \arrow[r] \arrow[d, dashed, "\beta"] & B\tilde{G} \arrow[r] \arrow[d, "\alpha", "\approx"'] & BS^3 = \mathbb{Q}P^\infty \arrow[d, equal] \\
S^3 \arrow[r, "120"', "\text{degree}"'] & F \arrow[r] & B\tilde{G}^+ \arrow[r, "\text{induced}"'] & BS^{3+}
\end{tikzcd}$$

$F = S^3$

$H_*(\alpha)$ is $\approx \Rightarrow H_*(\beta)$ is iso. $F = (S^3/\tilde{G})^+ = S^3$.

Conclude: $(B\tilde{G})^+$ is a fibre space over BS^3 with fibre

Let A be an abelian group, $K(A,n)$ Eilenberg-Maclane space.' n

$$\pi_q(K(A,n)) = \begin{cases} A & n=q \\ 0 & q \neq n \end{cases}$$

Hurewicz $\Rightarrow$

$$H_q(K(A,n)) = \begin{cases} 0 & q<n \\ A & q=n \\ H_{n+1}=0 & q=n+1 \end{cases}$$

$$[X, K(A,n)] \xrightarrow{\cong} H^n(X,A).$$

Lemma: Let $c: H_n(K(A,n)) \xrightarrow{\cong} A$ be canonical Hurewicz (invers iso. The map:

$$\begin{aligned} [X, K(A,n)] &\longrightarrow \mathrm{Hom}(H_n X, A) \\ f &\longmapsto c \cdot H_n(f) \end{aligned}$$

is onto with kernel $\approx \mathrm{Ext}^1(H_{n-1}X, A)$.

Proof: Universal coefficient formulas

$$0 \to Ext^1(H_{n-1}X, A) \to H^n(X,A) \xrightarrow{\rho} Hom(H_n X, A) \to 0$$

Let $X = K(A,n)$, $\exists$ unique cohomology class $u \in H^n(K(A,n),A)$. with $\rho(u) = c$.

Fact: $[X, K(A,n)] \xrightarrow{\sim} H^n(X,A)$
$$f \mapsto f^*(u).$$

$$\begin{tikzcd} H^n(X,A) \arrow[r] & Hom(H_n X, A) \\ f \in [X, K(A,n)] \arrow[u, "2"] \arrow[ur, "c \cdot (H_n f)"'] & \end{tikzcd}$$

Dror's constructions of an acyclic space AX starting from X.

$X_1 = X$

X_2 = the covering space of X with $\pi_1 X_2$ = largest perfect subgp of $\pi_1 X$.

We define inductively a tower: $\ldots \to X_3 \to X_2$ s.t.

(*) $\tilde{H}_q X_n = 0 \qquad q < n.$

(**) the fibre of $X_{n+1} \to X_n$ is $K(H_n X_n, n-1)$.

Suppose X_n is constructed: By Lemma :

$$[X_n, K(A,n)] \xrightarrow{\sim} Hom(H_n X_n, A).$$

Let $A = H_n(X_n)$: we get a unique map: $\chi_n : X_n \to K(H_n X_n, n)$. which induces the map:

$$\bar{c}^{-1} : H_n X_n \to H_n(K(H_n X_n, n)).$$

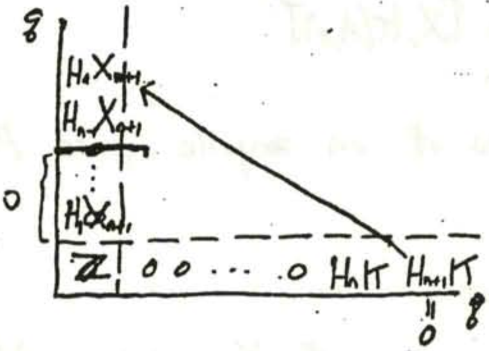
Put $X_{n+1} =$ fibre of χ_n.

$$\begin{tikzcd} X_{n+1} \arrow[r] & X_n \arrow[d] \\ & K(H_n X_n, n) = K \end{tikzcd}$$

$(**)$ is clear, since
$\Omega K(A,n) = K(A,n-1)$.

Look at Serre spectral sequence of fibration:

$$E^2_{pq} = H_p(K, H_q X_{n+1}) \Longrightarrow H_{p+q}(X_n).$$



E^R

$\tilde{H}_q X_{n+1} = 0 \quad q < n-1$

$0 \longrightarrow H_n X_{n+1} \longrightarrow H_n X_n \xrightarrow{\approx} H_n K \longrightarrow H_{n-1} X_{n+1} \longrightarrow 0$ by spectral-
$\qquad\qquad \overset{\shortparallel}{0} \qquad \leftarrow$ by choice of $\chi_n \rightarrow \overset{\shortparallel}{0}$

sequence chasing, obtain 5-term sequence.

$\therefore \; \tilde{H}_q X_{n+1} = 0 \quad q < n+1 \quad \Rightarrow (*)$.

$$\begin{tikzcd} & X_4 \arrow[d] & \\ & X_3 \arrow[r] \arrow[d] & K(H_3 X_3, 3) \\ & X_2 \arrow[r, "\chi_2"] \arrow[d] & K(H_2 X_2, 2) \\ T \arrow[r] \arrow[ur, dashed, "!"] \arrow[uur, dashed, "!"] & X = X_1 & \end{tikzcd}$$

Put $AX_\infty = \varprojlim X_n$. Then $\tilde{H}_*(AX) = 0 \; \therefore \; AX$ is acyclic

Proposition: For any acyclic space T, $[T, AX] \xrightarrow{\sim} [T, X]$.

Proof: see picture p. 12. $[T, K(A,n)] = H^n(T,A) = 0$.

$$[T, K(H_n X_n, n-1)] \to [T, X_{n+1}] \underset{\approx}{\to} [T, X_n] \to [T, K(H_n X_n, n)]$$
$$\overset{\shortparallel}{0} \qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{0}$$

Exercises: 1) Show $X^+ = \mathrm{Cone}\,(AX \to X)$.
2) Show $AX = $ Fibre of $(X \to X^+)$

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Schur multipliers and central extensions:

G group:

(E,p) is a central extension of G if (by A)

$$1 \to A \to E \to G \to 1$$

and $A \subset$ center of E.

(E,p) is a universal central extension of G if for any central extension (E',p') $\exists!$ homomorphism $h: E \to E'$ s.t.

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & G & \end{tikzcd}$$ commutes.

Proposition: (i) If (E,p) is a universal central extension then E is perfect: $E = (E,E)$.
(Hence if a universal central extension exists, G is perfect)
(ii) Any perfect G has a universal central extension.

Let $E^{ab} = E/(E,E)$.

uniqueness gives:

$$\begin{tikzcd} E \arrow[r, shift left, "{(p,0)}"] \arrow[r, shift right, "{(p,\pi)}"'] \arrow[dr, "p"'] & G\times E^{ab} \arrow[d, "pr_1"] \\ & G \end{tikzcd}$$

$\pi: E \longrightarrow E^{ab}$

so $E^{ab}=0 \Rightarrow E=(E,E)$.

Lemma: Let (E,p), (E',p') be central extensions. If E is perf[ect] then there exists at most one map $E\to E'$ (over G).

Proof: If $\alpha_1,\alpha_2 : E \rightrightarrows E'$, define: $E \xrightarrow{f} \ker(p')$. by

$f(e)=\alpha_1(e)\cdot\alpha_2(e)^{-1}$. Then f is a homomorphism.

$\alpha_1(e) = f(e)\cdot\alpha_2(e)$

$\alpha_1(ee') = \alpha_1(e)\cdot\alpha_1(e') = \underbrace{f(e)}_{\in \ker(p') \subseteq \text{center } E}\cdot\alpha_2(e)\cdot f(e')\alpha_2(e') = f(e)f(e')\alpha_2(e)\alpha_2(e')$

$f(e)\cdot f(e')\alpha_2(ee')$ ✓.

Then f is a homomorphism to an abelian group $\therefore f=0$.
$\therefore \alpha_1=\alpha_2$.

Proof of (ii): Write $G=F/R$, F free. Put $E=F/(F,R)$.
Put:

$$\begin{tikzcd} 1 \arrow[r] & R/(F,R) \arrow[r] & F/(F,R) \arrow[r] \arrow[d, equal] & F/R \arrow[r] \arrow[d, equal] & 1. \\ & & E \arrow[r, dashed, "p"'] & G & \end{tikzcd}$$

(a) E is a central extension of G. ✓

(b) E maps to any other central extension E' of G.

$$\begin{tikzcd} F \arrow[r, dashed, "\alpha"] \arrow[d, "\pi"'] & E' \arrow[d, "p'"] \\ R/F \arrow[r, "1"'] \arrow[ur, dashed] & G \end{tikzcd}$$

by freeness

since $\alpha((F,R))=0$.

(c) E is perfect.

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(c) (E,E) has properties (1),(2).

<u>Claim</u>: (E,E) is perfect.

Take $(e_1,e_2) \in (E,E)$ $\quad e_1 e_2 e_1^{-1} e_2^{-1}$.

Since $\quad (E,E) \longrightarrow (G,G) = G$. $\quad$ I know $e_i \equiv (e_i', e_i'') \cdot a_i \quad a_i \in \ker$

$(e_1, e_2) = ((e_1', e_1''), (e_2', e_2''))$. so: $(E,E) \subseteq ((E,E),(E,E))$

Def: If (E,p) is the universal central extension of a perfect group G, then $\ker(p)$ is called the Schur multiplier of G.

Schur: (1) Alternating group A_n, $n \geq 5$
Schur multiplier of $A_n = \mathbb{Z}/2\mathbb{Z}$.

(2) $SL_3(\mathbb{F}_2)$ size $(2^3-1)(2^3-2)(2^3-4)$
$7 \cdot 6 \cdot 4 = 168$

$PSL_2(\mathbb{F}_7) = SL_2(\mathbb{F}_7)/\{\pm id\} = \frac{(49-1)(49-7)}{6 \cdot 2} = 168$.

$SL_3(\mathbb{F}_2) \cong PSL_2(\mathbb{F}_7) = SL_2(\mathbb{F}_7)/\{\pm 1\}$.

universal central extension of $PSL_2(\mathbb{F}_7)$ is $SL_2(\mathbb{F}_7)$.
$\therefore$ Schur multiplier is $\mathbb{Z}/2\mathbb{Z}$.

G discrete group.

BG - classifying space of topological group. (classifying principal G-bundles).
in discrete case classifies covering spaces with G acting as deck transformations.

$$\begin{tikzcd} G \arrow[r] & \widetilde{BG} \arrow[d] \\ & BG \end{tikzcd}$$
universal covering $\sim$ pt.

$$\begin{tikzcd} & \arrow[d, "\text{fibre}"] & \\ BE = & X_3 \arrow[r] \arrow[d, "\text{fibre}"] & K(H_3 X_3, 3) \\ BN = & X_2 \arrow[r] \arrow[d] & K(H_2 X_2, 2). \\ BG = & X_1 & \end{tikzcd}$$

Dror construction:

N - largest perfect subgp of G.

Have fibration: $K(H_2 X_2, 1) \to X_3 \to X_2$. $X_2 = K(N,1)$.

Put $\pi_1 X_3 = E$.

$$1 \to H_2 BN \to E \to N \to 1$$

Proposition: E is the universal central extension of N.

Proof: Recall: $F \to E \to B$ fibration of pointed spaces. then $\pi_1 E$ acts on $\pi_* F$ as follows.

Loop $\alpha : [0,1] \to E$. Map

$$\begin{tikzcd} F \times [0,1] \arrow[r, "H"] \arrow[dr] & E \arrow[r] & B \\ & {[0,1]} \arrow[u] & \end{tikzcd}$$

$(f,t) \mapsto p(\alpha(t))$

$$\begin{tikzcd} F \cup_* [0,1] \arrow[rr] \arrow[d] & & E \arrow[d, "p"] \\ F \times [0,1] \arrow[r, "pr_2"'] \arrow[urr, dashed] & {[0,1]} \arrow[r, "p \circ \alpha"'] & B \end{tikzcd}$$

This describes the action.

Must use homotopy extension property.

$$\begin{tikzcd} K(H_2X_2,1) \arrow[r, "="] \arrow[d] & \Omega K(H_2X_2,2) = K(H_2X_2,1) \arrow[d] \\ X_3 \arrow[r] \arrow[d] & * = P[K(H_2X_2,2)] \arrow[d] \\ X_2 \arrow[r] & K(H_2X_2,2) \end{tikzcd}$$

The action of $\pi_1 X_3$ on $\pi_*(K(H_2X_2,1))$ is induced by action of $\pi_*(*)=1$. $\therefore$ $\pi_1 X_3$ acts trivially on $H_2X_2 = \pi_1$ and E is a central extension of N by $H_2X_2 = H_2(BN)$.

Lemma: If N is perfect s.t. $H_2(BN)=0$, then N has no non-trivial central extensions. (i.e. if $E \xrightarrow{p} N$ is a central extension, and E is perfect then $E \cong N$).

Proof: Suppose E is central extension of N.

$$1 \to A \to E \xrightarrow{p} N \to 1$$

Induces map:

$$\begin{tikzcd} BA \arrow[r] \arrow[d, equal] & BE \arrow[r, "Bp"] & BN \arrow[d, "u"] \\ K(A,1) \arrow[r] & * \arrow[r] & K(A,2) \end{tikzcd}$$

convert Bp to fibration $BA = K(A,1)$ is fibre. (u induces Bp).

