Some applications of algebraic surgery theory: 4-manifolds, triangular matrix rings and braids

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Doctoral thesis

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Declaration

This thesis, which was composed by the candidate himself, is submitted to the University of Edinburgh in partial fulfilment of the requirements for the degree of Doctor of Philosophy in the School of Mathematics.

The candidate hereby declares that the work presented in this thesis is, to the best of his knowledge and belief, original and his own, except where explicitly stated otherwise in the text. He further asserts that none of the material contained in this thesis has been submitted, either in part or whole, for any other degree or professional qualification.

Christopher Palmer
Edinburgh
18th September 2015

To my parents

Abstract

This thesis consists of three applications of Ranicki's algebraic theory of surgery to the topology of manifolds. The common theme is a decomposition of a global algebraic object into simple local pieces which models the decomposition of a global topological object into simple local pieces.

Part I: **Algebraic reconstruction of 4-manifolds.** We extend the product and glueing constructions for symmetric Poincaré complexes, pairs and triads to a thickening construction for a symmetric Poincaré representation of a quiver. Gay and Kirby showed that, subject to certain conditions, the fold curves and fibres of a Morse 2-function $F : M^4 \to \Sigma^2$ determine a quiver of manifold and glueing data which allows one to reconstruct M and F up to diffeomorphism. The Gay-Kirby method of reconstructing M glues the pre-images of disc neighbourhoods of cusps and crossings with thickenings of regular fibres and thickenings of cobordisms between regular fibres. We use our thickening construction for a symmetric Poincaré representation of a quiver to give an algebraic analogue of the Gay-Kirby result to reconstruct the symmetric Poincaré complex $(C(M), \phi_M)$ of M from a Morse 2-function.

Part II: **The L-theory of triangular matrix rings.** We construct a chain duality on the category of left modules over a triangular matrix ring $A = (A_1, A_2, B)$ where A_1, A_2 are rings with involution and B is an (A_1, A_2)-bimodule. We describe the resulting L-theory of A and relate it to the L-theory of A_1, A_2 and to the change of rings morphism $B \otimes_{A_2} - : A_2\text{-Mod} \to A_1\text{-Mod}$. By examining algebraic surgery over A we define a relative algebraic surgery operation on an $(n+1)$-dimensional symmetric Poincaré pair with data an $(n+2)$-dimensional triad. This gives an algebraic model for a half-surgery on a manifold with boundary. We then give an algebraic analogue of Borodzik, Némethi and Ranciki's half-handle decomposition of a relative manifold cobordism and show that every relative Poincaré cobordism is homotopy equivalent to a union of traces of elementary relative surgeries.

Part III: **Seifert matrices of braids with applications to isotopy and signatures.** Let β be a braid with closure $\widehat{\beta}$ a link. Collins developed an algorithm to find the Seifert matrix of the canonical Seifert surface Σ of $\widehat{\beta}$ constructed by Seifert's algorithm. Motivated by Collins' algorithm and a construction of Ghys, we define a 1-dimensional simplicial complex $K(\beta)$ and a bilinear form $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ such that there is an inclusion $K(\beta) \hookrightarrow \Sigma$ which is a homotopy equivalence inducing an isomorphism $H_1(\Sigma;\mathbb{Z}) \cong H_1(K(\beta);\mathbb{Z})$ such that $[\lambda_\beta] : H_1(K(\beta);\mathbb{Z}) \times H_1(K(\beta);\mathbb{Z}) \to \mathbb{Z} \subset \mathbb{Z}[\frac{1}{2}]$ is the Seifert form of Σ. We show that this chain level model is additive under the concatenation of braids and then verify that this model is chain equivalent to Banchoff's combinatorial model for the linking number of two

space polygons and Ranicki's surgery theoretic model for a chain level Seifert pairing. We then define the chain level Seifert pair (λ_β, d_β) of a braid β and equivalence relations, called A and $\widehat{A}$-equivalence. Two n-strand braids are isotopic if and only if their chain level Seifert pairs are A-equivalent and this yields a universal representation of the braid group. Two n-strand braids have isotopic link closures in the solid torus $D^2 \times S^1$ if and only if their chain level Seifert pairs are $\widehat{A}$-equivalent and this yields a representation of the braid group modulo conjugacy. We use the first representation to express the ω-signature of a braid β in terms of the chain level Seifert pair (λ_β, d_β).

Lay Summary

Imagine a sphere made out of rubber and suppose that you are allowed to squeeze, stretch or twist it as much as you want but you are not allowed to cut it or to glue parts of it together. Geometric properties of the sphere such as its surface area or the distance between two points may change drastically under these transformations (imagine inflating the sphere). Other more intrinsic properties may not change such as the fact that it a possible to draw a curve between any two points on the sphere or that a sphere has an inside and an outside.

One of the main goals of *topology* is the classification problem: when can one continuously deform one space to another or, slightly more generally, when can one continuously deform one space to another through a family of deformations? For example, a doughnut can be continuously deformed into to a coffee cup but it is not possible to continuously deform a sphere to a point without puncturing the sphere first. A central idea is to first find *topological invariants* of a space, namely properties of the space which do not change under continuous deformations. If two spaces have different values for the same invariant then it is not possible to continuously deform one into the other. *Algebraic topology* uses tools from algebra to help produce algebraic invariants to distinguish spaces.

A *manifold* is a topological space which looks flat around each point but may have a more complicated global structure. A hollow doughnut and the sphere are both 2-dimensional manifolds (imagine what a tiny ant sees if it walks on them). There are natural operations one can perform on two manifolds to produce a third manifold, such as taking a product or glueing them over a particular piece.

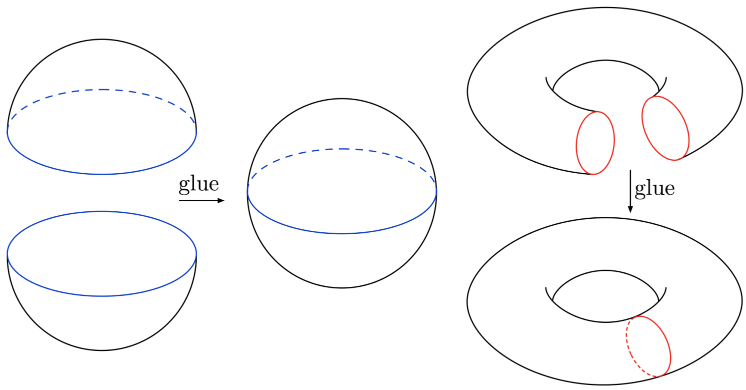

Figure 1: The sphere can be obtained by glueing two discs over their boundary circle and the doughnut can be obtained from a cylinder by gluing its two boundary circles.

Manifolds have a very rich set of invariants in algebraic topology and the invariants often have intricate structures. *Geometric surgery theory* is a collection of tools developed to answer the question of whether a topological space can be deformed into a manifold. *Algebraic surgery theory* is an algebraic model for geometric surgery theory in which manifolds and the various operations one can perform on them have precise algebraic analogues. The first part of this thesis examines how some of the basic tools of algebraic surgery theory can be extended to give an algebraic analogue of a geometric result which reconstructs a 4-dimensional manifold from simple pieces. The second part of this thesis gives an algebraic model for a geometric operation called a *half-surgery* and gives an algebraic analogue of a geometric result which decomposes a manifold with boundary into the traces of half-surgeries.

The third part of this thesis is concerned with knot theory. A *braid* is a collection of pieces of string travelling from left to right with the end points of each piece of string fixed on two vertical walls. The strings are allowed to intertwine but can never meet or reverse their direction of travel. Two braids can be *concatenated* by joining the right hand endpoints of the first braid to the left hand endpoints of the second braid. Every braid can be written as concatenation of *elementary braids* where each elementary braid has a single crossing.

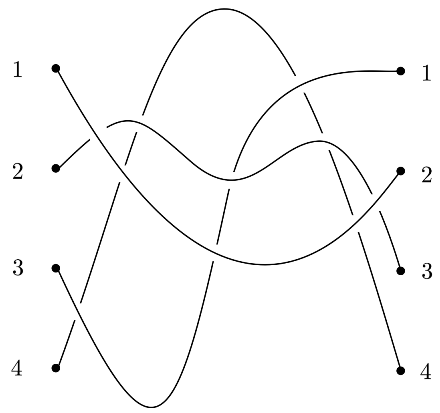

Figure 2: A 4-strand braid with 10 crossings.

The *closure* of a braid is an object in knot theory called a *link* and is formed by joining the end points of the braid as shown below.

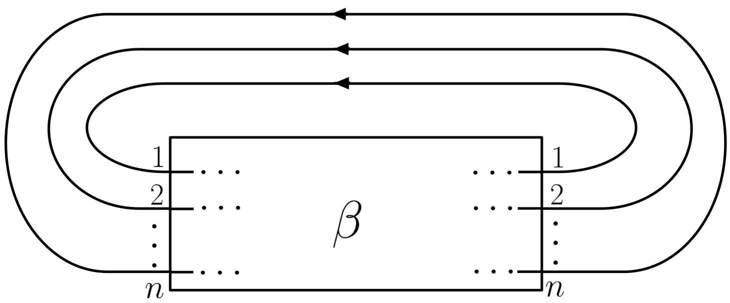

Figure 3: The closure of a braid.

A *Seifert surface* of a braid is a surface which has boundary equal to the closure of the braid. One can produce Seifert surfaces using *Seifert's algorithm*.

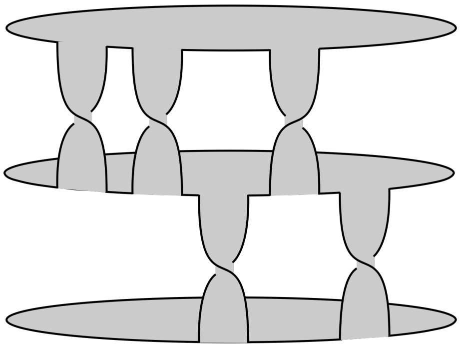

Figure 4: A Seifert surface constructed by Seifert's algorithm.

The *Seifert form* of a braid is an algebraic object which encodes geometric linking information about its Seifert surface. It is natural to ask how the Seifert form of a braid changes under the concatenation of braids.

To each braid we associate an algebraic object called a *chain level Seifert form.* We show that the chain level Seifert form of a braid determines its Seifert form and we then construct an algebraic glueing operation for chain level Seifert forms which models the geometric glueing of braids. This allows us to understand how the Seifert form of an arbitrary braid can be constructed from the chain level Seifert forms of elementary braids.

The common theme running through each of the three parts is an algebraic decomposition of a complicated algebraic object into simple pieces which models the decomposition of a complicated geometric object into simple pieces.

Acknowledgements

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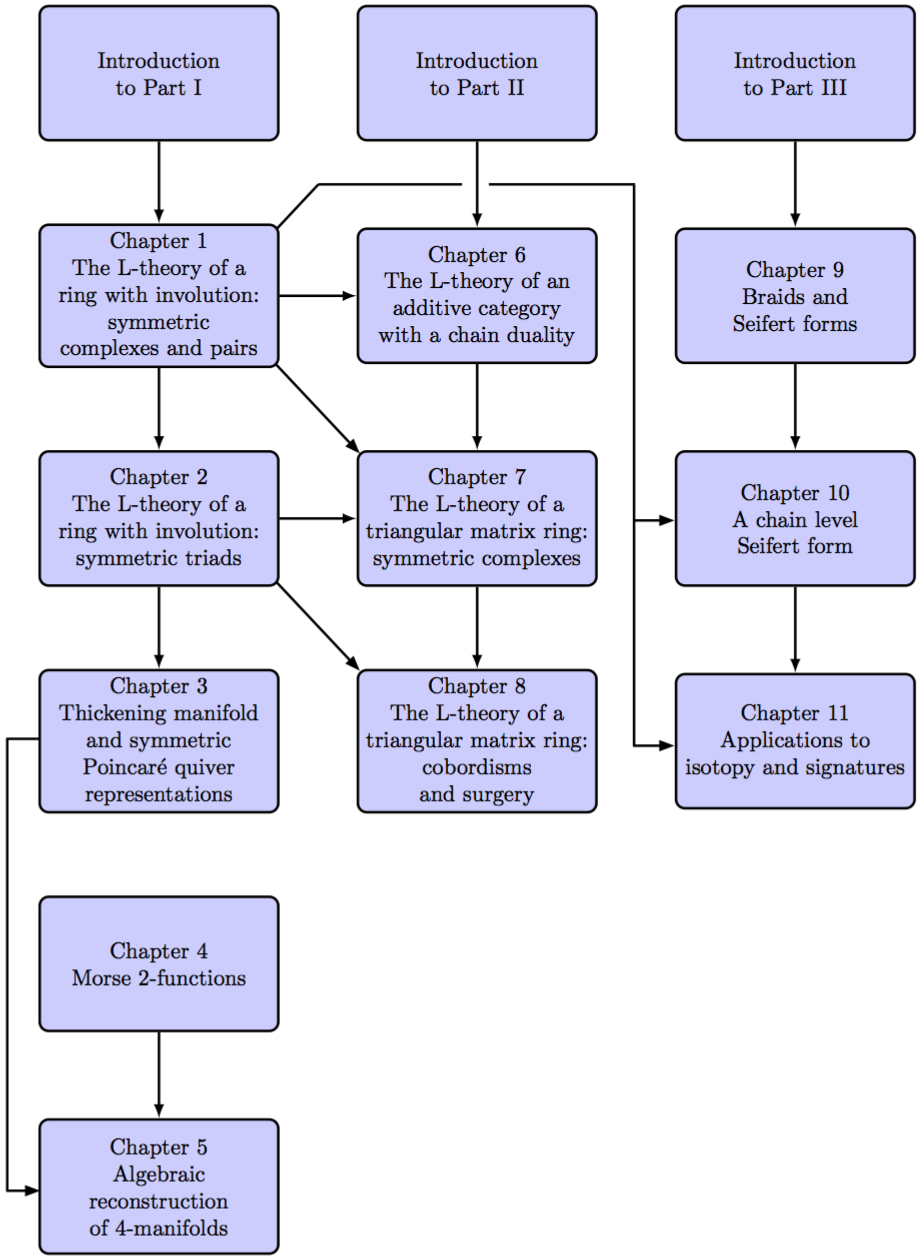
To the Edinburgh Surgery Theory study group I say thank you for the valuable knowledge I gained there and during the many surgery meetings we attended around Europe. Many thanks to Patrick Orson for discussions on algebraic surgery theory, to Maciej Borodzik for asking about quivers of symmetric complexes and to Étienne Ghys for suggesting the problem of finding a representation of the braid group by chain level Seifert forms.

Finally I would like to thank my parents and Hanyi, whose love and support has been with me every step of the way.

Organisation

This thesis is split into three parts. Each part has its own introduction and it is recommend to read the parts in order. The dependencies of the chapters in each of the parts is given by the following flow diagram.



Contents

Part I

Algebraic reconstruction of 4-manifolds

Introduction to Part I

Ranicki's algebraic theory of surgery [Ran80a],[Ran80b] is an algebraic model for geometric surgery theory in which manifold objects are modelled by chain complex objects with symmetric structures encoding various chain level dualities and symmetries.

An n-dimensional symmetric Poincaré complex (C, ϕ) over a ring with involution A is an algebraic model for a closed manifold. It consists of a finite dimensional chain complex C of finitely generated projective A-modules together with a symmetric structure $\phi = \{\phi_s | s \geq 0\}$ where $\phi_0 : C^{n-*} \to C$ is an abstract Poincaré duality and ϕ_{s+1} is a higher chain homotopy measuring the failure of ϕ_s to be symmetric. The symmetric construction associates to a commutative ring R and an oriented n-dimensional manifold M, an n-dimensional symmetric Poincaré complex $(C(M; R), \phi_M)$ such that if $[M] \in H_n(M; R)$ is the fundamental class of M determined by the orientation then $(\phi_M)_0 = [M] \cap - : C(M; R)^{n-*} \to C(M; R)$ is the Poincaré duality chain homotopy equivalence.

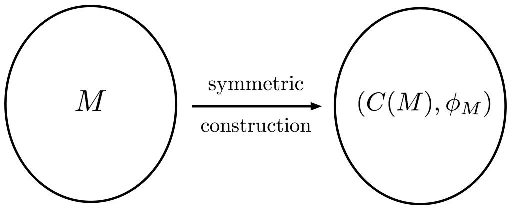


Figure 5: A schematic diagram from the symmetric construction.

The relative version of a symmetric Poincaré complex is a symmetric Poincaré pair $(f : C \to D, (\delta\phi, \phi))$. This is an algebraic model for a manifold with boundary and consists of a chain map $f : C \to D$ of finitely generated projective A-module chain complexes together with a relative symmetric structure $(\delta\phi, \phi)$ such that $(\delta\phi_0 \ \ f\phi_0) : \mathscr{C}(f)^{n+1-*} \to D$ is an abstract Poincaré-Lefschetz duality.

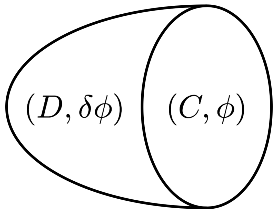


Figure 6: A schematic diagram for a symmetric pair.

The relative symmetric construction associates to a commutative ring R and an oriented $(n+$

1)-dimensional manifold with boundary (Σ, M), an $(n+1)$-dimensional symmetric Poincaré pair $(C(M;R) \to C(\Sigma;R), (\phi_\Sigma, \phi_M))$ such that if $[\Sigma] \in H_{n+1}(\Sigma, M; R)$ is the relative fundamental class then $((\phi_\Sigma)_0, i(\phi_M)_0) = [\Sigma] \cap - : C(\Sigma, M; R)^{n+1-*} \to C(\Sigma; R)$ is the Poincaré-Lefschetz duality chain homotopy equivalence.

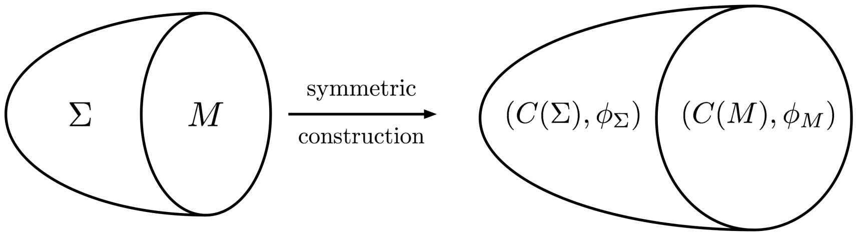


Figure 7: A schematic diagram for the relative symmetric construction.

A symmetric cobordism between two n-dimensional symmetric complexes (C, ϕ) and (C', ϕ') is a symmetric pair of the form $((f \ \ f') : C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi'))$ and is an algebraic model for a manifold cobordism. For a commutative ring R, the relative symmetric construction may be applied to an $(n+1)$-dimensional manifold cobordism $(W; M, M')$ to obtain an $(n+1)$-dimensional symmetric cobordism $(C(M;R) \oplus C(M';R) \to C(W;R), (\phi_W, \phi_M \oplus -\phi_M))$.

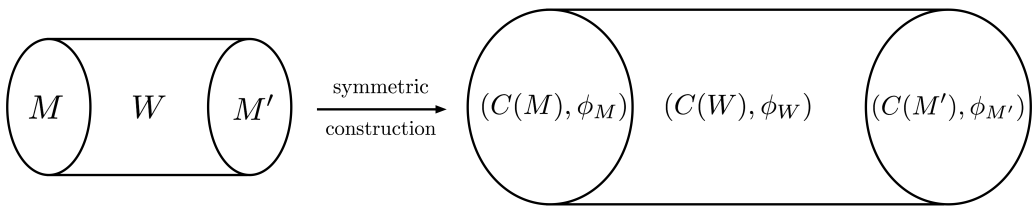


Figure 8: A schematic diagram for the relative symmetric construction applied to a manifold cobordism.

The standard operations which one can perform on manifolds, such as taking products and glueing adjoining cobordisms, also have algebraic models such that the symmetric construction commutes with these operations up to homotopy equivalence.

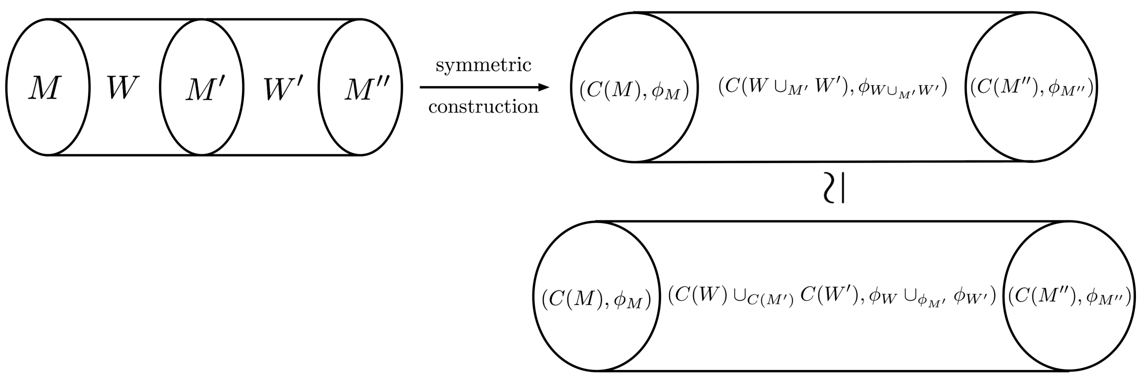


Figure 9: The effect of applying the symmetric construction to a union of adjoining manifold cobordisms.

A quiver $Q = (Q_0, Q_1; s, t : Q_1 \to Q_0)$ is a directed graph where each arrow $\alpha \in Q_1$ has a source vertex $s(\alpha) \in Q_0$ and a target vertex $t(\alpha) \in Q_0$. A representation of a quiver typically associates to each vertex $v \in Q_0$ a vector space and two each arrow $\alpha \in Q_1$ a linear map. We will work with n-dimensional oriented manifold (respectively n-dimensional symmetric Poincaré) representations where to each vertex we associate an n-dimensional oriented manifold M_v (respectively an n-dimensional symmetric Poincaré complex (C_v, ϕ_v)) and to each arrow α we associate an $(n+1)$-dimensional oriented manifold cobordism $(W_\alpha; M_{s(\alpha)}, M_{t(\alpha)})$ (respectively an $(n+1)$-dimensional symmetric cobordism $(C_{s(\alpha)} \oplus C_{t(\alpha)} \to D_\alpha, (\phi_\alpha, \phi_{s(\alpha)} \oplus -\phi_{t(\alpha)}))$. The manifold and symmetric Poincaré trinities of [BNR12a] are a special case of manifold and symmetric Poincaré quiver representations.

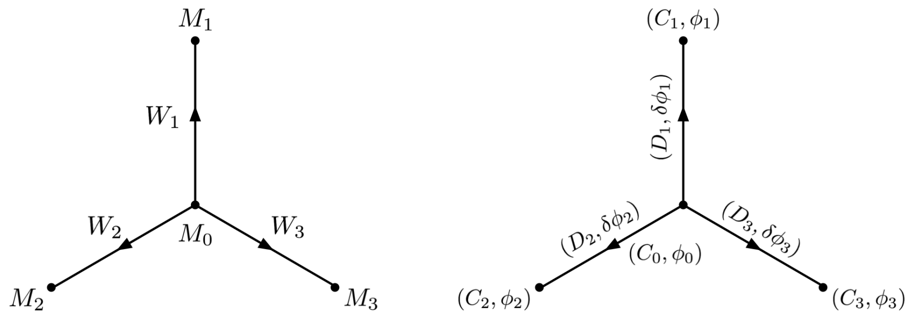


Figure 10: Representations of the trinity quiver.

We generalise the manifold and symmetric Poincaré trinity thickening operations of [BNR12a]

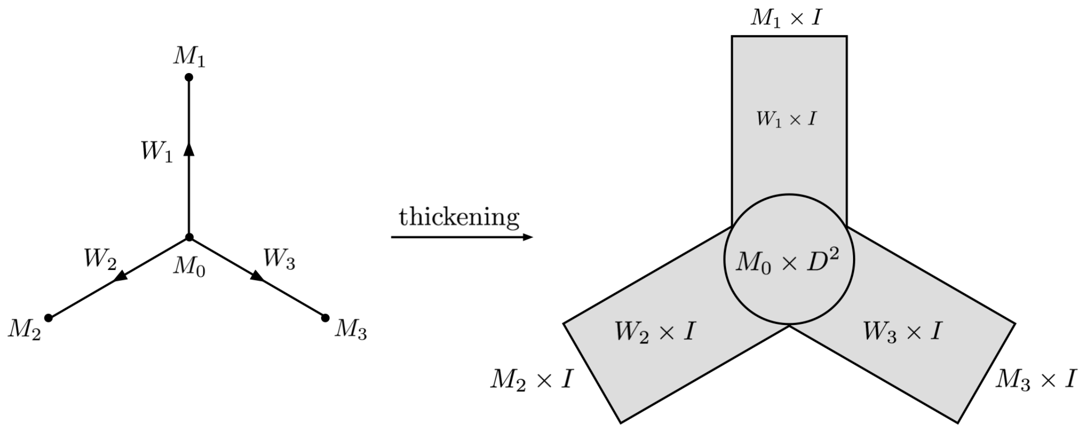


Figure 11: The manifold thickening operation

to thickening operations for oriented manifold and symmetric Poincaré representations of arbitrary quivers where we allow the thickening to be twisted by a self-homotopy equivalence of the data associated to the target vertex of each arrow. This yields:

Theorem 3.5.9. The symmetric construction commutes with the twisted thickening operations up to homotopy equivalence of the resulting symmetric pair, yielding a homotopy commutative diagram

$$\begin{array}{ccc}
(W_Q; M_Q, M'_Q) & \xrightarrow[\text{thickening}]{\text{twisted geometric}} & (\Omega, \partial\Omega) \\
\Big\downarrow {\scriptstyle \text{symmetric construction}} & & \Big\downarrow {\scriptstyle \text{symmetric construction}} \\
\begin{array}{c}(C(M_Q; R) \oplus C(M'_Q; R) \to C(W_Q; R), \\ (\phi_{W_Q}, \phi_{M_Q} \oplus -\phi_{M'_Q}))\end{array} & \xrightarrow[\text{thickening}]{\text{twisted algebraic}} & \begin{array}{c}(C(\partial\Omega; R) \to C(\Omega; R), (\phi_\Omega, \phi_{\partial\Omega})) \\ \wr\!\!\parallel \\ (\partial D \to D, (\phi_D, \phi_{\partial D}))\end{array}
\end{array}$$

A Morse 2-function is a smooth map $F : M^n \to \Sigma^2$ from a manifold to a surface which can be written locally as a generic homotopy of Morse functions $F(x) = (t, f_t(x))$. Each $f_t : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Morse function except at finitely many values of t where either two critical values cross or there is a birth-death singularity. Gay and Kirby [KG12] showed that if the fold curves of a Morse 2-function $F : M^4 \to \Sigma^2$ bound simply-connected regions and the fibres are connected, then the Morse 2-function determines a manifold representation of a quiver. One can then reconstruct M^4 up to diffeomorphism by thickening with a twist this representation and then glueing in disc neighbourhoods of cusps and crossings. We give an algebraic analogue of their result to show how, under the same hypotheses, a Morse 2-function $F : M^4 \to \Sigma^2$ can be used to reconstruct the symmetric Poincaré complex $(C(M; R), \phi_M)$ of M by thickening with a twist a symmetric Poincaré representation of a quiver and then glueing in the symmetric Poincaré pairs obtained by applying the symmetric construction to disk neighbourhoods of cusps and crossings. This also allows one to recover the signature of M. This yields:

Theorem 5.2.2. Let R be a commutative ring with identity. The symmetric Poincaré complex $(C(M; R), \phi_M)$ may be reconstructed up to homotopy equivalence by thickening with a twist a symmetric Poincaré representation induced from the 3-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q .

Theorem 5.2.3. In the case $R = \mathbb{Z}$ the signature of M may be recovered from the 3-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q and the twisted glueing data.

Part I is organised as follows.

In chapter 1 we recall from [Ran80a] the ϵ-symmetric complex, pair and cobordism objects which appear in the the L-theory of a ring with involution. We examine the symmetric construction and the glueing operation for adjoining ϵ-symmetric cobordisms to show that symmetric Poincaré complexes, pairs and cobordisms are algebraic models for closed manifolds, manifolds with boundary and cobordisms.

In chapter 2 we recall from [Ran81] the ϵ-symmetric triad objects and the triad definition of a homotopy equivalence of symmetric pairs which appear in the the L-theory of a ring with involution. We then examine a twisted glueing operation for ϵ-symmetric triads and show this is a model for the twisted glueing of manifolds with boundary and manifold triads.

In chapter 3 we extend the symmetric construction to a symmetric construction for an oriented manifold representation of a quiver where the vertices parametrise manifolds and the arrows parametrise cobordisms. This produces a symmetric Poincaré representation of a quiver where the vertices parametrise symmetric Poincaré complexes and the arrows parametrise symmetric Poincaré cobordisms. We also extend the definition of a symmetric pair to a symmetric pair with an ℓ-fold boundary splitting and show that this is an algebraic model for a manifold with boundary where the boundary can be written as a cyclic union of adjoining cobordisms. We then define algebraic thickening operations which are algebraic models for taking the product of a cobordism with an interval and for taking the product of a closed manifold with a disc D^2 where the boundary $S^1 = \partial D^2$ is split into ℓ pieces. We use the quiver symmetric construction together with the thickening operations to generalise the manifold and symmetric Poincaré trinity thickening operations of [BNR12a, p.44-46] to thickening operations for manifold and symmetric Poincaré representations of a quiver where parts of the data can be twisted by a self-homotopy equivalence. We then show that the twisted thickening operations commute with the symmetric construction up to homotopy equivalence.

In chapter 4 we examine Gay and Kirby's definitions of Morse 2-functions [KG13a] and of trisections of 4-manifolds [KG13b] as natural generalisations of Morse functions and Heegaard splittings of 3-manifolds. We use trisections to produce some examples of fold loci of Morse 2-functions.

In chapter 5 we give an algebraic analogue of the result of Gay and Kirby [KG12] to show how a Morse 2-function $F : M^4 \to S^2$ which has connected fibres and whose fold lines bound simply connected regions can be used to reconstruct the symmetric Poincaré complex $(C(M;R), \phi_M)$ of M.

Chapter 1

The L-theory of a ring with involution: symmetric complexes and pairs

In this chapter we recall from [Ran80a] the symmetric complex, pair and cobordism objects which appear in the the L-theory of a ring with involution. We examine the symmetric construction and the glueing operation for adjoining symmetric cobordisms to show that symmetric Poincaré complexes, pairs and cobordisms are algebraic models for closed manifolds, manifolds with boundary and manifold cobordisms.

1.1 Symmetric complexes

Definition 1.1.1. A *ring with involution* is a ring A with identity 1 together with a function $A \to A; a \mapsto \overline{a}$ such that

$$\overline{a+b} = \overline{a} + \overline{b},\ \ \overline{ab} = \overline{b} \cdot \overline{a},\ \ \overline{1} = 1,\ \ \overline{\overline{a}} = a \qquad (a, b \in A).$$

Example 1.1.2.

(i) Complex conjugation is an involution on $\mathbb{C}$.

(ii) The identity map of a commutative ring with identity is an involution.

From now on in Part I of this thesis let A denote a ring with involution and let all A-modules be left A-modules unless stated otherwise. Modules over a ring with involution have the following duals, transposes and tensor products.

Definition 1.1.3.

(i) The *dual* of an A-module M is the right A-module $M^* = \mathrm{Hom}_A(M, A)$ equipped with the scalar action

$$A \times M^* \to M^*; (a, f) \mapsto (x \mapsto f(x) \cdot \overline{a}).$$

(ii) The *dual* of an A-module morphism $f : M \to N$ is the A-module morphism

$$f^* : N^* \to M^*; g \mapsto (x \mapsto g(f(x))).$$

(iii) The *tensor product* of a right A-module M and a left A-module N is the $\mathbb{Z}$-module

$$M \otimes_A N = M \otimes_{\mathbb{Z}} N / \{xa \otimes y - x \otimes ay : x \in M, y \in N, a \in A\}.$$

(iv) The *transpose* of an A-module M is the right A-module M^t such that $M^t = M$ as an abelian group and M^t is equipped with the scalar action

$$M^t \times A \to M^t; (x, a) \mapsto \overline{a}x$$

such that if N is an A-module then

$$M^t \otimes_A N = M^t \otimes_{\mathbb{Z}} N / \{\overline{a}x \otimes y - x \otimes ay : x \in M, y \in N, a \in A\}.$$

(v) For A-modules M, N the *slant map* is the morphism

$$\backslash : M^t \otimes_A N \to \mathrm{Hom}_A(M^*, N); x \otimes y \mapsto (f \mapsto \overline{f(x)} \cdot y)$$

and is an isomorphism if M, N are finitely generated (f.g.) projective A-modules.

The tensor product and slant map have the following extensions to chain complexes over a ring with involution.

Definition 1.1.4. Let C, D be A-module chain complexes.

(i) The *tensor product* $C^t \otimes_A D$ is the $\mathbb{Z}$-module chain complex defined by

$$(C^t \otimes_A D)_r = \oplus_{p+q=r} C^t_p \otimes_A D_q$$

with differential

$$d_{C^t \otimes_A D} : (C^t \otimes D)_r \to (C^t \otimes_A D)_{r-1}; x \otimes y \mapsto x \otimes d_D(y) + (-)^q d_C(x) \otimes y \quad (x \in C^t_p, y \in D_q).$$

(ii) The *Hom* chain complex $\mathrm{Hom}_A(C, D)$ is the $\mathbb{Z}$-module chain complex by

$$\mathrm{Hom}_A(C, D)_r = \oplus_{q-p=r} \mathrm{Hom}_A(C_p, D_q)$$

with differential

$$d_{\mathrm{Hom}_A(C,D)} : \mathrm{Hom}_A(C, D)_r \to \mathrm{Hom}_A(C, D)_{r-1}; f \mapsto d_D f + (-)^q f d_C \quad (f \in \mathrm{Hom}_R(C_p, D_q)).$$

(iii) The *slant* map is the chain map

$$\backslash : C^t \otimes_A D \to \mathrm{Hom}_A(C^{-*}, D); x \otimes y \mapsto (f \mapsto \overline{f(x)} \cdot y)$$

and is an isomorphism if C is finite-dimensional. Here C^{-*} is the A-module chain complex defined by

$$C^{-*}_r = (C_{-r})^*, d_{C^{-*}} = d^*_C.$$

We now turn to chain complexes over a ring with involution. As we will later be working with chain complexes of manifolds which are homotopy equivalent to finite CW-complexes, it is useful to have a definition of the dimension of a chain complex which is only defined up to chain homotopy.

Definition 1.1.5. An A-module chain complex is *n-dimensional* ($n \in \mathbb{Z}_{\geq 0}$) if it is a chain complex of f.g. projective A-modules which is chain homotopy equivalent to a f.g. projective A-module chain complex of the form

$$C : \ldots \to 0 \to C_n \xrightarrow{d_C} C_{n-1} \xrightarrow{d_C} \ldots \xrightarrow{d_C} C_1 \xrightarrow{d_C} C_0 \to 0.$$

An A-module chain complex is *finite-dimensional* if it is n-dimensional for some $n \in \mathbb{Z}_{\geq 0}$.

The symmetric Q-groups of a finite-dimensional chain complex are defined in terms of the following $W^{\%}$ functor. The geometric motivation for this functor will become apparent in the proof of Theorem 1.1.11 where we discuss the symmetric construction.

Definition 1.1.6. ([Ran80a, Proposition 1.1]). Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$ resolution of $\mathbb{Z}$

$$W : \ldots \to W_3 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} W_2 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} W_1 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} W_0 = \mathbb{Z}[\mathbb{Z}_2]$$

and let C, D be finite-dimensional A-module chain complexes and let $\epsilon = \pm 1$.

(i) There is a $\mathbb{Z}_2$ action of T on $C^t \otimes_A C$ defined by

$$T_\epsilon(x \otimes y) = (-)^{pq} y \otimes \epsilon x \quad (x \in C^t_p, y \in C_q)$$

such that $C^t \otimes_A C$ is a finite-dimensional $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex.

(ii) The $\mathbb{Z}$-module chain complex $W^{\%}C = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C^t \otimes_A C)$ is such that under the slant isomorphism $\backslash : C^t \otimes_A C \cong \mathrm{Hom}_A(C^{-*}, C)$ a chain $\phi \in (W^{\%}C)_n$ can be identified with a collection of morphisms

$$\phi = \{\phi_s : C^{n-r+s} \to C_r | r \in \mathbb{Z}, s \geqslant 0\}$$

and the boundary $d_{W^{\%}C}\phi \in (W^{\%}C)_{n-1}$ may be identified with a collection of morphisms

$$d_{W^{\%}C}\phi = \{(d\phi)_s : C^{n-1-r+s} \to C_r | r \in \mathbb{Z}, s \geqslant 0\}$$

which satisfy

$$(d\phi)_s = d_C\phi_s + (-)^r \phi_s d^*_C + (-)^{n+s-1}(\phi_{s-1} + (-)^s T_\epsilon \phi_{s-1}) : C^{n-1-r+s} \to C_r \quad (r \in \mathbb{Z}, s \geqslant 0, \phi_{-1} = 0).$$

(iii) An A-module chain map $f : C \to D$ induces a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$f \otimes_A f : C^t \otimes_A C \to D^t \otimes_A D$$

and hence induces a $\mathbb{Z}$-module chain map

$$f^{\%} : W^{\%}C \to W^{\%}D; \quad \phi = \{\phi_s | s \geqslant 0\} \mapsto f^{\%}\phi = \{f\phi_s f^* | s \geqslant 0\}$$

such that a chain homotopy $k : f \simeq g : C \to D$ induces a chain homotopy

$$k^{\%} : f^{\%} \simeq g^{\%} : W^{\%}C \to W^{\%}D.$$

One can think of the chain complex $W^{\%}C = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C^t \otimes_A C)$ as the 'homotopy fixed points' of the involution T_ϵ.

Definition 1.1.7. Let C, D be finite-dimensional A-module chain complexes and let $\epsilon = \pm 1$.

(i) The *ϵ-symmetric Q-groups* of C are the $\mathbb{Z}$-module homology groups

$$Q^n(C, \epsilon) = H_n(W^{\%}(C)) \quad (n \in \mathbb{Z}).$$

(ii) The *morphism* of ϵ-symmetric Q-groups induced by a chain map $f : C \to D$ is the morphism

$$f^{\%} : Q^n(C, \epsilon) = H_n(W^{\%}C) \to Q^n(D, \epsilon) = H_n(W^{\%}D)$$

such that if f is a chain homotopy equivalence then $f^{\%}$ is an isomorphism.

Definition 1.1.8. ([Ran80a, p.102-3]). Let $\epsilon = \pm 1$.

(i) An *n-dimensional ϵ-symmetric complex* (C, ϕ) *over* A consists of an n-dimensional A-module chain complex C together with an element $\phi \in Q^n(C, \epsilon)$. In the case $\epsilon = 1$ we write $Q^n(C, 1) = Q^n(C)$ and we call a 1-symmetric complex a *symmetric complex*.

(ii) An n-dimensional ϵ-symmetric complex (C, ϕ) over A is *Poincaré* if the chain map $\phi_0 : C^{n-*} \to C$ is a chain homotopy equivalence.

(iii) A *morphism* $f : (C, \phi) \to (C', \phi')$ of n-dimensional ϵ-symmetric complexes over A is an A-module chain map $f : C \to C'$ such that $f^{\%}(\phi) = \phi' \in Q^n(C', \epsilon)$. A morphism $f : (C, \phi) \to (C', \phi')$ is a *homotopy equivalence* if $f : C \to C'$ is a chain homotopy equivalence.

Symmetric (Poincaré) complexes of dimension 0 are precisely (non-singular) symmetric forms.

Example 1.1.9. Let M be a f.g. projective A-module. A 0-dimensional ϵ-symmetric structure $\phi \in Q^0(M, \epsilon)$ is the same as a morphism $\phi_0 : M \to M^*$ which satisfies the ϵ-symmetry condition $\phi_0 = \epsilon\phi_0^* : M \to M^*$. It follows that (M, ϕ) is Poincaré if and only if $\phi_0 = \epsilon\phi_0^* : M \to M^*$ is an isomorphism, that is to say the ϵ-symmetric form (M, ϕ_0) is non-singular. If M' is another f.g projective A-module then a map of 0-dimensional ϵ-symmetric complexes $f : (M, \phi) \to (M', \phi')$ is the same as an A-module morphism $f : M \to M'$ which satisfies $\phi_0 = f^*\phi'_0 f : M \to M'$. In this case $f : (M, \phi) \to (M', \phi')$ is a homotopy equivalence if and only if $f : M \to M'$ is an A-module isomorphism.

For (finite-dimensional) A-module chain complexes C, C' we have the identity

$$(C \oplus C')^t \otimes_A (C \oplus C') = (C^t \otimes_A C) \oplus (C'^t \otimes_A C') \oplus (C^t \otimes_A C') \oplus (C'^t \otimes_A C)$$

and hence the symmetric Q groups fail to be additive under the direct sum of chain complexes. By examining the $\mathbb{Z}_2$ action we see that $W^{\%}(C \oplus C') = W^{\%}C \oplus W^{\%}C' \oplus (C \otimes_A C')$ and hence $Q^n(C \oplus C', \epsilon) = Q^n(C, \epsilon) \oplus Q^n(C', \epsilon) \oplus H_n(C^t \otimes_A C')$. This inclusion of Q-groups determines a direct sum operation.

Definition 1.1.10.

(i) The *direct sum* of n-dimensional ϵ-symmetric (Poincaré) complexes $(C, \phi \in Q^n(C, \epsilon)), (C', \phi' \in Q^n(C', \epsilon))$ over A is the n-dimensional ϵ-symmetric (Poincaré) complex over A

$$(C, \phi \in Q^n(C, \epsilon)) \oplus (C', \phi \in Q^n(C, \epsilon)') = (C \oplus C', \phi \oplus \phi' \in Q^n(C \oplus C', \epsilon))$$

determined by the inclusion

$$Q^n(C, \epsilon) \oplus Q^n(C', \epsilon) \hookrightarrow Q^n(C \oplus C', \epsilon).$$

(ii) The *zero* n-dimensional ϵ-symmetric Poincaré complex over A is $(0, 0 \in Q^n(0, \epsilon))$.

(iii) The *negative* of an n-dimensional ϵ-symmetric (Poincaré) complex $(C, \phi \in Q^n(C, \epsilon))$ over A is the n-dimensional ϵ-symmetric (Poincaré) complex over A

$$-(C, \phi \in Q^n(C, \epsilon)) = (C, -\phi \in Q^n(C, \epsilon)).$$

The most important symmetric complexes are those which arise from geometry. For a topological space X and a commutative ring R, the diagonal map $\Delta : X \to X \times X$ and the Eilenberg-Zilber chain homotopy equivalence $C(X \times X; R) \simeq C(X; R)^t \otimes_R C(X; R)$ may be used to produce an n-dimensional symmetric structure on the chain complex $C(X; R)$ from a homology class $[X] \in H_n(X; R)$.

Theorem 1.1.11. ([Ran80b, Proposition 1.2]). Let R be a commutative ring with identity and let X be a topological space whose singular chain complex $C(X; R)$ is of finite dimension (e.g. X is any space homotopy equivalent to a finite CW-complex).

(i) The symmetric construction is a chain map

$$\phi_X : C(X; R) \to W^{\%}C(X; R), \quad [X] \in C(X; R)_n \mapsto \phi_X([X]) \in (W^{\%}C(X; R))_n$$

which associates to each chain in $C(X; R)$ a natural chain homotopy class of R-module morphisms.

(ii) The induced homomorphisms in homology

$$\phi_X : H_*(X; R) \to Q^*(C(X; R)).$$

are such that a homology class $[X] \in H_n(X; R)$ determines an n-dimensional symmetric complex $(C(X; R), \phi_X([X]) \in Q^n(C(X; R))$ over R with the 0-dimensional part of the symmetric structure $\phi_X([X])$ given by the cap product with $[X]$

$$\phi_X([X])_0 = [X] \cap - : C(X; R)^{n-*} \to C(X; R).$$

(iii) The symmetric construction is natural in the sense that a map of spaces $f : X \to Y$ induces a commutative square of chain maps

$$\begin{array}{ccc} C(X;R) & \xrightarrow{\phi_X} & W^{\%}C(X;R) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f^{\%}} \\ C(Y;R) & \xrightarrow[\phi_Y]{} & W^{\%}C(Y;R) \end{array}$$

giving a commutative square

$$\begin{array}{ccc} H_*(X;R) & \xrightarrow{\phi_X} & Q^*(X;R) \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f^{\%}} \\ H_*(Y;R) & \xrightarrow[\phi_Y]{} & Q^*(Y;R) \end{array}$$

such that for a homology class $[X] \in H_n(X;R)$

$$(C(Y;R); f^{\%}(\phi_X([X]))) = (C(Y;R), \phi_Y(f_*([X]))).$$

Proof. We sketch the definition of ϕ_X in the case $R = \mathbb{Z}$. Recall that by the Eilenberg-Zilber theorem there is a natural chain choice of chain homotopy equivalence

$$\theta : C(X \times X;\mathbb{Z}) \simeq C(X;\mathbb{Z})^t \otimes_{\mathbb{Z}} C(X;\mathbb{Z})$$

such that θ is unique up to natural chain homotopy equivalence, see [Bre97, p.316]. If $\Delta : X \to X \times X$ is the diagonal map then there is a natural morphism

$$\Delta_0 : C(X;\mathbb{Z}) \xrightarrow{\Delta} C(X \times X;\mathbb{Z}) \xrightarrow{\theta} C(X;\mathbb{Z})^t \otimes_{\mathbb{Z}} C(X;\mathbb{Z}).$$

The composition with the slant map

$$C(X;\mathbb{Z}) \xrightarrow{\Delta_0} C(X;\mathbb{Z})^t \otimes_{\mathbb{Z}} C(X;\mathbb{Z}) \xrightarrow{\backslash} \mathrm{Hom}_{\mathbb{Z}}(C(X;\mathbb{Z})^{-*}, C(X;\mathbb{Z}))$$

sends a cycle $x \in C_n(X;\mathbb{Z})$ to the chain map

$$\phi_0 = \backslash\Delta_0(x) = x \cap - : C^{n-*}(X;\mathbb{Z}) \to C(X;\mathbb{Z}).$$

Since the map $T_\epsilon : C(X;\mathbb{Z}) \otimes_{\mathbb{Z}} C(X;\mathbb{Z}) \to C(X;\mathbb{Z}) \otimes_{\mathbb{Z}} C(X;\mathbb{Z})$ is an involution it follows that the composition $T_\epsilon\theta : C(X \times X;\mathbb{Z}) \to C(X;\mathbb{Z}) \otimes_{\mathbb{Z}} C(X;\mathbb{Z})$ is also a chain homotopy equivalence and hence there is a natural chain homotopy $\theta \simeq T_\epsilon\theta$. This determines a degree 1 chain map

$$\Delta_1 : C(X;\mathbb{Z})_* \to (C(X;\mathbb{Z}) \otimes_{\mathbb{Z}} C(X;\mathbb{Z}))_{*+1}.$$

providing a chain homotopy $\Delta_1 : \Delta_0 \simeq T_\epsilon\Delta_0$ which measures the failure of Δ_0 to be symmetric. If $x \in C_n(X;\mathbb{Z})$ is a cycle then the map

$$\phi_1 = \backslash\Delta_1(x) : C^{n+1-*}(X;\mathbb{Z}) \to C(X;\mathbb{Z})$$

is a chain homotopy between ϕ_0 and $T_\epsilon \phi_0$ so that

$$d_{C(X;\mathbb{Z})}\phi_1 + (-1)^r \phi_1 d^*_{C(X;\mathbb{Z})} + (-1)^n(\phi_0 - T_\epsilon \phi_0) = 0 : C(X;\mathbb{Z})^{n-*} \to C(X;\mathbb{Z}).$$

This process may be iterated to obtain a sequence $\Delta_i : C(X;\mathbb{Z})_* \to C(X;\mathbb{Z})^t \otimes C(X;\mathbb{Z})_{*+i}$ of degree i chain maps which satisfy the relation

$$d\Delta_{i+1} + (-)^i \Delta_{i+1} d = T_\epsilon \Delta_i + (-)^{i+1}\Delta_i \quad (i \geq 0)$$

and we can think of Δ_{i+1} as a measure of the failure of Δ_i to be symmetric. This sequence may be expressed as a natural degree 0 chain map

$$\Delta_X : W \otimes C(X;\mathbb{Z}) \to C(X;\mathbb{Z}) \otimes C(X;\mathbb{Z})$$

which has adjoint

$$\Delta_X : C(X;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(X;\mathbb{Z}) \otimes C(X;\mathbb{Z})) = W^{\%}(C(X;\mathbb{Z}))$$

such that the image of a cycle $x \in C_n(X;\mathbb{Z})$ is an n-dimensional symmetric structure with 0-dimensional component equal to $x \cap -$. □

The maps $\Delta_i : C(X;R)_* \to C(X;R)^t \otimes_R C(X;R)_{*+i}$ encode higher level information about the intersection properties of X. The cup product of two cocycles $x \in C^p(X;R), y \in C^p(X;R)$ may be expressed as $x \cup y = \Delta_0^*(x \otimes y)$ and the chain homotopy $\Delta_1 : \Delta_0 \simeq T\Delta_0$ expresses the failure of the cup product to commute on the cochain level. In the case $R = \mathbb{Z}_2$ the ith Steenrod square may be expressed as

$$Sq^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2); \quad x \mapsto \Delta^*_{i-n}(x \otimes x).$$

See [Bre97, chapter 4.16] for more details.

Example 1.1.12. Let R be a commutative ring with identity and let M be a closed, oriented n-dimensional manifold with fundamental class $[M] \in H_n(M;R)$ determined by the orientation. Applying the symmetric construction to $(M, [M] \in H_n(M;R))$ produces an n-dimensional symmetric complex $(C(M;R), \phi_M([M]))$ over R. The chain map of free R-module chain complexes

$$(\phi_M[M])_0 = [M] \cap - : C(M;R)^{n-*} \to C(M;R)$$

is a chain homotopy equivalence since it induces the Poincaré duality isomorphisms

$$[M] \cap - : H^{n-*}(M;R) \to H(M;R)$$

and hence $(C(M;R), \phi_M([M]))$ is Poincaré . Applying the symmetric construction to $(M, -[M] \in H_n(M;R))$ produces the n-dimensional symmetric Poincaré complex

$$(C(M;R), \phi_M(-[M])) = (C(M), -\phi_M([M])) = -(C(M), \phi_M([M])).$$

From now on we denote $\phi_M([M])$ by ϕ_M and we think of symmetric (Poincaré) complexes as algebraic models of closed (orientable) manifolds.

In Section 2.2 we will apply a relative version of the symmetric construction to a manifold cobordism to produce a symmetric cobordism. It is first necessary to understand the symmetric construction for a disjoint union.

Proposition 1.1.13. ([BNR12a, Proposition 4.4.3]). Let R be a commutative ring with identity and let X, Y be topological spaces whose singular chain complexes $C(X;R), C(Y;R)$ are of finite dimension (e.g. X, Y are any spaces homotopy equivalent to finite CW-complexes). The symmetric construction on $X \sqcup Y$

$$\phi_{X\sqcup Y} : H_*(X \sqcup Y; R) \to Q^*(C(X \sqcup Y); R)$$

is given by the composition

$$H_*(X\sqcup Y;R) = H_*(X;R)\oplus H_*(Y;R) \xrightarrow{\phi_X \oplus \phi_Y} Q^*(C(X;R))\oplus Q^*(C(Y;R)) \hookrightarrow Q^*(C(X;R)\oplus C(Y;R)).$$

The behaviour of the symmetric construction on the boundary of a manifold cobordism is as follows.

Example 1.1.14. Let R be a commutative ring with identity and let M and M' be disjoint closed, oriented n-dimensional manifolds with fundamental classes $[M] \in H_n(M;R)$ and $[M'] \in H_n(M';R)$. The disjoint union $M \sqcup M'$ is a closed, oriented n-dimensional manifold with fundamental class $([M],[M']) \in H_n(M \sqcup M';R) = H_n(M;R) \oplus H_n(M';R)$. The symmetric construction applied to $(M,[M]),(M',[M'])$ and $(M \sqcup M', ([M],[M']))$ produces three n-dimensional symmetric Poincaré complexes over R

$$\begin{aligned}
&(C(M;R), \phi_M \in Q^n(C(M;R)))\\
&(C(M';R), \phi_{M'} \in Q^n(C(M';R)))\\
&(C(M \sqcup M';R), \phi_{M\sqcup M'} \in Q^n(C(M \sqcup M';R)))
\end{aligned}$$

which satisfy

$$\begin{aligned}
&(C(M \sqcup M';R), \phi_{M\sqcup -M'} \in Q^n(C(M \sqcup -M';R)))\\
=&(C(M;R) \oplus C(M';R), \phi_{M\sqcup -M'} \in Q^n(C(M;R) \oplus C(M';R)))\\
=&(C(M;R) \oplus C(M';R), \phi_M \oplus -\phi_{M'} \in Q^n(C(M;R)) \oplus Q^n(C(M';R))\\
=&(C(M;R), \phi_M \in Q^n(C(M;R))) \oplus (C(M';R), -\phi_{M'} \in Q^n(C(M';R)))\\
=&(C(M;R), \phi_M \in Q^n(C(M;R))) \oplus -(C(M';R), \phi_{M'} \in Q^n(C(M';R))).
\end{aligned}$$

1.2 Symmetric pairs

Symmetric pairs are relative versions of symmetric complexes and are algebraic models of manifolds with boundary.

Definition 1.2.1. The *algebraic mapping cone* of an A-module chain map $f : C \to D$ is the A-module chain complex $\mathscr{C}(f)$ defined by

$$d_{\mathscr{C}(f)} = \begin{pmatrix} d_D & (-)^{n-1} f \\ 0 & 0 \end{pmatrix} : \mathscr{C}(f)_n = D_n \oplus C_{n-1} \to \mathscr{C}(f)_{n-1} = D_{n-1} \oplus C_{n-2} \quad (n \in \mathbb{Z})$$

with homology groups

$$H_n(f) = H_n(\mathscr{C}(f)) \quad (n \in \mathbb{Z}).$$

Example 1.2.2. Let R be a commutative ring and let $f : X \to Y$ be a cellular map of CW-complexes with geometric mapping cone $\mathscr{C}^{geo}(f)$. Cohen [Coh73, §3.9] showed that there is a chain homotopy equivalence $C(\mathscr{C}^{geo}(f); R) \simeq \mathscr{C}(f : C(X; R) \to C(Y; R))$ so we can think of algebraic mapping cones as a model for geometric mappings cones.

Definition 1.2.3. The *relative ϵ-symmetric Q-groups* of a chain map $f : C \to D$ of finite-dimensional A-module complexes are the relative $\mathbb{Z}$-module homology groups

$$Q^n(f, \epsilon) = H_n(\mathscr{C}(f^{\%} : W^{\%}C \to W^{\%}D)) \quad (n \in \mathbb{Z}).$$

The following long exact sequence of Q-groups is not needed for Part I of the thesis but will be used in Part II of the thesis.

Proposition 1.2.4. The relative ϵ-symmetric Q-groups of a chain map $f : C \to D$ of finite-dimensional A-module complexes fit into a long exact sequence of ϵ-symmetric Q-groups

$$\dots \to Q^{n+1}(C, \epsilon) \xrightarrow{f^{\%}} Q^{n+1}(D, \epsilon) \to Q^{n+1}(f, \epsilon) \to Q^n(C, \epsilon) \xrightarrow{f^{\%}} Q^n(D, \epsilon) \to \dots$$

with morphisms

$$Q^{n+1}(D, \epsilon) \to Q^{n+1}(f, \epsilon); \quad \delta\phi \mapsto (\delta\phi, 0)$$
$$Q^{n+1}(f, \epsilon) \to Q^n(C, \epsilon); \quad (\delta\phi, \phi) \mapsto \phi.$$

Proof. The A-module chain map $f : C \to D$ induces a $\mathbb{Z}$-module chain map $f^{\%} : W^{\%}C \to W^{\%}D$. The algebraic mapping cone $\mathscr{C}(f^{\%})$ determines a short exact sequence of $\mathbb{Z}$-module chain complexes

$$0 \to (W^{\%}D)_* \to \mathscr{C}(f^{\%})_* \to (W^{\%}C)_{*-1} \to 0$$

which induces a long exact sequence of homology groups

$$\dots \to Q^{n+1}(C, \epsilon) \xrightarrow{f^{\%}} Q^{n+1}(D, \epsilon) \to Q^n(f, \epsilon) \to Q^n(C, \epsilon) \xrightarrow{f^{\%}} Q^n(D, \epsilon) \to \dots$$

□

Definition 1.2.5.

(i) An $(n+1)$-*dimensional ϵ-symmetric pair* over A

$$(f : C \to D, (\delta\phi, \phi) \in Q^{n+1}(f, \epsilon))$$

consists of chain map $f : C \to D$ from an n-dimensional A-module chain complex C to an $(n+1)$-dimensional A-module chain complex D together with a cycle

$$(\delta\phi, \phi) \in \mathscr{C}(f^{\%} : W^{\%}C \to W^{\%}D)_{n+1}.$$

In the case $\epsilon = 1$ we write $Q^{n+1}(f, 1) = Q^{n+1}(f)$ and we call a 1-symmetric pair a *symmetric pair.*

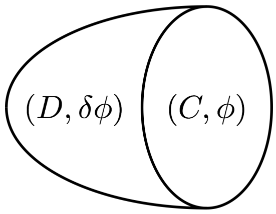


Figure 12: A schematic diagram for an ϵ-symmetric pair.

(ii) The *boundary* of an $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi, \phi) \in Q^{n+1}(f, \epsilon))$ over A is the n-dimensional ϵ-symmetric complex over A

$$\partial(f : C \to D, (\delta\phi, \phi) \in Q^{n+1}(f)) = (C, \phi \in Q^n(C, \epsilon)).$$

(iii) An $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi, \phi))$ over A is *Poincaré* if the A-module chain map

$$\begin{pmatrix} \delta\phi_0 & f\phi_0 \end{pmatrix} : \mathscr{C}(f)^{n+1-*} \to D$$

is a chain homotopy equivalence.

The symmetric construction for a topological space X extends to a relative symmetric construction which produces a symmetric pair from a map of topological spaces $f : X \to Y$ and a relative homology class $[Z] \in H_{n+1}(f; R)$.

Theorem 1.2.6. ([Ran80b, Proposition 6.1]). Let R be a commutative ring with identity and let X, Y be topological spaces with singular chain complexes $C(X; R), C(Y; R)$ of finite dimension (e.g. X, Y are any spaces homotopy equivalent to finite CW-complexes).

(i) By the naturality of the symmetric construction a map $f : X \to Y$ induces a natural chain homotopy class of chain maps

$$\phi_f : \mathscr{C}(f : C(X; R) \to C(Y; R)) \to \mathscr{C}(f^{\%} : W^{\%}C(X; R) \to W^{\%}C(Y; R)).$$

(ii) The induced morphisms in homology

$$\phi_f : H_*(f; R) \to Q^*(f), \quad [Z] \in H_{n+1}(f; R) \mapsto \phi_f([Z]) \in Q^{n+1}(f)$$

determine a morphism of long exact sequences

$$\begin{array}{ccccccccc}
\dots \longrightarrow & H_{n+1}(Y; R) & \longrightarrow & H_{n+1}(f) & \xrightarrow{\partial} & H_n(X; R) & \longrightarrow & H_n(Y; R) & \longrightarrow \dots \\
 & \downarrow{\scriptstyle \phi_Y} & & \downarrow{\scriptstyle \phi_f} & & \downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_Y} & \\
\dots \longrightarrow & Q^{n+1}(C(Y; R)) & \longrightarrow & Q^{n+1}(f) & \longrightarrow & Q^n(C(X; R)) & \longrightarrow & Q^n(C(Y; R)) & \longrightarrow \dots
\end{array}$$

such that for each homology class $[Z] \in H_{n+1}(f)$ there is an $(n+1)$-dimensional symmetric pair over R

$$(f : C(X; R) \to C(Y; R), \phi_f([Z]) \in Q^{n+1}(f))$$

with the 0-dimensional component of $\phi_f([Z]) \in Q^{n+1}(f)$ given by the cap product

$$\phi_f([Z])_0 = [Z] \cap - : \mathscr{C}(f)^{n+1-*} \to C(Y; R).$$

(iii) The boundary of the $(n+1)$-dimensional symmetric pair over R

$$(f : C(X;R) \to C(Y;R), \phi_f([Z]) \in Q^{n+1}(f))$$

is the n-dimensional symmetric complex over R

$$(C(X;R), \phi_X([X]) \in Q^n(C(X;R)), \quad [X] = \partial[Z] \in H_n(X;R))$$

obtained by applying the symmetric construction to $(X, [X] \in H_n(X;R))$.

Our interest lies in the case where (Y, X) is an oriented manifold with boundary and the map $f : X \to Y$ is the inclusion.

Example 1.2.7. Let R be a commutative ring with identity, let (Σ, M) be an oriented $(n+1)$-dimensional manifold with boundary and let $i : M \to \Sigma$ denote the inclusion so that $H_{n+1}(i) = H_{n+1}(\Sigma, M; R)$. Let $[\Sigma] \in H_{n+1}(\Sigma, M; R)$ and $[M] \in H_n(M;R)$ be the fundamental classes determined by the orientations of Σ and M so that $[M] = \partial[\Sigma] \in H_n(M;R)$. The symmetric construction applied to $(i : M \to \Sigma, [\Sigma] \in H_{n+1}(\Sigma, M; R))$ produces an $(n+1)$-dimensional symmetric pair $(i : C(M;R) \to C(\Sigma;R), \phi_i([\Sigma]) \in Q^{n+1}(i))$ which is Poincaré since the 0-dimensional component of $\phi_i([W])$ is given by the chain homotopy equivalence

$$\phi_i([\Sigma])_0 = [\Sigma] \cap - : C(\Sigma, M; R)^{n+1-*} \to C(\Sigma; R)$$

which induces the Poincaré-Lefschetz duality isomorphisms

$$\phi_i([\Sigma])_0 = [\Sigma] \cap - : H(\Sigma, M; R)^{n+1-*} \to H(\Sigma; R).$$

The boundary of the $(n+1)$-dimensional symmetric Poincaré pair $(i : C(M;R) \to C(\Sigma;R), \phi_i([\Sigma]) \in Q^{n+1}(i))$ is the n-dimensional symmetric Poincaré complex $(C(M;R), \phi_M([M]))$ obtained by applying the symmetric construction to $(M, [M] \in H_n(M;R))$. From now on we write

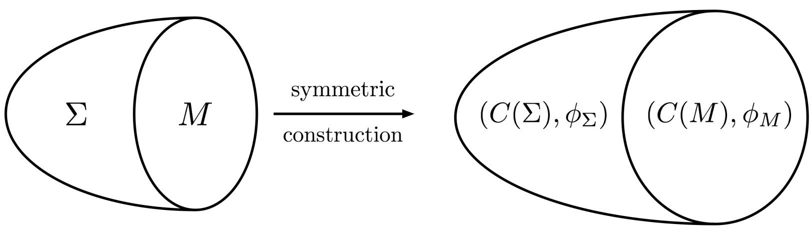


Figure 13: A schematic diagram for the passage from a manifold with boundary to a symmetric Poincaré pair.

$\phi_i([\Sigma]) = (\phi_\Sigma, \phi_M) \in Q^{n+1}(i)$ and we think of symmetric (Poincaré) pairs as algebraic models of (orientable) manifolds with boundary.

1.3 Symmetric cobordisms and unions

Symmetric cobordisms are algebraic models of manifold cobordisms with a glueing operation which models the glueing of manifold cobordisms.

Definition 1.3.1. The *union* of A-module chain complexes D, D' along A-module chain maps $f : C \to D, f' : C \to D'$ is the A-module chain complex

$$D \cup_C D' = \mathscr{C}\left(\begin{pmatrix} f \\ f' \end{pmatrix} : C \to D \oplus D'\right)$$

with differential

$$d_{D \cup_C D'} = \begin{pmatrix} d_D & (-)^{r-1} f & 0 \\ 0 & d_C & 0 \\ 0 & (-)^{r-1} f' & d_{D'} \end{pmatrix}$$
$$: (D \cup_C D')_r = D_r \oplus C_{r-1} \oplus D'_r \to (D \cup_C D')_{r-1} = D_{r-1} \oplus C_{r-2} \oplus D'_{r-1} \quad (r \in \mathbb{Z}).$$

Example 1.3.2. Let R be a commutative ring with identity and let X be a topological space with subsets $X_1, X_2 \subset X$ whose interiors cover X. Let $C(X_1 + X_2; R)$ denote the subcomplex of $C(X; R)$ consisting of sums of singular chains in X_1 and singular chains in X_2. Let $i_1 : X_1 \cap X_2 \to X_1$ and $i_2 : X_1 \cap X_2 \to X_2$ be the geometric inclusion maps and $j_1 : C(X_1; R) \to C(X_1 + X_2; R)$ and $j_2 : C(X_2; R) \to C(X_1 + X_2; R)$ be the algebraic inclusion maps. There is a short exact sequence of R-module chain complexes

$$0 \to C(X_1 \cap X_2; R) \xrightarrow{\begin{pmatrix} i_1 \\ i_2 \end{pmatrix}} C(X_1; R) \oplus C(X_2; R) \xrightarrow{\begin{pmatrix} j_1 & -j_2 \end{pmatrix}} C(X_1 + X_2; R) \to 0$$

with chain homotopy equivalences

$$C(X; R) \simeq C(X_1 + X_2; R) \simeq C(X_1; R) \cup_{C(X_1 \cap X_2; R)} C(X_2; R).$$

This shows that up to chain homotopy equivalence the algebraic union of chain complexes is an algebraic model for a geometric union of spaces.

A symmetric cobordism is a symmetric pair where the boundary is split into two disjoint pieces, just as for manifolds. The above glueing construction may be used to glue adjoining symmetric cobordisms along the common component of their boundaries.

Definition 1.3.3. ([Ran80a, p.135]).

(i) An *ϵ-symmetric cobordism* between two n-dimensional ϵ-symmetric Poincaré complexes $(C, \phi), (C', \phi')$ over A is an $(n+1)$-dimensional ϵ-symmetric Poincaré pair over A of the form

$$((f \;\; f') : C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi') \in Q^{n+1}((f \;\; f'), \epsilon)).$$

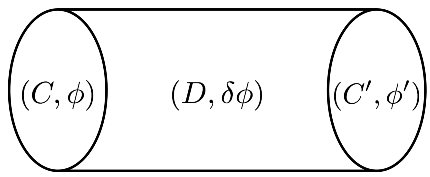


Figure 14: A schematic diagram for an ϵ-symmetric cobordism.

(ii) The *union* of adjoining $(n+1)$-dimensional ϵ-symmetric cobordisms over A

$$c = ((f_C \;\; f_{C'}) : C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi') \in Q^{n+1}((f_C \;\; f_{C'}), \epsilon))$$
$$c' = ((f'_{C'} \;\; f'_{C''}) : C' \oplus C'' \to D', (\delta\phi', \phi' \oplus -\phi'') \in Q^{n+1}(f'_{C'} \;\; f'_{C''}), \epsilon)$$

is the $(n+1)$-dimensional ϵ-symmetric cobordism over A

$$c \cup c' = ((f''_C \;\; f''_{C''}) : C \oplus C'' \to D'', (\delta\phi'', \phi \oplus -\phi'') \in Q^{n+1}((f''_C \;\; f''_{C''}), \epsilon))$$

with D'' the A-module chain complex

$$D'' = D \cup_{C'} D' = \mathscr{C}\left(\begin{pmatrix} f_{C'} \\ f'_{C'} \end{pmatrix} : C' \to D \oplus D'\right)$$

with differential

$$d_{D''} = \begin{pmatrix} d_D & (-)^{r-1} f_{C'} & 0 \\ 0 & d_{C'} & 0 \\ 0 & (-)^{r-1} f'_{C'} & d_{D'} \end{pmatrix}$$
$$: D''_r = D_r \oplus C'_{r-1} \oplus D'_r \to D''_{r-1} = D_{r-1} \oplus C'_{r-2} \oplus D'_{r-1} \qquad (r \in \mathbb{Z})$$

and A-module chain maps

$$f''_C = \begin{pmatrix} f_C \\ 0 \\ 0 \end{pmatrix} : C'_r \to D''_r = D_r \oplus C'_{r-1} \oplus D'_r \qquad (r \in \mathbb{Z})$$
$$f''_{C''} = \begin{pmatrix} 0 \\ 0 \\ f'_{C''} \end{pmatrix} : C''_r \to D''_r = D_r \oplus C'_{r-1} \oplus D'_r \qquad (r \in \mathbb{Z})$$

and the $\delta\phi''$ part of the relative symmetric structure $(\delta\phi'', \phi \oplus -\phi'')$ given by

$$\delta\phi''_s = \begin{pmatrix} \delta\phi_s & 0 & 0 \\ (-)^{n-r}\phi'_s f^*_{C'} & (-)^{n-r+s+1} T_\epsilon(\phi'_{s-1}) & 0 \\ 0 & (-)^s f'_{C'}\phi'_s & \delta\phi'_s \end{pmatrix}$$
$$: D''^{n+1-r+s} = D^{n+1-r+s} \oplus C'^{n-r+s} \oplus D'^{n+1-r+s} \to D''_r = D_r \oplus C'_{r-1} \oplus D'_r$$
$$(r \in \mathbb{Z}, s \geqslant 0, \phi'_{-1} = 0)$$

and from now on we denote $\delta\phi'' = \delta\phi \cup_{\phi'} \delta\phi'$ so that $\delta\phi''$ is obtained by glueing $\delta\phi$ and $\delta\phi'$ over ϕ'.

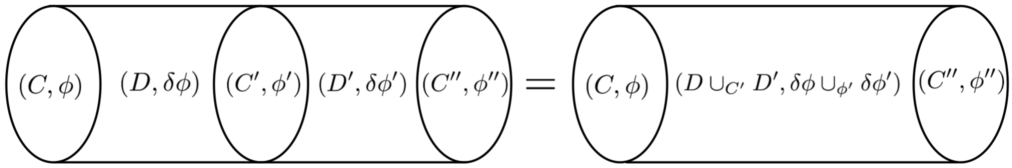


Figure 15: A schematic glueing diagram for adjoining ϵ-symmetric cobordisms.

Example 1.3.4. Let R be a commutative ring with identity and let $(W;M,M')$ be an oriented $(n+1)$-dimensional cobordism of closed, oriented n-dimensional manifolds M,M' with $\partial W = M\sqcup -M'$. Let $[W]\in H_{n+1}(W,M\sqcup M';R),[M]\in H_n(M;R),[M']\in H_n(M';R)$ be the fundamental classes determined by the orientation with

$$\partial[W] = ([M],-[M'])\in H_n(M;R)\oplus H_n(M';R) = H_n(M\sqcup M';R).$$

Example 1.1.14 and Example 1.2.7 imply that applying the symmetric construction to

$$(i = i_M\sqcup i_{M'} : \partial W = M\sqcup M\to W,[W]\in H_{n+1}(W,M\sqcup M';R))$$

produces an $(n+1)$-dimensional symmetric Poincaré pair over R

$$\begin{aligned}&(i: C(\partial W;R)\to C(W;R),(\phi_W,\phi_{\partial W})\in Q^{n+1}(i))\\ =&((i_M\ \ i_{M'}): C(M;R)\oplus C(M';R)\to C(W;R),(\phi_W,\phi_M\oplus -\phi_{M'})\in Q^{n+1}(i_M\ \ i_{M'}))\end{aligned}$$

which is a cobordism between the n-dimensional symmetric Poincaré complexes $(C(M;R),\phi_M)$ and $(C(M';R),\phi_{M'})$.

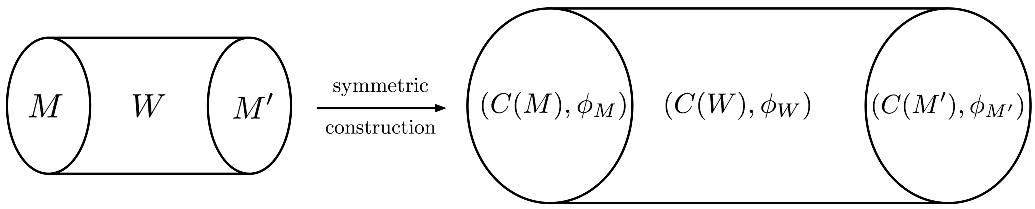


Figure 16: A schematic diagram for the passage from a cobordism of manifolds to a symmetric Poincaré cobordism.

We will examine the effect of applying the symmetric construction to a union of adjoining cobordisms $(W;M,M')\cup(W';M',M'')$ in Chapter 2 once we have established the notions of symmetric triads and of homotopy equivalences of symmetric pairs.

The cobordism of symmetric Poincaré complexes is an equivalence relation, just as for manifolds.

Lemma 1.3.5. Cobordism is an equivalence relation on n-dimensional ϵ-symmetric Poincaré complexes over A.

Proof. The identity map $1:(C,\phi)\to(C,\phi)$ determines an ϵ-symmetric cobordism

$$((1\ \ 1): C\oplus C\to C,(0,\phi\oplus-\phi)\in Q^{n+1}(f\ \ 1))$$

so we have reflexivity. An ϵ-symmetric cobordism

$$((f\ \ f') : C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi') \in Q^{n+1}(f\ \ f'))$$

between (C, ϕ) and (C', ϕ') determines an ϵ-symmetric cobordism

$$((f'\ \ f) : C' \oplus C \to D, (\delta\phi, \phi' \oplus -\phi) \in Q^{n+1}(f'\ \ f))$$

between (C', ϕ') and (C, ϕ) so this verifies symmetry. The glueing construction for adjoining ϵ-symmetric cobordisms then verifies transitivity. □

Definition 1.3.6. The *n-dimensional ϵ-symmetric L-group* $L^n(A, \epsilon)$ $(n \geqslant 0)$ of a ring A with involution is the abelian group of ϵ-symmetric cobordism classes of n-dimensional ϵ-symmetric Poincaré complexes over A with addition

$$(C, \phi \in Q^n(C, \epsilon)) + (C', \phi' \in Q^n(C', \epsilon)) = (C \oplus C', \phi \oplus \phi' \in Q^n(C \oplus C', \epsilon)) \in L^n(A, \epsilon)$$

and zero element $(0, 0 \in Q^n(0, \epsilon)) \in L^n(A, \epsilon)$ and additive inverses

$$-(C, \phi \in Q^n(C, \epsilon)) = (C, -\phi \in Q^n(C, \epsilon)) \in L^n(A, \epsilon).$$

In the case $\epsilon = 1$ we write $L^n(A, 1) = L^n(A)$ and we call the 1-symmetric L-groups of A the *symmetric L-groups of A*.

The symmetric L-groups of a ring with involution are a chain complex generalisation of the Witt group of symmetric bilinear forms. Recall from Example 1.1.9 that a symmetric Poincaré complex of dimension 0 is the same as a non-singular symmetric form. For a ring A with involution, one can identify [Ran80a, Proposition 5.1] the symmetric L-group $L^0(A)$ with the abelian group of equivalence classes of non-singular symmetric forms $(M, \lambda : M \otimes M \to A)$ over A. The equivalence relation identifies two non-singular symmetric forms up to stabilisation by a symmetric hyperbolic form of the form

$$H(L) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : L \oplus L^* \to (L \oplus L^*) = L^* \oplus L \right)$$

and the group addition is given by orthogonal direct sum

$$(M, \lambda) + (M', \lambda') = \left(M \oplus M', \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \right).$$

The symmetric L-groups of $\mathbb{Z}$ are as follows.

Proposition 1.3.7. ([Ran80a, Proposition 7.2]). The symmetric L-groups of $\mathbb{Z}$ are given by

$$L^n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} \text{ (signature)} & \text{if } n \equiv 0 \bmod 4 \\ \mathbb{Z}_2 \text{ (de Rham invariant)} & \text{if } n \equiv 1 \bmod 4 \\ 0 & \text{if } n \equiv 3 \bmod 4 \\ 0 & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

where the signature map is given by

$$\sigma(C,\phi\in Q^{4k}(C))=\sigma((H^{2k}(C),\phi_0:H^{2k}(C)\to H_{2k}(C)).$$

The symmetric L-groups of a ring with involution are not 4-periodic in general. Aside from the case $R=\mathbb{Z}$ the most general criterion for 4-periodicity is when the element $2\in A$ is invertible, see [Ran80a, Proposition 3.3] for more details.

The signature map allows one to recover the signature of a manifold from its symmetric Poincaré complex.

Example 1.3.8. Let M^n be a closed, oriented manifold of dimension n divisible by 4. If $[M]\in H_n(M;\mathbb{Z})$ is the fundamental class of M determined by the orientation then applying the symmetric construction to $(M,[M])$ produces an n-dimensional symmetric Poincaré complex $(C(M),\phi_M)$. In particular, the 0-dimensional component of the symmetric structure is give by $\phi_{M0}=[M]\cap -:C(M;\mathbb{Z})^{n-*}\to C(M;\mathbb{Z})$ and hence $(C(M),\phi_M)$ maps to the signature of M under the isomorphism $L^n(\mathbb{Z})\cong\mathbb{Z}$.

1.4 Algebraic surgery

There is an algebraic surgery operation on symmetric Poincaré complexes which is an algebraic model for a geometric surgery on a manifold. The definition of algebraic surgery is not needed for Part I of the thesis but will be used in Part II of the thesis.

Definition 1.4.1. ([Ran92, Definition 1.12, Proposition 4.1]).

(i) The effect of an *algebraic surgery* on an n-dimensional ϵ-symmetric complex (C,ϕ) over A with data an $(n+1)$-dimensional ϵ-symmetric pair $(f:C\to D,(\delta\phi,\phi))$ over A is the n-dimensional ϵ-symmetric complex (C',ϕ') over A with the chain complex C' defined by

$$d_{C'}=\begin{pmatrix} d_C & 0 & (-)^{n+1}\phi_0 f^* \\ (-)^r f_{C'} & d_D & (-)^r\delta\phi_0 \\ 0 & 0 & (-)^r d_D^* \end{pmatrix}$$
$$:C'_r=C_r\oplus D_{r+1}\oplus D^{n+1-r}\to C'_{r-1}=C_{r-1}\oplus D_r\oplus D^{n+2-r}\qquad (r\in\mathbb{Z})$$

and the symmetric structure ϕ' defined by

$$\phi'_0=\begin{pmatrix} \phi_0 & 0 & 0 \\ (-)^{n-r}fT_\epsilon\phi_1 & (-)^{n-r}T_\epsilon\delta\phi_1 & (-)^{r(n-r)}\epsilon \\ 0 & 1 & 0 \end{pmatrix}$$
$$:C'^{n-r}=C^{n-r}\oplus D^{n+1-r}\oplus D_{r+1}\to C'_r=C_r\oplus D_{r+1}\oplus D^{n+1-r}\qquad (r\in\mathbb{Z})$$
$$\phi'_s=\begin{pmatrix} \phi_s & 0 & 0 \\ (-)^{n-r}fT_\epsilon\phi_{s+1} & (-)^{n-r+s}T_\epsilon\delta\phi_{s+1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$:C'^{n-r+s}=C^{n-r+s}\oplus D^{n-r+s+1}\oplus D_{r-s+1}\to C'_r=C_r\oplus D_{r+1}\oplus D^{n+1-r}\qquad (r\in\mathbb{Z},s\geqslant 1).$$

(ii) The *trace* of such an algebraic surgery is the $(n+1)$-dimensional symmetric pair

$$((g\ \ g') : C \oplus C' \to D', (0, \phi' \oplus -\phi') \in Q^{n+1}(g\ \ g'))$$

defined by

$$d_{D'} = \begin{pmatrix} d_C & (-)^{n+1}\phi_0 f^* \\ 0 & (-)^r d_D^* \end{pmatrix} : D'_r = C_r \oplus D^{n+1-r} \to D'_{r-1} = C_{r-1} \oplus D^{n+2-r} \qquad (r \in \mathbb{Z})$$

$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C_r \to D'_r = C_r \oplus D^{n+1-r} \qquad (r \in \mathbb{Z})$$

$$g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \to D'_r = C_r \oplus D^{n-r+1} \qquad (r \in \mathbb{Z})$$

In fact, the cobordism relation on n-dimensional ϵ-symmetric Poincaré complexes is the equivalence relation generated by surgery and homotopy equivalence, see [Ran80a, Proposition 4.1].

Example 1.4.2. [Ran02b, p.4]. As in [Ran80a, Proposition 7.3] suppose that $(W; M, M')$ is an oriented $(n+1)$-dimensional cobordism arising as the trace of an index i geometric surgery on a closed n-dimensional manifold M. This surgery removes a framed embedding $S^i \times D^{n-i} \hookrightarrow M$ with effect

$$M' = \overline{M - S^i \times D^{n-i}} \cup_{S^i \times S^{n-i-1}} D^{i+1} \times S^{n-i-1}.$$

The trace of the surgery is the cobordism $(W; M, M')$ given by

$$W = M \times [0,1] \cup_{S^i \times S^{n-i-1}} D^{i+1} \times D^{n-i}$$

where W is obtained by attaching $D^{i+1} \times D^{n-i}$ to $M \times [0,1]$ along the framed embedding $S^i \times S^{n-i-1} \times \{1\} \hookrightarrow M \times \{1\}$.

Let R be a commutative ring with identity. As in Example 1.3.4, applying the symmetric construction to the oriented $(n+1)$-dimensional cobordism $(W; M, M')$ produces an $(n+1)$-dimensional symmetric cobordism over R

$$((i_M\ \ i_{M'}) : C(M;R) \oplus C(M';R) \to C(W;R), (\phi_W, \phi_M \oplus -\phi_{M'}) \in Q^{n+1}(i_M\ \ i_{M'}))$$

between $(C(M;R), \phi_M)$ and $(C(M';R), \phi_{M'})$. Note that the chain map

$$\begin{pmatrix} i_M \\ 0 \end{pmatrix} : C(M;R) \to \mathscr{C}(i_{M'})$$

may be identified with the composition of chain maps

$$C(M;R) \xrightarrow{i_M} C(W;R) \xrightarrow{\pi} C(W, M'; R) = \mathscr{C}(i_{M'})$$

so that the cobordism over R

$$((i_M\ \ i_{M'}) : C(M;R) \oplus C(M';R) \to C(W;R), (\phi_W, \phi_M \oplus -\phi_{M'}) \in Q^{n+1}(i_M\ \ i_{M'}))$$

induces an $(n+1)$-dimensional symmetric pair over R

$$(\pi i_M : C(M;R) \to C(W,M';R), (\phi_W/\phi_{M'}, \phi_M)).$$

This determines an algebraic surgery on $(C(M;R),\phi_M)$ with effect (C',ϕ') an n-dimensional symmetric Poincaré complex which is homotopy equivalent to $(C(M';R),\phi_{M'})$, see [Ran80b, Proposition 7.3].