Obstruction theory classifies fibrations with fibre $K(A,n)$ in terms of $H^{n+1}(-,A)$

Fact: Given a fibration $K(A,n) \to E \to B$ with $\pi_1(\ )$ acting trivially on A is induced from:

$$K(A,n) \longrightarrow * \longrightarrow K(A,n+1).$$

by a unique map $B \to K(A,n+1)$.

$$u \in [BN, K(A,2)] = H^2(BN, A) = \underset{\overset{\|}{0}}{\mathrm{Hom}(H_2 B_N, A)} \oplus \underset{=0}{\mathrm{Ext}(H_1 BN, A)}.$$

so u is trivial.

This implies Bp has a section and E is trivial extension $\cong N \times A$.

$\tilde{H}_q(X_3) = 0 \quad q \leq 2$ and $X_3 = BE$, we have that:

(a) E is perfect. $(H_1(BE) = (\pi_1(BE))^{ab} = E^{ab} = 0)$

(b) E has no non-trivial central extensions by Lemma.

Pf of Prop: Have shown E is a central extension of N. Show universality. Since E is perfect, it suffices to show. E map to any other central extension (E', p')

$$\begin{tikzcd} 1 \arrow[r] & A' \arrow[r] & E' \arrow[r, "p'"] & N \arrow[r] & 1 \\ 1 \arrow[r] & A' \arrow[r] \arrow[u, equal] & E \times_N E' \arrow[r] \arrow[u] & E \arrow[r] \arrow[u, "p"] \arrow[l, dashed, bend left] & 1 \end{tikzcd}$$

fibre product

But $E' \times_N E$ is a central extension of E via pr_2. By (b) above obtain section of pr_2, hence a map from $E \to E'$ over N.

Corollary: $H_2(BN) =$ Schur multiplier of N for any perfect group N.

$$\pi_1 X_{n+1} \longrightarrow \pi_1 X_n \qquad n \geq 3.$$

from fibration: $K(H_n X_n, n-1) \to X_{n+1} \longrightarrow X_n$.

$$\pi_1(AX) = \pi_1 X_n \quad \text{for} \quad n \geq 3$$
$$= E \quad \text{universal central extension of } N$$

Def: (Dror): G is <u>super-perfect</u> if it is perfect and every central extension of G is trivial. i.e. $H_1(BG) = H_2(BG) = 0$.

Exercise: (1) Show that if X is acyclic, then $\pi_1 X$ is super-perfect.
(2) Conversely if G is super perfect $\exists$ acyclic space $\pi_1 X = G$.
Let $X = A(BG)$.

Problem: Understand the functor: $X \mapsto X^+$.

e.g. Let G be finite perfect group.

Fact: $H_*(BG; \mathbb{Z})$ finite abelian groups killed by $|G|$.

Look at $(BG)^+$, a simply-connected space with

$$\tilde{H}_q(BG^+) = \tilde{H}_q(BG) \quad \text{finite, killed by } |G|$$

$\pi_{*_i}(BG^+)$ finite, killed by $|G|^{N_i}$

Homotopy theory $\Rightarrow$ $\quad BG^+ = \prod_{p \| |G|} X_p$.

can you talk about X_p in terms of groups structure of G. (e.g. p-subgroups of G).

9.23.74

Let A be an associative ring with 1.

Def: $K_0 A$ = Grothendieck group of f.g. projective A-modules.

$P = \operatorname{Im}(A^n \xrightarrow{e} A^n)$ $e^2 = e$, so correspond to idempotent matrices.

Two possible dfns of Grothendieck groups.

(1) Its the abelian group having one generator $[P]$ for each fin. gen. projective P and the relations: $[P] = [Q]$ iff $P \cong Q$.
$[P \oplus Q] = [P] + [Q]$ direct sum G. group

(2) Same generators [P] with one relation: exact sequence Gr.

$$[P] = [P'] + [P''] \quad \text{for each s.e.s.}$$

$$0 \to P' \to P \to P'' \to 0.$$

(2)$\Rightarrow$(1) easy. Remark: $\begin{pmatrix}\text{direct sum}\\ \text{Groth. gp}\end{pmatrix} \twoheadrightarrow \begin{pmatrix}\text{exact sequence}\\ \text{Groth. gp.}\end{pmatrix}$

and this map is iso in our case, since any s.e.s:

$$0 \to R \to E \to P \to 0 \quad \text{of } A\text{-modules}$$

with P projective splits.

Example of a category of A-modules where two Grothendieck groups differ: Let $A = \mathbb{Z}/p^2\mathbb{Z}$, take category of finitely generated A-modules. (= finite abelian groups killed by p^2). Call one M:

$$M \approx (\mathbb{Z}/p\mathbb{Z})^i \times (\mathbb{Z}/p^2\mathbb{Z})^j$$

Hence $\begin{pmatrix}\text{direct sum}\\ \text{Grothendieck gp}\end{pmatrix} \approx \mathbb{Z} \oplus \mathbb{Z}$ with generators $[\mathbb{Z}/p\mathbb{Z}], [\mathbb{Z}/p^2\mathbb{Z}]$

$\begin{pmatrix}\text{exact sequence}\\ \text{Grothendieck gp}\end{pmatrix} \approx \mathbb{Z}$ with generator $[\mathbb{Z}/p\mathbb{Z}]$

use Krull-Schmidt, Jordan Hölder to give most general result. $\quad 0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$

so $[\mathbb{Z}/p^2\mathbb{Z}] = 2[\mathbb{Z}/p\mathbb{Z}]$

Put $\mathcal{P}(A)$ = category of all fin-gen. projective A-modules. and A-module homomorphisms

$\mathrm{Iso}(\mathcal{P}(A))$ = iso. classes of $\mathcal{P}(A)$

This is an abelian monoid

Any abelian monoid M gives rise to an abelian $\overline{M}$ described as follows.

$$\overline{M} = M \times M / (m,s) = (m_1, s_1) \text{ if}$$

$$\exists t \ \text{ st. } \ m + s_1 + t = s + m_1 + t \qquad \text{(equivalence relation}$$

$+$ on $\overline{M}$ is $(m,s) + (m_1, s_1) = (m+m_1, s+s_1)$.

$\exists$ map: $M \to \overline{M}$ given by $m \mapsto (m,0)$ is universal for maps of M to a group.

Claim: $K_0 A = \overline{\mathrm{Iso}(\mathcal{P}(A))}$

Examples: (1) A division ring or field.

$\mathrm{Iso}(\mathcal{P}_A) = \mathbb{N} \Rightarrow K_0 A = \mathbb{Z}$.

$A = \mathrm{PID}$ $\quad \mathrm{Iso}(\mathcal{P}_A) = \mathbb{N}$

A = local ring (e.g. $\mathbb{Z}/p^n\mathbb{Z}$) every fin.gen. proj is free.

(2) $\mathcal{P}_{A \times B} = \mathcal{P}_A \times \mathcal{P}_B$. so $\mathrm{Iso}(\mathcal{P}_{A\times B}) = \mathrm{Iso}(\mathcal{P}_A) \times \mathrm{Iso}(\mathcal{P}_B)$

So $K_0(A \times B) = K_0 A \times K_0 B$.

(3) Dedekind domain A, $\mathrm{Pic}(A)$ = ideal class group of A.

$\mathrm{Iso}(\mathcal{P}_A) = \mathbb{N} \times \mathrm{Pic}(A) \qquad [P] \to (\mathrm{rank}(P), \Lambda^r P)$ not quite right.

So $\mathrm{Pic}(A)$ = iso classes of $P \in \mathcal{P}_A$ of rank 1.

$$Iso(\mathcal{P}_A) = (0,0) \cup (N - \{0\}) \times Pic(A) \subset N \times Pic(A)$$

$$[P] \mapsto (\text{rank } P, \Lambda^{rk(P)} P).$$

$\Rightarrow K_0 A = Z \times Pic(A)$ gives complete description.

(4) $K_0 A[T_1, \ldots, T_r] = K_0 A$ if A regular, Noetherian (eg. A a field) (Grothendieck-Serre)

Not known if f.g. projective module over $F[T_1, \ldots, T_r]$ is free (Serre's conjecture).

Let $GL_n(A) = GL(n, A) =$ group of invertible $(n \times n)$ matrices.
$= Aut(A^n)$.

$$E_n(A) = \text{subgp generated by } e_{ij}^a = I + a \begin{array}{cc} & j^{th} \text{ column} \\ & \begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} \leftarrow i^{th} \text{ r} \end{array}$$

$i \neq j$; $a \in A$.

Observe: (1) $e_{ij}^a e_{ij}^b = e_{ij}^{a+b}$

$(e_{ij}^a)^{-1} = e_{ij}^{-a}$

(2) $(e_{ij}^a, e_{kl}^b) = e_{ij}^a e_{kl}^b e_{ij}^{-a} e_{kl}^{-b}$

$= 1$ if $\{i,j\} \cap \{k,l\} = \phi$.

✓ $(e_{ij}^a, e_{jk}^b) = e_{ik}^{ab}$ i,j,k distinct.

$(e_{ij}^a, e_{ki}^b) = e_{kj}^{-ba}$

$GL_n(A) \to GL_{n+1}(A)$
$\alpha \to \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$

✓ - gives you $E_n(A)$ perfect, for $n \geq 3$.

$E(A) = \varinjlim E_n(A)$
$GL(A) = \varinjlim GL_n(A)$.

Whitehead Lemma: $E(A) = (E(A), E(A)) = (GL(A), GL(A)) \subseteq GL(A)$

To show $(GL(A), GL(A)) \subseteq E(A)$.

Let $\alpha, \beta \in GL(A)$, say $\alpha, \beta \in GL_n(A)$. Work in $GL_{3n}(A)$.

$$\begin{pmatrix} \alpha\beta\alpha^{-1}\beta^{-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} = \left(\begin{pmatrix} \alpha & & \\ & \alpha^{-1} & \\ & & 1 \end{pmatrix}, \begin{pmatrix} \beta & & \\ & 1 & \\ & & \beta^{-1} \end{pmatrix} \right)$$

Enough to show $\begin{pmatrix} \alpha & & \\ & \alpha^{-1} & \\ & & 1 \end{pmatrix}, \begin{pmatrix} \beta & & \\ & 1 & \\ & & \beta^{-1} \end{pmatrix} \in E_{3n}(A)$.

Proof that $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in E_{2n}(A)$.

① $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix}$ ③ $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

② $\begin{pmatrix} 1 & \\ -\alpha^{-1} & 1 \end{pmatrix}\begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ -1 & 0 \end{pmatrix}$

Collecting

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \alpha^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \text{———————— '' ————————} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} 1_k & * \\ 0 & 1_{n-k} \end{pmatrix}$ is a product of e_{ij}^{a} $k+1 \le i, j \le n-k$. <u>etc</u>.

Def: $K_1 A = GL(A)/E(A) = GL(A)^{ab}$.

Ex: A = skew-field F; Thm of Dieudonné: $GL_n(F)/E_n(F) = (F^{*})^{ab}$

1) A = field F

Thm: $(GL_n F, GL_n F) = E_n(F)$ = matrices of determinant 1. $(SL_n($
$n \geq 1$.
(except for $n=2$, $F = F_2$).

$K_1 F = F^*$, since $K_1 F = GL(F)/E(F) \xrightarrow[\det]{\simeq} F^*$

1') Dieudonne's theory of non-commutative determinants, for a skew field F:

$$GL_n(F)/E_n(F) \simeq (F^*)^{ab} \qquad \forall n \geq 1.$$

2) A Euclidean domain: $E_n A = SL_n(A) \quad \forall n \geq 2$

$$K_1 A = A^* = \text{units in } A.$$

$$K_1 \mathbb{Z} = \mathbb{Z}^* = \{\pm 1\}. \qquad (\text{local rings also})$$

Def: $St(A)$ = Steinberg group of A is the group with generators x_{ij}^a, $a \in A$, $i \neq j$, $1 \leq i, j \leq n$. and the relations:

$$x_{ij}^a x_{ij}^b = x_{ij}^{a+b}$$
$$(x_{ij}^a, x_{kl}^b) = 1 \quad \text{if} \quad i \neq l \text{ and } j \neq k.$$
$$3 \leq n \leq \infty \qquad\qquad x_{il}^{ab} \quad \text{if} \quad i, j = k, l \text{ distinct}$$

$St_\infty(A) = St(A)$

canonical homomorphism $St_n(A) \xrightarrow{\phi} E_n(A)$
$$x_{ij}^a \longmapsto e_{ij}^a.$$

Theorem (Milnor): $\phi: St(A) \to E(A)$ is universal central extensi[on] of $E(A)$

Def: $K_2 A = \ker\{\phi: St(A) \to E(A)\} = H_2(E(A))_{\mathbb{Z}} =$ Shur multiplier of

BG^+ ; Will show that one has following:

$K_i A = \pi_i \, BGL(A)^+$ $\quad i = 1, 2.$ (not $i=0$ since conned

This will be our definition $i \geqslant 1$.