There is a dual geometric $(n-i-1)$-surgery determined by the obvious embedding $D^{i+1} \times S^{n-i-1} \hookrightarrow M'$ and the trace of this surgery is an oriented $(n+1)$-dimensional cobordism

$$W' = M' \times [0,1] \cup_{D^{i+1}\times S^{n-i-1}} D^{i+1} \times D^{n-i}$$

satisfying $(W';M',M) = -(W;M,M')$. This induces a homotopy equivalence $W \simeq M' \cup_{S^{n-i-1}} D^{n-i}$ which induces a homotopy equivalence of pairs

$$(W,M') \simeq (W/M', M'/M') = (D^{n-i}/S^{n-i-1}, S^{n-i-1}/S^{n-i-1}) = (S^{n-i}, *)$$

and a hence there is a chain homotopy equivalence

$$C(W,M';R) \simeq C(S^{n-i},*;R) \simeq \dot{C}(S^{n-i};R) \simeq S^{n-i}R = (n-i) - \text{fold suspension of } R$$

with

$$C'_r = C(M;R)_r \oplus C(W,M)_{r+1} \oplus C(W,M)^{n+1-r} \simeq \begin{cases} C_r(M;R) \oplus R & \text{if } r = n-i-1, i+1 \\ C_r(M;R) & \text{otherwise.} \end{cases}$$

This shows that algebraic surgery on a symmetric Poincaré complex provides a model for geometric surgery on an oriented manifold such that the effect is orientable.

Not every manifold is orientable. For a path-connected topological space X with universal cover $\widetilde{X}$ the fundamental group $\pi_1(X)$ can be identified with the group of covering automorphisms. In particular, there is an action of $\pi_1(X)$ on the set of singular simplices in $\widetilde{X}$ so for a commutative ring R the singular chain complex of $\tilde{X}$ with R-coefficients can be viewed as an $R[\pi_1(X)]$-module chain complex. A CW-structure on X can be lifted to a CW-structure on $\widetilde{X}$ in such a way that if X is a finite CW-complex then $C(\tilde{X};R)$ is a finite-dimensional $R[\pi_1(X)]$-module chain complex. The symmetric construction can be generalised to produce a morphism $\phi_X : H_*(X;R) \to Q^*(C(\widetilde{X};R))$ such that if $[X] \in H_n(X;R)$ is a homology class then $\phi_X[X]$ is an n-dimensional symmetric structure on $C(\widetilde{X};R)$, see [Ran80b, Proposition 2.1]. For a manifold M it is then possible to produce a symmetric Poincaré structure on its universal cover $\widetilde{M}$, see [Wal99, Theorem 2.1] and it is also true that algebraic surgery is an algebraic model for any geometric surgery on a manifold, see [Ran80b, Proposition 7.3].

In Part I of this thesis we will only deal with oriented manifolds however. In chapter 5 we will apply the symmetric construction to oriented cobordisms arising as the trace of a geometric surgeries determined by a Morse 2-function.

Chapter 2

The L-theory of a ring with involution: symmetric triads

In this chapter we recall from [Ran81] the ϵ-symmetric triad objects and the triad definition of a homotopy equivalence of ϵ-symmetric pairs which appear in the the L-theory of a ring with involution. We then examine a twisted glueing operation for ϵ-symmetric triads and show this is a model for the twisted glueing of manifolds with boundary and manifold triads.

2.1 Symmetric triads

Symmetric triads are relative versions of symmetric pairs and are algebraic models for manifold triads.

Definition 2.1.1. ([Ran81, §1.3]).

(i) A *triad* Γ over A is a chain homotopy commutative square

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & \searrow^{k} & \downarrow{\scriptstyle h} \\ C' & \xrightarrow[f']{} & D' \end{array}$$

of A-module chain complexes. A triad is *commutative* if the above square of A-module chain maps is commutative, that is if we can choose $k = 0$.

(ii) The *chain complex* of a triad Γ over A is the algebraic mapping cone

$$C(\Gamma) = \mathscr{C}((g, h; k))$$

of the the A-module chain map of algebraic mapping cones

$$(g, h; k) : \begin{pmatrix} h & (-)^{r-1}k \\ 0 & g \end{pmatrix} : \mathscr{C}(f)_r = D_r \oplus C_{r-1} \to \mathscr{C}(f')_r = D'_r \oplus C'_{r-1}$$

with $\mathbb{Z}$-module homology groups the *triad homology groups*

$$H_n(\Gamma) = H_n((g,h;k)) \quad (n \in \mathbb{Z}).$$

Definition 2.1.2. ([Ran81, p.43]). Let $\epsilon = \pm 1$.

(i) The *ϵ-symmetric Q-groups* of a triad Γ of finite-dimensional A-module chain complexes

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & \overset{k}{\searrow} & \downarrow{\scriptstyle h} \\ C' & \xrightarrow[f']{} & D' \end{array}$$

are the relative homology groups

$$Q^n(\Gamma,\epsilon) = H_n((g,h;k)^{\%}) \quad (n \in \mathbb{Z})$$

of the $\mathbb{Z}$-module chain map

$$(g,h;k)^{\%} : \mathscr{C}(f^{\%} : W^{\%}C \to W^{\%}D) \to \mathscr{C}(f'^{\%} : W^{\%}C' \to W^{\%}D')$$

determined by $\mathbb{Z}$-module triad

$$\begin{array}{ccc} W^{\%}C & \xrightarrow{f^{\%}} & W^{\%}D \\ {\scriptstyle g^{\%}}\downarrow & \overset{k^{\%}}{\searrow} & \downarrow{\scriptstyle h^{\%}} \\ W^{\%}C' & \xrightarrow[f'^{\%}]{} & W^{\%}D' \end{array}$$

which is obtained by applying the functor $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,-)$ to the A-module triad

$$\begin{array}{ccc} C \otimes_A C & \xrightarrow{f\otimes f} & D \otimes_A D \\ {\scriptstyle g\otimes g}\downarrow & \overset{(hf)\otimes k + k\otimes(g'f)}{\searrow} & \downarrow{\scriptstyle h\otimes h} \\ C' \otimes_A C' & \xrightarrow[f'\otimes f']{} & D' \otimes_A D' \end{array}$$

(ii) An *$(n+2)$-dimensional ϵ-symmetric triad* $(\Gamma, \Phi \in Q^{n+2}(\Gamma,\epsilon))$ consists of a triad Γ over A

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & \overset{k}{\searrow} & \downarrow{\scriptstyle h} \\ C' & \xrightarrow[f']{} & D' \end{array}$$

together with an ϵ-symmetric structure $\Phi \in Q^{n+2}(\Gamma,\epsilon)$, subject to the condition that C is an n-dimensional chain complex, C', D are $(n+1)$-dimensional chain complexes and D' is an $(n+2)$-dimensional chain complex.

The following diagram of long exact sequences of the ϵ-symmetric Q-groups of the constituents of a triad is not needed for Part I of the thesis but will be used in Part II of the thesis.

Proposition 2.1.3. The ϵ-symmetric Q-groups of a triad Γ over A fit into a commutative diagram of $\mathbb{Z}$-modules with exact rows and columns

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & Q^{n+2}(\Gamma,\epsilon) & \longrightarrow & Q^{n+1}(g,\epsilon) & \longrightarrow & Q^{n+1}(h,\epsilon) & \longrightarrow & Q^{n+1}(\Gamma,\epsilon) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & Q^{n+1}(f,\epsilon) & \longrightarrow & Q^{n}(C,\epsilon) & \xrightarrow{f^{\%}} & Q^{n}(D,\epsilon) & \longrightarrow & Q^{n}(f,\epsilon) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow{\scriptstyle g^{\%}} & & \downarrow{\scriptstyle h^{\%}} & & \downarrow & & \\
\dots & \longrightarrow & Q^{n+1}(g,\epsilon) & \longrightarrow & Q^{n}(C',\epsilon) & \xrightarrow{f'^{\%}} & Q^{n}(D',\epsilon) & \longrightarrow & Q^{n}(g,\epsilon) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & Q^{n+1}(\Gamma,\epsilon) & \longrightarrow & Q^{n}(g,\epsilon) & \longrightarrow & Q^{n}(h,\epsilon) & \longrightarrow & Q^{n}(\Gamma,\epsilon) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Proof. The triad over $\mathbb{Z}$

$$\begin{array}{ccc}
W^{\%}C & \xrightarrow{f^{\%}} & W^{\%}D \\
{\scriptstyle g^{\%}}\downarrow & \overset{k^{\%}}{\searrow} & \downarrow{\scriptstyle h^{\%}} \\
W^{\%}C' & \xrightarrow[f'^{\%}]{} & W^{\%}D'
\end{array}$$

determines a commutative diagram of short exact sequences of chain complexes over $\mathbb{Z}$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (W^{\%}D')_* & \longrightarrow & \mathscr{C}(f'^{\%})_* & \longrightarrow & (W^{\%}C')_{*-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{C}(h^{\%})_* & \longrightarrow & \mathscr{C}((g,h;k)^{\%})_* & \longrightarrow & \mathscr{C}(g^{\%})_{*-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (W^{\%}D)_{*-1} & \longrightarrow & \mathscr{C}(f^{\%})_{*-1} & \longrightarrow & (W^{\%}C)_{*-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and we then take the long exact sequences associated to the mapping cones. $\square$

Homotopy equivalences of symmetric pair and cobordisms are defined in terms of triads.

Definition 2.1.4. ([Ran81, p.45]).

(i) A *homotopy equivalence* of $(n+1)$-dimensional ϵ-symmetric pairs over A

$$\Gamma : (f : C \to D, (\delta\phi, \phi) \in Q^{n+1}(f, \epsilon)) \simeq (f' : C' \to D', (\delta\phi', \phi') \in Q^{n+1}(f', \epsilon))$$

is a triad over A of the form

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & \searrow^{k} & \downarrow{\scriptstyle h} \\ C' & \xrightarrow[f']{} & D' \end{array}$$

such that the A-module chain maps $g : C \to C', h : D \to D'$ are chain homotopy equivalences and the morphism of relative ϵ-symmetric Q-groups respects the relative symmetric structures, that is

$$(g, h; k)^{\%}(\delta\phi, \phi) = (\delta\phi', \phi') \in Q^n(f', \epsilon).$$

(ii) A *homotopy equivalence* of $(n+1)$-dimensional ϵ-symmetric cobordisms over A

$$\Gamma : ((f_C \;\; f_{C'}) : C \oplus C' \to D, (\delta\phi, \phi_C \oplus -\phi_{C'})) \simeq ((f_{C''} \;\; f_{C'''}) : C'' \oplus C''' \to D', (\delta\phi', \phi_{C''} \oplus -\phi_{C'''}))$$

is a homotopy equivalence of the form

$$\Gamma = \begin{array}{ccc} C \oplus C' & \xrightarrow{\begin{pmatrix} f_C & f_{C'} \end{pmatrix}} & D \\ {\scriptstyle \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}}\downarrow & \searrow^{\begin{pmatrix} k & k' \end{pmatrix}} & \downarrow{\scriptstyle h} \\ C'' \oplus C''' & \xrightarrow[\begin{pmatrix} f_{C''} & f_{C'''} \end{pmatrix}]{} & D' \end{array}$$

Example 2.1.5. Let $(W; M, M'), (W'; M', M'')$ be two adjoining oriented $(n+1)$-dimensional cobordisms of manifolds. Glueing W and W' along M' produces an oriented $(n+1)$-dimensional cobordism $(W \cup_{M'} W'; M, M'')$. If R is a commutative ring with identity then applying the symmetric construction produces three $(n+1)$-dimensional symmetric cobordisms over R

$$((i_M \;\; i_{M'}) : C(M; R) \oplus C(M'; R) \to C(W; R), (\phi_W, \phi_M \oplus -\phi_{M'} \in Q^{n+1}(i_M \;\; i_{M'})))$$
$$((i'_{M'} \;\; i'_{M''}) : C(M'; R) \oplus C(M''; R) \to C(W'; R), (\phi_{W'}, \phi_{M'} \oplus -\phi_{M''} \in Q^{n+1}(i'_{M'} \;\; i'_{M''})))$$
$$((i''_M \;\; i''_{M''}) : C(M; R) \oplus C(M''; R) \to C(W \cup_{M'} W'; R), (\phi_W, \phi_M \oplus -\phi_{M'} \in Q^{n+1}(i''_M \;\; i''_{M''}))).$$

The triad

$$\begin{array}{ccc} C(M; R) \oplus C(M''; R) & \longrightarrow & C(W; R) \cup_{C(M'; R)} C(W'; R) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \simeq} \\ C(M; R) \oplus C(M''; R) & \longrightarrow & C(W \cup_{M'} W'; R) \end{array}$$

implies that the $(n+1)$-dimensional Poincaré cobordism, obtained by glueing the adjoining

$(n+1)$-dimensional Poincaré cobordisms

$$((i_M \;\; i_{M'}) : C(M;R) \oplus C(M';R) \to C(W;R), (\phi_W, \phi_M \oplus -\phi_{M'} \in Q^{n+1}(i_M \;\; i_{M'})))$$
$$((i'_{M'} \;\; i'_{M''}) : C(M';R) \oplus C(M'';R) \to C(W';R), (\phi_{W'}, \phi_{M'} \oplus -\phi_{M''} \in Q^{n+1}(i'_{M'} \;\; i'_{M''})))$$

over $(C(M';R), \phi_{M'})$, is homotopy equivalent to the $(n+1)$-dimensional Poincaré cobordism

$$((i''_M \;\; i''_{M''}) : C(M;R) \oplus C(M'';R) \to C(W \cup_{M'} W';R), (\phi_{W \cup_{M'} W'}, \phi_M \oplus -\phi_{M''} \in Q^{n+1}(i''_M \;\; i''_{M''})))$$

so that we may think of $\phi_{W \cup_{M'} W'} = \phi_W \cup_{\phi_{M'}} \phi_{W'}$.

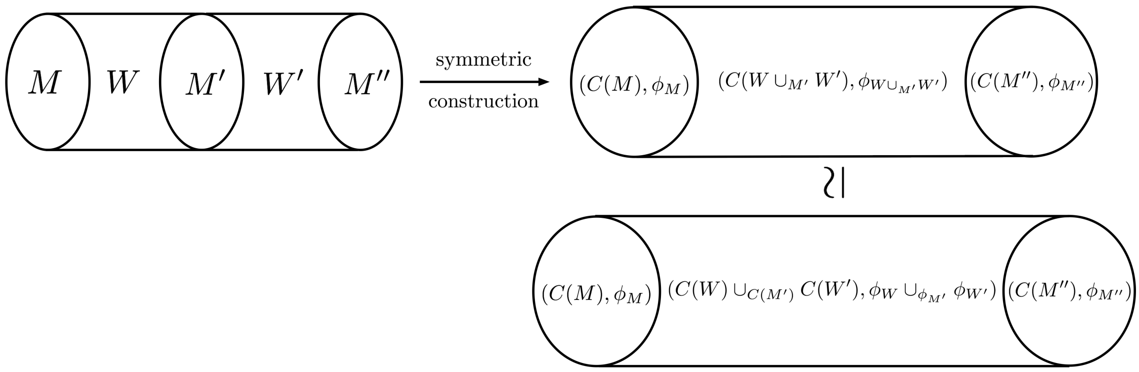


Figure 17: A schematic diagram of the homotopy equivalence.

This shows that the union of adjoining symmetric cobordisms is, up to homotopy equivalence, an algebraic model for the union of adjoining manifold cobordisms.

In chapter 3 we will work with symmetric pairs arising from a manifold with boundary (Σ^{n+1}, M^n) where Σ^{n+1} is contractible or deformation retracts onto a space of dimension at most n. The following lemma shows that Σ^{n+1} makes no contribution to the relative part of the symmetric structure.

Lemma 2.1.6. Let $(f : C \to D, (\delta\phi, \phi))$ be an $(n+1)$-dimensional ϵ-symmetric (Poincaré) pair over A. Suppose that D' is an A-module chain complex of dimension m such that $2m \leqslant n+1$ and there is a homotopy equivalence $h : D \to D'$. Then there is a homotopy equivalence of $(n+1)$-dimensional ϵ-symmetric (Poincaré) pairs over A

$$(f : C \to D, (\delta\phi, \phi)) \simeq (hf : C \to D', (0, \phi))$$

Proof. If $f' = hf : C \to D'$ then there is a commutative triad over A

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[f']{} & D' \end{array}$$

and applying the $W^{\%}$ functor produces a commutative triad

$$\begin{array}{ccc} W^{\%}C & \xrightarrow{f^{\%}} & W^{\%}D \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle h^{\%}} \\ W^{\%}C & \xrightarrow[f'^{\%}]{} & W^{\%}D' \end{array} .$$

Since D' is of dimension $2m \leqslant n+1$ it follows that the algebraic mapping cone of $f'^{\%}$ degenerates to dimensions $n, n+1, n+2$ to

$$\begin{array}{ccccc} \mathscr{C}(f'^{\%})_{n+2} & \xrightarrow{d_{\mathscr{C}(f'^{\%})}} & \mathscr{C}(f'^{\%})_{n+1} & \xrightarrow{d_{\mathscr{C}(f'^{\%})}} & \mathscr{C}(f'^{\%})_{n} \\ \| & & \| & & \| \\ 0 & \xrightarrow[0]{} & W^{\%}(C)_n & \xrightarrow[d_{W^{\%}C}]{} & W^{\%}(C)_{n-1} \end{array}$$

and hence

$$Q^{n+1}(f', \epsilon) = H_{n+1}(\mathscr{C}(f'^{\%})) = \ker(d_{W^{\%}C} : (W^{\%}C)_n \to (W^{\%}C)_{n-1}) = Q^n(C, \epsilon)$$

so that any element $(\delta\phi', \phi') \in Q^{n+1}(f', \epsilon)$ is necessarily of the form $(0, \phi')$ for some n-dimensional ϵ-symmetric structure $\phi' \in Q^n(C, \epsilon)$. It is clear then that $(1, h; 0)^{\%}(\delta\phi, \phi) = (0, \phi) \in Q^{n+1}(f', \epsilon)$ so that the triad Γ defines a homotopy equivalence

$$\Gamma : (f : C \to D, (\delta\phi, \phi)) \simeq (hf : C \to D', (0, \phi)).$$

□

Example 2.1.7.

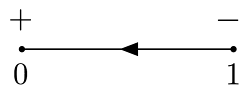
(i) Think of (D^1, S^0) as a CW-pair such that S^0 consists of two 0-cells and D^1 consists of one 1-cell in addition to the 0-cells of S^0. The constant map $h : D^1 \to \{*\}$ is a cellular homotopy equivalence determining a commutative diagram of CW-complexes and cellular maps

$$\begin{array}{ccc} S^0 & \xhookrightarrow{i} & D^1 \\ {\scriptstyle 1}\downarrow & & {\scriptstyle \simeq}\downarrow{\scriptstyle h} \\ S^0 & \xrightarrow[hi]{} & * \end{array}$$

such that if $R = \mathbb{Z}$ there is a commutative diagram of chain maps of cellular chain complexes

$$\begin{array}{ccc} C(S^0; \mathbb{Z}) & \xhookrightarrow{i} & C(D^1; \mathbb{Z}) \\ {\scriptstyle 1}\downarrow & & {\scriptstyle \simeq}\downarrow{\scriptstyle h} \\ C(S^0; \mathbb{Z}) & \xrightarrow[(1\ \ 1)]{} & C(*; \mathbb{Z}) = \mathbb{Z} \end{array}$$

Applying the symmetric construction to $(D^1, S^0) = ([0,1], \{0\} \sqcup \{1\})$ with the orientation



produces a 2-dimensional symmetric Poincaré pair over $\mathbb{Z}$

$$(i : C(S^0;\mathbb{Z}) \to C(D^1;\mathbb{Z}), (\phi_{D^1}, \phi_{S^0}))$$

which is homotopy equivalent to the 2-dimensional symmetric Poincaré pair over $\mathbb{Z}$

$$((1\ \ 1) : \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}, (0, 1\oplus -1)).$$

(ii) Think of (D^2, S^1) as a CW-pair such that S^1 has one 0-cell $*$ and one 1-cell and D^2 has one 2-cell in addition to cells of S^1. The constant map $h : D^2 \to \{*\}$ from D^2 to the 0-cell is a cellular map defining a homotopy equivalence such that there is a commutative diagram of CW-complexes and cellular maps

$$\begin{array}{ccc} S^1 & \xrightarrow{\ i\ } & D^2 \\ \downarrow{\scriptstyle 1} & & \simeq\downarrow{\scriptstyle h} \\ S^1 & \xrightarrow[hi]{} & * \end{array}$$

inducing a commutative diagram of cellular chain complexes and chain maps

$$\begin{array}{ccc} C(S^1;\mathbb{Z}) & \xrightarrow{\ i\ } & C(D^2;\mathbb{Z}) \\ \downarrow{\scriptstyle 1} & & \simeq\downarrow{\scriptstyle h} \\ C(S^1;\mathbb{Z}) & \xrightarrow[hi]{} & C(*;\mathbb{Z}) = \mathbb{Z} \end{array}$$

If $R = \mathbb{Z}$ then applying the symmetric construction to (D^2, S^1) with the standard orientation produces a 2-dimensional symmetric Poincaré pair over $\mathbb{Z}$

$$(i : C(S^1;\mathbb{Z}) \to C(D^2;\mathbb{Z}), (\phi_{D^2}, \phi_{S^1}))$$

which is homotopy equivalent to the 2-dimensional symmetric Poincaré pair over $\mathbb{Z}$

$$(hi : C(S^1;\mathbb{Z}) \to \mathbb{Z}, (0, \phi_{S^1}))$$

The definition from [Ran81, p.113] of the condition for an ϵ-symmetric triad to be Poincaré is somewhat unwieldy and the following alternative description can be used to circumvent this problem.

Proposition 2.1.8. There is a one-to-one correspondence between $(n+2)$-dimensional ϵ-symmetric triads (Γ, Φ) over A and quadruples consisting of:

(i) An n-dimensional ϵ-symmetric complex (C, ϕ) over A

(ii) An $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi, -\phi))$ over A

(iii) An $(n+1)$-dimensional ϵ-symmetric pair $(f' : C \to D', (\delta\phi', \phi))$ over A

(iv) An $(n+2)$-dimensional ϵ-symmetric pair $(e : D \cup_C D' \to E, (\phi', \delta\phi \cup_\phi \delta\phi'))$ over A

where

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle f'}\big\downarrow & \searrow^{k} & \big\downarrow{\scriptstyle g} \\ D' & \xrightarrow[g']{} & E \end{array}$$

$$\Phi = (\phi', \delta\phi', \delta\phi, \phi) \in Q^{n+2}(\Gamma)$$

$$e = \begin{pmatrix} g & (-)^{r-1}k & g' \end{pmatrix} : (D \cup_C D')_r = D_r \oplus C_{r-1} \oplus D'_r \to E_r \quad (r \in \mathbb{Z})$$

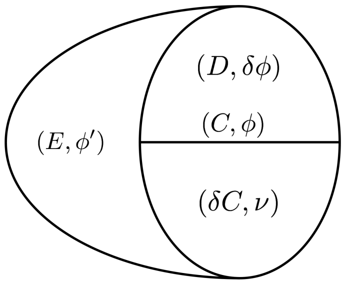


Figure 18: A schematic diagram for the data in an ϵ-symmetric triad.

Proof. See [Ran81, Proposition 2.1.1]. We have made a sign change in (ii) for convenience later. □

The Poincaré condition for a symmetric triad is then more naturally expressed in terms of the three symmetric pairs induced by the triad.

Definition 2.1.9. An $(n+2)$-dimensional ϵ-symmetric triad (Γ, Φ) over A

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle f'}\big\downarrow & \searrow^{k} & \big\downarrow{\scriptstyle g} \\ D' & \xrightarrow[g']{} & E \end{array}$$

$$\Phi = (\phi', \delta\phi', \delta\phi, \phi)$$

is *Poincaré* if and only if the following four conditions are all satisfied:

(i) The n-dimensional ϵ-symmetric complex (C, ϕ) is Poincaré .

(ii) The $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi, -\phi))$ is Poincaré .

(iii) The $(n+1)$-dimensional ϵ-symmetric pair $(f' : C \to D', (\delta\phi', \phi))$ is Poincaré .

(iv) The $(n+2)$-dimensional ϵ-symmetric pair $(e : D' \cup_C D' \to E, (\phi', \delta\phi \cup_\phi \delta\phi'))$ is Poincaré .

Example 2.1.10. Recall that an oriented $(n+2)$-dimensional manifold triad $(\Omega; \Sigma, \Sigma'; M)$ consists of an oriented $(n+2)$-dimensional manifold with boundary $(\Omega, \partial\Omega)$ such that there are two oriented codimension-0 submanifolds with boundary $(\Sigma, -M), (\Sigma', M)$ of $\partial\Omega$ such that $\Sigma \cap \Sigma' = M$ and $\partial\Omega = \Sigma \cup_M \Sigma'$. If R is a commutative ring with identity then applying the symmetric construction to $(\Sigma, -M), (\Sigma', M), (\Omega, \partial\Omega)$ produces two $(n+1)$-dimensional symmetric Poincaré pairs over R

$$(C(M;R) \to C(\Sigma;R), (\phi_\Sigma, -\phi_M))$$
$$(C(M;R) \to C(\Sigma';R), (\phi_{\Sigma'}, \phi_M))$$

and one $(n+2)$-dimensional symmetric Poincaré pair over R

$$(C(\partial\Omega;R) \to C(\Omega;R), (\phi_\Omega, \phi_{\partial\Omega}))$$

where the chain maps are induced from the inclusions of subspaces. This determines an $(n+2)$-dimensional commutative symmetric Poincaré triad over R

$$\Gamma = \begin{array}{ccc} C(M;R) & \longrightarrow & C(\Sigma;R) \\ \big\downarrow & & \big\downarrow \\ C(\Sigma';R) & \longrightarrow & C(\Omega;R) \end{array} \quad , \quad \Phi = (\phi_\Omega, \phi_{\Sigma'}, \phi_\Sigma, \phi_M) \in Q^{n+2}(\Gamma)$$

with the chain maps induced by inclusion. This shows that a symmetric Poincaré triad is an are algebraic model for an oriented manifold triad.

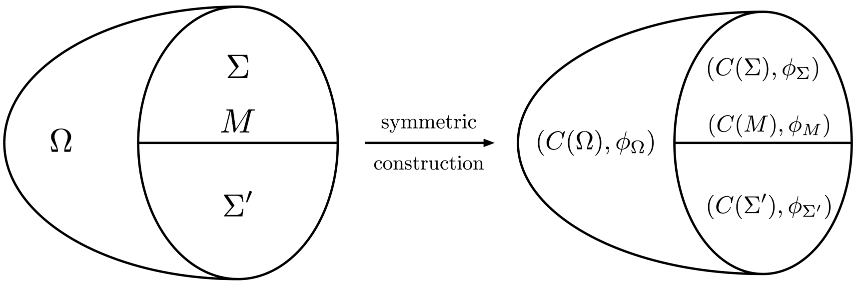


Figure 19: A schematic diagram of the passage from a triad of manifolds to a symmetric triad.

A cobordism of symmetric pairs is a symmetric triad which respects the boundary decomposition.

Definition 2.1.11. ([Ran81, p.114]). An *ϵ-symmetric cobordism* between $(n+1)$-dimensional ϵ-symmetric Poincaré pairs $(f : C \to D, (\delta\phi, \phi)), (f' : C' \to D', (\delta\phi', \phi'))$ over A is an $(n+2)$-dimensional ϵ-symmetric Poincaré triad (Γ, Φ) over A of the form

$$\Gamma = \begin{array}{ccc} C \oplus C' & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & D \oplus D' \\ \begin{pmatrix} g & g' \end{pmatrix} \big\downarrow & \overset{\begin{pmatrix} k & k' \end{pmatrix}}{\rightsquigarrow} & \big\downarrow \begin{pmatrix} h & h' \end{pmatrix} \\ \delta C & \xrightarrow[\delta f]{} & \delta D \end{array}$$

$$\Phi = (\delta\nu, \nu, \delta\phi \oplus -\delta\phi', \phi \oplus -\phi') \in Q^{n+2}(\Gamma, \epsilon).$$

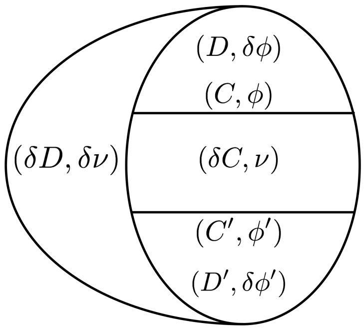


Figure 20: A schematic diagram for the data in an ϵ-symmetric cobordism of ϵ-symmetric pairs.

Example 2.1.12. Recall that an oriented $(n+2)$-dimensional relative cobordism $(\Omega;\Sigma,\Sigma',W;M,M')$ consists of an oriented $(n+2)$-dimensional manifold with boundary $(\Omega,\partial\Omega)$, oriented $(n+1)$-dimensional manifolds with boundary $(\Sigma,-M),(\Sigma',M')$ and an oriented $(n+1)$-dimensional cobordism $(W;M,M')$ such that $\partial\Omega = \Sigma\cup_M W\cup_{M'} -\Sigma'$. If R is a commutative ring with identity then applying the symmetric construction to $(\Sigma,M),(\Sigma',M'),(W;M,M')$, $(\Omega,\partial\Omega)$ produces three $(n+1)$-dimensional symmetric Poincaré pairs over R

$$\begin{aligned}
&(C(M;R)\to C(\Sigma;R),(\phi_\Sigma,-\phi_M))\\
&(C(M';R)\to C(\Sigma';R),(\phi_{\Sigma'},\phi_{M'}))\\
&(C(M;R)\oplus C(M';R)\to C(W;R),(\phi_W,\phi_M\oplus -\phi_{M'}))
\end{aligned}$$

and one $(n+2)$-dimensional Poincaré pair over R

$$(C(\partial\Omega;R)\to C(\Omega;R),(\phi_\Omega,\phi_{\partial\Omega}))$$

where the chain maps are all induced from the inclusions of subspaces. This determines an $(n+2)$-dimensional commutative Poincaré triad (Γ,Φ) over R with

$$\Gamma = \begin{array}{ccc}
C(M;R)\oplus C(M';R) & \longrightarrow & C(\Sigma;R)\oplus C(\Sigma';R)\\
\big\downarrow & & \big\downarrow\\
C(W;R) & \longrightarrow & C(\Omega;R)
\end{array}$$

$$\Phi = (\phi_\Omega,\phi_W,\phi_\Sigma\oplus -\phi_{\Sigma'},\phi_M\oplus -\phi_{M'})$$

which can be viewed as a cobordism between the $(n+1)$-dimensional symmetric Poincaré pairs $(C(M;R)\to C(\Sigma;R),(\phi_\Sigma,\phi_M))$ and $(C(M':R)\to C(\Sigma';R),(\phi_{\Sigma'},\phi_{M'}))$. This shows that a symmetric Poincaré cobordism of pairs is an algebraic model for a relative oriented manifold cobordism.

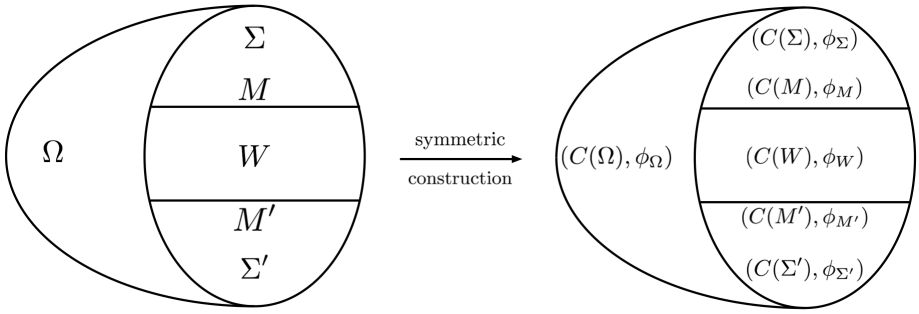


Figure 21: A schematic diagram for the passage from a relative cobordisms of manifolds to a symmetric Poincaré cobordism between symmetric pairs.

2.2 Unions of symmetric triads

The glueing operation for adjoining cobordisms from Chapter 1 may be extended to a glueing operation for adjoining symmetric triads and adjoining symmetric cobordisms of pairs. This gives an algebraic model for glueing adjoining manifold triads and adjoining relative cobordisms.

Definition 2.2.1. ([Ran81, p.481]). The *union* of two adjoining $(n+2)$-dimensional ϵ-symmetric (Poincaré) triads $(\Gamma, \Phi \in Q^{n+2}(\Gamma, \epsilon))$ and $(\Gamma', \Phi' \in Q^{n+2}(\Gamma', \epsilon))$ over A of the form

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & \overset{k}{\searrow} & \downarrow{\scriptstyle h} \\ \delta C & \xrightarrow[\delta f]{} & \delta D \end{array}, \quad \Phi = (\delta\nu, \nu, -\delta\phi, -\phi)$$

$$\Gamma' = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g'}\downarrow & \overset{k'}{\searrow} & \downarrow{\scriptstyle h'} \\ \delta C' & \xrightarrow[\delta f']{} & \delta D' \end{array}, \quad \Phi' = (\delta\nu', \nu', \delta\phi, \phi)$$

is the $(n+2)$-dimensional ϵ-symmetric (Poincaré) triad over A

$$(\Gamma, \Phi \in Q^{n+2}(\Gamma, \epsilon)) \cup (\Gamma', \Phi' \in Q^{n+2}(\Gamma', \epsilon)) = (\Gamma \cup \Gamma', \Phi \cup \Phi' \in Q^{n+2}(\Gamma \cup \Gamma', \epsilon))$$

with

$$\Gamma \cup \Gamma' = \begin{array}{ccc} C & \xrightarrow{f} & \delta C \\ {\scriptstyle g'}\downarrow & \overset{k''}{\searrow} & \downarrow{\scriptstyle \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix}} \\ \delta C' & \xrightarrow[\begin{pmatrix} 0 \\ 0 \\ \delta f' \end{pmatrix}]{} & \delta D \cup_{\delta C} \delta D' \end{array}, \quad \Phi \cup \Phi' = (\delta\nu \cup_{(\delta\phi, \phi)} \delta\nu', \nu', -\delta\phi, -\phi)$$

with chain homotopy

$$k'' = \begin{pmatrix} (-)^{r-1}k \\ g \\ (-)^{r-1}k' \end{pmatrix} : C_r \to (\delta D \cup_{\delta C} \delta D')_{r+1} = \delta D_{r+1} \oplus \delta C_r \oplus \delta D'_{r+1} \quad (r \in \mathbb{Z})$$

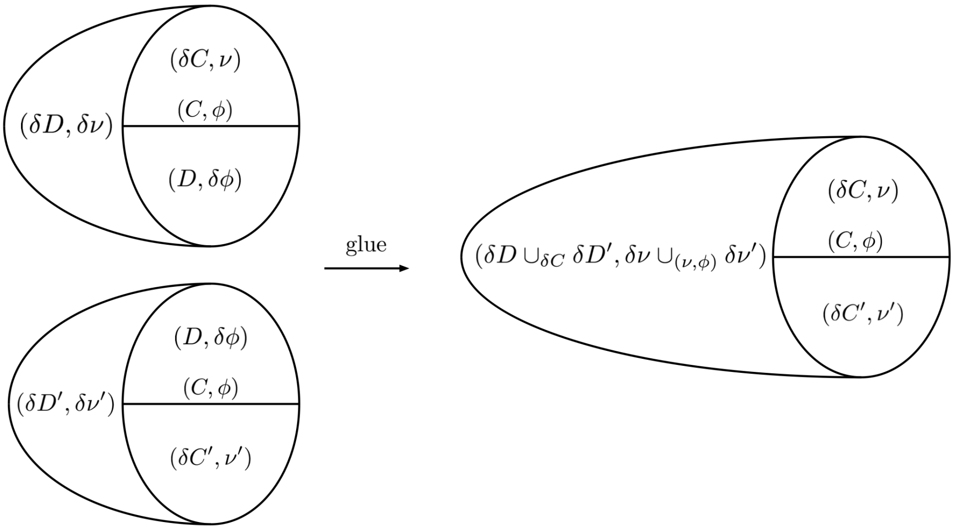


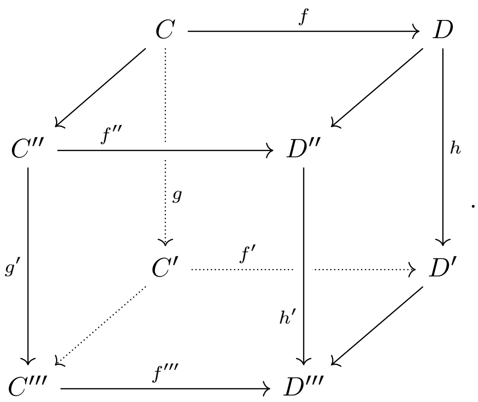
Figure 22: A schematic diagram glueing for the glueing of adjoining ϵ-symmetric triads.

By virtue of Proposition 2.1.8 the definition of a homotopy equivalence of symmetric pairs has the following extension to triads.

Definition 2.2.2. A *homotopy equivalence* of $(n+2)$-dimensional ϵ-symmetric triads $\Gamma \simeq \Gamma$' over A with

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & \searrow^{k} & \downarrow{\scriptstyle h} \\ C' & \xrightarrow[f']{} & D' \end{array}, \quad \Gamma' = \begin{array}{ccc} C'' & \xrightarrow{f''} & D'' \\ {\scriptstyle g'}\downarrow & \searrow^{k'} & \downarrow{\scriptstyle h'} \\ C''' & \xrightarrow[f''']{} & D''' \end{array}$$

is a cube of morphisms with chain homotopy commutative faces



inducing homotopy equivalences between the three ϵ-symmetric pairs determined by Γ and Γ'.

Example 2.2.3. Two adjoining oriented $(n+2)$-dimensional manifold triads $(\Omega;\Sigma,\Sigma';M),(\Omega';\Sigma',\Sigma'';M)$ may be glued over Σ' to produce an oriented $(n+2)$-dimensional manifold triad $(\Omega\cup_{\Sigma'}\Omega';\Sigma,\Sigma'';M)$. If R is a commutative ring with identity then applying the symmetric construction produces three $(n+2)$-dimensional symmetric commutative Poincaré triads over R

$$\Gamma = \begin{array}{ccc} C(M;R) & \longrightarrow & C(\Sigma;R) \\ \downarrow & & \downarrow \\ C(\Sigma';R) & \longrightarrow & C(\Omega;R) \end{array}, \qquad \Phi = (\phi_\Omega,\phi_{\Sigma'},\phi_\Sigma,\phi_M)$$

$$\Gamma' = \begin{array}{ccc} C(M;R) & \longrightarrow & C(\Sigma';R) \\ \downarrow & & \downarrow \\ C(\Sigma'';R) & \longrightarrow & C(\Omega';R) \end{array}, \qquad \Phi' = (\phi_{\Omega'},\phi_{\Sigma''},\phi_{\Sigma'},\phi_M)$$

$$\Gamma'' = \begin{array}{ccc} C(M;R) & \longrightarrow & C(\Sigma;R) \\ \downarrow & & \downarrow \\ C(\Sigma'';R) & \longrightarrow & C(\Omega\cup_{\Sigma'}\Omega';R) \end{array}, \quad \Phi'' = (\phi_{\Omega\cup_{\Sigma'}\Omega'},\phi_{\Sigma''},\phi_\Sigma,\phi_M).$$

There is a homotopy equivalence of $(n+2)$-dimensional symmetric Poincaré triads over R

$$(\Gamma'',\Phi'')\simeq(\Gamma,\Phi)\cup(\Gamma',\Phi')=(\Gamma\cup\Gamma',\Phi\cup\Phi')$$

with

$$\Gamma\cup\Gamma' = \begin{array}{ccc} C(M;R) & \longrightarrow & C(\Sigma;R) \\ \downarrow & \searrow & \downarrow \\ C(\Sigma'';R) & \longrightarrow & C(\Omega;R)\cup_{C(\Sigma';R)}C(\Omega';R) \end{array}$$

$$\Phi\cup\Phi' = (\phi_\Omega\cup_{(\phi_{\Sigma'},\phi_{M'})}\phi_{\Omega'},\phi_{\Sigma''},\phi_\Sigma,\phi_M)$$

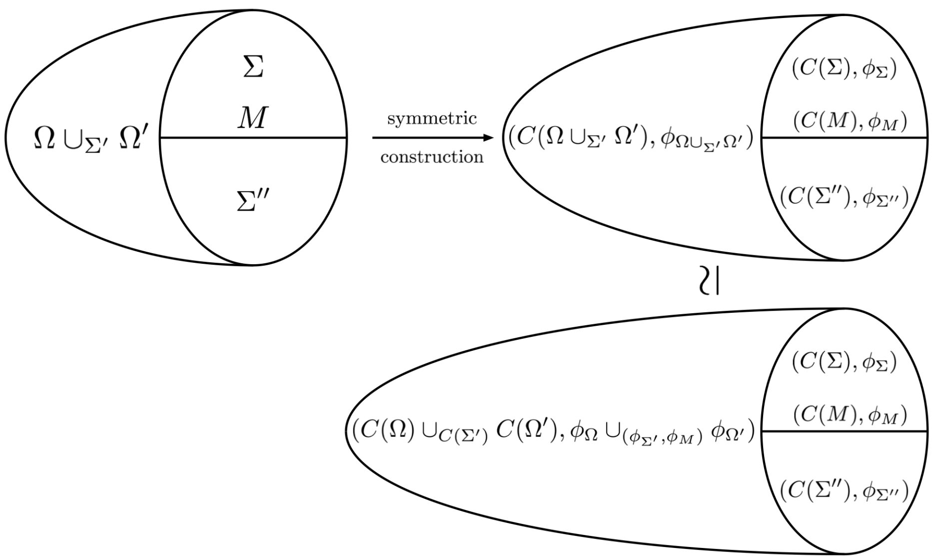


Figure 23: A schematic diagram for the homotopy equivalence.

The glueing of relative manifold cobordisms has the following algebraic model.

Definition 2.2.4. ([Ran81, p.117]). Let

$$\begin{aligned} &(f : C \to D, (\delta\phi, \phi) \in Q^{n+1}(f, \epsilon)) \\ &(f' : C' \to D', (\delta\phi', \phi')) \in Q^{n+1}(f', \epsilon)) \\ &(f'' : C'' \to D'', (\delta\phi'', \phi'') \in Q^{n+1}(f'', \epsilon)) \end{aligned}$$

be three $(n+1)$-dimensional ϵ-symmetric Poincaré pairs over A. The *union* of two adjoining $(n+2)$-dimensional ϵ-symmetric Poincaré cobordisms of pairs $(\Gamma, \Phi \in Q^{n+2}(\Gamma, \epsilon)), (\Gamma', \Phi' \in Q^{n+2}(\Gamma', \epsilon))$ over A of the form

$$\Gamma = \begin{array}{ccc} C \oplus C' & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & D \oplus D' \\ {\scriptstyle\begin{pmatrix} g & g' \end{pmatrix}}\Big\downarrow & \overset{\begin{pmatrix} k & k' \end{pmatrix}}{\searrow} & \Big\downarrow{\scriptstyle\begin{pmatrix} h & h' \end{pmatrix}} \\ \delta C & \xrightarrow[\delta f]{} & \delta D \end{array}$$

$$\Phi = (\delta\nu, \nu, \delta\phi \oplus -\delta\phi', \phi \oplus -\phi')$$

$$\Gamma' = \begin{array}{ccc} C' \oplus C'' & \xrightarrow{\begin{pmatrix} f' & 0 \\ 0 & f'' \end{pmatrix}} & D' \oplus D'' \\ {\scriptstyle\begin{pmatrix} \tilde{g}' & g'' \end{pmatrix}}\Big\downarrow & \overset{\begin{pmatrix} \tilde{k}' & k'' \end{pmatrix}}{\searrow} & \Big\downarrow{\scriptstyle\begin{pmatrix} \tilde{h}' & h'' \end{pmatrix}} \\ \delta C' & \xrightarrow[\delta f']{} & \delta D' \end{array}$$

$$\Phi' = (\delta\nu', \nu', \delta\phi' \oplus -\delta\phi'', \phi' \oplus -\phi'')$$

is the ϵ-symmetric Poincaré cobordism of pairs $(\Gamma'', \Phi'' \in Q^{n+2}(\Gamma'', \epsilon))$ over A defined by

$$\Gamma'' = \begin{array}{ccc} C \oplus C'' & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f'' \end{pmatrix}} & D' \oplus D'' \\ {\scriptstyle\begin{pmatrix} \tilde{g} & \tilde{g}'' \end{pmatrix}}\Big\downarrow & \overset{\begin{pmatrix} \tilde{k} & \tilde{k}'' \end{pmatrix}}{\searrow} & \Big\downarrow{\scriptstyle\begin{pmatrix} \tilde{h} & \tilde{h}'' \end{pmatrix}} \\ \delta C \cup_{C'} \delta C' & \xrightarrow[\delta f'']{} & \delta D \cup_{D'} \delta D' \end{array}$$

$$\Phi'' = (\delta\nu'', \nu'', \delta\phi \oplus -\delta\phi'', \phi \oplus -\phi'')$$

with $\delta C \cup_{C'} \delta C'$ and $\delta D \cup_{D'} \delta D'$ the A-module chain complexes

$$\begin{aligned} \delta C \cup_{C'} \delta C' &= \mathscr{C}\left(\begin{pmatrix} g' \\ \tilde{g}' \end{pmatrix} : C' \to \delta C \oplus \delta C' \right) \\ \delta D \cup_{D'} \delta D' &= \mathscr{C}\left(\begin{pmatrix} h' \\ \tilde{h}' \end{pmatrix} : D' \to \delta D \oplus \delta D' \right) \end{aligned}$$

with

$$d_{\delta C\cup_{C'}\delta C'} = \begin{pmatrix} d_{\delta C} & (-)^{r-1}g' & 0 \\ 0 & d_{C'} & 0 \\ 0 & (-)^{r-1}\tilde{g}' & d_{\delta C'} \end{pmatrix}$$
$$: (\delta C\cup_{C'}\delta C')_r = \delta C_r \oplus C'_{r-1} \oplus \delta C'_r \to (\delta C\cup_{C'}\delta C')_{r-1} = \delta C_{r-1} \oplus C'_{r-2} \oplus \delta C'_{r-1} \quad (r\in\mathbb{Z})$$

$$d_{\delta D\cup_{D'}\delta D'} = \begin{pmatrix} d_{\delta D} & (-)^{r-1}h' & 0 \\ 0 & d_{D'} & 0 \\ 0 & (-)^{r-1}\tilde{h}' & d_{\delta D'} \end{pmatrix}$$
$$: (\delta D\cup_{D'}\delta D')_r = \delta D_r \oplus D'_{r-1} \oplus \delta D'_r \to (\delta D\cup_{D'}\delta D')_{r-1} = \delta D_{r-1} \oplus D'_{r-2} \oplus \delta D'_{r-1} \quad (r\in\mathbb{Z})$$

and chain maps

$$\begin{pmatrix} \tilde{g} & \tilde{g}'' \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & 0 \\ 0 & g'' \end{pmatrix} : C_r \oplus C''_r \to (\delta C\cup_{C'}\delta C')_r = \delta C_r \oplus C'_{r-1} \oplus \delta C'_r \qquad (r\in\mathbb{Z})$$
$$\begin{pmatrix} \tilde{h} & \tilde{h}'' \end{pmatrix} = \begin{pmatrix} h & 0 \\ 0 & 0 \\ 0 & h'' \end{pmatrix} : D_r \oplus D''_r \to (\delta D\cup_{D'}\delta D')_r = \delta D_r \oplus D'_{r-1} \oplus \delta D'_r \qquad (r\in\mathbb{Z})$$

$$\delta f'' = \begin{pmatrix} \delta f & (-)^{r-1}k' & 0 \\ 0 & f' & 0 \\ 0 & (-)^{r-1}\tilde{k}' & \delta f' \end{pmatrix}$$
$$: (\delta C\cup_{C'}\delta C')_r = \delta C_r \oplus C'_{r-1} \oplus \delta C'_r \to (\delta D\cup_{D'}\delta D')_r = \delta D_r \oplus D'_{r-1} \oplus \delta D'_r \quad (r\in\mathbb{Z})$$

and a chain homotopy

$$\begin{pmatrix} \tilde{k} & \tilde{k}'' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & 0 \\ 0 & k'' \end{pmatrix} : C_r \oplus C''_r \to (\delta D\cup_{D'}\delta D')_{r+1} = \delta D_{r+1} \oplus D'_r \oplus \delta D'_{r+1} \qquad (r\in\mathbb{Z})$$

and symmetric structures $\nu'', \delta\nu''$

$$\nu''_s = \begin{pmatrix} \nu_s & 0 & 0 \\ (-)^{n-r}\phi'_s g'^* & (-)^{n-r+s+1}T\phi'_{s-1} & 0 \\ 0 & (-)^s g'\phi'_s & \nu'_s \end{pmatrix}$$
$$: (\delta C\cup_{C'}\delta C')^{n-r+s+1} = \delta C^{n-r+s+1} \oplus C'^{n-r+s} \oplus \delta C'^{n-r+s+1} \to \delta C''_r = \delta C_r \oplus C'_{r-1} \oplus \delta C'_r$$
$$(r\in\mathbb{Z}, s\geqslant 0, \phi'_{-1}=0)$$

$$\delta\nu''_s = \begin{pmatrix} \delta\nu_s & 0 & 0 \\ (-)^{n-r+1}\phi'_s h'^* & (-)^{n-r+s+2}T\delta\phi'_{s-1} & 0 \\ 0 & (-)^s\tilde{h}'\phi'_s & \delta\nu'_s \end{pmatrix}$$

$$: (\delta D \cup_{D'} \delta D')^{n-r+s+2} = \delta C^{n-r+s+2} \oplus D'^{n-r+s+1} \oplus \delta D'^{n-r+s+2} \to \delta D''_r = \delta D_r \oplus D'_{r-1} \oplus \delta D'_r$$
$$(r \in \mathbb{Z}, s \geqslant 0, \delta\phi'_{-1} = 0)$$

From now on we write

$$\begin{aligned}(\Gamma'', \Phi'') &= (\Gamma, \Phi) \cup (\Gamma', \Phi') \\ &= (\Gamma \cup \Gamma', \Phi \cup \Phi') \\ &= (\Gamma \cup \Gamma', (\delta\nu \cup_{(\delta\phi', \phi')} \delta\nu', \nu \cup_{\phi'} \nu', \delta\phi \oplus -\delta\phi'', \phi \oplus -\phi'')).\end{aligned}$$

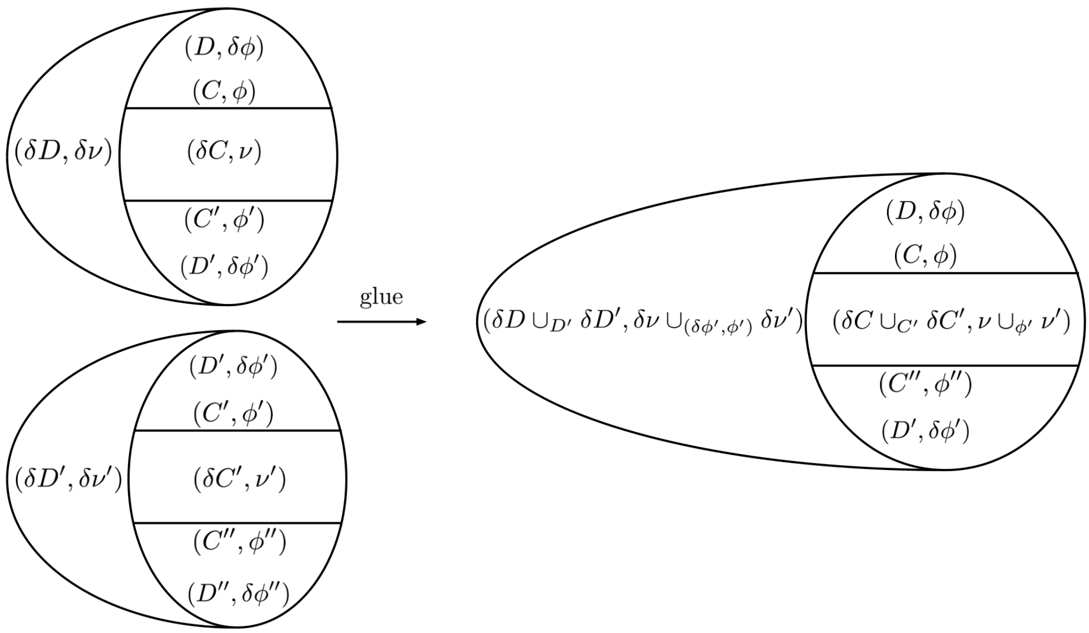


Figure 24: A schematic diagram for the glueing of adjoining ϵ-symmetric relative cobordisms.

Example 2.2.5. Two adjoining oriented $(n+2)$-dimensional relative cobordisms $(\Omega; \Sigma, \Sigma', W; M, M')$, $(\Omega'; \Sigma', \Sigma'', W'; M', M'')$ may be glued over (Σ', M') to produce an oriented $(n+2)$-dimensional relative cobordism $(\Omega \cup_{M'} \Omega; \Sigma, \Sigma'', W \cup_{M'} W'; M, M'')$. If R is a commutative ring with identity then applying the triad symmetric construction from Example 2.1.12 produces three $(n+2)$-dimensional symmetric Poincaré triads $(\Gamma, \Phi), (\Gamma', \Phi'), (\Gamma'', \Phi'')$ over R with

$$\Gamma = \begin{array}{ccc} C(M;R) \oplus C(M';R) & \longrightarrow & C(\Sigma;R) \oplus C(\Sigma';R) \\ \downarrow & & \downarrow \\ C(W;R) & \longrightarrow & C(\Omega;R) \end{array}$$

$$\Phi = (\phi_\Omega, \phi_W, \phi_\Sigma \oplus -\phi_{\Sigma'}, \phi_M \oplus -\phi_{M'})$$

$$\Gamma' = \begin{array}{ccc} C(M';R)\oplus C(M'';R) & \longrightarrow & C(\Sigma';R)\oplus C(\Sigma'';R) \\ \downarrow & & \downarrow \\ C(W';R) & \longrightarrow & C(\Omega';R) \end{array}$$

$$\Phi' = (\phi_{\Omega'}, \phi_{W'}, \phi_{\Sigma'} \oplus -\phi_{\Sigma''}, \phi_M \oplus -\phi_{M''})$$

$$\Gamma'' = \begin{array}{ccc} C(M;R)\oplus C(M'';R) & \longrightarrow & C(\Sigma;R)\oplus C(\Sigma'';R) \\ \downarrow & & \downarrow \\ C(W\cup_{M'} W;R) & \longrightarrow & C(\Omega\cup_{\Sigma'}\Omega;R) \end{array}$$

$$\Phi'' = (\phi_{\Omega\cup_{\Sigma'}\Omega'}, \phi_{W\cup_{M'}W'}, \phi_\Sigma \oplus -\phi_{\Sigma''}, \phi_M \oplus -\phi_{M''}).$$

There is a a homotopy equivalence of $(n+2)$-dimensional symmetric Poincaré triads over R

$$(\Gamma'', \Phi'') \simeq (\Gamma, \Phi) \cup (\Gamma', \Phi') = (\Gamma\cup\Gamma', \Phi\cup\Phi')$$

with

$$\Gamma\cup\Gamma' = \begin{array}{ccc} C(M;R)\oplus C(M'';R) & \longrightarrow & C(\Sigma;R)\oplus C(\Sigma'';R) \\ \downarrow & & \downarrow \\ C(W;R)\cup_{C(M';R)} C(W';R) & \longrightarrow & C(\Omega;R)\cup_{C(\Sigma';R)} C(\Omega';R). \end{array}$$

$$\Phi\cup\Phi' = (\phi_\Omega \cup_{(\phi_{\Sigma'},\phi_{M'})} \phi_{\Omega'}, \phi_W \cup_{\phi_{M'}} \phi_{W'}, \phi_\Sigma \oplus -\phi_{\Sigma''}, \phi_M \oplus -\phi_{M''})$$

This shows that, up to algebraic homotopy equivalence, the glueing of symmetric cobordisms of pairs is an algebraic model for the glueing of relative manifold cobordisms.

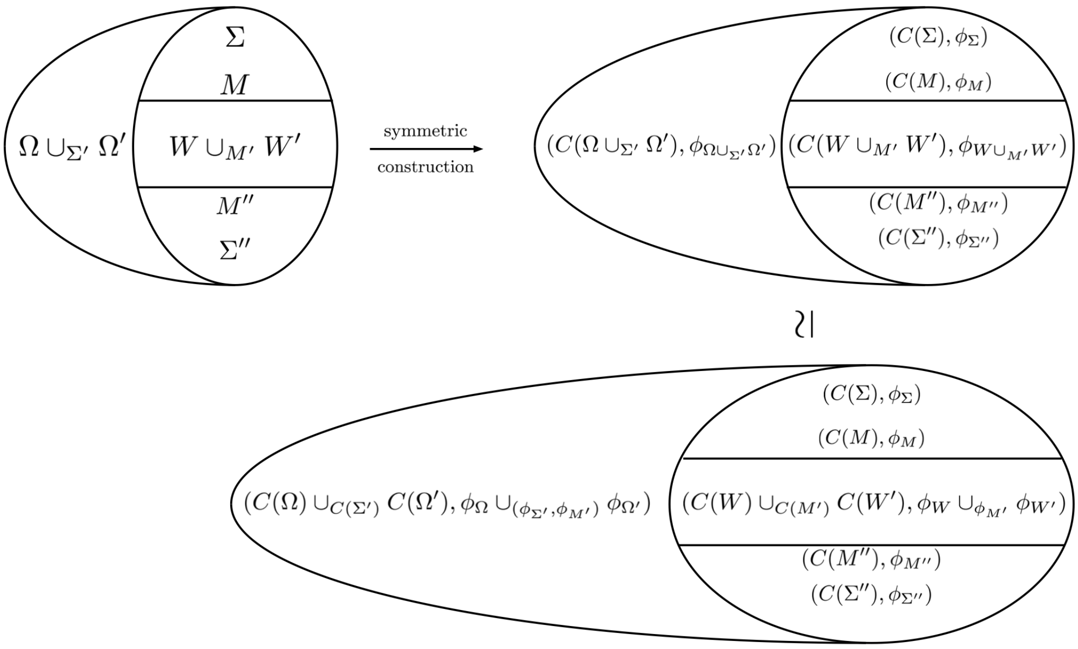


Figure 25: A schematic diagram for the glueing and homotopy equivalence.

2.3 Twisted unions of symmetric pairs and triads

Symmetric pairs and triads may also be glued with a twist. This gives an algebraic model for glueing manifolds with boundary and manifold triads with a twist.

Definition 2.3.1. ([Ran98, p.386-387]). Let

$$\begin{aligned} c &= (f : C \to D, (\delta\phi, -\phi) \in Q^{n+1}(f, \epsilon)) \\ c' &= (f' : C \to D', (\delta\phi', \phi) \in Q^{n+1}(f', \epsilon)) \end{aligned}$$

be two adjoining $(n+1)$-dimensional ϵ-symmetric (Poincaré) pairs over A and let $(h, \chi) : (C, \phi) \to (C, \phi)$ be a self-homotopy equivalence $h : (C, \phi) \to (C, \phi)$ together with a choice of coboundary $\chi \in (W^{\%}C)_{n+1}$ between $\phi \in (W^{\%}C)_n$ and $h^{\%}\phi \in (W^{\%}C)_n$.

(i) The *twist of* c *with respect to* (h, χ) is the $(n+1)$-dimensional ϵ-symmetric (Poincaré) pair over A

$$c(h, \chi) = (fh : C \to D, (\delta\phi + f^{\%}\chi, -\phi) \in Q^{n+1}(fh, \epsilon)).$$

(ii) The *twisted union of* c *and* c' *with respect to* (h, χ) is the $(n+1)$-dimensional ϵ-symmetric (Poincaré) complex over A

$$(D \cup_h D', \delta\phi \cup_\chi \delta\phi' \in Q^{n+1}(D \cup_h D', \epsilon)) = c \cup_{(h,\chi)} c' = c(h, \chi) \cup c'$$

obtained by glueing $c(h, \chi)$ to c', so that the chain complex

$$D \cup_h D' = \mathscr{C}\left(\begin{pmatrix} fh \\ f' \end{pmatrix} : C \to D \oplus D'\right)$$

is given by

$$d_{D \cup_h D'} = \begin{pmatrix} d_D & (-)^{r-1} fh & 0 \\ 0 & d_C & 0 \\ 0 & (-)^{r-1} f' & d_{D'} \end{pmatrix} :$$
$$(D \cup_h D')_r = D_r \oplus C'_{r-1} \oplus D'_r \to (D \cup_h D')_{r-1} = D_{r-1} \oplus C_{r-2} \oplus D'_{r-1} \quad (r \in \mathbb{Z})$$

and the symmetric structure $\delta\phi \cup_\chi \delta\phi'$ is given by

$$(\delta\phi \cup_\chi \delta\phi')_s = \begin{pmatrix} \delta\phi_s + f\chi_s f^* & 0 & 0 \\ (-)^{n-r}\phi_s h^* f^* & (-)^{n-r+s+1} T_\epsilon(\phi_{s-1}) & 0 \\ 0 & (-)^s f'\phi_s & \delta\phi'_s \end{pmatrix}$$
$$: (D \cup_h D')^{n+1-r+s} = D^{n+1-r+s} \oplus C^{n-r+s} \oplus D'^{n+1-r+s} \to$$
$$(D \cup_h D)_r = D_r \oplus C'_{r-1} \oplus D'_r \quad (s \geqslant 0, r \in \mathbb{Z}, \phi_{-1} = 0).$$

Example 2.3.2.

(i) Let (Σ, M) be an oriented $(n+1)$-dimensional manifold with boundary. Recall that the twisted double of (Σ, M) with respect to an orientation preserving homeomorphism $h : M \to M$ is the closed, oriented $(n+1)$-dimensional manifold $\Omega = \Sigma \cup_h -\Sigma$. If R is a commutative ring with identity then applying the symmetric construction produces an $(n+1)$-dimensional symmetric Poincaré pair $(C(M;R) \to C(\Sigma;R), (\phi_\Sigma, \phi_M))$ over R and an $(n+1)$-dimensional symmetric Poincaré complex $(C(\Omega;R), \phi_\Omega)$ over R.

Since $h : M \to M$ is orientation preserving it follows that $h^{\%}(\phi_M) = \phi_M \in Q^n(C(M;R))$. Moreover, as $C(M;R)$ is n-dimensional it follows that $(W^{\%}C(M;R))_{n+1} = 0$ and hence $h^{\%}(\phi_M) = \phi_M \in (W^{\%}C(M;R))_n$ so that $\chi = 0 \in (W^{\%}C(M;R))_{n+1}$. By [Ran98, p.387] there is a homotopy equivalence of $(n+1)$-dimensional symmetric Poincaré complexes over R

$$(C(\Omega;R), \phi_\Omega) \simeq (C(M;R) \to C(\Sigma;R), (\phi_\Sigma, \phi_M)) \cup_{(h,0)} (C(M;R) \to C(\Sigma;R), (-\phi_\Sigma, -\phi_M)).$$

This shows that the twisted glueing of adjoining symmetric pairs is an algebraic model for the twisted glueing of manifolds with boundary.

(ii) In the case that $h = 1 : M \to M$, the twisted geometric and algebraic unions degenerate to untwisted unions so that

$$\begin{aligned}
&(C(\Sigma \cup_M \Sigma; R), \phi_{\Sigma \cup_M \Sigma}) \\
\simeq &(C(M;R) \to C(\Sigma;R), (\phi_\Sigma, \phi_M)) \cup_{(1,0)} (C(M;R) \to C(\Sigma;R), (-\phi_\Sigma, -\phi_M)) \\
= &(C(M;R) \to C(\Sigma;R), (\phi_\Sigma, \phi_M)) \cup (C(M;R) \to C(\Sigma;R), (-\phi_\Sigma, -\phi_M)).
\end{aligned}$$

There is a relative notion of a self-homotopy equivalence of symmetric complex.

Definition 2.3.3. ([Ran98, p.393]). A *self-homotopy equivalence*

$$(\delta l, l, \delta\chi, \chi) : (f : C \to D, (\delta\phi, \phi)) \to (f : C \to D, (\delta\phi, \phi))$$

of an $(n+1)$-dimensional ϵ-symmetric (Poincaré) pair $(f : C \to D, (\delta\phi, \phi))$ over A is a quadruple $(\delta l, l, \delta\chi, \chi)$ consisting of two chain homotopy equivalences $l : C \to C$ and $\delta l : D \to D$ determining a commutative triad

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \delta l} \\ C & \xrightarrow[f]{} & D \end{array}$$

together with chains $\chi \in (W^{\%}C)_{n+1})$ and $\delta\chi \in (W^{\%}D)_{n+2}$ such that

$$l^{\%}(\phi) - \phi = d_{W^{\%}C}\chi \in (W^{\%}C)_n, \quad \delta l^{\%}(\delta\phi) - \delta\phi = d_{W^{\%}D}\delta\chi \in (W^{\%}D)_{n+1}$$

The twisted union of pairs has an extension to triads where we perform a twisted glueing of two triads by twisting the second triad and then glueing.

Definition 2.3.4. Let $(\Gamma, \Phi \in Q^{n+2}(\Gamma, \epsilon))$ and $(\Gamma', \Phi' \in Q^{n+2}(\Gamma', \epsilon))$ be two adjoining $(n+2)$-dimensional ϵ-symmetric (Poincaré) triads over A of the form

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & \searrow^{k} & \downarrow{\scriptstyle h} \\ \delta C & \xrightarrow[\delta f]{} & \delta D \end{array}, \quad \Phi = (\delta\nu, \nu, -\delta\phi, -\phi)$$

$$\Gamma' = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g'}\big\downarrow & \overset{k'}{\searrow} & \big\downarrow{\scriptstyle h'} \\ \delta C' & \xrightarrow[\delta f']{} & \delta D' \end{array}, \quad \Phi' = (\delta\nu', \nu', \delta\phi, \phi)$$

and let $(\delta l, l, \delta\chi, \chi)$ be a self-homotopy equivalence of $(f : C \to D, (-\delta\phi, \phi))$.

(i) The *twist of* $(\Gamma, \Phi \in Q^{n+2}(\Gamma, \epsilon))$ *with respect to* $(\delta l, l, \delta\chi, \chi)$ is the $(n+2)$-dimensional ϵ-symmetric (Poincaré) triad $(\Gamma_{(\delta l, l)}, \Phi_{(\delta\chi,\chi)} \in Q^{n+2}(\Gamma_l, \epsilon))$ over A with

$$\Gamma_{(\delta l,l)} = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle gl}\big\downarrow & \overset{k}{\searrow} & \big\downarrow{\scriptstyle h\delta l} \\ \delta C & \xrightarrow[\delta f]{} & \delta D \end{array}, \quad \Phi_{(\delta\chi,\chi)} = (\delta\nu + h\delta\chi h^*, \nu, -\delta\phi + f\chi f^*, -\phi).$$

(ii) The *twisted union of* $(\Gamma, \Phi \in Q^{n+2}(\Gamma, \epsilon))$ *and* $(\Gamma', \Phi' \in Q^{n+2}(\Gamma', \epsilon))$ *over* $(\delta l, l, \delta\chi, \chi)$ is the $(n+2)$-dimensional ϵ-symmetric (Poincaré) triad over A

$$(\Gamma \cup_{(\delta l,l)} \Gamma', \Phi \cup_{(\delta\chi,\chi)} \Phi' \in Q^{n+2}(\Gamma \cup_{(\delta l,l)} \Gamma', \epsilon))$$

obtained by glueing $(\Gamma_{(\delta l,l)}, \Phi_{(\delta\chi,\chi)})$ to (Γ', Φ') so that

$$(\Gamma, \Phi) \cup_{(\delta l,l,\delta\chi,\chi)} (\Gamma', \Phi') = (\Gamma_{(\delta l,l)}, \Phi_{(\delta\chi,\chi)}) \cup (\Gamma', \Phi') = (\Gamma \cup_{(\delta l,l)} \Gamma', \Phi \cup_{(\delta\chi,\chi)} \Phi').$$

Example 2.3.5.

(i) Let $(\Omega; \Sigma, \Sigma'; M), (\Omega'; \Sigma', \Sigma''; M)$ be two adjoining oriented $(n+2)$-dimensional manifold triads and let $\delta l : (\Sigma', M) \to (\Sigma', M)$ be an orientation preserving homeomorphism which restricts to an orientation preserving homeomorphism $l : M \to M$. The two triads may be glued over (Σ', M) with the glueing twisted by δl to produce an oriented $(n+2)$-dimensional manifold triad $(\Omega \cup_{\delta l} \Omega'; \Sigma, \Sigma''; M)$. If R is a commutative ring with identity then applying the symmetric construction produces three $(n+2)$-dimensional commutative Poincaré triads over R

$$\Gamma = \begin{array}{ccc} C(M;R) & \longrightarrow & C(\Sigma;R) \\ \big\downarrow & & \big\downarrow \\ C(\Sigma';R) & \longrightarrow & C(\Omega;R) \end{array}, \qquad \Phi = (\phi_\Omega, \phi_{\Sigma'}, \phi_\Sigma, \phi_M)$$

$$\Gamma' = \begin{array}{ccc} C(M;R) & \longrightarrow & C(\Sigma';R) \\ \big\downarrow & & \big\downarrow \\ C(\Sigma'';R) & \longrightarrow & C(\Omega';R) \end{array}, \qquad \Phi' = (\phi_{\Omega'}, \phi_{\Sigma''}, \phi_{\Sigma'}, \phi_M)$$

$$\Gamma'' = \begin{array}{ccc} C(M;R) & \longrightarrow & C(\Sigma;R) \\ \big\downarrow & & \big\downarrow \\ C(\Sigma'';R) & \longrightarrow & C(\Omega \cup_l \Omega';R) \end{array}, \qquad \Phi'' = (\phi_{\Omega\cup_{\delta l}\Omega'}, \phi_{\Sigma''}, \phi_\Sigma, \phi_M).$$

Since $(\delta l, l) : (\Sigma', M) \to (\Sigma', M)$ is orientation preserving it follows that

$$(l, \delta l; 0)^{\%}(\phi_{\Sigma'}, \phi_M) = (\phi_{\Sigma'}, \phi_M) \in Q^{n+1}(C(M;R) \to C(\Sigma';R))$$

and hence there are some chains $\delta\chi \in (W^{\%}C(\Sigma';R))_{n+2}, \chi \in (W^{\%}C(M;R))_{n+1}$ such that

$$\delta l^{\%}(\phi_{\Sigma'}) - \phi_{\Sigma'} = d_{W^{\%}C(\Sigma')}(\delta\chi) \in (W^{\%}C(\Sigma;R))_{n+1}, \quad l^{\%}(\phi_M) - \phi_M = d_{W^{\%}C(M)}(\chi) \in (W^{\%}C(M;R))_{n+1}$$

Since $C(\Sigma';R)$ is $(n+1)$-dimensional and $C(M;R)$ is n-dimensional it follows that $(W^{\%}C(\Sigma';R))_{n+2} = 0$ and $(W^{\%}C(M;R))_{n+2} = 0$ and hence $\delta\chi = 0$ and $\chi = 0$. Then there is a homotopy equivalence

$$(\Gamma'', \Phi'') \simeq (\Gamma, \Phi) \cup_{(\delta l, l, 0, 0)} (\Gamma', \Phi') = (\Gamma \cup_{(\delta l, l)} \Gamma', \Phi \cup \Phi')$$

with

$$\Gamma \cup_{(\delta l, l)} \Gamma' = \begin{array}{ccc} C(M;R) & \longrightarrow & C(\Sigma;R) \\ \downarrow & & \downarrow \\ C(\Sigma'';R) & \longrightarrow & C(\Omega;R) \cup_l C(\Omega';R) \end{array}$$

$$\Phi \cup \Phi' = (\phi_\Omega \cup \phi_{\Omega'}, \phi_{\Sigma''}, \phi_\Sigma, \phi_M).$$

This shows that, up to homotopy equivalence, the twisted glueing of symmetric triads is an algebraic model for the twisted glueing of triads.

(ii) In the case that $\delta l = 1 : \Sigma' \to \Sigma'$, the twisted geometric and algebraic unions degenerate to untwisted unions so that

$$(\Gamma, \Phi) \cup_{(\delta l, l, 0, 0)} (\Gamma', \Phi') = (\Gamma, \Phi) \cup (\Gamma', \Phi') \simeq (\Gamma'', \Phi'').$$

Chapter 3

Thickening manifold and symmetric Poincaré quiver representations

In this chapter we extend the definition of a symmetric pair to a symmetric pair with an ℓ-fold boundary splitting and show that this is an algebraic model for a manifold with boundary where the boundary can be written as a cyclic union of adjoining cobordisms. We define algebraic thickening operations which are algebraic models for taking the product of a cobordism with an interval and for taking the product of a closed manifold with a disc D^2 where the boundary ∂D^2 is split into ℓ pieces. We then extend the symmetric construction to a symmetric construction for an oriented manifold representation of a quiver where the vertices parametrise manifolds and the arrows parametrise cobordisms. This produces a symmetric Poincaré representation of a quiver where the vertices parametrise symmetric Poincaré complexes and the arrows parametrise symmetric Poincaré cobordisms. We use the quiver symmetric construction together with the thickening operations to generalise the manifold and symmetric Poincaré trinity thickening operations of [BNR12a, p.44-46] to thickening operations for manifold and symmetric Poincaré representations of a quiver where parts of the data can be twisted by a self-homotopy equivalence. We then show that the twisted thickening operations commute with the symmetric construction up to homotopy equivalence.

3.1 Products of symmetric complexes and pairs

There is a product operation for symmetric complexes and pairs which gives an algebraic model for products of manifolds and manifolds with boundary.

Definition 3.1.1. Let A, B be rings with involution.

(i) The *tensor product* of A and B is the ring with involution $A \otimes_{\mathbb{Z}} B$ where

$$\overline{a \otimes b} = \overline{a} \otimes \overline{b} \quad (a \in A, b \in B).$$

(ii) The *tensor product* of an A-module chain complex C and an B-module chain complex D is the $A \otimes_{\mathbb{Z}} B$-module chain complex $C \otimes_{\mathbb{Z}} D$ with the scalar action of $A \otimes_{\mathbb{Z}} B$ given by

$$A \otimes_{\mathbb{Z}} B \times C \otimes_{\mathbb{Z}} D \to C \otimes_{\mathbb{Z}} D; \quad (a \otimes b, x \otimes y) \mapsto ax \otimes by.$$

The product operation for symmetric structures

$$\otimes : Q^m(C, \epsilon) \otimes_{\mathbb{Z}} Q^n(D, \eta) \to Q^{m+n}(C \otimes_{\mathbb{Z}} D, \epsilon \otimes \eta)$$

arises from the following chain level construction.

Proposition 3.1.2. ([Ran80a, p.174-6]). Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$ resolution of $\mathbb{Z}$

$$W : \ldots \to W_3 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} W_2 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} W_1 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} W_0 = \mathbb{Z}[\mathbb{Z}_2] \to \ \to \ldots$$

let $\widehat{W}$ be the complete $\mathbb{Z}[\mathbb{Z}_2]$ resolution of $\mathbb{Z}$

$$\widehat{W} : \ldots \to \widehat{W}_3 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \widehat{W}_2 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \widehat{W}_1 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \widehat{W}_0 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \widehat{W}_{-1} = \mathbb{Z}[\mathbb{Z}_2] \to \ldots$$

and let $\epsilon, \eta = \pm 1$.

(i) For a finite-dimensional A-module chain complex C and a finite-dimensional B-module chain complex D there is a natural identification of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$(C^t \otimes_A C) \otimes_{\mathbb{Z}} (D^t \otimes_B D) \cong (C \otimes_{\mathbb{Z}} D)^t \otimes_{A \otimes_{\mathbb{Z}} B} (C \otimes_{\mathbb{Z}} D)$$

respecting the $\mathbb{Z}_2$-action given by $T_\epsilon \otimes T_\eta$ on the left and $T_{\epsilon \otimes \eta}$ on the right.

(ii) It is possible to construct a diagonal chain map $\Delta : \widehat{W} \to \widehat{W} \otimes_{\mathbb{Z}} \widehat{W}$ such that the restriction $\Delta : W \to W \otimes_{\mathbb{Z}} W$ defines a natural chain map

$$\otimes : W^{\%}(C) \otimes_{\mathbb{Z}} W^{\%}(D) \to W^{\%}(C \otimes_{\mathbb{Z}} D)$$

such that the product of chains

$$\phi = \{\phi_s : C^{m-r+s} \to C_r | s \geqslant 0, r \in \mathbb{Z}\} \in (W^{\%}C)_m$$
$$\theta = \{\theta_s : D^{n-r+s} \to D_r | s \geqslant 0, r \in \mathbb{Z}\} \in (W^{\%}D)_n$$

is the chain

$$\phi \otimes \theta = \{(\phi \otimes \theta)_s : (C \otimes_{\mathbb{Z}} D)^{m+n-r+s} \to (C \otimes_{\mathbb{Z}} D)_r | s \geqslant 0, r \in \mathbb{Z}\} \in W^{\%}(C \otimes_{\mathbb{Z}} D)_{m+n}$$

with

$$(\phi \otimes \theta)_s = \sum_{t=0}^{s} (-)^{(m+t)s} \phi_t \otimes T_\eta^r \theta_{s-t} : (C \otimes D)^{m+n-r+s} \to (C \otimes D)_r$$

(iii) The chain map $\otimes : W^{\%}(C) \otimes_{\mathbb{Z}} W^{\%}(D) \to W^{\%}(C \otimes_{\mathbb{Z}} D)$ induces a natural morphism of Q-groups

$$\otimes : Q^m(C, \epsilon) \otimes_{\mathbb{Z}} Q^n(D, \eta) \to Q^{m+n}(C \otimes_{\mathbb{Z}} D, \epsilon \otimes \eta)$$

defined by the composition

$$\begin{aligned} Q^m(C,\epsilon)\otimes_{\mathbb{Z}} Q^n(D,\eta) = H_m(W^{\%}(C))\otimes_{\mathbb{Z}} H_n(W^{\%}(D)) &\to H_{m+n}(W^{\%}(C)\otimes_{\mathbb{Z}} W^{\%}(D)) \\ &\xrightarrow{\otimes} H_{m+n}(W^{\%}(C\otimes_{\mathbb{Z}} D)) \\ &= Q^{m+n}(C\otimes_{\mathbb{Z}} D,\epsilon\otimes\eta). \end{aligned}$$

This construction allows us to take products of two symmetric pairs as follows.

Theorem 3.1.3. Let $\epsilon,\eta=\pm 1$. The product of an m-dimensional ϵ-symmetric (Poincaré) pair over A and an n-dimensional η-symmetric (Poincaré) pair over B

$$(f:C\to D,(\delta\phi,\phi)\in Q^m(f,\epsilon))\otimes(f':C'\to D',(\delta\phi',\phi')\in Q^n(f',\eta))$$

is an $(m+n)$-dimensional $\epsilon\otimes\eta$-symmetric (Poincaré) triad $(\Gamma,\Phi\in Q^{m+n}(\Gamma,\epsilon\otimes\eta))$ over $A\otimes_{\mathbb{Z}} B$ with

$$\Gamma = \begin{array}{ccc} C\otimes_{\mathbb{Z}} C' & \xrightarrow{1\otimes f'} & C\otimes_{\mathbb{Z}} D' \\ {\scriptstyle f\otimes 1}\downarrow & & \downarrow{\scriptstyle f\otimes 1} \\ D\otimes_{\mathbb{Z}} C' & \xrightarrow[1\otimes f']{} & D\otimes_{\mathbb{Z}} D' \end{array}$$

$$\Phi = (\delta\phi\otimes\delta\phi',\phi\otimes\delta\phi',\delta\phi\otimes\phi',\phi\otimes\phi')$$

Proof. The ϵ-symmetric (Poincaré) pair over A and the η-symmetric (Poincaré) pair over B

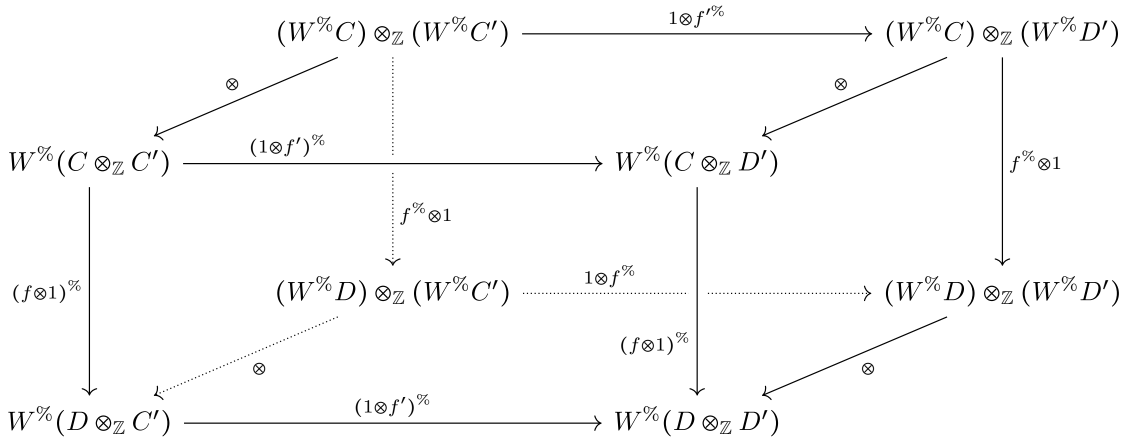
$$(f:C\to D,(\delta\phi,\phi)\in Q^m(f,\epsilon)),\quad (f':C'\to D',(\delta\phi',\phi')\in Q^n(f',\eta))$$

induce two commutative triads over $\mathbb{Z}$

$$\begin{array}{ccc} (W^{\%}C)\otimes_{\mathbb{Z}}(W^{\%}C') & \xrightarrow{1\otimes f'^{\%}} & (W^{\%}C)\otimes_{\mathbb{Z}}(W^{\%}D) \\ {\scriptstyle f^{\%}\otimes 1}\downarrow & & \downarrow{\scriptstyle f^{\%}\otimes 1} \\ (W^{\%}D)\otimes_{\mathbb{Z}}(W^{\%}C') & \xrightarrow[1\otimes f'^{\%}]{} & (W^{\%}D)\otimes_{\mathbb{Z}}(W^{\%}D') \end{array}$$

$$\begin{array}{ccc} W^{\%}(C\otimes_{\mathbb{Z}} C') & \xrightarrow{(1\otimes f')^{\%}} & W^{\%}(C\otimes_{\mathbb{Z}} D') \\ {\scriptstyle (f\otimes 1)^{\%}}\downarrow & & \downarrow{\scriptstyle (f\otimes 1)^{\%}} \\ W^{\%}(D\otimes_{\mathbb{Z}} C') & \xrightarrow[(1\otimes f')^{\%}]{} & W^{\%}(D\otimes_{\mathbb{Z}} D') \end{array}$$

The naturality of the chain level product determines a commutative cube



.

so that the cycles $(\delta\phi, \phi) \in \mathscr{C}(f^{\%}), (\delta\phi', \phi') \in \mathscr{C}(f'^{\%})$ determine a cycle

$$\Phi = (\delta\phi \otimes \delta\phi', \phi \otimes \delta\phi', \delta\phi \otimes \phi', \phi \otimes \phi')$$

representing an $(n+m)$-dimensional $\epsilon \otimes \eta$-symmetric structure on the triad

$$\begin{array}{ccc} C \otimes_{\mathbb{Z}} C' & \xrightarrow{1 \otimes f'} & C \otimes_{\mathbb{Z}} D' \\ {\scriptstyle f \otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle f \otimes 1} \\ D \otimes_{\mathbb{Z}} C' & \xrightarrow[1 \otimes f']{} & D \otimes_{\mathbb{Z}} D' \end{array}$$

over $A \otimes_{\mathbb{Z}} B$. □

The product of symmetric pairs then determines the product of a symmetric complex and a symmetric pair as a special case.

Corollary 3.1.4. The product of an m-dimensional ϵ-symmetric (Poincaré) complex $(C, \phi \in Q^n(C, \epsilon))$ over A and an n-dimensional η-symmetric (Poincaré) pair $(f' : C' \to D', (\delta\phi', \phi') \in Q^n(f, \eta))$ over B is the $(m+n)$-dimensional $\epsilon \otimes \eta$-symmetric (Poincaré) pair

$$\begin{aligned} &(C, \phi \in Q^n(C, \epsilon)) \otimes (f' : C' \to D', (\delta\phi', \phi') \in Q^n(f, \eta)) \\ =&(1 \otimes f' : C \otimes_{\mathbb{Z}} C' \to C \otimes_{\mathbb{Z}} D', (\phi \otimes \delta\phi', \phi \otimes \phi') \in Q^{m+n}(1 \otimes f', \epsilon \otimes \eta)) \end{aligned}$$

over $A \otimes_{\mathbb{Z}} B$.

Proof. The m-dimensional ϵ-symmetric (Poincaré) complex $(C, \phi \in Q^m(C, \epsilon))$ determines an m-dimensional ϵ-symmetric (Poincaré) pair $(0 : 0 \to C, (\phi, 0) \in Q^m(0, \epsilon))$ such that the product

$$(0 : 0 \to C, (\phi, 0) \in Q^m(0, \epsilon)) \otimes (f' : C' \to D', (\delta\phi', \phi') \in Q^n(f, \eta))$$

is the $(m+n)$-dimensional $\epsilon \otimes \eta$-symmetric (Poincaré) triad $(\Gamma, \Phi \in Q^{m+n}(\Gamma, \epsilon \otimes \eta))$ over $R \otimes_{\mathbb{Z}} S$

with

$$\Gamma = \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ C \otimes_{\mathbb{Z}} C' & \xrightarrow[1\otimes f]{} & C \otimes_{\mathbb{Z}} D' \end{array}$$

$$\Phi = (\phi \otimes \delta\phi', \phi \otimes \phi', 0, 0)$$

which determines the $(m+n)$-dimensional $\epsilon \otimes \eta$-symmetric (Poincaré) pair over $A \otimes_{\mathbb{Z}} B$

$$(1 \otimes f : C \otimes_{\mathbb{Z}} C' \to C \otimes_{\mathbb{Z}} D', (\phi \otimes \delta\phi', \phi \otimes \phi'))$$

□

Proposition 3.1.5. ([Ran80b, Proposition 8.1]). Let R be a commutative ring with identity. The symmetric construction is natural with respect to absolute and relative products.

Examples of products are given in the next section.

3.2 Manifold and symmetric Poincaré thickenings

We now use the naturality of the symmetric construction with respect to products to examine the symmetric pairs and triads obtained by thickening a manifold or cobordism by taking the product with an closed interval or a disc.

Definition 3.2.1.

(i) The *thickening* of a closed, oriented n-dimensional manifold M is the oriented $(n+1)$-dimensional cobordism

$$M \times (I : \{0\}, \{1\}) = (M \times I; M \times \{0\}, M \times \{1\}).$$

(ii) The *thickening* of an oriented $(n+1)$-dimensional cobordism $(W; M, M')$ is the oriented $(n+2)$-dimensional triad

$$\begin{array}{ccc} (M \sqcup M') \times \{0,1\} & \longrightarrow & (M \sqcup M') \times I \\ \downarrow & & \downarrow \\ W \times \{0,1\} & \longrightarrow & W \times I \end{array}$$

which is equal to the product of the oriented cobordisms $(W; M, M')$ and $(I; \{0\}, \{1\})$.

(iii) The *disc thickening* of a closed, oriented n-dimensional manifold M is the oriented $(n+2)$-dimensional manifold with boundary

$$M \times (D^2, S^1) = (M \times D^2; M \times S^1).$$

The effect of applying the symmetric construction to these geometric thickening operations is as follows.

Example 3.2.2. Let R be a commutative ring with identity and let $(W; M, M')$ be an $(n+1)$-dimensional oriented cobordism.

(i) Thickening M and applying the symmetric construction to the oriented $(n+1)$-dimensional cobordism $M\times(I;\{0\},\{1\}) = (M\times I; M\times\{0\}, M\times\{1\})$ produces an $(n+1)$-dimensional algebraic cobordism over R

$$((i_0\ \ i_1): C(M;R)\oplus C(M;R) \to C(M\times I;R), (\phi_{M\times I}, \phi_M \oplus -\phi_M))$$

with the chain map i_0 induced by the inclusion $i_0 : M = M\times\{0\} \hookrightarrow M\times I$ and the chain map i_1 induced by the inclusion $i_1 : M = M\times\{1\} \hookrightarrow M\times I$. The symmetric construction may be applied to M to produce an n-dimensional symmetric complex $(C(M;R), \phi_M)$ over R. The symmetric construction may be applied to $(I;\{0\},\{1\})$ to produce 1-dimensional symmetric Poincaré pair over $\mathbb{Z}$ and by Example 2.1.7 this symmetric pair is homotopy equivalent to $((1\ \ 1): \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}, (0, 1\oplus -1))$. The product of this symmetric complex and pair is then an $(n+1)$-dimensional symmetric pair over $R = R\otimes_{\mathbb{Z}}\mathbb{Z}$

$$\begin{aligned}&(C(M;R),\phi_M)\otimes((1\ \ 1):\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z},(0,1\oplus -1))\\ =&((1\ \ 1): C(M;R)\oplus C(M;R)\to C(M;R),(0,\phi_M\oplus -\phi_M)).\end{aligned}$$

This $(n+1)$-dimensional symmetric pair is homotopy equivalent to the $(n+1)$-dimensional symmetric pair

$$((i_0\ \ i_1): C(M;R)\oplus C(M;R) \to C(M\times I;R), (\phi_{M\times I}, \phi_M \oplus -\phi_M))$$

since the Eilenberg-Zilber chain homotopy equivalence $C(M\times I;R) \simeq C(M;R)\otimes_{\mathbb{Z}} C(I;\mathbb{Z})$ determines a triad over R

$$\begin{array}{ccc} C(M;R)\oplus C(M;R) & \xrightarrow{(i_0\ \ i_1)} & C(M\times I;R) \\ {\scriptstyle 1}\downarrow & \searrow & \downarrow \\ C(M;R)\oplus C(M;R) & \xrightarrow[(i_0\ \ i_1)]{} & C(M;R)\otimes_{\mathbb{Z}} C(I;\mathbb{Z}) \end{array}$$

where the two vertical maps are chain homotopy equivalences. This implies that up to homotopy equivalence, the symmetric pair induced by the cobordism $(M\times I; M\times\{0\}, M\times\{1\})$ can be obtained solely from the symmetric complex induced by M.

(ii) Thickening $(W; M, M')$ produces an oriented manifold triad

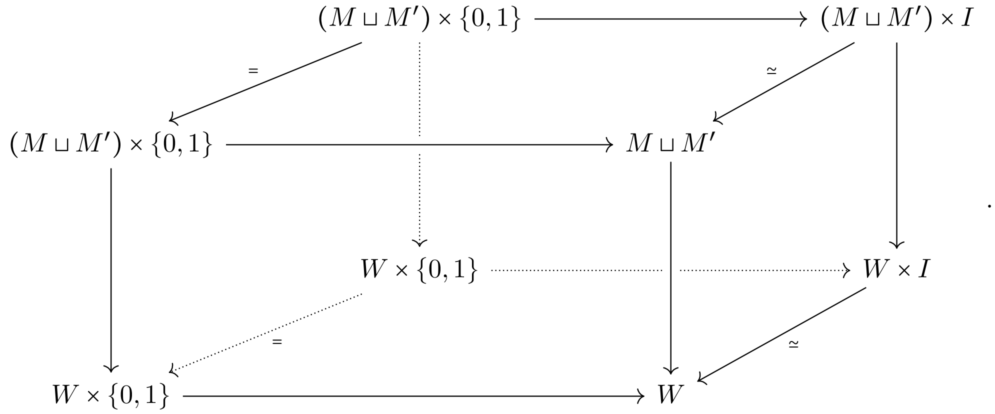
$$\begin{array}{ccc} (M\sqcup M')\times\{0,1\} & \longrightarrow & (M\sqcup M')\times I \\ \downarrow & & \downarrow \\ W\times\{0,1\} & \longrightarrow & W\times I \end{array}$$

If R is a commutative ring with identity then applying the symmetric construction produces an $(n+2)$-dimensional symmetric Poincaré triad (Γ, Φ) over R with

$$\Gamma = \begin{array}{ccc} C(M;R) \oplus C(M';R) \oplus C(M;R) \oplus C(M';R) & \longrightarrow & C(M \times I;R) \oplus C(M' \times I;R) \\ \downarrow & & \downarrow \\ C(W;R) \oplus C(W;R) & \longrightarrow & C(W \times I;R) \end{array}$$

$$\Phi = (\phi_{W \times I}, \phi_W \oplus -\phi_W, -\phi_{M \times I} \oplus \phi_{M' \times I}, \phi_M \oplus -\phi'_M \oplus -\phi_M \oplus \phi'_M).$$

There is a commutative cube



where each of the sloped arrows is a homotopy equivalence and all other arrows are inclusions. Example 3.2.2 implies that there is a homotopy equivalence of $(n+2)$-dimensional symmetric Poincaré triads $(\Gamma, \Phi) \simeq (\Gamma', \Phi')$ where

$$\Gamma' = \begin{array}{ccc} C(M;R) \oplus C(M';R) \oplus C(M;R) \oplus C(M';R) & \longrightarrow & C(M;R) \oplus C(M';R) \\ \downarrow & & \downarrow \\ C(W;R) \oplus C(W;R) & \longrightarrow & C(W;R) \end{array}$$

$$\Phi' = (0, \phi_W \oplus -\phi_W, 0, \phi_M \oplus -\phi'_M \oplus -\phi_M \oplus \phi'_M).$$

This shows that up to homotopy equivalence, the symmetric Poincaré triad induced from the thickening of $(W; M, M')$ may be obtained from the symmetric Poincaré cobordism induced from $(W; M, M')$.

(iii) A closed, oriented n-dimensional manifold M may be disc thickened to produce an oriented $(n+2)$-dimensional manifold with boundary $(M \times D^2, M \times S^1)$. If R is a commutative ring with identity then applying the symmetric construction produces an $(n+2)$-dimensional symmetric Poincaré pair over R

$$(C(M \times S^1; R) \to C(M \times D^2; R), (\phi_{M \times D^2}, \phi_{M \times S^1})).$$

The naturality of the symmetric construction with respect to products and Example 2.1.7 imply that there is a homotopy equivalence of $(n+2)$-dimensional symmetric Poincaré pairs over R

$$\begin{aligned}
&(C(M\times S^1;R)\to C(M\times D^2;R),(\phi_{M\times D^2},\phi_{M\times S^1}))\\
\simeq&(C(M;R),\phi_M)\otimes(C(S^1;\mathbb{Z})\to C(D^2;\mathbb{Z}),(\phi_{D^2},\phi_{S^1}))\\
\simeq&(C(M;R),\phi_M)\otimes(C(S^1;\mathbb{Z})\to\mathbb{Z},(0,\phi_{S^1}))\\
=&(C(M;R)\otimes_{\mathbb{Z}}C(S^1;\mathbb{Z})\to C(M;R),(0,\phi_M\otimes\phi_{S^1}))
\end{aligned}$$

so that up to homotopy equivalence, the symmetric Poincaré pair induced from $(M\times D^2, M\times S^1)$ may be obtained from the symmetric Poincaré complexes induced from M and S^1.

This motivates the following algebraic thickening operations.

Definition 3.2.3.

(i) The *thickening* of an n-dimensional ϵ-symmetric (Poincaré) complex (C,ϕ) over A is the $(n+1)$-dimensional ϵ-symmetric (Poincaré) pair over A

$$((1\ \ 1):C\oplus C\to C,(0,\phi\oplus-\phi))$$

which is homotopy equivalent to the (Poincaré) product pair over $A=A\otimes_{\mathbb{Z}}\mathbb{Z}$

$$(C,\phi)\otimes(C(S^0;\mathbb{Z})\to C(D^1;\mathbb{Z}),(\phi_{D^1},\phi_{S^0})).$$

(ii) The *disc thickening* of an n-dimensional ϵ-symmetric (Poincaré) complex (C,ϕ) over A is the $(n+2)$-dimensional ϵ-symmetric (Poincaré) pair over A

$$(C\otimes_{\mathbb{Z}}C(S^1;\mathbb{Z})\to C,(0,\phi\otimes\phi_{S^1}))$$

which is homotopy equivalent to the (Poincaré) product pair over $A=A\otimes_{\mathbb{Z}}\mathbb{Z}$.

$$(C,\phi)\otimes(C(S^1;\mathbb{Z})\to C(D^2;\mathbb{Z}),(\phi_{D^2},\phi_{S^1})).$$

(iii) The *thickening* of an $(n+1)$-dimensional ϵ-symmetric cobordism over A

$$((f\ \ f'):C\oplus C'\to D,(\delta\phi,\phi\oplus-\phi'))$$

is the $(n+2)$-dimensional ϵ-symmetric commutative (Poincaré) triad (Γ,Φ) over A with

$$\Gamma=\begin{array}{ccc}
C\oplus C'\oplus C\oplus C' & \xrightarrow{\begin{pmatrix}1&0&1&0\\0&1&0&1\end{pmatrix}} & C\oplus C'\\
{\scriptstyle\begin{pmatrix}f&f'&0&0\\0&0&f&f'\end{pmatrix}}\downarrow & & \downarrow{\scriptstyle(f\ \ f')}\\
D\oplus D & \xrightarrow[(1\ \ 1)]{} & D
\end{array}$$

$$\Phi=(0,\delta\phi\oplus-\delta\phi,0,\phi\oplus-\phi'\oplus-\phi\oplus\phi')$$

which is homotopy equivalent to the ϵ-symmetric commutative (Poincaré) product triad over

$A = A \otimes_{\mathbb{Z}} \mathbb{Z}$

$$((f\ \ f') : C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi')) \otimes (i : C(S^0; \mathbb{Z}) \to C(D^1; \mathbb{Z}), (\phi_{D^1}, \phi_{S^0})).$$

Proposition 3.2.4. Let R be a commutative ring with identity. For manifolds and cobordisms, the symmetric construction commutes, up to homotopy equivalence, with (disc) thickenings.

Proof. By Example 3.2.2. □

3.3 Manifold and symmetric Poincaré pairs with boundary splittings

We now extend the definition of a symmetric pair to a symmetric pair with an ℓ-fold boundary splitting and show that this is an algebraic model for a manifold with boundary where the boundary can be written as a cyclic union of adjoining cobordisms.

Definition 3.3.1. An *oriented $(n+1)$-dimensional manifold with an ℓ-fold boundary splitting*

$$(W, \partial W; \partial_1 W, \ldots, \partial_\ell W) \quad (\ell \geqslant 2)$$

consists of an oriented $(n+1)$-dimensional manifold with boundary $(W, \partial W)$ together with a collection of ℓ cyclically adjoining, oriented n-dimensional cobordisms $\{(\partial_i W; M_i, M_{i+1})\}_{i=1}^{\ell}$ such that:

(i) each $\partial_i W$ is a codimension 0-submanifold of ∂W

(ii) $\partial W = \cup_{i=1}^{\ell} \partial_i W$ where the orientations agree

(iii) $\partial_i W \cap \partial_{i+1} W = M_{i+1}$ where the index i is understood cyclically

(iv) $\partial_i W \cap \partial_j W = \varnothing$ if $i \neq j$ where the indices i, j are understood cyclically.

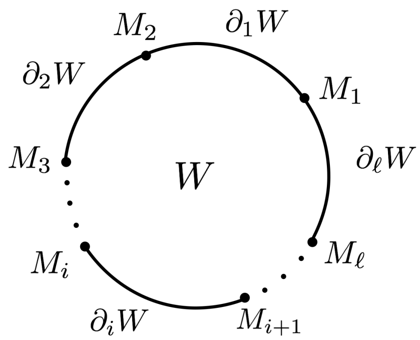


Figure 26: A schematic diagram for the boundary decomposition.

Example 3.3.2. If $\ell \geqslant 2$ then subdividing S^1 into arcs $I_1, I_2, \ldots, I_\ell \subset S^1$ as shown below induces an ℓ-fold boundary splitting $(D^2, S^1; I_1, \ldots, I_\ell)$ of (D^2, S^1)

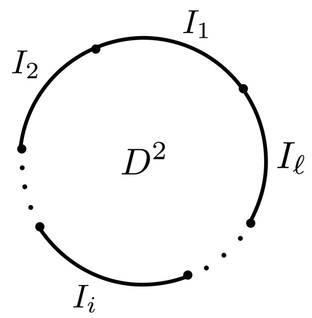


Figure 27: A schematic diagram for the boundary decomposition of D^2.

Definition 3.3.3. The *disc thickening with an ℓ-fold boundary splitting* of a closed, oriented n-dimensional manifold M is the ℓ-fold boundary splitting

$$(M \times D^2, M \times S^1; M \times I_1, M \times I_2, \dots, M \times I_\ell) = M \times (D^2; I_1, \dots, I_\ell) \quad (\ell \geqslant 2)$$

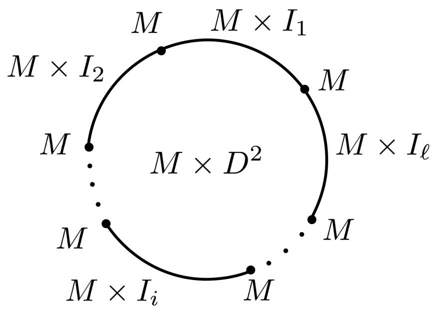


Figure 28: A schematic diagram for the boundary decomposition of $M \times D^2$.

This motivates the definition of a symmetric pair with a split boundary.

Definition 3.3.4.

(i) An $(n+1)$-dimensional ϵ-symmetric (Poincaré) pair over A with an ℓ-fold boundary splitting

$$(c; c_1, \dots, c_\ell) \quad (\ell \geq 2)$$

consists of an $(n+1)$-dimensional ϵ-symmetric (Poincaré) pair over A

$$c = (f : \partial D \to D, (\phi_D, \phi_{\partial D}))$$

together with a collection of ℓ cyclically adjoining n-dimensional ϵ-symmetric (Poincaré) pairs over A of the form

$$c_i = ((f_i^i \;\; f_{i+1}^i) : C_i \oplus C_{i+1} \to \partial_i D, (\phi_{\partial_i D}, \phi_{C_i} \oplus -\phi_{C_{i+1}})) \quad (1 \leqslant i \leqslant \ell)$$

such that there is a homotopy equivalence of n-dimensional ϵ-symmetric (Poincaré) pairs $(\partial D, \phi_{\partial D}) \simeq \cup_{i=1}^{\ell} c_i$ over A

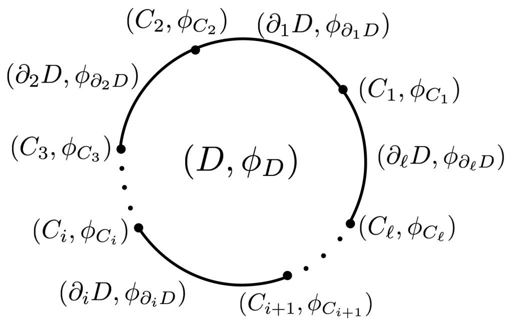


Figure 29: A schematic diagram for the boundary decomposition defined in the homotopy equivalence.

(ii) A *homotopy equivalence* of $(n+1)$-dimensional ϵ-symmetric pairs over A with ℓ-fold boundary splittings

$$(\Gamma; \Gamma_1, \ldots, \Gamma_\ell) : (c; c_1, \ldots, c_\ell) \simeq (c'; c'_1, \ldots, c'_\ell)$$

consists of a homotopy equivalence of $(n+1)$-dimensional ϵ-symmetric pairs over A

$$\Gamma = \left(\begin{array}{ccc} \partial D & \xrightarrow{f} & D \\ g \big\downarrow & \overset{k}{\searrow} & \big\downarrow h \\ \partial D' & \xrightarrow[f']{} & D' \end{array} \right) : c \simeq c'$$

together with a collection of homotopy equivalences of $(n+1)$-dimensional ϵ-symmetric cobordisms over A

$$\Gamma_i = \left(\begin{array}{ccc} C_i \oplus C_{i+1} & \xrightarrow{\begin{pmatrix} f^i_i & f^i_{i+1} \end{pmatrix}} & \partial_i D \\ \begin{pmatrix} g^i_i & 0 \\ 0 & g^i_{i+1} \end{pmatrix} \Big\downarrow & \overset{\begin{pmatrix} k^i_i & k^i_{i+1} \end{pmatrix}}{\searrow} & \Big\downarrow h_i \\ C'_i \oplus C'_{i+1} & \xrightarrow[\begin{pmatrix} f'^i_i & f'^i_{i+1} \end{pmatrix}]{} & \partial_i D' \end{array} \right) : c_i \simeq c'_i$$

such that there is a commutative diagram

$$\left(\begin{array}{ccc} (\partial D, \phi_{\partial D}) & \xrightarrow{\simeq} & \cup_{i=1}^{\ell} c_i \\ \simeq \big\downarrow g & & \cup_{i=1}^{\ell} \Gamma_i \big\downarrow \simeq \\ (\partial D', \phi_{\partial D'}) & \xrightarrow[\simeq]{} & \cup_{i=1}^{\ell} c'_i \end{array} \right)$$

respecting the ϵ-symmetric structures.

In the next section we will work with thickenings of manifolds and Poincaré pairs with split boundaries where the geometric and algebraic data is parametrised by a quiver. It will be more convenient to work with the schematic diagrams of the type in Definition 3.3.1 and Definition 3.3.4 rather than the schematic diagrams of chapter 1 of the type

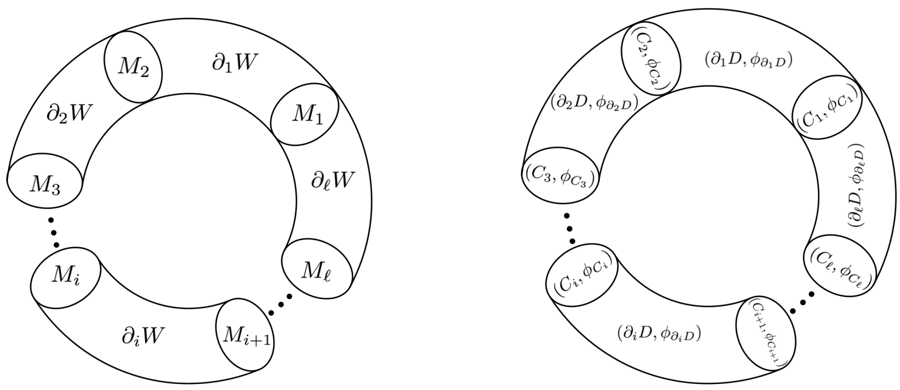


Figure 30: The schematic diagrams from chapter 1.

Example 3.3.5. Applying the symmetric construction over $\mathbb{Z}$ to the ℓ-fold boundary splitting $(D^2, S^1; I_1, \ldots, I_\ell)$ of (D^2, S^1) from Example 3.3.2 produces a 2-dimensional symmetric Poincaré pair over $\mathbb{Z}$ with an ℓ-fold boundary splitting $(c; c_1, \ldots, c_\ell)$ where

$$\begin{aligned} c &= (C(S^1;\mathbb{Z}) \to C(D^2;\mathbb{Z}), (\phi_{D^2}, \phi_{S^1})) \\ c_i &= (C(S^0;\mathbb{Z}) \to C(I_i;\mathbb{Z}), (\phi_{I_i}, \phi_{S^0})) \quad (1 \leqslant i \leqslant \ell). \end{aligned}$$

By Examples 2.1.7 and 2.1.7 there is a homotopy equivalence of 2-dimensional symmetric Poincaré pairs over $\mathbb{Z}$ with ℓ-fold boundary splittings $(c; c_1, \ldots, c_\ell) \simeq (c'; c'_1, \ldots, c'_\ell)$ where

$$\begin{aligned} c' &= (C(S^1;\mathbb{Z}) \to \mathbb{Z}, (0, \phi_{S^1})) \\ c'_i &= (\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}, (0, 1 \oplus -1)) \quad (1 \leqslant i \leqslant \ell). \end{aligned}$$

Let M be a closed, oriented n-dimensional manifold and let R be a commutative ring with identity. Applying the symmetric construction over R to the disc thickening of M with an ℓ-fold boundary splitting

$$(M \times D^2, M \times S^1; M \times I_1, M \times I_2, \ldots, M \times I_\ell)$$

produces an $(n+2)$-dimensional symmetric Poincaré pair over R with an ℓ-fold boundary splitting $(c''; c''_1, \ldots, c''_\ell)$ where

$$\begin{aligned} c'' &= (C(M \times S^1; R) \to C(M \times D^2; R), (\phi_{M \times D^2}, \phi_{M \times S^1})) \\ c''_i &= (C(M;R) \oplus C(M;R) \to C(M \times I_i; R), (\phi_{M \times I_i}, \phi_M \oplus -\phi_M)) \quad (1 \leqslant i \leqslant \ell). \end{aligned}$$

By the naturality of the symmetric construction with respect to products it follows that there is a homotopy equivalence of $(n+2)$-dimensional symmetric Poincaré pairs over R with ℓ-fold boundary splittings $(c''; c''_1, \ldots, c''_\ell) \simeq (c'''; c'''_1, \ldots, c'''_\ell)$ where

$$\begin{aligned} c''' &= (C(M;R) \otimes_{\mathbb{Z}} C(S^1;\mathbb{Z}) \to C(M;R), (0, \phi_M \otimes \phi_{S^1})) \\ c'''_i &= (C(M;R) \oplus C(M;R) \to C(M;R), (0, \phi_M \oplus -\phi_M)) \quad (1 \leqslant i \leqslant \ell). \end{aligned}$$

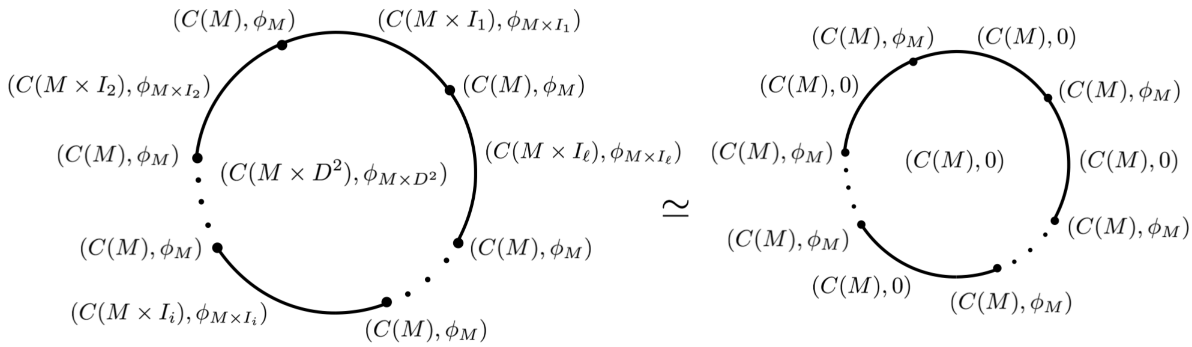


Figure 31: A schematic diagram for the homotopy equivalence.

The relative symmetric construction from Theorem 1.2.6 then extends to a symmetric construction for manifolds with split boundaries.

Definition 3.3.6. Let R be a commutative ring with identity and let $(W, \partial W; \partial_1 W, \dots, \partial_\ell W)$ be an $(n+1)$-dimensional oriented manifold with an ℓ-fold boundary splitting. The $(n+2)$-dimensional symmetric Poincaré pair over R with an ℓ-fold boundary splitting $(c; c_1, \dots, c_\ell)$ obtained by applying the *symmetric construction* over R is defined by

$$\begin{aligned} c &= (C(\partial W; R) \to C(W; R), (\phi_W, \phi_{\partial W})) \\ c_i &= (C(M_i; R) \oplus C(M_{i+1}; R) \to C(\partial_i W; R), (\phi_{\partial_i W}, \phi_{M_i} \oplus -\phi_{M_{i+1}})) \quad (1 \leqslant i \leqslant \ell). \end{aligned}$$

This motivates the definition of product operation for a symmetric pair with a split boundary.

Definition 3.3.7. The *product* of {an m-dimensional ϵ-symmetric (Poincaré) complex (E, ϕ_E) over A} with {an $(n+1)$-dimensional η-symmetric (Poincaré) pair over $\mathbb{Z}$ with an ℓ-fold boundary splitting $(c; c_1, \dots, c_\ell)$ $(\ell \geqslant 2)$} is the $(m+n+1)$-dimensional $\epsilon\eta$-symmetric (Poincaré) pair over $A = A \otimes_{\mathbb{Z}} \mathbb{Z}$ with an ℓ-fold boundary splitting

$$(E, \phi_E) \otimes (c; c_1, \dots, c_\ell) = (c'; c'_1, \dots, c'_\ell)$$

such that if

$$\begin{aligned} c &= (f : \partial D \to D, (\phi_D, \phi_{\partial D})) \\ c_i &= ((f_i^i \;\; f_{i+1}^i) : C_i \oplus C_{i+1} \to \partial_i D, (\phi_{\partial_i D}, \phi_{C_i} \oplus -\phi_{C_{i+1}})) \qquad (1 \leqslant i \leqslant \ell) \end{aligned}$$

then

$$\begin{aligned} c' &= (E, \phi_E) \otimes c \\ &= (1 \otimes f : E \otimes_{\mathbb{Z}} \partial D \to E \otimes_{\mathbb{Z}} D, (\phi_E \otimes \phi_D, \phi_E \otimes \phi_{\partial D})) \\ c'_i &= (E, \phi_E) \otimes c_i \\ &= ((1 \otimes f_i^i \;\; 1 \otimes f_{i+1}^i) : (E \otimes_{\mathbb{Z}} C_i) \oplus (E \otimes_{\mathbb{Z}} C_{i+1}) \to \\ &\qquad E \otimes_{\mathbb{Z}} \partial_i D, (\phi_E \otimes \phi_{\partial_i D}, (\phi_E \otimes \phi_{C_i}) \oplus -(\phi_C \otimes \phi_{C_{i+1}}))) \quad (1 \leqslant i \leqslant \ell). \end{aligned}$$

Definition 3.3.8. Let (C, ϕ) be an n-dimensional ϵ-symmetric (Poincaré) complex over A.

The *disc thickening with an ℓ-fold boundary splitting* ($\ell \geqslant 2$) of (C,ϕ) is the $(n+2)$-dimensional ϵ-symmetric (Poincaré) pair over $A = A \otimes_{\mathbb{Z}} \mathbb{Z}$ with an ℓ-fold boundary splitting $(c; c_1, \dots, c_\ell)$ where

$$c = (C \otimes_{\mathbb{Z}} C(S^1;\mathbb{Z}) \to C, (0, \phi \otimes \phi_{S^1}))$$
$$c_i = (C \oplus C \to C, (0, \phi \oplus -\phi))$$

such that there is a homotopy equivalence $(c; c_1, \dots, c_\ell) \simeq (C,\phi) \otimes (c'; c'_1, \dots, c'_\ell)$ where $(c'; c'_1, \dots, c'_\ell)$ is the 2-dimensional symmetric pair over $\mathbb{Z}$ with an ℓ-fold boundary splitting from Example 3.3.5 obtained by applying the symmetric construction to the ℓ-fold boundary splitting $(D^2, S^1; I_1, \dots, I_\ell)$.

Proposition 3.3.9. Let R be a commutative ring with identity. For closed, oriented manifolds the symmetric construction over R commutes, up to homotopy equivalence, with disc thickenings with an ℓ-fold boundary splitting.

Proof. By Example 3.3.5. □

3.4 Manifold and symmetric Poincaré quiver representations

We now extended the symmetric construction for oriented manifolds and cobordisms to a symmetric construction for an an oriented manifold representation of a quiver where the vertices parametrise manifolds and the arrows parametrise cobordisms. This produces a symmetric Poincaré representation of the quiver where the vertices parametrise symmetric Poincaré complexes and the arrows parametrise symmetric Poincaré cobordisms.

Definition 3.4.1. A *semi-groupoid* $\mathbb{A}$ consists of:

(i) A collection of *objects* $\mathrm{Obj}(\mathbb{A})$.

(ii) For each pair of objects $A, B \in \mathrm{Obj}(\mathbb{A})$ a collection of *morphisms* $\mathrm{Hom}_{\mathbb{A}}(A,B)$ from A to B.

(iii) For each triple of objects $A, B, C \in \mathrm{Obj}(\mathbb{A})$ a *composition map*

$$\mathrm{Hom}_{\mathbb{A}}(A,B) \times \mathrm{Hom}_{\mathbb{A}}(B,C) \to \mathrm{Hom}_{\mathbb{A}}(A,C); \quad (f,g) \to g \circ f$$

such that the composition of morphisms is associative.

Example 3.4.2.

(i) Every category is a semi-groupoid. One may think of semi-groupoids as categories without identity morphisms.

(ii) For each integer $n \geqslant 0$ there is a semi-groupoid $\mathbb{M}^n$ where the objects are closed, oriented n-dimensional manifolds and where a morphism from M^n to M'^n is an oriented $(n+1)$-dimensional cobordism $(W; M, M')$. The composition of morphisms $(W; M, M'), (W'; M', M'')$ is the morphism $(W \cup_{M'} W'; M, M'')$ obtained by glueing the cobordisms $(W; M, M')$ and $(W'; M', M'')$ along M'. In fact, we may form $\mathbb{M}^n$ into a category if we define a morphism from M to M' to be a cobordism between M and M' modulo oriented diffeomorphisms of the cobordism relative to the boundary, but we shall not use this here.

(iii) For each integer $n \geqslant 0$ and each ring with involution A there is a semi-groupoid $\mathbb{L}^n(A,\epsilon)$ where the objects are n-dimensional ϵ-symmetric Poincaré complexes over A and where a morphism from (C,ϕ) to (C',ϕ') is an $(n+1)$-dimensional ϵ-symmetric cobordism over A of the form

$$((f\ \ f') : C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi')).$$

The composition of morphisms

$$((f_C\ \ f_{C'}) : C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi'))$$
$$((f'_{C'}\ \ f'_{C''}) : C' \oplus C'' \to D', (\delta\phi', \phi' \oplus -\phi''))$$

is the morphism

$$((f''_C\ \ f''_{C''}) : C \oplus C'' \to D \cup_{C'} D', (\delta\phi \cup_{\phi'} \delta\phi', \phi \oplus -\phi''))$$

obtained by the glueing construction for adjoining ϵ-symmetric cobordisms from Definition 1.3.3. In the case $\epsilon = 1$ we write $\mathbb{L}^n(A,\epsilon) = \mathbb{L}^n(A)$.

Definition 3.4.3.

(i) A *quiver* $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ is a directed multi-graph consisting of a collection of vertices Q_0, a collection of arrows Q_1 and two functions $s : Q_1 \to Q_0$ respectively $t : Q_1 \to Q_0$, assigning to each arrow $\alpha \in Q_1$ its *source vertex* $s(\alpha) \in Q_0$ respectively *target vertex* $t(\alpha) \in Q_0$. A quiver is *finite* if it has finitely many vertices and arrows.

(ii) A *representation* $M = ((M_v)_{v\in Q_0}, (g_\alpha)_{\alpha\in Q_1})$ of a quiver Q in a semi-groupoid $\mathbb{A}$ is a collection of objects $M_v \in \mathbb{A}$ indexed by the vertices $v \in Q_0$ together with a collection of morphisms $g_\alpha \in \mathrm{Hom}_{\mathbb{A}}(M_{s(\alpha)}, M_{t(\alpha)})$ indexed by the arrows $\alpha \in Q_1$.

The most widely studied quiver representations are in the category of left R modules for some commutative ring R. The data we will parametrise by a quiver will come from two different sources: oriented manifolds and symmetric Poincaré complexes.

Definition 3.4.4.

(i) An $(n+1)$*-dimensional oriented manifold representation* $(W_Q; M_Q, M'_Q)$ of a quiver Q is a representation of Q in the semi-groupoid $\mathbb{M}^n$, consisting of a closed, oriented n-dimensional manifold M_v for each vertex $v \in Q_0$ and an $(n+1)$-dimensional cobordism $(W_\alpha; M_{s(\alpha)}, M'_{t(\alpha)})$ for each arrow $\alpha \in Q_1$.

(ii) An $(n+1)$*-dimensional* ϵ*-symmetric Poincaré representation* $(f_Q : C_Q \to D_Q, (\delta\phi_Q, \phi_Q))$ of a quiver Q over A is a representation of Q in the semi-groupoid $\mathbb{L}^n(A,\epsilon)$, consisting of an n-dimensional ϵ-symmetric Poincaré complex (C_v, ϕ_v) over A for each vertex $v \in Q_0$ and an $(n+1)$-dimensional ϵ-symmetric cobordism

$$((f_{s(\alpha)}\ \ f_{t(\alpha)}) : C_{s(\alpha)} \oplus C_{t(\alpha)} \to D_\alpha, (\delta\phi_\alpha, \phi_{s(\alpha)} \oplus -\phi_{t(\alpha)}))$$

over A for each arrow $\alpha \in Q_1$. In the case $\epsilon = 1$ an ϵ-symmetric Poincaré representation of Q over A is called a *symmetric Poincaré representation of Q over A.*

A quiver Q can also be viewed as a Δ-set where the 0-simplices are the vertices, the 1-simplices are the arrows and there are no simplices of higher dimension. The face maps from 1-simplices to 0-simplices are the source and target maps $s, t : Q_1 \to Q_0$. An $(n+1)$-dimensional ϵ-symmetric Poincaré representation of Q over A is then the same as a Δ-map from Q to the ϵ-symmetric $\mathbb{L}$-spectrum of A, see [Ran92, p.136] for more details.

Example 3.4.5.

(i) The trinity quiver T

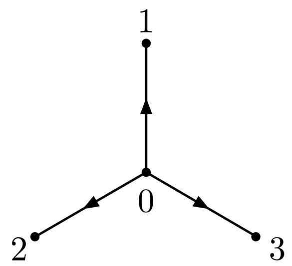


Figure 32: The trinity quiver.

has $(n+1)$-dimensional oriented manifold, respectively $(n+1)$-dimensional ϵ-symmetric Poincaré, representations of the form

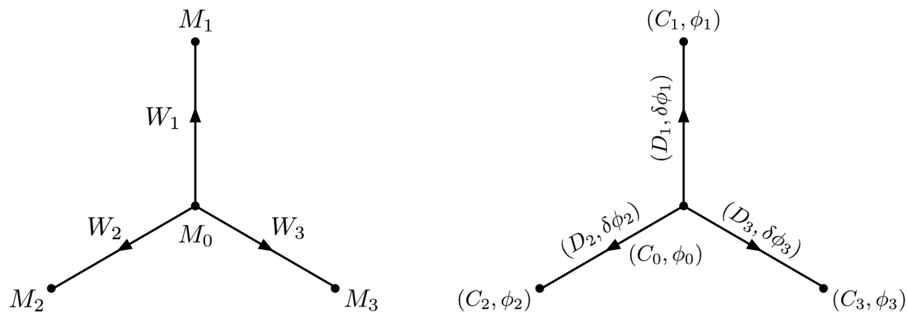


Figure 33: Manifold and symmetric Poincaré representations of the trinity quiver.

where by the data $(D_i, \delta\phi_i)$ over an arrow we mean an $(n+1)$-dimensional ϵ-symmetric cobordism over A of the form

$$((f_{0i}\ \ f_{ii}) : C_0 \oplus C_i \to D_i, (\delta\phi_i, \phi_0 \oplus -\phi_i) \in Q^{n+1}((f_{0i}\ \ f_{ii}), \epsilon)) \quad (1 \leqslant i \leqslant 3).$$

These representations are precisely the $(n+1)$-dimensional manifold, respectively ϵ-symmetric Poincaré, trinities of [BNR12a, p.44-46].

(ii) The Θ quiver

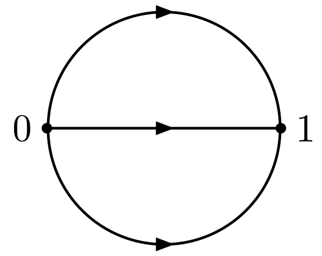


Figure 34: The Θ quiver.

has $(n+1)$-dimensional oriented manifold, respectively $(n+1)$-dimensional ϵ-symmetric Poincaré, representations of the form where by the data $(D_i, \delta\phi_i)$ over an arrow we mean an $(n+1)$-

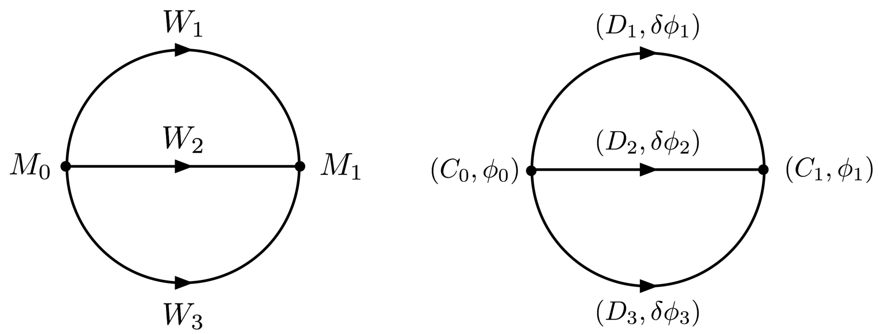


Figure 35: Manifold and symmetric Poincaré representations of the Θ quiver.

dimensional ϵ-symmetric cobordism over A of the form

$$((f_{0i}\ \ f_{1i}) : C_0 \oplus C_1 \to D_i, (\delta\phi_i, \phi_0 \oplus -\phi_1) \in Q^{n+1}((f_{0i}\ \ f_{1i}), \epsilon)) \quad (1 \leqslant i \leqslant 3).$$

The symmetric construction may be applied to oriented manifold representations in order to produce symmetric Poincaré representations.

Proposition 3.4.6. If Q is a quiver and if R is a commutative ring then the symmetric construction may be applied to an $(n+1)$-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q to produce an $(n+1)$-dimensional symmetric Poincaré representation of Q over R.

Proof. For each vertex $v \in Q_0$ applying the symmetric construction to the closed, oriented n-dimensional manifold M_v produces an n-dimensional symmetric Poincaré complex $(C(M_v; R), \phi_{M_v})$ over R. For each arrow $\alpha \in Q_1$ applying the symmetric construction to the oriented $(n+1)$-dimensional cobordism $(W_\alpha; M_{s(\alpha)}, M_{t(\alpha)})$ produces an $(n+1)$-dimensional cobordism

$$((i_{s(\alpha)}\ \ i_{t(\alpha)}) : C(M_{s(\alpha)}; R) \oplus C(M_{t(\alpha)}; R) \to C(W_\alpha; R), (\phi_{W(\alpha)}, \phi_{M_{s(\alpha)}} \oplus -\phi_{M_{t(\alpha)}}))$$

over R. The collection

$$((i_{s(\alpha)}\ \ i_{t(\alpha)}) : (C(M_{s(\alpha)}; R) \oplus C(M_{t(\alpha)}; R) \to C(W_\alpha; R), (\phi_{W(\alpha)}, \phi_{M_{s(\alpha)}} \oplus -\phi_{M_{t(\alpha)}}))_{\alpha \in Q_1}, \\ (C(M_v; R), \phi_{M_v})_{v \in Q_0})$$

is then an $(n+1)$-dimensional symmetric Poincaré representation of Q over R. From now on we denote this representation by

$$(C(M_Q;R) \oplus C(M'_Q;R) \to C(W_Q;R), (\phi_{W_Q}, \phi_{M_Q} \oplus -\phi_{M'_Q})).$$

□

Example 3.4.7. Let R be a commutative ring with identity.

(i) Applying the symmetric construction to the $(n+1)$-dimensional oriented manifold representation of the trinity quiver from Example 3.4.5 (i) produces an $(n+1)$-dimensional symmetric Poincaré representation over R

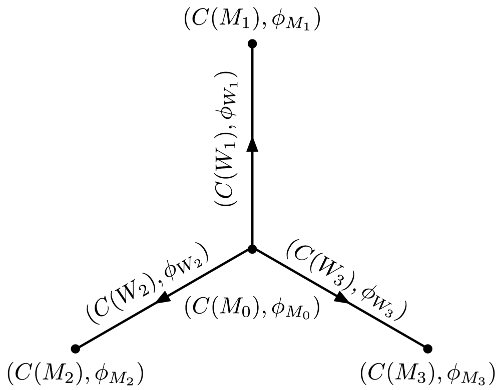


Figure 36: A symmetric Poincaré representation of the trinity quiver obtained by applying the symmetric construction to a manifold representation.

where by the data $(C(W_i;R), \phi_{W_i})$ $(1 \leqslant i \leqslant 3)$ over an arrow we mean the $(n+1)$-dimensional symmetric cobordism over A

$$(C(M_0;R) \oplus C(M_i;R) \to C(W_i;R), (\phi_{W_i}, \phi_{M_0} \oplus -\phi_{M_i})).$$

(ii) Applying the symmetric construction to the $(n+1)$-dimensional oriented manifold representation of the theta quiver from Example 3.4.5 (ii) produces an $(n+1)$-dimensional symmetric Poincaré representation over R

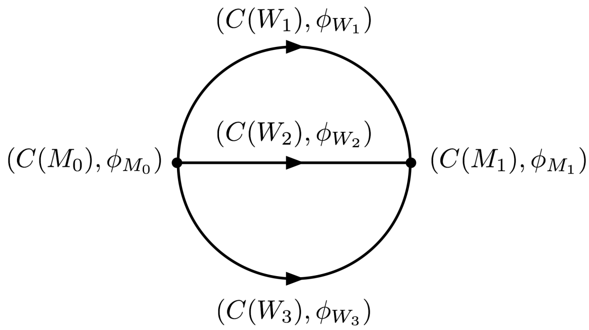


Figure 37: A symmetric Poincaré representation of the Θ quiver obtained by applying the symmetric construction to a manifold representation of the Θ quiver.

where by the data $(C(W_i;R), \phi_{W_i})$ over an arrow we mean the $(n+1)$-dimensional symmetric cobordism over R

$$(C(M_0;R)\oplus C(M_1;R)\to C(W_i;R),(\phi_{W_i},\phi_{M_0}\oplus -\phi_{M_1})).$$

In chapter 5 we will see that a Morse 2-function on an oriented 4-manifold yields a 3-dimensional oriented manifold quiver representation to which we may apply the symmetric construction.

3.5 Thickening manifold and symmetric Poincaré quiver representations

We extend the thickening operations from Section 3.2 to thickening operations of oriented manifold and symmetric Poincaré representations of a quiver in such a way that the symmetric construction and thickening operations commute up to homotopy equivalence.

Definition 3.5.1.

(i) The *degree* of a vertex $v\in Q_0$ of a quiver Q is the cardinality $\deg(v)$ of the set of arrows $\{\alpha\in Q_1 : s(\alpha)=v \text{ or } t(\alpha)=v\}$ with source or target vertex v.

(ii) An *ordered* quiver is a quiver Q such that at each vertex $v\in Q_0$ there is an ordering of the set of arrows $\{\alpha\in Q_1 : s(\alpha)=v \text{ or } t(\alpha)=v\}$ such that the arrows with source vertex v are ordered before the arrows with target vertex v.

Example 3.5.2. An ordered quiver where the vertices from left to right are of respective degrees $1,3,2,8,2$ and with the ordering of the edges at each respective vertex are denoted by natural numbers

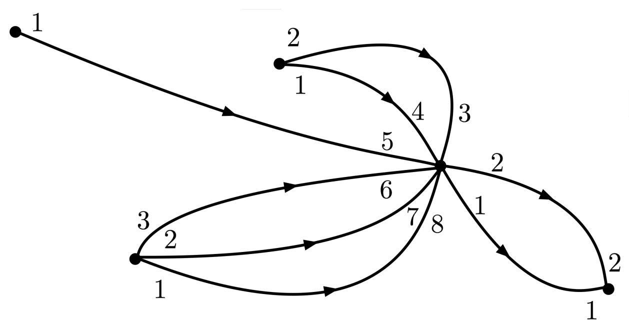


Figure 38: An ordered quiver.

The geometric thickening operations from Definition 3.2.1 have the following extension to quivers.

Definition 3.5.3. The *thickening* of an $(n+1)$-dimensional oriented manifold representation $(W_Q;M_Q,M'_Q)$ of an ordered quiver Q is the $(n+2)$-dimensional oriented manifold with boundary $(\Omega,\partial\Omega)$ constructed as follows. For each vertex $v\in Q_0$ with $\deg(v)\neq 1,2$ construct the disc thickening of the n-dimensional closed, oriented manifold M_v with a $\deg(v)$-fold boundary splitting

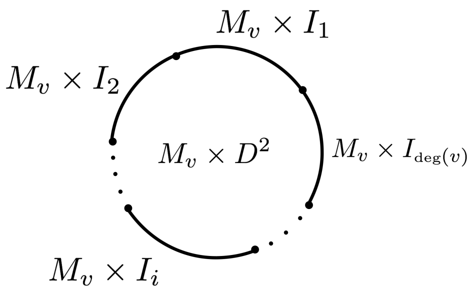


Figure 39: A schematic diagram for a deg(v)-fold boundary splitting of $M_v \times D^2$.

and form the disjoint union $\sqcup_{v \in Q_0} M_v \times D^2$. For each arrow $\alpha \in Q_1$ construct the thickening of the $(n+1)$-dimensional oriented cobordism $(W_\alpha; M_{s(\alpha)}, M_{t(\alpha)})$

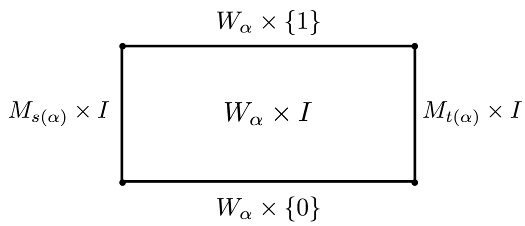


Figure 40: A schematic diagram for the thickening of $(W_\alpha; M_{s(\alpha)}, M_{t(\alpha)})$.

If α is an arrow from $s(\alpha)$ to $t(\alpha)$ where α is the ith arrow around $s(\alpha)$ and α is the jth arrow around $t(\alpha)$ according to the orderings, use the boundary splittings of $M_{s(\alpha)}$ and $M_{t(\alpha)}$ to glue $W_\alpha \times I$ to $M_{s(\alpha)} \times D^2 \sqcup M_{t(\alpha)} \times D^2 \subset \sqcup_{v \in Q_0} M_v \times D^2$ along $M_{s(\alpha)} \times I_i$ and $M_{t(\alpha)} \times I_j$

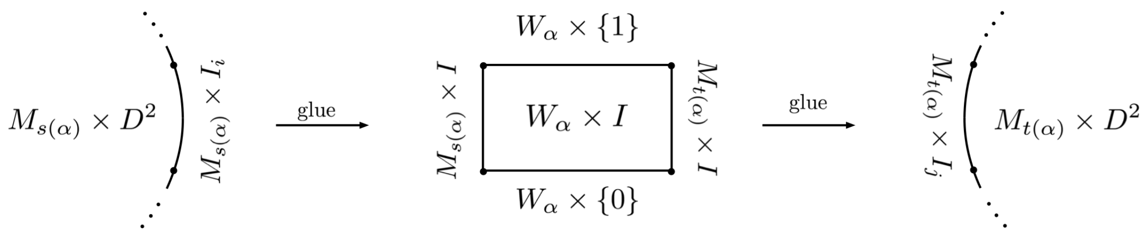


Figure 41: A schematic diagram for glueing $M_{s(\alpha)} \times D^2, W_\alpha \times I, M_{t(\alpha)} \times D^2$.

If v is a vertex with $\deg(v) = 2$ and if α, β are arrows such that $t(\alpha) = v = s(\beta)$ then we glue the adjoining cobordisms $(W_\alpha; M_{s(\alpha)}, M'_{t(\alpha)})$ and $(W_\beta; M_{s(\beta)}, M_{t(\beta)})$ in directly without disc thickening the manifold M_v. If v is a vertex with $\deg(v) = 1$ and if α is an arrow with $s(\alpha) = v$ or $t(\alpha) = v$ then we similarly glue the cobordism $(W_\alpha; M_{s(\alpha)}, M_{t(\alpha)})$ directly without disc thickening the manifold M_v and we may contract $M_v \times I$ to M_v so that $W_\alpha \times \{0\}$ and $W_\alpha \times \{1\}$ are glued directly.

Example 3.5.4. The thickening of the oriented manifold representation of the quiver from Example 3.5.2 is of the form

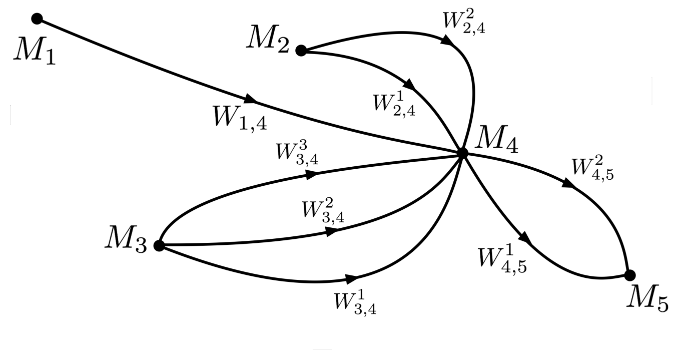


Figure 42: An oriented manifold representation of a quiver.

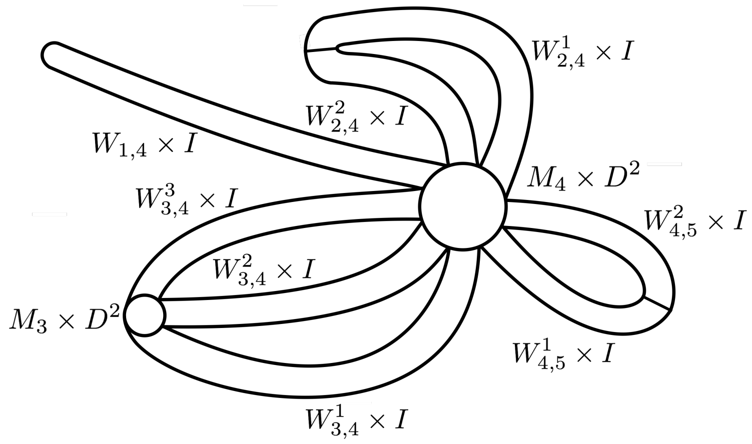


Figure 43: The thickening of the representation.

Note the procedure of thickening an oriented manifold representation of a quiver is only well defined once the quiver is ordered since otherwise the homeomorphism type of the resulting manifold may change unpredictably.

The geometric thickening operation for quivers, the algebraic thickening operations from Definition 3.2.3 together with the glueing operations for triads from Definition 2.2.1 and for cobordisms of symmetric pairs from Definition 2.2.4 motivate the following algebraic thickening operation for quivers.

Definition 3.5.5. The *thickening* of an $(n+1)$-dimensional ϵ-symmetric Poincaré representation of an ordered quiver Q over A

$$(f_Q : C_Q \to D_Q, (\phi_{D_Q}, \phi_{C_Q}))$$

is the $(n+2)$-dimensional ϵ-symmetric Poincaré pair over A

$$(f : \partial D \to D, (\phi_D, \phi_{\partial D}))$$

constructed as follows. For each vertex $v \in Q_0$ with $\deg(v) \neq 1, 2$ form the disc thickening with a $\deg(v)$-fold boundary splitting of the n-dimensional ϵ-symmetric Poincaré complex (C_v, ϕ_v) over A

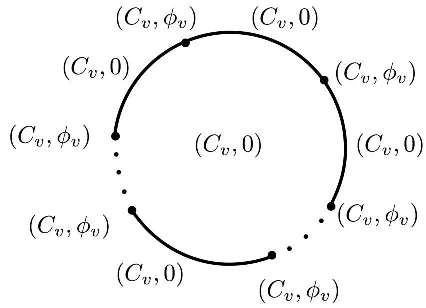


Figure 44: A schematic diagram for the disc thickening with a boundary splitting of the ϵ-symmetric complex associated to the vertex v.

and for each arrow $\alpha \in Q_1$ construct the thickening of the $(n+1)$-dimensional ϵ-symmetric cobordism over A

$$((f_{s(\alpha)} \ \ f_{t(\alpha)}) : C_{s(\alpha)} \oplus C_{t(\alpha)} \to D_\alpha, (\delta\phi_\alpha, \phi_{s(\alpha)} \oplus -\phi_{t(\alpha)}))$$

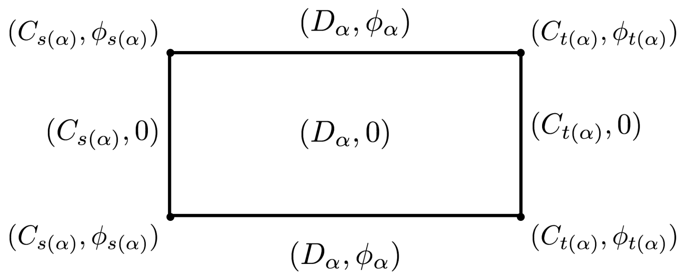


Figure 45: A schematic diagram for the thickening of the ϵ-symmetric cobordism associated to the arrow α.

and then form the direct sum

$$\oplus_{v \in Q_0}(C_v \otimes_{\mathbb{Z}} C(S^1; \mathbb{Z}) \to C_v, (0, \phi_v \otimes \phi_{S^1})).$$

If α is an arrow from $s(\alpha)$ to $t(\alpha)$, use the boundary splittings of $(C_{s(\alpha)}, \phi_{s(\alpha)})$ and $(C_{t(\alpha)}, \phi_{t(\alpha)})$, the ordering of the arrow α around the vertices $s(\alpha)$ and $t(\alpha)$, and the glueing operation for adjoining ϵ-symmetric triads to glue the thickening of

$$((f_{s(\alpha)} \ \ f_{t(\alpha)}) : C_{s(\alpha)} \oplus C_{t(\alpha)} \to D_\alpha, (\delta\phi_\alpha, \phi_{s(\alpha)} \oplus -\phi_{t(\alpha)}))$$

to

$$(C_{s(\alpha)} \otimes_{\mathbb{Z}} C(S^1; \mathbb{Z}) \to C_{s(\alpha)}, (0, \phi_{s(\alpha)} \otimes \phi_{S^1})) \oplus (C_{t(\alpha)} \otimes_{\mathbb{Z}} C(S^1; \mathbb{Z}) \to C_{t(\alpha)}, (0, \phi_{t(\alpha)} \otimes \phi_{S^1}))$$

along

$$\left(\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} : C_{s(\alpha)} \oplus C_{t(\alpha)} \oplus C_{s(\alpha)} \oplus C_{t(\alpha)} \to C_{s(\alpha)} \oplus C_{t(\alpha)}, (0, \phi_{s(\alpha)} \oplus -\phi_{t(\alpha)} \oplus -\phi_{s(\alpha)} \oplus \phi_{t(\alpha)})\right)$$

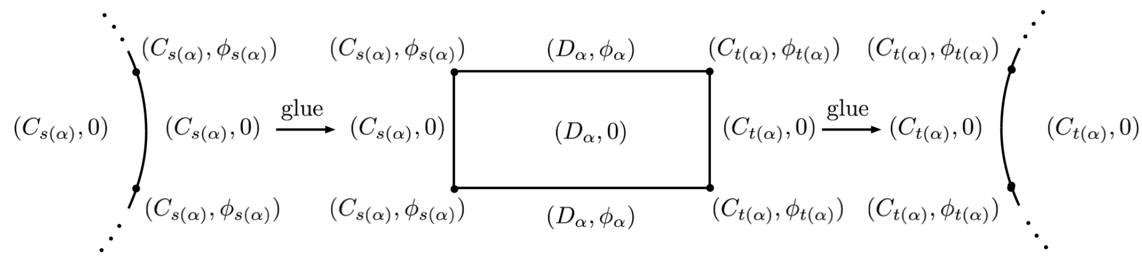


Figure 46: Glueing the three pieces.

If v is a vertex with $\deg(v) = 2$ and α, β are arrows such that $t(\alpha) = v = s(\alpha)$ then we glue the adjoining ϵ-symmetric cobordisms

$$((f_{s(\alpha)} \;\; f_{t(\alpha)}) : C_{s(\alpha)} \oplus C_{t(\alpha)} \to D_\alpha, (\delta\phi_\alpha, \phi_{s(\alpha)} \oplus -\phi_{t(\alpha)}))$$

and

$$((f_{s(\beta)} \;\; f_{t(\beta)}) : C_{s(\beta)} \oplus C_{t(\beta)} \to D_\beta, (\delta\phi_\beta, \phi_{s(\beta)} \oplus -\phi_{t(\beta)}))$$

directly without disc thickening the ϵ-symmetric Poincaré complex (C_v, ϕ_v). If v is a vertex with $\deg(v) = 1$ and α is an arrow with $s(\alpha) = v$ or $t(\alpha) = v$ then we similarly glue the ϵ-symmetric cobordism

$$((f_{s(\alpha)} \;\; f_{t(\alpha)}) : C_{s(\alpha)} \oplus C_{t(\alpha)} \to D_\alpha, (\delta\phi_\alpha, \phi_{s(\alpha)} \oplus -\phi_{t(\alpha)}))$$

in directly without disc thickening the ϵ-symmetric Poincaré complex (C_v, ϕ_v).

One could also define the thickening of an ϵ-symmetric Poincaré representation of a quiver in terms of the ϵ-symmetric $\mathbb{L}$-spectrum $\mathbb{L}^\bullet(A, \epsilon) = \{\mathbb{L}^n(A, \epsilon) | n \geq 0\}$ of [Ran92, Section 12]. This is an Ω-spectrum of pointed Kan Δ-sets with homotopy groups equal to the L-groups of A $\pi_n(\mathbb{L}^\bullet(A, \epsilon)) \cong L^n(A, \epsilon)$. The 0-simplices of the Δ-set $\mathbb{L}^n(A, \epsilon)$ are the n-dimensional ϵ-symmetric complexes over A and the 1-simplices of $\mathbb{L}^n(A, \epsilon)$ are the $(n+1)$-dimensional ϵ- symmetric cobordisms over A. One can thicken Q directly to produce a 2-dimensional manifold $(N, \partial N)$ which contains Q as a deformation retraction with a projection $p : N \to Q$ which is a homotopy equivalence. An n-dimensional symmetric Poincaré representation $(f_Q : C_Q \to D_Q, (\phi_{D_Q}, \phi_{C_Q}))$ of Q can then be regarded as a Δ-map $C : Q \to \mathbb{L}^n(A, \epsilon)$. Choosing a triangulation of $(N, \partial N)$ such that the projection map is a CW-map, and hence is a Δ-map, the composition $Cp : N \to Q \to \mathbb{L}^n(A, \epsilon)$ then represents a cohomology class $Cp \in H^{-n}(N; \mathbb{L}(A, \epsilon))$ which is Poincaré dual to a homology class $Cp^* \in H_{n+2}(N, \partial N; \mathbb{L}(A, \epsilon))$. This homology class can be described combinatorially as a cycle in the sense of [Ran92, Section 12] and represents the thickening of $(f_Q : C_Q \to D_Q, (\phi_{D_Q}, \phi_{C_Q}))$. This approach uses the Kan extension condition of $\mathbb{L}^n(A, \epsilon)$ to glue symmetric triads and relative symmetric cobordisms rather than using the glueing operations from chapter 1.

Example 3.5.6.

(i) Thickening the $(n+1)$-dimensional oriented manifold representation of the trinity quiver T from Example 3.4.5 (i) produces an oriented $(n+2)$-dimensional manifold with boundary $(\Omega, \partial\Omega)$

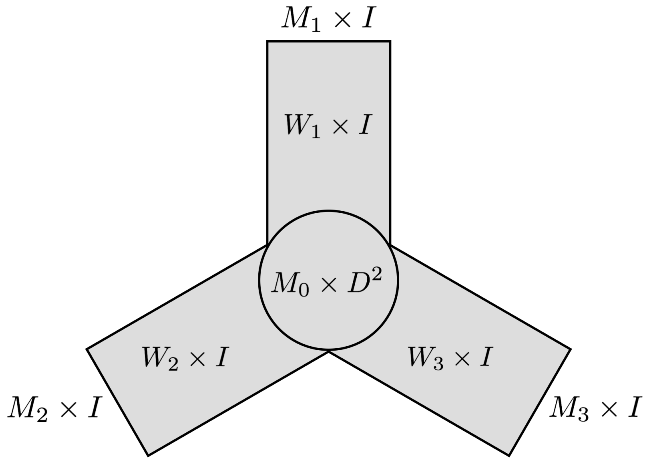


Figure 47: A schematic diagram for the thickening of the manifold representation of the trinity quiver.

The inclusion $W_1 \cup W_2 \cup W_3 \hookrightarrow \Omega$ is a homotopy equivalence and there is a homeomorphism

$$(W_1 \cup_{M_0} W_2) \cup (W_2 \cup_{M_0} W_3) \cup (W_3 \cup_{M_0} W_1) \cong \partial\Omega$$

as shown by the diagram

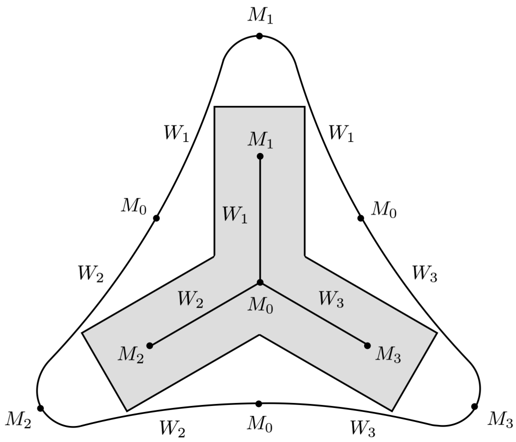


Figure 48: A schematic diagram for the homotopy equivalence and the boundary decomposition.

so that Ω and $\partial\Omega$ can be recovered, up to homotopy equivalence, by glueing copies of the cobordisms W_1, W_2, W_3. If R is a commutative ring with identity then this implies that there are chain homotopy equivalences $C(\Omega; R) \simeq E$ where

$$E = \mathscr{C}(C(M_0; R) \to C(W_1; R) \oplus C(W_2; R) \oplus C(W_3; R))$$

and $C(\partial\Omega;R)\simeq\partial E$ where

$$\begin{aligned}\partial E = \mathscr{C}(C(M_1;R)\oplus C(M_2;R)\oplus C(M_3;R)\to \quad & \mathscr{C}(C(M_0;R)\to C(W_1;R)\oplus C(W_2;R))\\ & \oplus\mathscr{C}(C(M_0;R)\to C(W_2;R)\oplus C(W_3;R))\\ & \oplus\mathscr{C}(C(M_0;R)\to C(W_3;R)\oplus C(W_1;R))).\end{aligned}$$

Applying the symmetric construction to $(\Omega,\partial\Omega)$ produces the $(n+2)$-dimensional symmetric pair $(C(\partial\Omega;R)\to C(\Omega;R),(\phi_\Omega,\phi_{\partial\Omega}))$ over R and by Example 2.1.5 and Example 3.2.2 there is a homotopy equivalence

$$(C(\partial\Omega;R)\to C(\Omega;R),(\phi_\Omega,\phi_{\partial\Omega}))\simeq(\partial E\to E,(\phi_E,\partial\phi_E))$$

where $\phi_{\partial E}=(-\phi_{W_1}\cup\phi_{W_2})\cup(-\phi_{W_2}\cup\phi_{W_3})\cup(-\phi_{W_3}\cup\phi_{W_1})$ If $(\partial D\to D,(\phi_D,\partial\phi_D))$ is the $(n+2)$-dimensional symmetric pair over R obtained by thickening the $(n+1)$-dimensional symmetric Poincaré representation of Example 3.4.5 (i) then there is a homotopy equivalence

$$(C(\partial\Omega;R)\to C(\Omega;R),(\phi_\Omega,\phi_{\partial\Omega}))\simeq(\partial E\to E,(\phi_E,\partial\phi_E))\simeq(\partial D\to D,(\phi_D,\partial\phi_D)).$$

(ii) Thickening the $(n+1)$-dimensional oriented manifold representation of the theta quiver Θ from Example 3.4.5 (ii) produces an $(n+2)$-dimensional oriented manifold with boundary $(\Omega,\partial\Omega)$

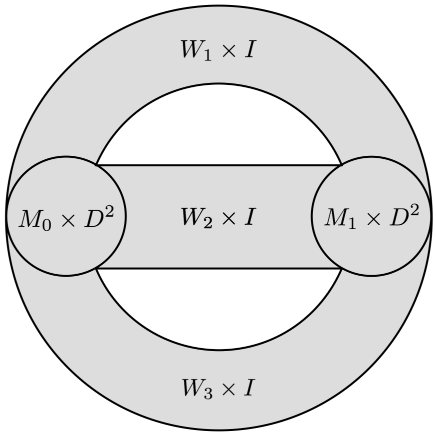


Figure 49: A schematic diagram for the thickening of the manifold representation of the Θ quiver.

such that the inclusion $W_1\cup W_2\cup W_3\hookrightarrow\Omega$ is a homotopy equivalence and there is a homeomorphism

$$(W_1\cup_{M_0\sqcup M_1}W_3)\sqcup(W_1\cup_{M_0\sqcup M_1}W_2)\sqcup(W_2\cup_{M_0\sqcup M_1}W_3)\cong\partial\Omega$$

as shown by the diagram

so that Ω and $\partial\Omega$ can be recovered, up to homotopy equivalence, by glueing copies of the cobordisms W_1,W_2,W_3. If R is a commutative ring with identity then this implies that there are chain homotopy equivalences $C(\Omega;R)\simeq E$ where

$$E=\mathscr{C}(C(M_0;R)\oplus C(M_1;R)\to C(W_1;R)\oplus C(W_2;R)\oplus C(W_3;R))$$

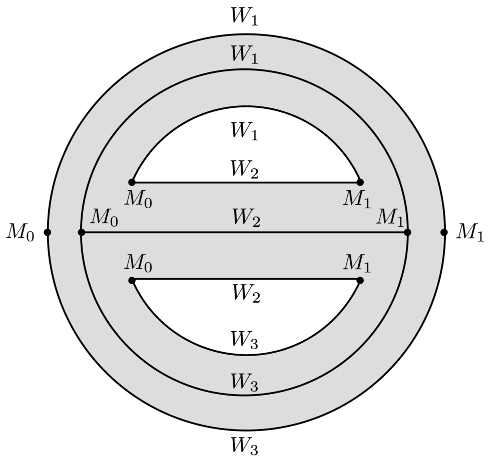


Figure 50: A schematic diagram for the homotopy equivalence and the boundary decomposition.

and $C(\partial\Omega; R) \simeq \partial E$ where

$$\begin{aligned}\partial E = \quad & \mathscr{C}(C(M_0; R) \oplus C(M_1; R) \to C(W_1; R) \oplus C(W_2; R)) \\ & \oplus \mathscr{C}(C(M_0; R) \oplus C(M_1; R) \to C(W_2; R) \oplus C(W_3; R)) \\ & \oplus \mathscr{C}(C(M_0; R) \oplus C(M_1; R) \to C(W_3; R) \oplus C(W_1; R)).\end{aligned}$$

Applying the symmetric construction to $(\Omega, \partial\Omega)$ produces the $(n+2)$-dimensional symmetric Poincaré pair $(C(\partial\Omega; R) \to C(\Omega; R), (\phi_\Omega, \phi_{\partial\Omega}))$ over R and by Example 2.1.5 and Example 3.2.2 there is a homotopy equivalence

$$(C(\partial\Omega; R) \to C(\Omega; R), (\phi_\Omega, \phi_{\partial\Omega})) \simeq (\partial E \to E, (\phi_E, \phi_{\partial E}))$$

where $\phi_{\partial E} = (\phi_{W_1} \cup -\phi_{W_2}) \oplus (\phi_{W_2} \cup -\phi_{W_3}) \oplus (\phi_{W_3} \cup -\phi_{W_1})$ If $(\partial D \to D, (\phi_D, \partial\phi_D))$ is the $(n+2)$-dimensional symmetric pair over R obtained by thickening the $(n+1)$-dimensional symmetric Poincaré representation from Example 3.4.5 (iii) there is a homotopy equivalence

$$((C(\partial\Omega; R) \to C(\Omega; R), (\phi_\Omega, \phi_{\partial\Omega})) \simeq (\partial E \to E, (\phi_E, \phi_{\partial E})) \simeq (\partial D \to D, (\phi_D, \phi_{\partial D})).$$

Theorem 3.5.7. The symmetric construction commutes with the thickening operations up to homotopy equivalence of the resulting symmetric pair.

Proof. Let R be a commutative ring with identity and let $(W_Q; M_Q, M'_Q)$ be an $(n+1)$-dimensional oriented manifold representation of an ordered quiver Q. If $(\Omega, \partial\Omega)$ be the thickening of $(W_Q; M_Q, M'_Q)$ then let $(\partial D \to D, (\phi_D, \phi_{\partial D}))$ is the thickening of the $(n+1)$-dimensional symmetric Poincaré representation

$$(C(M_Q; R) \oplus C(M'_Q; R) \to C(W_Q; R), (\phi_{W_Q}, \phi_{M_Q} \oplus -\phi_{M'_Q}))$$

obtained by applying the symmetric construction to $(W_Q; M_Q, M'_Q)$ over R. By Propositions 3.2.4 and 3.3.9 there is a homotopy equivalence of $(n+2)$-dimensional symmetric Poincaré pairs over R

$$(C(\partial\Omega; R) \to C(\Omega; R), (\phi_\Omega, \phi_{\partial\Omega})) \simeq (\partial D \to D, (\phi_D, \phi_{\partial D})).$$

The following diagram is commutative up to the homotopy type of the resulting symmetric pair.

$$
\begin{array}{ccc}
(W_Q; M_Q, M'_Q) & \xrightarrow[\text{thickening}]{\text{geometric}} & (\Omega, \partial\Omega) \\
\Big\downarrow{\scriptstyle \text{symmetric construction}} & & \Big\downarrow{\scriptstyle \text{symmetric construction}} \\
\begin{array}{c}(C(M_Q; R) \oplus C(M'_Q; R) \to C(W_Q; R), \\ (\phi_{W_Q}, \phi_{M_Q} \oplus -\phi_{M'_Q}))\end{array} & \xrightarrow[\text{thickening}]{\text{algebraic}} & \begin{array}{c}(C(\partial\Omega; R) \to C(\Omega; R), (\phi_\Omega, \phi_{\partial\Omega})) \\ \wr\!\! \mid \\ (\partial D \to D, (\phi_D, \phi_{\partial D}))\end{array}
\end{array}
$$

□

Note that in Definition 3.5.3 one could choose to glue in the thickening of each cobordism $(W_\alpha; M_{s(\alpha)}, M'_{t(\alpha)})$ with a twist on the right hand side

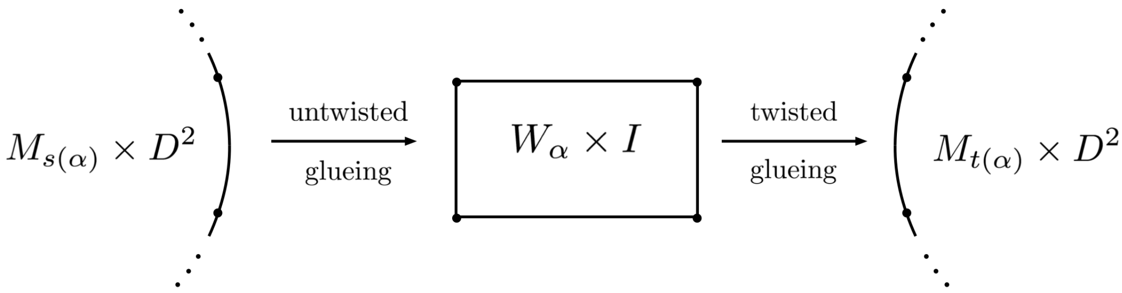


Figure 51: A schematic diagram for a twisted geometric glueing.

and using the twisted glueing operations for triads from Definition 2.3.4 one could choose to glue in the thickening of each cobordism $((f_{s(\alpha)} \;\; f_{t(\alpha)}) : C_{s(\alpha)} \oplus C_{t(\alpha)} \to D_\alpha, (\delta\phi_\alpha, \phi_{s(\alpha)} \oplus -\phi_{t(\alpha)}))$ with a twist on the right hand side

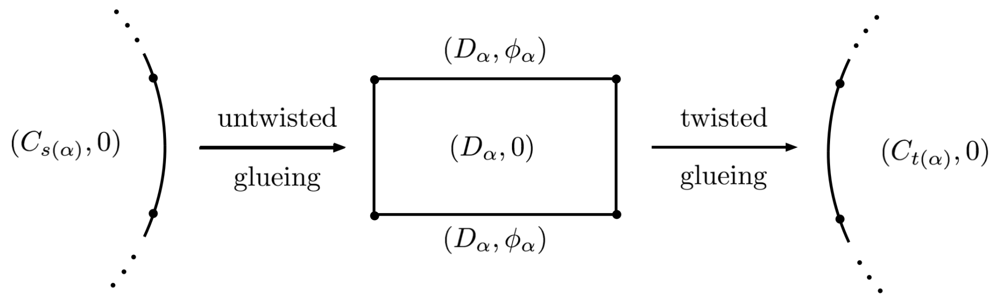


Figure 52: A schematic diagram for a twisted algebraic glueing.

Definition 3.5.8. Let Q be a finite quiver.

(i) A *twisted $(n+1)$-dimensional oriented manifold quiver representation* of Q consists of an $(n+1)$-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q together with twisted glueing data at the right hand side of each cobordism $(W_\alpha; M_{s(\alpha)}, M'_{t(\alpha)})$ determined by each arrow $\alpha \in Q_1$.

(ii) A *twisted $(n+1)$-dimensional ϵ-symmetric Poincaré quiver representation* of Q consists of an $(n+1)$-dimensional ϵ-symmetric Poincaré representation $(f_Q : C_Q \to D_Q, (\delta\phi_Q, \phi_Q))$ of Q together with twisted glueing data at the right hand side of each symmetric cobordism $((f_{s(\alpha)}\ \ f_{t(\alpha)}) : C_{s(\alpha)} \oplus C_{t(\alpha)} \to D_\alpha, (\delta\phi_\alpha, \phi_{s(\alpha)} \oplus -\phi_{t(\alpha)}))$ determined by each arrow $\alpha \in Q_1$.

Since the twisted union of adjoining triads is an algebraic model for the twisted union of adjoining manifold triads we have

Theorem 3.5.9. The symmetric construction commutes with the twisted thickening operations up to a homotopy equivalence of the resulting symmetric pair with a homotopy commutative diagram.

$$\begin{array}{ccc}
(W_Q; M_Q, M'_Q) & \xrightarrow[\text{thickening}]{\text{twisted geometric}} & (\Omega, \partial\Omega) \\
\text{symmetric construction}\big\downarrow & & \big\downarrow\text{symmetric construction} \\
\begin{array}{c}(C(M_Q;R) \oplus C(M'_Q;R) \to C(W_Q;R), \\ (\phi_{W_Q}, \phi_{M_Q} \oplus -\phi_{M'_Q}))\end{array} & \xrightarrow[\text{thickening}]{\text{twisted algebraic}} & \begin{array}{c}(C(\partial\Omega;R) \to C(\Omega;R), (\phi_\Omega, \phi_{\partial\Omega})) \\ \wr\!\! \\ (\partial D \to D, (\phi_D, \phi_{\partial D}))\end{array}
\end{array}$$

In chapter 5 we will show that a Morse 2-function on a 4-manifold M determines a twisted quiver representation which can be used to reconstruct the symmetric Poincaré complex $(C(M), \phi_M)$. Part of the reconstruction uses the twisted thickening operation.

Chapter 4

Morse 2-functions

In this chapter we examine Gay and Kirby's definition of Morse 2-functions [KG13a] and trisections of 4-manifolds [KG13b]. These are natural generalisations of Morse functions and Heegaard splittings of 3-manifolds. The geometric results in this chapter will be applied in chapter 5 in order to produce a symmetric Poincaré analogue of the Gay and Kirby's technique [KG13b] to reconstruct a 4-manifold M^4 from a 3-dimensional manifold representation of a quiver determined by a Morse 2-function.

4.1 Morse Functions, Heegaard Splittings and Heegaard Diagrams

As a warm up, we first recall the relationship between a Morse function on a 3-manifold and a Heegaard decomposition.

Definition 4.1.1. The *standard index k Morse model in dimension m* is the function

$$\mu_k^m : \mathbb{R}^m \to \mathbb{R}, \quad (x_1, \dots, x_m) \mapsto -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots x_m^2.$$

Let W be a smooth manifold with boundary and let N be a smooth 1-manifold. A smooth function $f : W \to N$ is *locally Morse* if f is locally of the form $f(x) = \mu_k^m(x)$ around each critical point $p \in W$ and in this case we say that k is the *index* of p. A *Morse function* on a smooth cobordism $(W; M_0, M_1)$ is a smooth map $f : W \to [0,1]$ which is locally Morse such that $f^{-1}(0) = M_0$ and $f^{-1}(1) = M_1$ and all critical points of f occur in the interior of W.