9.24.70

G group, N = largest perfect subgroup.

$$\begin{array}{ll} X_3 = B\tilde{N} \longrightarrow K(H_3 B\tilde{N}, 3) & \\ \quad \downarrow \alpha_2 & \\ X_2 = BN \to K(H_2 BN, 2) & \tilde{N} = \text{universal central extension of } N \\ \quad \downarrow \alpha_1 & \\ X_1 = BG & \end{array}$$

tower of fibrations.

$$\begin{array}{ccc} B\tilde{N} & \longrightarrow & PK(H_2BN, 2) \\ \vdots & & \downarrow \\ BN & \longrightarrow & K(H_2 BN, 2) \end{array}$$

$\varprojlim = A(BG)$.

$$X_{n+1} \to X_n \to K(H_n X_n, n)$$

$$\pi_q X_{n+1} \to \pi_q X_n \to \pi_q(K(H_n X_n, n)) \longrightarrow \pi_{q-1}(X_{n+1}).$$

$$\pi_q(B\tilde{N}) = \begin{cases} 0 & \text{o.w} \\ \tilde{N} & q=1 \end{cases} \qquad \pi_q(X_4) = \begin{cases} \tilde{N} & q=1 \\ H_3 X_3 & q=2 \\ 0 & q \geqslant 3 \end{cases}$$

$$\pi_q X_n = \begin{cases} \tilde{N} & q=1 \\ H_{q+1} X_{q+1} & 2 \leqslant q \leqslant n-2 \\ 0 & \text{ow} \end{cases}$$

$$\pi_q(A(BG)) = \begin{cases} \tilde{N} & q=1 \\ H_{q+1} X_{q+1} & q \geqslant 2. \end{cases}$$ by above result.

$A(BG)$ = homotopy fibre of $BG \to BG^+$ $\qquad ABG \to BG \to BG^+$

$$\ldots \to \pi_q A(BG) \to \begin{Bmatrix} G & q=1 \\ 0 & q\neq 1 \end{Bmatrix} \to \pi_q BG^+ \xrightarrow{\partial} \pi_{q-1}(A(BG)).$$

$$\pi_q(BG^+) = \begin{cases} G/N & q=1 \\ H_2(BN) & q=2 \\ H_q X_q & q \geq 2 \end{cases}$$

Exercise: Show the Dror tower is $AX \to \ldots \to X_2 \to X_1 = X$ is the Postnikov tower for $X \to X^+$.

Take $G = GL(A)$, $E(A)$ generated by e_{ij}^a.

$E(A) = (E(A), E(A)) = (GL(A), GL(A))$. So $E(A)$ is largest perfect subgroup of $GL(A)$.

$St(A) =$ group with generators x_{ij}^a $i\neq j$ $1 \leq i,j < \infty$. $a \in A$ and relations above.

Bass: $K_1 A = GL(A)/E(A)$

Milnor $K_2 A = \ker\{\phi : St(A) \to E(A)\}$
$= H_2(BE(A), \mathbb{Z}) = H_2(K(E(A),1), \mathbb{Z})$.,

$\pi_1(BGL(A)^+) = GL(A)/E(A)$ by above.

$\pi_2(BGL(A)^+) = H_2(BEA), \mathbb{Z})$. So we define

Def: $K_n A \simeq \pi_n(BGL(A))^+$. $n \geq 1$

$$\begin{tikzcd}
X_4 \arrow[d] & \\
BSt(A) \arrow[r] \arrow[d] & K(H_3 BSt(A), 3) \\
BE(A) \arrow[r] \arrow[d] & K(H_2 BEA, 2) \\
BGL(A) \arrow[r] & K(H_1 BGLA^{+}, 1)
\end{tikzcd}$$

Dror construction for $BGL(A)$

$K_3 A = H_3(St(A), \mathbb{Z})$. $K_3 \mathbb{Z}$ is ~~unknown~~ ($\mathbb{Z}/48$, Lee-Szczarba

Universal property of $BGL(A)^+$:

1)
$$\begin{tikzcd}
BGL(A) \arrow[r, "f"] \arrow[dr, "u"'] & BGL(A)^+ \arrow[d, dashed, "\exists! h"] \\
& Z
\end{tikzcd}$$

use obstruction theory.

If $\pi_1(u)(E(A)) = 0$, then $\exists! h \ni h\cdot f \sim u$.

Recall
$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X^+ \arrow[r, dashed] & Y^+
\end{tikzcd}$$

$A \to B$ ring homomorphism induces

$GL(A) \to GL(B)$

$BGL(A)^+ \to BGL(B)^+$.

$u_* : K_n A \longrightarrow K_n B$.

K_n is a functor from rings to abelian groups.

product of rings $A \times A'$. $GL(A\times A') = GL(A) \times GL(A')$.

Fact: $B(G\times G') \sim BG \times BG'$.

$$BGL(A\times A') \sim BGL(A)\times BGL(A').$$

Fact: $(X\times Y)^+ \sim X^+ \times Y^+$

$$BGL(A\times A')^+ \sim BGL(A)^+ \times BGL(A')^+$$

Hence: $K_n(A\times A') = K_nA \times K_nA'$.

Theorem: $BGL(A)^+$ is a homotopy associative and commutative H-space.

Recall $GL_n(A) \times GL_p(A) \to GL_{n+p}(A)$

$$\alpha\oplus\beta \mapsto \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} \quad \text{Whitney sum.}$$

$$\alpha\oplus(\beta\oplus\gamma) = (\alpha\oplus\beta)\oplus\gamma$$

$\alpha\oplus\beta \neq \beta\oplus\alpha$. they are conjugate in $GL_{n+p}(A)$

Choose $N = \{1,2,3,\dots\}$ and choose $N \amalg N \xhookrightarrow{\varepsilon} N$.

$$\varepsilon : GL(A)\times GL(A) \to GL(A).$$

$\varepsilon_*(\alpha,\beta)_{k,l} =$ etc. $\quad BGL(A)^+ \times BGL(A)^+ \to BGL(A)^+$.

Reduction of theorem to:

Lemma: Given an embedding $u: N \to N$ $N = \{1,2,\dots\}$ then the induced map $\mu_*: BGL(A)^+ \to BGL(A)^+$ is homotopic to the identity.

Sublemma1: u_* is a homotopy equivalence.

Lemma2: Let M be the monoid of embeddings $N \hookrightarrow N$. Then any homomorphism $M \xrightarrow{f} G$ with G a group is trivial.

(2) Proof: Given any $u, v \in M$, define an embedding $v_*(u)$

$$V_*(u)(vn) = v(un)$$
$$V_*(u)(n) = n \qquad n \notin \mathrm{Im}(v).$$
$$\Rightarrow v_*(u) \cdot v = v \cdot u.$$

Choose v so that complement of $\mathrm{Im}(v)$ is ∞, whence $\exists w \in M$ st. $\mathrm{Im}(w) \cap \mathrm{Im}(v) \neq \phi$ then $v_*(u) \cdot w = w$.

$$\begin{array}{ccc} \rho(v_*(u) \cdot w) & = & \rho(w) \\ \| & \diagup\!\!\diagup & \\ \rho(v_*(u)) \cdot \rho(w) & \Rightarrow & \rho(v_*(u)) = 1. \end{array}$$

but: $V_*(u) \cdot v = v \cdot u$. $\rho(v_*(u))\rho(v) = \rho(v)\rho(u) \to \rho(u) = 1$.

(1) Proof: Step (1) u_* induces homology isomotphism.

Step (2) Show $\pi_1(BGL(A)^+)$ acts trivially on $H_*(\widetilde{BGLA^+})$

Then can apply a suitable version of Whitehead Theorem.

$$H_*(BGL(A)^+) = H_*(BGL(A)) = \varinjlim H_*(BGL_n(A))$$

$$\begin{tikzcd} H_*(BGL_n(A)) \arrow[r] \arrow[dr, "u^n_*"'] & H_*(BGL(A)) \arrow[d, "u_*"] \\ & H_*(BGL(A)) \end{tikzcd}$$

Fact: Inner automorphisms of G act trivially on $H_*(BG)$

use embedding

$u^n : \{1, \dots n\} \mapsto \{u_1, \dots, u_n\}$ $\quad u^n_* = id^n_*$ $\quad \forall n \Rightarrow u_* = id.$

9.30.74

Theorem. $BGL(A)^+$ is a homotopy commutative and associative H-spa

Key point in proof is to show $u_* \sim id$ on $BGL(A)^+$ for any embedding $u : N \hookrightarrow N$. Follows from above lemmas.

Proof of Lemma 1: By Whitehead Thm., it suffices to show u_* induces iso on $\pi_1(BGL(A)^+) = GL(A)/E(A) = H_1(BGL(A))$
and on H_* of universal cover $\widetilde{BGL(A)^+} = BE(A)$

But: Remark: $\widetilde{BGL(A^+)} = BE(A)^+$

$$\begin{tikzcd} T \arrow[r, "f'"] \arrow[d] & \widetilde{BGL(A)^+} \arrow[d] \\ BGL(A) \arrow[r, "f"'] & BGL(A)^+ \end{tikzcd}$$

pullback square.

T must be the covering of $BGL(A)$ with $\pi_1(T) = E(A)$

So: $\quad T = BE(A)$.

f acyclic $\Rightarrow$ f' acyclic, so $H_*(f')$ is iso. $\Rightarrow BE(A)^+ = \widetilde{BGL}$

Exercise: Dror tower:

$$\begin{tikzcd} X_4 \arrow[r] \arrow[d] & X_4 \arrow[d] \\ X_3 \arrow[r] \arrow[d] & BX_3^+ = BSt(A)^+ \; P_3 \\ BE(A) \arrow[r] \arrow[d] & BE(A)^+ = P_2 \arrow[d] \\ BGL(A) \arrow[r] & BGL(A)^+ = P_1 \end{tikzcd}$$

Show the tower of H spaces is the Postnikov system of $BGL(A)^+$.

$E(A) = \bigcup E_n A.$

converts maps to cofibrations with map co

$\Rightarrow \quad BE(A) = \varinjlim BE_n(A) \qquad BE_1 \hookrightarrow BE_2 \hookrightarrow BE_3 \hookrightarrow \ldots \qquad \to \bigcup B$

∞ mapping telescope.

$\Rightarrow \quad H_*(BE(A)) = \varinjlim H_*(BE_n(A)) \qquad E_n = E_n(A).$

$$\begin{tikzcd} E_n \arrow[r, "i_n"] & E \arrow[d, "u_*"] \\ & E \end{tikzcd}$$

To show u_* induces identity on $H_*(BE)$ it is enough to show $u_* \cdot i_n$ and i_n induce same map from $H_*(BE_n) \to H_*(BE)$.

$$\{1, \ldots n\} \longmapsto \{u(1), \ldots, u(n)\} \leq N$$

· $u_* i_n$ and i_n are conjugate by an element $\sigma \in \Sigma_N \subseteq E_l L_N$

$$\begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \in E_{2N}.$$

FACT: Conjugation acts trivially on $H_*(BG)$

Eilenberg-Maclane group cohomology. G group, M $\mathbb{Z}[G]$-module

$H_i(G, M) = H_i\{P_* \otimes_{\mathbb{Z}[G]} M\}$ where

$P_*: \qquad \ldots \to P_1 \to P_0 \to \mathbb{Z} \to 0$

is a free $\mathbb{Z}[G]$-module resolution of $\mathbb{Z}$ (trivial G-actio

One can take $P_* = C_*(\widetilde{BG})$.

$$\begin{tikzcd} \widetilde{BG} \sim 0 \arrow[d] \\ BG \end{tikzcd}$$

$C_i(BG, M) = C_i(\widetilde{BG}) \otimes_{\mathbb{Z}[G]} M$

One can show that: $C_*(BG, M) = C_*(\widetilde{BG}) \otimes_{\mathbb{Z}[G]} M$

$$\Rightarrow \quad H_*(BG, M) = H_*(-) = H_*(G, M)$$

$$\begin{array}{c}(X,*)^{(Z,*)}\\ \downarrow\\ X^Z \xrightarrow[\text{at } * \in Z]{\text{evaluation}} X\end{array}$$

Consider $X^Z \xrightarrow[\text{at } * \in Z]{\text{evaluation}} X$ is a fibration

X, Z connected based CW

$$\pi_0\left[((X,*)^{(Z,*)}\right]_{\pi_1 X} = \pi_0(X^Z)$$

$$\begin{array}{ccc} \| & & \| \\ [Z,X]_{\pi_1 X} & = & [Z \amalg pt, X]_{\text{(free homotopy classes)}} \end{array}$$

gives

$\pi_1 X$ acts on $[Z,X]$

Suppose $X = BG$; $\quad [Z, BG] \to \mathrm{Hom}(\pi_1 Z, G)$ obstruction theo

$$[Z \amalg pt, BG] \xrightarrow{\approx} \mathrm{Hom}(\pi_1 Z, G)/_{\text{conjugation by elts of } G}$$

by above remarks.

An inner automorphism of G induces a map on BG which is homotopic to the identity, (not base pt preserving
σ induces id on $H_*(BG)$ QED

Theorem (Milnor & Moore): Suppose M is a connected H-space.

$$\mathbb{Q} \underset{\mathbb{Z}}{\otimes} \pi_i M \xrightarrow{\approx} \mathrm{Prim}\{H_i(M,\mathbb{Q})\}$$

$$\| $$

$$\{z \in H_i(M,\mathbb{Q}) : H_i(M,\mathbb{Q}) \xrightarrow{\Delta} H_i(M \times M, \mathbb{Q}) \quad \Delta z = z\otimes 1 + 1 \otimes z\}$$

$H_*(M,\mathbb{Q}) \underset{\downarrow}{\cong}$ [Sym alg on $\pi_{ev}(M)\otimes\mathbb{Q}$] $\otimes$ [Ext alg. on $\pi_{odd}(M)\otimes$

canonical algebra iso if M is homotopy commutative and associat

See appendix to Milnor-Moore paper on "Hopf Algebras."