Example 4.1.2.

(i) The height function on the torus above a tangential plane as shown below is a Morse function with four critical points p, q, r, s of respective indices $0, 1, 1, 2$.

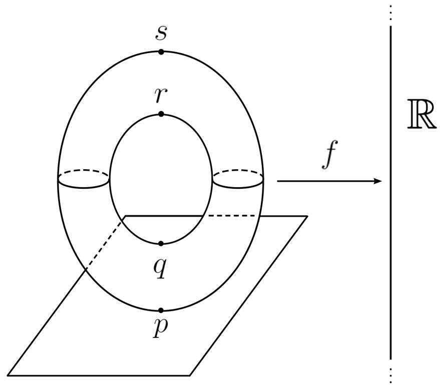


Figure 53: A Morse function on the torus.

(ii) The solid torus

$$S^1 \times D^2 = \{(\theta, x, y) \in S^1 \times \mathbb{R}^2 : x^2 + y^2 = 1\}$$

has a Morse function

$$f : S^1 \times D^2 \to [0,1], \quad (\theta, x, y) \mapsto \frac{1}{2}\left(x^2 + y^2 + \frac{1}{2}\cos(\theta)(1 - x^2 - y^2) + 1\right)$$

such that $f^{-1}(0) = \varnothing$ and $f^{-1}(1) = S^1 \times S^1$ and f has one critical point of index 0 at $(\pi, 0, 0)$ with critical value $\frac{1}{4}$ and one critical point of index of 1 at $(0,0,0)$ with critical value $\frac{3}{4}$.

If W is a finite-dimensional smooth manifold then space of Morse functions of W is a dense, locally stable subspace of the space of all smooth functions on W so that generically every smooth function is Morse. See [BH04, §5.5] for more details.

Definition 4.1.3.

(i) A *Heegaard splitting* of a closed, oriented, connected 3-manifold M is a triple (U_1, U_2, Σ) such that $M = U_1 \cup_\Sigma U_2$ where U_1, U_2 are 3-dimensional handlebodies with boundary $\Sigma = \partial U_1 = -\partial U_2$. The closed orientable surface Σ is called a *Heegaard surface* for M and the *genus* of the splitting is the genus of Σ .

(ii) Two Heegaard splittings $(U_1, U_2, \Sigma), (U_1', U_2', \Sigma')$ of a closed, oriented, connected 3-manifold M are *equivalent* if there is an orientation preserving homeomorphism $f : M \to M$ such that $f(U_1) = U_1', f(U_2) = U_2', f(\Sigma) = \Sigma'$.

(iii) The *stabilisation* of a genus g Heegaard splitting (U_1, U_2, Σ) of a closed, oriented, connected 3-manifold M is the genus $g+1$ Heegaard splitting $(U_1 \natural (S^1 \times D^2), U_2 \natural (D^2 \times S^1), \Sigma \# (S^1 \times S^1))$ obtained by taking the connected sum of the Heegaard splitting (U_1, U_2, Σ) of M with the genus 1 Heegaard splitting $(S^1 \times D^2, D^2 \times S^1, S^1 \times S^1)$ of S^3.

Morse functions can be used to yield Heegaard decompositions of 3-manifolds.

Theorem 4.1.4. Every closed, oriented, connected 3-manifold M admits a Heegaard splitting.

Proof. By [Sma61, Theorem C] one can find a Morse function $f : M \to \mathbb{R}$ which is self-indexing (the value of each critical point $p \in M$ is equal to its index) such that f has exactly one critical point of index 0 and one critical point of index 3. Then $\frac{3}{2}$ is a regular value of f

and $f^{-1}([0,\frac{3}{2}]), f^{-1}([\frac{3}{2},3])$ are handlebodies with boundary a closed oriented surface $f^{-1}(\frac{3}{2})$ of genus g equal to the number of index 1 critical points. □

A Heegaard splitting (U_1, U_2, Σ) may be expressed combinatorially in terms of attaching curves on the Heegaard surface Σ.

Definition 4.1.5. A *set of attaching circles* on a closed, oriented surface Σ of genus g is a collection of simple closed curves $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_g)$ embedded in Σ such that

(i) The curves are disjoint from each other.

(ii) The homology classes $[\gamma_1], [\gamma_2], \dots, [\gamma_g] \in H_1(\Sigma; \mathbb{Z})$ are linearly independent.

The *handlebody determined by* $\boldsymbol{\gamma}$ is the result of attaching 3-dimensional 2-handles to $\Sigma \times [-1,1]$ along the curves $\gamma_1, \gamma_2, \dots, \gamma_g \subset \Sigma \times \{1\}$ to produce a cobordism between Σ and a disjoint union of $k - g + 1$ spheres and then capping these spherical boundaries with disks.

A *Heegaard diagram* is a triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ where Σ is a closed oriented surface of genus g and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_g), \boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_g)$ are sets of attaching circles. This determines a Heegaard splitting $(U_{\boldsymbol{\alpha}}, U_{\boldsymbol{\beta}}, \Sigma)$ of $M = U_{\boldsymbol{\alpha}} \cup_{\Sigma} U_{\boldsymbol{\beta}}$ where $U_{\boldsymbol{\alpha}}$ is the handlebody determined by $\boldsymbol{\alpha}$ and $U_{\boldsymbol{\beta}}$ is the handlebody determined by $\boldsymbol{\beta}$.

Example 4.1.6. A Heegaard diagram for the genus 1 Heegaard splitting of S^3

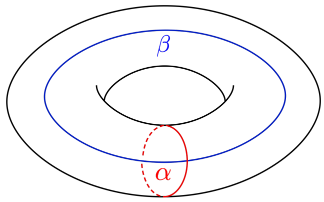


Figure 54: A genus 1 Heegaard diagram for S^3.

Certain standard moves can be performed on the attaching circles to transform one Heegaard diagram into another.

Definition 4.1.7. Let $\gamma_1, \gamma_2, \dots, \gamma_g$ be a set of attaching circles for a handle body U of genus g.

(i) An *isotopy* is performed by moving $\gamma_1, \gamma_2, \dots, \gamma_g$ through a one-parameter family of curves parametrised by $t \in [0,1]$ such that the curves remain mutually disjoint from each other at each instant.

(ii) A *handle-slide* is performed by choosing two distinct curves γ_i and γ_j and replacing γ_i with a new simple closed curve γ_i' embedded in ∂U in such a way that γ_i' is disjoint from $\gamma_1, \gamma_2, \dots, \gamma_g$ and $\gamma_i, \gamma_i', \gamma_j$ bound an embedded pair of pants.

(iii) Attaching curves $\boldsymbol{\alpha}, \boldsymbol{\beta}$ for a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ of genus g are in *standard position* if it is possible to perform a finite sequence of isotopies and handle slides such they are of the form

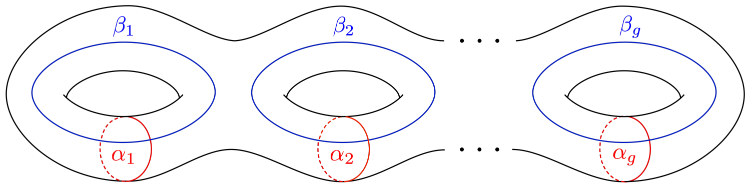


Figure 55: Attaching curves in standard position.

(iv) Two attaching curves α, β are *cancelling* if they intersect transversely in a single point

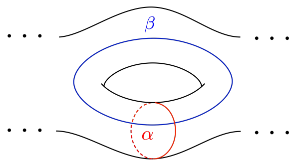


Figure 56: Cancelling attaching curves.

(v) Two attaching curves are *parallel* if they are of the form

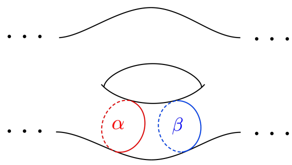


Figure 57: Parallel attaching curves.

Example 4.1.8.

(i) An isotopy between two attaching circles (γ_1, γ_2) and (γ_1', γ_2') for a handle body of genus 2.

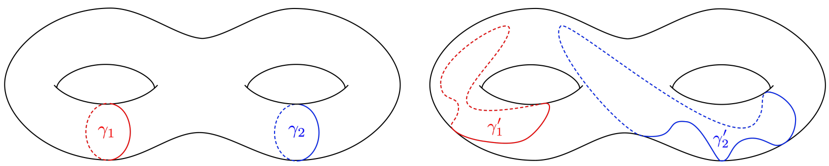


Figure 58: An isotopy of attaching circles.

(ii) A handle-slide between two attaching circles (γ_1, γ_2) and (γ_1', γ_2') for a handle body of genus 2 with the embedded pair of pants cobordism bounded by $\gamma_1, \gamma_2, \gamma_1'$.

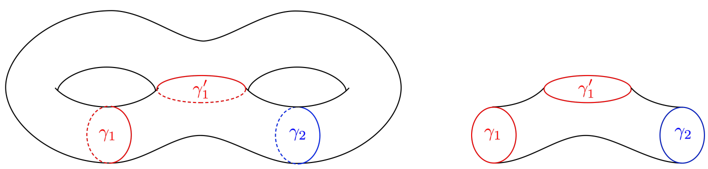


Figure 59: A handle-slide of attaching circles.

Theorem 4.1.9. ([Sin33]) Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ be Heegaard decompositions of a closed, oriented, connected 3-manifold M. It is possible to apply finitely many stabilisations, isotopies and handle slides so that the resulting Heegaard diagrams are equivalent.

4.2 Generic homotopies and Morse 2-functions

Gay and Kirby's Morse 2-functions are smooth maps from a manifold to a surface which look locally like a generic homotopy of Morse functions but with no global time direction.

Definition 4.2.1. Let M be a smooth manifold and let N be a smooth 1-manifold.

(i) A *homotopy* of Morse functions $f_0, f_1 : M \to N$ is a smooth map $f : I \times M \to N$ such that $f(0,x) = f_0(x)$ and $f(1,x) = f_1(x)$. We write $f_t = f(t,-) : M \to N$.

(ii) An *arc* of Morse functions is a homotopy $f : I \times M \to N$ such that each $f_t : M \to N$ is Morse.

(iii) Let $f : I \times M \to N$ be a smooth homotopy between Morse functions $f_0, f_1 : M \to N$. The *singular locus* of the map

$$F : I \times M \to I \times N, \quad (t,x) \mapsto (t, f_t(x))$$

is the set

$$Z_F = \{(t,x) : x \text{ is a critical point of } f_t\}$$

with image

$$F(Z_F) = \{(t, f_t(x)) : x \text{ is a critical point of } f_t\}$$

the *Cerf graphic* of F.

Cerf graphics allow one to track the evolution of critical values in a homotopy of Morse functions as time passes.

Definition 4.2.2. The *standard index k birth-death singularity model in dimension m* is the function

$$\sigma_m^k : \mathbb{R}^m \to \mathbb{R}, \quad (x_1, \ldots, x_m) \mapsto -x_1^2 - \ldots - x_k^2 + x_{k+1}^3 + x_{k+2}^2 + \ldots x_m^2.$$

Definition 4.2.3. ([Cer70] [KG13a, Definition 2.3]). Let M be a smooth m-dimensional manifold and let N be a smooth 1-dimensional manifold. A homotopy $f : I \times M \to N$ of Morse functions $f_0, f_1 : M \to N$ is called *generic* if each $f_t : M \to N$ is a Morse function except at finitely many values of t. At those values t_* where f_{t_*} is not Morse, exactly one of the following events should occur:

(i) Two critical values cross at t_*. More precisely, f_{t_*} is locally Morse but not Morse and there is a small $\epsilon > 0$ such that such $Z_F \cap ([t_* - \epsilon, t_* + \epsilon] \times M)$ is a collection of arcs on which F is an embedding except for exactly one transverse double point where the images of two arcs cross. This event is called a *crossing*

(ii) A pair of cancelling critical points are born (or die). More precisely, there is a small $\epsilon > 0$ such that for all $t \in [t_* - \epsilon, t_* + \epsilon]$, f_t is a Morse function outside of a ball and inside that ball there are local coordinates (which may depend on t) such that

$$f_t(x_1, \ldots, x_m) = -x_1^2 - \ldots - x_k^2 + x_{k+1}^3 - (t - t_*)x_{k+1} + x_{k+2}^2 + \ldots + x_m^2$$

and f_t has no other critical values near 0. This event is called a *birth singularity*.

In particular f_{t_*} has a birth singularity and f_t is Morse when $t \neq t_*$ with no critical points in this ball when $t < t_*$ and are precisely two critical points in this ball, one of index k and one of index $k+1$, when $t > t_*$. Moreover, F is injective on $Z_F \cap ([t_* - \epsilon, t_* + \epsilon] \times M)$ and $Z_F \cap ([t_* - \epsilon, t_* + \epsilon] \times M)$ is a collection of arcs. All but one arc has end points at $t_* - \epsilon$ and $t_* + \epsilon$ and is smoothly embedded via F, and the remaining one arc has both end points at $t_* + \epsilon$ and is mapped via F to a semi-cubical cusp in $[t_* - \epsilon, t_* + \epsilon] \times N$. The effect of reversing the direction of t is a *death singularity*.

Example 4.2.4. The figure below is a typical Cerf graphic for a generic homotopy of Morse functions

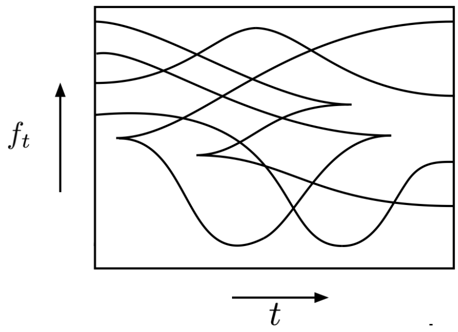


Figure 60: A Cerf graphic.

Definition 4.2.5. ([KG13a, Definition 2.7]). Let X be a smooth, closed n-dimensional manifold and let Σ be a smooth, closed 2-dimensional manifold. A smooth, proper map $F : X \to \Sigma$ is called a *Morse 2-function* if for each $q \in \Sigma$ there is a compact neighbourhood S of q with a diffeomorphism $\psi : S \to I \times I$ and a diffeomorphism $\phi : F^{-1}(S) \to I \times W$ for a smooth $(n-1)$-dimensional cobordism $(W; M, M')$ such that the coordinate representation $\psi \circ F \circ \phi^{-1} : I \times W \to I \times I$ is of the form $(t, x) \mapsto (t, f_t(x))$ for some generic homotopy of Morse functions $f_t : (W; M, M') \to (I; \{0\}, \{1\})$. A singular point of F is called a *fold point* if the homotopy used to model F at that point is an arc of Morse functions. A singular point of F is called a *cusp point* if the homotopy used to model F at that point has a birth or death at that point.

Example 4.2.6. The map

$$F : S^1 \times \mathbb{R}^2 \to \mathbb{R}^2 \subset S^2, \quad (t, x, y) \mapsto (t, \frac{1 + x^2 + y^2}{2})$$

is a Morse 2-function on the (open) manifold $S^1 \times \mathbb{R}^2$. Here we are using polar coordinates on the codomain. The function $(x, y) \mapsto \frac{1+x^2+y^2}{2}$ is Morse with a single critical point of index 0 at $(0, 0)$ so that F has singular set $S^1 \times \{(0, 0)\}$. The function $f_t(x, y) = \frac{1+x^2+y^2}{2}$ is Morse for each fixed time t so that each singular point of F is a fold point. The image of the singular set is a circle of folds in $\mathbb{R}^2$ of radius $\frac{1}{2}$ centred at the origin.

4.3 Trisections, trisection diagrams and the existence of Morse 2-functions

We now examine Kirby and Gay's result that every 4-manifold admits a Morse 2-function in order to produce some examples of fold loci of Morse 2-functions . The existence of Morse 2-functions on a 4-manifold be connected to the existence of a trisection of a 4-manifold and is a generalisation of the relationship between a Morse function on a 3-manifold and a Heegaard decomposition.

Definition 4.3.1. ([KG13b, Definition 1]). For integers $0 \leqslant k \leqslant g$ let $Z_k = \natural_k(S^1 \times D^3)$ be the 4-manifold with boundary $Y_k = \#_k(S^1 \times S^2)$ and let $(Y^+_{k,g}, Y^-_{k,g}, \#_g(S^1 \times S^1))$ be the genus g Heegaard splitting of Y_k obtained by stabilising $(g-k)$ times the genus k Heegaard splitting $(\natural_k(S^1 \times D^2), \natural_k(S^1 \times D^2), \#_k(S^1 \times S^1))$ of Y_k.

A (g,k)-*trisection* of a smooth, closed, connected, oriented 4-manifold X is a decomposition $X = X_1 \cup X_2 \cup X_3$ of X into 3 codimension 0-submanifolds X_i with boundary satisfying the following properties:

(i) For each $i = 1,2,3$ there is a diffeomorphism $\phi_i : X_i \to Z_k$.

(ii) For each $i = 1,2,3$ $\phi_i(X_{i-1} \cap X_i) = Y^+_{k,g}$ and $\phi_i(X_i \cap X_{i+1}) = Y^-_{k,g}$ where the index i is understood modulo 3.

(iii) $X_1 \cap X_2 \cap X_3$ is an orientable surface of genus g.

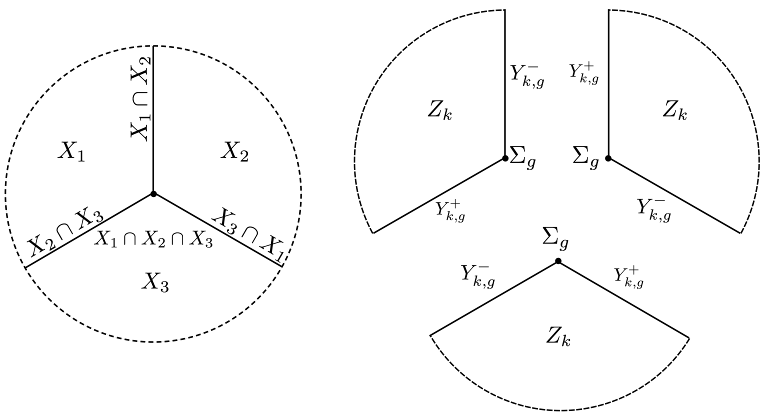


Figure 61: A schematic diagram for the trisection of X into the submanifolds X_1, X_2, X_3 or into the pieces Z_1, Z_2, Z_3.

Example 4.3.2.

(i) The 4-sphere

$$S^4 = \{(z, x_3, x_4, x_5) \in \mathbb{C} \times \mathbb{R}^3 : |z|^2 + x_3^2 + x_4^2 + x_5^2 = 1\}$$

has a $(0,0)$-trisection with

$$\begin{aligned} X_j &= \left\{ (re^{i\theta}, x_3, x_4, x_5) : r^2 + x_3^2 + x_4^2 + x_5^2 = 1, 0 \leqslant r, \frac{2\pi j}{3} \leqslant \theta \leqslant \frac{2\pi(j+1)}{3} \right\} \\ &\cong \left\{ (r, x_3, x_4, x_5, \theta) | r^2 + x_3^2 + x_4^2 + x_5^2 = 1, 0 \leqslant r, \frac{2\pi j}{3} \leqslant \theta \leqslant \frac{2\pi(j+1)}{3} \right\} \\ &= D_+^3 \times \left[\frac{2\pi j}{3}, \frac{2\pi(j+1)}{3} \right] \\ &\cong D^4 \\ &= \natural_0 (S^1 \times D^3) \end{aligned}$$

where D_+^3 is the northern hemisphere of

$$S^3 = \{(r, x_3, x_4, x_5) | r^2 + x_3^2 + x_4^2 + x_5^2 = 1\}.$$

(ii) The clutching function ω written in complex coordinates

$$\omega : S^1 \to SO(2), (x \mapsto (y \mapsto xy))$$

determines a vector bundle η_ω over S^2 whose sphere bundle $S(\eta_\omega)$ is the Hopf Bundle $S^1 \to S^3 \to S^2$. It is shown in [GS99, Example 4.24] that η_ω has a disc bundle

$$(D^2, S^1) \to (D(\eta_\omega), S(\eta_\omega)) = (\overline{\mathbb{C}P^2 - D^4}, S^3) \to S^2.$$

Decompose the base sphere $S^2 = D_+^2 \cup_{S^1} D_-^2$ as the union of a northern and southern hemisphere which intersect in the equator. The disc bundle $D(\eta_\omega)$ can be expressed as a union of the induced bundles over the contractible spaces D_+^2 and D_-^2. The induced bundles are the trivial disc bundles $D^2 \times D_+^2, D^2 \times D_-^2$ and hence we may decompose $\mathbb{C}P^2$ as the union of three balls D^4.

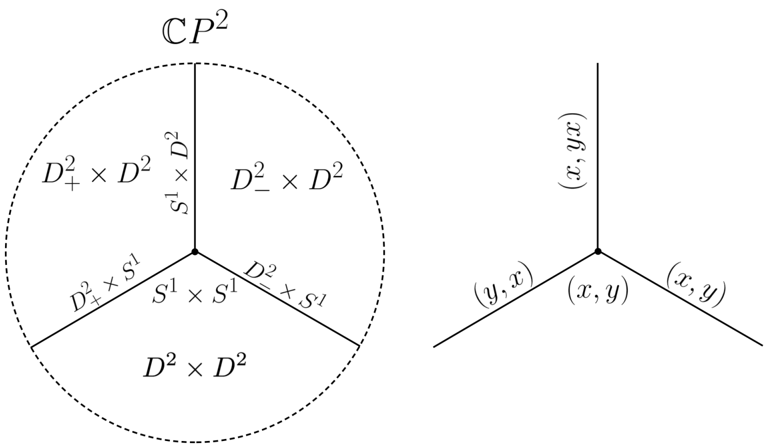


Figure 62: A trisection of $\mathbb{C}P^2$.

The diagram on the right shows how the central torus $S^1 \times S^1$ sits inside of each of the three pieces $D_+^2 \times S^1, D_-^2 \times S^1 S^1 \times D^2$. This determines a $(1,0)$-trisection of $\mathbb{C}P^2$.

The 4-manifold analogue of a Heegaard diagram is a trisection diagram which allows one to combinatorially record a trisection of a 4-manifold.

Definition 4.3.3. ([KG13b, p.8]). For integers $0 \leqslant k \leqslant g$ a (g,k) *trisection diagram* is a quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ where Σ is a closed oriented surface of genus g and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}), (\Sigma, \boldsymbol{\gamma}, \boldsymbol{\alpha})$ are genus g Heegaard diagrams for $\#_k(S^1 \times S^2)$. This determines a 4-manifold $X(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ obtained by attaching 4-dimensional 2-handles to $\Sigma \times D^2$ along

$$\boldsymbol{\alpha} \times \{1\}, \boldsymbol{\beta} \times \{e^{\frac{2\pi i}{3}}\}, \boldsymbol{\gamma} \times \{e^{\frac{4\pi i}{3}}\} \subset \Sigma \times S^1 = \partial(\Sigma \times D^2)$$

and filling in the rest with 3 and 4-handles (by [LP72] there is only one way to fill in with 3 and 4-handles). Equivalently, we may think of $X(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ as being obtained by glueing three copies of $\natural_k(S^1 \times D^3)$ over their boundaries, using the Heegaard decompositions

$$\#_k(S^1 \times S^2) = U_{\boldsymbol{\alpha}} \cup_\Sigma U_{\boldsymbol{\beta}} = U_{\boldsymbol{\beta}} \cup_\Sigma U_{\boldsymbol{\gamma}} = U_{\boldsymbol{\gamma}} \cup_\Sigma U_{\boldsymbol{\alpha}}$$

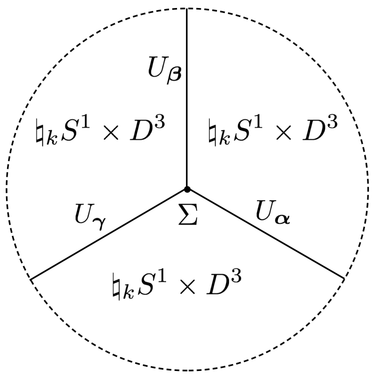


Figure 63: The trisection determined by a trisection diagram.

Example 4.3.4. ([KG13b, p.7-8]).

(i) A $(1,0)$-trisection diagram which determines the $(1,0)$ trisection of $\mathbb{C}P^2$ from Example 4.3.2

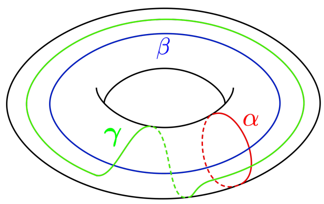


Figure 64: A trisection diagram for $\mathbb{C}P^2$.

(ii) A $(2,0)$ trisection diagram for $S^2 \times S^2$

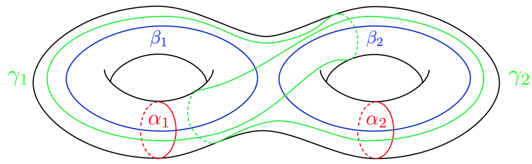


Figure 65: A trisection diagram for $S^2 \times S^2$.

(iii) A $(1, 1)$ trisection diagram for $S^1 \times S^3$

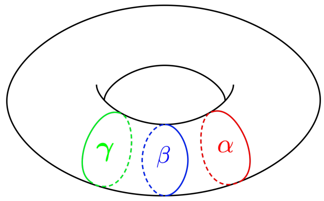


Figure 66: A trisection diagram for $S^1 \times S^3$.

The connection between Morse 2-functions and trisections is as follows.

Theorem 4.3.5. ([KG13b, Theorem 4, §3]). If X is a smooth, closed, connected, oriented 4-manifold then there exists a Morse 2-function $F : X \to \mathbb{R}^2 \subset S^2$ with a fold locus of the form

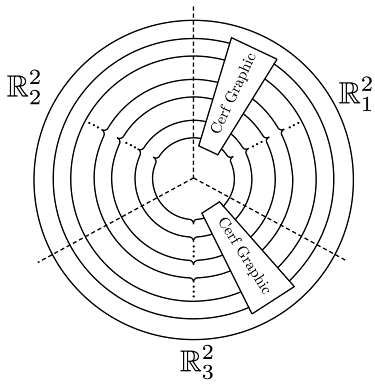


Figure 67: The prescribed fold locus.

where each sector

$$\mathbb{R}^2_j = \left\{ (r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 : 0 \leqslant r, \frac{2\pi j}{3} \leqslant \theta \leqslant \frac{2\pi(j+1)}{3} \right\} \quad (1 \leqslant j \leqslant 3)$$

contains the same number of fold curves. If k is the number of fold curves without cusps in each sector $\mathbb{R}^2_j$ then this induces a (g, k)-trisection of X with $X_j = F_3^{-1}(\mathbb{R}^2_j)$ and $g = \chi(X) + 2k - 2$.

The key idea is that by Cerf's 1-parameter theorem [Cer70] any two Morse functions $f_0, f_1 : M \to [0, 1]$ on a 3-manifold can be connected by a generic homotopy $f_t : f_0 \simeq f_1$. The homotopy can be chosen to keep the index one critical values below $\frac{1}{2}$ and the index two critical values above. During the homotopy are births and deaths of cancelling pairs of index one and two critical points but these stabilise the Heegaard splittings of M induced by f_0 and f_1 such that the induced handle slides taken one Heegaard splitting to the other. One can start with a trisection diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ of X and use Cerf's 1-parameter theorem to extend a Morse function on $U_\alpha, U_\beta, U_\gamma$, which realises the Heegaard decompositions in the trisection diagram, to a Morse 2-function on X. In terms of handles this corresponds to performing handle slides and handle cancellations to transform any one of the Heegaard diagrams of $U_\alpha, U_\beta, U_\gamma$ into any of the other Heegaard diagrams.

Example 4.3.6. We now show that there exists a Morse 2-function $F : \mathbb{C}P^2 \to \mathbb{R}^2 \hookrightarrow S^2$ such that the image of the fold locus in Theorem 4.3.5 is of the form

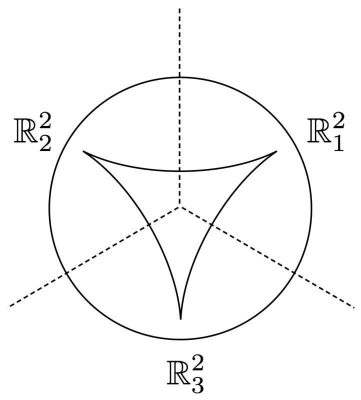


Figure 68: The fold locus for a Morse 2-function on $\mathbb{C}P^2$.

By Example 4.3.4 there is a genus 1 trisection diagram inducing a genus 1 trisection of $\mathbb{C}P^2$ into three discs D^4

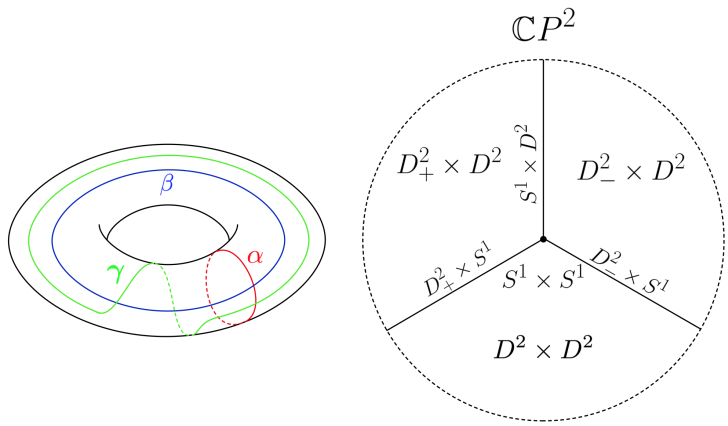


Figure 69: Trisection and trisection diagram for $\mathbb{C}P^2$.

By Example 4.1.2 there is a Morse function

$$f : S^1 \times D^2 \to I, \quad (\theta, x, y) \mapsto \frac{1}{2}\left(x^2 + y^2 + \frac{1}{2}\cos(\theta)(1 - x^2 - y^2) + 1\right)$$

which has exactly one critical point of index 0 and one critical point of index 1. Glueing the two of $S^1 \times D^2, D^2_+ \times S^1, D^2_- \times S^1$ over $S^1 \times S^1$ produces a sphere S^3 with a genus 1 Heegaard decomposition arising from one of the Heegaard diagrams

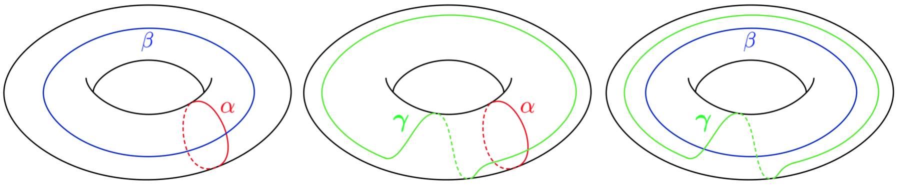


Figure 70: Three heegaard diagrams for S^3.

and glueing the two Morse functions produces a Morse function on S^3 which has two critical points of index 0 and two critical points of index 1 and the results may be glued together

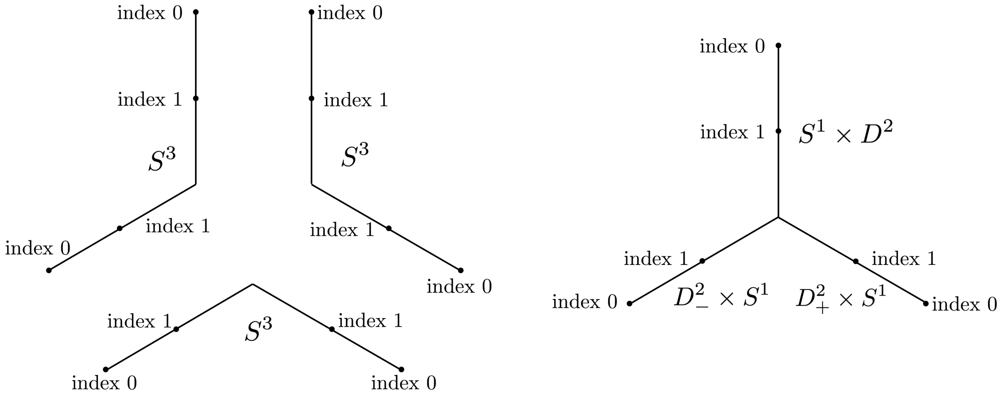


Figure 71: The critical points on the union.

We may thicken each of the three copies of $S^1 \times D^2, D^2_+ \times S^1, D^2_- \times S^1$ and their Morse functions by crossing each of the three copies of $S^1 \times D^2$ and their Morse functions with the closed unit interval I to produce a Morse 2-function

$$F : S^1 \times D^2 \times I \to \mathbb{R} \times I \hookrightarrow \mathbb{R} \times \mathbb{R}, \quad (\theta, x, y, t) \mapsto (f(\theta, x, y), t)$$

in such a way that after glueing a critical point of f becomes a fold of F, as shown below

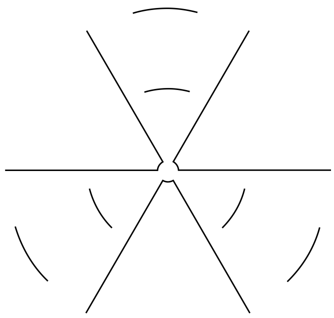


Figure 72: Extending the Morse function.

The resulting Morse 2-function then arises by performing handle slides and cancellations to transform any one of the Heegaard diagrams of S^3 into any of the other of the Heegaard diagrams.

Example 4.3.7. We now show that there exists a Morse 2-function $F : S^2 \times S^2 \to \mathbb{R}^2 \hookrightarrow S^2$ such that the fold locus has an image of the form

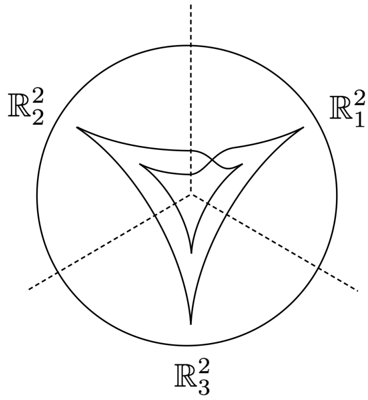


Figure 73: The fold locus for a Morse 2-function on $S^2 \times S^2$.

By Example 4.3.4 there is a genus 2 trisection diagram inducing a genus 2 trisection of $S^2 \times S^2$

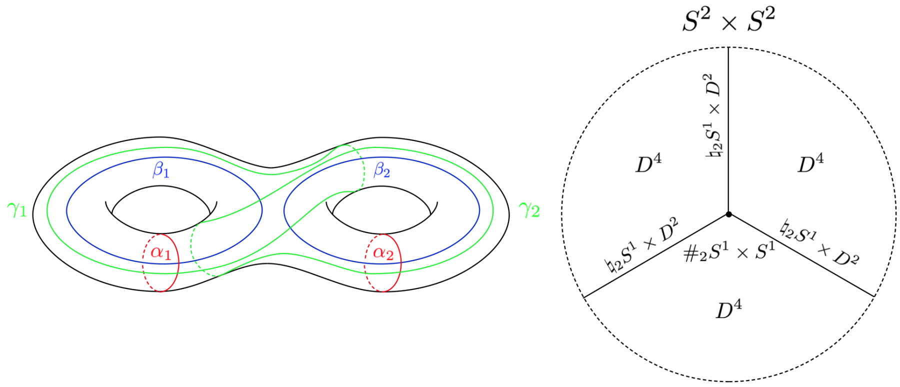


Figure 74: Trisection and trisection diagram for $S^2 \times S^2$.

The Morse function

$$S^1 \times D^2 \to I, \quad (\theta, x, y) \mapsto \frac{1}{2}\left(x^2 + y^2 + \frac{1}{2}\cos(\theta)(1 - x^2 - y^2) + 1\right)$$

from Example 4.1.2 with one critical point of index 0 and one critical point of index 1 induces a Morse function $S^1 \times D^2 \sqcup S^1 \times D^2 \to I$ with two critical points of index 0 and two critical points of index 1. This in turn induces a Morse function on the connected sum $(S^1 \times D^2) \natural (S^1 \times D^2)$ with two critical points of index 0 and three critical points of index 1. The 1-handle $I \times D^2$ attached to $S^1 \times D^2 \sqcup S^1 \times D^2$ to form the connected sum cancel with the 0-handle in the second copy of $S^1 \times D^2$ and so it is possible to a Morse function $f : (S^1 \times D^2) \natural (S^1 \times D^2) \to I$ with one index 0 critical point and two index 1 critical points. Glueing two copies of $(S^1 \times D^2) \natural (S^1 \times D^2)$ over $(S^1 \times S^1) \#(S^1 \times S^1)$ produces a sphere S^3 with a genus 2 Heegaard decomposition arising from one of the Heegaard diagrams

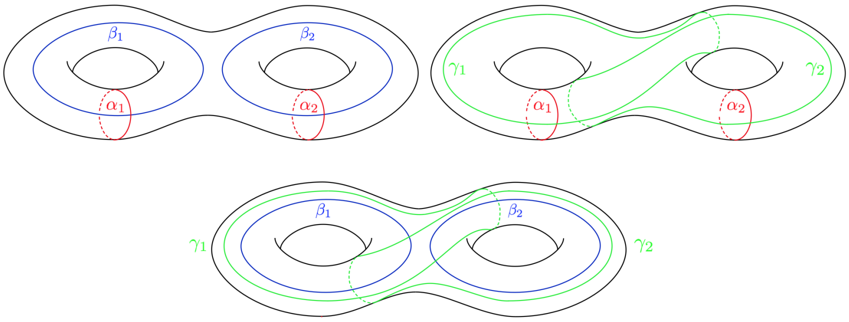


Figure 75: Three Heegaard diagrams for S^3.

and also produces a Morse function on S^3 which has two critical points of index 0 and four critical points of index 1 We may thicken by crossing with the closed unit interval I to produce a Morse 2-function

$$F : ((S^1 \times D^2) \natural (S^1 \times D^2)) \times I \to \mathbb{R} \times I \hookrightarrow \mathbb{R} \times \mathbb{R}, \quad (\theta, x, y, t) \mapsto (f(\theta, x, y), t)$$

in such a way that a critical point of f becomes a fold of F as shown below

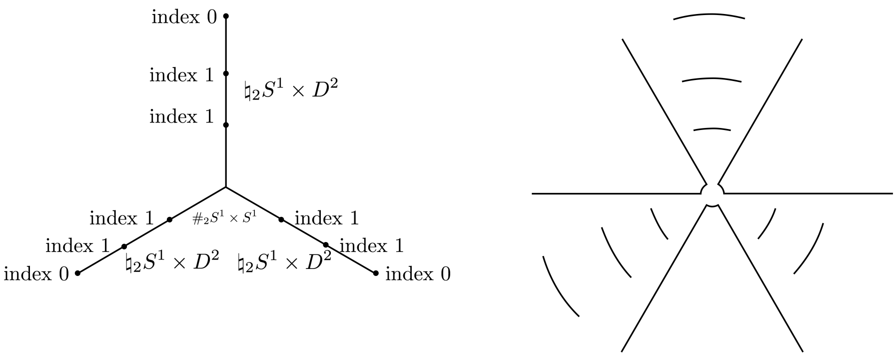


Figure 76: Critical points on the union.

The resulting Morse 2-function then arises by performing handle slides and cancellations to transform any one of the Heegaard diagrams of S^3 into any of the other of the Heegaard diagrams.

Example 4.3.8. The $(1, 1)$ trisection diagram for $S^1 \times S^3$ from Example 8

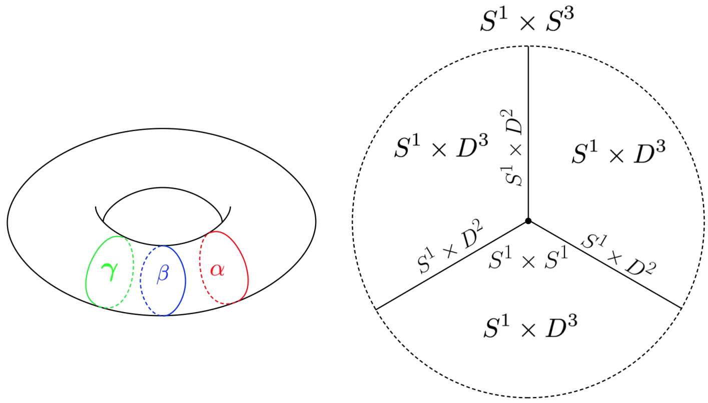


Figure 77: Trisection diagram and trisection for $S^1 \times S^3$.

induces a Morse 2-function such that the fold locus has an image consisting of no cusps, no crossings and two circles.

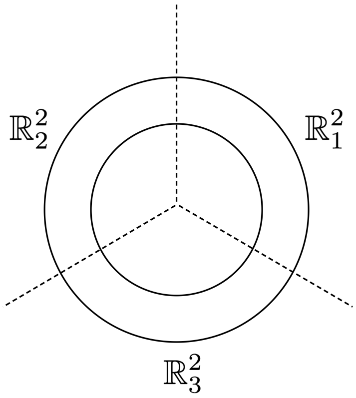


Figure 78: The fold locus of a Morse 2-function on $S^1 \times S^3$.

Chapter 5

Algebraic reconstruction of 4-manifolds

Gay and Kirby [KG12] showed that, subject to certain conditions, the fold curves and fibres of a Morse 2-function $F : M^4 \to S^2$ determine a manifold representation of a quiver from which one to reconstruct M^4 up to diffeomorphism. The reconstructions thickens with a twist the quiver representation and then glues in disc neighbourhoods of cusps and crossings. We give an algebraic analogue of their result by applying the symmetric construction at each step to show how a Morse 2-function $F : M^4 \to S^2$ can be used to reconstruct the symmetric Poincaré complex $(C(M; R), \phi_M)$ of M. The algebraic reconstruction thickens with a twist the symmetric Poincaré representation of a quiver and then glues the result to the symmetric Poincaré pairs obtained by applying the symmetric construction to disk neighbourhoods of cusps and crossings. This section follows closely the method in [KG12] and we recall the geometric reconstruction method presented there in order to motivate the algebraic reconstruction method.

5.1 Determining the quiver and its representations

From now on let M^n and Σ^2 be smooth, closed, connected oriented manifolds of dimension n and 2 respectively and let $F : M^n \to \Sigma^2$ be a Morse 2-function with connected fibres. By definition each point $p \in M$ has a Euclidean neighbourhood $\mathbb{R} \times \mathbb{R}^{n-1}$ and $f(p) \in \Sigma$ has a Euclidean neighbourhood $\mathbb{R} \times \mathbb{R}$ so that locally $F(x) = F(t, x) = (t, f_t(x))$ where $f_t : \mathbb{R}^{n-1} \to \mathbb{R}$ is a generic homotopy of Morse functions. It follows for each point $p \in M$ the rank of the differential $DF(p) : T_pM \to T_{F(p)}\Sigma$ is at least 1 so F has no critical points. The set of points in M for which the rank of the differential DF is precisely 1 is an embedded 1-dimensional submanifold of M with image under F an immersed 1-manifold in Σ^2, see [AGZV12, p.28].

Definition 5.1.1. The *fold graph* of F is the embedded graph $\Gamma \subset \Sigma^2$ which is formed from the immersed 1-manifold in Σ^2 by placing a degree 2 vertex at each cusp and a degree 4 vertex at each crossing.

The fold graph Γ then divides Σ into a collection of finitely many regions which we label $R_1, R_2, \ldots, R_m$. For each region R_i fix a representative point $y_i \in R_i$ in the interior of R_i so that y_i does not lie on any of the edges surrounding R_i.

Example 5.1.2. A fold graph $\Gamma \subset \mathbb{R}^2$ in the case where $\Sigma = S^2 = \mathbb{R}^2 \cup \{\infty\}$.

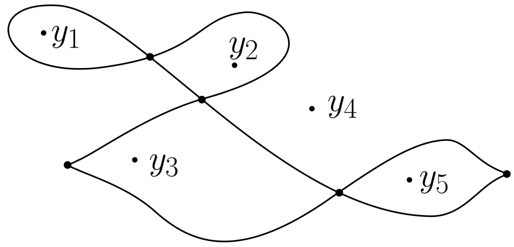


Figure 79: A fold graph.

Lemma 5.1.3.

(i) Let $M_i = F^{-1}(y_i)$ be the preimage of the representative point y_i under f. Then M_i is a smooth, closed, oriented $(n-2)$-dimensional embedded submanifold of M.

(ii) In the interior of each region R_i the map F is the projection of a locally trivial fibration, so that for any point y in the interior of R_i there is a small disc neighbourhood D^2 of y with a homeomorphism $h : F^{-1}(D^2) \to D^2 \times F^{-1}(y)$ such that there is a commutative diagram

$$\begin{array}{ccc} F^{-1}(D^2) & \xrightarrow{h} & D^2 \times F^{-1}(y) \\ {\scriptstyle f}\downarrow & \swarrow_{\pi} & \\ D^2 & & \end{array} .$$

(iii) If each region R_i is simply connected then F is a trivial fibre bundle over the interior of R_i.

Proof.

(i) The compactness of M_i follows from the fact that M is compact and M_i is a closed subset of M. By assumption y_i lies in the interior of the region A_i and hence is a regular value of F. By the Thom transversality theorem [Tho54] F is transverse regular to $\{y_i\} \subset \Sigma^2$ and hence M_i is smooth, closed $(n-2)$-dimensional submanifold of M. There is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{F} & \Sigma^2 \\ \uparrow & & \uparrow \\ M_i & \xrightarrow{F|} & \{y_i\} \end{array}$$

such that the normal bundle of M_i in M is the pullback of the normal bundle of $\{y_i\}$ in Σ and hence the normal bundle of M_i is trivial. The bundle isomorphism $T_{M_i} \oplus \nu_{M_i \hookrightarrow M} \cong T_M|_{M_i}$ then implies that $T_{M_i} \oplus \epsilon^2 = T_M|_{M_i}$ so that the tangent bundle of M_i is oriented and hence M_i is oriented.

(ii) The map F is proper and each point in the interior of R_i is a regular value so by Ehresmann's fibration theorem [Ehr51] it follows that F is a locally trivial fibration on the interior of R_i

(iii) The interior of R_i is then a simply-connected open subset of Σ which embeds into $\mathbb{R}^2$ and hence is contractible. This implies that hence F is a trivial bundle over the interior of R_i.

□

Suppose now that R_i and R_j are adjacent regions with the points y_i and y_j chosen close to an edge which separates R_i and R_j. Choose a path $\alpha : [0,1] \to M$ from y_i to y_j which intersects the separating edge between R_i and R_j transversely at a single point p which is neither a cusp nor a crossing, as shown below

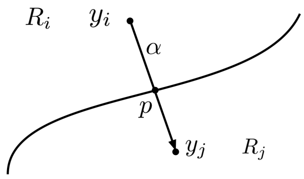


Figure 80: An edge between regions separated by a fold curve.

Lemma 5.1.4. Let $M_i = F^{-1}(y_i), M_j = F^{-1}(M_j), W_{i,j} = F^{-1}(\alpha_{i,j})$. The points p, y_i, y_j and the arc α can be chosen such that $(W_{i,j}; M_i, M_j)$ is an oriented $(n-1)$-dimensional cobordism which arises as the trace of a surgery on M_i with effect M_j.

Proof. Since F is a Morse 2-function there is a compact neighbourhood of p diffeomorphic to $I \times I$ and an $(n-1)$-dimensional cobordism W such that locally F is of the form

$$I \times W \to I \times I, \quad (t, x) \mapsto (t, f_t(x))$$

for some generic homotopy of Morse functions $f_t : W \to I$. Since p is a fold point the generic homotopy f_t can be chosen to be an arc of Morse functions. Choosing y_i, y_j and the path α such that the image of α has the same t coordinate as p then implies that there is a Morse function $f : W \to [0,1]$ such that y_i and y_j are regular values of f and p is the only critical value of f between y_i and y_j. Ordinary Morse theory then implies that $(W_{i,j}; M_i, M_j)$ is a cobordism which arises as the trace of a surgery on M_i with effect M_j. □

We now specialise to the case $n = 4$ and assume that each regular fibre is connected so that each M_i is homeomorphic to a a standard, closed, oriented, connected surface F_i of genus g_i. In the context Lemma 5.1.4 it follows that F_i and F_j differ by one in their genus since the surgery relates the Euler characteristics by $\chi(F_j) = \chi(F_i) \pm 2$, see Proposition 4.33 [Ran02a]. This naturally determines a quiver.

Definition 5.1.5. The *ordered quiver of* F is the ordered quiver $Q = (Q_0, Q_1, s, t; Q_1 \to Q_0)$ defined as follows:

(i) The vertices Q_0 are the representative points $\{y_i\}_{i=1}^m$.

(ii) If R_i and R_j are adjacent regions such that F_i has higher genus than F_j then there is an arrow $\alpha \in Q_1$ with source y_i and target y_j for each edge of the fold graph Γ which separates R_i and R_j.

The quiver contains no cycles because the regular fibres F_i are all assumed to be connected the genus of the regular fibres is strictly decreasing along the directed edges.

Example 5.1.6. The fold graph in Example 5.1.2 determines the quiver

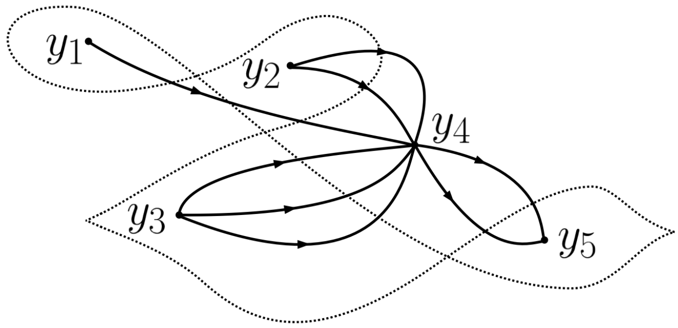


Figure 81: The quiver determined by a fold graph.

where, for example, F_1, F_2, F_3 have genus 3, F_4 has genus 2 and F_5 has genus 1 as shown below

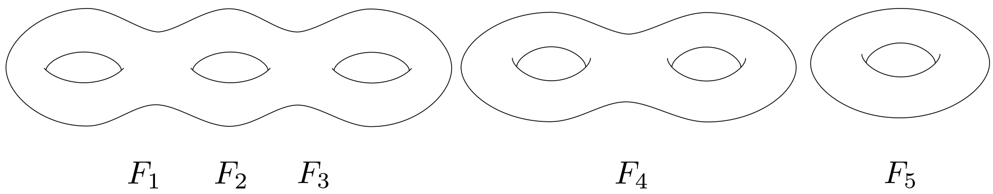


Figure 82: Regular fibres.

Lemma 5.1.4 then determines a 3-dimensional oriented manifold representation of the quiver.

Definition 5.1.7. The 3-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q associates to each vertex $y_i \in Q_0$ the 2-dimensional oriented manifold M_i and associates to each arrow $\alpha \in Q_1$ from y_i to y_j the 3-dimensional oriented cobordism $(W_{i,j}; M_i, M_j)$.

Example 5.1.8. The quiver from Example 5.1.6 has a 3-dimensional oriented manifold representation of the form

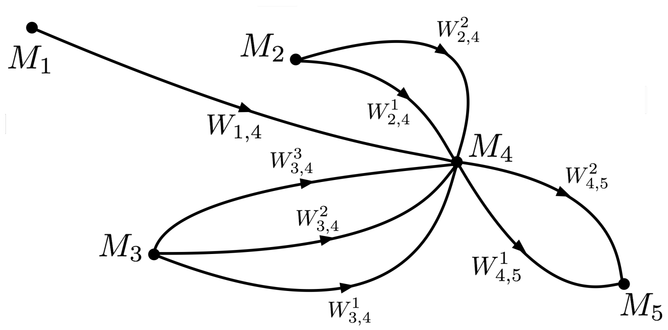


Figure 83: A manifold representation of a quiver.

as in Example 3.5.4 where each cobordism $(W; M_i, M_j)$ arises at the trace of a surgery.

This determines a symmetric Poincaré representation.

Definition 5.1.9. Let R be a commutative ring with identity. The 3-dimensional symmetric Poincaré representation

$$(C(M_Q;R) \oplus C(M'_Q;R) \to C(W_Q;R), (\phi_{W_Q}, \phi_{M_Q} \oplus -\phi_{M'_Q}))$$

of Q over R is obtained by applying the symmetric construction over R from Proposition 3.4.6 to the 3-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q.

5.2 Algebraic Reconstruction

The 3-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q together with glueing data is enough to reconstruct M and F geometrically.

Theorem 5.2.1. ([KG12, p.6-8]). Let $F : M^4 \to \Sigma^2$ be a Morse 2-function and suppose that all the regions bounded by F are simply-connected and the regular fibres of F are all of genus at least one and are connected. Then the following data suffice to reconstruct M and F up to diffeomorphism:

(i) The fold graph Γ.

(ii) The standard fibre over an interior point in each region R_i, that is a drawing of a standard closed, oriented surface F_i of genus g_i.

(iii) An attaching circle C_α for each arrow α of the quiver Q such that passing from from F_i to F_j along the arrow α is achieved by attaching attaching a 2-handle to F_i via C_α.

(iv) Glueing data for each arrow α of the quiver Q , that is a collection of $2(g_i - 1)$ simple closed curves on F_i

$$a_{1,\alpha}, b_{1,\alpha}, \ldots, a_{g_i-1,\alpha}, b_{g_i-1,\alpha}$$

which are disjoint from C_α such that $a_{k,\alpha} \cap b_{k,\alpha} = \{*\}$ is a single transverse point of intersection and $(a_{k,\alpha} \cup b_{k,\alpha}) \cap a_{l,\alpha} \cap b_{l,\alpha} = \varnothing$ if $k \neq l$, so that the genus $g_i - 1$ surface obtained from F_i by surgery along C_α should be identified with F_j in such a way that the curves

$$a_{1,\alpha}, b_{1,\alpha}, \ldots, a_{g_i-1,\alpha}, b_{g_i-1,\alpha}$$

map to the standard basis for F_j.

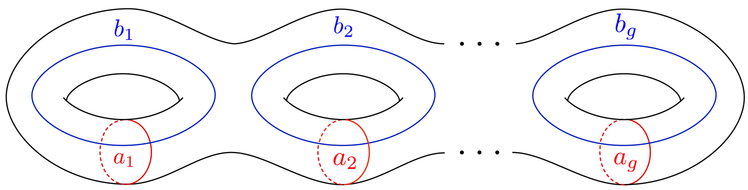


Figure 84: The standard basis for the standard surface of genus g.

Proof. The full proof of existence and uniqueness is given in [KG12]. We only sketch the existence part of a reconstruction of M and use it later as a template for an an algebraic analogue to reconstruct $(C(M), \phi_M)$. Part (iii) of Lemma 5.1.3 implies that F is a trivial fibre bundle over the interior of each region R_i. It follows that the thickening of the 3-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q from Definition 5.1.7 is an oriented 4-manifold with boundary $(\Omega, \partial\Omega)$ which determines M outside of the preimage of disc neighbourhoods of cusps and crossings. We invite the reader to look the sequence of diagrams from Examples 5.1.2, 5.1.6, 5.1.8 and 3.5.4 to see this process. If we wish to work with the standard fibre over an interior point y_i in each region R_i as in part (ii) of Theorem 5.2.1, then instead of working with the preimage $F^{-1}(y_i)$ directly as in $(W_Q; M_Q, M'_Q)$, we must use the glueing data from part (iv). By assumption M is an extension of $(\Omega, \partial\Omega)$ over the preimages of disk neighbourhoods of cusps and crossings. Gay and Kirby then show that this extension is unique. The preimage of a disc neighbourhood of a cusp z between regions R_i and R_j is a 4-manifold $\Omega_{i,j}$ with boundary $\partial\Omega_{i,j} = W^1_{i,j} \cup_{M_i \sqcup M_j} -W^2_{i,j}$ such that $\Omega_{i,j}$ deformation retracts onto the central fibre $F^{-1}(z)$ which has the same topological type as M_i.

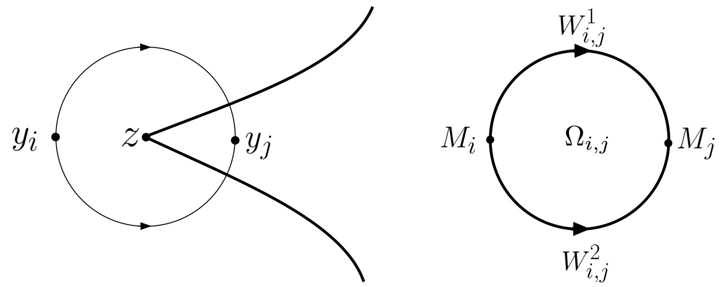


Figure 85: The pre-image of the disc neighbourhood of a cusp.

The preimage of a disc neighbourhood of a crossing z between regions R_i, R_j, R_k, R_l is a 4-manifold $\Omega_{i,j,k,l}$ with boundary $\partial\Omega_{i,j,k,l} = W_{i,j} \cup W_{j,l} \cup -W_{k,l} \cup -W_{i,k}$ such that $\Omega_{i,j,k,l}$ deformation retracts onto the central fibre $F^{-1}(z)$. The central fibre is homotopy equivalent to the space obtained from M_i by separately collapsing the disjoint framed embeddings $S^{m_i} \times D^{4-m_i} \hookrightarrow M_i$ and $S^{m_j} \times D^{4-m_j} \hookrightarrow M_i$ on which we do surgery to obtain M_j and M_k.

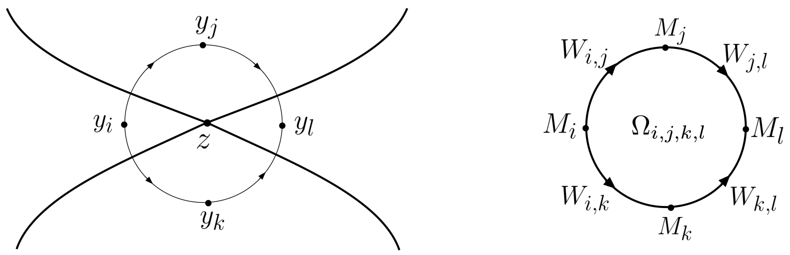


Figure 86: The pre-image of the disc neighbourhood of a crossing.

The symmetric Poincaré complex $(C(M), \phi_M)$ may then be reconstructed by applying the symmetric construction to the twisted thickening of the 3-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q and to the preimages of disc neighbourhoods of cusps and crossings, and then glueing the results. □

Theorem 5.2.2. Let R be a commutative ring with identity. The symmetric Poincaré complex $(C(M; R), \phi_M)$ may be reconstructed up to homotopy equivalence from the 3-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q with twisted glueing data.

Proof. By Theorem 3.5.9 the symmetric construction commutes with the twisted thickening operations up to homotopy equivalence of the resulting symmetric pair, as expressed in the diagram

$$\begin{array}{ccc}
(W_Q; M_Q, M'_Q) & \xrightarrow[\text{thickening}]{\text{twisted geometric}} & (\Omega, \partial\Omega) \\
\Big\downarrow {\scriptstyle \text{symmetric construction}} & & \Big\downarrow {\scriptstyle \text{symmetric construction}} \\
\begin{array}{c}(C(M_Q; R) \oplus C(M'_Q; R) \to C(W_Q; R), \\ (\phi_{W_Q}, \phi_{M_Q} \oplus -\phi_{M'_Q}))\end{array} & \xrightarrow[\text{thickening}]{\text{twisted algebraic}} & \begin{array}{c}(C(\partial\Omega; R) \to C(\Omega; R), (\phi_\Omega, \phi_{\partial\Omega})) \\ \wr\!\! \mid \\ (\partial D \to D, (\phi_D, \phi_{\partial D}))\end{array}
\end{array}$$

This implies that up to homotopy equivalence, the 4-dimensional symmetric Poincaré pair

$$(C(\partial\Omega; R) \to C(\Omega; R), (\phi_\Omega, \phi_{\partial\Omega}))$$

can be reconstructed by thickening the algebraic Morse 2-function determined by the 3-dimensional symmetric Poincaré representation

$$(C(M_Q; R) \oplus C(M'_Q; R) \to C(W_Q; R), (\phi_{W_Q}, \phi_{M_Q} \oplus -\phi_{M'_Q}))$$

and twisted algebraic glueing data obtained by applying the symmetric construction to the 3-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q and twisted geometric glueing data.

The symmetric construction may be applied to the pre-images of disk neighbourhoods of cusps and crossings described in the sketch of the proof of Theorem 5.2.1. In the case of a cusp the resulting 4-dimensional symmetric Poincaré pair has a 2-fold split boundary. Since $\Omega_{i,j}$ deformations retracts onto M_i and M_i is 2-dimensional, it follows by Lemma 2.1.6 that the relative part of the symmetric pair is given up to homotopy equivalence by $(C(M_i), 0)$.

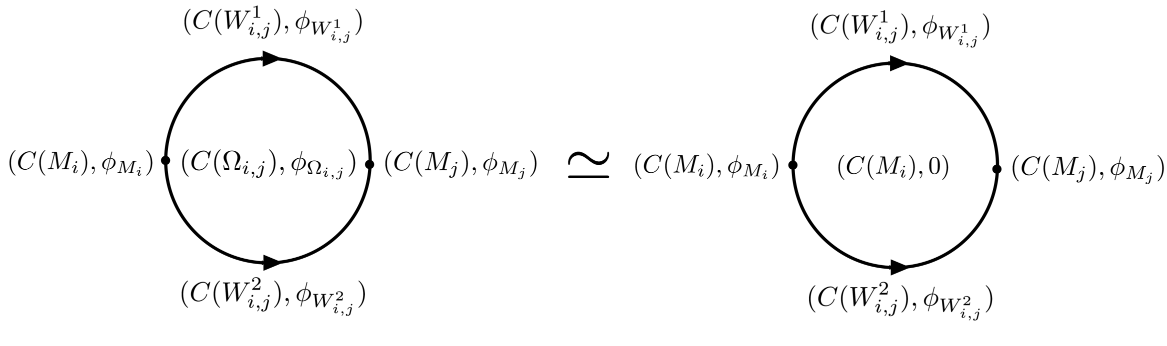


Figure 87: A schematic diagram for a homotopy equivalence of the algebraic data associated to a cusp.

In the case of a crossing, the resulting 4-dimensional symmetric Poincaré pair has a 4-fold split boundary. Since $\Omega_{i,j,k,l}$ deformations retracts onto M_i, it again follows that the relative part of the symmetric pair is given, up to homotopy equivalence by $(C(M_i), 0)$.

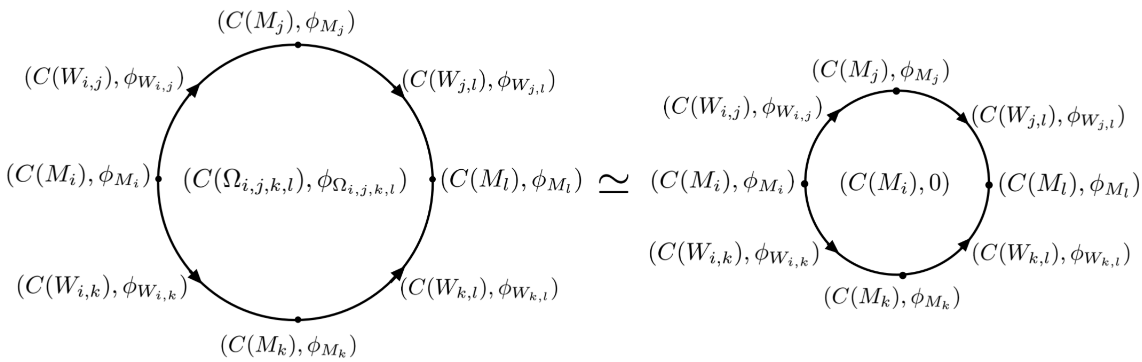


Figure 88: A schematic diagram for a homotopy equivalence of the algebraic data associated to a crossing.

Glueing in the symmetric Poincaré pairs obtained by applying the symmetric construction to each cusp and crossing produces a 4-dimensional symmetric complex which is homotopy equivalent to $(C(M), \phi_M)$. □

Corollary 5.2.3. The signature of M may be recovered from the 3-dimensional oriented manifold representation $(W_Q; M_Q, M'_Q)$ of Q and the twisted glueing data.

Proof. If $R = \mathbb{Z}$ then the isomorphism $L^4(\mathbb{Z}) \cong \mathbb{Z}$ from Proposition 1.3.7 sends $(C(M;\mathbb{Z}), \phi_M)$ to $\sigma(M)$. □

5.3 An open question

Ranicki showed [Ran80a, Proposition 4.7] that every $(n+1)$-dimensional ϵ-symmetric cobordism is homotopy equivalent to a union of elementary ϵ-symmetric cobordisms arising as the traces of elementary surgeries. This is an exact algebraic analogue of the result of Thom [Tho49] and Milnor [Mil61] that every manifold cobordism has a handle decomposition as a union of elementary cobordisms which arise as the traces of elementary surgeries. We may think of this manifold and algebraic data as being parametrised by quiver representations of the form

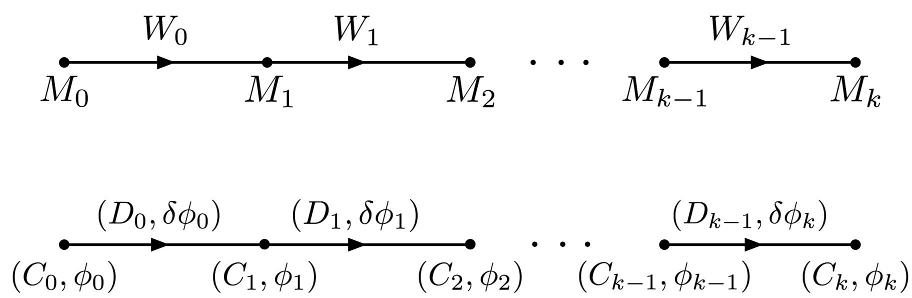


Figure 89: Quiver representations for a sequence of adjoining cobordisms.

In [BNR12a, Theorem 4.5.6] it was shown that every relative symmetric Poincaré cobordism is homotopy equivalent to the thickening of a symmetric Poincaré representation of the trinity quiver.

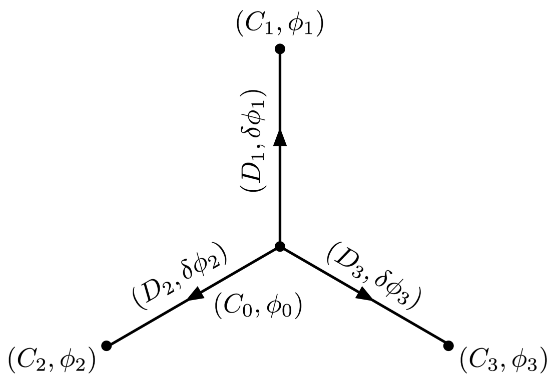


Figure 90: A symmetric Poincaré representation of the trinity quiver.

This suggests the following:

Open question: Is every $(n+2)$-dimensional symmetric Poincaré pair $(f : C \to D, (\delta\phi, \phi))$ homotopy equivalent to a twisted thickening of an n-dimensional symmetric Poincaré representation of a quiver?

Open question: Does every $(n+2)$-dimensional symmetric Poincaré complex (C, ϕ) arise from data which can parametrised in an n-dimensional symmetric Poincaré representation of a quiver. In the sense of Meyer [Mey73], is there an explicit cocycle on the space of n-dimensional symmetric Poincaré representations of a quiver which realises the signature of (C, ϕ)?

Part II

The L-theory of a triangular matrix ring

Introduction to Part II

Recall from Part I that the algebraic model for a surgery on a closed n-dimensional manifold M with surgery data a framed embedding $S^i \times D^{n-i} \hookrightarrow M$ and effect an n-dimensional manifold M' is an algebraic surgery on an n-dimensional ϵ-symmetric complex (C, ϕ) with surgery data an $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi, \phi))$ with effect an n-dimensional ϵ-symmetric complex (C', ϕ'). The algebraic model for the trace $(W; M, M')$ of the geometric surgery is the trace of the algebraic surgery which is an $(n+1)$-dimensional ϵ-symmetric cobordism $((g\ \ g') : C \oplus C' \to D', (0, \phi \oplus -\phi'))$. Milnor [Mil61] and Thom [Tho49] used Morse theory to show that every $(n+1)$-dimensional cobordism $(W; M, M')$ can be expressed as a union of elementary cobordisms which arise as the traces of surgeries. Ranicki [Ran80a, Proposition 4.7] gave a precise algebraic analogue of this decomposition and showed that every $(n+1)$-dimensional ϵ-symmetric cobordism is homotopy equivalent to a union of elementary ϵ-symmetric cobordisms arising as the traces of elementary algebraic surgeries.

Borodzik, Némethi and Ranicki [BNR12a] generalised the ordinary surgery operation on a closed manifold M to a half-surgery operation on an $(n+1)$-dimensional manifold with boundary (Σ, M) as follows:

(i) The effect of an index $i+1$ right half-surgery with surgery data a framed embedding $S^i \times D^{n-i} \hookrightarrow M$ is the $(n+1)$-dimensional manifold with boundary

$$(\Sigma', M') = (\Sigma \cup_{S^i \times D^{n-i}} D^{i+1} \times D^{n-i}, \overline{M - S^i \times D^{n-i}} \cup_{S^i \times S^{n-i-1}} D^{i+1} \times S^{n-i-1}).$$

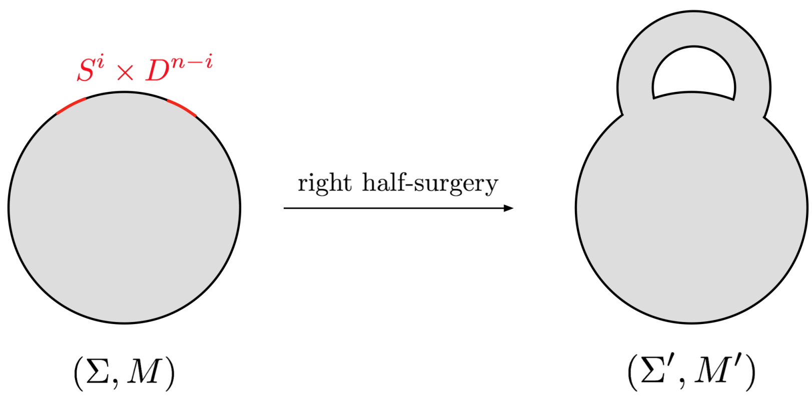


Figure 91: The effect of a right half-surgery.

If $(W; M, M')$ is the trace of the ordinary surgery on M removing the framed embedding

$S^i \times D^{n-i} \hookrightarrow M$, then the trace of the right-half surgery is the $(n+2)$-dimensional relative cobordism

$$(\Sigma' \times I; \Sigma \times \{0\}, \Sigma' \times \{1\}, W; M, M')$$

between (Σ, M) and (Σ', M').

(ii) The effect of an index $i+1$ left half-surgery with surgery data a framed embedding $(D^{i+1} \times D^{n-i}, S^i \times D^{n-i}) \hookrightarrow (\Sigma, M)$ is the $(n+1)$-dimensional manifold with boundary

$$(\Sigma', M') = (\overline{\Sigma - D^{i+1} \times D^{n-i}}, \overline{M - S^i \times D^{n-i}} \cup_{S^i \times S^{n-i-1}} D^{i+1} \times S^{n-i-1}).$$

Figure 92: The effect of a left half-surgery.

If $(W; M, M')$ is the trace of the ordinary surgery on M removing is the framed embedding $S^i \times D^{n-i} \hookrightarrow M$, then the trace of the left-half surgery is the $(n+2)$-dimensional relative cobordism

$$(\Sigma \times I; \Sigma \times \{0\}, \Sigma' \times \{1\}, W; M, M')$$

between (Σ, M) and (Σ', M').

Borodzik, Némethi and Ranicki [BNR12b, Theorem 4.18] used Morse theory on a manifold with boundary to show that every $(n+2)$-dimensional relative cobordism $(\Omega; \Sigma, \Sigma', W; M, M')$, such that Σ, Σ', Ω have no closed connected components, can be expressed as a union of adjoining elementary relative cobordisms

$$\Omega = \Omega_0 \cup \Omega_{\frac{1}{2}} \cup \Omega_1 \cup \Omega_{\frac{3}{2}} \cup \ldots \cup \Omega_{n+\frac{3}{2}} \cup \Omega_{n+2}$$

where Ω_0 arises as the effect of an index 0 handle attachment, Ω_i arises as the trace of an index i right-half surgery, $\Omega_{i+\frac{1}{2}}$ arises as the trace of an index i left half-surgery and Ω_{n+2} arises as the effect of an index $(n+2)$-handle attachment.

This suggests that the algebraic model for a half-surgery on an $(n+1)$-dimensional manifold with boundary should be a relative algebraic surgery on an $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi, \phi))$ with algebraic surgery data an $(n+2)$-dimensional ϵ-symmetric triad

(Γ, Φ) of the form

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \delta C & \xrightarrow[\delta f]{} & \delta D \end{array}$$

$$\Phi = (\phi'', \delta\phi'', \delta\phi, \phi).$$

The effect of the algebraic surgery should be an $(n+1)$-dimensional ϵ-symmetric pair $(f' : C' \to D', (\delta\phi', \phi'))$ with trace an $(n+2)$-dimensional ϵ-symmetric relative cobordism (Γ', Φ') between $(f : C \to D, (\delta\phi, \phi))$ and $(f' : C' \to D', (\delta\phi', \phi'))$ of the form

$$\Gamma' = \begin{array}{ccc} C \oplus C' & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & D \oplus D' \\ {\scriptstyle \begin{pmatrix} g' & g'' \end{pmatrix}}\big\downarrow & & \big\downarrow{\scriptstyle \begin{pmatrix} h' & h'' \end{pmatrix}} \\ \delta C' & \xrightarrow[\delta f']{} & \delta D' \end{array}$$

$$\Phi' = (0, 0, \delta\phi \oplus -\delta\phi', \phi \oplus -\phi').$$

Moreover, (C', ϕ') should be the effect of an algebraic surgery on (C, ϕ) with algebraic surgery data the $(n+1)$-dimensional symmetric pair $(g : C \to \delta C, (\delta\theta, \phi))$ and trace the $(n+1)$-dimensional ϵ-symmetric cobordism $((g'\ g'') : C \oplus C' \to \delta C', (0, \phi \oplus -\phi'))$. In addition, every $(n+2)$-dimensional commutative ϵ-symmetric Poincaré relative cobordism should be homotopy equivalent to a union of traces of elementary relative surgeries.

The obstruction to doing an algebraic surgery on an $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi, \phi))$ is that the symmetric structure $(\delta\phi, \phi)$ is a relative cycle in the algebraic mapping cone of $f^{\%} : W^{\%}C \to W^{\%}D$. There is no chain homotopy equivalence between $\mathscr{C}(f^{\%})$ and $W^{\%}\mathscr{C}(f)$ and so we cannot interpret $(\delta\phi, \phi)$ as a non-relative symmetric structure directly.

The triangular matrix ring A determined by two rings with identity A_1, A_2 and an (A_1, A_2)-bimodule B is the matrix ring

$$A = \left\{ \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} : a_1 \in A_1, a_2 \in A_2, b \in B \right\}.$$

By the work of Green [Gre82] an n-dimensional A-module chain complex $\mathbf{C}$ can be identified with a triple $(C, C', \mu : B \otimes_{A_2} C' \to C)$ where C is an n-dimensional A_1-module chain complex, C' is an n-dimensional A_2-module chain complex and $\mu : B \otimes_{A_2} C' \to C$ is an A_1-module chain map. A chain map $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ can be identified with a pair of chain maps $(f : C \to D, f' : C' \to D'$ such that there is a commutative triad

$$\begin{array}{ccc} B \otimes_{A_2} C' & \xrightarrow{\mu} & C \\ {\scriptstyle 1 \otimes f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B \otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

This suggest that the relative surgery problem could be solved by working over a triangular matrix ring. However, if A_1, A_2 are rings with involution then the extension of the involutions of A_1, A_2 to A given by

$$-:\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \to \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}; \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} \mapsto \overline{\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}} = \begin{pmatrix} \overline{a_1} & b \\ 0 & \overline{a_2} \end{pmatrix}$$

is in fact not an involution since

$$\overline{\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} a_1' & b' \\ 0 & a_2' \end{pmatrix}} = \begin{pmatrix} \overline{a_1 a_1'} & a_1 b' + b a_2 \\ 0 & \overline{a_2 a_2'} \end{pmatrix} = \begin{pmatrix} \overline{a_1'}\,\overline{a_1} & a_1 b' + b a_2 \\ 0 & \overline{a_2'}\,\overline{a_2} \end{pmatrix}$$

whereas

$$\overline{\begin{pmatrix} a_1' & b' \\ 0 & a_2' \end{pmatrix}} \cdot \overline{\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}} = \begin{pmatrix} \overline{a_1'} & b' \\ 0 & \overline{a_2'} \end{pmatrix}\begin{pmatrix} \overline{a_1} & b \\ 0 & \overline{a_2} \end{pmatrix} = \begin{pmatrix} \overline{a_1'}\,\overline{a_1} & \overline{a_1} b' + b\overline{a_2} \\ 0 & \overline{a_2'}\,\overline{a_2} \end{pmatrix}.$$

It follows that the techniques from chapter 1 cannot be applied directly to determine the symmetric L-theory of A. To resolve this problem, one must construct a chain duality on the additive category A-Mod of A-modules which allows to dual of an object in A-Mod to be a chain complex in A-Mod.

The L-theory of triangular matrix rings was first considered [Ran06] in connection with generalised free products and noncommutative localisation $A \to \Sigma^{-1}A$ of a ring with involution A. Ranicki made a suggestion [Ran06, Section 2.5] for a chain duality on A-Mod and gave a claim that there was a long exact sequence

$$\ldots \to L^{n+1}(A_2, \epsilon) \to L^{n+1}(A_1, \epsilon) \to L^n(A, \epsilon) \to L^n(A_2, \epsilon) \to L^n(A_1, \epsilon) \to \ldots$$

relating the L-theory of A to the L-theory of A_1 and A_2 via the change of rings morphism $B \otimes_{A_2} - : A_2\text{-Mod} \to A_1\text{-Mod}$.

Under the assumption that B is equipped with a non-degenerate bilinear pairing $\beta : B \times B \to A_1$ which is symmetric modulo the involution on A_1, we show that:

Theorem 7.2.7. The contravariant additive functor

$$\begin{gathered} \mathbf{T} : A\text{-Mod} \to A - \text{Chain} \\ \mathbf{M} = (M, M', \mu : B \otimes_{A_2} M' \to M) \mapsto \mathbf{C} = (C, C', \hat{\mu} : B \otimes_{A_2} C' \to C) \end{gathered}$$

where $\mathbf{C}$ is the 1-dimensional chain complex

$$\mathbf{C} = (C, C', \hat{\mu}) = \left(\begin{array}{c} M^* \\ \big\downarrow (\beta^{-1}\otimes 1)\mu^* \\ B \otimes_{A_2} M'^* \end{array}, \begin{array}{c} 0 \\ \big\downarrow 0 \\ M'^* \end{array}, \begin{array}{ccc} 0 & \xrightarrow{0} & M^* \\ \big\downarrow 0 & & \big\downarrow (\beta^{-1}\otimes 1)\mu^* \\ B \otimes_{A_2} M'^* & \xrightarrow[1]{} & B \otimes_{A_2} M'^* \end{array} \right)$$

determines a 1-dimensional local chain duality (T, e) on A-Mod.

We then examine the resulting L-theory of A-Mod to show that ϵ-symmetric complexes over A may be described in terms of symmetric pairs over A_1:

Theorems 7.4.2, 7.4.4. A locally n-dimensional symmetric Poincaré complex $\mathbf{C} = (C, C', \mu)$ over A determines an $(n+1)$-dimensional symmetric Poincaré pair $(\mu : B \otimes_{A_2} C' \to C, (\delta\phi, \beta^{-1} \otimes \phi))$ over A_1.

We then describe ϵ-symmetric pairs over A in terms of ϵ-symmetric triads over A_1 and ϵ-symmetric cobordisms over A in terms of ϵ-symmetric relative cobordisms over A_1:

Theorems 8.1.4, 8.2.3. A locally $(n+1)$-dimensional symmetric pair $\mathbf{f} = (f, f') : \mathbf{C} = (C, C', \mu) \to \mathbf{D} = (D, D', \nu)$ over A determines an $(n+2)$-dimensional symmetric Poincaré triad (Γ, ϕ) over A_1 where

$$\Gamma = \begin{array}{ccc} B \otimes_{A_2} C' & \xrightarrow{\mu} & C \\ {\scriptstyle 1\otimes f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B \otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

$$\Phi = (\phi', \beta^{-1} \otimes \delta\phi', \delta\phi'', \beta^{-1} \otimes \phi).$$

Theorem 8.2.5. A locally $(n+1)$-dimensional ϵ-symmetric cobordism $((\mathbf{f}\ \mathbf{f}') : \mathbf{C} \oplus \mathbf{C}' \to \mathbf{D}, (\boldsymbol{\Delta\Phi}, \boldsymbol{\Phi} \oplus -\boldsymbol{\Phi}'))$ over A determines an $(n+2)$-dimensional symmetric relative cobordism (Γ, Φ) over A_1

$$\Gamma = \begin{array}{ccc} (B \otimes_{A_2} C') \oplus (B \otimes_{A_2} C''') & \xrightarrow{\begin{pmatrix} \mu & 0 \\ 0 & \mu' \end{pmatrix}} & C \oplus C'' \\ {\scriptstyle (1\otimes f'\ \ 1\otimes f''')}\big\downarrow & & \big\downarrow{\scriptstyle (f\ \ f'')} \\ B \otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

$$\Phi = (\phi'', \beta^{-1} \otimes \delta\phi'', \delta\phi \oplus -\delta\phi', \beta^{-1} \otimes (\phi \oplus -\phi')).$$

The description of cobordisms yields a long exact sequence of ϵ-symmetric L-groups which recovers [Ran81, Proposition 2.2]:

Theorem 8.2.8. For a triangular matrix ring $A = (A_1, A_2, B)$ there is a long exact sequence of ϵ-symmetric L-groups

$$\ldots \to L^{n+1}(A_2, \epsilon) \to L^{n+1}(A_1, \epsilon) \to L^n(A, \epsilon) \to L^n(A_2, \epsilon) \to L^n(A_1, \epsilon) \to \ldots$$

such that an element in $L^n(A, \epsilon)$ is a pair

$$((C', \phi \in Q^n_{A_2}(C', \epsilon)), (\mu : B \otimes_{A_2} C' \to C, (\delta\phi, \beta^{-1} \otimes \phi) \in Q^{n+1}_{A_1}(\mu, \epsilon)))$$

consisting of an n-dimensional ϵ-symmetric Poincaré pair (C', ϕ) over A_2 and an $(n+1)$-dimensional ϵ-symmetric Poincaré pair $(\mu : B \otimes_{A_2} C' \to C, (\delta\phi, \beta^{-1} \otimes \phi))$ over A_1 subject to the equivalence relation

$$((C',\phi),(\mu: B\otimes_{A_2} C' \to C,(\delta\phi,\beta^{-1}\otimes\phi))) \sim ((C''',\phi'),(\mu': B\otimes_{A_2} C''' \to C'',(\delta\phi',\beta^{-1}\otimes\phi')))$$

if and only if there exists an $(n+1)$-dimensional ϵ-symmetric cobordism over A_2 of the form

$$((f' \;\; f''') : C'\oplus C''' \to D', (\delta\phi'', \phi\oplus -\phi'))$$

and an $(n+2)$-dimensional ϵ-symmetric Poincaré triad (Γ,Φ) over A_1 of the form

$$\Gamma = \begin{array}{ccc} B\otimes_{A_2}(C'\oplus C''') & \xrightarrow{(\mu \;\; \mu')} & C\oplus C'' \\ {\scriptstyle 1\otimes(f' \;\; f''')}\big\downarrow & & \big\downarrow{\scriptstyle (f \;\; f'')} \\ B\otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

$$\Phi = (\phi'', \beta^{-1}\otimes\delta\phi'', \delta\phi\oplus -\delta\phi', \beta^{-1}\otimes(\phi\oplus -\phi'))$$

We then examine the effect of algebraic surgery on an ϵ-symmetric complex in A-Mod with surgery data an ϵ-symmetric pair in A-Mod and as an application we consider the special case $A = (R,R,R)$ where R is a ring with involution. This allows us to define a relative algebraic surgery operation on an $(n+1)$-dimensional symmetric Poincaré pair over R with surgery data an $(n+2)$-dimensional symmetric triad over R and is an algebraic model for geometric half-surgeries. This is used to give an algebraic analogue of Borodzik, Némethi and Ranicki's half-handle decomposition theorem:

Theorem 8.3.5. Every $(n+2)$-dimensional commutative ϵ-symmetric Poincaré relative cobordism over a ring with involution is homotopy equivalent to a union of traces of elementary relative surgeries.