Take $M = BGL(A)^+$.

Corollary: $K_i A \otimes \mathbb{Q} \simeq \mathrm{Prim}\, H_i(BGL(A), \mathbb{Q})$

Pf: $\pi_i(BGL(A)^+) \otimes \mathbb{Q} = \mathrm{Prim}(H_i(BGL(A)^+, \mathbb{Q}))$

$H_*(BGL(\mathbb{Z}), \mathbb{Q})$ unknown. (?

Theorem of Borel: F number field with r_1 real, r_2 complex absolute values.

$$[F:\mathbb{Q}] = r_1 + 2r_2 \qquad F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

Let A = ring of integers in F = int closure$_F(\mathbb{Z})$.

$$\dim(K_i A \otimes \mathbb{Q}) = \begin{cases} 1 & i = 0 \\ r_1 + r_2 - 1 & i = 1 \\ 0 & i \equiv 2 \\ r_2 & i \equiv 3 \\ 0 & i \equiv 0 \\ r_1 + r_2 & i \equiv 1 \end{cases} \pmod 4$$

$K_0 A = \mathbb{Z} \oplus \underset{\substack{\text{finite ideal} \\ \text{class group}}}{\underset{\downarrow}{\mathrm{Pic}(A)}}$

Ex 1: $A = \mathbb{Z}$

$K_0 \mathbb{Z} = \mathbb{Z}$

$K_1 \mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$ (units)

$K_2 \mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$ (Milnor)

$\dim(K_3 \mathbb{Z} \otimes \mathbb{Q}) = 0 = r_2$

$\dim(K_4 \mathbb{Z} \otimes \mathbb{Q}) = 0$

$\dim(K_5 \mathbb{Z} \otimes \mathbb{Q}) = 1$

Ex. 2. $F = \mathbb{Q}[\sqrt{d}]$ $\quad$ 1, 1, 0002, 0002, ...

$r_1 = 2$

$r_2 = 0$

$F = \mathbb{Q}[\sqrt{-d}]$ $\quad$ 1, 0, 0101, 0101,

$r_1 = 0$

$r_2 = 1$

Let A be a fixed ring, $GL_n = GL_n(A)$, n fixed.

$$G_n = \left\{ \begin{pmatrix} 1_r & * \\ 0 & GL_n \end{pmatrix} \right\} \subseteq GL_{r+n}(A)$$

= group of automorphisms of exact sequence:

$$0 \to A^r \to A^{r+n} \to A^n \to 0$$

which identity on ~~quotient~~ subspace A^r.

$$G_n = GL_n(A) \ltimes \mathrm{Hom}(A^n, A^r) \longleftrightarrow GL_n(A)$$

$$\begin{tikzcd} G_n \arrow[r, "p"] & GL_n(A) \arrow[l, dashed, bend left, "s"] \end{tikzcd} \qquad s(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

$$p\begin{pmatrix} 1 & * \\ 0 & \alpha \end{pmatrix} = \alpha.$$

$$H_*(G_n) = H_*(GL_n) \oplus [\ker(p_*)]$$

Theorem: $\varinjlim_{n\to\infty} H_*(G_n) \cong \varinjlim_n H_*(GL_n)$.

Topological analogue: take $GL_n(\mathbb{C})$ with its topology.

$$B(G_n\mathbb{C}) \xleftarrow{Bs} BGL_n(\mathbb{C})$$ infinite Grassmannian

$H \subseteq G$ fibration: $G/H \to BH \to BG$

$$\begin{pmatrix} 1 & M_{rn}(\mathbb{C}) \\ & GL_n\mathbb{C} \end{pmatrix} \Big/ GL_n\mathbb{C} \cong M_{rn}(\mathbb{C}) \quad \text{contractible.}$$

Corresponds to statement in vector bundle theory that:

short exact sequences of vector bundles split, use Riemannian metric in standard way.

Example: $A = \mathbb{F}_p$. $\quad GL_1 = \mathbb{F}_p^* \subseteq \begin{pmatrix} 1 & \mathbb{F}_p \\ 0 & \mathbb{F}_p^* \end{pmatrix} = G_1$

$H_*(\mathbb{F}_p^*)$ has no p-torsion

$H_n(G_1)$ has p-torsion for ∞ many n.

Show theorem needs $\varinjlim$ above.

Remarks:
$$GL_p \times GL_q \to GL_{p+q}$$
$$(\alpha, \beta) \xrightarrow{\oplus} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad \text{Whitney sum}$$

associative, commutative up to conjugacy.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\oplus$ induces products:

$$\begin{tikzcd} H_*(GL_p) \otimes H_*(GL_q) \arrow[r, "\text{canonical}"'] \arrow[dr, "\mu_{p,q}"'] & H_*(GL_p \times GL_q) \arrow[d, "\oplus_*"] \\ & H_*(GL_{p+q}) \end{tikzcd}$$

bc a commutative $H_*(X) = H_*(X, \Lambda)$.

$$\begin{tikzcd} H_*(GL_p) \otimes H_*(GL_q) \arrow[r, "\mu_{pq}"] \arrow[d] & H_*(GL_{p+q}) \arrow[d] \\ H_*(GL_{p+1}) \otimes H_*(GL_q) \arrow[r, "\mu_{p+1,q}"'] & H_*(GL_{p+q+1}) \end{tikzcd}$$

one direction $\begin{pmatrix} \alpha & \\ & \beta_1 \end{pmatrix}$ conjugate.
other $\begin{pmatrix} \alpha_1 & \\ & \beta \end{pmatrix}$ alright.

$\cong H_*(BGL^+)$ ↓ ~ Pontryagin product for the H-space.

One gets a product in $H_*(GL) = \varinjlim H_*(GL_n)$, get ring structure or Λ-algebra, as above

Return to Theorem: (homology with coefficients in Λ.)

Proof: Reduce to case where $\Lambda = \mathbb{F}_p$, $\mathbb{Q}$.
Consider the category $\mathcal{C}$ of abelian gps for which thm holds.

$$0 \to \Lambda' \to \Lambda \to \Lambda'' \to 0$$

1) If two are in $\mathcal{C}$, so is the third. Use homology exact sequence, $\varinjlim$ preserves exactness and "5" Lemma.

2) $\mathcal{C}$ is closed under filtered inductive limits.

Lemma: Any $\mathcal{C}$ with properties (1) and (2) containing $\mathbb{F}_p$, $\mathbb{Q}$ must be all abelian groups.

Take any abelian group Λ : $0 \to t\Lambda \to \Lambda \to \Lambda/t\Lambda \to 0$

$$0 \to \Lambda/t\Lambda \to \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \to \text{torsion gp} \to 0$$

From now on $H_* = H_*(-, \Lambda)$; Λ field. Künneth formula:

$$H_*(X) \otimes H_*(Y) \xrightarrow{\approx} H_*(X \times Y)$$

Consequently for any space X ; $H_*(X)$ is a coalgebra coproduct induced by diagonal $\Delta : X \to X \times X$.

$$H_*(X) \xrightarrow{\Delta_*} H_*(X \times X) \xrightarrow{\approx} H_*(X) \otimes H_*(X)$$

If X is connected; $H_0(X) = \Lambda$, so $\exists$ distinguished generator of $H_0(X)$ denoted 1.

Have algebra structures on $H_*(GL) = \varinjlim H_*(GL_n)$ induced

Easily seen that $\Delta : GL_p \to GL_p \times GL_p$ are compatible with $\oplus : GL_p \times G_q \to GL_{p+q}$, i.e.

$$\begin{tikzcd}
GL_p \times GL_q \arrow[rr, "\oplus"] \arrow[d, "{(\Delta,\Delta)}"'] & & GL_{p+q} \arrow[d, "\Delta"] \\
GL_p \times GL_p \times GL_q \times GL_q \arrow[rr, "\oplus"'] & & GL_{p+q} \times GL_{p+q}
\end{tikzcd}$$

$$\alpha \quad \beta \quad \gamma \quad \delta \qquad\qquad (\alpha+\gamma,\ \beta+\delta)$$

commutes

Hence: $\Delta : H_*(GL) \to H_*(GL \times GL) = H_*(GL) \otimes H_*(GL)$ is an algebra homomorphism $\therefore$ $H_*(GL)$ is a Hopf algebra.

Define $\perp : G_p \times G_q \to G_{p+q}$ $\qquad \begin{pmatrix} 1 & * \\ 0 & \alpha \end{pmatrix} \perp \begin{pmatrix} 1 & *' \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & * & x' \\ & \alpha & \\ & & \beta \end{pmatrix}$

Check that this operation ~~induces~~ on these G-matrices induces an

Hopf algebra structure on $H_*(G_\infty) = \varinjlim H_*(G_n)$ $\qquad G_\infty = \bigcup G_n$

$p : G_n \to GL_n$ is compatible with $\perp$.

$\therefore$ $p_* : H_*(G_\infty) \to H_*(GL_\infty)$ is algebra homomorphism

$\therefore$ $s_* : H_*(GL_\infty) \to H_*(G_\infty)$ " " " "

Hence $p_* \cdot s_* = 1$ want $s_* \cdot p_* = 1_{H_*(G_\infty)}$

<u>**Lemma:**</u> $\begin{pmatrix} 1 & u \\ 0 & \alpha \end{pmatrix} \perp \begin{pmatrix} 1 & u \\ 0 & \alpha \end{pmatrix}$ is conjugate to $\begin{pmatrix} 1 & u \\ 0 & \alpha \end{pmatrix} \perp \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$.

Proof:
$$\begin{pmatrix} 1 & u & u \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{P} \begin{pmatrix} 1 & u & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{P^{-1}}$$

$$G_p \xrightarrow{\Delta} G_p \times G_p \underset{1\times sp}{\overset{1\times 1}{\rightrightarrows}} G_p \times G_p \xrightarrow{\oplus} G_{2p}$$

Lemma shows these two homomorphisms are conjugate, so:

$$H_*(G_p) \xrightarrow{\Delta_*} H_*(G_p \times G_p) \approx H_*(G_p) \otimes H_*(G_p) \underset{id\otimes s\cdot p.}{\overset{id\otimes id}{\rightrightarrows}} H_*(G_p) \otimes H_*(G_p) \xrightarrow[\text{product}]{\mu} H_*(G_{2p}).$$

are the same maps. Take limit, replace p by ∞.

Now we can prove $s_* \cdot p_* = 1$ on $H_n(G_\infty)$ by induction on n. Assume true for degrees $< n$; let $x \in H_n(G_\infty)$

$$\Delta_* x = 1 \otimes x + \sum_{\deg(x_i'') < n} x_i' \otimes x_i''$$

$$\mu(id \otimes id)(\Delta_* x) = x + \sum_{\deg(x_i'') < n} x_i' \cdot x_i'' \in H_*(G_\infty).$$

$\|$ by above

$$\mu(id \otimes s_* \cdot p_*)(\Delta_* x) = s_* p_* x + \sum_{\deg(\,) < n} x_i' s_* p_* x_i''$$

$\Sigma = \Sigma$, so $x = s_* p_* x$, completes induction. QED

General fact about Hopf algebras: If C is a Hopf coalgebra and A is an algebra, one can make $Hom(C, A)$ into a monoid.

Given $u, v : C \rightrightarrows A$ define convolution:

$$u * v : C \xrightarrow{\Delta} C \otimes C \xrightarrow{u \otimes v} A \otimes A \xrightarrow{\mu} A.$$

If C is connected, then the algebra C has an inversion, so this monoid is a group.

$id * id = id * s_* p_*$ if group can cancel

Let $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2 A \right\} = \begin{pmatrix} A & A \\ & A \end{pmatrix}$.

$$\underset{\substack{\| \\ 2}}{GL_n} B = \left(\begin{array}{cc|cc} \cdot & \cdot & \cdot & \cdot \\ & \cdot & & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ & \cdot & & \cdot \end{array}\right) \quad \text{conjugate get} \quad \left(\begin{array}{cc|cc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline & & \cdot & \cdot \\ & & \cdot & \cdot \end{array}\right)$$

$$GL_n B \simeq \left(\begin{array}{c|c} GL_n A & M_n A \\ \hline & GL_n A \end{array}\right)$$

Assertion: $K_n B \simeq K_n A \oplus K_n A$.

Corollary: $$H_*\begin{pmatrix} GL_r & M_{rn} \\ 0 & GL_n \end{pmatrix} \longleftarrow H_*\begin{pmatrix} GL_r & 0 \\ 0 & GL_n \end{pmatrix}$$

induced by inclusion is isomorphism, in the limit as $n \to \infty$.