Part II is organised as follows.

In chapter 6 we present the basic constructions of [Ran92] needed to determine the L-theory of an additive category with a chain duality. This is a generalisation of the L-theory of a ring with involution where the dual of an object in $\mathbb{A}$ is allowed to be a finite chain complex in $\mathbb{A}$ rather than just an object in $\mathbb{A}$.

In chapter 7 we use the techniques of chapter 6 to construct a local chain duality on the additive category of left modules over a triangular matrix ring $A = (A_1, A_2, B)$. We then use the results of chapter 1 to show that an ϵ-symmetric (Poincaré) structure on an A-module chain complex $\mathbf{C} = (C, C', \mu)$ over A can be described in terms of a relative ϵ-symmetric (Poincaré) structure on the A_1-module chain map $\mu : B\otimes_{A_2} C' \to C$.

In chapter 8 we extend the description of ϵ-symmetric complexes over a triangular matrix ring $A = (A_1, A_2, B)$ to ϵ-symmetric pairs, cobordisms and surgery on ϵ-symmetric complexes over A_1. We then use the results of chapter 2 to show that a relative ϵ-symmetric (Poincaré) structure on an A-module chain map can be described in terms of an ϵ-symmetric (Poincaré) structure on a commutative A_1-module triad in such a way that an ϵ-symmetric cobordism over

A can be viewed as a relative ϵ-symmetric cobordism over A_1. We then describe the effect of a surgery on an ϵ-symmetric complex over A and examine the special case $A = (R, R, R)$ to prove the relative surgery decomposition theorem.

Chapter 6

The L-theory of an additive category with a chain duality

The L-theory of an additive category $\mathbb{A}$ with a chain duality is a generalisation of the L-theory of a ring with involution where the dual of an object in $\mathbb{A}$ is allowed to be a finite chain complex in $\mathbb{A}$ rather than just an object in $\mathbb{A}$. This chapter is based on [Ran92] and presents the basic constructions needed to determine the L-theory of an additive category with a chain duality.

6.1 Chain complexes in an additive category

Definition 6.1.1. An *additive category* is a category $\mathbb{A}$ satisfying the follow properties:

(i) For any pair of objects $A, B \in \mathbb{A}$, the set of morphisms $\mathrm{Hom}_{\mathbb{A}}(A,B)$ has the structure of an abelian group such that for any object $C \in \mathbb{A}$ the composition of morphisms

$$\mathrm{Hom}_{\mathbb{A}}(A,B) \times \mathrm{Hom}_{\mathbb{A}}(B,C) \to \mathrm{Hom}_{\mathbb{A}}(A,C); \quad (f,g) \mapsto gf$$

is bilinear over $\mathbb{Z}$. The zero element in $\mathrm{Hom}_{\mathbb{A}}(A,B)$ is the *zero morphism.*

(ii) There is a zero object $0 \in \mathbb{A}$ such that for each object $A \in \mathbb{A}$ the groups $\mathrm{Hom}_{\mathbb{A}}(A,0)$ and $\mathrm{Hom}_{\mathbb{A}}(0,A)$ are trivial and contain only the zero morphism.

(iii) For any pair of objects $A, B \in \mathbb{A}$ there is an object $C \in \mathbb{A}$ together with morphisms

$$A \underset{\pi_A}{\overset{i_A}{\rightleftarrows}} C \underset{\pi_B}{\overset{i_B}{\leftrightarrows}} B$$

such that $\pi_A i_A = 1_A, \pi_B i_B = 1_B, i_A\pi_A + i_B\pi_B = 1_C$. Such an object C is a *biproduct* of A and B and we write $C = A \oplus B$ and it follows that C is necessarily both a product and coproduct of A and B.

Example 6.1.2. Let A be a ring with involution. The category of f.g. free left A-modules is a full additive subcategory of the additive additive category of f.g. projective left A-modules.

Chain complexes, chain maps and chain homotopies in $\mathbb{A}$ are defined as follows.

Definition 6.1.3. Let $\mathbb{A}$ be an additive category.

(i) A *chain complex* in $\mathbb{A}$ is a collection of objects $C = \{C_r\}_{r\in\mathbb{Z}}$ in $\mathbb{A}$ together with a collection of morphisms

$$d_C = \{d_C \in \mathrm{Hom}_{\mathbb{A}}(C_r, C_{r-1})\}_{r\in\mathbb{Z}}$$

such that

$$d_C d_C = 0 \in \mathrm{Hom}_{\mathbb{A}}(C_r, C_{r-2}) \qquad (r \in \mathbb{Z}).$$

(ii) A *chain map* of chain complexes C, D in $\mathbb{A}$ is a collection of morphisms

$$f = \{f_r \in \mathrm{Hom}_{\mathbb{A}}(C_r, D_r)\}_{r\in\mathbb{Z}}$$

such that the following diagram is commutative

$$\begin{array}{ccc} C_r & \xrightarrow{f_r} & D_r \\ {\scriptstyle d_C}\downarrow & & \downarrow{\scriptstyle d_D} \\ C_{r-1} & \xrightarrow{f_{r-1}} & D_{r-1} \end{array} \qquad (r \in \mathbb{Z})$$

that is

$$d_D f_r = f_{r-1} d_C \in \mathrm{Hom}_{\mathbb{A}}(C_r, D_{r-1}) \qquad (r \in \mathbb{Z}).$$

(iii) A *chain homotopy* $k : f \simeq g : C \to D$ between chain maps $f, g : C \to D$ in $\mathbb{A}$ is a collection of morphisms

$$k = \{k_r \in \mathrm{Hom}_{\mathbb{A}}(C_r, D_{r+1})\}_{r\in\mathbb{Z}}$$

such

$$f_r - g_r = k_{r-1} d_C + d_D k_r \in \mathrm{Hom}_{\mathbb{A}}(C_r, D_r) \quad (r \in \mathbb{Z}).$$

(iv) A chain map $f : C \to D$ in $\mathbb{A}$ is a *chain homotopy equivalence* if there exists a chain map $g : D \to C$ in $\mathbb{A}$ such that there are chain homotopies k and h with

$$k : gf \simeq 1 : C \to C, \quad h : fg \simeq 1 : D \to D.$$

As in the case of a ring with involution, it is useful to have a definition of the dimension of a chain complex which is only defined up to chain homotopy.

Definition 6.1.4. A chain complex C in an additive category $\mathbb{A}$ is

(i) *finite* if $C_r = 0$ for all but finitely many $r \in \mathbb{Z}$.

(ii) *strictly n-dimensional* if $n \geq 0$ and $C_r = 0$ except possibly when $0 \leq r \leq n$.

(iii) *n-dimensional* if it is chain homotopy equivalent to a strictly n-dimensional chain complex in $\mathbb{A}$.

The additive category of finite-dimensional chain complexes in $\mathbb{A}$ and chain maps is denoted by $\mathbb{B}(\mathbb{A})$.

Example 6.1.5. For an additive category $\mathbb{A}$ there is a natural embedding $1 : \mathbb{A} \hookrightarrow \mathbb{B}(\mathbb{A})$ which identifies an object $M \in \mathbb{A}$ with the 0-dimensional chain complex M given by

$$M_r = \begin{cases} M & \text{if } r = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Double complexes in $\mathbb{A}$ and total complexes thereof are defined as follows.

Definition 6.1.6. Let $\mathbb{A}$ be an additive category.

(i) A *double complex* in $\mathbb{A}$ is a collection of objects $C = \{C_{p,q}\}_{p,q\in\mathbb{Z}}$ in $\mathbb{A}$ together with two collections of morphisms

$$d' = \{d'_{p,q} \in \mathrm{Hom}_{\mathbb{A}}(C_{p,q}, C_{p-1,q})\}_{p,q\in\mathbb{Z}}, d'' = \{d''_{p,q} :\in \mathrm{Hom}_{\mathbb{A}}(C_{p,q}, C_{p,q-1})\}_{p,q\in\mathbb{Z}}$$

such that

$$\begin{aligned} d'd' &= 0 \in \mathrm{Hom}_{\mathbb{A}}(C_{p,q}, C_{p-2,q}) \\ d''d'' &= 0 \in \mathrm{Hom}_{\mathbb{A}}(C_{p,q}, C_{p,q-2}) \\ d'd'' &= d''d' \in \mathrm{Hom}_{\mathbb{A}}(C_{p,q}, C_{p-1,q-1}) \quad (p, q \in \mathbb{Z}). \end{aligned}$$

(ii) The *total complex* of a double complex C in $\mathbb{A}$ is the chain complex C in $\mathbb{A}$ defined by

$$d = \bigoplus_{p+q=r} (d'' + (-1)^q d') : C_r = \bigoplus_{p+q=r} C_{p,q} \to C_{r-1} = \bigoplus_{p+q=r-1} C_{p,q} \quad (r \in \mathbb{Z}).$$

Example 6.1.7. Let C, D be chain complexes in an additive category $\mathbb{A}$. There is a double complex of $\mathbb{Z}$-modules $\mathrm{Hom}_{\mathbb{A}}(C, D)$ with chain groups

$$\mathrm{Hom}_{\mathbb{A}}(C, D)_{p,q} = \mathrm{Hom}_{\mathbb{A}}(C_{-p}, D_q) \quad (p, q \in \mathbb{Z})$$

and differentials

$$\begin{aligned} d'(f) &= f d_C : C_{-p+1} \to D_q \\ d''(f) &= d_D f : C_{-p} \to D_{q-1} \quad (f \in \mathrm{Hom}_{\mathbb{A}}(C_{-p}, D_q)). \end{aligned}$$

The total complex $\mathrm{Hom}_{\mathbb{A}}(C, D)$ is given by

$$\mathrm{Hom}_{\mathbb{A}}(C, D)_r = \bigoplus_{p+q=r} \mathrm{Hom}_{\mathbb{A}}(C_{-p}, D_q) \quad (r \in \mathbb{Z})$$

with differential

$$\begin{aligned} d_{\mathrm{Hom}_{\mathbb{A}}(C,D)} : \mathrm{Hom}_{\mathbb{A}}(C, D)_r &\to \mathrm{Hom}_{\mathbb{A}}(C, D)_{r-1} \\ f &\mapsto d_D f + (-1)^q f d_C \quad (f \in \mathrm{Hom}_{\mathbb{A}}(C_{-p}, D_q)). \end{aligned}$$

The total complex of a double complex is used in the following extension construction.

Definition 6.1.8. Let $\mathbb{A}$ be an additive category. The *standard extension* of a contravariant additive functor

$$T : \mathbb{A} \to \mathbb{B}(\mathbb{A}); \quad M \mapsto T(M)$$

is the contravariant additive functor

$$T : \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbb{A}); \quad C \mapsto T(C)$$

defined to send a finite chain complex C in $\mathbb{A}$ to the total complex $T(C)$ of the double complex $T(C)$ in $\mathbb{A}$ with

$$T(C)_{p,q} = T(C_{-p})_q, \quad d' = T(d_C), \quad d'' = d_{T(C_{-p})}$$

so that

$$d_{T(C)} = \oplus_{p+q=r}(d_{T(C_{-p})} + (-)^q T(d_C)) : T(C)_r = \oplus_{p+q=r} T(C_{-p})_q \to \oplus_{p+q=r-1} T(C_{-p})_q \qquad (r \in \mathbb{Z})$$

6.2 A chain duality on an additive category

A chain duality on an additive category $\mathbb{A}$ is a generalisation of the functor from Definition 1.1.3 in Part I which sends a left A-module M to its dual M^*.

Definition 6.2.1. ([Ran92, p.27]). A *chain duality* (T, e) on an additive category $\mathbb{A}$ is a contravariant additive functor

$$T : \mathbb{A} \to \mathbb{B}(\mathbb{A})$$

together with a natural transformation of covariant functors

$$e : T^2 \to 1 : \mathbb{A} \to \mathbb{B}(\mathbb{A}) \to T\mathbb{B}(\mathbb{A})$$

such that for each object $M \in \mathbb{A}$

(i) $e(T(M))T(e(M)) = 1 : T(M) \to T^3(M) \to T(M)$

(ii) $e(M) : T^2(M) \to M$ is a chain equivalence.

A chain duality (T, e) on an additive category $\mathbb{A}$ is *n-dimensional* if for each object $M \in \mathbb{A}$ the chain complex $T(M)$ is strictly n-dimensional.

A chain duality is used to define duels of objects and chain complexes in $\mathbb{A}$ as follows.

Definition 6.2.2. Let $(\mathbb{A}, T, e)$ be an additive category with chain duality and let C, D be finite chain complexes in $\mathbb{A}$ and let $f : C \to D$ be a chain map:

(i) The *dual* of C is the finite chain complex in $\mathbb{D}$ defined by $C^* = T(C)_{-*}$.

(ii) The *dual* of f is the chain map of finite chain complexes in $\mathbb{A}$ defined by

$$f^* = T(f) : D^* = T(D)_{-*} \to C^* = T(C)_{-*}$$

(iii) The *n-dual* of C is the finite chain complex C^{n-*} in $\mathbb{A}$ defined by

$$d_{C^{n-*}} = (-1)^r d^*_{T(C)} : C^{n-r} = T(C)_{r-n} \to C^{n-r+1} = T(C)_{r-1-n} \quad (r \in \mathbb{Z}).$$

(iv) The *n-dual* of f is the the chain map of finite chain complexes in $\mathbb{A}$ defined by

$$f^* = T(f) : D^{n-*} \to C^{n-*}.$$

Example 6.2.3.

(i) Let A be a ring with involution and denote by A-Mod the additive category of f.g. projective left A-modules and by A-Chain the additive category of finite chain complexes in A-Mod. Recall that the the *dual* of a A-module M is the left A-module $M^* = \mathrm{Hom}_A(M, A)$ with action

$$A \times M^* \to M^*; (a, f) \mapsto (x \mapsto f(x) \cdot \overline{a})$$

such that if M if f.g. projective then there is a natural isomorphism

$$e(M)^{-1} : M \to M^{**}; \quad x \mapsto (f \mapsto \overline{f(x)})$$

This implies that the contravariant additive functor

$$T : A\text{-Mod} \to A - \text{Chain}; \quad M \mapsto T(M) = M^*$$

and the natural transformation

$$e(M) : T^2(M) \xrightarrow{\cong} M$$

define a 0-dimensional chain duality (T, e) on A-Mod.

(ii) In chapter 7 we will construct a 1-dimensional chain duality on the category of left modules over a triangular matrix ring $A = (A_1, A_2, B)$. This will resolve the difficulty of the non-existence of a 0-dimensional chain duality as mentioned in the introduction.

A chain duality (T, e) on an additive category $\mathbb{A}$ determines a tensor product of finite chain complexes complexes C, D in such a way that a slant equality holds $C \otimes_{\mathbb{A}} D = \mathrm{Hom}_{\mathbb{A}}(T(C), D)$ and there is an ϵ-duality involution $T_{C,\epsilon} : C \otimes_{\mathbb{A}} C \to C \otimes_{\mathbb{A}} C$.

Definition 6.2.4. Let $(\mathbb{A}, T, e)$ be an additive category with chain duality.

(i) The *tensor product* of two objects $M, N \in \mathbb{A}$ is the finite $\mathbb{Z}$-module chain complex

$$M \otimes_{\mathbb{A}} N = \mathrm{Hom}_{\mathbb{A}}(T(M), N)$$

defined to be the total complex of the double complex $\mathrm{Hom}_{\mathbb{A}}(T(M), N)$ such that

$$(M \otimes_{\mathbb{A}} N)_r = \mathrm{Hom}_{\mathbb{A}}(T(M)_{-r}, N) \quad (r \in \mathbb{Z}).$$

(ii) The *tensor product* of two finite chain complexes C, D in $\mathbb{A}$ is the finite $\mathbb{Z}$-module chain complex

$$C \otimes_{\mathbb{A}} D = \mathrm{Hom}_{\mathbb{A}}(T(C), D)$$

defined to be the total complex of the double complex $\mathrm{Hom}_{\mathbb{A}}(T(C), D)$ such that

$$(C \otimes_{\mathbb{A}} D)_n = \oplus_{p+q+r=n} (C_p \otimes_{\mathbb{A}} D_q)_r \quad (r \in \mathbb{Z}).$$

Proposition 6.2.5. ([Ran92, p.28-29]). Let $(\mathbb{A}, T, e)$ be an additive category with chain duality and let $\epsilon = \pm 1$.

(i) For objects $M, N \in \mathbb{A}$ there is an ϵ-duality isomorphism

$$T_\epsilon = T_{M,N,\epsilon} : M \otimes_{\mathbb{A}} N \xrightarrow{\cong} N \otimes_{\mathbb{A}} M$$

of $\mathbb{Z}$-module chain complexes with

$$T_{M,N,\epsilon} : (M \otimes_{\mathbb{A}} N)_r = \mathrm{Hom}_{\mathbb{A}}(T(M)_{-r}, N) \to (N \otimes_{\mathbb{A}} M)_r = \mathrm{Hom}_{\mathbb{A}}(T(N)_{-r}, M)$$

given by

$$(f : T(M)_{-r} \to N) \mapsto (T_{M,N}(f) : T(N)_{-r} \to M)$$

with

$$T_{M,N,\epsilon}(f) = \epsilon e(M) T(f) : T(N)_{-r} \xrightarrow{T(f)} (T(M)_{-r})_{-r} \hookrightarrow T^2(M)_0 \xrightarrow{\epsilon e(M)} M_0 = M$$

and inverse

$$T^{-1}_{M,N,\epsilon} = T_{N,M,\epsilon} : N \otimes_{\mathbb{A}} M \to M \otimes_{\mathbb{A}} N$$

(ii) For finite chain complexes C, D in $\mathbb{A}$ there is an ϵ-duality isomorphism

$$T_\epsilon = T_{C,D,\epsilon} : C \otimes_{\mathbb{A}} D \xrightarrow{\cong} D \otimes_{\mathbb{A}} C$$

of $\mathbb{Z}$-module chain complexes with

$$\begin{aligned} T_{C,D,\epsilon} = \bigoplus (-)^{pq} T_{C_p, D_q, \epsilon} : (C \otimes_A D)_n = \bigoplus_{p+q+r=n} (C_p \otimes_{\mathbb{A}} D_q)_r \to \\ (D \otimes_A C)_n = \bigoplus_{p+q+r=n} (D_q \otimes_{\mathbb{A}} C_p)_r \quad (n \in \mathbb{Z}) \end{aligned}$$

and inverse

$$T^{-1}_{C,D,\epsilon} = T_{D,C,\epsilon} : D \otimes_{\mathbb{A}} C \to C \otimes_{\mathbb{A}} D.$$

(iii) For finite chain complex C in $\mathbb{A}$ the ϵ-duality isomorphism

$$T_\epsilon = T_{C,\epsilon} = T_{C,C,\epsilon} : C \otimes_{\mathbb{A}} C \to C \otimes_{\mathbb{A}} C$$

is an involution which defines a $\mathbb{Z}_2$-action on $C \otimes_{\mathbb{A}} C$.

6.3 Symmetric complexes in an additive category

The generalisation of the $W^{\%}$ functor from a ring with involution to an additive category with chain duality is as follows.

Definition 6.3.1. ([Ran92, p.29-30]). Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$ resolution of $\mathbb{Z}$

$$W : \ldots \to W_3 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} W_2 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} W_1 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} W_0 = \mathbb{Z}[\mathbb{Z}_2].$$

Let $(\mathbb{A}, T, e)$ be an additive category with chain duality and let C, D be finite-dimensional chain complexes in $\mathbb{A}$ and let $\epsilon = \pm 1$.

(i) The ϵ-duality isomorphism

$$T_\epsilon : C \otimes_{\mathbb{A}} C \to C \otimes_{\mathbb{A}} C$$

defines a $\mathbb{Z}_2$-action on $C \otimes_{\mathbb{A}} C$ so that $C \otimes_{\mathbb{A}} C$ is a finite $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex.

(ii) The $\mathbb{Z}$-module chain complex

$$W^{\%}C = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C)$$

is such that under the definition

$$C \otimes_{\mathbb{A}} C = \mathrm{Hom}_{\mathbb{A}}(T(C), C)$$

a chain $\phi \in (W^{\%}C)_n$ can be identified with a collection of morphisms

$$\phi = \{\phi_s \in \mathrm{Hom}_{\mathbb{A}}(C^{n-r+s}, C_r) | r \in \mathbb{Z}, s \geqslant 0\}$$

such that the boundary $d_{W^{\%}C}\phi \in (W^{\%}C)_{n-1}$ can be identified with a collection of morphisms

$$d_{W^{\%}C}\phi = \{(d\phi)_s \mathrm{Hom}_{\mathbb{A}}(C^{n-1-r+s}, C_r) | r \in \mathbb{Z}, s \geqslant 0\}$$

satisfying

$$(d\phi)_s = d_C\phi_s + (-)^r \phi_s d_C^* + (-)^{n+s-1}(\phi_{s-1} + (-)^s T_\epsilon \phi_{s-1}) \in \mathrm{Hom}_{\mathbb{A}}(C^{n-1-r+s}, C_r)$$
$$(r \in \mathbb{Z}, s \geqslant 0, \phi_{-1} = 0).$$

(iii) An chain map $f : C \to D$ induces a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$f \otimes_{\mathbb{A}} f : C \otimes_{\mathbb{A}} C \to D \otimes_{\mathbb{A}} D$$

and hence induces a $\mathbb{Z}$-module chain map

$$f^{\%} : W^{\%}C \to W^{\%}D; \quad \phi = \{\phi_s | s \geqslant 0\} \mapsto f^{\%}\phi = \{f\phi_s f^* | s \geqslant 0\}$$

in such a way that a chain homotopy $k : f \simeq g : C \to D$ induces a chain homotopy

$$k^{\%} : f^{\%} \simeq g^{\%} : W^{\%}C \to W^{\%}D.$$

The ϵ-symmetric Q-groups of a finite chain complex C over $\mathbb{A}$ are then defined in the same way as for a ring with involution.

Definition 6.3.2. Let $(\mathbb{A}, T, e)$ be an additive category with chain duality and let $\epsilon = \pm 1$.

(i) The *ϵ-symmetric Q-groups* of a finite chain complex C in $\mathbb{A}$ are the $\mathbb{Z}$-module homology groups

$$Q^n(C, \epsilon) = H_n(W^{\%}C) \quad (n \in \mathbb{Z}).$$

(ii) The morphism of ϵ-symmetric Q-groups induced by a chain map $f : C \to D$ of finite chain complexes in $\mathbb{A}$ is the morphism of $\mathbb{Z}$-module homology groups

$$f^{\%} : Q^n(C,\epsilon) = H_n(W^{\%}C) \to Q^n(D,\epsilon) = H_n(W^{\%}D).$$

Similarly, ϵ-symmetric Poincaré complexes and homotopy equivalences thereof are defined as for a ring with involution.

Definition 6.3.3. Let $(\mathbb{A},T,e)$ be an additive category with chain duality.

(i) An *n-dimensional ϵ-symmetric complex* $(C,\phi \in Q^n(C,\epsilon)))$ in $\mathbb{A}$ is an n-dimensional chain complex C in $\mathbb{A}$ with a cycle $\phi \in (W^{\%}C)_n$

(ii) An n-dimensional ϵ-symmetric complex (C,ϕ) in $\mathbb{A}$ is *Poincaré* if the chain map $\phi_0 : C^{n-*} \to C$ is a chain equivalence.

(iii) A *morphism* of n-dimensional ϵ-symmetric complexes $f : (C,\phi) \to (C',\phi')$ in $\mathbb{A}$ is a chain map $f : C \to C'$ in $\mathbb{A}$ such that $f^{\%}(\phi) = \phi'$. A morphism of n-dimensional ϵ-symmetric complexes $f : (C,\phi) \to (C',\phi')$ is a *homotopy equivalence* if the chain map $f : C \to C'$ is a chain homotopy equivalence.

The ϵ-symmetric Q-groups again have the same failure to be additive.

Proposition 6.3.4. Let $(\mathbb{A},T,e)$ be an additive category with chain duality and let C, C' be finite-dimensional chain complexes in $\mathbb{A}$. The ϵ-symmetric Q-groups of the finite-dimensional chain complex $C \oplus C'$ are given by

$$Q^n(C \oplus C',\epsilon) = Q^n(C,\epsilon) \oplus Q^n(C',\epsilon) \oplus H_n(C \otimes_{\mathbb{A}} C',\epsilon)$$

so that there is an inclusion

$$Q^n(C,\epsilon) \oplus Q^n(C',\epsilon) \hookrightarrow Q^n(C \oplus C',\epsilon).$$

One may also form direct sums and negatives of ϵ-symmetric Poincaré complexes.

Definition 6.3.5. Let $(\mathbb{A},T,e)$ be an additive category with chain duality.

(i) The *direct sum* of n-dimensional ϵ-symmetric (Poincaré) complexes $(C,\phi \in Q^n(C,\epsilon)), (C',\phi' \in Q^n(C',\epsilon))$ in $\mathbb{A}$ is the n-dimensional ϵ-symmetric (Poincaré) complex in $\mathbb{A}$

$$(C,\phi \in Q^n(C,\epsilon)) \oplus (C',\phi \in Q^n(C',\epsilon)) = (C \oplus C',\phi \oplus \phi' \in Q^n(C \oplus C',\epsilon))$$

determined by the inclusion

$$Q^n(C,\epsilon) \oplus Q^n(C',\epsilon) \hookrightarrow Q^n(C \oplus C',\epsilon).$$

(ii) The *zero* n-dimensional ϵ-symmetric Poincaré complex in $\mathbb{A}$ is $(0, 0 \in Q^n(0,\epsilon))$.

(iii) The *negative* of an n-dimensional -symmetric (Poincaré) complex $(C,\phi \in Q^n(C,\epsilon))$ in $\mathbb{A}$ is the n-dimensional ϵ-symmetric (Poincaré) complex in $\mathbb{A}$

$$-(C,\phi \in Q^n(C,\epsilon)) = (C,-\phi \in Q^n(C,\epsilon)).$$

If we wish to emphasise the additive category $\mathbb{A}$ then we write $W^{\%}C = W^{\%}_{\mathbb{A}}C$ and $Q^n(C,\epsilon) = Q^n_{\mathbb{A}}(C,\epsilon)$. In chapter 7 when $A = (A_1, A_2, B)$ is a triangular matrix ring we will work with the additive category $\mathbb{A} = A$-Mod. We will relate the Q-groups of a chain complex $\mathbf{C} = (C_1, C_2, \mu)$ over A to the Q-groups of C_1 over A_1 and C_2 over A_2 and it will be necessary decorate the Q-groups with the subscripts A, A_1, A_2 to keep track of which category we are working over.

6.4 Symmetric pairs and cobordisms in an additive category

An ϵ-symmetric pair in an additive category with chain duality is a generalisation of an ϵ-symmetric complex over a ring with involution where a relative symmetric structure of a chain map $f : C \to D$ is defined in terms of a cycle of the mapping cone of $f^{\%} : W^{\%}C \to W^{\%}D$.

Definition 6.4.1. The *algebraic mapping cone* of a chain map $f : C \to D$ in an additive category $\mathbb{A}$ is the chain complex $\mathscr{C}(f)$ in $\mathbb{A}$ defined by

$$d_{\mathscr{C}(f)} = \begin{pmatrix} d_D & (-)^{n-1}f \\ 0 & d_C \end{pmatrix} : \mathscr{C}(f)_n = D_n \oplus C_{n-1} \to \mathscr{C}(f)_{n-1} = D_{n-1} \oplus C_{n-2} \quad (n \in \mathbb{Z})$$

and homology groups

$$H_n(f) = H_n(\mathscr{C}(f)) \quad (n \in \mathbb{Z}).$$

It will be useful in chapter 7 to have a definition of an algebraic mapping cone with different signs.

Definition 6.4.2. The *algebraic mapping cone* of a chain map $f : C \to D$ in an additive category $\mathbb{A}$ is the chain complex $\widetilde{\mathscr{C}}(f)$ in $\mathbb{A}$ defined by

$$d_{\widetilde{\mathscr{C}}(f)} = \begin{pmatrix} d_D & f \\ 0 & -d_C \end{pmatrix} : \widetilde{\mathscr{C}}(f)_n = D_n \oplus C_{n-1} \to \widetilde{\mathscr{C}}(f)_{n-1} = D_{n-1} \oplus C_{n-2} \quad (n \in \mathbb{Z})$$

and homology groups

$$\widetilde{H}_n(f) = H_n(\widetilde{\mathscr{C}}(f)) \quad (n \in \mathbb{Z}).$$

These two definitions of mapping cones are in fact the same up to isomorphism.

Lemma 6.4.3. If $f : C \to D$ is a chain map in an additive category $\mathbb{A}$ then the algebraic mapping cones $\mathscr{C}(f)$ and $\widetilde{\mathscr{C}}(f)$ are isomorphic.

Proof. The commutative diagram

$$\begin{array}{ccc} D_r \oplus C_{r-1} & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & (-)^{r-1} \end{pmatrix}} & D_r \oplus C_{r-1} \\ {\scriptstyle \begin{pmatrix} d_D & (-)^{r-1}f \\ 0 & d_C \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle \begin{pmatrix} d_D & f \\ 0 & -d_C \end{pmatrix}} \\ D_{r-1} \oplus C_{r-2} & \xrightarrow[\begin{pmatrix} 1 & 0 \\ 0 & (-)^{r} \end{pmatrix}]{} & D_{r-1} \oplus C_{r-2} \end{array}$$

implies that there is a chain map $\theta : \mathscr{C}(f) \to \widetilde{\mathscr{C}}(f)$ defined by

$$\theta = \begin{pmatrix} 1 & 0 \\ 0 & (-)^{r-1} \end{pmatrix} : \mathscr{C}(f)_r \to \widetilde{\mathscr{C}}(f)_r \quad (r \in \mathbb{Z})$$

and it is clear that θ is an isomorphism. □

Relative symmetric structures are expressed in terms cycles of the algebraic mapping cone $\mathscr{C}(f^{\%} : W^{\%}C \to W^{\%}D)$.

Proposition 6.4.4. ([Ran92, p.31]). Let $(\mathbb{A}, T, e)$ be an additive category with chain duality and let $f : C \to D$ be a chain map of finite-dimensional chain complexes in $\mathbb{A}$. The $\mathbb{Z}$-module chain map $f^{\%} : W^{\%}C \to W^{\%}D$ has an algebraic mapping cone $\mathscr{C}(f^{\%})$ such that a cycle $(\delta\phi, \phi) \in \mathscr{C}(f^{\%})_{n+1}$ consists precisely of a cycle

$$\phi = \{\phi_s \in \mathrm{Hom}_{\mathbb{A}}(C^{n-r+s}, C_r) | s \geqslant 0, r \in \mathbb{Z}\} \in (W^{\%}C)_n$$

together with a chain

$$\delta\phi = \{\delta\phi_s \in \mathrm{Hom}_{\mathbb{A}}(D^{n+1-r+s}, D_r) | s \geqslant 0, r \in \mathbb{Z}\} \in (W^{\%}D)_{n+1}$$

such that

$$d_D \delta\phi_s + (-)^r \delta\phi_s d_D^* + (-)^{n+s}(\delta\phi_{s-1} + (-)^s T_\epsilon(\delta\phi_{s-1})) + (-)^n f \phi_s f^* = 0 \in \mathrm{Hom}_{\mathbb{A}}(D^{n-r+s}, D_r)$$
$$(r \in \mathbb{Z}, s \geqslant 0, \delta\phi_{-1} = 0, \phi_{-1} = 0).$$

Definition 6.4.5. Let $(\mathbb{A}, T, e)$ be an additive category with chain duality and let $\epsilon = \pm 1$. The *relative ϵ-symmetric Q-groups* of a chain map $f : C \to D$ of finite-dimensional chain complexes in $\mathbb{A}$ are the relative $\mathbb{Z}$-module homology groups

$$Q^n(f, \epsilon) = H_n(\mathscr{C}(f^{\%} : W^{\%}C \to W^{\%}D)) \quad (n \in \mathbb{Z}).$$

If we wish to emphasis the additive category $\mathbb{A}$ over which we are working we will decorate the relative Q-groups with the symbol $\mathbb{A}$ as in the absolute case.

An ϵ-symmetric Poincaré pair is then defined as follows.

Definition 6.4.6. Let $(\mathbb{A}, T, e)$ be an additive category with chain duality and let $\epsilon = \pm 1$.

(i) An $(n+1)$*-dimensional ϵ-symmetric pair* $(f : C \to D, (\delta\phi, \phi) \in Q^{n+1}(f, \epsilon))$ in $\mathbb{A}$ consists of chain map $f : C \to D$ from an n-dimensional chain complex C in $\mathbb{A}$ to an $(n+1)$-dimensional chain complex D in $\mathbb{A}$, together with a cycle $(\delta\phi, \phi) \in \mathscr{C}(f^{\%} : W^{\%}C \to W^{\%}D)_{n+1}$.

(ii) An $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi, \phi))$ over $\mathbb{A}$ is *Poincaré* if the chain map

$$(\delta\phi_0 \quad f\phi_0) : \mathscr{C}(f)^{n+1-*} \to D$$

is a chain homotopy equivalence.

A chain map $f : C \to D$ determines a long exact sequence of ϵ-symmetric Q-groups as follows.

Proposition 6.4.7. Let $(\mathbb{A}, T, e)$ be an additive category with chain duality. The relative ϵ-symmetric Q-groups of a chain map $f : C \to D$ of finite-dimensional chain complexes in $\mathbb{A}$ fit into a long exact sequence of ϵ-symmetric Q-groups

$$\ldots \to Q^{n+1}(f,\epsilon) \xrightarrow{f^{\%}} Q^{n+1}(D,\epsilon) \to Q^n(f,\epsilon) \to Q^n(C,\epsilon) \xrightarrow{f^{\%}} Q^n(D,\epsilon) \to \ldots$$

with

$$Q^{n+1}(f,\epsilon) \to Q^n(C,\epsilon); (\delta\phi, \phi) \mapsto \phi$$
$$Q^n(D,\epsilon) \to Q^n(f,\epsilon); \delta\phi \mapsto (\delta\phi, 0).$$

Proof. As in the case for a ring with involution using the long exact sequence associated to the chain map $f^{\%} : W^{\%}C \to W^{\%}D$. See the proof of Proposition 1.2.4 in Section 2.2. □

An ϵ-symmetric cobordism in an additive category with chain duality is a generalisation of an ϵ-symmetric Poincaré complex over a ring with involution.

Definition 6.4.8. Let $(\mathbb{A}, T, e)$ be an additive category with chain duality. An *ϵ-symmetric cobordism* between n-dimensional ϵ-symmetric Poincaré complexes $(C,\phi), (C',\phi')$ in $\mathbb{A}$ is an $(n+1)$-dimensional ϵ-symmetric Poincaré pair in $\mathbb{A}$ of the form

$$((f\ \ f') : C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi') \in Q^{n+1}((f\ \ f'),\epsilon)).$$

The *n-dimensional ϵ-symmetric L-group* $L^n(\mathbb{A},\epsilon)$ of $\mathbb{A}$ is the abelian group of cobordism classes of n-dimensional ϵ-symmetric Poincaré complexes in $\mathbb{A}$ with addition

$$(C, \phi \in Q^n(C,\epsilon)) + (C', \phi' \in Q^n(C',\epsilon)) = (C \oplus C', \phi \oplus \phi' \in Q^n(C \oplus C',\epsilon)) \in L^n(\mathbb{A},\epsilon)$$

and zero element

$$(0, 0 \in Q^n(0,\epsilon)) \in L^n(\mathbb{A},\epsilon)$$

and additive inverses

$$-(C, \phi \in Q^n(C,\epsilon)) = (C, -\phi \in Q^n(C,\epsilon)) \in L^n(\mathbb{A},\epsilon).$$

Example 6.4.9.

(i) Let A be a ring with involution and equip A-Mod with the 0-dimensional chain duality (T,e) from Example 6.2.3. Then the L-theory of the ring A is the same as the L-theory of the additive category $(A - \mathrm{Mod}, T, e)$.

(ii) In chapter 8 we will show in Theorem 8.2.8 that for a triangular matrix ring $A = (A_1, A_2, B)$ there is a long exact sequence of L-groups

$$\ldots \to L^{n+1}(A_2,\epsilon) \xrightarrow{(\mu,\beta)^{\%}} L^{n+1}(A_1,\epsilon) \to L^n(A,\epsilon) \to L^n(A_2,\epsilon) \xrightarrow{(\mu,\beta)^{\%}} L^n(A_1,\epsilon) \to \ldots$$

6.5 Algebraic surgery in an additive category

Algebraic surgery in an additive category with a chain duality is a generalisation of algebraic surgery over a ring with involution with an analogous trace construction constructing a cobordism between the input and the output of the surgery.

Definition 6.5.1. ([Ran92, Definition 1.12]). Let $(\mathbb{A}, T, e)$ be an additive category with chain duality.

(i) The effect of *algebraic surgery* on an n-dimensional ϵ-symmetric Poincaré complex (C, ϕ) in $\mathbb{A}$ with data an $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi, \phi))$ in $\mathbb{A}$ is the n-dimensional ϵ-symmetric Poincaré complex (C', ϕ') in $\mathbb{A}$ with the chain complex C' defined by

$$d_{C'} = \begin{pmatrix} d_C & 0 & (-)^{n+1}\phi_0 f^* \\ (-)^r f & d_D & (-)^r \delta\phi_0 \\ 0 & 0 & (-)^r d_D^* \end{pmatrix}$$
$$: C'_r = C_r \oplus D_{r+1} \oplus D^{n+1-r} \to C'_{r-1} = C_{r-1} \oplus D_r \oplus D^{n+2-r} \qquad (r \in \mathbb{Z})$$

and the ϵ-symmetric structure ϕ' defined by

$$\phi'_0 = \begin{pmatrix} \phi_0 & 0 & 0 \\ (-)^{n-r} f T_\epsilon \phi_1 & (-)^{n-r} T_\epsilon \delta\phi_1 & (-)^{r(n-r)} e \\ 0 & 1 & 0 \end{pmatrix}$$
$$: C'^{n-r} = C^{n-r} \oplus D^{n+1-r} \oplus (T^2 D)_{r+1} \to \quad C'_r = C_r \oplus D_{r+1} \oplus D^{n+1-r} \qquad (r \in \mathbb{Z})$$
$$\phi'_s = \begin{pmatrix} \phi_s & 0 & 0 \\ (-)^{n-r} f T_\epsilon \phi_{s+1} & (-)^{n-r+s} T_\epsilon \delta\phi_{s+1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$: C'^{n-r+s} = C_r \oplus D_{r+1} \oplus (T^2 D)_{r-s+1} \to C'_r = C_r \oplus D_{r+1} \oplus D^{n+1-r} \qquad (r \in \mathbb{Z}, s \geqslant 1).$$

(ii) The *trace* of such an algebraic surgery is the $(n+1)$-dimensional ϵ-symmetric pair in $\mathbb{A}$

$$((g\ \ g') : C \oplus C' \to D', (0, \phi' \oplus -\phi') \in Q^{n+1}(g\ \ g'))$$

defined by

$$d_{D'} = \begin{pmatrix} d_C & (-)^{n+1}\phi_0 f^* \\ 0 & (-)^r d_D^* \end{pmatrix} : D'_r = C_r \oplus D^{n+1-r} \to D'_{r-1} = C_{r-1} \oplus D^{n+2-r} \qquad (r \in \mathbb{Z})$$
$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C_r \to D'_r = C_r \oplus D^{n+1-r} \qquad (r \in \mathbb{Z})$$
$$g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \to D'_r = C_r \oplus D^{n-r+1} \qquad (r \in \mathbb{Z})$$

Proof. The proof is the same as in the case of a ring with involution, see [Ran80a, Proposition]. Note the presence of summand involving T^2 in the C'^{n-r} and the natural transformation e in ϕ'_0. □

Chapter 7

The L-theory of a triangular matrix ring: symmetric complexes

As mentioned in the introduction to Part II a triangular matrix ring $A = (A_1, A_2, B)$ does not admit a natural involution. Using the techniques of chapter 6 we construct a local chain duality on the additive category of left modules over A. We take the candidate contravariant additive functor $\mathbf{T} : A\text{-Mod} \to A - \text{Chain}$ of [Ran06], find its standard extension $\mathbf{T} : A\text{-Chain} \to A\text{-Chain}$ and show there exists a natural transformation $\mathbf{e} : \mathbf{T}^2 \to \mathbf{1} : A\text{-Mod} \to A\text{-Chain}$ such that the triple $(\mathbb{A}, \mathbf{T}, \mathbf{e})$ satisfies a weakened version Definition 6.2.1. We then use the results of chapter 1 to show that an ϵ-symmetric (Poincaré) structure on an A-module chain complex $\mathbf{C} = (C, C', \mu)$ over A can be described in terms of a relative ϵ-symmetric (Poincaré) structure on the A_1-module chain map $\mu : B \otimes_{A_2} C' \to C$ in such a way that there is a long exact sequence of ϵ-symmetric Q-groups

$$\ldots \to Q^{n+1}_{A_1}(C, \epsilon) \to Q^n_A(\mathbf{C}, \epsilon) \to Q^n_{A_2}(C', \epsilon) \to Q^n_{A_1}(C, \epsilon) \to \ldots.$$

7.1 Chain complexes and chain maps over a triangular matrix ring

Definition 7.1.1. The *triangular matrix ring* A determined by rings A_1, A_2 with identities $1_{A_1}, 1_{A_2}$ and an (A_1, A_2)-bimodule B is the matrix ring

$$A = \left\{ \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} : a_1 \in A_1, a_2 \in A_2, b \in B \right\}$$

with addition

$$+ : \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \times \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \to \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$$
$$\left(\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} a_1' & b' \\ 0 & a_2' \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 + a_1' & b + b' \\ 0 & a_2 + a_2' \end{pmatrix}$$

and multiplication

$$* : \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \times \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \to \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$$
$$\left(\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} a_1' & b' \\ 0 & a_2' \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 a_1' & a_1 b' + b a_2 \\ 0 & a_2 a_2' \end{pmatrix}$$

and identity

$$1_A = \begin{pmatrix} 1_{A_1} & 0 \\ 0 & 1_{A_2} \end{pmatrix}$$

For brevity we write $A = (A_1, A_2, B)$.

In order to describe symmetric complexes over a triangular matrix ring A it is first necessary to describe modules and chain complexes over A. Left A-modules and morphisms thereof have the following descriptions in terms of the data A_1, A_2, B.

Proposition 7.1.2. ([Gre82]).

(i) A left A-module $\mathbf{M}$ can be identified with a triple

$$(M_1, M_2, \mu : B \otimes_{A_2} M_2 \to M_1)$$

where M_1 is an A_1 module, M_2 is an A_2 module and μ is an A_1-module homomorphism.

(ii) A left A-module morphism

$$\mathbf{f} : \mathbf{M} = (M_1, M_2, \mu : B \otimes_{A_2} M_2 \to M_1) \to \mathbf{N} = (N_1, N_2, \nu : B \otimes_{A_2} N_2 \to N_1)$$

can be identified with a pair of morphisms

$$(f_1 \in \mathrm{Hom}_{A_1}(M_1, N_1), f_2 \in \mathrm{Hom}_{A_2}(M_2, N_2))$$

such that the following diagram is commutative

$$\begin{array}{ccc} B \otimes_{A_2} M_2 & \xrightarrow{\mu} & M_1 \\ {\scriptstyle 1 \otimes f_2} \downarrow & & \downarrow {\scriptstyle f_1} \\ B \otimes_{A_2} N_2 & \xrightarrow[\nu]{} & N_1 \end{array} .$$

(iii) The addition of morphisms

$$\mathbf{f} = (f \in \mathrm{Hom}_{A_1}(M, N), f' \in \mathrm{Hom}_{A_2}(M', N')) \in \mathrm{Hom}_A(\mathbf{M}, \mathbf{N})$$
$$\mathbf{g} = (g \in \mathrm{Hom}_{A_1}(M, N), g' \in \mathrm{Hom}_{A_2}(M', N')) \in \mathrm{Hom}_A(\mathbf{M}, \mathbf{N})$$

is given by

$$\mathbf{f}+\mathbf{g} = (f+g \in \mathrm{Hom}_{A_1}(M,N), f'+g' \in \mathrm{Hom}_{A_2}(M',N')) \in \mathrm{Hom}_A(\mathbf{M},\mathbf{N})$$

and the composition of morphisms

$$\mathbf{f} = (f \in \mathrm{Hom}_{A_1}(M,N), f' \in \mathrm{Hom}_{A_2}(M',N')) \in \mathrm{Hom}_A(\mathbf{M},\mathbf{N})$$
$$\mathbf{g} = (g \in \mathrm{Hom}_{A_1}(N,P), g' \in \mathrm{Hom}_{A_2}(N',P')) \in \mathrm{Hom}_A(\mathbf{N},\mathbf{P})$$

is given by

$$\mathbf{gf} = (gf \in \mathrm{Hom}_{A_1}(M,P), g'f' \in \mathrm{Hom}_{A_2}(M',N')) \in \mathrm{Hom}_A(\mathbf{M},\mathbf{P}).$$

(iv) A left A-module $\mathbf{M} = (M_1, M_2, \mu : B \otimes_{A_2} M_2 \to M_1)$ is a (f.g.) projective if and only if M_1 if (f.g.) projective, coker(μ) is (f.g.) projective and μ is split injective.

The above description of A-modules extends to one of A-module chain complexes.

Proposition 7.1.3. Let $A = (A_1, A_2, B)$ be a triangular matrix ring.

(i) An n-dimensional A-module chain complex $\mathbf{C}$ can be identified with a triple

$$\mathbf{C} = (C, C', \mu : B \otimes_{A_2} C' \to C)$$

where C is an n-dimensional A_1-module chain complex, C' is an n-dimensional A_2-module chain complex and $\mu : B \otimes_{A_2} C' \to C$ is a chain map of A_1-module chain complexes with

$$\mathbf{C}_r = (C_r, C'_r, \mu : B \otimes_{A_2} C'_r \to C_r) \quad (0 \leqslant r \leqslant n).$$

(ii) Let (C, C', μ) and $\mathbf{D} = (D, D', \nu)$ be A-module chain complexes. A chain map $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ can be identified with a pair of chain maps

$$(f : C \to D, f' : C' \to D')$$

such that the following diagram is commutative

$$\begin{array}{ccc} B \otimes_{A_2} C'_r & \xrightarrow{\mu_r} & C_r \\ {\scriptstyle 1\otimes f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B \otimes_{A_2} D'_r & \xrightarrow[\nu_r]{} & D_r \end{array} \qquad (r \in \mathbb{Z}).$$

(iii) The addition and composition of morphisms is componentwise.

(iv) An A-module chain complex $\mathbf{C} = (C, C', \mu : B \otimes_{A_2} C' \to C)$ is (f.g.) projective if and only if C' is (f.g.) projective, coker(μ) is (f.g.) projective and μ is split injective.

Proof.

(i) Suppose that $\mathbf{C}$ has chain groups $\mathbf{C}_r$ and differential $d_{\mathbf{C}} : \mathbf{C}_r \to \mathbf{C}_{r-1}$. Write

$$\mathbf{C}_r = (C_r, C'_r, \mu_r : B \otimes_{A_2} C'_r \to C_r)$$

so that the differential $d_{\mathbf{C}} : \mathbf{C}_r \to \mathbf{C}_{r-1}$ may be identified with a pair

$$d_{\mathbf{C}} = (d_C \in \mathrm{Hom}_{A_1}(C_r, C_{r-1}), d_{C'} \in \mathrm{Hom}_{A_2}(C'_r, C'_{r-1})).$$

The composition $d^2_{\mathbf{C}} \in \mathrm{Hom}_A(\mathbf{C}_r, \mathbf{C}_{r-2})$ is then identified with the pair

$$d^2_{\mathbf{C}} = (d^2_C \in \mathrm{Hom}_{A_1}(C_r, C_{r-2}), d^2_{C'} \in \mathrm{Hom}_{A_2}(C'_r, C'_{r-2}))$$

and hence

$$d^2_{\mathbf{C}} = 0 \in \mathrm{Hom}_A(\mathbf{C}_r, \mathbf{C}_{r-2}) \iff d^2_C = 0 \in \mathrm{Hom}_A(C_r, C_{r-2}) \text{ and } d^2_{C'} = 0 \in \mathrm{Hom}_A(C'_r, C'_{r-2}).$$

It follows that $(\mathbf{C}, d_{\mathbf{C}})$ is an n-dimensional A-module chain complex if and only if (C, d_C) is an n-dimensional A_1-module chain complex and $(C', d_{C'})$ is an n-dimensional A_2-module chain complex and $\mu : B \otimes_{A_2} C' \to C$ is a chain map of A_1-module chain complexes.

(ii) As in (i) we may identify the morphism $\mathbf{f}_r \in \mathrm{Hom}_A(\mathbf{C}_r, \mathbf{C}_r)$ with a pair of morphisms

$$\mathbf{f}_r = (f_r \in \mathrm{Hom}_{A_1}(C_r, D_r), f'_r \in \mathrm{Hom}_{A_2}(C'_r, D'_r))$$

and it remains to show that the diagram

$$\begin{array}{ccc} \mathbf{C}_r & \xrightarrow{\mathbf{f}_r} & \mathbf{D}_r \\ {\scriptstyle d_{\mathbf{C}}}\downarrow & & \downarrow{\scriptstyle d_{\mathbf{D}}} \\ \mathbf{C}_{r-1} & \xrightarrow[\mathbf{f}_{r-1}]{} & \mathbf{D}_{r-1} \end{array} \qquad (0 \leqslant r \leqslant n)$$

is commutative. The composition $d_{\mathbf{D}}\mathbf{f}_r \in \mathrm{Hom}_A(\mathbf{C}_r, \mathbf{D}_{r-1})$ is represented by the composition of the commutative diagrams

$$\left(\begin{array}{ccc} B\otimes_{A_2} D_r & \xrightarrow{\nu_r} & D'_r \\ {\scriptstyle 1\otimes d_D}\downarrow & & \downarrow{\scriptstyle d_{D'}} \\ B\otimes_{A_2} D_{r-1} & \xrightarrow[\nu_{r-1}]{} & D'_{r-1} \end{array}\right)\left(\begin{array}{ccc} B\otimes_{A_2} C'_r & \xrightarrow{\mu_r} & C_r \\ {\scriptstyle 1\otimes f'_r}\downarrow & & \downarrow{\scriptstyle f_r} \\ B\otimes_{A_2} D'_r & \xrightarrow[\nu_r]{} & D_r \end{array}\right) = \begin{array}{ccc} B\otimes_{A_2} C'_r & \xrightarrow{\mu_r} & C'_r \\ {\scriptstyle 1\otimes(d_{D'}f'_r)}\downarrow & & \downarrow{\scriptstyle d_{D'}f_r} \\ B\otimes_{A_2} D'_{r-1} & \xrightarrow[\nu_{r-1}]{} & D'_{r-1} \end{array}$$

and the composition $\mathbf{f}_{r-1}\mathbf{d}_C \in \mathrm{Hom}_A(\mathbf{C}_r, \mathbf{D}_{r-1})$ is represented by the composition of the commutative diagrams

$$\left(\begin{array}{ccc} B\otimes_{A_2} C'_{r-1} & \xrightarrow{\mu_{r-1}} & C_{r-1} \\ {\scriptstyle 1\otimes f'_{r-1}}\downarrow & & \downarrow{\scriptstyle f_{r-1}} \\ B\otimes_{A_2} D'_{r-1} & \xrightarrow[\nu_{r-1}]{} & D_{r-1} \end{array}\right)\left(\begin{array}{ccc} B\otimes_{A_2} C'_r & \xrightarrow{\mu_r} & C_r \\ {\scriptstyle 1\otimes d_{C'}}\downarrow & & \downarrow{\scriptstyle d_C} \\ B\otimes_{A_2} C'_{r-1} & \xrightarrow[\mu_{r-1}]{} & C_{r-1} \end{array}\right) = \begin{array}{ccc} B\otimes_{A_2} C'_r & \xrightarrow{\mu_r} & C_r \\ {\scriptstyle 1\otimes(f'_{r-1}d_{C'})}\downarrow & & \downarrow{\scriptstyle f_{r-1}d_C} \\ B\otimes_{A_2} D'_{r-1} & \xrightarrow[\nu_{r-1}]{} & D_{r-1} \end{array}$$

It follows that we have commutativity of the required diagram if and only if $d_{D'}f' = f'd_{C'}$ and $d_D f = f d_C$, that is if and only if $f : C \to D$ is an A_1-module chain map and $f' : C' \to D'$ is an A_2-module chain map.

(iii) Follows from part (iii) of Proposition 7.1.3.

(iv) Follows from part (iv) of Proposition 7.1.3.

□

7.2 A local chain duality for a triangular matrix ring

We now take the candidate contravariant additive functor $\mathbf{T} : A\text{-Mod} \to A - \text{Chain}$ of [Ran06], find its standard extension $\mathbf{T} : A\text{-Chain} \to A\text{-Chain}$ and show there exists a natural transformation $\mathbf{e} : \mathbf{T}^2 \to \mathbf{1} : A\text{-Mod} \to A\text{-Chain}$ such that the triple $(\mathbb{A}, \mathbf{T}, \mathbf{e})$ satisfies a weakened version Definition 6.2.1.

From now on we assume that B is f.g. as a left A_1-module and B is equipped with a pairing $\beta : B \times B \to A_1$ such that for all $a_1, a_1' \in A_1, a_2 \in A_2, b, b', b'' \in B$

(i) $\beta(b, b' + b'') = \beta(b, b') + \beta(b, b'') \in A_1$

(ii) $\beta(b', b) = \overline{\beta(b, b')} \in A_1$

(iii) $\beta(b, b'a_2) = \beta(b\overline{a_2}, b')$

(iv) $\beta(a_1 b, a_1' b_1') = a_1' \beta(b, b_1') \overline{a_1}$

(v) The A_1-module morphism

$$\beta : B \to B^* = \mathrm{Hom}_{A_1}(B, A_1); \quad (b \mapsto (b' \mapsto \beta(b, b')))$$

is an isomorphism such that $\beta = \beta^* \in \mathrm{Hom}_{A_1}(B, B^*)$.

Proposition 7.2.1. There is a contravariant additive functor

$$\mathbf{T} : A\text{-Mod} \to A - \text{Chain}$$
$$\mathbf{M} = (M, M', \mu : B \otimes_{A_2} M' \to M) \mapsto \mathbf{C} = (C, C', \hat{\mu} : B \otimes_{A_2} C' \to C)$$

where $\mathbf{C}$ is the strictly 1-dimensional chain complex

$$\mathbf{C} = (C, C', \hat{\mu}) = \left(\begin{array}{c} M^* \\ \downarrow{\scriptstyle (\beta^{-1}\otimes 1)\mu^*} \\ B \otimes_{A_2} M'^* \end{array}, \begin{array}{c} 0 \\ \downarrow{\scriptstyle 0} \\ M'^* \end{array}, \begin{array}{ccc} 0 & \xrightarrow{0} & M^* \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (\beta^{-1}\otimes 1)\mu^*} \\ B \otimes_{A_2} M'^* & \xrightarrow[1]{} & B \otimes_{A_2} M'^* \end{array} \right)$$

and $\hat{\mu} : B \otimes C' \to C$ the inclusion chain map of A_1-module chain complexes.

Proof. We first check the functoriality of $\mathbf{T}$ and then check the additivity of $\mathbf{T}$.

(i) Proposition 7.1.3 implies that C is a chain complex of A-modules so that $\mathbf{T}$ sends objects to objects.

(ii) Let $\mathbf{N} = (N, N', \nu : B \otimes_{A_2} N' \to N)$ be a second A-module with

$$\mathbf{D} = (D, D', \hat{\nu}) = \left(\begin{array}{c} N^* \\ \downarrow{\scriptstyle (\beta^{-1}\otimes 1)\nu^*} \\ B \otimes_{A_2} N'^* \end{array}, \begin{array}{c} 0 \\ \downarrow{\scriptstyle 0} \\ N'^* \end{array}, \begin{array}{ccc} 0 & \xrightarrow{0} & N^* \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (\beta^{-1}\otimes 1)\nu^*} \\ B \otimes_{A_2} N'^* & \xrightarrow[1]{} & B \otimes_{A_2} N'^* \end{array} \right).$$

A morphism $\mathbf{f} \in \mathrm{Hom}_A(\mathbf{M},\mathbf{N})$ may be identified with a pair of morphisms

$$\mathbf{f} = (f \in \mathrm{Hom}_{A_1}(M,N), f' \in \mathrm{Hom}_{A_2}(M',N'))$$

such that the following diagram commutes

$$\begin{array}{ccc} B\otimes_{A_2} M' & \xrightarrow{\mu} & M \\ {\scriptstyle 1\otimes f_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ B\otimes_{A_2} N' & \xrightarrow[\nu]{} & N \end{array}$$

The dual of this square with respect to A_1 is the commutative square

$$\begin{array}{ccc} N^* & \xrightarrow{f^*} & M^* \\ {\scriptstyle \nu^*}\downarrow & & \downarrow{\scriptstyle \mu^*} \\ B^*\otimes_{A_2} N'^* & \xrightarrow[1\otimes f'^*]{} & B^*\otimes_{A_2} M'^* \end{array}$$

which when composed with the commutative square

$$\begin{array}{ccc} B^*\otimes_{A_2} N' & \xrightarrow{1\otimes f'^*} & B^*\otimes_{A_2} M' \\ {\scriptstyle \beta^{-1}\otimes 1}\downarrow & & \downarrow{\scriptstyle \beta^{-1}\otimes 1} \\ B\otimes_{A_2} N'^* & \xrightarrow[1\otimes f'^*]{} & B\otimes_{A_2} M'^* \end{array}$$

yields the commutative square

$$\begin{array}{ccc} N^* & \xrightarrow{f^*} & M^* \\ {\scriptstyle (\beta\otimes 1)^{-1}\nu^*}\downarrow & & \downarrow{\scriptstyle (\beta\otimes 1)^{-1}\mu^*} \\ B\otimes_{A_2} N'^* & \xrightarrow[1\otimes f'^*]{} & B\otimes_{A_2} M'^* \end{array}$$

which then defines a chain map $T(\mathbf{f}) : D \to C$ of A_1-module chain complexes. The dual morphism $f'^* \in \mathrm{Hom}_{A_2}(N'^*, M'^*)$ then defines an A_2-module chain map $T(\mathbf{f})' : D' \to C'$. It follows that the pair $(T(\mathbf{f}), T(\mathbf{f})')$ defines a chain map $\mathbf{T}(\mathbf{f}) : \mathbf{T}(\mathbf{D}) \to \mathbf{T}(\mathbf{C})$ of A-module chain complexes if and only if the following diagram is commutative

$$\begin{array}{ccc} B\otimes_{A_2} D' & \xrightarrow{\hat{\mu}} & D \\ {\scriptstyle 1\otimes T(\mathbf{f})'}\downarrow & & \downarrow{\scriptstyle T(\mathbf{f})} \\ B\otimes_{A_2} C' & \xrightarrow[\hat{\nu}]{} & C \end{array}$$

Since D' is 0-dimensional, commutativity needs to only be checked in dimension 0 and this follows by the trivial commutativity of the diagram

$$\begin{array}{ccc} B\otimes_{A_2} N'^* & \xrightarrow{1} & B\otimes_{A_2} N'^* \\ {\scriptstyle 1\otimes f'^*}\downarrow & & \downarrow{\scriptstyle 1\otimes f'^*} \\ B\otimes_{A_2} M'^* & \xrightarrow[1]{} & B\otimes_{A_2} M'^* \end{array}$$

so $\mathbf{T}$ sends morphisms to morphisms.

(iii) Note that

$$T'(1_M) = (1 : M'^* \to M'^*) = 1_{T'(M)}$$

and

$$T(1_M) = \left(\begin{array}{ccc} M^* & \xrightarrow{\quad 1 \quad} & M^* \\ {\scriptstyle (\beta^{-1}\otimes 1)\mu^*}\downarrow & & \downarrow{\scriptstyle (\beta^{-1}\otimes 1)\mu^*} \\ B \otimes_{A_2} M'^* & \xrightarrow[\quad 1 \quad]{} & B \otimes_{A_2} M'^* \end{array} \right) = 1_{T(M)}$$

so that $\mathbf{T(1_M)} = \mathbf{1_{T(M)}}$ as required.

(iv) Let $\mathbf{P} = (P, P', \xi : B \otimes_{A_2} P' \to P)$ be a third A-module and let $\mathbf{g} \in \mathrm{Hom}_A(\mathbf{N}, \mathbf{P})$ be a morphism. Under the identification

$$\mathbf{g} = (g \in \mathrm{Hom}_{A_1}(N, P), g' \in \mathrm{Hom}_{A_2}(N', P'))$$

the composition $\mathbf{gf} \in \mathrm{Hom}_A(\mathbf{M}, \mathbf{P})$ is identified with the pair

$$\mathbf{gf} = (gf \in \mathrm{Hom}_{A_1}(M, P), g'f' \in \mathrm{Hom}_{A_2}(M', P')).$$

It is clear from (ii) that $T(\mathbf{gf}) = T(\mathbf{f})T(\mathbf{g})$ and $T(\mathbf{gf})' = T(\mathbf{f})'T(\mathbf{g})'$ and so that $\mathbf{T(gf)} = \mathbf{T(f)T(g)}$ as required and hence $\mathbf{T}$ is contravariant.

(v) For A-modules $\mathbf{M}, \mathbf{N}$ it is clear that the map

$$\mathbf{T} : \mathrm{Hom}_A(\mathbf{M}, \mathbf{N}) \to \mathrm{Hom}_A(T(\mathbf{N}), T(\mathbf{N})), \quad \mathbf{f} \mapsto \mathbf{T(f)}$$

is a homomorphism of abelian groups so that $\mathbf{T}$ is additive.

□

We may now write down explicitly the standard extension of $\mathbf{T}$.

Proposition 7.2.2. The standard extension of the contravariant additive functor

$$\mathbf{T} : A\text{-Mod} \to A - \text{Chain}; \quad \mathbf{M} \mapsto \mathbf{T(M)}$$

is the contravariant additive functor

$$\mathbf{T} : A - \text{Chain} \to A - \text{Chain}; \quad \mathbf{C} \mapsto \mathbf{T(C)}$$

such that if $\mathbf{C} = (C, C', \mu)$ is a finite chain complex of A-modules then

$$\mathbf{T(C)}_* = \left(\widetilde{\mathscr{C}}(\mu(\beta^{-1} \otimes 1))^{1-*}, C'^{-*}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} : B \otimes_{A_2} C'^{-*} \to \widetilde{\mathscr{C}}(\mu(\beta^{-1} \otimes 1))^{1-*} \right)$$

where $\mu(\beta^{-1} \otimes 1)$ is the A_1-module chain map

$$\mu(\beta^{-1} \otimes 1) : B^* \otimes_{A_2} C' \to B \otimes_{A_2} C' \to C$$

so that $\mathbf{T}(\mathbf{C})$ has chain groups given by

$$\mathbf{T}(\mathbf{C})_r = \left((B \otimes_{A_2} C'^{-r}) \oplus C^{1-r}, C'^{-r}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} : B \otimes_{A_2} C'^{-r} \to (B \otimes_{A_2} C'^{-r}) \oplus C^{1-r}\right)$$

and differential $\mathbf{d}_{\mathbf{T}(\mathbf{C})} : \mathbf{T}(\mathbf{C})_r \to \mathbf{T}(\mathbf{C})_{r-1}$ given by

$$\left(\begin{array}{c} (B \otimes_{A_2} C'^{-r}) \oplus C^{1-r} \\ \Big\downarrow \begin{pmatrix} 1 \otimes d^*_{C'} & (\beta^{-1} \otimes 1)\mu^* \\ 0 & -d^*_C \end{pmatrix} \\ (B \otimes_{A_2} C'^{1-r}) \oplus C'^{2-r} \end{array}, \begin{array}{c} C'^{-r} \\ \Big\downarrow d^*_{C'} \\ C'^{1-r} \end{array}, \begin{array}{ccc} B \otimes_{A_2} C'^{-r} & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & (B \otimes_{A_2} C'^{-r}) \oplus C^{1-r} \\ \Big\downarrow 1 \otimes d^*_{C'} & & \Big\downarrow \begin{pmatrix} 1 \otimes d^*_{C'} & (\beta^{-1} \otimes 1)\mu^* \\ 0 & -d^*_C \end{pmatrix} \\ B \otimes_{A_2} C'^{1-r} & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & (B \otimes_{A_2} C'^{1-r}) \oplus C^{2-r} \end{array} \right)$$

Proof. Recall from Definition 6.1.8 that $\mathbf{T}(\mathbf{C})$ is the total complex of the double complex $\mathbf{T}(\mathbf{C})_{*,*}$ with chain groups

$$\begin{aligned} \mathbf{T}(\mathbf{C})_{p,q} &= \mathbf{T}(\mathbf{C}_{-p})_q \\ \mathbf{d}'_{p,q} &= \mathbf{T}(\mathbf{d}_{\mathbf{C}} : \mathbf{C}_{1-p} \to \mathbf{C}_{-p}) : \mathbf{T}(\mathbf{C}_{-p})_q \to \mathbf{T}(\mathbf{C}_{1-p})_q \\ \mathbf{d}''_{p,q} &= \mathbf{d}_{\mathbf{T}(\mathbf{C}_{-p})} : \mathbf{T}(\mathbf{C}_{-p})_q \to \mathbf{T}(\mathbf{C}_{-p})_{q-1} \qquad (p, q \in \mathbb{Z}) \end{aligned}$$

so that

$$\begin{aligned} \mathbf{T}(\mathbf{C})_r &= \mathbf{T}(\mathbf{C}_{-r})_0 \oplus \mathbf{T}(\mathbf{C}_{1-r})_1 \\ &= \mathbf{T}(C_{-r}, C'_{-r}, \mu : B \otimes_{A_2} C'_{-r} \to C_{-r})_0 \oplus \mathbf{T}(C_{1-r}, C'_{1-r}, \mu : B \otimes_A C'_{1-r} \to C_{1-r})_1 \\ &= (B \otimes_{A_2} C'^{-r}, C'^{-r}, 1 : B \otimes_A C'^{-r} \to B \otimes_{A_2} C'^{-r}) \oplus (C^{1-r}, 0, 0 : 0 \to C^{1-r}) \\ &= \left((B \otimes_{A_2} C'^{-r}) \oplus C^{1-r}, C''^{-r}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} : B \otimes_{A_2} C'^{-r} \to (B \otimes_{A_2} C'^{-r}) \oplus C^{1-r}\right) \end{aligned}$$

The chain complex $\mathbf{T}(\mathbf{C})$ has differential

$$\mathbf{d}_{\mathbf{T}(\mathbf{C})} = \begin{pmatrix} \mathbf{d}'_{r,0} & \mathbf{d}''_{r-1,1} \\ 0 & -\mathbf{d}'_{r-1,1} \end{pmatrix} : \mathbf{T}(\mathbf{C})_r = \mathbf{T}(\mathbf{C}_{-r})_0 \oplus \mathbf{T}(\mathbf{C}_{1-r})_1 \to \mathbf{T}(\mathbf{C})_{r-1} = \mathbf{T}(\mathbf{C}_{1-r})_0 \oplus \mathbf{T}(\mathbf{C}_{2-r})_1$$

where

$$\begin{aligned} \mathbf{d}''_{r-1,1} &= ((\beta^{-1} \otimes 1)\mu^* \in \mathrm{Hom}_{A_1}(C'^{1-r}, B \otimes_{A_2} C''^{1-r}), 0 \in \mathrm{Hom}_{A_2}(0, C''^{1-r})) \\ \mathbf{d}'_{r,0} &= (1 \otimes d^*_{C''} \in \mathrm{Hom}_{A_1}(B \otimes_A C''^{-r}, B \otimes_A C''^{1-r}), d^*_{C''} \in \mathrm{Hom}_{A_2}(C''^{-r}, C''^{1-r})) \\ \mathbf{d}'_{r-1,1} &= (d^*_{C'} \in \mathrm{Hom}_{A_1}(C'^{1-r}, C'^{2-r}), 0 \in \mathrm{Hom}_{A_2}(0, 0)) \end{aligned}$$

so that $\mathbf{d}_{\mathbf{T}(\mathbf{C})} : \mathbf{T}(\mathbf{C})_r \to \mathbf{T}(\mathbf{C})_{r-1}$ is represented by the pair of morphisms

$$\left(\begin{array}{l} \begin{pmatrix} 1 \otimes d^*_{C'} & (\beta^{-1} \otimes 1)\mu^* \\ 0 & -d^*_C \end{pmatrix} \in \mathrm{Hom}_{A_1}((B \otimes_{A_2} C'^{-r}) \oplus C^{1-r}, (B \otimes_{A_2} C'^{1-r}) \oplus C^{2-r}) \\ \qquad\qquad\qquad\qquad\quad d^*_{C'} \in \mathrm{Hom}_{A_2}(C'^{-r}, C'^{1-r}) \end{array} \right)$$

so that

$$\mathbf{T}(\mathbf{C})_* = \left(\widetilde{\mathscr{C}}(\mu(\beta^{-1}\otimes 1))^{1-*}, C'^{-*}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} : B\otimes_{A_2} C'^{-*} \to \widetilde{\mathscr{C}}(\mu(\beta^{-1}\otimes 1))^{1-*}\right)$$

as required. By comparison with Proposition 7.2.1 is clear that $\mathbf{T} : A\text{-Chain} \to A\text{-Chain}$ is an extension of $\mathbf{T} : A\text{-Mod} \to A\text{-Chain}$. It then remains to check the functoriality and additivity of $\mathbf{T}$.

(i) Proposition 7.1.3 implies that $\mathbf{T}(\mathbf{C})$ is indeed an A-module chain complex so that $\mathbf{T}$ sends objects to objects.

(ii) Let $\mathbf{D} = (D, D', \nu : B\otimes_{A_2} D' \to D)$ be another A-module chain complex. Proposition 7.1.3 implies that an A-module chain map $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ may be identified with a pair of chain maps

$$\mathbf{f} = (f : C \to D, f' : C' \to D')$$

such that the following diagram commutes

$$\begin{array}{ccc} B\otimes_{A_2} C' & \xrightarrow{\mu} & C \\ {\scriptstyle 1\otimes f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B\otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

Note that $f'^* : D'^{-*} \to C'^{-*}$ defines an A_2-module chain map. Since $f\mu = \nu(1\otimes f)$ the following diagram is commutative

$$\begin{array}{ccc} (B\otimes_{A_2} D'^{-r})\oplus D^{1-r} & \xrightarrow{\begin{pmatrix} 1\otimes f'^* & 0 \\ 0 & f^* \end{pmatrix}} & (B\otimes_{A_2} D'^{1-r})\oplus D^{2-r} \\ {\scriptstyle \begin{pmatrix} 1\otimes d^*_{D'} & (\beta^{-1}\otimes 1)\nu^* \\ 0 & -d^*_D \end{pmatrix}}\big\downarrow & & \big\downarrow{\scriptstyle \begin{pmatrix} 1\otimes d^*_{C'} & (\beta^{-1}\otimes 1)\mu^* \\ 0 & -d^*_C \end{pmatrix}} \\ (B\otimes_{A_2} C'^{-r})\oplus C^{1-r} & \xrightarrow[\begin{pmatrix} 1\otimes f'^* & 0 \\ 0 & f^* \end{pmatrix}]{} & (B\otimes_{A_2} C'^{1-r})\oplus C^{2-r} \end{array}$$

so that

$$\begin{pmatrix} 1\otimes f'^* & 0 \\ 0 & f^* \end{pmatrix} : \widetilde{\mathscr{C}}(\nu(\beta^{-1}\otimes 1))^{1-*} \to \widetilde{\mathscr{C}}(\mu(\beta^{-1}\otimes 1))^{1-*}$$

is an A_1-module chain map. The commutative diagram

$$\begin{array}{ccc} B\otimes_{A_2} D'^{-r} & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & (B\otimes_{A_2} D'^{-r})\oplus D^{1-r} \\ {\scriptstyle 1\otimes f'^*}\big\downarrow & & \big\downarrow{\scriptstyle \begin{pmatrix} 1\otimes f'^* & 0 \\ 0 & f^* \end{pmatrix}} \\ B\otimes_{A_2} C'^{-r} & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & (B\otimes_{A_2} C'^{-r})\oplus C^{1-r} \end{array}$$

and Proposition 7.1.3 implies that the pair

$$\mathbf{T}(\mathbf{f}) = \left(\begin{pmatrix} 1\otimes f'^* & 0 \\ 0 & f^* \end{pmatrix}, f'^*\right) : \mathbf{T}(\mathbf{D}) \to \mathbf{T}(\mathbf{C})$$

defines an A-module chain map and hence $\mathbf{T}$ sends morphisms to morphisms.

(iii) It is clear from (ii) that $\mathbf{T}(\mathbf{1_C}) = \mathbf{1_{T(C)}}$

(iv) Let $\mathbf{E} = (E, E', \xi : B \otimes_{A_2} E' \to E)$ be third A-module chain complex and let $\mathbf{g} \in \mathrm{Hom}_A(\mathbf{N}, \mathbf{P})$ be a morphism. Under the identification

$$\mathbf{g} = (g : D \to E, g' : D' \to E')$$

the composition $\mathbf{gf}$ is identified with the pair

$$\mathbf{gf} = (gf : C \to E, g'f' : C' \to E')$$

and it is clear from (ii) that $\mathbf{T}(\mathbf{gf}) = \mathbf{T}(\mathbf{f})\mathbf{T}(\mathbf{g})$ so $\mathbf{T} : A - \mathrm{Chain} \to A - \mathrm{Chain}$ defines a contravariant functor.

(v) For finite A-module chain complexes $\mathbf{C}, \mathbf{D}$ it is clear from (ii) that the map

$$\mathbf{T} : \mathrm{Hom}_A(\mathbf{C}, \mathbf{D}) \to \mathrm{Hom}_A(\mathbf{T}(\mathbf{C}), \mathbf{T}(\mathbf{D})), \quad \mathbf{f} \mapsto \mathbf{T}(\mathbf{f})$$

is a homomorphism of $\mathbb{Z}$-modules.

□

We next need to find a natural transformation $\mathbf{e} : \mathbf{T}^2 \to 1 : A\text{-Mod} \to A\text{-Chain}$. The effect of $\mathbf{T}^2$ on an object in A-Mod is as follows.

Corollary 7.2.3. For an A-module $\mathbf{M} = (M, M', \mu)$, the chain complex $\mathbf{T}^2(\mathbf{M})$ is a strictly 1-dimensional A-module chain complex with

$$\mathbf{T}^2(\mathbf{M}) = \left(\begin{array}{c} B^* \otimes_{A_2} M' \\ \Big\downarrow \begin{pmatrix} \beta^{-1} \otimes 1 \\ -\mu(\beta^{-1} \otimes 1) \end{pmatrix} \\ (B \otimes_{A_2} M') \oplus M \end{array}, \begin{array}{c} 0 \\ \Big\downarrow 0 \\ M' \end{array}, \begin{array}{ccc} 0 & \xrightarrow{\;0\;} & B^* \otimes_{A_2} M' \\ \Big\downarrow 0 & & \Big\downarrow \begin{pmatrix} \beta^{-1} \otimes 1 \\ -\mu(\beta^{-1} \otimes 1) \end{pmatrix} \\ B \otimes_{A_2} M' & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & (B \otimes_{A_2} M') \oplus M \end{array} \right)$$

Proof. Use Proposition 7.2.1 to write down $\mathbf{T}(M)$ and then use Proposition 7.2.2 to write down $\mathbf{T}^2(\mathbf{M}) = \mathbf{T}(\mathbf{T}(M))$. □

In order for the chain map $\mathbf{e}(\mathbf{M}) : \mathbf{T}^2(\mathbf{M}) \to \mathbf{M}$ to be a chain homotopy equivalence it is necessary to work with the weaker class of morphisms in the category A-Mod.

Definition 7.2.4. Let $\mathbf{C} = (C, C', \mu)$ and $\mathbf{D} = (D, D', \nu)$ be A-module chain complexes.

(i) A *local A-module chain map* is a pair of chain maps

$$\mathbf{f} = (f \in \mathrm{Hom}_{A_1}(C, D), f' \in \mathrm{Hom}_{A_2}(C', D')).$$

(ii) An A-module chain map

$$\mathbf{f} = (f \in \mathrm{Hom}_{A_1}(C, D), f' \in \mathrm{Hom}_{A_2}(C', D'))$$

is a *local chain equivalence* if both f and f' are chain homotopy equivalences.

Note that the forgetful functor

$$\begin{gathered}(A\text{-Mod}, A\text{-module chain maps}) \to (A\text{-Mod}, \text{ local } A\text{-module chain maps})\\ (f_1, f_2) \mapsto (f_1, f_2)\end{gathered}$$

is not in general surjective since a local chain map $\mathbf{f} = (f \in \mathrm{Hom}_{A_1}(C,D), f' \in \mathrm{Hom}_{A_2}(C',D'))$ does not necessarily determine a commutative diagram

$$\begin{array}{ccc} B \otimes_{A_2} C' & \xrightarrow{\mu} & C \\ {\scriptstyle 1\otimes f'}\downarrow & & \downarrow{\scriptstyle f} \\ B \otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

Proposition 7.2.5. There is a natural transformation of covariant functors

$$\mathbf{e} : \mathbf{T}^2 \to \mathbf{1} : A\text{-Mod} \to A - \text{Chain}$$

such that for each A-module $\mathbf{M}$ the A-module chain map

$$\mathbf{e}(\mathbf{M}) : \mathbf{T}^2(\mathbf{M}) \to \mathbf{M}$$

is a local A-module chain equivalence.

Proof. Let $\mathbf{M} = (M, M', \mu)$ be an A-module which we may view as a strictly 0-dimensional A-module chain complex. By Corollary 7.2.3 we may write

$$\mathbf{T}^2(\mathbf{M}) = \left(\begin{array}{c} B^* \otimes_{A_2} M' \\ \Big\downarrow \begin{pmatrix} \beta^{-1}\otimes 1 \\ -\mu(\beta^{-1}\otimes 1) \end{pmatrix} \\ (B \otimes_{A_2} M') \oplus M \end{array}, \begin{array}{c} 0 \\ \Big\downarrow 0 \\ M' \end{array}, \begin{array}{ccc} 0 & \xrightarrow{0} & B^* \otimes_{A_2} M' \\ \Big\downarrow 0 & & \Big\downarrow \begin{pmatrix} \beta^{-1}\otimes 1 \\ -\mu(\beta^{-1}\otimes 1) \end{pmatrix} \\ B \otimes_{A_2} M' & \xrightarrow[\binom{1}{0}]{} & (B \otimes_{A_2} M') \oplus M \end{array} \right)$$

Each component of the chain map $\mathbf{e}(\mathbf{M}) : \mathbf{T^2}(\mathbf{M}) \to \mathbf{M}$ is necessarily zero apart from the 0-dimensional component $\mathbf{e}(\mathbf{M}) : \mathbf{T^2}(\mathbf{M})_\mathbf{0} \to \mathbf{M_0} = \mathbf{M}$. The pair of morphisms

$$\mathbf{e}(\mathbf{M}) = \Big(\begin{pmatrix} \mu & 1 \end{pmatrix} \in \mathrm{Hom}_{A_1}((B \otimes_{A_2} M') \oplus M, M), 1 \in \mathrm{Hom}_{A_2}(M', M') \Big)$$

determine a commutative diagram

$$\begin{array}{ccc} B \otimes_{A_2} M' & \xrightarrow{\binom{1}{0}} & (B \otimes_{A_2} M') \oplus M_1 \\ {\scriptstyle 1\otimes 1}\downarrow & & \downarrow \begin{pmatrix} \mu & 1 \end{pmatrix} \\ B \otimes_{A_2} M' & \xrightarrow[\mu]{} & M \end{array}$$

so that $\mathbf{e}(\mathbf{M}) : \mathbf{T}^2(\mathbf{M})_0 \to \mathbf{M}_0$ is an A-module morphism. The commutative diagram

$$\begin{array}{ccc} B^* \otimes_{A_2} M' & \xrightarrow{\quad 0 \quad} & 0 \\ {\scriptstyle \begin{pmatrix} \beta^{-1}\otimes 1 \\ -\mu(\beta^{-1}\otimes 1) \end{pmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle 0} \\ (B \otimes_{A_2} M') \oplus M & \xrightarrow[\begin{pmatrix} \mu & 1 \end{pmatrix}]{} & M \end{array}$$

implies that there is a commutative diagram

$$\begin{array}{ccc} \mathbf{T}^2(\mathbf{M})_1 & \xrightarrow{\quad \mathbf{0} \quad} & \mathbf{0} \\ {\scriptstyle \mathbf{d}_{\mathbf{T}^2(\mathbf{M})}} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathbf{0}} \\ \mathbf{T}^2(\mathbf{M})_0 & \xrightarrow[\mathbf{e}(\mathbf{M})]{} & \mathbf{M} \end{array}$$

so that $\mathbf{e}(\mathbf{M}) : \mathbf{T}^2(\mathbf{M}) \to \mathbf{M}$ is an A-module chain map. It remains to show that $\mathbf{e}(\mathbf{M})$ is a local chain equivalence. We explain after the proof why it is not possible to choose $\mathbf{e}(\mathbf{M})$ to be a genuine chain equivalence.

For ease of notation, let $\mathbf{f} = \mathbf{e}(\mathbf{M})$. Since $\mathbf{M}$ is a strictly 0-dimensional A-module chain complex, a local chain map $\mathbf{g} = (g_1, g_2) : \mathbf{M} \to \mathbf{T}^2(\mathbf{M})$ is uniquely determined by an arbitrary local morphism of A-modules $\mathbf{g} : \mathbf{M}_0 = \mathbf{M}_0 \to \mathbf{T}^2(\mathbf{M})_0$ with no commutativity requirements . Define a local A-module chain map $\mathbf{g} : \mathbf{M} \to \mathbf{T}^2(\mathbf{M})$ by the pair of morphisms

$$\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathrm{Hom}_{A_1}(M, (B \otimes_{A_2} M') \oplus M), 1 \in \mathrm{Hom}_{A_2}(M', M') \right).$$

The composition $\mathbf{fg} : \mathbf{M} \to \mathbf{T}^2(\mathbf{M}) \to \mathbf{M}$ is represented by the pair of morphisms

$$(1 \in \mathrm{Hom}_{A_1}(M, M), 1 \in \mathrm{Hom}_{A_2}(M', M'))$$

so that $\mathbf{fg} = \mathbf{1_M}$ and hence there is a chain homotopy $\mathbf{0} : \mathbf{fg} \simeq \mathbf{1_M}$.

Denote by C the strictly 1-dimensional chain complex

$$\begin{pmatrix} C_1 \\ \Big\downarrow {\scriptstyle d_C} \\ C_0 \end{pmatrix} = \begin{pmatrix} B^* \otimes_{A_2} M' \\ \Big\downarrow \begin{pmatrix} \beta^{-1}\otimes 1 \\ -\mu(\beta^{-1}\otimes 1) \end{pmatrix} \\ (B \otimes_{A_2} M') \oplus M \end{pmatrix}$$

The composition $\mathbf{h} = \mathbf{gf} : T^2(\mathbf{M}) \to \mathbf{M} \to T^2(\mathbf{M})$ is represented by the pair of morphisms

$$\mathbf{h} = (h, h') = \left(\begin{pmatrix} 0 & 0 \\ \mu & 1 \end{pmatrix} \in \mathrm{Hom}_{A_1}((B \otimes_{A_2} M') \oplus M, (B \otimes_{A_2} M') \oplus M), 1 \in \mathrm{Hom}_{A_2}(M', M') \right).$$

Since $\mathbf{h}$ is only required to be a local chain equivalence, it is then enough to find a chain homotopy $k : 1 \simeq h : C \to C$. The morphism

$$k = \begin{pmatrix} \beta \otimes 1 & 0 \end{pmatrix} \in \mathrm{Hom}_{A_1}((B \otimes_{A_2} M') \oplus M, B^* \otimes_{A_2} M')$$

defines a degree one morphism $k : C_* \to C_{*+1}$ satisfying

$$\begin{aligned} kd_C &= \begin{pmatrix} \beta \otimes 1 & 0 \end{pmatrix} \begin{pmatrix} \beta^{-1} \otimes 1 \\ -\mu(\beta^{-1} \otimes 1) \end{pmatrix} \\ &= 1 - h \in \mathrm{Hom}_{A_1}(C_1, C_1) \end{aligned}$$

$$\begin{aligned} d_C k &= \begin{pmatrix} \beta^{-1} \otimes 1 \\ -\mu(\beta^{-1} \otimes 1) \end{pmatrix} \begin{pmatrix} \beta \otimes 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\mu & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \mu & 1 \end{pmatrix} \\ &= 1 - h \in \mathrm{Hom}_{A_1}(C_0, C_0) \end{aligned}$$

so that $k : 1 \simeq h : C \to C$ is a chain homotopy as required. □

Note that if we require that $\mathbf{e}(\mathbf{M})$ to be a genuine A-module chain map then necessarily

$$\mathbf{e}(\mathbf{M}) = \pm \Big(\begin{pmatrix} \mu & 1 \end{pmatrix} \in \mathrm{Hom}_{A_1}((B \otimes_{A_2} M') \oplus M, M), 1 \in \mathrm{Hom}_{A_2}(M', M') \Big)$$

so that $\mathbf{e}(\mathbf{M})$ is unique up to sign. A genuine A-module chain map $\mathbf{g} : \mathbf{M} \to \mathbf{T}^2(\mathbf{M})$ is uniquely determined by a genuine morphism of A-modules $\mathbf{g} : \mathbf{M}_0 = \mathbf{M}_0 \to \mathbf{T}^2(\mathbf{M})_0$ determining a commutative square

$$\begin{array}{ccc} B \otimes_{A_2} M' & \xrightarrow{\quad \mu \quad} & M \\ {\scriptstyle 1 \otimes g_3} \Big\downarrow & & \Big\downarrow {\scriptstyle \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} \\ B \otimes_{A_2} M' & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & (B \otimes_{A_2} M') \oplus M \end{array}$$

Given the data available, the only natural choice for g_3 is $g_3 = 1$ so that necessarily $g_2\mu = 0$ and $g_1\mu = 1$. Proposition 7.1.2 implies that $\mu : B \otimes_{A_2} M' \to M$ is split-injective since $\mathbf{M}$ is projective. Choose a morphism $\kappa : M \to B \otimes_{A_2} M'$ which splits μ and then set $g_1 = \kappa$. Recall that we want $\mathbf{g}$ to be a chain homotopy inverse to $\mathbf{f}$. Note that the composition $\mathbf{fg} : \mathbf{M} \to \mathbf{M}$ is represented by $(\mu\kappa, 1)$. Since $\mathbf{M}$ is a zero-dimensional chain complex, the requirement that $\mathbf{fg} \simeq 1 : \mathbf{M} \to \mathbf{M}$ is equivalent to the requirement that $\mathbf{fg} = 1 : \mathbf{M} \to \mathbf{M}$. However κ is only a left inverse for μ so it is not necessarily true that $\mu\kappa = 1$ unless μ is invertible and this is too strong of a restriction.

This difficulty can be resolved if we weaken the requirement that $\mathbf{g}$ is a chain map to $\mathbf{g}$ is a local chain map. See [Ran92, chapters 4,7] for other examples of local morphisms and local homotopy equivalences used elsewhere in L-theory.

We now check that $(\mathbf{T}, \mathbf{e})$ satisfies the remaining requirements of Definition 6.2.1.

Proposition 7.2.6. For each A-module $\mathbf{M}$

$$\mathbf{e}(\mathbf{T}(\mathbf{M}))\mathbf{T}(\mathbf{e}(\mathbf{M})) = \mathbf{1} : \mathbf{T}(\mathbf{M}) \to \mathbf{T}^3(\mathbf{M}) \to \mathbf{T}(\mathbf{M})$$

Proof. If $\mathbf{M} = (M, M', \mu)$ then let $\mathbf{C} = \mathbf{T}(\mathbf{M}), \mathbf{D} = \mathbf{T}^2(\mathbf{M})$. By Proposition 7.2.1

$$\mathbf{C} = \left(\begin{array}{c} M^* \\ \Big\downarrow{\scriptstyle (\beta^{-1}\otimes 1)\mu^*} \\ B\otimes_{A_2} M'^* \end{array}, \begin{array}{c} 0 \\ \Big\downarrow{\scriptstyle 0} \\ M'^* \end{array}, \begin{array}{ccc} 0 & \xrightarrow{0} & M^* \\ \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle (\beta^{-1}\otimes 1)\mu^*} \\ B\otimes_{A_2} M'^* & \xrightarrow[1]{} & B\otimes_{A_2} M'^* \end{array} \right)$$

and since $\mathbf{C}$ is a strictly 1-dimensional chain complex it is enough to verify the identity $\mathbf{e}(\mathbf{T}(\mathbf{M}))\mathbf{T}(\mathbf{e}(\mathbf{M})) = \mathbf{1}$ only in dimensions $0, 1$. By Corollary 7.2.3 the chain complex $\mathbf{D}$ is strictly 1-dimensional

$$\mathbf{D} = \left(\begin{array}{c} B^*\otimes_{A_2} M' \\ \Big\downarrow{\scriptstyle \begin{pmatrix} \beta^{-1}\otimes 1 \\ -\mu(\beta^{-1}\otimes 1) \end{pmatrix}} \\ (B\otimes_{A_2} M')\oplus M \end{array}, \begin{array}{c} 0 \\ \Big\downarrow{\scriptstyle 0} \\ M' \end{array}, \begin{array}{ccc} 0 & \xrightarrow{0} & B\otimes_{A_2} M' \\ \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle \begin{pmatrix} \beta^{-1}\otimes 1 \\ -\mu(\beta^{-1}\otimes 1) \end{pmatrix}} \\ B\otimes_{A_2} M' & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & (B\otimes_{A_2} M')\oplus M \end{array} \right)$$

and by Proposition 7.2.2 the chain complex $\mathbf{T}(\mathbf{D})$ is given in dimensions 0 and 1 by

$$\left(\begin{array}{c} (B^*\otimes_{A_2} M'^*)\oplus M^* \\ \Big\downarrow{\scriptstyle \begin{pmatrix} \beta^{-1}\otimes 1 & 0 \\ \beta^{-1}\otimes 1 & -(\beta^{-1}\otimes 1)\mu^* \end{pmatrix}} \\ (B\otimes_{A_2} M'^*)\oplus(B\otimes_{A_2} M'^*) \end{array}, \begin{array}{c} 0 \\ \Big\downarrow{\scriptstyle 0} \\ M'^* \end{array}, \begin{array}{ccc} 0 & \xrightarrow{0} & (B\otimes_{A_2} M'^*)\oplus M^* \\ \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle \begin{pmatrix} \beta^{-1}\otimes 1 & 0 \\ \beta^{-1}\otimes 1 & -(\beta^{-1}\otimes 1)\mu^* \end{pmatrix}} \\ (B\otimes_{A_2} M'^*)\oplus M^* & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & (B\otimes_{A_2} M'^*)\oplus(B\otimes_{A_2} M'^*) \end{array} \right)$$

Dimension 0**:** By Proposition 7.2.5 we have

$$\mathbf{e}(\mathbf{M}) = (\begin{pmatrix} \mu & 1 \end{pmatrix} \in \mathrm{Hom}_{A_1}((B\otimes_{A_2} M')\oplus M, M), 1 \in \mathrm{Hom}_{A_2}(M', M')) \in \mathrm{Hom}_A(\mathbf{T}^2(\mathbf{M})_0, \mathbf{M})$$

and hence by Proposition 7.2.2 the morphism $\mathbf{T}(\mathbf{e}(\mathbf{M}))_0 \in \mathrm{Hom}_A(\mathbf{T}(\mathbf{M})_0, \mathbf{T}^3(\mathbf{M})_0)$ is identified with the pair of morphisms

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathrm{Hom}_{A_1}(B\otimes_{A_2} M'^*, (B\otimes_{A_2} M'^*)\oplus(B\otimes_{A_2} M'^*)), 1 \in \mathrm{Hom}_{A_2}(M'^*, M'^*)\right).$$

Similarly the morphism $\mathbf{e}(\mathbf{T}(\mathbf{M})) \in \mathrm{Hom}_A(\mathbf{T}^3(\mathbf{M})_0, \mathbf{T}(\mathbf{M})_0)$ is identified with the pair of morphisms

$$\left(\begin{pmatrix} 1 & 1 \end{pmatrix} \in \mathrm{Hom}_{A_1}((B \otimes_{A_2} M'^*) \oplus (B \otimes_{A_2} M'^*), B \otimes_{A_2} M'^*), 1 \in \mathrm{Hom}_{A_2}(M'^*, M'^*) \right)$$

so that the composition $\mathbf{e}(\mathbf{T}(\mathbf{M}))\mathbf{T}(\mathbf{e}(\mathbf{M})) \in \mathrm{Hom}_A(\mathbf{T}(\mathbf{M})_0, \mathbf{T}(\mathbf{M})_0)$ is identified with the pair of morphisms

$$(1 \in \mathrm{Hom}_{A_1}(B \otimes_{A_2} M'^*, B \otimes_{A_2} M'^*), 1 \in \mathrm{Hom}_{A_2}(M'^*, M'^*))$$

which is the identity morphism $\mathbf{1} \in \mathrm{Hom}_A(\mathbf{T}(\mathbf{M})_0, \mathbf{T}(\mathbf{M})_0)$.

Dimension 1: By Proposition 7.2.2 the morphism $\mathbf{T}(\mathbf{e}(\mathbf{M}))_1 \in \mathrm{Hom}_A(\mathbf{T}(\mathbf{M})_1, \mathbf{T}^3(\mathbf{M})_1)$ is identified with the pair of morphisms

$$\left(\begin{pmatrix} \mu^* \\ 1 \end{pmatrix} \in \mathrm{Hom}_{A_1}(M^*, (B^* \otimes_{A_2} M'^*) \oplus M^*), 0 \in \mathrm{Hom}_{A_2}(0,0) \right)$$

and the morphism $\mathbf{e}(\mathbf{T}(\mathbf{M})) \in \mathrm{Hom}_A(\mathbf{T}^3(\mathbf{M})_1, \mathbf{T}(\mathbf{M})_1)$ is identified with the pair of morphisms

$$\left(\begin{pmatrix} 0 & 1 \end{pmatrix} \in \mathrm{Hom}_{A_1}((B \otimes_{A_2} M'^*) \oplus M^*), B \otimes_{A_2} M'^*), 0 \in \mathrm{Hom}_{A_2}(0,0) \right)$$

so that the composition $\mathbf{e}(\mathbf{T}(\mathbf{M}))\mathbf{T}(\mathbf{e}(\mathbf{M})) \in \mathrm{Hom}_A(\mathbf{T}(\mathbf{M})_1, \mathbf{T}(\mathbf{M})_1)$ is identified with the pair of morphisms

$$(1 \in \mathrm{Hom}_{A_1}(M^*, M^*), 0 \in \mathrm{Hom}_{A_2}(0,0))$$

which is the identity morphism $\mathbf{1} \in \mathrm{Hom}_A(\mathbf{T}(\mathbf{M})_1, \mathbf{T}(\mathbf{M})_1)$ as required. □

Theorem 7.2.7. The pair $(\mathbf{T}, \mathbf{e})$ defines a local chain duality on the additive category A-Mod.

Proof. By Propositions 7.2.1, 7.2.2, 7.2.5, 7.2.6. □

7.3 Symmetric complexes over a triangular matrix ring

Having established the existence of a local chain duality $(\mathbf{T}, \mathbf{e})$ on the additive category A-Mod we may now describe the ϵ-symmetric Q-groups of a triangular matrix ring $A = (A_1, A_2, B)$. We start by obtaining a description of the tensor products of objects $\mathbf{M} \otimes_A \mathbf{N}$ and of the ϵ-duality involution $\mathbf{T}_{\mathbf{M},\mathbf{N},\epsilon} : \mathbf{M} \otimes_A \mathbf{N} \to \mathbf{N} \otimes_A \mathbf{M}$ for A-modules $\mathbf{M}, \mathbf{N}$.

Lemma 7.3.1. Let $\mathbf{M} = (M, M', \mu), \mathbf{N} = (N, N', \nu)$ be A-modules. The tensor product $\mathbf{M} \otimes_A \mathbf{N}$ is the $\mathbb{Z}$-module chain complex concentrated in dimensions 0 and -1 with chain groups

$$(\mathbf{M} \otimes_A \mathbf{N})_0 = \mathrm{Hom}_A((B \otimes_{A_2} M'^*, M'^*, 1 : B \otimes_{A_2} M'^* \to B \otimes_{A_2} M'^*), (N, N', \nu : B \otimes_{A_2} N' \to N))$$
$$(\mathbf{M} \otimes_A \mathbf{N})_{-1} = \mathrm{Hom}_A((M^*, 0, 0), (N, N', \nu : B \otimes_{A_2} N' \to N))$$

and differential

$$\mathbf{d}_{\mathbf{M} \otimes_A \mathbf{N}} : (\mathbf{M} \otimes_A \mathbf{N})_0 \to (\mathbf{M} \otimes_A \mathbf{N})_{-1}$$

which sends a pair of morphisms

$$\mathbf{f} = (f = \nu(1 \otimes f') \in \mathrm{Hom}_{A1}(B \otimes_{A_2} M'^*, N), f' \in \mathrm{Hom}_{A_2}(M'^*, N')) \in (\mathbf{M} \otimes_A \mathbf{N})_0$$

to the pair of morphisms

$$\mathbf{d}_{\mathbf{M}\otimes_A\mathbf{N}}(\mathbf{f}) = (\nu(\beta^{-1} \otimes f')\mu^* \in \mathrm{Hom}_{A_1}(M^*, N), 0 \in \mathrm{Hom}_{A_2}(0, M')) \in (\mathbf{M} \otimes_A \mathbf{N})_{-1}.$$

Proof. Recall from Definition 6.2.4 that

$$(\mathbf{M} \otimes_A \mathbf{N})_r = \mathrm{Hom}_A(\mathbf{T}(\mathbf{M}), \mathbf{N})_r = \oplus_{p+q=r} \mathrm{Hom}_A(\mathbf{T}(\mathbf{M})_{-p}, \mathbf{N}_q) = \mathrm{Hom}_A(\mathbf{T}(\mathbf{M})_{-r}, \mathbf{N}).$$

Since $\mathbf{T}$ is a 1-dimensional chain duality it follows that $(\mathbf{M} \otimes_A \mathbf{N})_r$ is zero except for

$$(\mathbf{M} \otimes_A \mathbf{N})_0 = \mathrm{Hom}_A((B \otimes_{A_2} M'^*, M'^*, 1 : B \otimes_{A_2} M'^* \to B \otimes_{A_2} M'^*), (N, N', \nu : B \otimes_{A_2} N' \to N))$$
$$(\mathbf{M} \otimes_A \mathbf{N})_{-1} = \mathrm{Hom}_A(M^*, 0, 0), (N, N', \nu : B \otimes_{A_2} N' \to N))$$

The chain complex $\mathbf{T}(\mathbf{M})$ has differential

$$\mathbf{d}_{\mathbf{T}(\mathbf{M})} = ((\beta^{-1} \otimes 1)\mu^* \in \mathrm{Hom}_{A_1}(M^*, B \otimes_{A_2} M'^*), 0 \in \mathrm{Hom}_{A_2}(0, M'^*)))$$

and the differential of the chain complex $\mathbf{M} \otimes_A \mathbf{N}$ is given by the composition

$$\mathrm{Hom}_A(\mathbf{T}(\mathbf{M})_0, \mathbf{N}) \to \mathrm{Hom}_A(\mathbf{T}(\mathbf{M})_1, \mathbf{N}), \quad \mathbf{f} \mapsto \mathbf{f}\mathbf{d}_{\mathbf{T}(\mathbf{M})}$$

so that if

$$\mathbf{f} = (\nu(1 \otimes f') \in \mathrm{Hom}_{A_1}(B \otimes_{A_2} M'^*, N), f' \in \mathrm{Hom}_{A_2}(M'^*, N'))$$

then

$$\mathbf{d}_{\mathbf{M}\otimes_A\mathbf{N}}(\mathbf{f}) = (\nu(\beta^{-1} \otimes f')\mu^* \in \mathrm{Hom}_{A_1}(M^*, N), 0 \in \mathrm{Hom}_{A_2}(0, M'))$$

as required. □

Lemma 7.3.2. Let $\mathbf{M} = (M, M', \mu), \mathbf{N} = (N, N', \nu)$ be A-modules. The abelian group chain complex isomorphism

$$\mathbf{T}_{\mathbf{M},\mathbf{N},\epsilon} : \mathbf{M} \otimes_A \mathbf{N} \to \mathbf{N} \otimes_A \mathbf{M}$$

is zero in all dimensions apart from dimension 0 where it is given by

$$(\nu(1 \otimes f') \in \mathrm{Hom}_{A_1}(B \otimes_{A_2} M'^*, N), f' \in \mathrm{Hom}_{A_2}(M'^*, N')) \mapsto$$
$$(\epsilon\mu(1 \otimes f'^*) \in \mathrm{Hom}_{A_1}(B \otimes_{A_2} N'^*, M), \epsilon f'^* \in \mathrm{Hom}_{A_2}(N'^*, M'))$$

and in dimension -1 where it is given by

$$(f \in \mathrm{Hom}_{A_1}(M^*, N), 0 \in \mathrm{Hom}_{A_2}(0, N')) \mapsto$$
$$(\epsilon f^* \in \mathrm{Hom}_{A_1}(N^*, M), 0 \in \mathrm{Hom}_{A_2}(0, M'))$$

Proof. Recall from part (i) of Proposition 6.2.5 that the abelian group homomorphism $\mathbf{T}_{\mathbf{M},\mathbf{N},\epsilon}$:

$(\mathbf{M}\otimes_A \mathbf{N})_r \to (\mathbf{N}\otimes_A \mathbf{N})_r$ sends an element

$$\mathbf{f}\in(\mathbf{M}\otimes_A\mathbf{N})_r = \mathrm{Hom}_A(T(\mathbf{M}),\mathbf{N})_r = \mathrm{Hom}_A(T(\mathbf{M})_{-r},\mathbf{N})$$

to the composition

$$(\mathbf{T}(\mathbf{N})_{-r} \xrightarrow{\mathbf{T(f)}} \mathbf{T}(\mathbf{T}(\mathbf{M})_{-r})_{-r} \xrightarrow{\textbf{inclusion}} \mathbf{T}^2(\mathbf{M})_0 \xrightarrow{\epsilon\mathbf{e(M)}} \mathbf{M}_0 = \mathbf{M}) \in \mathrm{Hom}_A(\mathbf{T}(\mathbf{N})_{-r},\mathbf{M}).$$

By Lemma 7.3.1 it follows that $\mathbf{T}_{\mathbf{M},\mathbf{N},\epsilon} = \mathbf{0} : (\mathbf{M}\otimes_A\mathbf{N})_r \to (\mathbf{N}\otimes_A\mathbf{M})_r$ unless $r=0$ or $r=-1$.

Dimension 0**:** If $\mathbf{f}\in(\mathbf{M}\otimes_A\mathbf{N})_0$ is identified with

$$\mathbf{f} = (\nu(1\otimes f') \in \mathrm{Hom}_{A_1}(B\otimes_{A_2} M'^*, N), f' \in \mathrm{Hom}_{A_2}(M'^*, N'))$$

then $\mathbf{T(f)} : \mathbf{T(N)}_0 \to \mathbf{T}(\mathbf{T}(\mathbf{M})_0)_0$ is identified with

$$\mathbf{T(f)} = (1\otimes f'^* \in \mathrm{Hom}_{A_1}(B\otimes_{A_2} N'^*, B\otimes_{A_2} M'), f'^* \in \mathrm{Hom}_{A_2}(N'^*, M')).$$

The inclusion $\mathbf{T}(\mathbf{T}(\mathbf{M})_0)_0 \hookrightarrow \mathbf{T}^2(\mathbf{M})_0$ is identified with

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathrm{Hom}_{A_1}(B\otimes_{A_2} M', (B\otimes_{A_2} M')\oplus M), 1 \in \mathrm{Hom}_{A_2}(M',M')\right)$$

and $\mathbf{T}^2(\mathbf{M})_0 \to \mathbf{M}_0 = \mathbf{M}$ is identified with

$$\mathbf{e(M)} = (\begin{pmatrix} \mu & 1 \end{pmatrix} \in \mathrm{Hom}_{A_1}((B\otimes_{A_2} M')\oplus M, M), 1 \in \mathrm{Hom}_{A_2}(M',M'))$$

so the composition

$$\mathbf{T(M)}_0 \xrightarrow{\mathbf{T(f)}} \mathbf{T}(\mathbf{T}(\mathbf{M})_0)_0 \xrightarrow{\textbf{inclusion}} \mathbf{T}^2(\mathbf{M})_0 \xrightarrow{\epsilon\mathbf{e(M)}} \mathbf{M}_0 = \mathbf{M}$$

is identified with

$$(\epsilon\mu(1\otimes f'^*) \in \mathrm{Hom}_{A_1}(B\otimes_{A_2} N'^*, M), \epsilon f'^* \in \mathrm{Hom}_{A_2}(N'^*, M'))$$

Dimension -1**:** If $\mathbf{f}\in(\mathbf{M}\otimes_A\mathbf{N})_{-1}$ is identified with

$$\mathbf{f} = (f \in \mathrm{Hom}_{A_1}(M^*, N), 0 \in \mathrm{Hom}_{A_2}(0, N'))$$

then $\mathbf{T(f)} : \mathbf{T(N)}_1 \to \mathbf{T}(\mathbf{T}(\mathbf{M})_1)_1$ is identified with

$$\mathbf{T(f)} = (f^* \in \mathrm{Hom}_{A_1}(N^*, M), 0 \in \mathrm{Hom}_{A_2}(0,0))$$

and the inclusion $\mathbf{T}(\mathbf{T}(\mathbf{M})_1)_1 \hookrightarrow \mathbf{T}^2(\mathbf{M})_0$ is identified with

$$\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathrm{Hom}_{A_1}(M, (B\otimes_{A_2} M')\oplus M), 0 \in \mathrm{Hom}_{A_2}(0, M')\right)$$

and $\mathbf{e}(\mathbf{M}) : \mathbf{T}^2(\mathbf{M})_0 \to \mathbf{M}_0 = \mathbf{M}$ is identified with

$$\mathbf{e}(\mathbf{M}) = \Big(\begin{pmatrix} \mu & 1 \end{pmatrix} \in \mathrm{Hom}_{A_1}((B \otimes_{A_2} M') \oplus M, M), 1 \in \mathrm{Hom}_{A_2}(M', M') \Big)$$

so the composition

$$\mathbf{T}(\mathbf{N})_1 \xrightarrow{\mathbf{T}(\mathbf{f})} \mathbf{T}(\mathbf{T}(\mathbf{M})_1)_1 \xrightarrow{\textbf{inclusion}} \mathbf{T}^2(\mathbf{M})_0 \xrightarrow{\epsilon \mathbf{e}(\mathbf{M})} \mathbf{M}_0 = \mathbf{M}$$

is identified with

$$(\epsilon f^* \in \mathrm{Hom}_{A_1}(N^*, M), 0 \in \mathrm{Hom}_{A_2}(0, M'))$$

as required.

□

We may now describe the tensor product $\mathbf{C} \otimes_A \mathbf{C}$ and the ϵ-duality involution $\mathbf{T}_{\mathbf{C},\epsilon} : \mathbf{C} \otimes_A \mathbf{C} \to \mathbf{C} \otimes_A \mathbf{C}$ for an A-module chain complex $\mathbf{C}$.

Proposition 7.3.3. Let $\mathbf{C} = (C, C', \mu)$ be a finite-dimensional A-module chain complex. The $\mathbb{Z}$-module chain complex $\mathbf{C} \otimes_A \mathbf{C}$ is such that there is a one-to-one correspondence between chains $\mathbf{\Phi} \in (\mathbf{C} \otimes_A \mathbf{C})_n$ and collection of pairs of morphisms

$$(\delta\phi, \phi) = \{(\delta\phi_r \in \mathrm{Hom}_{A_1}(C^{n+1-r}, C_r), \phi_r \in \mathrm{Hom}_{A_2}(C'^{n-r}, C'_r)) | r \in \mathbb{Z}\}$$

such that each pair $(\delta\phi_r, \phi_r)$ determines a commutative diagram

$$\begin{array}{ccc} B \otimes_{A_2} C'^{n-r} & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & (B \otimes_{A_2} C'^{n-r}) \oplus C'^{n+1-r} \\ {\scriptstyle 1\otimes\phi_r}\downarrow & & \downarrow{\scriptstyle (\mu_r(1\otimes\phi_r) \quad \delta\phi_r)} \\ B \otimes_{A_2} C'_r & \xrightarrow[\mu_r]{} & C_r \end{array}$$

Proof. Proposition 7.2.2 implies that we may write

$$\mathbf{T}(\mathbf{C})_{r-n} = \left((B \otimes_{A_2} C'^{n-r}) \oplus C^{n+1-r}, C'^{n-r}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} : B \otimes_{A_2} C'^{n-r} \to (B \otimes_{A_2} C'^{n-r}) \oplus C^{n+1-r} \right)$$

and by Definition 6.2.4 the tensor product $(\mathbf{C} \otimes_A \mathbf{C})$ has chain groups

$$(\mathbf{C} \otimes_A \mathbf{C})_n = \mathrm{Hom}_A(\mathbf{T}(\mathbf{C}), \mathbf{C})_n = \oplus_r \mathrm{Hom}_A(\mathbf{T}(\mathbf{C})_{r-n}, \mathbf{C}_r).$$

It follows that a chain $\mathbf{\Phi} \in (\mathbf{C} \otimes_A \mathbf{C})_n$ can be identified with a collection of pairs

$$\Big\{\Big(\begin{pmatrix} \mu(1 \otimes \phi_r) & \delta\phi_r \end{pmatrix} \in \mathrm{Hom}_{A_1}((B \otimes_{A_2} C'^{n-r}) \oplus C^{n+1-r}, C_r), \phi_r \in \mathrm{Hom}_{A_2}(C'^{n-r}, C'_r) \Big) | r \in \mathbb{Z} \Big\}$$

with the data of each pair determining a commutative diagram

$$\begin{array}{ccc} B \otimes_{A_2} C'^{n-r} & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & (B \otimes_{A_2} C'^{n-r}) \oplus C'^{n+1-r} \\ {\scriptstyle 1\otimes\phi_r}\downarrow & & \downarrow{\scriptstyle (\mu_r(1\otimes\phi_r) \quad \delta\phi_r)} \\ B \otimes_{A_2} C'_r & \xrightarrow[\mu_r]{} & C_r \end{array} .$$

Such a pair

$$\Big(\begin{pmatrix} \mu(1\otimes\phi_r) & \delta\phi_r \end{pmatrix} \in \mathrm{Hom}_{A_1}((B\otimes_{A_2} C'^{n-r})\oplus C^{n+1-r}, C_r), \phi_r \in \mathrm{Hom}_{A_2}(C'^{n-r}, C'_r)\Big)$$

is uniquely determined by, and uniquely determines, a pair

$$(\delta\phi_r \in \mathrm{Hom}_{A_1}(C^{n+1-r}, C_r), \phi_r \in \mathrm{Hom}_{A_2}(C'^{n-r}, C'_r)).$$

□

We may then write the ϵ-duality involution on A-Mod in terms of the ϵ-duality involutions on A_1-Mod and A_2-Mod.

Corollary 7.3.4. The ϵ-duality involution $\mathbf{T}_{\mathbf{C},\epsilon} : \mathbf{C}\otimes_A \mathbf{C} \to \mathbf{C}\otimes_A \mathbf{C}$ is given by

$$\mathbf{T}_{\mathbf{C},\epsilon}(\delta\phi,\phi) = (T_{C,\epsilon}(\delta\phi), T_{C',\epsilon}(\phi)).$$

Proof. As Proposition 6.2.5 write

$$\mathbf{T}_{\mathbf{C},\epsilon} = \begin{pmatrix} \oplus_r(-)^{(n-r)r}T_{\mathbf{C}_{n-r},\mathbf{C}_r,\epsilon} & 0 \\ 0 & \oplus_r(-)^{(n+1-r)r}T_{\mathbf{C}_{n+1-r},\mathbf{C}_r,\epsilon} \end{pmatrix} : (\mathbf{C}\otimes_A \mathbf{C})_n \to (\mathbf{C}\otimes_A \mathbf{C})_n$$

where the domain is decomposed as

$$[\oplus_r(\mathbf{C}_{n-r}\otimes_A \mathbf{C}_r)_0]\oplus[\oplus_r(\mathbf{C}_{n+1-r}\otimes_A \mathbf{C}_r)_{-1}]$$

and the codomain is decomposed as

$$[\oplus_r(\mathbf{C}_r\otimes_A \mathbf{C}_{n-r})_0]\oplus[\oplus_r(\mathbf{C}_r\otimes_A \mathbf{C}_{n+1-r})_{-1}].$$

By Proposition 7.3.3 we may identify an element $\mathbf{\Phi} \in (\mathbf{C}\otimes_A \mathbf{C})_n$ with a collection of pairs

$$(\delta\phi,\phi) = \{(\delta\phi_r \in \mathrm{Hom}_{A_1}(C^{n+1-r}, C_r), \phi_r \in \mathrm{Hom}_{A_2}(C'^{n-r}, C'_r))| r\in\mathbb{Z}\}$$

so that

$$\begin{aligned} &\mathbf{T}_{\mathbf{C}_{n-r},\mathbf{C}_r,\epsilon}(\mu(1\otimes\phi_r)\in \mathrm{Hom}_{A_1}(B\otimes_{A_2} C'^{n-r}, C_r), \phi_r \in \mathrm{Hom}_{A_2}(C'^{n-r}, C'_r)) \\ &\quad =(\epsilon\mu(1\otimes\phi_r^*)\in \mathrm{Hom}_{A_1}(B\otimes_{A_2} C'^{r}, C_{n-r}), \epsilon\phi_r^* \in \mathrm{Hom}_{A_2}(C'^{r}, C'_{n-r}))\end{aligned}$$

and

$$\begin{aligned} &\mathbf{T}_{\mathbf{C}_{n+1-r},\mathbf{C}_r,\epsilon}(\delta\phi_r \in \mathrm{Hom}_{A_1}(C^{n+1-r}, C_r), 0 \in \mathrm{Hom}_{A_2}(0, C'_r)) \\ &\quad =(\epsilon\delta\phi_r^* \in \mathrm{Hom}_{A_1}(C^{r}, C_{n+1-r}), 0 \in \mathrm{Hom}_{A_2}(0, C'_{n+1-r})).\end{aligned}$$

Under the identifications of Proposition 7.3.3 it follows that $\mathbf{T}_{\mathbf{C},\epsilon}(\delta\phi,\phi) = (T_{C,\epsilon}(\delta\phi), T_{C',\epsilon}(\phi))$ as required. From now on we write $\mathbf{T}_\epsilon = \mathbf{T}_{\mathbf{C},\epsilon} = (T_{C,\epsilon}, T_{C',\epsilon}) = (T_\epsilon, T_\epsilon)$. □

The description of the ϵ-duality involution on A-Mod may then be used to determine the chain complex $W^{\%}_A\mathbf{C}$ and its differential.

Proposition 7.3.5. Let $\mathbf{C} = (C, C', \mu)$ be a finite A-module chain complex and let $\epsilon = \pm 1$. A chain $\mathbf{\Phi} \in (W^{\%}_A \mathbf{C})_n$ may be identified with a collection of pairs

$$(\delta\phi, \phi) = \{(\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r), \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r)) | s \geqslant 0, r \in \mathbb{Z}\}$$

such that if $\mathbf{\Phi}$ has differential the chain $\boldsymbol{\chi} \in (W^{\%}_A \mathbf{C})_{n-1}$ then $\boldsymbol{\chi}$ is identified with the collection of pairs

$$(\delta\chi, \chi) = \{\delta\phi'_s \in \mathrm{Hom}_{A_1}(C^{n-r+s}, C_r), \phi'_s \in \mathrm{Hom}_{A_2}(C'^{n-1-r+s}, C'_r) | s \geqslant 0, r \in \mathbb{Z}\}$$

where

$$\begin{aligned}\delta\chi_s &= d_C \delta\phi_s + (-)^{r+1}\delta\phi_s d_C^* + (-)^{n+s-1}(\delta\phi_{s-1} + (-)^s T_\epsilon(\delta\phi_{s-1})) + (-)^r \mu(\beta^{-1} \otimes \phi_s)\mu^* \\ \chi_s &= d_{C'}\phi_s + (-)^r \phi_s d_{C'}^* + (-)^{n+s-1}(\phi_{s-1} + (-)^s T_\epsilon(\phi_{s-1})).\end{aligned}$$

Proof. By Definition 6.3.1 the chain $\mathbf{\Phi} = \{\mathbf{\Phi}_s \in \mathrm{Hom}_A(\mathbf{C}^{n-r+s}, \mathbf{C}_r) | s \geqslant 0, r \in \mathbb{Z}\}$ has differential

$$\boldsymbol{\chi} = \{\mathbf{d_C}\mathbf{\Phi}_s + (-)^r \mathbf{\Phi}_s \mathbf{d}^*_{\mathbf{C}} + (-)^{n+s-1}(\mathbf{\Phi}_{s-1} + (-)^s \mathbf{T}_\epsilon(\mathbf{\Phi}_{s-1})) \in \mathrm{Hom}_A(\mathbf{C}^{n-1-r+s}, \mathbf{C}_r) | s \geqslant 0, r \in \mathbb{Z}\}.$$

From Proposition 7.3.3 we may identify

$$\mathbf{\Phi}_s = (\delta\phi_s, \phi_s) = \{(\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r), \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r)) | r \in \mathbb{Z}\}$$

where we have now dropped the r indices on $\delta\phi_s$ and ϕ_s. The composition $\mathbf{d_C}\mathbf{\Phi}_s \in \mathrm{Hom}_A(\mathbf{C}^{n-1-r+s}, \mathbf{C}_r)$ is then identified with

$$(d_C \delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r), d_{C'}\phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r)).$$

Note that

$$\begin{aligned}&\left(\begin{pmatrix} \mu(1\otimes\phi_s) & \delta\phi_s \end{pmatrix}, \phi_s\right)\left(\begin{pmatrix} 1 \otimes d^*_{C'} & (\beta^{-1}\otimes 1)\mu^* \\ 0 & -d^*_C \end{pmatrix}, d^*_{C'}\right) \\ =&\left(\begin{pmatrix} \mu(1\otimes\phi_s) & \delta\phi_s \end{pmatrix}\begin{pmatrix} 1 \otimes d^*_{C'} & (\beta^{-1}\otimes 1)\mu^* \\ 0 & -d^*_C \end{pmatrix}, \phi_s d^*_{C'}\right) \\ =&\left(\begin{pmatrix} \mu(1\otimes\phi_s d^*_{C'}) & \mu(\beta^{-1}\otimes\phi_s)\mu^* - \delta\phi_s d^*_C \end{pmatrix}, \phi_s d^*_{C'}\right)\end{aligned}$$

so by Proposition 7.2.2 the composition $\mathbf{\Phi}_s \mathbf{d}^*_{\mathbf{C}} \in \mathrm{Hom}_A(\mathbf{C}^{n-r+s-1}, \mathbf{C}_r)$ is identified with

$$\left(\mu(\beta^{-1}\otimes\phi_s)\mu^* - \delta\phi_s d^*_C \in \mathrm{Hom}_{A_1}(C^{n-r+s}, C_r), \phi_s d^*_{C'} \in \mathrm{Hom}_{A_2}(C'^{n-1-r+s}, C'_r)\right).$$

By Corollary 7.3.4 it follows that we may identify $\mathbf{\Phi}_{s-1} + (-)^s \mathbf{T}_\epsilon(\mathbf{\Phi}_{s-1}) \in \mathrm{Hom}_A(\mathbf{C}^{n-r+s-1}, \mathbf{C}_r)$ with

$$(\delta\phi_{s-1} + (-)^s T_\epsilon(\delta\phi_{s-1}) \in \mathrm{Hom}_{A_1}(C^{n-r+s}, C_r), \phi_{s-1} + (-)^s T_\epsilon(\phi_{s-1}) \in \mathrm{Hom}_{A_2}(C'^{n-1-r+s}, C'_r))$$

and hence we may identify

$$\mathbf{d_C}\mathbf{\Phi}_s + (-)^r\mathbf{\Phi}_s\mathbf{d}^*_{\mathbf{C}} + (-)^{n+s-1}(\mathbf{\Phi}_{s-1} + (-)^s\mathbf{T}_\epsilon(\mathbf{\Phi}_{s-1})) \in \mathrm{Hom}_A(\mathbf{C}^{n-r+s-1}, \mathbf{C}_r)$$

with

$$\{\delta\chi_s \in \mathrm{Hom}_{A_1}(C^{n-r+s}, C_r), \chi_s \in \mathrm{Hom}_{A_2}(C'^{n-1-r+s}, C'_r) | r \in \mathbb{Z}\}$$

where

$$\begin{aligned}\delta\chi_s &= d_C\delta\phi_s + (-)^{r+1}\delta\phi_s d^*_C + (-)^{n+s-1}(\delta\phi_{s-1} + (-)^s T_\epsilon(\delta\phi_{s-1})) + (-)^r\mu(\beta^{-1}\otimes\phi_s)\mu^*\\ \chi_s &= d_{C'}\phi_s + (-)^r\phi_s d^*_{C'} + (-)^{n+s-1}(\phi_{s-1} + (-)^s T_\epsilon(\phi_{s-1}))\end{aligned}$$

as required. □

We now work towards showing that each A-module chain complex $\mathbf{C} = (C, C', \mu)$ induces a long exact sequence of ϵ-symmetric Q-groups

$$\ldots \to Q^{n+1}_{A_2}(C', \epsilon) \xrightarrow{(\mu,\beta)^{\%}} Q^{n+1}_{A_1}(C, \epsilon) \to Q^n_A(\mathbf{C}, \epsilon) \to Q^n_{A_2}(C', \epsilon) \xrightarrow{(\mu,\beta)^{\%}} Q^n_{A_1}(C, \epsilon) \to \ldots$$

The key idea is to identify the ϵ-symmetric Q-groups $Q^n_A(\mathbf{C}, \epsilon)$ with the relative ϵ-symmetric Q-groups of a chain map induced by μ and β.

Proposition 7.3.6. A finite-dimensional A-module chain complex $\mathbf{C} = (C, C', \mu)$ induces a $\mathbb{Z}$-module chain map $(\mu, \beta)^{\%} = \mu^{\%}(\beta^{-1} \otimes -) : W^{\%}_{A_2}C' \to W^{\%}_{A_1}C$ such that there is an isomorphism of $\mathbb{Z}$-module chain complexes $W^{\%}_A(\mathbf{C}, \epsilon)_* \cong \mathscr{C}((\mu, \beta)^{\%})_{*+1}$.

Proof. If

$$\phi = \{\phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r) | s \geqslant 0, r \in \mathbb{Z}\} \in (W^{\%}_{A_2}C')_n$$

then define

$$\beta^{-1}\otimes\phi = \{\beta^{-1}\otimes\phi_s \in \mathrm{Hom}_{A_1}(B\otimes_{A_2}C'^{n-r+s}, B\otimes_{A_2}C'_r)s \geqslant 0, r \in \mathbb{Z}\} \in W^{\%}_{A_1}(B\otimes_{A_2}C').$$

Since the differential of the chain complex $B\otimes_{A_2}C'$ is given by $1\otimes d_{C'}$ it is clear that there is a commutative diagram

$$\begin{array}{ccc}(W^{\%}_{A_2}C')_n & \xrightarrow{\beta^{-1}\otimes-} & (W^{\%}_{A_1}(B\otimes_{A_2}C'))_n\\ \Big\downarrow d_{W^{\%}_{A_2}C'} & & \Big\downarrow d_{W^{\%}_{A_1}(B\otimes_{A_2}C')}\\ (W^{\%}_{A_2}C')_{n-1} & \xrightarrow[\beta^{-1}\otimes-]{} & (W^{\%}_{A_1}(B\otimes_{A_2}C'))_{n-1}\end{array}$$

so that $\beta^{-1}\otimes-$ defines a $\mathbb{Z}$-module chain map. Since $\mu^{\%}$ is a $\mathbb{Z}$-module chain map the composition $(\mu, \beta)^{\%}$ is a $\mathbb{Z}$-module chain map. By Proposition 7.3.5 an element $\mathbf{\Phi} \in (W^{\%}_A\mathbf{C})_n$ is identified with a collection

$$(\delta\phi, \phi) = \{(\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r), \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r)) | s \geqslant 0, r \in \mathbb{Z}\}$$

and there is a bijection

$$\theta_n : W_A^{\%}(\mathbf{C})_n \to \mathscr{C}((\mu,\beta)^{\%})_{n+1}$$
$$\{(\delta\phi_s,\phi_s)|s\geqslant 0, r\in\mathbb{Z}\} \mapsto \{((-)^{n+1-r}\delta\phi_s,\phi_s)|s\geqslant 0, r\in\mathbb{Z}\} \quad (n\geq 0).$$

Clearly θ_n is an isomorphism of $\mathbb{Z}$-modules so it is enough to show that the following diagram is commutative

$$\begin{array}{ccc} (W_A^{\%}\mathbf{C})_n & \xrightarrow{\theta_n} & \mathscr{C}((\mu,\beta)^{\%})_{n+1} \\ {\scriptstyle d_{W_A^{\%}\mathbf{C}}}\downarrow & & \downarrow{\scriptstyle d_{\mathscr{C}((\mu,\beta)^{\%})}} \\ (W_A^{\%}\mathbf{C})_{n-1} & \xrightarrow[\theta_{n-1}]{} & \mathscr{C}((\mu,\beta)^{\%})_n \end{array}$$

By Proposition 7.3.5 the differential $\boldsymbol{\chi} = d_{W_A^{\%}(\mathbf{C})}\boldsymbol{\Phi} \in (W_A^{\%}\mathbf{C})_{n-1}$ is identified with the collection

$$(\delta\chi,\chi) = \{\delta\chi_s \in \mathrm{Hom}_{A_1}(C^{n-r+s},C_r), \chi_s \in \mathrm{Hom}_{A_2}(C'^{n-1-r+s},C'_r)|s\geqslant 0, r\in\mathbb{Z}\}$$

where

$$\begin{aligned} \delta\chi_s &= d_C\delta\phi_s + (-)^{r+1}\delta\phi_s d_C^* + (-)^{n+s-1}(\delta\phi_{s-1} + (-)^s T_\epsilon(\delta\phi_{s-1})) + (-)^r\mu(\beta^{-1}\otimes\phi_s)\mu^* \\ \chi_s &= d_{C'}\phi_s + (-)^r\phi_s d_{C'}^* + (-)^{n+s-1}(\phi_{s-1} + (-)^s T_\epsilon(\phi_{s-1})). \end{aligned}$$

If $\boldsymbol{\psi} = \theta_{n-1}(\boldsymbol{\chi}) \in \mathscr{C}((\mu,\beta)^{\%})_n$ is identified by the collection

$$(\delta\psi,\psi) = \{\delta\psi_s \in \mathrm{Hom}_{A_1}(C^{n-r+s},C_r), \psi_s \in \mathrm{Hom}_{A_2}(C'^{n-1-r+s},C'_r)|s\geqslant 0, r\in\mathbb{Z}\}$$

then

$$\begin{aligned} \delta\psi_s &= (-)^{n-r}(d_C\delta\phi_s + (-)^{r+1}\delta\phi_s d_C^* + (-)^{n+s-1}(\delta\phi_{s-1} + (-)^s T_\epsilon(\delta\phi_{s-1})) + (-)^r\mu(\beta^{-1}\otimes\phi_s)\mu^*) \\ &= (-)^{n-r}d_C\delta\phi_s + (-)^{n+1}\delta\phi_s d_C^* + (-)^{s-r-1}(\delta\phi_{s-1} + (-)^s T_\epsilon(\delta\phi_{s-1})) + (-)^n\mu(\beta^{-1}\otimes\phi_s)\mu^* \\ \psi_s &= d_{C'}\phi_s + (-)^r\phi_s d_{C'}^* + (-)^{n+s-1}(\phi_{s-1} + (-)^s T_{C',\epsilon}(\phi_{s-1})). \end{aligned}$$

On the other hand, if $\boldsymbol{\upsilon} = \theta_n(\boldsymbol{\Phi}) \in \mathscr{C}((\mu,\beta)^{\%})_{n+1}$ then $\boldsymbol{\upsilon}$ is identified with the collection

$$(\delta\upsilon,\upsilon) = \{\delta\upsilon_s = (-)^{n+1-r}\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s},C_r), \upsilon_s = \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s},C'_r)|s\geqslant 0, r\in\mathbb{Z}\}.$$

The algebraic mapping cone $\mathscr{C}((\mu,\beta)^{\%})$ has differential

$$\begin{aligned} d_{\mathscr{C}((\mu,\beta)^{\%})} &= \begin{pmatrix} d_{W_{A_1}^{\%}(C)} & (-)^n(\mu,\beta)^{\%} \\ 0 & d_{W_{A_2}^{\%}(C')} \end{pmatrix} \\ &: \mathscr{C}((\mu,\beta)^{\%})_{n+1} = W_{A_1}^{\%}(C)_{n+1}\oplus W_{A_2}^{\%}(C') \to \mathscr{C}((\mu,\beta)^{\%})_n = W_{A_1}^{\%}(C)_n \oplus W_{A_2}^{\%}(C')_{n-1} \end{aligned}$$

so that if $\boldsymbol{\omega} = d_{\mathscr{C}((\mu,\beta)^{\%})}\boldsymbol{\upsilon}$ is identified with the collection

$$(\delta\omega,\omega) = \{\delta\omega_s \in \mathrm{Hom}_{A_1}(C^{n-r+s},C_r), \omega_s \in \mathrm{Hom}_{A_2}(C'^{n-1-r+s},C'_r)|s\geqslant 0, r\in\mathbb{Z}\}$$

then

$$\begin{aligned}\delta\omega_s &= d_{C'}\delta\nu_s + (-)^r\delta\nu_s d^*_{C'} + (-)^{n+s}(\delta\nu_{s-1} + (-)^s T_\epsilon(\delta\phi_{s-1})) + (-)^n\mu(\beta^{-1}\otimes\nu_s)\mu^* \\ &= (-)^{n-r}d_{C'}\delta\phi_s + (-)^{n+1}\delta\phi_s d^*_{C'} + (-)^{s+r+1}(\delta\nu_{s-1} + (-)^s T_\epsilon(\delta\phi_{s-1})) + (-)^n\mu(\beta^{-1}\otimes\phi_s)\mu^* \\ \omega_s &= d_{C'}\phi_s + (-)^r\phi_s d^*_{C'} + (-)^{n+s-1}(\phi_{s-1} + (-)^s T_{C',\epsilon}(\phi_{s-1})).\end{aligned}$$

It follows that $\boldsymbol{\psi} = \boldsymbol{\omega}$ and hence the diagram is commutative. □

Theorem 7.3.7. An A-module chain complex $\mathbf{C} = (C, C', \mu)$ induces a long exact sequence of ϵ-symmetric Q-groups

$$\ldots \to Q^{n+1}_{A_2}(C',\epsilon) \xrightarrow{(\mu,\beta)^\%} Q^{n+1}_{A_1}(C,\epsilon) \to Q^n_A(\mathbf{C},\epsilon) \to Q^n_{A_2}(C',\epsilon) \xrightarrow{(\mu,\beta)^\%} Q^n_{A_1}(C,\epsilon) \to \ldots$$

Proof. The $\mathbb{Z}$-module chain map $(\mu,\beta)^\% : W^\%_{A_2}C' \to W^\%_{A_1}C$ induces a short exact sequence of $\mathbb{Z}$-module chain complexes

$$0 \to (W^\%_{A_1}C)_* \to \mathscr{C}((\mu,\beta)^\%)_* \to (W^\%_{A_2}C')_{*-1} \to 0$$

which induces a long exact sequence of ϵ-symmetric Q-groups

$$\ldots \to Q^{n+1}_{A_2}(C',\epsilon) \xrightarrow{(\mu,\beta)^\%} Q^{n+1}_{A_1}(C,\epsilon) \to Q^{n+1}((\mu,\beta),\epsilon) \to Q^n_{A_2}(C',\epsilon) \xrightarrow{(\mu,\beta)^\%} Q^n_{A_1}(C,\epsilon) \to \ldots$$

The isomorphism of $\mathbb{Z}$-module chain complexes $W^\%_A(\mathbf{C},\epsilon)_* \cong \mathscr{C}((\mu,\beta)^\%)_{*+1}$ induces an isomorphism of homology groups

$$Q^n_A(\mathbf{C},\epsilon) = H_n(W^\%_A(\mathbf{C},\epsilon)) \cong H_{n+1}(\mathscr{C}((\mu,\beta)^\%)) = Q^{n+1}((\mu,\beta),\epsilon)$$

and hence there is a long exact sequence of ϵ-symmetric Q-groups

$$\ldots \to Q^{n+1}_{A_2}(C',\epsilon) \xrightarrow{(\mu,\beta)^\%} Q^{n+1}_{A_1}(C,\epsilon) \to Q^n_A(\mathbf{C},\epsilon) \to Q^n_{A_2}(C',\epsilon) \xrightarrow{(\mu,\beta)^\%} Q^n_{A_1}(C,\epsilon) \to \ldots$$

as required. □

Example 7.3.8.

(i) Let A_1 and A_2 be rings with involution determining the triangular matrix ring $A = (A_1, A_2, 0)$. A finite-dimensional A-module chain complex $\mathbf{C}$ is then the same as a pair (C, C') with C an A_1-module chain complex and C' an A_2-module chain complex. Since $\beta = 0$ it follows that $(\mu,\beta)^\% = 0 : W^\%_{A_2}C' \to W^\%_{A_1}C$ and hence

$$\mathscr{C}((\mu,\beta)^\%)_* = (W^\%_{A_1}C)_* \oplus (W^\%_{A_2}C')_{*-1}.$$

Proposition 7.3.6 implies that there is an isomorphism

$$Q^n_A(\mathbf{C},\epsilon) \cong Q^{n+1}_{A_1}(C,\epsilon) \oplus Q^n_{A_2}(C',\epsilon)$$

and by Theorem 7.3.7 there is a long exact sequence of Q-groups

$$\ldots \to Q^{n+1}_{A_2}(C',\epsilon) \xrightarrow{0} Q^{n+1}_{A_1}(C,\epsilon) \to Q^n_A(\mathbf{C},\epsilon) \cong Q^{n+1}_{A_1}(C,\epsilon) \oplus Q^n_{A_2}(C',\epsilon)$$
$$\to Q^n_{A_2}(C',\epsilon) \xrightarrow{0} Q^n_{A_1}(C,\epsilon) \to \ldots$$

(ii) Let R be a ring with involution determining the triangular matrix ring $A = (R,R,R)$ where the third copy of R with R is viewed as an (R,R)-bimodule in the standard way. Let $\mathbf{C} = (C,C',\mu)$ be a finite-dimensional A-module chain complex. The isomorphism of rings

$$R \to \mathrm{Hom}_R(R,R), \quad x \mapsto (y \mapsto xy)$$

gives an identification such that the map

$$\beta : R \times R \to R; \quad (x,y) \mapsto xy$$

has adjoint

$$\widehat{\beta} = 1 : R \to R = \mathrm{Hom}_R(R,R)$$

and hence by Proposition 7.3.6 there is an isomorphism of Q-groups $Q^n_A(\mathbf{C},\epsilon) \cong Q^{n+1}_R(\mu,\epsilon)$ and there is a long exact sequence of Q-groups

$$\ldots \to Q^{n+1}_R(C',\epsilon) \xrightarrow{\mu^\%} Q^{n+1}_R(C,\epsilon) \to Q^n_A(\mathbf{C},\epsilon) \cong Q^{n+1}_R(\mu,\epsilon) \to Q^n_R(C',\epsilon) \xrightarrow{\mu^\%} Q^n_R(C,\epsilon) \to \ldots$$

This recovers the standard long exact sequence of Q-groups associated to the chain map $\mu : C' \to C$ of R-module chain complexes from Proposition 1.2.4 from Part I.

We may now interpret a symmetric complex over A as a pair consisting of a symmetric complex over A_2 and a symmetric pair over A_1.

Theorem 7.3.9. Let $\mathbf{C} = (C,C',\mu)$ be a finite-dimensional A-module chain complex. An n-dimensional ϵ-symmetric structure

$$\mathbf{\Phi} = \{(\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r), \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r)) | s \geqslant 0, r \in \mathbb{Z}\} \in Q^n_A(\mathbf{C},\epsilon)$$

determines

(i) An n-dimensional ϵ-symmetric complex $(C', \phi \in Q^n_{A_2}(C',\epsilon))$ over A_2.

(ii) An $(n+1)$-dimensional ϵ-symmetric pair over A_1

$$(\mu : B \otimes_{A_2} C' \to C, (\delta\phi', \beta^{-1} \otimes \phi) \in Q^{n+1}(\mu,\epsilon))$$

where

$$\delta\phi'_s = (-)^{n+1-r}\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r).$$

Proof. This follows from the long exact sequence of Theorem 7.3.7 □

7.4 The Poincaré condition for a symmetric complex

Having described the ϵ-symmetric Q-groups of a triangular matrix ring $A = (A_1, A_2, B)$ we now describe the Poincaré condition for symmetric complexes over A.

Definition 7.4.1. An n-dimensional ϵ-symmetric complex $(\mathbf{C}, \mathbf{\Phi} \in Q^n_A(\mathbf{C}, \epsilon))$ over A is *Poincaré* if and only if $\mathbf{\Phi}_0 : \mathbf{C}^{n-*} \to \mathbf{C}$ is a local A-module chain equivalence.

Theorem 7.4.2. Let $\mathbf{C} = (C, C', \mu)$ be a n-dimensional A-module chain complex. An n-dimensional ϵ-symmetric structure

$$\mathbf{\Phi} = \{(\delta\phi_s \in \operatorname{Hom}_{A_1}(C^{n+1-r+s}, C_r), \phi_s \in \operatorname{Hom}_{A_2}(C'^{n-r+s}, C'_r))|s \geqslant 0, r \in \mathbb{Z}\} \in Q^n_A(\mathbf{C}, \epsilon)$$

is Poincaré if and only if both of the following conditions hold:

(i) (C', ϕ) is an n-dimensional ϵ-symmetric Poincaré complex over A_2.

(ii) $(\mu : B \otimes_{A_2} C' \to C, (\delta\phi', \beta^{-1} \otimes \phi))$ is an $(n+1)$-dimensional ϵ-symmetric Poincaré pair over A_1, where

$$\delta\phi'_s = (-)^{n+1-r}\delta\phi_s \in \operatorname{Hom}_{A_1}(C^{n+1-r+s}, C_r).$$

Proof. Definition 7.2.4 and Proposition 7.3.3 imply that $\mathbf{\Phi}$ is Poincaré if and only if both of the following conditions hold:

(1)$\phi_0 : C'^{n-*} \to C'$ is an A_2-module chain homotopy equivalence.

(1)$\begin{pmatrix} \mu(1 \otimes \phi_0) & \delta\phi_0 \end{pmatrix} : \mathscr{C}((\mu, \beta)^{\%})^{n+1-*} \to C$ is an A_1-module chain homotopy equivalence.

The commutative diagram of A_1-module chain maps

$$\begin{array}{ccc} B^* \otimes_{A_2} C' & \xrightarrow{\mu(\beta^{-1}\otimes 1)} & C \\ {\scriptstyle \beta^{-1}\otimes 1}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle 1}\big\downarrow{\scriptstyle \cong} \\ B \otimes_{A_2} C' & \xrightarrow[\mu]{} & C \end{array}$$

induces an isomorphism of algebraic mapping cones

$$\begin{pmatrix} \beta^{-1} \otimes 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathscr{C}((\mu, \beta)^{\%}) \to \mathscr{C}(\mu^{\%})$$

with dual an isomorphism of A_1-module chain complexes

$$\begin{pmatrix} \beta^{-1} \otimes 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathscr{C}(\mu^{\%})^{n+1-*} \to \mathscr{C}((\mu, \beta)^{\%})^{n+1-*}.$$

Hence the A_1-module chain map

$$(\mu(1 \otimes \phi_0), \delta\phi_0) : \mathscr{C}((\mu, \beta)^{\%})^{n+1-*} \to C$$

is a chain homotopy equivalence if and only if the A_1-module chain map

$$\begin{pmatrix} \mu(\beta^{-1} \otimes \phi_0) & \delta\phi_0 \end{pmatrix} : \mathscr{C}(\mu^{\%})^{n+1-*} \to C$$

is a chain homotopy equivalence. By Proposition 7.3.6 it follows that $\mathbf{\Phi}$ is Poincaré if and only if both of the following conditions hold:

(i) (C,ϕ) is an n-dimensional ϵ-symmetric Poincaré complex over A_2.

(ii) $(\mu : B \otimes_{A_2} C' \to C, (\delta\phi', \beta^{-1} \otimes \phi))$ is an $(n+1)$-dimensional ϵ-symmetric Poincaré pair over A_1.

□

Recall from Definition 6.1.4 that a chain complex in an additive category $\mathbb{A}$ is n-dimensional if and only it is chain homotopy equivalent to a strictly n-dimensional chain complex in $\mathbb{A}$. This implies that any n-dimensional chain complex can be viewed as an $(n+1)$-dimensional chain complex. In the context of Theorem 7.4.2 where $\mathbb{A} = A$-Mod, if $\mathbf{C} = (C, C', \mu)$ is an n-dimensional A-module chain complex then C is an n-dimensional chain complex over A_1 and C' is an n-dimensional chain complex over A_2. By the above remark we can view C as an $(n+1)$-dimensional chain complex over A_1 so that $(\mu : B \otimes_{A_2} C' \to C, (\delta\phi', \beta^{-1} \otimes \phi))$ is an $(n+1)$-dimensional symmetric pair over A_1. As such it is useful to introduce a weaker notion of dimensionality to allow C to be a chain complex which is $(n+1)$-dimensional but not necessarily n-dimensional.

Definition 7.4.3. A chain complex $\mathbf{C} = (C, C', \mu)$ over A is *locally n-dimensional* if $\mathbf{C}$ is an $(n+1)$-dimensional chain complex over A such that C' is an n-dimensional chain complex over A_2.

Theorem 7.4.4. Under the assumption that $\mathbf{C} = (C, C', \mu)$ is a locally n-dimensional A-module chain complex, Theorem 7.3.9 gives a one-to-one correspondence and Theorem 7.4.2 still holds.

Chapter 8

The L-theory of a triangular matrix ring: symmetric pairs and surgery

Using the results of chapter 2 and chapter 7 we now extend the description of ϵ-symmetric complexes over a triangular matrix ring $A = (A_1, A_2, B)$ to ϵ-symmetric pairs, cobordisms and surgery on ϵ-symmetric complexes over A. We show that a relative ϵ-symmetric (Poincaré) structure on an A-module chain map $\mathbf{f} = (f, f') : \mathbf{C} = (C, C', \mu), \to \mathbf{D} = (D, D', \nu)$ can be described in terms of an ϵ-symmetric (Poincaré) structure on the commutative A_1-module triad

$$\begin{array}{ccc} B \otimes_{A_2} C' & \xrightarrow{\mu} & C \\ {\scriptstyle 1\otimes f'}\downarrow & & \downarrow{\scriptstyle f} \\ B \otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

in such a way that an ϵ-symmetric cobordism over A can be viewed as a relative ϵ-symmetric cobordism over A_1. This determines a long exact sequence of L-groups

$$\ldots \to L^{n+1}(A_2, \epsilon) \to L^{n+1}(A_1, \epsilon) \to L^n(A, \epsilon) \to L^n(A_2, \epsilon) \to L^n(A_1, \epsilon) \to \ldots$$

We then describe the effect of a surgery on an ϵ-symmetric complex over A and examine the special case $A = (R, R, R)$ to define a relative algebraic surgery operation on an an ϵ-symmetric pair over R. This is an algebraic model for geometric half-surgeries on a manifold with boundary. We then show that every ϵ-symmetric relative cobordism over R is homotopy equivalent to a union of relative algebraic surgeries.

8.1 Symmetric pairs and cobordisms over a triangular matrix ring

The first step is to understand relative symmetric structures over A is to examine the morphism $\mathbf{f}^{\%} : W_A^{\%}(\mathbf{C}) \to W_A^{\%}(\mathbf{D})$ induced by a chain map $\mathbf{f} : \mathbf{C} \to \mathbf{D}$.

Proposition 8.1.1. Let $\mathbf{f} = (f, f') : \mathbf{C} = (C, C', \mu) \to \mathbf{D} = (D, D', \nu)$ be a chain map of finite dimensional A-module chain complexes and let $\epsilon = \pm 1$.

(i) The $\mathbb{Z}$-module chain complex map $\mathbf{f}^{\%} : W_A^{\%}(\mathbf{C}) \to W_A^{\%}(\mathbf{D})$ is given by

$$\mathbf{\Phi} = \{(\delta\phi_s, \phi_s) | s \geqslant 0, r \in \mathbb{Z}\} \mapsto \mathbf{f\Phi f}^* = \{(f\delta\phi_s f^*, f'\phi_s f'^*) | s \geqslant 0, r \in \mathbb{Z}\}.$$

(ii) A chain $(\mathbf{\Delta\Phi}, \mathbf{\Phi}) \in \mathscr{C}(\mathbf{f}^{\%})_{n+1}$ is represented by a collection of morphisms

$$\begin{aligned}\mathbf{\Delta\Phi} &= \{(\phi'_s \in \mathrm{Hom}_{A_1}(D^{n+2-r+s}, D_r), \delta\phi'_s \in \mathrm{Hom}_{A_2}(D'^{n+1-r+s}, D'_r)) | s \geqslant 0, r \in \mathbb{Z}\} \\ \mathbf{\Phi} &= \{(\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r), \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r)) | s \geqslant 0, r \in \mathbb{Z}\}\end{aligned}$$

and has differential $(\mathbf{\Delta\chi}, \mathbf{\chi}) \in \mathscr{C}(\mathbf{f}^{\%})_n$ represented by the collection of morphisms

$$\begin{aligned}\mathbf{\Delta\chi}' &= \{(\chi'_s \in \mathrm{Hom}_{A_1}(D^{n+1-r+s}, D_r), \delta\chi'_s \in \mathrm{Hom}_{A_2}(D'^{n-r+s}, D'_r)) | s \geqslant 0, r \in \mathbb{Z}\} \\ \mathbf{\chi}' &= \{(\delta\chi_s \in \mathrm{Hom}_{A_1}(C^{n-r+s}, C_r), \chi_s \in \mathrm{Hom}_{A_2}(C'^{n-1-r+s}, C'_r)) | s \geqslant 0, r \in \mathbb{Z}\}\end{aligned}$$

where

$$\begin{aligned}\chi'_s &= d_D\phi'_s + (-)^r\phi'_s d_D^* + (-)^{n+s}(\phi'_{s-1} + (-)^s T_\epsilon(\phi'_{s-1})) + (-)^n f\delta\phi_s f^* + (-)^{n+1+s}\nu(\beta^{-1} \otimes \delta\phi'_s)\nu^* \\ \delta\chi'_s &= d_{D'}\delta\phi'_s + (-)^r\delta\phi'_s d_{D'}^* + (-)^{n+s}(\delta\phi'_{s-1} + (-)^s T_\epsilon(\delta\phi'_{s-1})) + (-)^n f'\phi_s f'^* \\ \delta\chi_s &= d_C\delta\phi_s + (-)^r\delta\phi_s d_C^* + (-)^{n+s-1}(\delta\phi_{s-1} + (-)^s T_\epsilon(\delta\phi_{s-1})) + (-)^{n+s}\mu(\beta^{-1} \otimes \phi_s)\mu^* \\ \chi_s &= d_{C'}\phi_s + (-)^r\phi_s d_{C'}^* + (-)^{n+s-1}(\phi_{s-1} + (-)^s T_\epsilon(\phi_{s-1}))\end{aligned}$$

Proof.

(i) Proposition 7.3.3 implies that an element $\mathbf{\Phi} \in W_A^{\%}(\mathbf{C})_n$ can be identified with a collection of pairs

$$\{(\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r), \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r)) | s \geqslant 0, r \in \mathbb{Z}\}.$$

By Proposition 7.2.2 the map $\mathbf{f}^* : \mathbf{D}^{n-r+s} \to \mathbf{C}^{n-r+s}$ is then identified with

$$\left(\begin{array}{c}\begin{pmatrix} 1 \otimes f'^* & 0 \\ 0 & f^* \end{pmatrix} \in \mathrm{Hom}_{A_1}((B \otimes_{A_2} D'^{n-r+s}) \oplus D^{n+1-r+s}, (B \otimes_{A_2} C'^{n-r+s}) \oplus C^{n+1-r+s}) \\ f'^* \in \mathrm{Hom}_{A_2}(D'^{n-r+s}, C'^{n-r+s})\end{array}\right).$$

The composition $\mathbf{\Phi}_s\mathbf{f}^* \in \mathrm{Hom}_A(\mathbf{D}^{n-r+s}, \mathbf{C}_r)$ is then identified with the collection of pairs

$$\left(\begin{pmatrix} \mu(1 \otimes \phi_s) & \delta\phi_s \end{pmatrix}\begin{pmatrix} 1 \otimes f'^* & 0 \\ 0 & f^* \end{pmatrix}, \phi_s f'^*\right) = \left(\begin{pmatrix} \mu(1 \otimes \phi_s f'^*) & \delta\phi_s f^* \end{pmatrix}, \phi_s f'^*\right)$$

so that we may identify $\mathbf{\Phi}_s\mathbf{f}^*$ with the collection of pairs

$$\{(\delta\phi_s f^* \in \mathrm{Hom}_{A_1}(D^{n+1-r+s}, C_r), \phi_s f'^* \in \mathrm{Hom}_{A_2}(D'^{n-r+s}, C'_r)) | r \in \mathbb{Z}\}$$

and hence the composition $\mathbf{f}\Phi_s\mathbf{f}^*$ is then identified with the collection of pairs

$$\{(f\delta\phi_s f^* \in \mathrm{Hom}_{A_1}(D^{n+1-r+s}, D_r), f'\phi_s f'^* \in \mathrm{Hom}_{A_2}(D'^{n-r+s}, D'_r)) | r \in \mathbb{Z}\}$$

as required.

(ii) The chain map $\mathbf{f}^{\%}$ of $\mathbb{Z}$-module chain complexes has an algebraic mapping cone with differential

$$\mathbf{d}_{\mathscr{C}(\mathbf{f}^{\%})} = \begin{pmatrix} d_{W_A^{\%}(\mathbf{D})} & (-)^n \mathbf{f}^{\%} \\ 0 & d_{W_A^{\%}(\mathbf{C})} \end{pmatrix}$$
$$: \mathscr{C}(\mathbf{f}^{\%})_{n+1} = W_A^{\%}(\mathbf{D})_{n+1} \oplus W_A^{\%}(\mathbf{C})_n \to \mathscr{C}(\mathbf{f}^{\%})_n = W_A^{\%}(\mathbf{D})_n \oplus W_A^{\%}(\mathbf{C})_{n-1}.$$

The result then follows from part (i) and Proposition 7.3.5.

□

We now show that a relative ϵ-symmetric structure $(\mathbf{\Delta\Phi}, \mathbf{\Phi}) \in Q_A^{n+1}(\mathbf{f})$ determines an ϵ-symmetric structure on a commutative triad.

Theorem 8.1.2. An $(n+1)$-dimensional ϵ-symmetric pair $(\mathbf{f} : \mathbf{C} \to \mathbf{D}, (\mathbf{\Delta\Phi}, \mathbf{\Phi}) \in Q_A^{n+1}(\mathbf{f}, \epsilon))$ over A with

$$\mathbf{\Delta\Phi} = \{(\phi'_s \in \mathrm{Hom}_{A_1}(D^{n+2-r+s}, D_r), \delta\phi'_s \in \mathrm{Hom}_{A_2}(D'^{n+1-r+s}, D'_r)) | s \geqslant 0, r \in \mathbb{Z}\}$$
$$\mathbf{\Phi} = \{(\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r), \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r)) | s \geqslant 0, r \in \mathbb{Z}\}$$

determines

(i) An n-dimensional ϵ-symmetric complex (C', ϕ) over A_2.

(ii) An $(n+1)$-dimensional ϵ-symmetric pair $(\mu : B \otimes_{A_2} C' \to C, (\delta\phi'', \beta^{-1} \otimes \phi))$ over A_1.

(iii) An $(n+1)$-dimensional ϵ-symmetric pair $(f' : C' \to D', (\delta\phi', \phi))$ over A_2.

(iv) An $(n+2)$-dimensional ϵ-symmetric triad (Γ, Φ) over A_1 where

$$\Gamma = \begin{array}{ccc} B \otimes_{A_2} C' & \xrightarrow{\mu} & C \\ {\scriptstyle 1\otimes f'}\downarrow & & \downarrow{\scriptstyle f} \\ B \otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$
$$\Phi = (\phi', \beta^{-1} \otimes \delta\phi', \delta\phi'', \beta^{-1} \otimes \phi)$$

where

$$\delta\phi''_s = (-)^{n+1-r} \delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r)$$

Proof. Note that if $(\mathbf{f} : \mathbf{C} \to \mathbf{D}, (\mathbf{\Delta\Phi}, \mathbf{\Phi}) \in Q_A^{n+1}(\mathbf{f}, \epsilon))$ is an $(n+1)$-dimensional ϵ-symmetric pair then necessarily $(\mathbf{C}, \mathbf{\Phi} \in Q_A^n(\mathbf{C}, \epsilon))$ is an n-dimensional ϵ-symmetric complex so that

(i) Follows from Theorem 7.3.9.

(ii) Follows from Theorem 7.3.9.

(iii) Follows from part (ii) of Proposition 8.1.1.

(iv) Follows from part (ii) of Proposition 8.1.1.

□

As in the case of ϵ-symmetric complexes over A, it is useful to introduce a weaker notion of dimensionality for ϵ-symmetric pairs over A.

Definition 8.1.3. A *locally* $(n+1)$*-dimensional* ϵ*-symmetric pair* is an ϵ-symmetric pair $(\mathbf{f} : \mathbf{C} \to \mathbf{D}, (\mathbf{\Delta\Phi}, \mathbf{\Phi}) \in Q^{n+1}(\mathbf{f}, \epsilon))$ such that the chain complex $\mathbf{C} = (C, C', \mu)$ is locally n-dimensional and the chain complex $\mathbf{D} = (D, D', \nu)$ is locally $(n+1)$-dimensional.

The following is an analogue of Proposition 2.1.8 from Section 2.5 of Part I.

Theorem 8.1.4. Under the assumption that $(\mathbf{f} : \mathbf{C} \to \mathbf{D}, (\mathbf{\Delta\Phi}, \mathbf{\Phi}) \in Q_A^{n+1}(\mathbf{f}, \epsilon))$ is a locally $(n+1)$-dimensional ϵ-symmetric pair, Theorem 8.1.2 gives an one-to-one correspondence.

We now describe the long exact sequence for ϵ-symmetric pair structures over A and relate it to the long exact sequences for ϵ-symmetric triad structures.

Proposition 8.1.5. A chain map $\mathbf{f} = (f, f') : \mathbf{C} = (C, C', \mu) \to \mathbf{D} = (D, D', \nu)$ of finite dimensional A-module chain complexes determines:

(i) A commutative triad Γ of A_1-module chain complexes

$$\Gamma = \begin{array}{ccc} B \otimes_{A_2} C' & \xrightarrow{\mu} & C \\ {\scriptstyle 1\otimes f'}\downarrow & & \downarrow{\scriptstyle f} \\ B \otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

(ii) A commutative triad $\widehat{\Gamma}$ of $\mathbb{Z}$-module chain complexes

$$\widehat{\Gamma} = \begin{array}{ccc} W^{\%}_{A_2} C' & \xrightarrow{(\mu,\beta)^{\%}} & W^{\%}_{A_1} C \\ {\scriptstyle f'^{\%}}\downarrow & & \downarrow{\scriptstyle f^{\%}} \\ W^{\%}_{A_2} D' & \xrightarrow[(\nu,\beta)^{\%}]{} & W^{\%}_{A_1} D \end{array}$$

(iii) A $\mathbb{Z}$-module chain map

$$(f, f')^{\%} = \begin{pmatrix} f^{\%} & 0 \\ 0 & f'^{\%} \end{pmatrix} : \mathscr{C}((\mu, \beta)^{\%}) \to \mathscr{C}((\nu, \beta)^{\%})$$

given by

$$\{(\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r), \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r)) | s \geqslant 0, r \in \mathbb{Z}\} \mapsto$$
$$\{(f\delta\phi_s f^* \in \mathrm{Hom}_{A_1}(D^{n+1-r+s}, D_r), f'\phi_s f'^* \in \mathrm{Hom}_{A_2}(D'^{n-r+s}, D'_r)) | s \geqslant 0, r \in \mathbb{Z}\}.$$

(iv) A $\mathbb{Z}$-module chain map

$$(\mu, \nu, \beta)^{\%} = \begin{pmatrix} (\mu, \beta)^{\%} & 0 \\ 0 & (\nu, \beta)^{\%} \end{pmatrix} : \mathscr{C}(f'^{\%}) \to \mathscr{C}(f^{\%})$$

given by

$$\{(\delta\phi_s \in \mathrm{Hom}_{A_2}(D'^{n+1-r+s}, D'_r), \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r))|s \geqslant 0, r \in \mathbb{Z}\} \mapsto$$
$$\{(\nu(\beta^{-1} \otimes \delta\phi_s)\nu^* \in \mathrm{Hom}_{A_1}(D^{n+1-r+s}, D_r), \mu(\beta^{-1} \otimes \phi_s)\mu^* \in \mathrm{Hom}_{A_1}(C^{n-r+s}, C_r))|s \geqslant 0, r \in \mathbb{Z}\}.$$

(v) A commutative diagram of chain maps of $\mathbb{Z}$-module chain complexes

$$\begin{array}{ccc} W_A^{\%}(\mathbf{C})_* & \xrightarrow{\mathbf{f}^{\%}} & W_A^{\%}(\mathbf{D})_* \\ \theta_{\mathbf{C}}\downarrow & & \theta_{\mathbf{D}}\downarrow \\ \mathscr{C}((\mu,\beta)^{\%})_{*+1} & \xrightarrow[(f,f')^{\%}]{} & \mathscr{C}((\nu,\beta)^{\%})_{*+1} \end{array}$$

where the two vertical maps are isomorphisms.

(vi) A commutative diagram of ϵ-symmetric Q-groups

$$\begin{array}{ccc} Q_A^*(\mathbf{C},\epsilon) & \xrightarrow{\mathbf{f}^{\%}} & Q_A^*(\mathbf{D},\epsilon) \\ \theta_{\mathbf{C}}\downarrow & & \theta_{\mathbf{D}}\downarrow \\ Q^{*+1}((\mu,\beta),\epsilon) & \xrightarrow[(f,f')^{\%}]{} & Q^{*+1}((\nu,\beta),\epsilon). \end{array}$$

where the two vertical maps are isomorphisms.

(vii) An isomorphism of $\mathbb{Z}$-module chain complexes

$$\Theta = \begin{pmatrix} \theta_{\mathbf{D}} & 0 \\ 0 & \theta_{\mathbf{C}} \end{pmatrix} : \mathscr{C}(\mathbf{f}^{\%})_* \xrightarrow{\cong} \mathscr{C}((f,f')^{\%})_{*+1}.$$

(viii) An isomorphism $\Theta : Q_A^*(\mathbf{f},\epsilon) \cong H_{*+1}(\widehat{\Gamma})$.

Proof.

(i) Follows from part (iv) of Theorem 8.1.2.

(ii) Follows from Proposition 7.3.6.

(iii) Immediate from the commutativity of $\hat{\Gamma}$.

(iv) Immediate from the commutativity of $\hat{\Gamma}$.

(v) Follows from part (iii) and Propositions 7.3.6, 8.1.1.

(vi) Follows from part (v) by taking homology groups.

(vii) The chain map Θ of algebraic mapping cones is an isomorphism with inverse

$$\Theta^{-1} = \begin{pmatrix} \theta_{\mathbf{D}}^{-1} & 0 \\ 0 & \theta_{\mathbf{C}}^{-1} \end{pmatrix} : \mathscr{C}((f,f')^{\%}) \to \mathscr{C}(\mathbf{f}^{\%}).$$

(viii) Follows from (vii) and the definition

$$H_*(\hat{\Gamma}) = H_*((f^{\%}, f'^{\%}; 0)) = H_*(\mathscr{C}((f,f')^{\%})$$

of the homology of a symmetric triad from Definition 2.1.2 of Section

□

Corollary 8.1.6. A chain map $\mathbf{f} = (f, f') : \mathbf{C} = (C, C', \mu) \to \mathbf{D} = (D, D', \nu)$ of finite dimensional A-module chain complexes determines:

(i) A commutative diagram of short exact sequences of $\mathbb{Z}$-module chain complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W^{\%}_{A_1}(D)_* & \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & \mathscr{C}((\nu,\beta)^{\%})_* & \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & W^{\%}_{A_2}(D')_{*-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & & \downarrow{\scriptstyle\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & & \downarrow{\scriptstyle\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & & \\
0 & \longrightarrow & \mathscr{C}(f^{\%})_* & \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & \mathscr{C}((f,f')^{\%})_* & \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & \mathscr{C}(f'^{\%})_{*-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & & \downarrow{\scriptstyle\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} & & \downarrow{\scriptstyle\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & & \\
0 & \longrightarrow & W^{\%}_{A_1}(C)_{*-1} & \xrightarrow[\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)]{} & \mathscr{C}((\mu,\beta)^{\%})_{*-1} & \xrightarrow[\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)]{} & W^{\%}_{A_2}(C')_{*-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(ii) A commutative diagram of long exact sequences of $\mathbb{Z}$-modules

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & Q^{n+2}(\Gamma,\epsilon) & \longrightarrow & Q^{n+1}_{A_2}(f',\epsilon) & \xrightarrow{(\mu,\nu,\beta)^{\%}} & Q^{n+1}_{A_1}(f,\epsilon) & \longrightarrow & Q^{n+1}(\Gamma,\epsilon) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & Q^{n+1}(\mu,\beta,\epsilon) & \longrightarrow & Q^{n}_{A_2}(C',\epsilon) & \xrightarrow{(\mu,\beta)^{\%}} & Q^{n}_{A_1}(C,\epsilon) & \longrightarrow & Q^{n}(\mu,\beta,\epsilon) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle (f,f')^{\%}} & & \downarrow{\scriptstyle f'^{\%}} & & \downarrow{\scriptstyle f^{\%}} & & \downarrow{\scriptstyle (f,f')^{\%}} & & \\
\dots & \longrightarrow & Q^{n+1}(\nu,\beta,\epsilon) & \longrightarrow & Q^{n}_{A_2}(D',\epsilon) & \xrightarrow{(\nu,\beta)^{\%}} & Q^{n}_{A_1}(D,\epsilon) & \longrightarrow & Q^{n}(\nu,\beta,\epsilon)^{\%} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & Q^{n+1}(\Gamma,\epsilon) & \longrightarrow & Q^{n}_{A_2}(f',\epsilon) & \xrightarrow{(\mu,\nu,\beta)^{\%}} & Q^{n}_{A}(f,\epsilon) & \longrightarrow & Q^{n}(\Gamma,\epsilon) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

(iii) A commutative diagram of long exact sequences of $\mathbb{Z}$-module chain complexes

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & Q_A^{n+1}(\mathbf{f},\epsilon) & \longrightarrow & Q_{A_2}^{n+1}(f',\epsilon) & \xrightarrow{(\mu,\nu,\beta)^{\%}} & Q_{A_1}^{n+1}(f,\epsilon) & \longrightarrow & Q_A^{n}(\mathbf{f},\epsilon) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & Q_A^{n}(\mathbf{C},\epsilon) & \longrightarrow & Q_{A_2}^{n}(C',\epsilon) & \xrightarrow{(\mu,\beta)^{\%}} & Q_{A_1}^{n}(C,\epsilon) & \longrightarrow & Q_A^{n}(\mathbf{C},\epsilon) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \mathbf{f}^{\%}} & & \downarrow{\scriptstyle f'^{\%}} & & \downarrow{\scriptstyle f^{\%}} & & \downarrow{\scriptstyle \mathbf{f}^{\%}} & & \\
\dots & \longrightarrow & Q_A^{n}(\mathbf{D},\epsilon) & \longrightarrow & Q_{A_2}^{n}(D',\epsilon) & \xrightarrow{(\nu,\beta)^{\%}} & Q_{A_1}^{n}(D,\epsilon) & \longrightarrow & Q_A^{n}(\mathbf{D},\epsilon) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & Q_A^{n}(\mathbf{f},\epsilon) & \longrightarrow & Q_{A_2}^{n}(f',\epsilon) & \xrightarrow{(\mu,\nu,\beta)^{\%}} & Q_{A}^{n+1}(f,\epsilon) & \longrightarrow & Q_A^{n-1}(\mathbf{f},\epsilon) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Proof.