Proof:
$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_n & \longrightarrow & \begin{pmatrix} GL_r & M_{rn} \\ & GL_n \end{pmatrix} & \longrightarrow & GL_r & \longrightarrow & 1 \\ & & \cup & & \cup & & \| & & \\ 1 & \longrightarrow & GL_n & \longrightarrow & \begin{pmatrix} GL_r & 0 \\ 0 & GL_n \end{pmatrix} & \longrightarrow & GL_r & \longrightarrow & 1 \end{array}$$

Write down group cohomology spectral sequence:

$$E^2_{pq} = H_p(GL_r, H_q(G_n)) \Rightarrow H_*\left(\bigcup_n \begin{pmatrix} GL_r & M_{rn} \\ & GL_n \end{pmatrix}\right)$$
$$\quad\quad \cong \text{Thm:}$$
$$E^2_{pq} = H_p(GL_r, H_q(GL_n)) \Rightarrow H_*\left(\bigcup_n \begin{pmatrix} GL_r & 0 \\ 0 & GL_n \end{pmatrix}\right) \qquad \text{limits preserve exactness.}$$

$\Rightarrow$ isomorphism desired, by Comparison Thm.

Corollary: Now let $r \to \infty$. $$H_*\left(\bigcup_{r,n} \begin{pmatrix} GL_r & M_{rn} \\ & GL_n \end{pmatrix}\right) \approx H_*\begin{pmatrix} GL & 0 \\ 0 & GL \end{pmatrix}.$$
$$\| \\ GL(B)$$

Thus we obtain embedding: $A \times A = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \subset \begin{pmatrix} A & A \\ 0 & A \end{pmatrix} = B$ induces iso.

of $H_*(GL(A\times A))$ with $H_*(GL(B))$.

$BGL(A\times A)^+ \longrightarrow BGL(B)^+$ is a map of H-spaces ($\Rightarrow$ simple) which is a homology isomorphism, hence a homotopy equivalence. QED. for assertion.

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A, r fixed

Thm: $H_*\begin{pmatrix} 1 & 0 \\ 0 & GL_n A\end{pmatrix} \hookrightarrow H_*\begin{pmatrix} 1 & M_{rn}A \\ 0 & GL_n A\end{pmatrix}$ iso as $n\to\infty$.

Corollary: $H_*\begin{pmatrix} GL_r A & 0 \\ 0 & GL_n A\end{pmatrix} \longrightarrow H_*\begin{pmatrix} GL_r A & M_{rn}A \\ 0 & GL_n A\end{pmatrix}$ as $n\to\infty$.

$$\begin{array}{cc} \| & \| \\ H_*\begin{pmatrix} GL_r A & 0 \\ 0 & GL(A)\end{pmatrix} & H_*\begin{pmatrix} GL_r A & \mathcal{M}_{r,\infty}A \\ 0 & GL(A)\end{pmatrix}\end{array}$$

Corollary $K_*\begin{pmatrix} A & 0 \\ 0 & A\end{pmatrix} \longrightarrow K_*\begin{pmatrix} A & A \\ 0 & A\end{pmatrix}$

$$\begin{array}{c} \wr\wr \\ K_* A \oplus K_*(A). \end{array} \qquad GL_n\begin{pmatrix} A & A \\ 0 & A\end{pmatrix} \simeq \begin{pmatrix} GL_n A & M_{rn} A \\ 0 & GL_n A\end{pmatrix}$$

Swan's counterexample:

Recall Milnor's Theorem: If
$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "g'"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$
is a cartesian (pullback) square of rings and if either f or g is epic we have a Mayer-Vietoris sequence.

$$K_1 D \to K_1 A \oplus K_1 C \to K_1 B \xrightarrow{\partial} K_0 D \to K_0 A \oplus K_0 C \to K_0$$

If both f, g are epic then: can write in the K_2 terms.

$$K_2 D \to K_2 A \oplus K_2 C \to K_2 B \xrightarrow{\partial} K_1 D.$$

Swan's Theorem: 2^{nd} conclusion is false if only one is surjective.
There is no functor $\bar{K}_2$ s.t. for every cartesian square with g (split epic) surjective one has:

$$\bar{K}_2 D \to \bar{K}_2 A \oplus \bar{K}_2 C \to \bar{K}_2 B \to K_1 D \to K_1 A \oplus K_1 C \to K_1$$

exact. $\quad (K_1 = \text{Bass}'\ K_1)$

Example:

$$\begin{tikzcd} A[\varepsilon] \arrow[r, "f'"] \arrow[d, "g'"'] & \begin{pmatrix} 0 & A \\ A & A \end{pmatrix} \arrow[d, "g = \text{projection}"] \\ A \arrow[r, "f"', "\Delta"] & \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \end{tikzcd}$$

here $A[\varepsilon] = A + A\varepsilon \quad \varepsilon^2 = 0 \quad$ = ring of dual numbers $/A$.
elts: $a + b\varepsilon$

$$f'(a+b\varepsilon) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$
$$g'(a+b\varepsilon) = a$$

Because g has a section $\Rightarrow \bar{K}_2 C \twoheadrightarrow \bar{K}_2 B$, hence:

$$0 \to K_1(A[\varepsilon]) \to K_1 A \oplus K_1 \begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \to K_1 \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \to 0.$$

is exact:

$$\Rightarrow \quad K_1(A[\varepsilon]) \underset{g'_*}{\simeq} K_1 A. \ (\dagger)$$

Recall if A is commutative then $K_1 A \approx A^* \oplus \ker(\det: K_1 A \to A^*)$
$\approx A^* \oplus SK_1(A)$.

$$0 \to SK_1 A \to K_1 A \xrightarrow[\det]{} A^* \to 0$$

$(\dagger) \Longrightarrow \quad A[\varepsilon]^* \simeq A^*. \quad$ but this is false $\Rightarrow\!\Leftarrow$.
$1 + \varepsilon A$
are units

Let $\rho : G \longrightarrow GL_n A = \mathrm{Aut}(A^n)$.

$$
\begin{tikzcd}
H_* G \arrow[r] \arrow[rr, bend right, "\rho_*"'] & H_* GL_n A \arrow[r] & H_* GL(A).
\end{tikzcd}
$$

Cor: $$G = \begin{pmatrix} GL_r & M_{rn} \\ 0 & GL_n \end{pmatrix} \underset{j}{\overset{i}{\rightrightarrows}} GL_{r+n}$$

$$i : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$j : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

Then $i_* = j_* : H_*\begin{pmatrix} GL_r & M_{rn} \\ 0 & GL_n \end{pmatrix} \longrightarrow H_*(GL)$.

Proof:
$$
\begin{tikzcd}
\begin{pmatrix} GL_r & M_{rn} \\ 0 & GL_n \end{pmatrix} \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] \arrow[d, phantom, "\cap" description] & GL_{r+n} \arrow[d, phantom, "\cap" description] \\
\begin{pmatrix} GL_r & M_{r\infty} \\ 0 & GL \end{pmatrix} \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & GL_\infty
\end{tikzcd}
$$

enough to show $i_* = j_*$ as maps on bottom of square.
But we know:
$$H_*\begin{pmatrix} GL_r & 0 \\ 0 & GL \end{pmatrix} \xrightarrow{\sim} H_*\begin{pmatrix} GL_r & M_{r\infty} \\ 0 & GL \end{pmatrix}$$

and $i = j$ on $\begin{pmatrix} GL_r & 0 \\ 0 & GL \end{pmatrix}$. QED.

Theorem: $\mathbb{F}_q$ = finite field with $q = p^d$ elements, p a prime.
Then:
$$H_i(GL(\mathbb{F}_q), \mathbb{Z}/p\mathbb{Z}) = 0 \qquad i \geq 1.$$

Proof: Enough to show : $GL_n(\mathbb{F}_q) \hookrightarrow GL(\mathbb{F}_q)$ induces zero map on $\mathbb{Z}_p$-homology. $\forall n$. Recall that $H_*(P) \twoheadrightarrow H_*(G)$ if $[G:P] \not\equiv 0 \pmod p$, (transfer argument), and Sylow p-subgp of $GL_n(\mathbb{F}_q)$ is $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

Enough to show $H_*\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \longrightarrow H_*(GL(\mathbb{F}_q))$ is zero map. $* > 0$

$$\begin{tikzcd} T_{n-1} = \left(\begin{array}{ccc|c} 1 & & * & \\ & \ddots & & * \\ & & 1 & \\ \hline & & & 1 \end{array}\right) \arrow[r, phantom, "\subseteq"] \arrow[d, shift right, "i"'] \arrow[d, shift left, "j"] & \left(\begin{array}{c|c} GL_{n-1} & * \\ \hline & GL_1 \end{array}\right) \arrow[d, shift right, "i"'] \arrow[d, shift left, "j"] \\ {} & GL(\mathbb{F}_q) \end{tikzcd}$$

$\leftarrow$ these induce same map on H_*, by corollary.

$\left(\begin{array}{ccc|c} 1 & & * & \\ & \ddots & & * \\ & & 1 & \\ \hline & & & 1 \end{array}\right) \subseteq GL$ has same effect has killing last list column. Iterate this, use above facts.

Let G be a group. Recall $\mathcal{P}_A$.

By a representation of G over A we mean a pair (P, ρ) $P \in \mathcal{P}_A$; $G \xrightarrow{\rho} \mathrm{Aut}(P)$ is a homomorphism

Let $\mathrm{Rep}(P, A)$ denote the iso classes of representations of G over A, under $+$ abelian monoid

Let $S = \mathrm{Iso}(\mathcal{P}_A)$, P_s representative of $s \in S$.

$\mathrm{Hom}(G, \mathrm{Aut}(P))_{\mathrm{Aut}(P)}$ = iso classes of representations of G on projectives $P \approx P_s$. $(P \in s)$

Two representations ρ, ρ' are equivalent if $\theta \rho(g) \theta^{-1} = \rho'(g)$.

$\mathrm{Rep}(G, A) = \coprod_{s \in S} \mathrm{Hom}(G, \mathrm{Aut}(P_s))_{\mathrm{Aut}(P_s)}$.

Goal: To any representation $E=(P,\rho)$ of G I want a map (canonical).

$$[E]: BG \to BGLA^+.$$

(as based homotopy class of map).

$$\rho: G \to \mathrm{Aut}(P) \qquad P\oplus Q \overset{\theta}{\cong} A^n$$

$$g \mapsto \rho(g)\oplus 1_Q \overset{\theta}{\mapsto} \in \mathrm{Aut}(A^n) = GL_n(A)$$

Choosing $\underline{Q}, \underline{\theta}$ we get a homomorphism:

$$BG \to BGL_n A \to BGLA \to BGLA^+.$$

Exercise: · Verify independent of choices, well-defined.

Point: $\pi_1(BGL(A)^+) = K_1 A$ acts trivially on $BGL(A)^+$, because $BGLA^+$ is an H-space, hence simple.

Properties of: $\mathrm{Rep}(G;A) \to [BG, BGL(A^+)]$
$E \mapsto [E]$.

① If T is a trivial representation, $\rho(g) = id_P\ \forall g\in G$.
then. $T=(P,\rho)$
$[(P,\rho)] =$ 0-map.

② $[E] + [E'] = [E\oplus E']$ $\quad$ $+$ = operation on $[BG, BGLA^+]$ deduced from H-space structure of $BGLA^+$.

③ If $0\to E' \to E \to E'' \to 0$ is a s.e.s. of representations.
then:
$$[E] = [E'] + [E''].$$

Proof of ②:

By adding trivial representations to E, E', E'' we can suppose underlying A-module of E', E, E'' are $\simeq A^p, A^q, A^{p+q}$.

$$0 \to A^p \longrightarrow A^{p+q} \longrightarrow A^q \to 0.$$

Can replace G by auto group of this exact sequence:

i.e. $G = \begin{pmatrix} GL_q & M_{qp} \\ 0 & GL_p \end{pmatrix}$ Have to show:

Enough to show $[E] = [E' \oplus E'']$ assuming ②.

So we want:
$$G = \begin{pmatrix} GL_q & M_{qp} \\ 0 & GL_{p+q} \end{pmatrix} \underset{j \;\; E'\oplus E''}{\overset{i \;\; E}{\rightrightarrows}} GL.$$

that $i_* = j_* : BG \to BGLA^+$.

Enough to show: $p \to \infty$
$$\begin{pmatrix} GL_q & M_{q\infty} \\ 0 & GL_\infty \end{pmatrix} \underset{j}{\overset{i}{\rightrightarrows}} GL$$
induce same maps to $BGLA^+$.

But i, j coincide on subgp: $\begin{pmatrix} GL_q & 0 \\ 0 & GL \end{pmatrix}$

and we know that
$$B\begin{pmatrix} GL_q & \\ & GL \end{pmatrix} \xrightarrow{\downarrow} B\begin{pmatrix} GL_q & M_{q\infty} \\ & GL \end{pmatrix} \rightrightarrows BGL(A)^+$$
is homology iso.

Lemma: $X \xrightarrow{f} Y$ homology iso $\Rightarrow$ M H-space connected
$[Y, M] \hookrightarrow [X, M]$.

Proof: Puppe: $[X, M] \leftarrow [Y, M] \leftarrow [C_f, M] \leftarrow [\Sigma X, M] \leftarrow \ldots$

Show $[C_f, M] = 0$, but Cf is acyclic, so there are no non-trivial maps, by obstruction theory, to any space having no non-trivial perfect subgroups of π_1

$K_* A = \pi_* (BGL(A)^+)$

Problem (vague): Describe elements of $[X, BGL(A)^+]$ as some sort of "geometric structures" over X. (analogous to BPL etc)

$\widetilde{KU}(X) = [X, BU] = \varinjlim [X, BU_n] =$ iso classes of n-dimensional complex vector bundles over

X a finite complex.

Attempt:

$$\begin{tikzcd} X \arrow[r, "u"] & BGL_n(A)^+ \\ X' \arrow[u, "\text{acyclic}"] \arrow[r] & BGL_n(A) \arrow[u, "\text{acyclic}"'] \end{tikzcd} \quad \text{pullback}$$

· u thus induces map $X' \longrightarrow X$ acyclic together with an element of $[X', BGL_n(A)] = \mathrm{Hom}(\pi_1 X', GL_n(A))$

If X is finite complex, then:

$$[X, BGL(A)^+] = \varinjlim [X, BGL_n(A)^+]$$

Assertion: X finite. Then an element of $[X, BGL(A)^+]$ gives rise to

$X' \longrightarrow X$ acyclic.

$\pi_1 X' \to GL_n A$.

If you have $X' \to X$ acyclic then, $[X', BGL(A)^+] \cong [X, BGL(A)^+]$

Use the fact that $\pi_1(BGL(A)^+)$ is abelian.
& universal property of acyclic maps. (on killing perfect subgp).

Conclusion: (For X finite) every pair $(X' \xrightarrow{g} X, \pi_1 X' \to GL_n A)$ with g acyclic determines an element of $[X, BGL(A)^+]$.
If finite every element of group is obtained this way.

Trouble – I don't know when two pairs give the same element of $[X, BGL(A)^+]$. Case of S^n:

$$\begin{tikzcd} & X''' \arrow[dl] \arrow[dr] & \\ X' \arrow[dr] & & X'' \arrow[dl] \\ & S^n & \end{tikzcd}$$

pullback allows one to assume $\begin{tikzcd} X' \arrow[d] \\ S^n \end{tikzcd}$ $\quad \pi_1 X' \rightrightarrows GL_N A.$

This reduces to the question of when two homomorphisms $\pi_1 Y \to GL_N$ determine the same map as $[Y, BGL(A)^+]$.

We have seen this is the case if α, β are Jordan-Hölder isomorphic. (This is the theorem: $0 \to E' \to E \to E'' \to 0$ is s.e.s. of representations of $G \Rightarrow [E] = [E' \oplus E''] \in [BG, BGL(A)^+]$.

Problem: converse.

Recall $Rep(G, A) = \coprod_s Hom(G, Aut(P_s))_{Aut(P_s)}$.

$R(G; A) =$ Grothendieck group of representations with:

$$[E] = [E'] \oplus [E''] \quad \text{if } 0 \to E' \to E \to E'' \to 0 \quad \text{s.e.s.}$$

$R'(G, A) =$ larger Grothendieck with relations:

better notation.

$R_\oplus(G, A)$. $\quad [E' \oplus E''] = [E'] + [E'']$. $\quad$ large use direct sume

= abelian group generated by $Rep(G, A)$.

$$R'(G, A) \twoheadrightarrow R(G, A).$$

$$R'(G, A) \underset{\text{trivial action}}{\overset{\text{forget}}{\rightleftarrows}} R'(e, A) = K_0 A$$

Gives splitting: $R'(G,A) = K_0 A \oplus \widetilde{R}'(G,A)$

Similarly: $R(G,A) = K_0 A \oplus \widetilde{R}(G,A)$.

We have map: $R_{ep}(G,A) \longrightarrow [BG, BGL(A)^+]$

$$E \longmapsto [E]$$

① trivial repr. go to 0 ② $0 \to E' \to E \to E'' \to \Rightarrow [E] = [E'] + [E'']$

Lemma: M connected H-space. (~ CW complex). Then M has a homotopy inverse (so $[X,M]$ is a group).

$$\begin{array}{ccc} (m_1, m_2) & \longmapsto & (m_1, m_1 m_2) \\ g: M \times M & \xrightarrow[h.\,equiv]{} & M \times M \\ \downarrow pr_1 & & \downarrow pr_2 \\ M & = & M \end{array}$$

since h. equiv on fibre and base space

g is a map of fibrations, so by long exact homotopy sequence: g is weak homotopy equivalence ⇒ g hom. eq

$g_*: [X,M] \times [X,M] \longrightarrow [X,M] \times [X,M]$ gives inverse.

By universal property of $R(G,A)$ we get:

$$\widetilde{R}(G,A) = R(G,A)/K_0 A \twoheadleftarrow R(G,A) \longrightarrow [BG, BGL(A)^+]$$

Given any X then we have: $X \longrightarrow B\pi_1(X)$.

so you get a canonical map: $\widetilde{R}(\pi_1 X, A) \xrightarrow{\alpha_*} [B\pi_1 X, BGL(A)^+] \longrightarrow [X, BGL(A)^+]$

Theorem: Let X range over finite complexes. (ptd connected)

Then any natural transformation: $h: \bar{R}(\pi_1 X, A) \longrightarrow [X, Z]$ with Z having no non-trivial perfect subgps of $\pi_1 Z$, extends uniquely to a nat. trans:

$$\tilde{h}: [X, BGL(A)^+] \longrightarrow [X, Z].$$

Example, λ-ring structure on $\bar{R}(\pi_1 X, A)$ induces operations:

$$[X, BGL(A)^+] \xrightarrow{\tilde{\lambda}^k} [X, BGL(A)^+].$$

A representation of $\pi_1 X$ over A may be identified with a fibre bundle with fibre$_\wedge$ discrete a $\underline{P}$ in $\mathcal{P}_A$.

10.16.74

Theorems as above

Let X range over the category of pointed finite complexes, morphisms are homotopy classes of base point preserving maps.

Consider a natural transformation $\gamma: F(\underset{-}{?}) \to G(\underset{-}{?})$ F, G map to Sets. Say γ has property $(*)$ if:

given $$F(X) \xrightarrow{\alpha} [X, Z]$$

where Z is a space s.t. $\pi_1 Z$ has no nontrivial perfect subgroups, then:

$$\begin{tikzcd} F(X) \arrow[r, "\alpha"] \arrow[d, "\gamma(X)"'] & {[X, Z]} \\ G(X) \arrow[ur, dashed, "\exists ! \theta"'] & \end{tikzcd}$$

Theorem: The canonical natural transformation

$$[X, BGL(A)] \longrightarrow [X, BGL(A)^+]$$

has property $(*)$.

Proof: Fact: If Y is a CW complex and $F_\alpha \subseteq Y$, $\alpha \in J$ is a directed system of finite subcomplexes st. $\bigcup F_\alpha = Y$, then:

$$[X,Y] \xleftarrow{\approx} \varinjlim_{J} [X, F_\alpha]. \qquad X \text{ finite, image caught up in some } F$$

Take F_0 to be the 2-skeleton of $B\mathfrak{A}_5 \subseteq BGL_5(A)$.
By choosing $B\mathfrak{A}_5$ suitably (Milnor model), obtain finite com

$$\pi_1(F_0) = \pi_1(B\mathfrak{A}_5) = \mathfrak{A}_5 \quad \text{perfect group.}$$

By attaching a single 2- & 3-cell to F_0 we obtain $F_0 \hookrightarrow F_0^+$.
(go over proof). Form:

$$BGLA \cup_{F_0} F_0^+ = BGL(A)^+$$

$$\begin{array}{ccc} F_0 & \xrightarrow{\text{acyclic}} & F_0^+ \\ \downarrow & & \downarrow \\ BGL(A) & \xrightarrow[\text{acyclic}]{} & BGL(A) \cup_{F_0} F_0^+ . \end{array}$$

Van Kampen: $\pi_1(BGL(A) \cup_{F_0} F_0^+) = GL(A)/\text{normal sub gen by } \mathfrak{A}_5$.
this is $E(A)$

Because it contains $\mathfrak{A}_\infty$ hence given matrices $\alpha, \beta \in GL(A)$ mod this normal subgp, α, β commute.

Now with $Y = BGL(A)$, take F_α = all finite subcomplexes of Y containing F_0.

$$BGL(A) = \bigcup F_\alpha.$$

$$\Rightarrow [X, BGL(A)] = \varinjlim [X, F_\alpha]$$

Hence: $BGL(A)^+ = \bigcup (F_\alpha \cup_{F_0} F_0^+)$

$\Rightarrow [X, BGL(A)^+] = \varinjlim [X, F_\alpha \cup_{F_0} F_0^+]$.

$\text{Nat. Trans.}\left([X, BGL(A)^+], T(X)\right) = \varprojlim_\alpha \text{Nat Transf}\left([X, F_\alpha], T\right)$

$= \varprojlim_\alpha T(F_\alpha)$. Yoneda's Lemma.

similarly $\text{Nat Trans.}\left([X, BGL(A)^+], T(X)\right) = \varprojlim T(F_\alpha \cup_{F_0} F_0^+)$.

Take $T = [\ , Z]$ so:

$$\begin{array}{ccc} \text{Nat. Trans.}\left([X, BGL(A)^+], [X, Z]\right) & = \varprojlim & [F_\alpha \cup_{F_0} F_0^+, Z]. \\ \downarrow & & \downarrow \\ \text{Nat. Trans}\left([X, BGL(A)], [X, Z]\right) & = \varprojlim & [F_\alpha, Z]. \end{array}$$

We know $\pi_1(Z)$ has no nontrivial perfect subgroups. $\Rightarrow$

$[F_\alpha, Z] \xleftarrow{\ \cong\ } [F_\alpha \cup_{F_0} F_0^+, Z]$. by acyclicity of map $F_\alpha \longrightarrow F_\alpha \cup_{F_0} F_0^+$.

We assume finite CW complexes since now:

X finite $\Rightarrow \pi_1 X$ fin. gen. gp. $[X, BGL(A)] = \text{Hom}(\pi_1 X, GL(A))$
$= \varinjlim \text{Hom}(\pi_1 X, GL_n A)$

$$\begin{array}{c} R'(\pi_1 X, A) \longrightarrow R(\pi_1 X, A) \\ \parallel \end{array}$$

abelian group generated by the monoid of iso classes of rep. of $\pi_1 X$ in $\mathcal{P}(A)$

$\tilde{R}'(\pi_1 X; A) = \bar{R}'(\pi_1 X, A) / R'(e, A) = K_0 A$.

Exercise: Show $\tilde{R}'(\pi_1 X, A)$ = abelian group generated by the monoid $\varinjlim \underbrace{\mathrm{Hom}(\pi_1 X, GL_n A)_{GL_n A}}_{\text{iso classes of rep. of } \pi_1 X \text{ on } A^n}$.

$$
\begin{tikzcd}
\varinjlim \mathrm{Hom}(\pi_1 X, GL_n A) = [X, BGL(A)] \arrow[r] \arrow[d] & {[X, BGL(A)^+]} \\
\oplus \quad \tilde{R}'(\pi_1 X, A) \arrow[ur, dashed, "h"] \arrow[d] & \\
\tilde{R}^{\#}(\pi_1 X, A) \arrow[uur, dashed, "k"'] \arrow[d] & \\
0 &
\end{tikzcd}
$$

Theorem: h, k have property $(*)$

Suffices to show it for h, by diagram chasing.

Fact. Suppose M is an abelian monoid; $\bar{M}$ = abelian group gen by M. Grothendieck construction:

$$
M \times M \times M \overset{a}{\underset{b}{\rightrightarrows}} \overset{a,\ b}{M \times M} \longrightarrow \overset{b-a}{\bar{M}} \qquad \text{exact.}
$$

$$
(m_1, m_2, m_3) \overset{a}{\longmapsto} (m_1 + m_3, m_3 + m_2), \quad \overset{b}{\longmapsto} (m_1, m_2) \quad m_2 + m_3
$$

$$
(m_1', m_2') \sim (m_1, m_2) \Leftrightarrow \exists t \text{ s.t. } m_1' + m_2 + t = m_2' + m_1 + t.
$$

exact means: given $\varphi : M \times M \longrightarrow S$ $\varphi a = \varphi b$ then φ factors uniquely thru $\bar{M}$.

$$
\begin{tikzcd}
M \times M \times M \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[dr, shift left] \arrow[dr, shift right] & M \times M \arrow[r] \arrow[d, "\varphi"] & \bar{M} \arrow[dl, dashed, "!"] \\
& S &
\end{tikzcd}
$$

Proof that h satisfies $(*)$.

Put: $M(X) = \varinjlim \mathrm{Hom}(\pi_1 X, GL_n A)_{GL_n A}$

$$\begin{tikzcd}
M(X)^3 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[d, "u"] & M(X)^2 \arrow[r] \arrow[d, "v"] & \tilde{R}'(\pi_1 X; A) \arrow[d] \arrow[dr, "N.T."] & \\
{[X, BGL(A)^+]^3} \arrow[r, shift left] \arrow[r, shift right] & {[X, BGL(A)^+]^2} \arrow[r] \arrow[rr, dashed, bend right, "\text{H-space multiplication}"'] & {[X, BGL(A)^+]} \arrow[r, dashed, "!"] & {[X, Z]}
\end{tikzcd}$$

Both rows exact.
u, v have property $(*)$ (apply previous thm to $A \times A$, $A \times A \times A$).