(i) The three vertical and three horizontal short exact sequences of chain complexes are those associated to algebraic mapping cones.

(ii) Follows by taking the homology groups in part (i).

(iii) Follows from part (ii) and the isomorphisms $Q_A^n(\mathbf{C}) \cong Q^{n+1}(\mu,\beta)$ and $Q_A^n(\mathbf{D}) \cong Q^{n+1}(\nu,\beta)$ from Proposition 7.3.6 and the isomorphism $Q_A^n(\mathbf{f}) \cong Q^{n+1}(\Gamma)$ from part (vii) of Proposition 8.1.5.

□

Example 8.1.7. As in Example 7.3.8 let R be a ring with involution determining the triangular matrix ring $A = (R,R,R)$ and let $\mathbf{f} = (f,f') : \mathbf{C} = (C,C',\mu : C' \to C) \to \mathbf{D} = (D,D',\nu : D' \to D)$ be a chain map of finite A-module chain complexes. The commutative triad of A-module chain complexes from part (i) of Proposition 8.1.5 reduces to the commutative triad of R-module chain complexes

$$\Gamma = \begin{array}{ccc} C' & \xrightarrow{\mu} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ D' & \xrightarrow[\nu]{} & D \end{array}$$

such that $Q^*(\Gamma) = H^*(\widehat{\Gamma})$ and the commutative diagram of exact sequence from part (ii) of

Corollary 8.1.6 reduces to the commutative diagram of exact sequences

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & Q^{n+2}(\Gamma,\epsilon) & \longrightarrow & Q^{n+1}_{A_2}(f',\epsilon) & \xrightarrow{(\mu,\nu,1)^{\%}} & Q^{n+1}_{A_1}(f,\epsilon) & \longrightarrow & Q^{n+1}(\Gamma,\epsilon) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & Q^{n+1}(\mu,\epsilon) & \longrightarrow & Q^{n}_{A_2}(C',\epsilon) & \xrightarrow{\mu^{\%}} & Q^{n}_{A_1}(C,\epsilon) & \longrightarrow & Q^{n}(\mu,\epsilon) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle (f,f')^{\%}} & & \downarrow{\scriptstyle f'^{\%}} & & \downarrow{\scriptstyle f^{\%}} & & \downarrow{\scriptstyle (f,f')^{\%}} & & \\
\ldots & \longrightarrow & Q^{n+1}(\nu,\epsilon) & \longrightarrow & Q^{n}_{A_2}(D',\epsilon) & \xrightarrow{\nu^{\%}} & Q^{n}_{A_1}(D,\epsilon) & \longrightarrow & Q^{n}(\nu,\epsilon) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & Q^{n+1}(\Gamma,\epsilon) & \longrightarrow & Q^{n}_{A_2}(f',\epsilon) & \xrightarrow{(\mu,\nu,1)^{\%}} & Q^{n}_{A}(f,\epsilon) & \longrightarrow & Q^{n}(\Gamma,\epsilon) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

This is the same diagram for the ϵ-symmetric Q-groups of a triad of R-module chain complexes as Proposition 2.1.3 from Section 1.5 of Part I.

8.2 The Poincaré condition for a symmetric pair

We now examine the Poincaré condition for an ϵ-symmetric pair and show that every ϵ-symmetric Poincaré pair determines a commutative ϵ-symmetric Poincaré triad.

Definition 8.2.1. Let $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ be a chain map of finite dimensional A-module chain complexes. A relative ϵ-symmetric structure $(\boldsymbol{\Delta\Phi}, \boldsymbol{\Phi}) \in Q^{n+1}_A(\mathbf{f}, \epsilon)$ is *Poincaré* if and only if

$$\begin{pmatrix} \mathbf{f}\boldsymbol{\Phi}_0 & \boldsymbol{\Delta\Phi}_0 \end{pmatrix} : \mathscr{C}(\mathbf{f})^{n+1-*} \to \mathbf{D}$$

is a local A-module chain equivalence.

We first need a technical lemma to compute the $(n+1)$-dual of a chain complex $\mathbf{C}$ in the sense of Definition 6.2.2.

Lemma 8.2.2. A chain map $\mathbf{f} = (f, f') : \mathbf{C} = (C, C', \mu) \to \mathbf{D} = (D, D', \nu)$ of finite dimensional A-module chain complexes determines:

(i) An A_1-module chain map

$$\begin{pmatrix} \nu & \mu \end{pmatrix} : B \otimes_{A_2} \mathscr{C}(f') \to \mathscr{C}(f)$$

given by

$$\begin{pmatrix} \nu & 0 \\ 0 & \mu \end{pmatrix} : (B \otimes_{A_2} D'_r) \oplus (B \otimes_{A_2} C'_{r-1}) \to D_r \oplus C'_{r-1} \quad (r \in \mathbb{Z})$$

(ii) An algebraic mapping cone

$$\mathscr{C}(\mathbf{f})_* = (\mathscr{C}(f)_*, \mathscr{C}(f')_*, \begin{pmatrix} \nu & \mu \end{pmatrix} : B \otimes_{A_2} \mathscr{C}(f')_* \to \mathscr{C}(f)_*)$$

which is a finite A-module chain complex with differential

$$\mathbf{d}_{\mathscr{C}(\mathbf{f})} : \mathscr{C}(\mathbf{f})_r \to \mathscr{C}(\mathbf{f})_{r-1}$$

given by

$$\left(\begin{array}{c} \mathscr{C}(f)_r \\ \downarrow d_{\mathscr{C}(f)} \\ \mathscr{C}(f)_{r-1} \end{array}, \begin{array}{c} \mathscr{C}(f')_r \\ \downarrow d_{\mathscr{C}(f')} \\ \mathscr{C}(f')_{r-1} \end{array}, \begin{array}{ccc} B \otimes_{A_2} \mathscr{C}(f')_r & \xrightarrow{\begin{pmatrix} \nu & \mu \end{pmatrix}} & \mathscr{C}(f)_r \\ \downarrow 1\otimes d_{\mathscr{C}(f')} & & \downarrow d_{\mathscr{C}(f)} \\ B \otimes_{A_2} \mathscr{C}(f')_{r-1} & \xrightarrow[\begin{pmatrix} \nu & \mu \end{pmatrix}]{} & \mathscr{C}(f)_{r-1} \end{array} \right)$$

(iii) An A_1-module chain map

$$\begin{pmatrix} \nu & \mu \end{pmatrix} (\beta^{-1} \otimes 1) : B^* \otimes_{A_2} \mathscr{C}(f') \to B \otimes_{A_2} \mathscr{C}(f') \to \mathscr{C}(f).$$

(iv) A dual algebraic mapping cone

$$\mathscr{C}(\mathbf{f})^{-*} = (\mathscr{C}(\begin{pmatrix} \nu & \mu \end{pmatrix}(\beta^{-1}\otimes 1))^{1-*}, \mathscr{C}(f')^{-*}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} : B\otimes_{A_2}\mathscr{C}(f')^{-*} \to \mathscr{C}(\begin{pmatrix} \nu & \mu \end{pmatrix}(\beta^{-1}\otimes 1))^{1-*})$$

which is a finite A-module chain complex with differential

$$\mathbf{d}_{\mathscr{C}(\mathbf{f})^{-*}} : \mathscr{C}(\mathbf{f})^{-r} \to \mathscr{C}(\mathbf{f})^{1-r}$$

given by

$$\left(\begin{array}{c} \mathscr{C}(\begin{pmatrix} \nu & \mu \end{pmatrix}(\beta^{-1}\otimes 1))^{1-r} \\ \downarrow d^*_{\mathscr{C}((\nu\ \mu)(\beta^{-1}\otimes 1))} \\ \mathscr{C}(\begin{pmatrix} \nu & \mu \end{pmatrix}(\beta^{-1}\otimes 1))^{2-r} \end{array}, \begin{array}{c} \mathscr{C}(f')^{-r} \\ \downarrow d^*_{\mathscr{C}(f')} \\ \mathscr{C}(f')^{1-r} \end{array}, \begin{array}{ccc} B \otimes_{A_2} \mathscr{C}(f')^{-r} & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & \mathscr{C}(\begin{pmatrix} \nu & \mu \end{pmatrix}(\beta^{-1}\otimes 1))^{1-r} \\ \downarrow 1\otimes d^*_{\mathscr{C}(f')} & & \downarrow d^*_{\mathscr{C}((\nu\ \mu)(\beta^{-1}\otimes 1))} \\ B \otimes_{A_2} \mathscr{C}(f')_{2-r} & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & \mathscr{C}(\begin{pmatrix} \nu & \mu \end{pmatrix}(\beta^{-1}\otimes 1))^{2-r} \end{array} \right)$$

Proof.

(i) By assumption $\mu : B \otimes_{A_2} C' \to C$ and $\nu : B \otimes_{A_2} D' \to D$ are A_1-module chain maps and so the commutative diagrams

$$\begin{array}{ccc} B \otimes_{A_2} C'_r & \xrightarrow{\mu} & C_r \\ {\scriptstyle 1\otimes d_{C'}}\downarrow & & \downarrow {\scriptstyle d_C} \\ B \otimes_{A_2} C'_{r-1} & \xrightarrow[\mu]{} & C_{r-1} \end{array} \qquad \begin{array}{ccc} B \otimes_{A_2} D'_r & \xrightarrow{\nu} & D_r \\ {\scriptstyle 1\otimes d_{D'}}\downarrow & & \downarrow {\scriptstyle d_D} \\ B \otimes_{A_2} D'_{r-1} & \xrightarrow[\nu]{} & D_{r-1} \end{array}$$

imply that there is a commutative diagram

$$\begin{array}{ccc}
B \otimes_{A_2} (D'_r \oplus C'_{r-1}) & \xrightarrow{\begin{pmatrix} \nu & 0 \\ 0 & \mu \end{pmatrix}} & D_r \oplus C_{r-1} \\
{\scriptstyle 1\otimes\begin{pmatrix} d_{D'} & (-)^{r-1}f' \\ 0 & d_{C'} \end{pmatrix}} \big\downarrow & & \big\downarrow {\scriptstyle \begin{pmatrix} d_D & (-)^{r-1}f \\ 0 & d_C \end{pmatrix}} \\
B \otimes_{A_2} (D'_{r-1} \oplus C'_{r-2}) & \xrightarrow[\begin{pmatrix} \nu & 0 \\ 0 & \mu \end{pmatrix}]{} & D_{r-1} \oplus C_{r-2}
\end{array}$$

which can be written as

$$\begin{array}{ccc}
B \otimes_{A_2} \mathscr{C}(f')_r & \xrightarrow{\begin{pmatrix} \nu & \mu \end{pmatrix}} & \mathscr{C}(f)_r \\
{\scriptstyle 1\otimes d_{\mathscr{C}(f')}} \big\downarrow & & \big\downarrow {\scriptstyle d_{\mathscr{C}(f')}} \\
B \otimes_{A_2} \mathscr{C}(f')_{r-1} & \xrightarrow[\begin{pmatrix} \nu & \mu \end{pmatrix}]{} & \mathscr{C}(f)_{r-1}
\end{array}$$

as required. Note that we are using $\begin{pmatrix} \nu & \mu \end{pmatrix}$ as a shorthand for $\begin{pmatrix} \nu & 0 \\ 0 & \mu \end{pmatrix}$.

(ii) By definition the algebraic mapping cone $\mathscr{C}(\mathbf{f})$ has chain groups

$$\begin{aligned}
\mathscr{C}(\mathbf{f})_r &= \mathbf{D}_r \oplus \mathbf{C}_{r-1} \\
&= (D_r, D'_r, \nu : B \otimes_{A_2} D'_r \to D_r) \oplus (C_r, C'_r, \mu : B \otimes_{A_2} C'_r \to C_r) \\
&= (D_r \oplus C_{r-1}, D'_r \oplus C'_{r-1}, \begin{pmatrix} \nu & \mu \end{pmatrix} : B \otimes_{A_2} (D'_r \oplus C'_{r-1} \to D_r \oplus C_{r-1}))
\end{aligned}$$

and differential

$$d_{\mathscr{C}(\mathbf{f})} = \begin{pmatrix} d_{\mathbf{D}} & (-)^{r-1}\mathbf{f} \\ 0 & d_{\mathbf{C}} \end{pmatrix} : \mathscr{C}(\mathbf{f})_r = \mathbf{D}_r \oplus \mathbf{C}_{r-1} \to \mathscr{C}(\mathbf{f})_{r-1} = \mathbf{D}_{r-1} \oplus \mathbf{C}_{r-2}$$

given by

$$\left(\begin{array}{c}
D_r \oplus C_{r-1} \\
\big\downarrow {\scriptstyle \begin{pmatrix} d_D & (-)^{r-1}f \\ 0 & d_C \end{pmatrix}} \\
D_{r-1} \oplus C_{r-2}
\end{array},
\begin{array}{c}
D'_r \oplus C'_{r-1} \\
\big\downarrow {\scriptstyle \begin{pmatrix} d_{D'} & (-)^{r-1}f' \\ 0 & d_{C'} \end{pmatrix}} \\
D'_{r-1} \oplus C'_{r-2}
\end{array},
\begin{array}{ccc}
B \otimes_{A_2} (D'_r \oplus C'_{r-1}) & \xrightarrow{\begin{pmatrix} \nu & \mu \end{pmatrix}} & D_r \oplus C_{r-1} \\
\big\downarrow {\scriptstyle 1\otimes\begin{pmatrix} d_D & (-)^{r-1}f \\ 0 & d_C \end{pmatrix}} & & \big\downarrow {\scriptstyle \begin{pmatrix} d_D & (-)^{r-1}f \\ 0 & d_C \end{pmatrix}} \\
B \otimes_{A_2} (D_r \oplus C_{r-1}) & \xrightarrow[\begin{pmatrix} \nu & \mu \end{pmatrix}]{} & D_{r-1} \oplus C_{r-2}
\end{array}\right)$$

which by part (ii) can be written as

$$\left(\begin{array}{c}
\mathscr{C}(f)_r \\
\big\downarrow {\scriptstyle d_{\mathscr{C}(f)}} \\
\mathscr{C}(f)_{r-1}
\end{array},
\begin{array}{c}
\mathscr{C}(f')_r \\
\big\downarrow {\scriptstyle d_{\mathscr{C}(f')}} \\
\mathscr{C}(f')_{r-1}
\end{array},
\begin{array}{ccc}
B \otimes_{A_2} \mathscr{C}(f')_r & \xrightarrow{\begin{pmatrix} \nu & \mu \end{pmatrix}} & \mathscr{C}(f)_r \\
\big\downarrow {\scriptstyle 1\otimes d_{\mathscr{C}(f')}} & & \big\downarrow {\scriptstyle d_{\mathscr{C}(f)}} \\
B \otimes_{A_2} \mathscr{C}(f)_r & \xrightarrow[\begin{pmatrix} \nu & \mu \end{pmatrix}]{} & \mathscr{C}(f)_{r-1}
\end{array}\right)$$

as required.

(iii) This is the composition of the A_1-module chain map $\beta^{-1} \otimes 1 : B^* \otimes_{A_2} C' \to B \otimes_{A_2} C'$ with the A_1-module chain map $\begin{pmatrix} \nu & \mu \end{pmatrix} : B \otimes_{A_2} \mathscr{C}(f') \to \mathscr{C}(f)$ from part (i).

(iv) The dual algebraic mapping cone is $\mathscr{C}(\mathbf{f})^{-*} = \mathbf{T}(\mathscr{C}(\mathbf{f}))_*$ where the chain duality $\mathbf{T}$ is as in Proposition 7.2.2. The rest follows from part (iii) and Proposition 7.2.2.

□

It then follows that every ϵ-symmetric Poincaré pair over A determines a commutative ϵ-symmetric Poincaré triad over A_1.

Theorem 8.2.3. An $(n+1)$-dimensional ϵ-symmetric pair $(\mathbf{f} : \mathbf{C} \to \mathbf{D}, (\mathbf{\Delta\Phi}, \mathbf{\Phi}))$ over A with

$$\begin{aligned}\mathbf{\Delta\Phi} &= \{(\phi'_s \in \mathrm{Hom}_{A_1}(D^{n+2-r+s}, D_r), \delta\phi'_s \in \mathrm{Hom}_{A_2}(D'^{n+1-r+s}, D'_r)) | s \geqslant 0, r \in \mathbb{Z}\} \\ \mathbf{\Phi} &= \{(\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r), \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s}, C'_r)) | s \geqslant 0, r \in \mathbb{Z}\}\end{aligned}$$

is Poincaré if and only if

(i) (C', ϕ) is an n-dimensional ϵ-symmetric Poincaré complex over A_2.

(ii) $(\mu : B \otimes_{A_2} C' \to C, (\delta\phi'', \beta^{-1} \otimes \phi))$ is an $(n+1)$-dimensional ϵ-symmetric Poincaré pair over A_1

(iii) $(f' : C' \to D', (\delta\phi', \phi))$ is an $(n+1)$-dimensional ϵ-symmetric Poincaré pair over A_2.

(iv) (Γ, Φ) is an $(n+2)$-dimensional ϵ-symmetric Poincaré triad over A_1 where

$$\Gamma = \begin{array}{ccc} B \otimes_{A_2} C' & \xrightarrow{\mu} & C \\ {\scriptstyle 1 \otimes f'} \big\downarrow & & \big\downarrow {\scriptstyle f} \\ B \otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

$$\Phi = (\phi', \beta^{-1} \otimes \delta\phi', \delta\phi'', \beta^{-1} \otimes \phi)$$

with

$$\begin{aligned}\delta\phi''_s &= (-)^{n+1-r}\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r) \\ \delta\phi'''_s &= (-)^{n+1-r}\delta\phi'_s \in \mathrm{Hom}_{A_2}(D'^{n+1-r+s}, D_r)\end{aligned}$$

Proof. Recall that a relative ϵ-symmetric structure $(\mathbf{\Delta\Phi}, \mathbf{\Phi})$ is Poincaré if and only if the A-module chain map $(\mathbf{\Delta\Phi}_0 \quad \mathbf{f\Phi}_0) : \mathscr{C}(\mathbf{f})^{n+1-*} \to \mathbf{D}$ is a local chain equivalence. In the case that $(\mathbf{f} : \mathbf{C} \to \mathbf{D}, (\mathbf{\Delta\Phi}, \mathbf{\Phi}))$ is Poincaré then necessarily $(\mathbf{C}, \mathbf{\Phi})$ is Poincaré. Theorem 7.4.2 implies that parts (i) and (ii) are equivalent to $(\mathbf{C}, \mathbf{\Phi})$ being Poincaré.

The chain groups and differential of the dual algebraic mapping cone $\mathscr{C}(\mathbf{f})^{n+1-*}$ were identified in part (iii) of Lemma 8.2.2. Note that $\mathbf{f\Phi}_0 : \mathbf{C}^{n-*} \to \mathbf{D}$ is identified with the collection

$$\{(f\delta\phi_0 \in \mathrm{Hom}_{A_1}(C^{n+1-r}, D_r), f'\phi_0 \in \mathrm{Hom}_{A_2}(C'^{n-r}, D'_r)) | r \in \mathbb{Z}\}$$

and $\mathbf{\Delta\Phi}_0 : \mathbf{D}^{n+1-*} \to \mathbf{D}$ is identified with the collection

$$\{(\delta\phi'_0 \in \mathrm{Hom}_{A_1}(D^{n+2-r}, D_r), \phi'_0 \in \mathrm{Hom}_{A_2}(D'^{n+1-r}, D'_r))|r \in \mathbb{Z}\}$$

so that $(\mathbf{\Delta\Phi}_0 \quad \mathbf{f\Phi}_0) : \mathscr{C}(\mathbf{f})^{n+1-*} \to \mathbf{D}$ is identified with the collection

$$\{(\delta\phi'_0 \quad f\delta\phi_0) \in \mathrm{Hom}_{A_1}(D^{n+2-r} \oplus C^{n+1-r} \to D_r), (\phi'_0 \quad f'\phi_0) \in \mathrm{Hom}_{A_2}(D'^{n+1-r} \oplus C'^{n-r} \to D'_r)|r \in \mathbb{Z}\}.$$

By Definition 7.2.4 and Proposition 7.3.3 it follows that $(\mathbf{\Delta\Phi}, \mathbf{\Phi}) \in Q^{n+1}_A(\mathbf{f})$ is Poincaré if and only if both of the following conditions hold:

(a)$\begin{pmatrix} \delta\phi'_0 & f'\phi_0 \end{pmatrix} : \mathscr{C}(f')^{n+1-*} \to D'$ is an A_2-module chain homotopy equivalence.

(a)$\begin{pmatrix} \nu(1 \otimes f'\phi_0) & \nu(1 \otimes \delta\phi'_0) & f\delta\phi_0 & \phi'_0 \end{pmatrix} : \mathscr{C}(\begin{pmatrix} \nu & \mu \end{pmatrix}(\beta^{-1} \otimes 1))^{n+2-*} \to D$ is an A_1-module chain homotopy equivalence.

It is clear (iii) is equivalent to (a). The commutative diagram of A_1-module chain maps

$$\begin{array}{ccc} B^* \otimes_{A_2} \mathscr{C}(f') & \xrightarrow{(\nu \ \mu)(\beta^{-1} \otimes 1)} & \mathscr{C}(f) \\ {\scriptstyle \beta^{-1}\otimes 1} \downarrow \cong & & {\scriptstyle 1} \downarrow \cong \\ B \otimes_{A_2} \mathscr{C}(f') & \xrightarrow[(\nu \ \mu)]{} & \mathscr{C}(f) \end{array}$$

induces an isomorphism of algebraic mapping cones

$$\begin{pmatrix} \beta^{-1} \otimes 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathscr{C}(\begin{pmatrix} \nu & \mu \end{pmatrix}(\beta^{-1} \otimes 1)) \to \mathscr{C}\begin{pmatrix} \nu & \mu \end{pmatrix}$$

with dual an isomorphism of dual algebraic mapping cones

$$\begin{pmatrix} \beta^{-1} \otimes 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathscr{C}\begin{pmatrix} \nu & \mu \end{pmatrix}^{n+2-*} \to \mathscr{C}\Big(\begin{pmatrix} \nu & \mu \end{pmatrix}(\beta^{-1} \otimes 1)\Big)^{n+2-*}$$

so that (b) is equivalent to

(b')$\begin{pmatrix} \delta\phi_0 & f\delta\phi_0 & \nu(\beta^{-1} \otimes f'\phi_0) & \nu(\beta^{-1} \otimes \phi_0) \end{pmatrix} : \mathscr{C}\begin{pmatrix} \nu & \mu \end{pmatrix}^{n+2-*} \to D$ is an A_1-module chain homotopy equivalence.

If parts (i), (ii) and (iii) hold then (iv) is equivalent to

(iv')$\begin{pmatrix} \delta\phi'_0 & f\delta\phi_0 & \nu(\beta^{-1} \otimes f'\phi_0) & \beta^{-1} \otimes \phi_0 \end{pmatrix} : \mathscr{C}\begin{pmatrix} f & 0 & \nu \end{pmatrix}^{n+2-*} \to D$ is an A_1-module chain homotopy equivalence.

and it is then enough to show that (iv$'$) is equivalent to (b$'$). Indeed, the chain map

$$\begin{pmatrix} f & 0 & -\nu \end{pmatrix} : C \cup_{B \otimes_{A_2} C'} (B \otimes_{A_2} D') \to D$$

has an algebraic mapping cone with chain groups

$$\begin{aligned} \mathscr{C}\begin{pmatrix} f & 0 & -\nu \end{pmatrix}_r &= D_r \oplus (C \cup_{B \otimes_{A_2} C'} (B \otimes_{A_2} D'))_{r-1} \\ &= D_r \oplus C_{r-1} \oplus (B \otimes_{A_2} C'_{r-2}) \oplus (B \otimes_{A_2} D'_{r-1}) \end{aligned}$$

and differential

$$d_{\mathscr{C}(f\ \ 0\ \ \nu)} = \begin{pmatrix} d_D & (-)^{r-1}f & 0 & (-)^{r-1}\nu \\ 0 & d_C & (-)^{r-1}\mu & (-)^r 1\otimes f' \\ 0 & 0 & 1\otimes d_{C'} & 0 \\ 0 & 0 & 0 & 1\otimes d_{D'} \end{pmatrix}$$
$$: D_r \oplus C_{r-1} \oplus (B\otimes_{A_2} C'_{r-2}) \oplus (B\otimes_{A_2} D'_{r-1}) \to D_{r-1}\oplus C_{r-2}\oplus (B\otimes_{A_2} C'_{r-3})\oplus (B\otimes_{A_2} D'_{r-2}).$$

On the other hand the chain map

$$\begin{pmatrix} \nu & \mu \end{pmatrix} : B\otimes_{A_2}\mathscr{C}(f') \to B\otimes_{A_2}\mathscr{C}(f) \to \mathscr{C}(f)$$

has an algebraic mapping cone with chain groups

$$\mathscr{C}\begin{pmatrix} \nu & \mu \end{pmatrix}_r = \mathscr{C}(f)_r \oplus (B\otimes_{A_2}\mathscr{C}(f')_{r-1}) = D_r \oplus C_{r-1}\oplus (B\otimes_{A_2} D'_{r-1})\oplus (B\otimes_{A_2} C'_{r-2})$$

and differential

$$d_{\mathscr{C}(\nu\ \ \mu)} = \begin{pmatrix} d_D & (-)^{r-1}f & (-)^{r-1}\nu & 0 \\ 0 & d_C & 0 & (-)^{r-1}\mu \\ 0 & 0 & 1\otimes d_{D'} & (-)^r 1\otimes f' \\ 0 & 0 & 0 & 1\otimes d_{C'} \end{pmatrix}$$
$$: D_r \oplus C_{r-1} \oplus (B\otimes_{A_2} D'_{r-1}) \oplus (B\otimes_{A_2} C'_{r-2}) \to D_{r-1}\oplus C_{r-2}\oplus (B\otimes_{A_2} D'_{r-2})\oplus (B\otimes_{A_2} C'_{r-3}).$$

and hence the algebraic mapping cones of $\begin{pmatrix} f & 0 & -\nu \end{pmatrix}$ and $\begin{pmatrix} \nu & \mu \end{pmatrix}$ are isomorphic so (iv′) is equivalent to (b′). □

Theorem 8.2.4. Theorem 8.2.3 holds under the assumption that $(\mathbf{f}:\mathbf{C}\to\mathbf{D},(\mathbf{\Delta\Phi},\mathbf{\Phi}))$ is a locally $(n+1)$-dimensional ϵ-symmetric pair over A.

It then follows that every ϵ-symmetric cobordism over A determines a commutative ϵ-symmetric relative cobordism over A_1.

Theorem 8.2.5. A $(n+1)$-dimensional ϵ-symmetric cobordism $((\mathbf{f}\ \mathbf{f}'):\mathbf{C}\oplus\mathbf{C}'\to\mathbf{D},(\mathbf{\Delta\Phi},\mathbf{\Phi}\oplus -\mathbf{\Phi}'))$ over A with

$$\begin{aligned}
\mathbf{C} &= (C, C', \mu)\\
\mathbf{C}' &= (C'', C''', \mu')\\
\mathbf{D} &= (D, D', \nu)\\
\mathbf{f} &= (f\in \mathrm{Hom}_{A_1}(C,D), f'\in\mathrm{Hom}_{A_2}(C',D'))\\
\mathbf{f}' &= (f''\in \mathrm{Hom}_{A_1}(C'',D''), f'''\in\mathrm{Hom}_{A_2}(C''',D'''))\\
\mathbf{\Phi} &= \{(\delta\phi_s\in\mathrm{Hom}_{A_1}(C^{n+1-r+s},C_r),\phi_s\in\mathrm{Hom}_{A_2}(C'^{n-r+s},C'_r))\,|\,s\geqslant 0, r\in\mathbb{Z}\}\\
\mathbf{\Phi}' &= \{(\delta\phi'_s\in\mathrm{Hom}_{A_1}(C''^{n+1-r+s},C''_r),\phi'_s\in\mathrm{Hom}_{A_2}(C'''^{n-r+s},C'''_r))\,|\,s\geqslant 0, r\in\mathbb{Z}\}\\
\mathbf{\Delta\Phi} &= \{(\phi''_s\in\mathrm{Hom}_{A_1}(D^{n+2-r+s},D_r),\delta\phi''_s\in\mathrm{Hom}_{A_2}(D'^{n+1-r+s},D'_r))\,|\,s\geqslant 0, r\in\mathbb{Z}\}
\end{aligned}$$

determines an $(n+2)$-dimensional ϵ-symmetric Poincaré triad (Γ,Φ) over A_1 with

$$\Gamma = \begin{array}{ccc} B\otimes_{A_2}(C'\oplus C''') & \xrightarrow{(\mu\ \ \mu')} & C\oplus C'' \\ {\scriptstyle 1\otimes(f'\ \ f''')}\big\downarrow & & \big\downarrow{\scriptstyle (f\ \ f'')} \\ B\otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

$$\Phi = (\phi'', \beta^{-1}\otimes\delta\phi'', \delta\omega\oplus -\delta\omega', \beta^{-1}\otimes(\phi\oplus -\phi'))$$

where

$$\delta\omega_s = (-)^{n+1-r}\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s}, C_r)$$
$$\delta\omega'_s = (-)^{n+1-r}\delta\phi'_s \in \mathrm{Hom}_{A_2}(C''^{n+1-r+s}, C''_r)$$

such that (Γ,Φ) can be viewed as a relative ϵ-cobordism

$$\Gamma = \begin{array}{ccc} (B\otimes_{A_2} C')\oplus(B\otimes_{A_2} C''') & \xrightarrow{\begin{pmatrix}\mu & 0\\ 0 & \mu'\end{pmatrix}} & C\oplus C'' \\ {\scriptstyle (1\otimes f'\ \ 1\otimes f''')}\big\downarrow & & \big\downarrow{\scriptstyle (f\ \ f'')} \\ B\otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

$$\Phi = (\phi'', \beta^{-1}\otimes\delta\phi'', \delta\omega\oplus -\delta\omega', \beta^{-1}\otimes(\phi\oplus -\phi'))$$

between the $(n+1)$-dimensional ϵ-symmetric Poincaré pairs $(\mu : B\otimes_{A_2} C' \to C, (\delta\omega, \beta^{-1}\otimes\phi))$ and $(\mu' : B\otimes_{A_2} C''' \to C'', (\delta\omega', \beta^{-1}\otimes\phi))$ over A_1.

Proof. The statements about the symmetry conditions follow from Theorem 8.1.2 and the statements about the Poincaré conditions follow from Theorem 7.3.9. □

Theorem 8.2.6. Theorem 8.2.5 holds under the assumption that $((\mathbf{f}\ \mathbf{f}') : \mathbf{C}\oplus\mathbf{C}' \to \mathbf{D}, (\mathbf{\Delta\Phi}, \mathbf{\Phi}\oplus -\mathbf{\Phi}'))$ is a locally $(n+1)$-dimensional ϵ-symmetric pair over A.

We may now use the description of cobordisms to obtain a long exact sequence of ϵ-symmetric L-groups.

Definition 8.2.7. The *n-dimensional ϵ-symmetric L-group* of a triangular matrix ring $A = (A_1, A_2, B)$ is the abelian group $L^n(A,\epsilon)$ of cobordism classes of locally n-dimensional chain complexes $\mathbf{C} = (C, C', \mu)$ over A.

Theorem 8.2.8. For a triangular matrix ring $A = (A_1, A_2, B)$ there is a long exact sequence of L-groups

$$\ldots \to L^{n+1}(A_2,\epsilon) \to L^{n+1}(A_1,\epsilon) \to L^n(A,\epsilon) \to L^n(A_2,\epsilon) \to L^n(A_1,\epsilon) \to \ldots$$

such that an element in $L^n(A,\epsilon)$ is a pair

$$((C', \phi\in Q^n_{A_2}(C',\epsilon)), (\mu : B\otimes_{A_2} C' \to C, (\delta\phi, \beta^{-1}\otimes\phi) \in Q^{n+1}_{A_1}(\mu,\epsilon)))$$

consisting of an n-dimensional ϵ-symmetric Poincaré pair (C',ϕ) over A_2 and an $(n+1)$-dimensional ϵ-symmetric Poincaré pair $(\mu : B\otimes_{A_2} C' \to C, (\delta\phi, \beta^{-1}\otimes\phi))$ over A_1 subject to the equivalence relation

$$((C',\phi),(\mu : B\otimes_{A_2} C' \to C,(\delta\phi,\beta^{-1}\otimes\phi))) \sim ((C''',\phi'),(\mu' : B\otimes_{A_2} C''' \to C'',(\delta\phi',\beta^{-1}\otimes\phi')))$$

if and only if there exists an $(n+1)$-dimensional ϵ-symmetric cobordism over A_2 of the form

$$((f' \;\; f''') : C'\oplus C''' \to D', (\delta\phi'', \phi\oplus -\phi'))$$

and an $(n+2)$-dimensional ϵ-symmetric Poincaré triad (Γ,Φ) of the form

$$\Gamma = \begin{array}{ccc} B\otimes_{A_2}(C'\oplus C''') & \xrightarrow{(\mu \;\; \mu')} & C\oplus C'' \\ {\scriptstyle 1\otimes(f' \;\; f''')}\downarrow & & \downarrow{\scriptstyle (f \;\; f'')} \\ B\otimes_{A_2} D' & \xrightarrow[\nu]{} & D \end{array}$$

$$\Phi = (\phi'', \beta^{-1}\otimes\delta\phi'', \delta\phi\oplus -\delta\phi', \beta^{-1}\otimes(\phi\oplus -\phi'))$$

Proof. The long exact sequence follows from Theorems 7.3.7, 7.4.2 and Theorem 8.2.5 and the equivalence relation follows from Theorem 8.2.3. □

Example 8.2.9.

(i) When $(A_1,A_2,B) = (B,A_2,B)$ and $\beta = 1 : B \to B \cong B^*$ then the long exact sequence of ϵ-symmetric Q-groups

$$\ldots \to Q^{n+1}_{A_2}(C',\epsilon) \xrightarrow{\mu^{\%}} Q^{n+1}_{B}(C,\epsilon) \to Q^{n+1}(\mu,\epsilon) \to Q^{n}_{A_2}(C',\epsilon) \xrightarrow{\mu^{\%}} Q^{n}_{B}(C,\epsilon) \to \ldots$$

induces the long exact sequence of L-groups

$$\ldots \to L^{n+1}(A_2,\epsilon) \xrightarrow{\mu^{\%}} L^{n+1}(B,\epsilon) \to L^{n+1}(\mu,\epsilon) \to L^{n}(A_2,\epsilon) \xrightarrow{\mu^{\%}} L^{n}(B,\epsilon) \to \ldots$$

This recovers the long exact sequence of ϵ-symmetric Q-groups associated to the change of rings morphism $\mu : A_2 \to B, a_2 \mapsto 1.a_2$ from [Ran81, Proposition 2.2]. The description of cobordisms over A from Theorem 8.2.5 is a generalisation of the equivalence relation [Ran81, p.123] used to define the relative L-groups of the change of rings morphism $\mu : A_2 \to B$.

(ii) The long exact sequence of ϵ-symmetric Q-groups from Example 7.3.8 (i) determines the long exact sequence of L-groups

$$\ldots \to L^{n+1}(A_2,\epsilon) \xrightarrow{0} L^{n+1}(A_1,\epsilon) \to L^{n}(A,\epsilon) \to L^{n}(A_2,\epsilon) \xrightarrow{0} L^{n}(A_1,\epsilon) \to \ldots$$

with an isomorphism of L-groups $L^n(A,\epsilon) \cong L^{n+1}(A_1,\epsilon)\oplus L^n(A_2,\epsilon)$.

8.3 Algebraic surgery over a triangular matrix ring

Having established a description of symmetric complexes, pairs and cobordisms over A we may now describe the effect of algebraic surgery over A. We will be particularly interested in the special case $A = (R, R, R)$ to obtain a definition of algebraic surgery on an ϵ-symmetric Poincaré pair $(f : C \to D, (\delta\phi, \phi))$ over R with data an ϵ-symmetric triad over R. This will provide an algebraic model for geometric half-surgeries on a manifold with boundary.

Proposition 8.3.1. The effect of surgery on an n-dimensional ϵ-symmetric Poincaré complex $(\mathbf{C},\mathbf{\Phi})$ over A with data an $(n+1)$-dimensional ϵ-symmetric pair $(\mathbf{f}:\mathbf{C}\to\mathbf{D},(\mathbf{\Delta\Phi},\mathbf{\Phi}))$ over A with

$$\begin{aligned}
\mathbf{f} &= (f \in \mathrm{Hom}_{A_1}(C,D), f' \in \mathrm{Hom}_{A_2}(C',D')) : \mathbf{C} = (C,C',\mu) \to \mathbf{D} = (D,D',\nu) \\
\mathbf{\Phi} &= \{(\delta\phi_s \in \mathrm{Hom}_{A_1}(C^{n+1-r+s},C_r), \phi_s \in \mathrm{Hom}_{A_2}(C'^{n-r+s},C'_r))|s \geqslant 0, r \in \mathbb{Z}\} \\
\mathbf{\Delta\Phi} &= \{(\phi'_s \in \mathrm{Hom}_{A_1}(D^{n+2-r+s},D_r), \delta\phi'_s \in \mathrm{Hom}_{A_2}(D'^{n+1-r+s},D'_r))|s \geqslant 0, r \in \mathbb{Z}\}
\end{aligned}$$

is an n-dimensional ϵ-symmetric Poincaré complex $(\mathbf{C}',\mathbf{\Phi}')$ over A with

$$\begin{aligned}
\mathbf{C}' &= (C'',C''',\mu') \\
\mathbf{\Phi}' &= \{(\delta\phi''_s \in \mathrm{Hom}_{A_1}(C''^{n+1-r+s},C''_r), \phi''_s \in \mathrm{Hom}_{A_2}(C'''^{n-r+s},C'''_r))|s \geqslant 0, r \in \mathbb{Z}\}
\end{aligned}$$

where

$$\begin{aligned}
C''_r &= C_r \oplus D_{r+1} \oplus (B \otimes_{A_2} D'^{n+1-r}) \oplus D^{n+2-r} \\
C'''_r &= C'_r \oplus D'_{r+1} \oplus D'^{n+1-r} \\
\mu' &= \begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} : B \otimes_{A_2} (C'_r \oplus D'_{r+1} \oplus D'^{n+1-r}) \to C_r \oplus D_{r+1} \oplus (B \otimes_{A_2} D'^{n+1-r}) \oplus D^{n+2-r} \\
\delta\phi''_0 &= \begin{pmatrix} \mu(1\otimes\phi_0) & \delta\phi_0 & 0 & 0 & 0 & 0 & 0 \\ (-)^{n-r} f\mu(1\otimes T_\epsilon(\phi_1)) & (-)^{n-r} fT_\epsilon(\delta\phi_1) & (-)^{n-r}\nu(1\otimes T_\epsilon(\delta\phi'_1)) & (-)^{n-r}T_\epsilon(\phi'_1) & (-)^{r(n-r)}\mu & (-)^{r(n-r)+1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\delta\phi''_s = \begin{pmatrix} \mu(1\otimes\phi_s) & \delta\phi_s & 0 & 0 & 0 & 0 & 0 \\ (-)^{n-r} f\mu(1\otimes T_\epsilon(\phi_{s+1})) & (-)^{n-r} fT_\epsilon(\delta\phi_{s+1}) & (-)^{n-r+s}\nu(1\otimes T_\epsilon(\delta\phi'_{s+1})) & (-)^{n-r+s}T_\epsilon(\phi'_{s+1}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (s \geqslant 1)$$

$$\phi''_0 = \begin{pmatrix} \phi_0 & 0 & 0 \\ (-)^{n-r} f'T_\epsilon(\phi_1) & (-)^{n-r} f'T_\epsilon(\delta\phi'_1) & (-)^{r(n-r)} \\ 0 & 1 & 0 \end{pmatrix}$$

$$\phi''_s = \begin{pmatrix} \phi_s & 0 & 0 \\ (-)^{n-r} f'T_\epsilon(\phi_{s+1}) & (-)^{n-r+s}T_\epsilon(\delta\phi'_{s+1}) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (s \geqslant 1).$$

Proof. By the proof of Proposition 7.2.2 we may write

$$\mathbf{D}^{n+1-r} = \left((B\otimes_{A_2} D'^{n+1-r})\oplus D^{n+2-r}, D'^{n+1-r}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} : B\otimes_{A_2} D'^{n+1-r} \to (B\otimes_{A_2} D'^{n+1-r})\oplus D^{n+2-r}\right)$$

and by Definition 6.5.1 the chain complex $\mathbf{C}'$ has chain groups

$$\begin{aligned} \mathbf{C}'_r &= \mathbf{C}_r \oplus \mathbf{D}_{r+1} \oplus \mathbf{D}^{n+1-r} \\ &= (C_r \oplus D_{r+1} \oplus (B\otimes_{A_2} D'^{n+1-r})\oplus D^{n+2-r}, C'_r \oplus D'_{r+1} \oplus D'^{n+1-r}, \mu') \\ &= (C''_r, C'''_r, \mu') \end{aligned}$$

where

$$\mu' = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} : B\otimes_{A_2} (C'_r \oplus D'_{r+1} \oplus D'^{n+1-r}) \to C_r \oplus D_{r+1} \oplus (B\otimes_{A_2} D'^{n+1-r})\oplus D^{n+2-r})$$

and the differential of $\mathbf{C}'$

$$\mathbf{d}_{\mathbf{C}'} = \begin{pmatrix} \mathbf{d}_{\mathbf{C}} & \mathbf{0} & (-)^{n+1}\mathbf{\Phi}_0\mathbf{f}^* \\ (-)^r\mathbf{f} & \mathbf{d}_{\mathbf{D}} & (-)^r\mathbf{\Delta\Phi}_0 \\ \mathbf{0} & \mathbf{0} & (-)^r\mathbf{d}_{\mathbf{D}}^* \end{pmatrix} : \mathbf{C}'_r = \mathbf{C}_r \oplus \mathbf{D}_{r+1} \oplus \mathbf{D}^{n+1-r} \to \mathbf{C}'_{r-1} = \mathbf{C}_{r-1} \oplus \mathbf{D}_r \oplus \mathbf{D}^{n+2-r}$$

is identified with the collection

$$\left(\begin{pmatrix} d_C & 0 & (-)^{n+1}\mu(1\otimes\phi_0 f'^*) & (-)^{n+1}\delta\phi_0 f^* \\ (-)^r f & d_D & (-)^r\nu(1\otimes\phi'_0) & (-)^r\delta\phi'_0 \\ 0 & 0 & (-)^r 1\otimes d^*_{D'} & (-)^n(\beta^{-1}\otimes 1)\nu^* \\ 0 & 0 & 0 & (-)^r d^*_D \end{pmatrix} \in \mathrm{Hom}_{A_1}(C''_r, C''_{r-1}), \begin{pmatrix} d_{C'} & 0 & (-)^{n+1}\phi_0 f'^* \\ (-)^r f' & d_D & (-)^r\phi'_0 \\ 0 & 0 & (-)^r d^*_{D'} \end{pmatrix} \in \mathrm{Hom}_{A_2}(C'''_r, C'''_{r-1})\right).$$

By the proof of Proposition 7.2.2 we may also write

$$\mathbf{C}^{n-r} = \left((B\otimes_{A_2} C'^{n-r})\oplus C^{n+1-r}, C'^{n-r}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} : B\otimes_{A_2} C'^{n-r} \to (B\otimes_{A_2} C'^{n-r})\oplus C^{n+1-r}\right)$$

$$\mathbf{T}^2(\mathbf{D})_{r+1} = \left((B\otimes_{A_2} D'_{r+1})\oplus D_{r+1}\oplus(B^*\otimes_{A_2} D'_r), D'_{r+1}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : B\otimes_{A_2} D'_{r+1} \to (B\otimes_{A_2} D'_{r+1})\oplus D_{r+1}\oplus(B^*\otimes_{A_2} D'_r)\right)$$

so that

$$\mathbf{C}'^{n-r} = \mathbf{C}^{n-r} \oplus \mathbf{D}^{n+1-r} \oplus \mathbf{T}^2(\mathbf{D})_{r+1} = (C''''^{n-r}, C'''''^{n-r}, \mu'')$$

with

$$C''''^{n-r} = (B\otimes_{A_2} C'^{n-r})\oplus C^{n+1-r}\oplus(B\otimes_{A_2} D'^{n+1-r})\oplus D^{n+2-r}\oplus(B\otimes_{A_2} D'_{r+1})\oplus D_{r+1}\oplus(B^*\otimes_{A_2} D'_r)$$
$$C'''''^{n-r} = C'^{n-r}\oplus D'^{n+1-r}\oplus D'_{r+1}$$

$$\mu'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : B \otimes_{A_2} C'''''^{n-r} \to C''''^{n-r}.$$

By Proposition 7.2.5 the natural transformation $\mathbf{e}(\mathbf{D}) : \mathbf{T}^2(\mathbf{D})_{r+1} \to \mathbf{D}_{r+1}$ is identified with the pair of morphisms

$$\Big(\begin{pmatrix} \mu & 1 & 0 \end{pmatrix} \in \mathrm{Hom}_{A_1}(B \otimes_{A_2} D'_{r+1}) \oplus D_{r+1} \oplus (B^* \otimes_{A_2} D'_r), 1 \in \mathrm{Hom}_{A_2}(D'_{r+1}, D'_{r+1}) \Big).$$

The symmetric structure $\mathbf{\Phi}'$ has 0-dimensional component

$$\mathbf{\Phi}'_0 = \begin{pmatrix} \mathbf{\Phi}_0 & \mathbf{0} & \mathbf{0} \\ (-)^{n-r}\mathbf{f}\mathbf{T}_\epsilon(\mathbf{\Phi}_1) & (-)^{n-r}\mathbf{T}_\epsilon(\mathbf{\Delta\Phi}_1) & (-)^{r(n-r)}\mathbf{e}(\mathbf{D}) \\ \mathbf{0} & 1 & \mathbf{0} \end{pmatrix} : \mathbf{C}'^{n-r} = \mathbf{C}^{n-r} \oplus \mathbf{D}^{n+1-r} \oplus \mathbf{T}^2(\mathbf{D})_{r+1} \to \mathbf{C}'_r = \mathbf{C}_r \oplus \mathbf{D}_{r+1} \oplus \mathbf{D}^{n+1-r}$$

represented by the pair of morphisms $(\delta\phi''_0 \in \mathrm{Hom}_{A_1}(C''''^{n-r}, C''_r,), \phi''_0 \in \mathrm{Hom}_{A_2}(C'''''^{n-r}, C''_r))$ where

$$\delta\phi''_0 = \begin{pmatrix} \mu(1 \otimes \phi_0) & \delta\phi_0 & 0 & 0 & 0 & 0 & 0 \\ (-)^{n-r} f\mu(1 \otimes T_\epsilon(\phi_1)) & (-)^{n-r} fT_\epsilon(\delta\phi_1) & (-)^{n-r}\nu(1 \otimes T_\epsilon(\phi'_1)) & (-)^{n-r}T_\epsilon(\delta\phi'_1) & (-)^{r(n-r)}\mu & (-)^{r(n-r)+1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\phi''_0 = \begin{pmatrix} \phi_0 & 0 & 0 \\ (-)^{n-r} f' T_\epsilon(\phi_1) & (-)^{n-r}T_\epsilon(\phi'_1) & (-)^{r(n-r)} \\ 0 & 1 & 0 \end{pmatrix}$$

The higher dimensional components of $\mathbf{\Phi}'$ are given by

$$\mathbf{\Phi}'_s = \begin{pmatrix} \mathbf{\Phi}_s & \mathbf{0} & \mathbf{0} \\ (-)^{n-r}\mathbf{f}\mathbf{T}_\epsilon(\mathbf{\Phi}_{s+1}) & (-)^{n-r+s}\mathbf{T}_\epsilon(\mathbf{\Delta\Phi}_{s+1}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} : \mathbf{C}'^{n-r+s} = \mathbf{C}^{n-r+s} \oplus \mathbf{D}^{n+1-r+s} \oplus \mathbf{T}^2(\mathbf{D})_{r+1-s} \to \mathbf{C}'_r = \mathbf{C}_r \oplus \mathbf{D}_{r+1} \oplus \mathbf{D}^{n+1-r} \quad (s \geqslant 1)$$

represented by the pair of morphisms $(\delta\phi''_s \in \mathrm{Hom}_{A_1}(C''''^{n-r+s}, C''_r), \phi''_s \in \mathrm{Hom}_{A_2}(C'''''^{n-r+s}, C'''_r))$ where

$$\delta\phi''_s = \begin{pmatrix} \mu(1\otimes\phi_s) & \delta\phi_s & 0 & 0 & 0 & 0 & 0 \\ (-)^{n-r} f\mu(1\otimes T_\epsilon(\phi_{s+1})) & (-)^{n-r} fT_\epsilon(\delta\phi_{s+1}) & (-)^{n-r+s}\nu(1\otimes T_\epsilon(\phi'_{s+1})) & (-)^{n-r+s}T_\epsilon(\delta\phi'_{s+1}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\phi''_s = \begin{pmatrix} \phi_s & 0 & 0 \\ (-)^{n-r} f' T_\epsilon(\phi_{s+1}) & (-)^{n-r+s}T_\epsilon(\phi'_{s+1}) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

□

Example 8.3.2. Let R be a ring with involution determining the triangular matrix ring $A = (R, R, R)$ as in Example 7.3.8.

(i) An $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi, \phi))$ over R can be identified with an $(n+1)$-dimensional ϵ-symmetric pair $(\mathbf{f} : \mathbf{C} \to \mathbf{D}, (\mathbf{\Delta\Phi}, \mathbf{\Phi}))$ over A where $\mathbf{f} = (0, f), \mathbf{C} = (0, C, 0), \mathbf{D} = (0, D, 0), \mathbf{\Delta\Phi} = (0, \delta\phi), \mathbf{\Phi} = (0, \phi)$. If (C, ϕ) is Poincaré then $(\mathbf{C}, \mathbf{\Phi})$ is Poincaré. The effect $(\mathbf{C}', \mathbf{\Phi}')$ of an algebraic surgery on $(\mathbf{C}, \mathbf{\Phi})$ with surgery data $(\mathbf{f} : \mathbf{C} \to \mathbf{D}, (\mathbf{\Delta\Phi}, \mathbf{\Phi}))$ can be identified with the effect of algebraic surgery on (C, ϕ) with surgery data $(f : C \to D, (\delta\phi, \phi))$.

Moreover, the trace of the algebraic surgery is an $(n+1)$-dimensional ϵ-symmetric cobordism over A $((\mathbf{g}\ \ \mathbf{g}') : \mathbf{C} \oplus \mathbf{C}' \to \mathbf{D}', (\mathbf{0}, \mathbf{\Phi} \oplus -\mathbf{\Phi}'))$ which can be identified with the $(n+2)$-dimensional ϵ-symmetric relative cobordism (Γ,Φ) over R where

$$\Gamma = \begin{array}{ccc} C\oplus C' & \xrightarrow{\quad 0 \quad} & 0 \\ {\scriptstyle (g\ \ g')}\big\downarrow & & \big\downarrow{\scriptstyle 0} \\ \delta C' & \xrightarrow[\quad 0 \quad]{} & 0 \end{array}$$

$$\Phi = (0,0,0,\phi\oplus -\phi')$$

where the $(n+1)$-dimensional ϵ-symmetric cobordism $((g\ \ g') : C\oplus C' \to \delta C', (0,\phi\oplus -\phi'))$ can be identified with the trace of the surgery on (C,ϕ) with surgery data $(f : C \to D, (\delta\phi,\phi))$. This shows that algebraic surgery over the triangular matrix ring (R,R,R) recovers algebraic surgery over R.

(ii) An $(n+2)$-dimensional ϵ-symmetric commutative triad (Γ,Φ) over R with

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{\ f\ } & D \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \delta C & \xrightarrow[\ \delta f\ ]{} & \delta D \end{array}$$

$$\Phi = (\phi',\delta\phi',\delta\phi,\phi)$$

can be identified with an $(n+1)$-dimensional ϵ-symmetric pair $(\mathbf{f} : \mathbf{C} \to \mathbf{D}, (\mathbf{\Delta\Phi},\mathbf{\Phi}))$ over A where $\mathbf{f} = (h,g), \mathbf{C} = (D,C,f), \mathbf{D} = (\delta D,\delta C,\delta f), \mathbf{\Delta\Phi} = (\phi',\delta\phi'), \mathbf{\Phi} = (\delta\phi,\phi)$. If the $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi,\phi))$ is Poincaré then $(\mathbf{C},\mathbf{\Phi})$ is Poincaré. The effect of an algebraic surgery on $(\mathbf{C},\mathbf{\Phi})$ with surgery data $(\mathbf{f} : \mathbf{C} \to \mathbf{D}, (\mathbf{\Delta\Phi},\mathbf{\Phi}))$ is the n-dimensional ϵ-symmetric Poincaré complex $(\mathbf{C}',\mathbf{\Phi}')$ such that if $\mathbf{C}' = (D',C',f'), \mathbf{\Phi}' = (\delta\phi'',\phi'')$

then

$$C'_r = C_r \oplus \delta C_{r+1} \oplus \delta C^{n+1-r}$$
$$D'_r = D_r \oplus \delta D_{r+1} \oplus \delta C^{n+1-r} \oplus \delta D^{n+2-r}$$
$$f' = \begin{pmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} : C_r \oplus \delta C_{r+1} \oplus \delta C^{n+1-r} \to D_r \oplus \delta D_{r+1} \oplus \delta C^{n+1-r} \oplus \delta D^{n+2-r}$$
$$\delta\phi''_0 = \begin{pmatrix} f\phi_0 & \delta\phi_0 & 0 & 0 & 0 & 0 & 0 \\ (-)^{n-r} g f T_\epsilon(\phi_1)) & (-)^{n-r} g T(\delta\phi_1) & (-)^{n-r}\delta f T_\epsilon(\delta\phi'_1) & (-)^{n-r} T_\epsilon(\phi'_1) & (-)^{r(n-r)} f & (-)^{r(n-r)+1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$
$$\delta\phi''_s = \begin{pmatrix} f\phi_s & \delta\phi_s & 0 & 0 & 0 & 0 & 0 \\ (-)^{n-r} h f T_\epsilon(\phi_{s+1}) & (-)^{n-r} g T_\epsilon(\delta\phi_{s+1}) & (-)^{n-r+s} f T_\epsilon(\delta\phi'_{s+1}) & (-)^{n-r+s} T_\epsilon(\phi'_{s+1}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (s \geqslant 1)$$

$$\phi''_0 = \begin{pmatrix} \phi_0 & 0 & 0 \\ (-)^{n-r} g T_\epsilon(\phi_1) & (-)^{n-r} T_\epsilon(\delta\phi'_1) & (-)^{r(n-r)} \\ 0 & 1 & 0 \end{pmatrix}$$
$$\phi''_s = \begin{pmatrix} \phi_s & 0 & 0 \\ (-)^{n-r} g T_\epsilon(\phi_{s+1}) & (-)^{n-r+s} T_\epsilon(\delta\phi'_{s+1}) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (s \geqslant 1).$$

Moreover, the trace of the algebraic surgery is an $(n+1)$-dimensional ϵ-symmetric cobordism $((\mathbf{g}\ \mathbf{g}') : \mathbf{C} \oplus \mathbf{C}' \to \mathbf{D}', (\mathbf{0}, \mathbf{\Phi} \oplus -\mathbf{\Phi}'))$ over A which can be identified with the $(n+2)$-dimensional ϵ-symmetric relative cobordism (Γ', Φ') over R between $(f : C \to D, (\delta\phi, \phi))$ and $(f' : C' \to D', (\delta\phi', \phi'))$ with

$$\Gamma' = \begin{array}{ccc} C \oplus C' & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & D \oplus D' \\ {\scriptstyle \begin{pmatrix} g' & g'' \end{pmatrix}} \big\downarrow & & \big\downarrow {\scriptstyle \begin{pmatrix} h' & h'' \end{pmatrix}} \\ \delta C' & \xrightarrow[\delta f']{} & \delta D' \end{array}$$

$$\Phi' = (0, 0, \delta\phi \oplus -\delta\phi'', \phi \oplus -\phi'').$$

In addition (C', ϕ') is the effect of an algebraic surgery on (C, ϕ) with algebraic surgery data the $(n+1)$-dimensional symmetric pair $(g : C \to \delta C, (\delta\phi', \phi))$ and trace the $(n+1)$-dimensional ϵ-symmetric cobordism $((g' \; g'') : C \oplus C' \to \delta C', (0, \phi \oplus -\phi'))$.

This recovers the formula from [BNR12a, p.49] for algebraic surgery on the Poincaré pair $(f : C \to D, (\delta\phi, \phi))$ over R with input a triad over R

$$\Gamma = \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g} \big\downarrow & & \big\downarrow {\scriptstyle h} \\ \delta C & \xrightarrow[\delta f]{} & \delta D \end{array}$$

$$\Phi = (\phi', \delta\phi', \delta\phi, \phi).$$

We now show that the definition of algebraic surgery on a symmetric Poincaré pair over R with data a symmetric triad over R provides an algebraic model for geometric half-surgeries of Borodzik, Némethi and Ranicki on a manifold with boundary.

Definition 8.3.3. [BNR12a, p.5-10] Let (Σ, M) be an $(n+1)$-dimensional manifold with boundary.

(i) The *effect* an index $i+1$ right half-surgery removing a framed embedding $S^i \times D^{n-i} \hookrightarrow M$ is the $(n+1)$-dimensional manifold with boundary

$$(\Sigma', M') = (\Sigma \cup_{S^i \times D^{n-i}} D^{i+1} \times D^{n-i}, \overline{M - S^i \times D^{n-i}} \cup_{S^i \times S^{n-i-1}} D^{i+1} \times S^{n-i-1}).$$

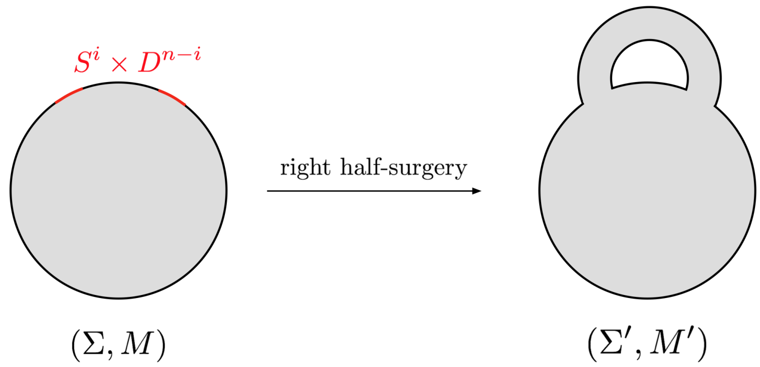


Figure 93: The effect of a right half-surgery.

If $(W; M, M')$ is the trace of the ordinary surgery on M removing the framed embedding $S^i \times D^{n-i} \hookrightarrow M$, then the *trace* of the right half-surgery is the $(n+2)$-dimensional relative cobordism

$$(\Sigma' \times I; \Sigma \times \{0\}, \Sigma' \times \{1\}, W; M, M')$$

between (Σ, M) and (Σ', M').

(ii) The *effect* of an index $i+1$ left half-surgery removing a framed embedding $(D^{i+1} \times D^{n-i}, S^i \times D^{n-i}) \hookrightarrow (\Sigma, M)$ is the $(n+1)$-dimensional manifold with boundary

$$(\Sigma', M') = (\overline{\Sigma - D^{i+1} \times D^{n-i}}, \overline{M - S^i \times D^{n-i}} \cup_{S^i \times S^{n-i-1}} D^{i+1} \times S^{n-i-1}).$$

Figure 94: The effect of a left half-surgery.

If $(W;M,M')$ is the trace of the ordinary surgery on M removing is the framed embedding $S^i \times D^{n-i} \hookrightarrow M$, then the *trace* of the left half- surgery is the $(n+2)$-dimensional relative cobordism

$$(\Sigma \times I; \Sigma \times \{0\}, \Sigma' \times \{1\}, W; M, M')$$

between (Σ, M) and (Σ', M').

Example 8.3.4. ([BNR12a, p.49-51]). Write the half-surgery traces as $(\Omega;\Sigma,\Sigma',W;M,M')$. If R is a commutative ring with identity then applying the symmetric construction to the $(n+2)$-dimensional relative cobordism $(\Omega;\Sigma,\Sigma',W;M,M')$ produces an $(n+2)$-dimensional commutative symmetric Poincaré triad (Γ,Φ) over R with

$$\Gamma = \begin{array}{ccc} C(M;R)\oplus C(M';R) & \longrightarrow & C(\Sigma;R)\oplus C(\Sigma';R) \\ \big\downarrow & & \big\downarrow \\ C(W;R) & \longrightarrow & C(\Omega;R) \end{array}$$

$$\Phi = (\phi_\Omega, \phi_W, \phi_\Sigma \oplus -\phi_{\Sigma'}, \phi_M \oplus -\phi_{M'})$$

which can be viewed as a cobordism between the $(n+1)$-dimensional symmetric Poincaré pairs $(C(M;R) \to C(\Sigma;R), (\phi_\Sigma, \phi_M))$ and $(C(M':R) \to C(\Sigma';R), (\phi_{\Sigma'}, \phi_{M'}))$.

In the case of a left half-surgery the inclusion $(W,M') \hookrightarrow (\Omega,\Sigma')$ is a homotopy equivalence so there is a chain homotopy equivalence

$$C(\Omega,\Sigma';R) \simeq C(W,M';R) \simeq S^{n-i}R = (n-i)\text{-fold suspension of } R$$

so that

$$C'_r \simeq \begin{cases} C(M;R)_r \oplus R & \text{if } r = i+1, n-i-1 \\ C(M;R)_r & \text{otherwise} \end{cases} \qquad (r \in \mathbb{Z}).$$

$$D'_r \simeq \begin{cases} C(\Sigma;R)_r \oplus R & \text{if } r = i+1, i+2, n-i-1 \\ C(\Sigma;R)_r & \text{otherwise} \end{cases} \qquad (r \in \mathbb{Z}).$$

In the case of a right half-surgery the inclusion $\Sigma' \hookrightarrow \Omega$ is a homotopy equivalence so there is a chain homotopy equivalence

$$C(\Omega,\Sigma';R) \simeq C(\Sigma',\Sigma';R) \simeq \dot{C}(*;R) \simeq S^0R \ = 0\text{-fold suspension of } R$$

so that

$$C(M';R)_r \simeq \begin{cases} C(M;R)_r \oplus R & \text{if } r = i+1, n-i-1 \\ C(M;R)_r & \text{otherwise} \end{cases} \qquad (r \in \mathbb{Z})$$

$$C(\Sigma':R)_r \simeq \begin{cases} C(\Sigma;R)_r \oplus R & \text{if } r = i+1, i+2, n-i-1 \\ C(\Sigma;R)_r & \text{otherwise} \end{cases} \qquad (r \in \mathbb{Z}).$$

In fact, the $(n+1)$-dimensional symmetric Poincaré pair $(C(M':R) \to C(\Sigma';R), (\phi_{\Sigma'}, \phi_{M'}))$ is homotopy equivalent to the effect $(C' \to D', (\delta\phi, \phi))$ of relative algebraic surgery on the

$(n+1)$-dimensional symmetric Poincaré pair $(C(M;R) \to C(\Sigma;R), (\phi_\Sigma, \phi_M))$ with data the $(n+2)$-dimensional symmetric triad (Γ', Φ') over R with

$$\Gamma' = \begin{array}{ccc} C(M;R) & \longrightarrow & C(\Sigma;R) \\ \big\downarrow & & \big\downarrow \\ C(W,M';R) & \longrightarrow & C(\Omega,\Sigma';R) \end{array}$$

$$\Phi' = (\phi_\Omega/\phi_{\Sigma'}, \phi_W/\phi_{M'}, \phi_\Sigma, \phi_M)$$

with

$$C'_r = C(M)_r \oplus C(W,M')_{r+1} \oplus C(W,M')^{n+1-r} \qquad (r \in \mathbb{Z})$$
$$D'_r = C(\Sigma)_r \oplus C(\Omega,\Sigma')_{r+1} \oplus C(W,M')^{n+1-r} \oplus C(\Omega,\Sigma')^{n+2-r} \qquad (r \in \mathbb{Z}).$$

such that the triad (Γ, Φ) arises as the trace of this relative algebraic surgery. This shows that relative algebraic surgery gives an algebraic model for a geometric half-surgeries.

Milnor [Mil61] and Thom [Tho49] used Morse theory to show that every $(n+1)$-dimensional cobordism $(W;M,M')$ can be expressed as a union of elementary cobordisms

$$(W;M,M') = (W_0;M_0;M_1) \cup (W_1;M_1;M_2) \cup \ldots \cup (W_\ell;M_\ell;M_{\ell+1}) \quad (M_0 = M, M_{\ell+1} = M')$$

where $(W_i;M_i,M_{i+1})$ arises the trace of a surgery on M_i with effect M_{i+1}. Ranicki [Ran80a, Proposition 4.7] gave a precise algebraic analogue of this and showed that every $(n+1)$-dimensional ϵ-symmetric cobordism over a ring with involution is homotopy equivalent to a union of elementary ϵ-symmetric cobordisms arising as the traces of elementary algebraic surgeries.

Borodzik, Némethi and Ranicki [BNR12b, Theorem 4.18] used Morse theory on a manifold with boundary to show that every $(n+2)$-dimensional relative cobordism $(\Omega;\Sigma,\Sigma',W;M,M')$, such that Σ and Σ' have no closed connected components and Ω has no connected components, can be expressed as a union of adjoining elementary relative cobordisms

$$\Omega = \Omega_0 \cup \Omega_{\frac{1}{2}} \cup \Omega_1 \cup \Omega_{\frac{3}{2}} \cup \ldots \cup \Omega_{n+\frac{3}{2}} \cup \Omega_{n+2}$$

where Ω_0 arises as the effect of an index 0 handle attachment, Ω_i arises as the trace of an index i right-half surgery, $\Omega_{i+\frac{1}{2}}$ arises as the trace of an index i left half-surgery and Ω_{n+2} arises as the effect of an index $(n+2)$-handle attachment. Ranicki [BNR12b, Theorem 4.71] gave an algebraic analogue and showed that every $(n+2)$-dimensional ϵ-symmetric relative Poincaré cobordism is homotopy equivalent to a union of traces of algebraic half-surgeries. Whereas Ranicki's proof was indirect and made use of the thickening operation for algebraic trinities, we can give a more direct proof using triangular matrix rings.

Theorem 8.3.5. Let R be a ring with involution. Every $(n+2)$-dimensional commutative ϵ-symmetric Poincaré relative cobordism is homotopy equivalent to a union of traces of elementary relative surgeries.

Proof. Let $A = (R,R,R)$ be the triangular matrix ring determined by R. By Theorem 8.2.5 an

$(n+2)$-dimensional ϵ-symmetric relative Poincaré cobordism (Γ,Φ) over R with

$$\Gamma = \begin{array}{ccc} C\oplus C' & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & D\oplus D' \\ {\scriptstyle (g' \;\; g'')}\downarrow & & \downarrow{\scriptstyle (h' \;\; h'')} \\ \delta C & \xrightarrow[\delta f']{} & \delta D \end{array}$$

$$\Phi = (\phi'', \delta\phi'', \delta\phi \oplus -\delta\phi', \phi \oplus -\phi').$$

determines a locally $(n+1)$-dimensional ϵ-symmetric cobordism $((\mathbf{f}\ \mathbf{f}') : \mathbf{C}\oplus\mathbf{C}' \to \mathbf{D}, (\mathbf{\Delta\Phi}, \mathbf{\Phi}\oplus -\mathbf{\Phi}'))$ over A. It follows from [Ran92, Proposition 1.13] that $((\mathbf{f}\ \mathbf{f}') : \mathbf{C}\oplus\mathbf{C}' \to \mathbf{D}, (\mathbf{\Delta\Phi}, \mathbf{\Phi}\oplus -\mathbf{\Phi}'))$ can be realised, up to homotopy equivalence, as the trace of a surgery over A and it follows from the additive category with chain duality generalisation of [Ran80a, Proposition 4.7] that this surgery can be decomposed as a sequence of elementary surgeries over A. By Example 8.3.2 the trace of an elementary surgery over A can be interpreted as the trace of an elementary relative surgery over R. □

8.4 An open question

We end with an open question reality relating the L-theory of the triangular matrix ring $A = (R,R,R)$ to the L-theory of (R,K)-modules.

Definition 8.4.1. Let R be a ring and let K be a simplicial complex.

(i) An (R,K)*-module* is a f.g. projective R-module M with a *fracturing* over K, that is a choice direct sum decomposition $M = \oplus_{\sigma\in K} M(\sigma)$ such that each summand $M(\sigma)$ is a f.g. projective R-module and at most finitely many summands $M(\sigma)$ are non-zero.

(ii) A *morphism* of (R,K)-modules $f : M = \oplus_{\sigma\in K} M(\sigma) \to N = \oplus_{\tau\in K} N(\tau)$ is a collection of morphisms

$$f = \{f(\tau,\sigma) \in \mathrm{Hom}_R(M(\sigma), N(\tau)) | \sigma,\tau \in K | f(\tau,\sigma) = 0 \text{ unless } \sigma \geq \tau\}$$

between the summands.

Example 8.4.2. Let K be a locally finite simplicial complex. The simplicial chain complex $\Delta(K;R)$ of K with R-coefficients is an (R,K)-module chain complex where a simplex $\sigma \in K$ contributes the summand

$$\Delta(K,R)(\sigma) = S^{|\sigma|}R = |\sigma|\text{-fold suspension of } R$$

to $\Delta(K;R)$.

Ranicki and Weiss [RW90] examined when an R-module chain complex admits a fracturing over a simplicial complex K. The L-theory of (R,K)-modules was determined by Ranicki by constructing a chain duality on the category of (R,K)-modules, see [Ran92, chapter 5].

Example 8.4.3. ([Ran92, Example 5.4]) Let R be a ring with involution and let $\Delta^n = [0,1,\dots,n]$ denote the standard n-simplex.

(i) An n-dimensional ϵ-symmetric Poincaré complex over Δ^0 is the same as an n-dimensional ϵ-symmetric Poincaré complex over R.

(ii) An n-dimensional ϵ-symmetric Poincaré complex over $\Delta^1 = [0,1]$ is the same as an n-dimensional ϵ-symmetric Poincaré cobordism over R which is fragmented over $[0,1]$ as shown below

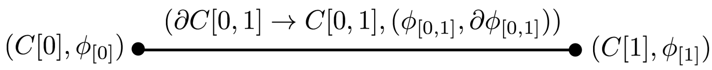


Figure 95: An ϵ-symmetric complex fragmented over a 1-simplex.

(iii) An n-dimensional ϵ-symmetric Poincaré complex over $\Delta^2 = [0,1,2]$ is the same as an n-dimensional ϵ-symmetric Poincaré pair over R with a 3-fold boundary splitting (recall Definition 3.3.4 of Section 3.1 of Part I) and is fragmented over $[0,1,2]$ as shown below

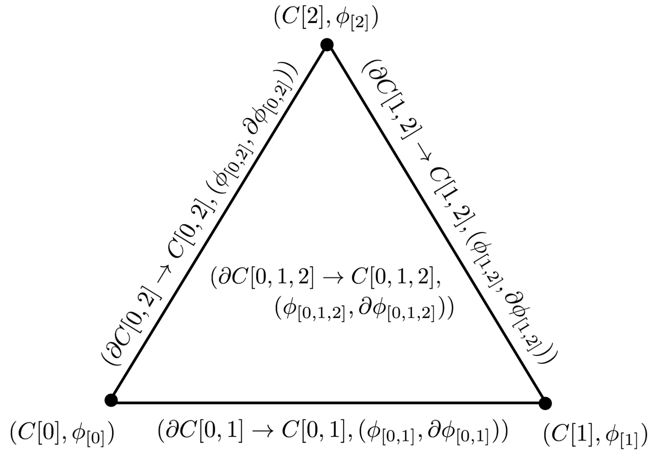


Figure 96: An ϵ-symmetric complex fragmented over a 2-simplex.

Note that the category of (R, K)-modules is a small abelian category and hence by the Freyd-Mitchell embedding theorem [Fre03] there exists some ring A such that the category of (R, K)-modules can be embedded as a full subcategory of the category of A-modules. In Example 8.4.3 (i) we can take $A = R$ and in Example 8.4.3 (ii) we can take A to be the triangular matrix ring (R, R, R). In general, one would wish to find a simple model for A.

Open Question: Given a simplicial complex K and a ring R with involution, does there exist a generalisation of a triangular matrix ring A such that the L-theory of (R, K)-modules is equivalent to the L-theory of A-modules?

Part III

Seifert matrices of braids with applications to isotopy and signatures

Introduction to Part III

Let β be a braid with closure $\widehat{\beta}$ a link. The canonical Seifert surface of β constructed by Seifert's algorithm resolves each crossing of $\widehat{\beta}$

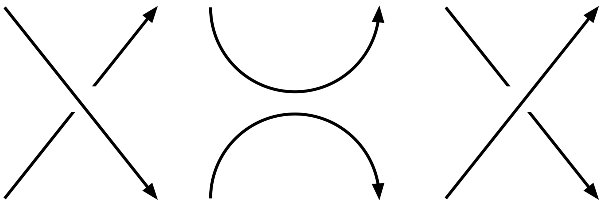


Figure 97: Resolving overcrossings and undercrossings.

to produce a collection of disjoint, oriented, simple, planar circles called Seifert circles. Each Seifert circle bounds a planar disc and we may push the planar disks vertically to make them disjoint. Attaching a twisted band between the Seifert circles for each resolution of a crossing, with the twist matching the type of the crossing, then produces the canonical Seifert surface of $\widehat{\beta}$, which is a closed orientable surface of genus $g \geq 0$ with boundary $\widehat{\beta}$.

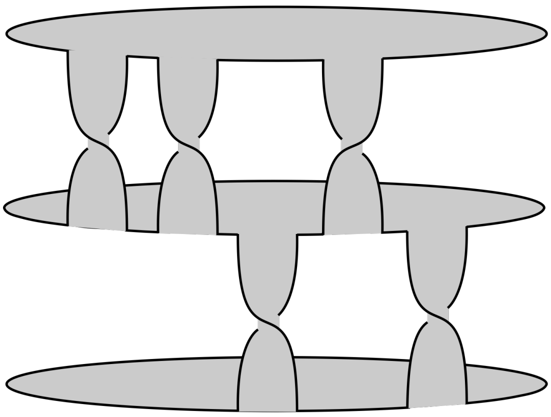


Figure 98: A Seifert surface produced by Seifert's algorithm.

Choosing an ordered basis $\{[\gamma_i]\}_{i=1}^{2g}$ of $H_1(\Sigma;\mathbb{Z})$, with each basis homology class $[\gamma_i]$ represented by simple, closed curve $\gamma_i \subset \Sigma$, we may push each γ_i in the positive normal direction to produce a simple closed curve γ_i^+ which lies in $S^3 - \Sigma$. The Seifert form $V : H_1(\Sigma;\mathbb{Z}) \times H_1(\Sigma;\mathbb{Z}) \to \mathbb{Z}$ is the bilinear form determined on the basis homology classes by the linking numbers $\mathrm{Lk}(\gamma_i, \gamma_j^+)$.

Two n-strand braids β, β' may be concatenated to produce an n-strand $\beta\beta'$. The effect of the concatenation of braids is a gluing of Seifert surfaces along parts of their boundaries. The Mayer-Vietoris sequence then provides an obstruction for the Seifert form to be additive under the concatenation of braids. This suggests that one could try to find a chain level Seifert form, expressed in terms of partial linking numbers, which is additive on the chain level under the concatenation of braids and descends to the Seifert form on the homology level.

Banchoff [Ban76] gave a combinatorial linking formula for two disjoint space polygons expressed in terms of partial linking numbers of pairs of edges as follows. Let $X = \{X_0, X_1, \ldots, X_{m-1}\}$ respectively $Y = \{Y_0, Y_1, \ldots, Y_{n-1}\}$ be a set of points in general position in $\mathbb{R}^3$. For a unit vector $\xi \in S^2$ let $p_\xi : \mathbb{R}^3 \to P$ denote the projection map from $\mathbb{R}^3$ onto the plane P orthogonal to ξ. A vector $\xi \in S^2$ in called general for X and Y if the projections $p_\xi(X), p_\xi(Y) \subset \mathbb{R}^2$ are in general position. For a vector $\xi \in S^2$ which is general for X and Y, define $C_{i,j}(X,Y,\xi)$ to be the sign of $P_\xi(Y_{j+1} - Y_j) \times P_\xi(X_{i+1} - X_i).(\overline{X_i} - \overline{Y_j})$ if there are interior points $\overline{X_i}$ of the edge X_iX_{i+1} and $\overline{Y_j}$ of the edge Y_jY_{j+1} such that $p_\xi(\overline{X_i}) = p_\xi(\overline{Y_j})$ and define $C_{i,j}(X,Y,\xi)$ to be zero otherwise.

The linking number of two space polygons is then expressible as the sum of the partial linking numbers of all edge pairs.

Theorem [Ban76, p.1176-1177] For disjoint polygonal knots $X, Y \subset \mathbb{R}^3$ the value

$$C(X,Y,\xi) = \sum_{\substack{0 \leqslant i \leqslant m-1 \\ 0 \leqslant j \leqslant n-1}} C_{i,j}(X,Y,\xi) \in \mathbb{Z}$$

is independent of the choice of general vector $\xi \in S^2$. The linking number of the polygonal knots determined by X and Y is the average value of $C(X,Y,\xi)$, that is

$$\mathrm{Lk}(X,Y) = \frac{1}{4\pi} \int_{\xi \in S^2} C(X,Y,\xi) d\omega = \frac{1}{4\pi} \sum_{\substack{0 \leqslant i \leqslant m-1 \\ 0 \leqslant j \leqslant n-1}} \int_{\xi \in S^2} C_{i,j}(X,Y,\xi) d\omega \in \mathbb{Z}$$

where ω is the volume form on S^2. Moreover this integral may be expressed in terms of dihedral angles of tetrahedra.

The closure of an n-strand braid with ℓ-crossings arises as the trace of ℓ 0-surgeries on a disjoint union of n circles. Ranicki [Ran14] applied the algebraic theory of surgery to the geometric surgeries to obtain a chain level formula which is defined inductively in terms of Seifert graphs. The Seifert graph of a braid β is the 1-dimensional CW-complex $X(\beta)$ constructed from the canonical Seifert surface of β by collapsing each Seifert disc to a point and collapsing each twisted band to its core. If β is an n-strand braid with ℓ-crossings then the Seifert graph $X(\beta)$ has ℓ 1-cells and n 0-cells and has a cellular chain complex of the form

$$d : C_1(X(\beta);\mathbb{Z}) \cong \mathbb{Z}^\ell \to C_0(X(\beta);\mathbb{Z}) \cong \mathbb{Z}^n$$

where d is a signed incidence matrix. If β' is another n-strand braid with ℓ' crossings then the

Seifert graph $X(\beta')$ has a cellular chain complex of the form

$$d' : C_1(X(\beta');\mathbb{Z}) \cong \mathbb{Z}^{\ell'} \to C_0(X(\beta');\mathbb{Z}) \cong \mathbb{Z}^n.$$

The Seifert graph of the concatenated braid $\beta\beta'$ is a CW-complex which can be formed from the Seifert graphs of β, β' by identifying the 0-cells so that $X(\beta\beta')$ has $(\ell + \ell')$ 1-cells, n 0-cells and a cellular chain complex of the form

$$d'' = \begin{pmatrix} d & d' \end{pmatrix} : C_1(X(\beta\beta');\mathbb{Z}) \cong \mathbb{Z}^{\ell} \oplus \mathbb{Z}^{\ell'} \to C_0(X(\beta\beta');\mathbb{Z}) \cong \mathbb{Z}^n.$$

Ranicki defined the canonical generalised Seifert matrices of the elementary regular n-strand braids σ_i, σ_i^{-1} to be the 1×1 matrices

$$\psi_{\sigma_i} = \begin{pmatrix} 1 \end{pmatrix}, \quad \psi_{\sigma_i^{-1}} = \begin{pmatrix} -1 \end{pmatrix}$$

and inductively defined the generalised Seifert matrix of the concatenated braid $\beta\beta'$ to be the matrix

$$\psi_{\beta\beta'} = \begin{pmatrix} \psi_\beta & -d^*\chi d' \\ 0 & \psi_{\beta'} \end{pmatrix} : C_1(X(\beta\beta');\mathbb{Z}) \times C_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}$$

where χ is the lower triangular $n \times n$ matrix with ones below the diagonal.