Cor from above: $\prod_{i=1}^{n} [X, BGL(A_i)] \longrightarrow \prod_{i=1}^{k} [X, BGL(A_i)^+]$

done now by diagram chases

10.21.74

$\tilde{K}(X, A) = [X, BGL(A)^+]$ analogous to $\tilde{K}(X) = [X, BU]$

① $A \xrightarrow{u} B$ ring-homomorphism:

$u^*: \mathcal{P}_A \longrightarrow \mathcal{P}_B$. $u^*(P) = P \otimes_A B$. also,

$$\begin{tikzcd}
\tilde{R}(\pi_1 X, A) \arrow[r] \arrow[d] & \tilde{R}(\pi_1 X, B) \arrow[d] \\
\tilde{K}(X; A) \arrow[r, dashed] & \tilde{K}(X; B)
\end{tikzcd}$$

$A \mapsto \tilde{K}(X, A)$ is covariant functor. Rings $\longrightarrow$ Ab.

② transfer: $u: A \longrightarrow B$ is a ring homomorphism st. $B \in \mathcal{P}_A$.
Then one has "restriction of scalars".

$u_*: \mathcal{P}_B \longrightarrow \mathcal{P}_A$. $u_*(Q) = Q|_A$.

You get

$$\begin{tikzcd} \tilde{R}(\pi, X, B) \arrow[r] \arrow[d] & \tilde{R}(\pi, X, A). \arrow[d] \\ \tilde{K}(X, B) \arrow[r, dashed] & \tilde{K}(X; A) \end{tikzcd}$$

In particular: $X = S^n$: $u_*: K_n B \to K_n A$.
called the transfer

③ Products: $K_0 A \otimes \tilde{K}(X;A) \to \tilde{K}(X;A)$, A commutative.

Proposition: (Projection formula): Suppose $u: A \to B$ as in (2), $B \in$ ...
then:

$$u_* u^*: \tilde{K}(X, A) \to \tilde{K}(X, A).$$ Forget this.

Remark: If B has a finite projective resolution.

$$0 \to P_n \to \dots \to P_1 \to P_0 \to B \to 0 \qquad \text{with } P_i \in \mathcal{P}_A$$

then one can define a transfer $\tilde{K}_*(X, B) \to \tilde{K}_*(X, A)$.
(using other methods).

③ Products again: $\mathcal{P}_A \times \mathcal{P}_B \longrightarrow \mathcal{P}_{A \otimes_{\mathbb{Z}} B}$

$$(P, Q) \longmapsto P \otimes_{\mathbb{Z}} Q.$$

$$R(\pi, X, A) \otimes R(\pi, X, B) \longrightarrow R(\pi, X, A \otimes B).$$
$$[E], [F] \mapsto [E \otimes F].$$

Let $\varepsilon: R(\pi, X, A) \to K_0 A$. the map to underlying module with trivial actions.

$$\begin{tikzcd} R(\pi, X, A) \arrow[r] \arrow[d, "\text{forgets action}"] & K_0 A \arrow[dl] \\ R(\pi, X, A) & \end{tikzcd}$$

Then: $\tilde{R}(\pi, X; A) \otimes \tilde{R}(\pi, X, B) \to \tilde{R}(\pi, X, A \otimes B)$

$[E]-[\varepsilon E]$ typic element. $([E]-[\varepsilon E])\otimes([F]-[\varepsilon F]) = [E\otimes F] - [E]\otimes[\varepsilon F] - [\varepsilon F]\otimes[E] + [\varepsilon E \otimes \varepsilon F]$

$$[E]^{-\varepsilon E}, [F]^{-\varepsilon F} \longmapsto [E\otimes F] - [E\otimes \varepsilon F] - [\varepsilon E \otimes F] + [\varepsilon E \otimes \varepsilon F] \quad (*)$$

Thus by sending a representation E of $\pi_1 X$ over X and a rep. F of $\pi_1 X$ over B to above $(*)$ one gets a (representation) natural transformation

$$\begin{array}{ccc} \tilde{R}(\pi_1 X, A) \otimes \tilde{R}_1(\pi_1 X, B) & \longrightarrow & \tilde{R}(\pi_1 X, A\otimes B) \\ \downarrow & & \downarrow \\ \tilde{K}(X,A) \times \tilde{K}(X,B) & \overset{\mu}{\dashrightarrow} & \tilde{K}(X; A\otimes B) \end{array}$$

Properties of μ:

Lemma ① μ is bilinear, associative, commutative.

Proof: $\mu(\alpha+\beta, \gamma) = \mu(\alpha,\gamma) + \mu(\beta,\gamma)$.

$$\begin{array}{ccc} \tilde{K}(X,A)^2 \times \tilde{K}(X,B) & \overset{\alpha,\beta,\gamma \mapsto \mu(\alpha+\beta,\gamma)}{\underset{\alpha,\beta,\gamma \mapsto \mu(\alpha,\gamma)+\mu(\beta,\gamma)}{\rightrightarrows}} & \tilde{K}(X, A\otimes B) \\ \uparrow & \nearrow & \uparrow \\ \tilde{R}(\pi_1 X, A) \times \tilde{R}(\pi_1 X, B) & \rightrightarrows & \tilde{R}(\pi_1 X, A\otimes B) \end{array}$$

Because for $\tilde{R}$ one has $(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma$. Then same holds for $\tilde{K}$ by uniqueness part of theorem.

associativity: $\tilde{K}(X,A) \times \tilde{K}(X,B) \times \tilde{K}(X,C) \rightrightarrows \tilde{K}(X, A\otimes B\otimes C)$

$$\begin{array}{ccc} \tilde{K}(X;B) \times \tilde{K}(X,A) & \longrightarrow & \tilde{K}(X, B\otimes A) \\ \| & & \\ \tilde{K}(X,A) \times \tilde{K}(X,B) & = & \tilde{K}(X, A\otimes B) \end{array}$$

Can also define products: $\widetilde{K}_0 A \otimes \widetilde{K}(X,B) \to \widetilde{K}(X, A\otimes B)$.

$$\widetilde{K}(X,A) \otimes \widetilde{K}(B) \to \widetilde{K}(X, A\otimes B)$$

so that putting $K(X,A) = K_0 A \times \widetilde{K}(X,A)$,

one gets products: $K(X,A) \times K(X,B) \to K(X, A\otimes B)$.

which are associative, commutative, unitary.

Take A to be commutative, so we have $A\otimes A \to A$, $a\otimes b \mapsto$
Then $K(X,A)$ is commutative ring and $\widetilde{K}(X,A)$ is an ideal in $K(X,A)$.

Products in K-groups..

$$\begin{array}{c} \widetilde{K}(pt^A) \leftarrow \overset{x}{\widetilde{K}(X,A)} \times \overset{y}{\widetilde{K}(Y,B)} \to \widetilde{K}(Y,B) \\ \downarrow \\ \widetilde{K}(X\times Y, A) \times \widetilde{K}(X\times Y, B) \\ pr_1^*(x) \downarrow \quad pr_2^*(y) \\ \widetilde{K}(pt\times Y, A\otimes B) \longleftarrow \widetilde{K}(X\times Y, A\otimes B) \ni \\ \mu(pr_1^* x, pr_2^* y) \end{array}$$

$\mu(pr_1^* x, pr_2^* y)$ dies on $* \times Y$ and $X \times *$ on $X\times Y$, $X \vee Y$

In general.

$$0 \to [X\wedge Y, Z] \to [X\times Y, Z] \longrightarrow [X\vee Y, Z] \to 0$$

$$[X\vee Y, Z] = [X,Z] \times [Y,Z] \text{ --- splitting} \dashrightarrow [X\times Y, Z]$$

for H-space Z.

Conclude $\mu(pr_1^* x, pr_2^* y) \in \widetilde{K}(X\wedge Y, A\otimes B)$
$= \text{subgp of } \xi \in \widetilde{K}(X\times Y, A\otimes B)$ dying on $X\times pt \vee pt \times Y$

So one gets natural transformation:

$$\widetilde{K}(X,A) \times \widetilde{K}(Y,B) \longrightarrow \widetilde{K}(X\wedge Y, A\otimes B)$$

Take $X = S^p$, $Y = S^q$, $X\wedge Y = S^{p+q}$ and set $K_i A \otimes K_j B \to K_{i+j}(A\otimes B)$

Exercise: extend this to p or $q=0$. & check following properties:

① associative ② commutative.

$$\begin{array}{ccc} K_pA \otimes K_qB & \longrightarrow & K_{p+q}(B\otimes A) \\ T \big\| & & \text{\scriptsize $\wr$} \\ K_qB\otimes K_pA & \longrightarrow & K_{p+q}(B\otimes A) \end{array} \qquad (-1)^{pq}$$

A commutative ⇒ K_*A graded anticommutative ring.

λ-operations and Adams operations in K_*A.

Suppose A commutative.

Recall (Atiyah: "K-Theory"):

$K(X) = [X, \mathbb{Z}\times BU]$

$$\begin{array}{c} E \\ \downarrow \\ \Lambda^k E \end{array} \qquad R(G,A) \qquad E \mapsto \Lambda^k E.$$

one proves this operation induces on bundles operations

$$\lambda^k : K(X) \to K(X)$$

satisfying: let t be indeterminate $\lambda_t x = 1 + (\lambda^1 x)t + (\lambda^2 x)t^2 + \dots \in K(X)[[t]]$

→ ① $\lambda_t(x+y) = \lambda_t(x)\cdot\lambda_t(y)$

② $\lambda^k(xy) = P_k(\lambda^1x,\dots,\lambda^k x, \lambda^1 y,\dots,\lambda^k y)$

③ $\lambda^k\lambda^\ell(x) = P_{k,\ell}(\lambda^1 x, \dots\dots)$ this is a λ-ring.

Adams Ψ-operations: $\lambda_t(L) = 1+tL$ L line bundle

So $\lambda_{-t}(L) = 1-tL$

$$\ln\left(\frac{1}{1-tL}\right) = tL + \frac{t^2L^2}{2} + \frac{t^3L^3}{3} \longleftarrow \frac{1}{\lambda_{-t}(L)} = \frac{1}{1-tL}$$

$\Psi^k L = L^k \qquad -\ln(\lambda_{-t}(L)) = tL + \frac{t^2L^2}{2} + \ldots$

$$t\frac{d}{dt}\left(\frac{1}{\lambda_{-t}(L)}\right) = tL + t^2L^2 + t^3L^3 + \ldots$$

Put $\sum_{k\geq 1} t^k \Psi^k x = t\frac{d}{dt}\ln\left(\frac{1}{\lambda_{-t}(x)}\right)$ $\qquad \Psi^0 = id$ by convention

Identities: (i) $\Psi^k(x+y) = \Psi^k(x) + \Psi^k y$.
(ii) $\Psi^k(xy) = \Psi^k(x)\cdot\Psi^k(y)$.
(iii) $\Psi^k \circ \Psi^\ell(x) = \Psi^{k\ell}(x)$.

Theorem: On $\tilde{K}(X,A)$ one has Adams operations, A commutative
satisfying (i)—(iii)

$$\tilde{K}(X,A)\otimes\mathbb{Q} = \bigoplus_{n\geq 1} V_n \text{ (eigenspaces)}$$

$x \in V^n$ iff $x \in \tilde{K}(X,A)\otimes\mathbb{Q}$ & $\Psi^k x = k^n x \quad \forall k \geq 1$. eigenvalue.

topological analogue: $\tilde{K}(X)\otimes\mathbb{Q} \approx \bigoplus_{n\geq 1} H^{2n}(X,\mathbb{Q})$

? algebraic J-groups,

10.03.74.

A = commutative ring. $\quad \otimes, \Lambda^k$.

$R(G,A)$ is a commutative ring with identity with
$$\lambda^k : R(G,A) \longrightarrow R(G,A).$$

satisfying:

① $\lambda_t(x+y) = \lambda_t(x)\cdot\lambda_t(y) \qquad \lambda_t(x) = \sum_{k\geq 0}\lambda^k(x)t^k$.
$\lambda_t(0) = 1$

② $\lambda^k(xy) = P_n(\lambda^1 x, \ldots \lambda^k x, \lambda^1 y, \ldots, \lambda^k y)$

③ $\lambda^j \lambda^k(x) = Q_{jk}(\lambda^1 x, \dots, \lambda^{jk})$

④ L 1-dimensional $\lambda_t(L) = 1 + t[L]$

Procedure for calculating P_k $\quad X_1, \dots, X_n, Y_1, \dots, Y_m$.

② $\lambda^k(xy) \longleftrightarrow k^{th}$ elementary symmetric function of $X_i \times Y_j$. (n^2 var
$1 \le i \le n$, $1 \le j \le m$.
$= P_k$ (elem. sym. ftns of X_i, elem. sym. ftns of Y_i).

$P_k(\lambda^1 x, .. \lambda^k x, \lambda^1 y . \lambda^k y) \longleftarrow$

② $\lambda^k(x) \leftrightarrow k^{th}$ elementary symettric function of X_i's
$$\sum_{1 \le i_1 < \dots < i_k \le n} X_{i_1} \cdots X_{i_k}$$

Take j^{th} symmetric function of $X_{i_1} \cdots X_{i_k}$ $\quad 1 \le i_1 < \dots < i_k \le n$.
and write it as a polynomial in the elem sym. functions of X_i.

$Q_{j,k}$ (elem. sym. functions of X_i's)
$$\updownarrow$$
$$Q_{j,k}(\lambda^1 x, \dots, \lambda^{jk} x)$$

Adams operations $\Psi^k : R(G, A) \to R(G, A)$

$$\frac{1}{\lambda_{-t}(x)} = \exp\left(\sum_{m \ge 1} \Psi^m(x) \frac{t^m}{m}\right) \qquad \Psi^0 = 1.$$

(looks like Weil zeta function)

1/2) Ψ is a ring homomorphism
③ $\Psi^k(\Psi^j x) = \Psi^{kj}$.
④ $\Psi^k \ell = \underbrace{\ell \otimes \dots \otimes \ell}_{k}$ $\quad \dim \ell = 1$.

SGA 6: Fact: If A is of characteristic p, then $\exists$ Frobenius endomorphism of A: $\quad Fa = a^p$, which induces map F on $R(G, A)$.
Assertion: $\Psi^p = F$, on $R(G, A)$.

γ-operations. $\qquad x_1^2 + x_2^2 =$

$p=2 \qquad \Psi^2 x = x^2 - 2\lambda^2 x =$

$$0 \to \Lambda^2 P \to P \otimes P \to S^2 P \to 0$$

$$0 \to K \to S^2 P \to \Lambda^2 P \to 0$$

$$K = \langle p_1 p_1 \rangle = F(P). \qquad \langle p_1 p_1 \rangle \mapsto p_1 \wedge p_2.$$

$E = L_1 + \dots + L_n$. want $\gamma_k(E-n) = k^{th}$ elem sym fcn of $L_i - 1$.

want $\gamma_t(E-n) = \prod (1 + t(L_i - 1))$.

$\gamma_t = \sum \gamma^k t^k$.

want γ_t to satisfy: $\gamma_t(x+y) = \gamma_t(x)\cdot\gamma_t(y)$

$$\gamma_t(L-1) = 1 + t(L-1) = (1-t) + tL.$$
$$\| $$
$$\gamma_t(L)/\gamma_t(1).$$

$$\gamma_t(L) = (1-t+tL)\gamma_t(1). \qquad \gamma_t(1) = \frac{1}{1-t}$$
$$= 1 + \left(\frac{t}{1-t}\right) L.$$
$$= \lambda_{\left(\frac{t}{1-t}\right)} L. \qquad \Rightarrow \boxed{Def: \quad \gamma_t x = \lambda_{\left(\frac{t}{1-t}\right)}(x).}$$

γ-filtration: of $R(G,A)$ is defined by:

$$F_p^{\gamma}(R(G;A)) = \text{subgp gen. by } \gamma^{i_1}(x_1)\dots\gamma^{i_k}(x_k), \quad i_1 + \dots + i_k \geq p.$$

Fact: On $F_i^{\gamma}/F_{i+1}^{\gamma}$ one has $\Psi^k = k^i$.

Fact: In a λ-ring one has:

Ψ^p

Atiyah's book

Theorem: X finite complex. On $\tilde{K}(X,A)$, the γ-filtration is locally nilpote
i.e. $\forall x \in \tilde{K}(X,A)$, $\exists N_x$ s.t. $\gamma^{i_1}(x)\dots\gamma^{i_r}(x)=0$ $\quad i_1+\dots+i_r \geq N$.

Corollary: For any $x \in \tilde{K}(X;A)$, $k \geq 1$: $\exists N$ s.t.:

$$(\Psi^k-1)(\Psi^k-k)\dots(\Psi^k-k^N)x = 0.$$

$\exists$ formula for $(\Psi^k-1)\dots(\Psi^k-k^N)$ (in terms of γ^i's of high weight)
in terms of $P(\gamma^1,\dots,\gamma^k)$, involves monomials of weight $\geq N$.

Theorem: A perfect of char. $p \Rightarrow K_* A$ uniquely p-divisible. i:

Pf: Ψ^p is an automorphism on $\tilde{K}(X,A)$ (because $\Psi^p = \text{Frob.}$)

Lemma: If A is an abelian gp with auto. Ψ satisfying:
$$A = \bigcup_n \ker(\Psi-p)\dots(\Psi-p^n) \quad \text{then } p: A \xrightarrow{\cong} A$$

seem to need: $A^\Psi \xrightarrow[\cong]{p} A^\Psi$

$$X_n = \ker(\Psi-p)\dots(\Psi-p^n)$$

$$\begin{tikzcd}
0 \arrow[r] & X_{n-1} \arrow[r] \arrow[d, "\Psi", "\wr"'] & X_n \arrow[r] \arrow[d, "\Psi", "\wr"'] & X_n/X_{n-1} \arrow[r] \arrow[d, "\Psi = p^n", "\wr"'] & 0 \\
0 \arrow[r] & X_{n-1} \arrow[r] & X_n \arrow[r] & X_n/X_{n-1} \arrow[r] & 0
\end{tikzcd}$$

use induction
take union.

$x \in \tilde{K}(X,A) = \varinjlim [X, BGL_n(A)^+]$

$\lambda^k(x+n) = 0 \quad k > n.$

say $x \in [X, BGL_n(A)^+]$

proof next time.

10.30.74

A commutative ring.

λ operations on $\tilde{K}(X;A)$.

$$\log\left(\frac{1}{\lambda_{-t}(x)}\right) = \sum_{m\geq 1} \Psi^m(x) \frac{t^m}{m}.$$

$$= -\log(\lambda_{-t}(x))$$

Differentiate this:

$$\frac{1}{\lambda_{-t}(x)}\left(\lambda'_{-t}(x)\right) = \sum_{m\geq} \Psi^m(x)\, t^{m-1}.$$

$$\lambda'_{-t}(x) = \lambda_{-t}(x) \cdot \sum_{m\geq 1} \Psi^m x\, t^{m-1} \qquad \lambda_t(x) = 1 + tx + t^2\lambda^2 x$$

$$\lambda'_t(x) = x + 2t\lambda^2 x + \dots \qquad \|$$

$$\lambda'_{-t}(x) = x - 2t\lambda^2 x + 3t^2\lambda^3 x -$$

$$= (1 - tx + t^2\lambda^2 x - \dots)(\Psi^1 x\, t^0 + \Psi^2 x\, t^1 + \Psi$$

$\Psi^1 x = x$

$-2\lambda^2 x = \Psi^2 x - x\Psi^1 x$

$+3\lambda^3 x = \Psi^3 x - x\Psi^2 x t + \lambda^2 x \cdot x.$

$-4\lambda^4 x = \Psi^4 - x\Psi^3 x + \lambda^2 x \cdot \Psi^2 x - \lambda^3 x \cdot x.$

Newton $(-1)^{k-1} k \lambda^k x = \Psi^k x - x\Psi^{k-1} x + \lambda^2 x \cdot \Psi^{k-2} x - \dots + (-1)^{k-1}\lambda^{k-1}(x)\cdot x.$

think of $\lambda^k x = \text{elem} \sum_{i_1 < \dots < i_k} x_{i_1} \cdots x_{i_k}; \; \Psi^k x = \sum x_i^{\,k}.$

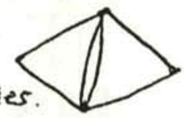
Theorem: A perfect of char. p $\Rightarrow K_n A$ uniquely p-divisible; $p: K_n A \xrightarrow{\cong} K_n A$ $n \geq$

Proof: (1) $\Psi^p =$ Frobenius.

Hence Ψ^p is isomorphism on $\tilde{K}(X,A)$, for any X. Let $X = S^n$.
$= SS^n$

(2) cup-products in $\tilde{K}(SY;A)$ vanish.

general fact about multiplicative cohomology theories.



drag cocycles to diff. sides.

Hence $(-1)^{p-1} p \cdot \lambda^p(x) = \Psi^p(x)$ in $\tilde{K}(SY;A)$. QED.

λ^p is homo now since cup-products vanish.

$$\gamma_t(x) = 1 + t\gamma^1 x + t^2\gamma^2 x + \dots = \lambda_{\frac{t}{1-t}}(x).$$

$$\gamma_t(L) = 1 + \left(\frac{t}{1-t}\right)L.$$

$$\gamma_t(L-1) = \frac{\left(1 + \frac{t}{1-t}L\right)}{\left(1 + \frac{t}{1-t}\right)} = 1 + t(L-1).$$

Thm: $\forall x \in \tilde{K}(X,A)$, $\exists n$ s.t. $\gamma^{i_1}(x)\cdots\gamma^{i_r}(x) = 0$ $\quad i_1 + \dots + i_r \geq n$.

Cor: $\forall x \in \tilde{K}(X,A)$, $\exists n$ s.t. $\prod_{i=1}^{n}(\Psi^k - k^i)(x) = 0$.

Because formally one has identities: $\left[\prod_{i=1}^{n}(\Psi^k - k^i)\right](x) \equiv 0$ mod ideal generated by $\gamma^{i_1}(x)\cdots\gamma^{i_r}(x)$, $\quad i_1 + \dots + i_r \geq m$.

Case $k=2$: $\Psi^k L = L^k = (1 + (L-1))^k = 1 + k(L-1) + \binom{k}{2}(L-1)^2 + \dots$

$$\Psi^k\left[\sum(L_i - 1)\right] = \sum \Psi^k(L_i - 1) = \sum(\Psi^k L_i - 1)$$

$$= k\left[\sum(L_i - 1) + \binom{k}{2}\sum(L_i - 1)^2 + \dots\right].$$

Proof of Theorem: First step is to show $\gamma_t(x)$ is a polynomial in t.

$$\tilde{K}(X,A) = [X, BGL(A)^+].$$

i.e. $\gamma^n(x)=0$, n large.

$$\begin{tikzcd}
\mathrm{hom}(\pi_1 X, GL_n A) = [X, BGL_n A] \arrow[r, "①"'] \arrow[d] & {[X, BGL_n(A)^+]} \arrow[d, "②"] & \\
\tilde{R}(\pi_1 X; A) \arrow[r] \arrow[d, "\gamma^k"] & {[X, BGL(A)^+]} = \tilde{K}(X,A) \arrow[r, "\gamma^k"] & {[X, BGL(A)^+]} \\
\tilde{R}(\pi_1 X; A) \arrow[urr, bend right] & &
\end{tikzcd}$$

$[A^n] \mapsto [E]-n$

Lemma: If $x \in \tilde{K}(X,A)$ comes from $[X, BGL_n(A)^+]$, then $\gamma^k x = 0$, $k >$

Proof: "Universal property" theorem shows the nat. trans. $\gamma^k \circ ②$ is determined by $\gamma^k \circ ② \circ ①$.

It suffices to show that: $\gamma^k([E]-n) = 0$ in $\tilde{R}(\pi_1 X; A)$. if E is a representation on A^n, for $k>n$.

$$\gamma_t(1) = \frac{1}{1-t}, \qquad \gamma_t([E]-n) = \gamma_t(E)/\gamma_t(n) = \gamma_t(E)(1-t)^n$$

$$= \lambda_{\frac{t}{1-t}}(E)(1-t)^n =$$

$$= \left(1 + \lambda^1(E)\frac{t}{1-t} + \dots + \lambda^n E\left(\frac{t}{1-t}\right)^n\right)(1-t)^n.$$

because $\Lambda^k E = 0$, for $k>n$, $\dim E = n$

s. $\gamma_t([E]-n)$ is a polynomial of degree $\le n$.

$$\gamma_t(x) \cdot \gamma_t(-x) = \gamma_t(0) = 1.$$

Lemma: If $(f,g) = 1$, then coefficients of x_i, $i>0$, are nilpotent

γ-filtration is Locally nilpotent

$\Rightarrow$ (as in Atiyah's book) one gets $\widetilde{K}(X,A)\otimes\mathbb{Q} \simeq \bigoplus_{i\geq 1} V_{(i)}$

where $V_{(i)} = \{x \in \widetilde{K}(X,A)\otimes\mathbb{Q} : \Psi^k x = k^i x \ \forall k\}$.

So for algebraic K-groups: $K_n(A)\otimes\mathbb{Q} \cong \bigoplus V_{(i)_n}$.

$$A^* \to K_1 A \qquad \Psi^k\alpha = k\alpha, \quad \alpha\in A^* \subseteq K_1 A.$$

F field: $\alpha \in K_2 F \qquad \Psi^k\alpha = k^2\alpha.$

Adams operations in algebraic K-groups are not well-understood.

11.3.74

define suspension of ring A, SA s.t. $K_1(SA) = K_0 A$.

Let $\pi : A \twoheadrightarrow B$ ring epimorphism.

Consider triples (E,F,α) $\quad E,F \in \mathcal{P}(A) \quad \alpha : \pi E \xrightarrow{\approx} \pi F$

$\pi E = E/(\ker\pi)E$.

The set of iso classes of such triples is a monoid. M

$$(E,F,\alpha)\oplus(E',F',\alpha') = (E\oplus E', F\oplus F', \alpha\oplus\alpha').$$

Lemma 1: Any triple (E,F,α) in M, has "inverse" (E',F',α') s.t.

$\exists n$: $\quad (E,F,\alpha)\oplus(E',F',\alpha') = (A^n, A^n, id_{B^n})$. $\quad (A,A,id_B)$ is called basic element of M

Proof: Choose E_1, F_1 s.t. $E\oplus E_1 \approx A^m \quad F\oplus F_1 \approx A^m$.

$$\pi E_1 \oplus B^m = \pi E_1 \oplus (\pi F \oplus \pi F_1) \simeq \pi E_1 \oplus \pi E \oplus \pi F_1 \ \text{(using } \alpha\text{)}$$
$$\simeq B^m \oplus \pi_0 F_1.$$