Theorem [Ran14, p.37-38] Let β, β' be regular n-strand braids. The generalised Seifert matrix

$$\psi_{\beta\beta'} : C_1(X(\beta\beta');\mathbb{Z}) \times C_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}$$

induces the Seifert form of $\beta\beta'$

$$\psi_{\beta\beta'} : H_1(X(\beta\beta');\mathbb{Z}) \times H_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}.$$

Motivated by the space polygon linking formula of Banchoff [Ban76] and the surgery-theoretic linking formula of Ranicki [Ran14] we construct a new chain level Seifert form. Following a suggestion of Étienne Ghys, to each braid β we associate a 1-dimensional simplicial complex $K(\beta)$ called a fence. The fence of an elementary n-strand braid $\sigma_i^{\pm 1}$ with a single crossing between strand i and strand $i+1$ is the oriented 1-dimensional simplicial complex $K(\beta)$ with $2n$ 0-simplices and $(n+1)$ 1-simplices as shown below

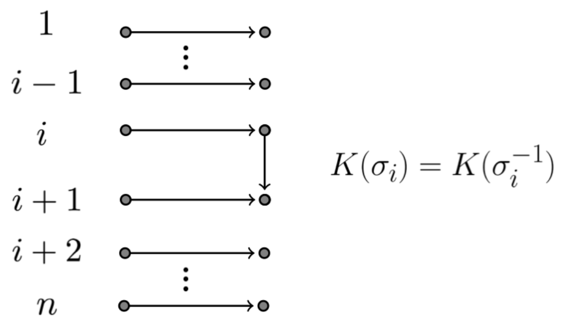


Figure 99: The fences associated to the elementary n-strand braids $\sigma_i^{\pm 1}$.

The fence of a regular n-strand braid $\beta = \beta_1\beta_2\ldots\beta_\ell$ with ℓ crossings is the concatenation of the fences of the elementary braids from left to right and there is a natural embedding of the fence of β into the canonical Seifert surface of β.

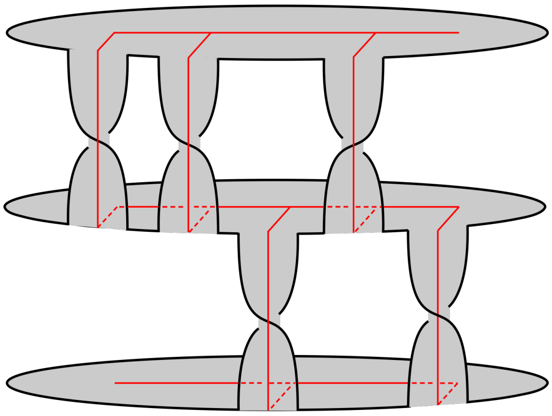


Figure 100: The embedding of the fence $K(\beta)$ in to the canonical Seifert surface for $\widehat{\beta}$.

By examining how a fence links with itself when it is pushed in the positive normal direction to the canonical Seifert surface

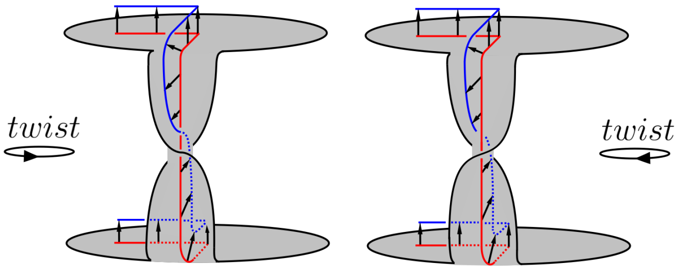
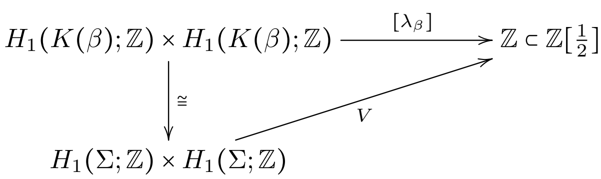


Figure 101: Pushing part of the fences in the positive normal direction.

we can associate to each fence a $\mathbb{Z}[\frac{1}{2}]$-valued bilinear form $\lambda_\beta : C_1(K(\beta);\mathbb{Z})\times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ which encodes partial self-linking information. This descends to the Seifert form of β on the homology level:

Theorem 10.2.1. The embedding $K(\beta) \hookrightarrow \Sigma$ is a homotopy equivalence inducing an isomorphism $H_1(K(\beta);\mathbb{Z}) \cong H_1(\Sigma;\mathbb{Z})$ with a commutative diagram



Moreover, this chain level Seifert form is additive under the concatenation of braids:

Theorem 10.3.4. Let $\beta = \beta_1\beta_2\ldots\beta_\ell$ be a braid where each β_i is an elementary braid. The chain level pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ can be represented by a block diagonal matrix

$$\begin{pmatrix} \lambda_{\beta_1} & 0 & \dots & 0 \\ 0 & \lambda_{\beta_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_{\beta_\ell} \end{pmatrix}$$

We then compare our model to Banchoff's and Ranicki's. Our model has the advantage that the partial linking numbers are $\mathbb{Z}[\frac{1}{2}]$-valued and not $\mathbb{R}$-valued as in Banchoff's model. Moreover, the concatenation behaviour in our model is additive and gives an instant chain level Seifert form whereas Ranicki's model is inductively defined.

Propositions 10.4.7, 10.4.8. Our model is chain equivalent to Banchoff's combinatorial model for the linking number of two space polygons and chain equivalent to Ranicki's surgery-theoretic chain level Seifert pairing model.

We give two applications of this chain level Seifert form to the isotopy of braids and to the signature of braids.

Two n-strand braids β, β' are isotopic if β can be continuously deformed to β' through a family of n-strand braids. Isotopy is an equivalence relation on the set of n-strand braids and the set of isotopy classes form a group B_n called the n-strand braid group. Artin [Art47] showed that there is a presentation of the braid group B_n with generators the elementary n-strand braids $\{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$ and relations of the form

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geqslant 2, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1, \quad \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1.$$

We define the chain level Seifert pair (λ_β, d_β) of a braid β and two equivalence relations, called A and $\widehat{A}$-equivalence, such that:

Propositions 11.1.7, 11.1.13. The A-equivalence class of the chain level Seifert pair of an n-strand braid β is a complete isotopy invariant. The $\widehat{A}$-equivalence class of the chain level Seifert pair of an n-strand geometric braid β is an isotopy invariant of the closure $\widehat{\beta}$ inside the solid torus.

The A-equivalence relation yields a universal representation of the braid group and the $\widehat{A}$-equivalence relation yields a representation of the braid group modulo conjugacy:

Theorems 11.1.8, 11.1.14. Let $n \geqslant 2$ and denote by F_n the free group on the set of elementary n-strand braids $\{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$ and denote by B_n denote the braid group. The map

$$(\lambda, d) : F_n \to \{\text{chain level Seifert pairs}\}, \quad \beta \mapsto (\lambda_\beta, d_\beta)$$

is a bijection such that words $\beta, \beta' \in F_n$ differ by the relations in the braid group if and only if the chain level Seifert pairs $(\lambda_\beta, d_\beta), (\lambda_{\beta'}, d_\beta)$ are A-equivalent. Moreover two words $\beta, \beta' \in B_n$

are conjugate if and only if the chain level Seifert pairs $(\lambda_\beta, d_\beta), (\lambda_{\beta'}, d_\beta)$ are $\widehat{A}$-equivalent. This induces an isomorphism of groups

$$(\lambda, d) : B_n \to \frac{\{\text{chain level Seifert pairs}\}}{A - \text{equivalence}}, \quad [\beta] \mapsto [(\lambda_\beta, d_\beta)]$$

and descends to a bijection

$$(\lambda, d) : \frac{B_n}{\text{conjugacy}} \to \frac{\{\text{chain level Seifert pairs}\}}{\widehat{A} - \text{equivalence}}, \quad [\beta] \mapsto [(\lambda_\beta, d_\beta)]$$

such that there is a commutative diagram

$$\begin{array}{ccc}
F_n & \xrightarrow[\cong]{(\lambda,d)} & \{\text{chain level Seifert pairs}\} \\
\downarrow & & \downarrow \\
B_n & \xrightarrow[\cong]{(\lambda,d)} & \frac{\{\text{chain level Seifert pairs}\}}{A\text{-equivalence}} \\
\downarrow & & \downarrow \\
\frac{B_n}{\text{conjugacy}} & \xrightarrow[\cong]{(\lambda,d)} & \frac{\{\text{chain level Seifert pairs}\}}{\widehat{A}\text{-equivalence}}
\end{array}$$

For a unit complex number $\omega \neq 1$ the ω-signature of a braid β with Seifert matrix V is the signature $\sigma_\omega(\beta)$ of the hermitian form $(H_1(\Sigma;\mathbb{Z}), (1-\omega)V + (1-\overline{\omega})V^t)$. We can express the ω-signature of a braid in terms of its chain level Seifert pair:

Theorem 11.2.6. Let β be a braid with chain level Seifert pair (λ_β, d_β) and let $\omega \neq 1$ be a unit complex number. The ω-signature of β is the signature of the hermitian pair

$$\left(C^1(K(\beta);\mathbb{C}) \oplus C_0(K(\beta);\mathbb{C}), \begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix} \right)$$

so that

$$\sigma_\omega(\beta) = \sigma\left(\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix} \right).$$

Part III is organised as follows.

In chapter 9 we introduce the basic operations one can perform on braids such as concatenation, taking the closure, performing an isotopy and constructing a Seifert form from a canonical Seifert surface.

In chapter 10 we define the 1-dimensional simplicial complex $K(\beta)$ and the $\mathbb{Z}[\frac{1}{2}]$-valued bilinear form $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$. We show that there is an embedding of $K(\beta) \hookrightarrow \Sigma$ with image a deformation retract of the canonical Seifert surface Σ of $\widehat{\beta}$ constructed by Seifert's algorithm. We examine how the image $K(\beta)$ is pushed along the positive normal to the Seifert surface to show that λ_β descends to the Seifert form on the homology level. We then show that the bilinear form λ_β is additive under the concatenation of braids and we compare our chain level Seifert form to the space polygon linking model of Banchoff and the surgery-theoretic Seifert form of Ranicki.

In chapter 11 we define the A and $\widehat{A}$-equivalence relations and use the chain level Seifert pair (λ_β, d_β) of a braid β to produce a representation of the braid group and of the braid group modulo conjugacy. We then construct a chain level formula for the ω-signature of a braid.

Chapter 9

Braids and Seifert forms

In this chapter we introduce the basic operations one can perform on braids such as concatenation, taking the closure, performing an isotopy and constructing a Seifert form from a Seifert surface.

9.1 Links and linking numbers

Definition 9.1.1. An *n-component link* is an embedding $L : \sqcup_n S^1 \hookrightarrow S^3$ of n disjoint, piecewise smooth, simple, closed curves. A *knot* is a one-component link. Let $P \subset \mathbb{R}^3$ be a 2-dimensional subspace of $\mathbb{R}^3$ and let $p : \mathbb{R}^3 \to P$ be the orthogonal projection map onto P. We say that $p : \mathbb{R}^3 \to \mathbb{R}$ is a *regular projection* of a link L if for each $x \in P$ the intersection $p^{-1}(x) \cap L$ consists of at most two points, in which case the *link diagram* is the image $p(L) \subset P$ with the over and under crossings recorded. An *oriented link* is a link for which each connected component has been given an orientation and this is recorded on a link diagram by a choice of arrow on each component of the link diagram. Two links L, L' are *ambient isotopic* if there is a homotopy of orientation preserving homeomorphisms $f_t : \sqcup_n S^1 \hookrightarrow S^3$ with $(0 \leq t \leq 1)$such that f_0 is the identity and $f_1(L) = L'$.

We will abuse the terminology in the standard way, with the word 'link' sometimes referring to the embedding and sometimes referring to the image of the embedding.

Example 9.1.2. Regular projections of an oriented trefoil knot and oriented Hopf link.

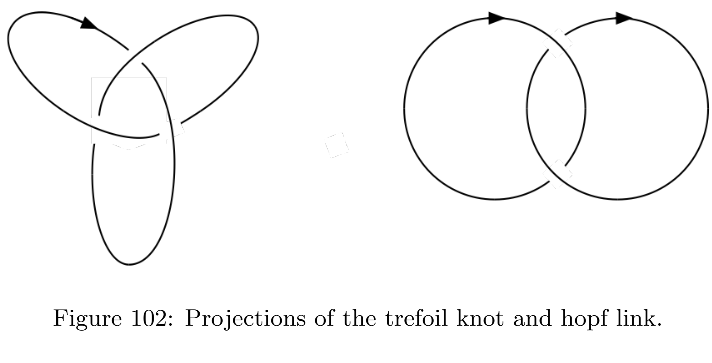


Figure 102: Projections of the trefoil knot and hopf link.

The linking number of two knots is an important numerical invariant in knot theory and may be defined in any of the following ways.

Definition 9.1.3. Let J, K be two disjoint oriented knots in S^3.

(i) Let $p : \mathbb{R}^3 \to P$ be a regular projection of the link $J \sqcup K \subset \mathbb{R}^3$. The linking number is half the sum of the signed crossings $\text{Lk}_1(J,K) = \frac{1}{2}\sum_{x\in p(J)\cap p(K)} \epsilon_x \in \mathbb{Z}$ where each crossing $x \in p(J) \cap p(K)$ is assigned a sign $\epsilon_x = \pm 1$ as follows

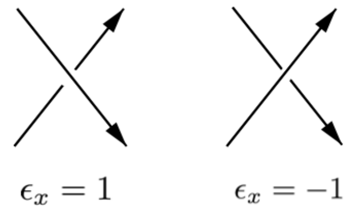


Figure 103: The signs associated to an overcrossing and an undercrossing.

(ii) Orienting $S^1 \times S^1$ and S^2, the linking number $\text{Lk}_2(J,K) \in \mathbb{Z}$ is the degree of the Gauss map

$$f : S^1 \times S^1 \to S^2; \quad f(u,v) = \frac{J(u) - K(v)}{\|J(u) - K(v)\|}.$$

(iii) The linking number $\text{Lk}_3(J,K)$ is the Gauss integral

$$\frac{1}{4\pi}\int_J\int_K \frac{(x'-x)(dydz'-dzdy') + (y'-y)(dzdx'-dxdz') + (z'-z)(dxdy'-dydx')}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{3/2}} \in \mathbb{Z}$$

Theorem 9.1.4. [Rol90, p.132-135]. The above definitions of linking numbers agree and the linking number is an ambient isotopy invariant.

9.2 Seifert surfaces and Seifert matrices of links

Definition 9.2.1. A *Seifert surface* for an oriented link L is a compact oriented surface $\Sigma \subset S^3$ with oriented boundary $\partial\Sigma = L$ such that the normal bundle $\nu_{\Sigma\subset S^3}$ is trivial.

Seifert's algorithm [Sei35] produces a Seifert surface for an oriented link L in the following way. Fix a regular projection of L and resolve each crossing as shown below.

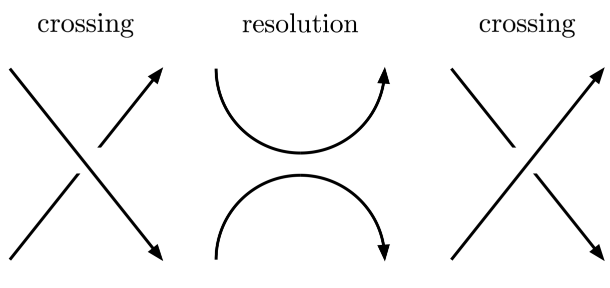


Figure 104: Resolving an overcrossing and an undercrossing.

Doing so produces a collection of disjoint, oriented, simple, planar circles called Seifert circles. Each Seifert circle bounds a planar disc. If some of the discs are not disjoint, because the corresponding Seifert circles are nested, we may push some the discs in a direction perpendicular to the plane to make them disjoint. We then attach a twisted band between the Seifert circles for each resolution of crossing with the twist matching the type of the crossing.

Example 9.2.2. Seifert surfaces for an oriented trefoil knot and oriented Hopf link constructed by Seifert's algorithm. We have labelled the Seifert circles to keep track of them when we move the discs they bound.

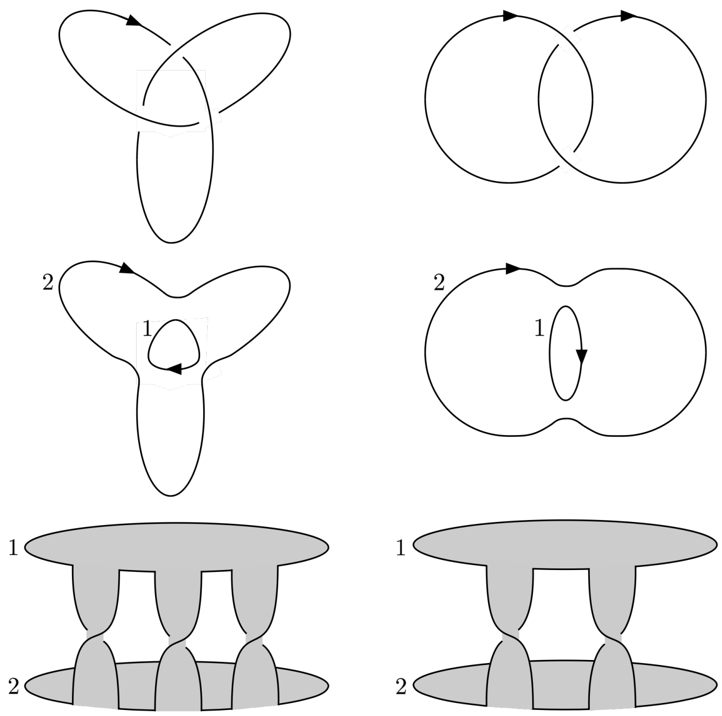


Figure 105: Seifert's algorithm performed on a trefoil knot and Hopf link.

A link has many regular projections so the Seifert surfaces constructed by Seifert's algorithm are highly non-unique. A Seifert surface for a link is however unique up to a certain relation called S-equivalence.

Definition 9.2.3. Two compact surfaces with boundary $(\Sigma_1, \partial\Sigma_1)$ and $(\Sigma_2, \partial\Sigma_2)$ are S-*equivalent* if $(\Sigma_2, \partial\Sigma_2)$ can be obtained from $(\Sigma_1, \partial\Sigma_1)$ by a combination of ambient isotopy and adding or subtracting finitely many handles by ambient surgery.

Theorem 9.2.4. [Kaw96, Lemma 5.2.4] Any two Seifert surfaces of a link L are S-equivalent.

Let L be an oriented link with Seifert surface Σ of genus g. Then $H_1(\Sigma;\mathbb{Z})$ is a f.g. free abelian group of rank $2g$. Choose a basis $\{[\gamma_i]\}_{i=1}^{2g}$ of $H_1(\Sigma;\mathbb{Z})$ with each basis homology classes $[\gamma_i]$ represented by simple, closed curve $\gamma_i \subset \Sigma$. Use the triviality of the normal bundle $\nu_{\Sigma\subset S^3}$ to define a small bi-collar $\Sigma\times[-1,1]$ of $\Sigma \subset S^3$ and for each $1 \leqslant i \leqslant n$ define $\gamma_i^+ = \gamma_i \times \{1\} \subset \Sigma\times[-1,1]$ to be the simple, closed curve in S^3 obtained by pushing γ_i in the positive normal direction to Σ.

Definition 9.2.5. The *Seifert matrix* of Σ with respect to this bi-collar and this choice of basis is the $2g \times 2g$ matrix V defined by

$$V_{i,j} = \mathrm{Lk}(\gamma_i, \gamma_j^+), \quad (1 \leqslant i, j \leqslant 2g)$$

and the *Seifert form* of Σ is the bilinear form

$$V : H_1(\Sigma;\mathbb{Z}) \times H_1(\Sigma;\mathbb{Z}) \to \mathbb{Z}.$$

The ambiguity in the choice of Seifert surface for a link means that the Seifert matrix of a link is only unique up to an algebraic S-equivalence relation.

Definition 9.2.6. Two $n \times n$ integral matrices are *S-equivalent* if one can be transformed into the other by a finite sequence of the following operations:

(i) $V \mapsto PVP^t$ with P integral and unimodular.

(ii) $V \mapsto \left(\begin{array}{c|cc} V & \xi & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$

(iii) $V \mapsto \left(\begin{array}{c|cc} V & 0 & 0 \\ \hline \xi & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$.

Theorem 9.2.7. [Mur65, Theorem 3.1] The S-equivalence class of the Seifert matrix of a link is an isotopy invariant.

In chapter 10 we will develop a chain level lift of the Seifert matrix for a link which can be expressed as the closure of a braid. In chapter 11 we will develop equivalence relations, called A- and $\widehat{A}$-equivalence, such that the A-equivalence class of the chain level lift is an isotopy invariant of the braid and the $\widehat{A}$-equivalence class of the chain level lift is an isotopy invariant of the closure of the braid.

9.3 Regular braids, geometric braids and closures

We are particularly interested in those links which can be written as the closure of a braid.

Definition 9.3.1. For $1 \leqslant i \leqslant n-1$ the *elementary n-strand braid* σ_i is the n-strand braid of polygonal arcs with a single crossing of strand i over strand $i+1$ and no crossings between any other pairs of adjacent stands and the *elementary n-strand braid* σ_i^{-1} is the n-strand braid with a single crossing of strand i under strand $i+1$ and no crossings between any other pairs of adjacent strands. The *trivial n-strand braid* 1 is the n-strand braid of polygonal arcs with no crossings.

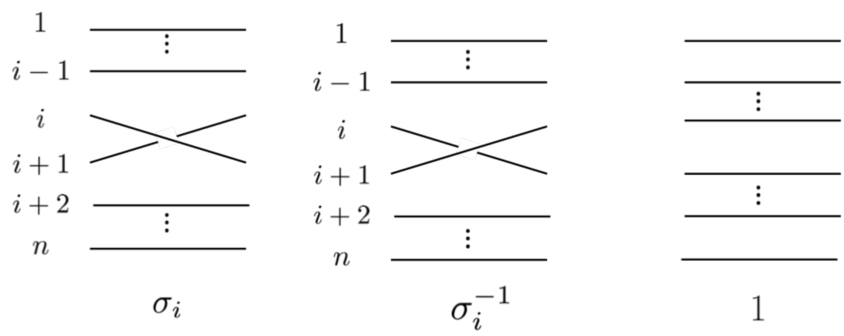


Figure 106: The elementary n-strand braids.

A *regular n-strand braid* $\beta = \beta_1\beta_2\dots\beta_\ell$ is the concatenation from left to right of finitely many elementary n-strand braids and trivial n-strand braids.

Regular braids are combinatorial models for geometric braids.

Definition 9.3.2. Let $n \geqslant 1$. A *geometric n-strand braid* β with permutation $\sigma \in S_n$ of the set $\{1, 2, \dots, n\}$ is an embedding

$$\beta : \{1, 2, \dots, n\} \times [0,1] \hookrightarrow \mathbb{R}^2 \times [0,1]; \quad (k,t) \mapsto \beta(k,t)$$

such that

$$\begin{aligned} \beta(k,0) &= (k,0,0) \in \mathbb{R}^2 \times \{0\} \qquad (1 \leqslant k \leqslant n) \\ \beta(k,1) &= (\sigma(k),0,1) \in \mathbb{R}^2 \times \{1\} \quad (1 \leqslant k \leqslant n) \end{aligned}$$

and each composition

$$[0,1] \xrightarrow{\beta(k,-)} \mathbb{R}^2 \times [0,1] \xrightarrow{\text{projection}} [0,1] \quad (1 \leqslant k \leqslant n)$$

is a homeomorphism.

Example 9.3.3. A geometric 4-strand braid with permutation $\sigma = (123)(4) \in S_4$

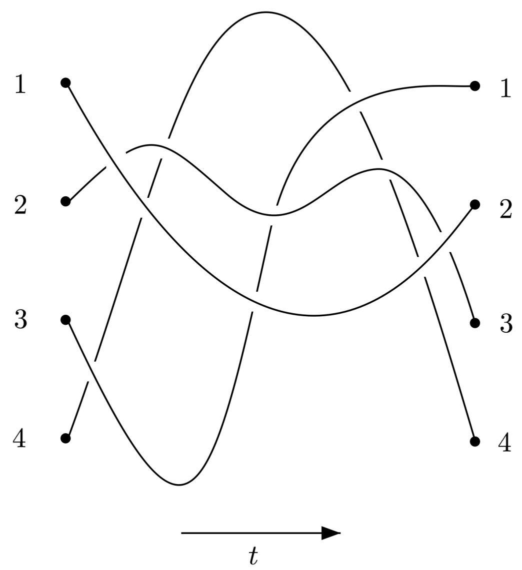


Figure 107: A 4-strand braid.

Definition 9.3.4. The *concatenation* of geometric n-strand braids β with permutation $\sigma \in S_n$ and β' with permutation $\sigma' \in S_n$ is the geometric n-strand braid

$$\beta\beta' : \{1, 2, \dots, n\} \times [0,1] \hookrightarrow \mathbb{R}^2 \times [0,1]$$

with permutation $\sigma\sigma' \in S_n$ defined by

$$\beta\beta'(k,t) = \begin{cases} \beta'(k,2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \beta(k,2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Definition 9.3.5. Two geometric n-strand braids β, β' are *isotopic* if there exists a family of geometric n-strand braids

$$\beta_s : \{1, 2, \ldots, n\} \times [0,1] \hookrightarrow \mathbb{R}^2 \times [0,1] \quad (s \in [0,1])$$

such that $\beta_0 = \beta$ and $\beta_1 = \beta'$ and each function function

$$\{1, 2, \ldots, n\} \times [0,1] \times [0,1] \to \mathbb{R}^2 \times [0,1]; \quad (k,t,s) \mapsto \beta_s(k,t) \quad (1 \leqslant k \leqslant n)$$

is continuous.

Lemma 9.3.6. Isotopy of geometric n-strand braids is an equivalence relation. The set of isotopy classes of geometric n-strand braids is a group with:

(i) The composition of the isotopy classes $[\beta], [\beta']$ of geometric n-strand braids β, β' equal to the isotopy class $[\beta\beta']$ of the geometric n-strand braid $\beta\beta'$.

(ii) The identity element equal to the isotopy class of the geometric n-strand braid

$$\{1, 2, \ldots, n\} \times [0,1] \hookrightarrow \mathbb{R}^2 \times [0,1]; \quad (k,t) \mapsto (k,0,t)$$

(iii) The inverse of the isotopy class $[\beta]$ of a geometric n-strand braid β

$$\beta : \{1, 2, \ldots, n\} \times [0,1] \hookrightarrow \mathbb{R}^2, \quad (k,t) \mapsto \beta(k,t)$$

equal to the isotopy class of the geometric n-strand braid

$$\{1, 2, \ldots, n\} \times [0,1] \hookrightarrow \mathbb{R}^2, \quad (k,t) \mapsto \beta(k, 1-t).$$

Regular braids can be used to give a presentation of the braid group.

Theorem 9.3.7. [Art47] Each geometric n-strand braid is isotopic to a regular n-strand braid so that the braid group B_n of isotopy classes of geometric n-strand braids has a presentation

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geqslant 2, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1\rangle.$$

In particular, two geometric n-strand braids β, β' are isotopic if and only if they are isotopic to regular n-strand braids determined by braid words β, β' from the alphabet $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_{n-1}^{-1}\}$ such that β' can be obtained from β by applying finitely many of the relations

(i) $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$

(ii) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geqslant 2$

(iii) $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i-j| = 1$.

Every braid β determines a link $\widehat{\beta}$ by a closure operation.

Proposition 9.3.8. [KT08, p.18] Let $U \subset \mathbb{R}^2$ be an open disc containing the set of points $\{(1,0), (2,0), \ldots, (n,0)\}$.

(i) Any geometric n-strand braid

$$\beta : \{1, 2, \ldots, n\} \times [0,1] \hookrightarrow \mathbb{R}^2 \times [0,1]$$

is isotopic to a geometric n-strand braid

$$\beta' : \{1, 2, \ldots, n\} \times [0,1] \hookrightarrow U \times [0,1] \hookrightarrow \mathbb{R}^2 \times [0,1]$$

with image contained in $U \times [0,1]$.

(ii) Any two geometric n-strand braids which are isotopic in $\mathbb{R}^2 \times [0,1]$ and have image in $U \times [0,1]$ are isotopic in $U \times [0,1]$.

(iii) The quotient map

$$D^2 \times [0,1] \to D^2 \times S^1 = \frac{D^2 \times [0,1]}{(x,0) \sim (x,1)}$$

sends a geometric n-strand braid β' contained in $U \times [0,1] \subset D^2 \times [0,1] \subset \mathbb{R}^2 \times [0,1]$ to a canonically oriented link $\widehat{\beta}$ contained in $U \times S^1 \subset D^2 \times S^1$.

(iv) Given a geometric n-strand braid β, the isotopy class of the link $\widehat{\beta}$ in $D^2 \times S^1$ relative to the boundary $S^1 \times S^1$ depends only on the isotopy class of β.

Definition 9.3.9. The *closure* of a regular n-strand braid β is the isotopy class of the link $\widehat{\beta}$ formed from any geometric n-strand braid isotopic to the regular n-strand braid β.

Proposition 9.3.8 ensures that the closure operation is well-defined. It is often convenient to picture the closure of a braid, which is oriented from left to right, as follows

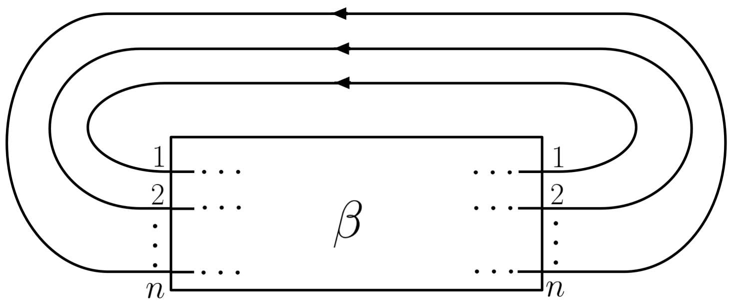


Figure 108: The closure of a braid.

Theorem 9.3.10. [Ale23] Every oriented link in S^3 is isotopic to the closure of a regular braid.

The choice of such a braid is highly non-canonical. However, by Markov's theorem [Mar] any two such braids (with the same braid axis) differ only by a braid isotopy and a finite number of braid stabilisations and destabilisations.

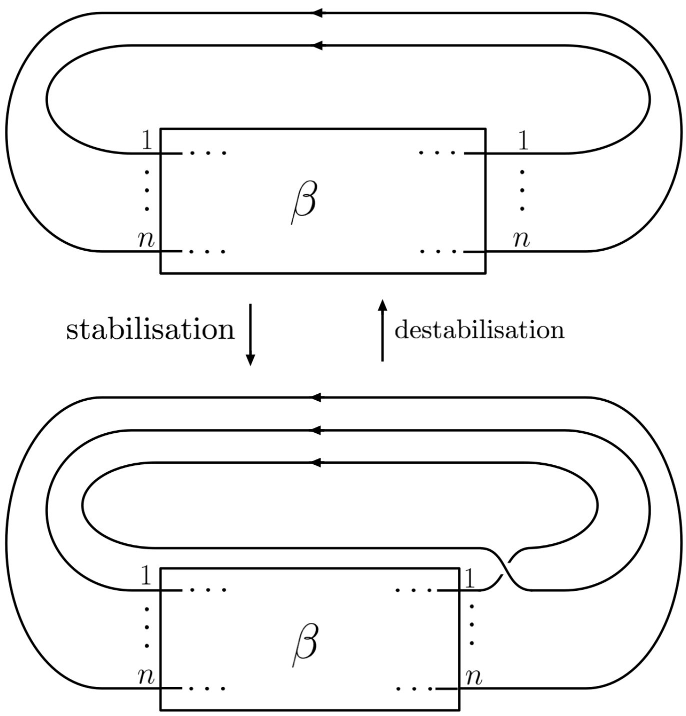


Figure 109: The stabilisation and destabilisation operations.

Chapter 10

A chain level Seifert form

In this chapter we associate to each braid β a 1-dimensional simplicial complex $K(\beta)$ and a $\mathbb{Z}[\frac{1}{2}]$-valued bilinear form $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$. We show that there is an embedding $K(\beta) \hookrightarrow \Sigma$ with image a deformation retract of the canonical Seifert surface Σ of $\widehat{\beta}$ constructed by Seifert's algorithm. We examine how the image $K(\beta)$ is pushed along the normal to the Seifert surface to show that λ_β descends to the Seifert form on the homology level. We then show that the bilinear form λ_β is additive under the concatenation of braids and we compare our chain level Seifert form to the space polygon linking model of Banchoff and the inductive surgery-theoretic Seifert form of Ranicki.

10.1 Pushing fences

Definition 10.1.1. The *fence* of the elementary n-strand braid $\sigma_i^{\pm 1}$ with a single crossing between strand i and strand $i+1$ is the oriented 1-dimensional simplicial complex $K(\beta)$ with $2n$ 0-simplices and $(n+1)$ 1-simplices as shown below. The *fence* of the trivial n-strand braid 1 is the 0-dimensional simplicial complex $K(1)$ with n 0-simplices as shown below.

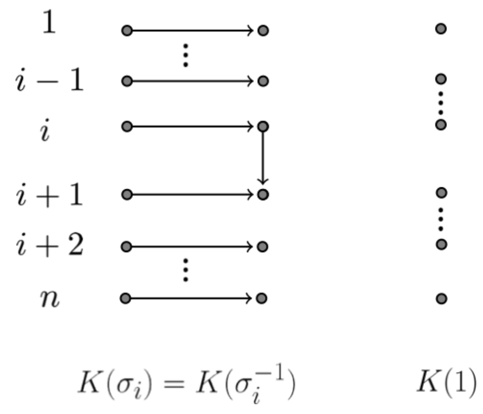


Figure 110: The fences associated to the elementary braids $\sigma_i^{\pm 1}$ and the trivial braid.

The *fence* of a regular n-strand braid $\beta = \beta_1\beta_2\ldots\beta_\ell$ is the concatenation of the fences $K(\beta_1), K(\beta_2), \ldots, K(\beta_\ell)$ from left to right so that $K(\beta_1\beta_2\ldots\beta_\ell) = \cup_{i=1}^{\ell} K(\beta_i)$ where $K(\beta_i)$ intersects $K(\beta_{i+1})$ in the right hand vertex set of $K(\beta_i)$ and the left hand vertex set of $K(\beta_{i+1})$.

Example 10.1.2. The 3-strand braid $\beta = \sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2$

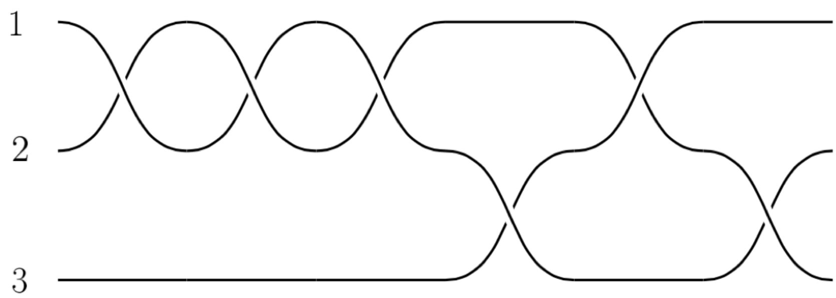


Figure 111: The braid $\sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2$.

has the fence

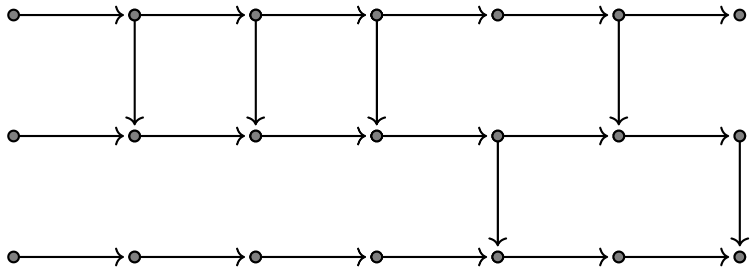


Figure 112: The fence $K(\sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2)$.

Proposition 10.1.3. For a regular braid β with closure $\widehat{\beta}$ let Σ be the canonical Seifert surface of $\widehat{\beta}$ constructed by Seifert's algorithm. There is an inclusion $K(\beta) \hookrightarrow \Sigma$ which is a homotopy equivalence.

Proof. Suppose that $\beta = \beta_1\beta_2\ldots\beta_\ell$ is a regular n-strand braid with ℓ crossings where each β_i is an elementary n-strand braid . The orientation of the n strands of the braid from left to right induces an orientation of the link $\widehat{\beta}$ in a natural way. Seifert's algorithm resolves the ℓ crossings of $\hat{\beta}$ to produce n Seifert circles. The Seifert circles may be labelled $1, 2, \ldots, n$, stacked one below the other with 1 at the top and n at the bottom and then filled in with discs. For each $1 \leqslant k \leqslant \ell$ we then attach a twisted band between the Seifert circles corresponding the crossing encoded by β_k. The order in which the bands are attached from left to right is determined by the order in the braid word $\beta_1\beta_2\ldots\beta_\ell$.

Firstly suppose that Σ is connected. A deformation retraction of Σ onto an embedding of $K(\beta)$ is obtained by pushing the left and right most parts of the discs to meet the ends of $K(\beta)$ and then contracting each of the twisted bands to its central vertical core and contracting each of the discs to a part of its horizontal diameter. The inclusion $K(\beta) \hookrightarrow \Sigma$ is a homotopy inverse. The reader should try to visualise this in the case $\beta = \sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2$.

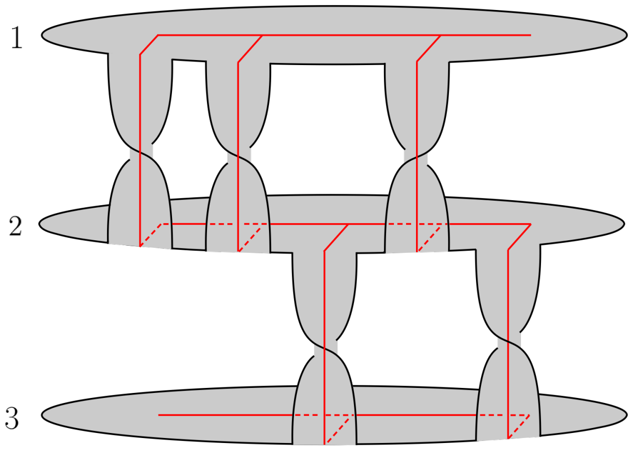


Figure 113: The inclusion of $K(\beta)$ into Σ.

Now suppose that $\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \ldots \sqcup \Sigma_k$ is disconnected with k connected components. It is then possible to write $\beta = \beta_1' \sqcup \beta_2' \sqcup \ldots \sqcup \beta_k'$ for sub-braids $\beta_i' \subset \beta$ such that Σ_i is the connected Seifert surface for the closure of the braid β_i'. Similarly we may write $K(\beta) = K(\beta_1') \sqcup K(\beta_2') \sqcup \ldots K(\beta_k')$. It follows from the connected case that the inclusion $K(\beta_i) \hookrightarrow \Sigma_i$ is a homotopy equivalence and the inclusion $K(\beta) \hookrightarrow \Sigma$ is a homotopy equivalence. □

Definition 10.1.4. For a regular n-strand braid β with a fence $K(\beta)$ define a bilinear form $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ with values on the basis oriented 1-simplices as follows:

$$\lambda_\beta(x,y) = \begin{cases} -\frac{1}{2} & \text{if if } x = y = \downarrow = \text{ a vertical simplex corresponding to a } \sigma_i \\ \frac{1}{2} & \text{if } x = y = \downarrow = \text{ a vertical simplex corresponding to a } \sigma_i^{-1} \\ \frac{1}{2} & \text{if } (x,y) = (\to,\downarrow) \text{ are adjacent simplices meeting like } \\ \frac{1}{2} & \text{if } (x,y) = (\downarrow,\to) \text{ are adjacent simplices meeting like } \\ 0 & \text{otherwise.} \end{cases}$$

Example 10.1.5. If the 1-simplices in the fence $K(\beta)$ from Example 1 are labelled as follows

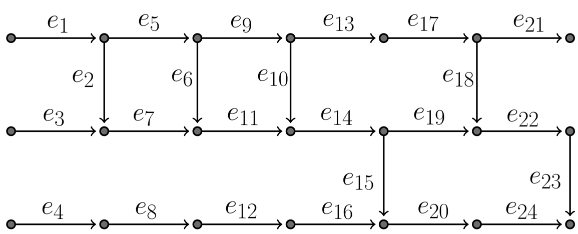


Figure 114: Labelled 1-simplices in the fence $K(\sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2)$.

then $C_1(K(\beta);\mathbb{Z}) = \mathbb{Z}\langle e_1, e_2, \ldots, e_{24}\rangle$ and for pairs of basis elements $(x,y) \in \{e_1, e_2 \ldots, e_{24}\}^2$

we have

$$\lambda_\beta(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,y) \in \{(e_1,e_2),(e_2,e_3),(e_5,e_6),(e_6,e_7)(e_9,e_{10}),(e_{10},e_{11}),(e_{14},e_{15}), \\ & (e_{15},e_{15}),(e_{15},e_{16}),(e_{17},e_{18}),(e_{18},e_{19}),(e_{22},e_{23}),(e_{23},e_{24})\} \\ -\frac{1}{2} & \text{if } (x,y) \in \{(e_2,e_2),(e_6,e_6),(e_{10},e_{10}),(e_{18},e_{18}),(e_{23},e_{23})\} \\ 0 & \text{otherwise.} \end{cases}$$

The motivation for the chain level pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ is as follows. Let $\beta = \sigma_i$ be the elementary n-strand braid with a single crossing of strand i over strand $i+1$. The Seifert surface for $\hat{\beta}$ consists of a disjoint union of n disks, stacked one above the other with a single twisted band attached from disc i to disc $i+1$. Smooth the corners of Σ and choose the positive normal direction to the smoothed Seifert surface to be in the upwards direction. Let K_i be the embedded part of K between disc i and disc $i+1$. If K_i^+ is obtained by pushing K_i in the direction of the positive normal, then 'reversing' the embeddings produces disjoint simplicial complexes with crossings of the following type

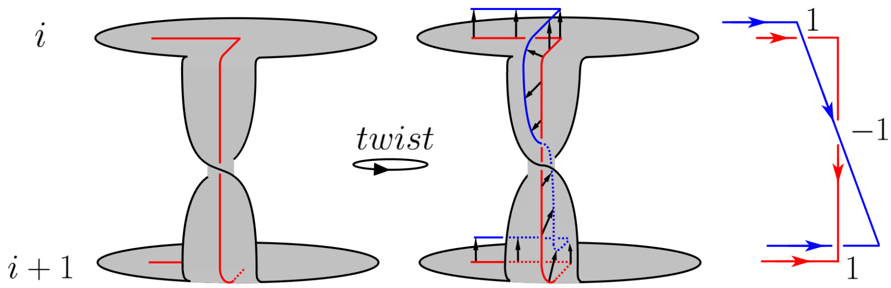


Figure 115: Pushing K_i in the normal direction.

The twist in the diagram refers to the direction of the twist in the attached band and the resulting twist of the positive normal vector to the Seifert surface along the vertical part of the red curve. In the case $\beta = \sigma_i^{-1}$ we obtain crossings of the type

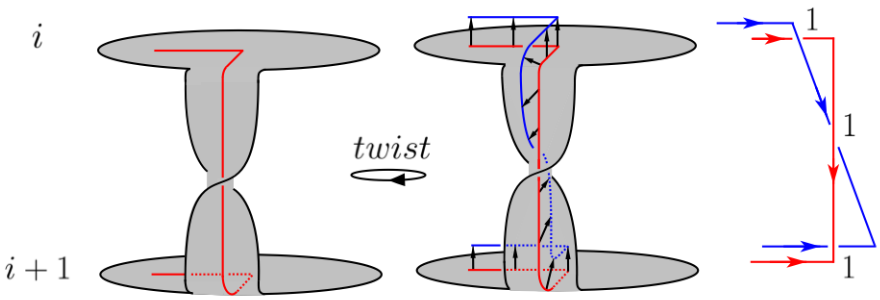


Figure 116: Pushing K_i in the normal direction.

Recall that from Definition 9.1.3 that the linking number of the components of a two component oriented link may be computed as one half of the sum of the signed crossings between one component and the other. The crossings above define a pairing $\lambda : C_1(K_i;\mathbb{Z}) \times C_1(K_i^+;\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$. Since K_i and K_i^+ are simplicially isomorphic we may equivalently think of this as a pairing $\lambda : C_1(K_i;\mathbb{Z}) \times C_1(K_i;\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ which is given by

$$\lambda(x,y)=\begin{cases}-\frac{1}{2} & \text{if if } x=y=\downarrow=\text{ a vertical simplex corresponding to a } \sigma_i\\ \frac{1}{2} & \text{if } x=y=\downarrow=\text{ a vertical simplex corresponding to a } \sigma_i^{-1}\\ \frac{1}{2} & \text{if } (x,y)=(\rightarrow,\downarrow) \text{ are adjacent simplices meeting like}\\ \frac{1}{2} & \text{if } (x,y)=(\downarrow,\rightarrow) \text{ are adjacent simplices meeting like}\\ 0 & \text{otherwise.}\end{cases}$$

10.2 Descending to homology

We now show that the chain level formula gives the Seifert form on the homology level.

Theorem 10.2.1. Let β be a braid with Seifert surface Σ constructed by Seifert's algorithm and Seifert form $V : H_1(\Sigma;\mathbb{Z})\times H_1(\Sigma;\mathbb{Z})\to\mathbb{Z}$. If K is the fence of β then inclusion $K\hookrightarrow\Sigma$ induces an isomorphism $H_1(K;\mathbb{Z})\cong H_1(\Sigma;\mathbb{Z})$ with a commutative diagram

$$\begin{array}{ccc} H_1(K;\mathbb{Z})\times H_1(K;\mathbb{Z}) & \xrightarrow{[\lambda]} & \mathbb{Z}\subset\mathbb{Z}[\frac{1}{2}] \\ \downarrow\cong & \nearrow_{V} & \\ H_1(\Sigma;\mathbb{Z})\times H_1(\Sigma;\mathbb{Z}) & & \end{array}$$

Proof. Suppose that $\beta=\beta_1\beta_2\ldots\beta_\ell$ is a regular n-strand braid with ℓ crossings where each β_i is an elementary n-strand braid. Proposition 10.1.3 implies that there is an inclusion $K\hookrightarrow\Sigma$ which is a homotopy equivalence and hence $H(K;\mathbb{Z})\cong H(\Sigma;\mathbb{Z})$. Suppose that Σ has k connected components. For $1\leqslant i\leqslant n-1$ let l_i denote the number of crossings between strand i and strand $i+1$. By [Col12, Lemma 3.1] we may write

$$b_1(K;\mathbb{Z})=b_1(\Sigma;\mathbb{Z})=\sum_{i=1}^{n-1}(l_i-1)=l-k+n$$

and Collins shows that there is a basis of $H_1(\Sigma;\mathbb{Z})$ with one basis element for each pair of consecutive crossings between between adjacent strands. More explicitly, a pair of consecutive crossings between strand i and strand $i+1$

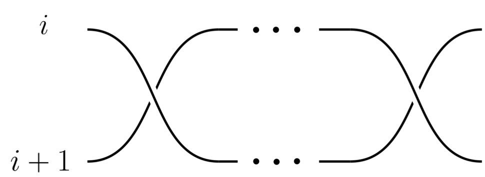


Figure 117: A pair of consecutive crossings between strand i and strand $i+1$.

determines a 1-cycle, shown in red below as an embedded polygonal circle oriented in the clockwise direction, in the part of Σ which is created by attaching to two Seifert disc two twisted bands corresponding to the two crossings between the same strands.

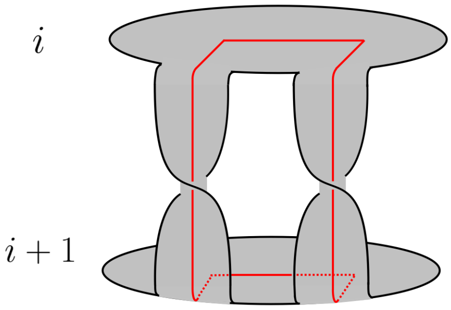


Figure 118: The 1-cycle.

The cycles may be labelled $c_1, c_2, \ldots, c_{\ell-n+k} \in Z_1(\Sigma;\mathbb{Z})$ according to their positions from left to right along the braid diagram. The set of homology classes $[c_1], [c_2], \ldots, [c_{\ell-n+k}]$ is then a basis for $H_1(\Sigma)$. The cycles $c_1, c_2, \ldots, c_{\ell-n+k} \in Z_1(\Sigma;\mathbb{Z})$ induce cycles $c'_1, c'_2, \ldots, c'_{\ell-n+k} \in Z_1(K;\mathbb{Z})$ giving a basis $[c'_1], [c'_2], \ldots, [c'_{\ell-n+k}]$ of $H_1(K;\mathbb{Z})$. The homology class $[c'_i] \in H_1(K;\mathbb{Z})$ maps to the homology class $[c_i] \in H_1(\Sigma;\mathbb{Z})$ under the isomorphism $H_1(K;\mathbb{Z}) \cong H_1(\Sigma;\mathbb{Z})$ induced by the inclusion $K \hookrightarrow \Sigma$. If c_j^+ is the push of the cycle c_j in the positive normal to Σ then it suffices to show that $\lambda(c'_i, c'_j) = Lk(c_i, c_j^+)$ for $1 \leqslant i, j \leqslant \ell - n + k$. Note that since the linking number $Lk(c_i, c_j^+)$ is always an integer and $H_1(K;\mathbb{Z})$ is a free abelian group this implies that $\lambda : H_1(K;\mathbb{Z}) \times H_1(K;\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ factors through a map $H_1(K;\mathbb{Z}) \times H_1(K;\mathbb{Z}) \to \mathbb{Z}$. The proof now proceeds by cases.

Diagonal Entries: Suppose that $i = j$. The diagram below shows c_i in red and its push off c_i^+ in blue

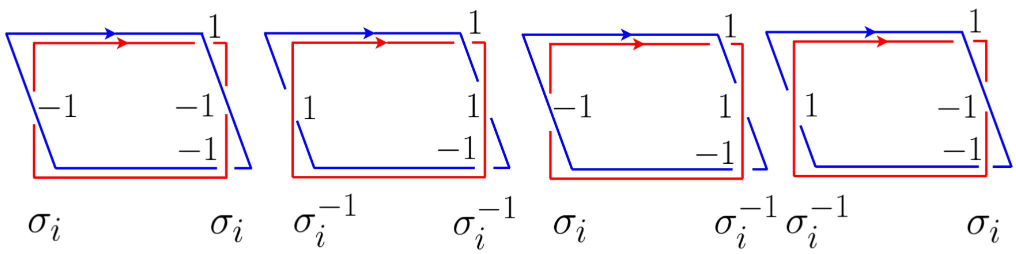


Figure 119: The cycle c_i and its pushoff c_i^+.

so that the self-linking numbers are given by

$$Lk(c_i, c_i^+) = \begin{cases} -1 & \text{if both crossings correspond to a } \sigma_i \\ \ 1 & \text{if both crossings correspond to a } \sigma_i^{-1} \\ \ 0 & \text{otherwise.} \end{cases}$$

The cycle $c_i \in Z_1(\Sigma;\mathbb{Z})$ corresponds to a cycle $c'_i \in Z_1(K;\mathbb{Z})$ which may be written as $c'_i = -e_1 + (\sum_{p=2}^{s+2} e_p) - (\sum_{p=s+3}^{2s+2} e_p)$ as in the diagram

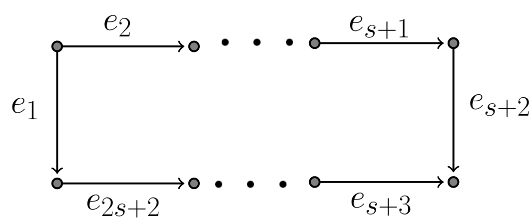


Figure 120: Labelled 1-simplices in the cycle c_i'.

It follows that

$$\begin{aligned}\lambda(c_i', c_i') &= \lambda(-e_1, -e_1) + \lambda(e_{s+1}, e_{s+2}) + \lambda(e_{s+2}, e_{s+2}) + \lambda(e_{s+2}, -e_{s+3}) \\ &= \lambda(e_1, e_1) + \frac{1}{2} + \lambda(e_{s+2}, e_{s+2}) - \frac{1}{2} \\ &= \lambda(e_1, e_1) + \lambda(e_{s+2}, e_{s+2})\end{aligned}$$

and hence

$$\lambda(c_i', c_i') = \begin{cases} -1 & \text{if both crossings correspond to a } \sigma_i \\ 1 & \text{if both crossings correspond to a } \sigma_i^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Non-Diagonal Entries: Suppose that $1 \leqslant i < j \leqslant \ell - n + k$. . Let the cycle c_i' be written as in the diagonal case and let the cycle c_j' be written as $c_j' = -f_1 + (\sum_{q=2}^{t+2} f_q) - (\sum_{i=t+3}^{2t+2} f_q)$ as in the diagram

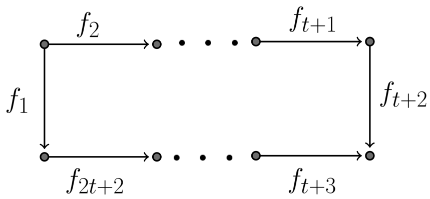


Figure 121: Labelled 1-simplices in the cycle c_j'.

Let $E = \{e_p\}_{p=1}^{2s+2}$ and $F = \{f_q\}_{q=1}^{2t+2}$. It is enough to consider the five cases of the relative positions of the cycles as in [Col12, Section 3.3]:

1. Either $E \cap F = \{e_{s'}, e_{s'+1}, \ldots, e_{s''}\} = \{f_{t+3}, f_{t+4}, \ldots, f_{2t+2}\}$ for some $2 < s' < s'' < s+1$ with $e_{s'} = f_{2t+2}$ and $e_{s''} = f_{t+3}$ as in

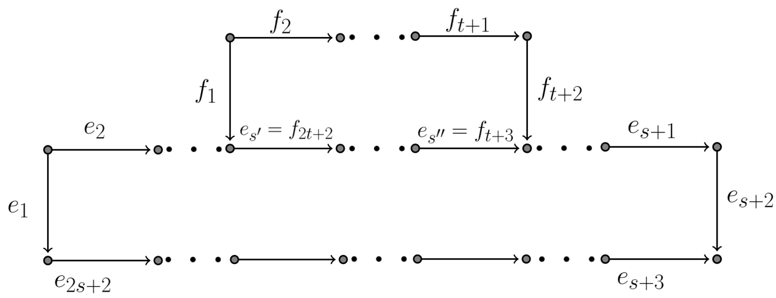


Figure 122: The cycles c_i' and c_j'.

or $E \cap F = \{e_2, e_3, \ldots, e_{s+1}\} = \{f_{t'}, f_{t'+1}, \ldots, f_{t''}\}$ for some $t + 3 < t' < t'' < 2t + 2$ with $e_2 = f_{t''}$ and $e_{s+1} = f_{t'}$ as in

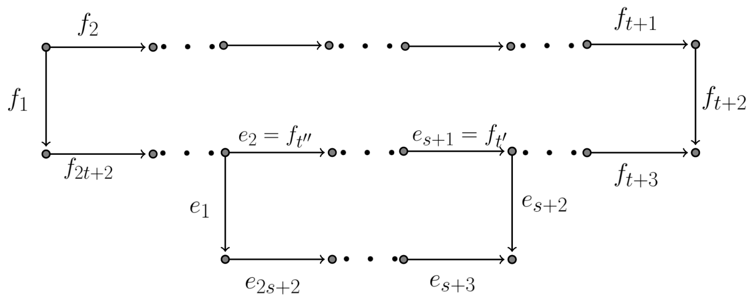


Figure 123: The cycles c'_i and c'_j.

so that in the first case

$$\lambda(c'_i, c'_j) = 0$$

$$\lambda(c'_j, c'_i) = \lambda(-f_1, e_{s'-1}) + \lambda(f_{t+2}, e_{s''}) = -\frac{1}{2} + \frac{1}{2} = 0$$

and in the second case

$$\lambda(c'_i, c'_j) = 0$$

$$\lambda(c'_j, c'_i) = \lambda(-f_{t''+1}, -e_1) + \lambda(-f_{t'}, e_{s+2}) = \frac{1}{2} - \frac{1}{2} = 0.$$

The push-off c_j^+ of c_j in relation to c_i is given in the first (respectively second) case by

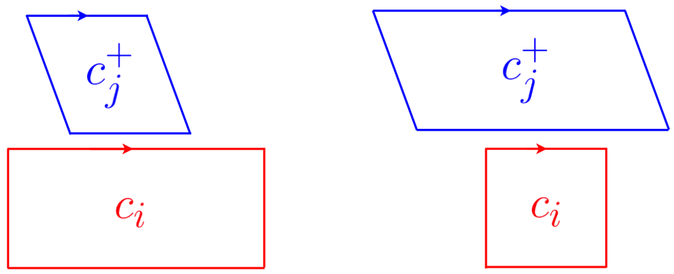


Figure 124: The cycles c_j^+ and c_i.

and in either case $\mathrm{Lk}(c_i, c_j^+) = 0$. The push-off c_i^+ of c_i in relation to c_j is given in the first (respectively second) case by and in either case $\mathrm{Lk}(c_j, c_i^+) = 0$.

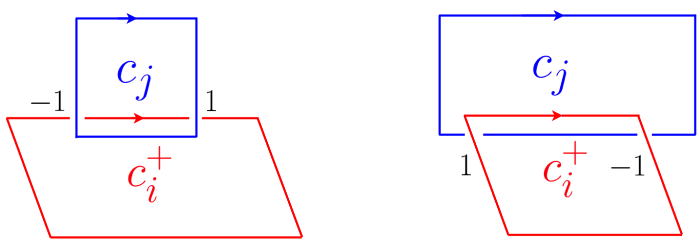


Figure 125: The cycles c_i^+ and c_j.

2. In this case E and F are disjoint as in

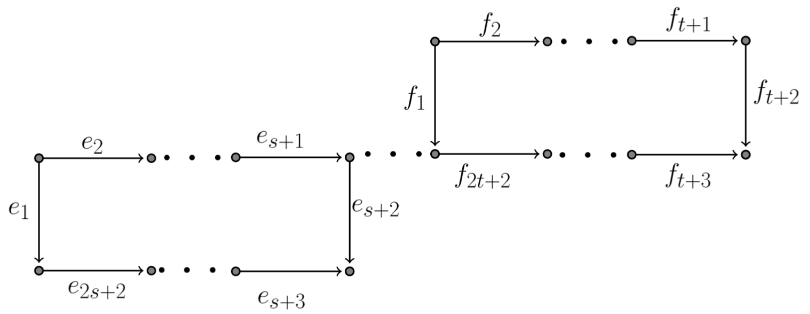
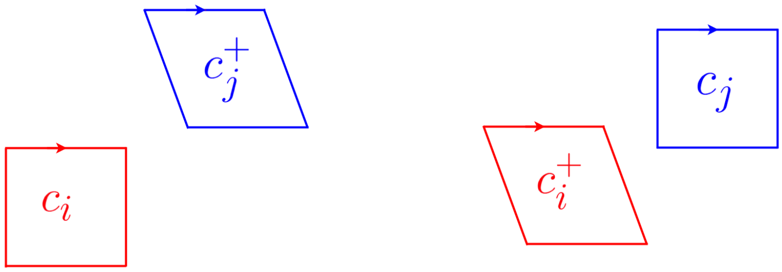


Figure 126: The cycles c_i' and c_j'.

so it is immediate that $\lambda(c_i', c_j') = \lambda(c_j', c_i') = 0$. The push-off c_j^+ of c_j in relation to c_i (respectively the push-off c_i^+ of c_i in relation to c_j) is given by



so that $\mathrm{Lk}(c_j, c_i^+) = \mathrm{Lk}(c_i^+, c_j) = 0$.

3. In this case $E \cap F = \{e_{s+2}\} = \{f_1\}$ as in

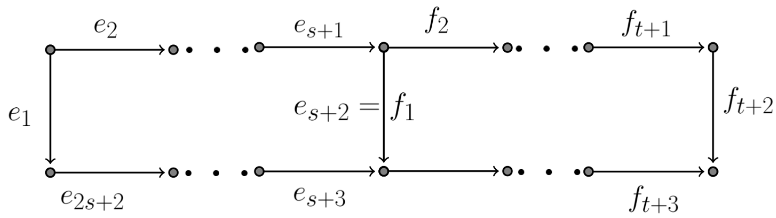


Figure 127: The cycles c_i' and c_j'.

and it follows that

$$\lambda(c_i', c_j') = \lambda(e_{s+1}, -f_1) + \lambda(e_{s+2}, -f_1) = -\frac{1}{2} - \lambda(f_1, f_1)$$

and hence

$$\lambda(c_i', c_j') = \begin{cases} 0 & \text{if } f_1 \text{ corresponds to a } \sigma_i \\ -1 & \text{if } f_1 \text{ corresponds to a } \sigma_i^{-1}. \end{cases}$$

The push-off c_j^+ of c_j in relation to c_i is given in the first (respectively second) case by

so that

$$\mathrm{Lk}(c_i, c_j^+) = \begin{cases} 0 & \text{if } f_1 \text{ corresponds to a } \sigma_i \\ -1 & \text{if } f_1 \text{ corresponds to a } \sigma_i^{-1}. \end{cases}$$

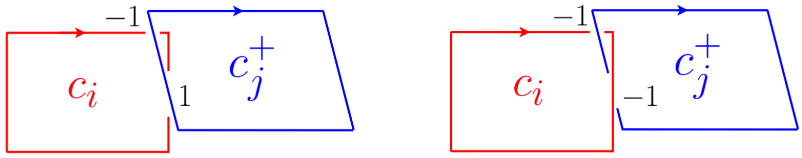


Figure 128: The cycles c_j^+ and c_i.

Similarly

$$\lambda(c_j', c_i') = \lambda(-f_1, e_{s+2}) + \lambda(-f_1, -e_{s+3}) = -\lambda(f_1, f_1) + \frac{1}{2}$$

and hence

$$\lambda(c_j', c_i') = \begin{cases} 1 & \text{if } f_1 \text{ corresponds to a } \sigma_i \\ 0 & \text{if } f_1 \text{ corresponds to a } \sigma_i^{-1}. \end{cases}$$

The push-off c_i^+ of c_i in relation to c_j is given in the first (respectively second case) by

Figure 129: The cycles c_i^+ and c_j.

so that

$$\mathrm{Lk}(c_j, c_i^+) = \begin{cases} 1 & \text{if } f_1 \text{corresponds to a } \sigma_i \\ 0 & \text{if } f_1 \text{ corresponds to a } \sigma_i^{-1}. \end{cases}$$

4. In this case E and F are disjoint as in

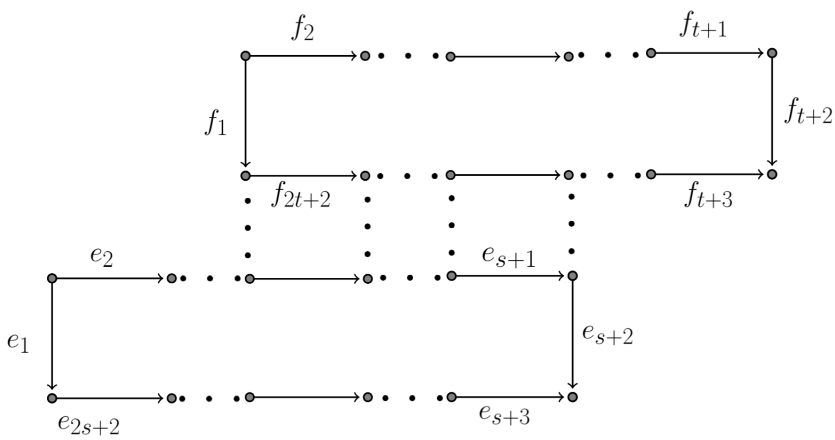


Figure 130: The cycles c_i' and c_j'.

so it is immediate that $\lambda(c_i', c_j') = \lambda(c_j', c_i') = 0$. The push-off c_j^+ in relation to c_i and c_i^+ in relation to c_j are given by the similar figures as in case 2 and it follows that $\mathrm{Lk}(c_j, c_i^+) = \mathrm{Lk}(c_i^+, c_j) = 0$.

5. Either $E \cap F = \{e_{s+3}, e_{s+4}, \ldots, e_{s'}\} = \{f_2, f_3, \ldots, f_{t'}\}$ for some $s+3 \leqslant s' < 2s+2$ and $2 \leqslant t' < t+1$ with $e_{s'} = f_2$ and $e_{s+3} = f_{t'}$ as in

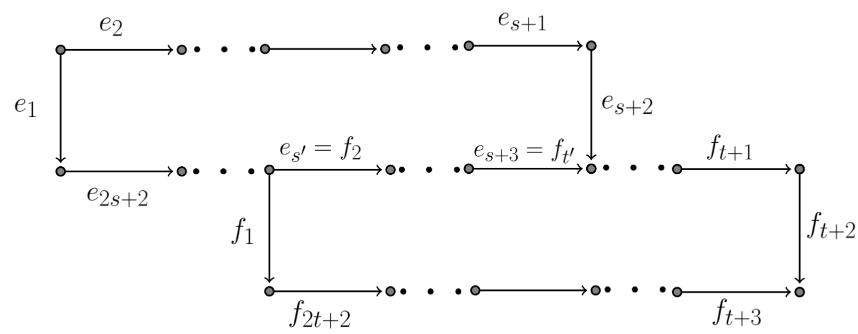


Figure 131: The cycles c'_i and c'_j.

or $E \cap F = \{e_{s'}, e_{s'+1}, \ldots, e_{s+1}\} = \{f_{t'}, f_{t'+1}, \ldots, f_{2t+2}\}$ for some $2 < s' \leqslant s+1$ and some $t+3 < t' \leqslant 2t+2$ with $e_{s'} = f_{2t+2}$ and $e_{s+1} = f_{t'}$ as in

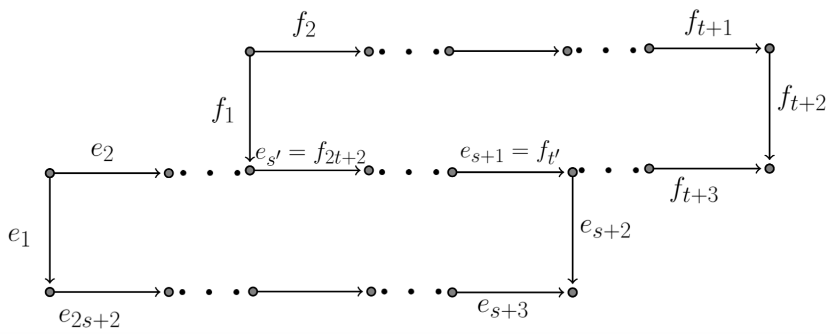


Figure 132: The cycles c'_i and c'_j.

In the first case

$$\lambda(c'_i, c'_j) = \lambda(e_{s+2}, f_{t'-1}) + \lambda(-e_{s'+1}, -f_1) = \frac{1}{2} + \frac{1}{2} = 1$$
$$\lambda(c'_j, c'_i) = -\lambda(f_{t'}, e_{s+2}) = 0.$$

and in the second case

$$\lambda(c'_i, c'_j) = 0$$
$$\lambda(c'_j, c'_i) = \lambda(-f_1, e_{s'-1}) + \lambda(-f_{t'}, e_{s+2}) = -\frac{1}{2} - \frac{1}{2} = -1$$

The push-off c_j^+ of c_j in relation to c_i is given in the first (respectively second case) by

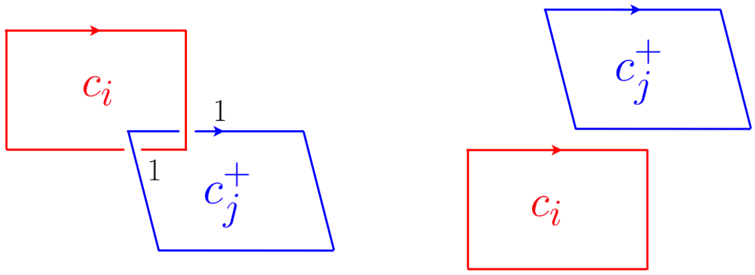


Figure 133: The cycles c_j^+ and c_i.

so that

$$\mathrm{Lk}(c_i, c_j^+) = \begin{cases} 1 & \text{in the first case} \\ 0 & \text{in the second case.} \end{cases}$$

The push-off c_i^+ of c_i in relation to c_j is given in the first (respectively second) case by so

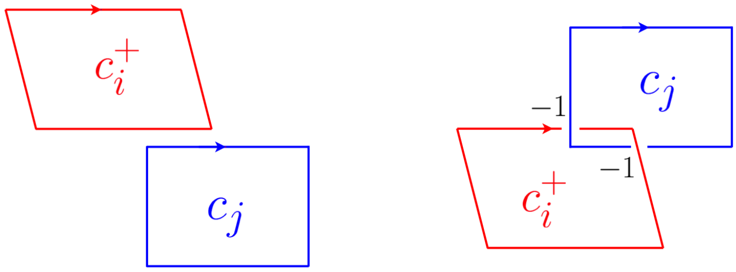


Figure 134: The cycles c_i^+ and c_j.

that

$$\mathrm{Lk}(c_j, c_i^+) = \begin{cases} 0 & \text{in the first case} \\ -1 & \text{in the second case.} \end{cases}$$

□

This motivates the following definition.

Definition 10.2.2. The *chain level Seifert pair* of a regular n-strand braid β is the pair

$$(\lambda_\beta, d_\beta) = (\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}], d_\beta : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z}))$$

Corollary 10.2.3. A regular n-strand braid β with chain level Seifert pair (λ_β, d_β) has Seifert form

$$\lambda_\beta : \ker(d_\beta) \times \ker(d_\beta) \to \mathbb{Z} \subset \mathbb{Z}[\frac{1}{2}].$$

Proof. The fence $K(\beta)$ is a 1-dimensional simplicial complex and hence

$$H_1(\Sigma) \cong H_1(K(\beta)) = \ker(d_\beta : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z}))$$

□

Example 10.2.4. The 2-strand braid $\beta = \sigma_1\sigma_1\sigma_1$ with closure $\hat{\beta}$ the trefoil knot

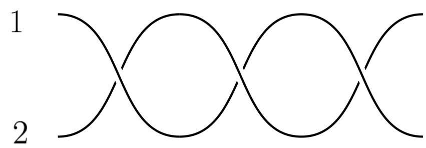


Figure 135: The braid $\sigma_1\sigma_1\sigma_1$.

has the fence $K(\beta)$

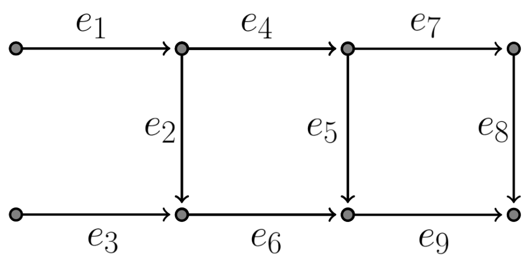


Figure 136: The fence $K(\sigma_1\sigma_1\sigma_1)$.

such that $C_1(K(\beta);\mathbb{Z})$ is a free abelian group of rank 9 with a basis $\{e_1, e_2, \ldots, e_9\}$. The bilinear pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ is represented with respect to the ordered basis $(e_1, e_2, \ldots, e_9)$ by the upper triangular matrix

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If $\gamma = e_4 + e_5 - e_6 - e_2$ and $\delta = e_7 + e_8 - e_9 - e_5$ then

$$H_1(K(\beta);\mathbb{Z}) = \ker(d_\beta : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z})$$

is a free abelian group of rank 2 with a basis $\{\gamma, \delta\}$. One then checks that the Seifert matrix with respect to the ordered basis (γ, δ) of $H_1(K(\beta);\mathbb{Z})$ is given by

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

as usual.

10.3 The effect of concatenation

We now examine the effect of the concatenation of braids on Seifert surfaces and fences to obtain an inductive formula for the chain level pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$. We first construct the Seifert surface of a closure of a braid in a way which mirrors more closely the decomposition of a braid into a concatenation of elementary braids.

Definition 10.3.1. The *open Seifert surface* $\Sigma_{\sigma_i^{\pm 1}}$ of the elementary n-strand braid $\sigma_i^{\pm 1}$ with a single crossing between strand i and strand $i+1$ is the disjoint union of a single twisted band and $n-1$ line segments, stacked vertically one above the other, as shown below

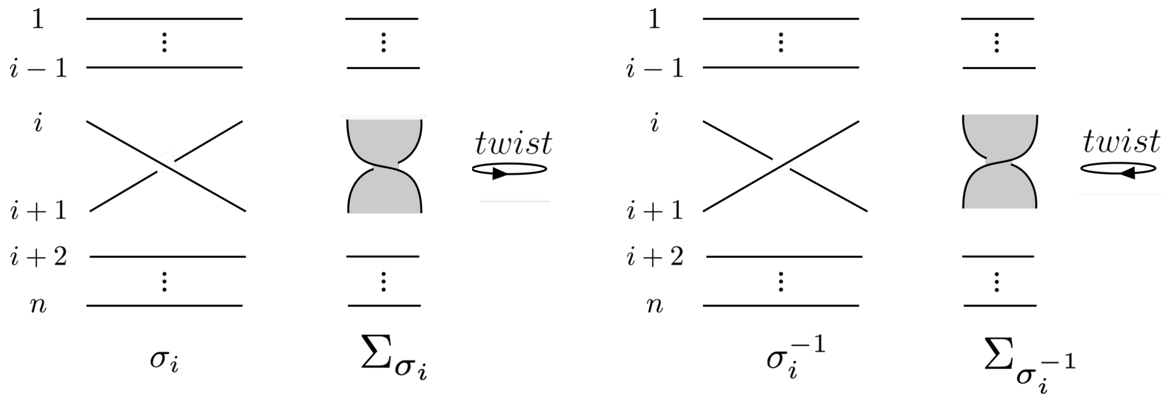


Figure 137: The open Seifert surfaces associated to the elementary braids $\sigma_i^{\pm 1}$.

The *open Seifert surface* Σ_β of a regular n-strand braid $\beta = \beta_1\beta_2\ldots\beta_\ell$ is the concatenation of the open Seifert surfaces $\Sigma_{\beta_1}, \Sigma_{\beta_2}, \ldots, \Sigma_{\beta_\ell}$ from left to right so that $\Sigma_\beta = \cup_{i=1}^{\ell}\Sigma_{\beta_i}$ where Σ_{β_i} intersects $\Sigma_{\beta_{i+1}}$ in the right hand part of Σ_{β_i} and the left hand part of $\Sigma_{\beta_{i+1}}$ as shown below

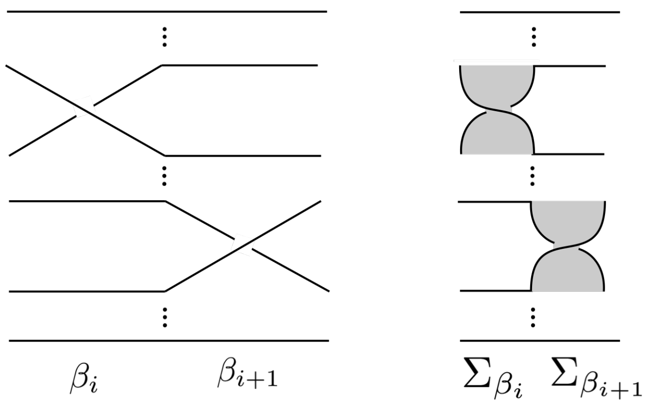


Figure 138: The concatenation of open Seifert surfaces associated to two adjacent elementary braids.

The *closure* $\widehat{\Sigma}_\beta$ of the open Seifert surface of a regular n-strand braid $\beta = \beta_1\beta_2\ldots\beta_\ell$ is the union of the open Seifert surface Σ_β with n horizontal discs as shown in the diagram below

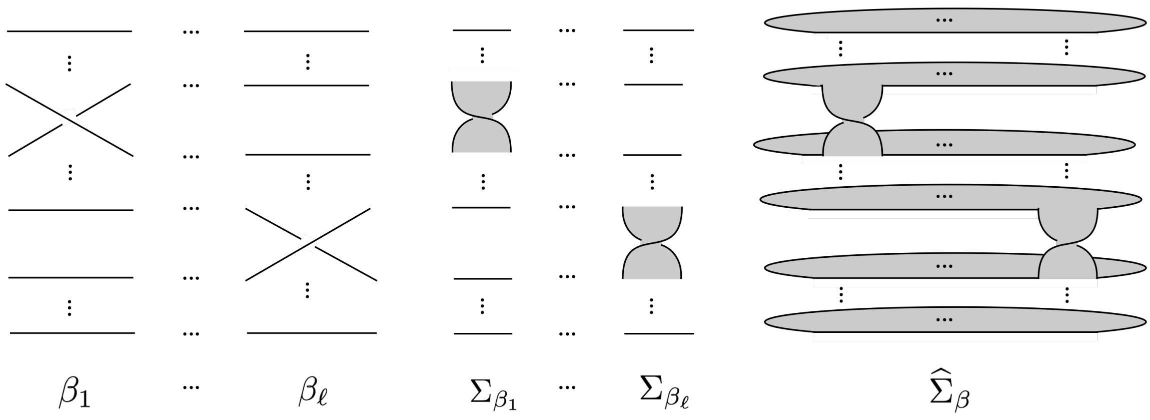


Figure 139: The closure of an open Seifert surface.

Proposition 10.3.2. Let β be a regular n-strand braid. The closure of the open Seifert surface for β is the Seifert surface for the closure of β constructed by Seifert's algorithm, that is $\widehat{\Sigma}_\beta = \Sigma_{\widehat{\beta}}$.

Proof. By induction on the length of the braid.

□

In order to obtain an inductive formula for the chain level pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ we first consider the effect of concatenating a braid β with an elementary braid β_i.

Proposition 10.3.3. Let β be a regular n-strand braid with ℓ crossings and fence K. Let β_i be an elementary n-strand braid with a single crossing between strand i and strand $i+1$. Define an $(n+1)\times(n+1)$-matrix λ_{β_i} by

$$(\lambda_{\beta_i})_{j,k} = \begin{cases} -\frac{1}{2} & \text{if } j = k = i \text{ and } \beta_i = \sigma_i \\ \frac{1}{2} & \text{if } j = k = i \text{ and } \beta_i = \sigma_i^{-1} \\ \frac{1}{2} & \text{if } j = i \text{ and } k = i+1 \\ \frac{1}{2} & \text{if } j = i+1 \text{ and } k = i+2 \\ 0 & \text{otherwise.} \end{cases}$$

Then the chain level Seifert pairing for $\beta\beta_i$ is represented by the matrix

$$\lambda_{\beta\beta_i} = \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \lambda_{\beta_i} \end{pmatrix}.$$

Proof. The fence $K(\beta_i)$ is a simplicial complex with n 0-simplices, n-horizontal simplices and a single vertical 1-simplex as shown below. With respect to the ordered basis $(f_1, f_2 \ldots, f_{n+1})$, the pairing $\lambda_{\beta_i} : C_1(K(\beta_i);\mathbb{Z}) \times C_1(K(\beta_i);\mathbb{Z}) \to \mathbb{R}$ is represented by the $(n+1)\times(n+1)$-matrix

λ_{β_i} with

$$(\lambda_{\beta_i})_{j,k} = \begin{cases} -\frac{1}{2} & \text{if } j = k = i \text{ and } \beta_i = \sigma_i \\ \frac{1}{2} & \text{if } j = k = i \text{ and } \beta_i = \sigma_i^{-1} \\ \frac{1}{2} & \text{if } j = i \text{ and } k = i+1 \\ \frac{1}{2} & \text{if } j = i+1 \text{ and } k = i+2 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that we have a matrix representation $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{R}$ with respected to an ordered basis $\mathbf{e}_{K(\beta)}$ of $C_1(K(\beta);\mathbb{Z})$. The fence $K(\beta\beta_i)$ of $\beta\beta_i$ is obtained from $K(\beta)$ by the fence $K(\beta_i)$ as follows

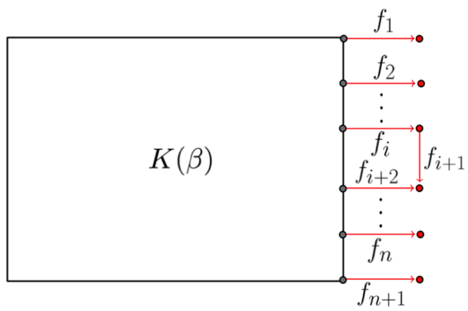


Figure 140: The fences of β, β_i and $\beta\beta_i$.

Here the red simplices are simplices added to $K(\beta)$ and the 1-simplices of $K(\beta)$ and $K(\beta_i)$ are disjoint. This gives an ordered basis $\mathbf{e}_{K(\beta\beta_i)} = (\mathbf{e}_k, f_1, f_2, \ldots, f_{n+1})$ of $C_1(K(\beta\beta_i);\mathbb{Z})$ and it follows that with respect to $\mathbf{e}_{K(\beta\beta_i)}$ that the pairing $\lambda_{\beta\beta_i} : C_1(K(\beta\beta_i);\mathbb{Z}) \times C_1(K(\beta\beta_i);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ is represented by the block diagonal matrix

$$\lambda_{\beta\beta_i} = \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \lambda_{\beta_i} \end{pmatrix}$$

as required.

□

Theorem 10.3.4. Let $\beta = \beta_1\beta_2\ldots\beta_\ell$ be a regular n-strand braid with ℓ crossings, where each β_i is an elementary n-strand braid with a single crossing between strand j_i and j_{i+1}. The chain level pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ can be represented by a block diagonal matrix

$$\begin{pmatrix} \lambda_{\beta_1} & 0 & \ldots & 0 \\ 0 & \lambda_{\beta_2} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_{\beta_\ell} \end{pmatrix}$$

where

$$(\lambda_{\beta_i})_{j,k} = \begin{cases} -\frac{1}{2} & \text{if } j = k = j_i \text{ and } \beta_i = \sigma_{j_i} \\ \frac{1}{2} & \text{if } j = k = j_i \text{ and } \beta_i = \sigma_{j_i}^{-1} \\ \frac{1}{2} & \text{if } j = j_i \text{ and } k = j_i + 1 \\ \frac{1}{2} & \text{if } j = j_i + 1 \text{ and } k = j_i + 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the definition of the concatenation of fences we may write $K(\beta) = \cup_{i=1}^{\ell} K(\beta_i)$. Since $K(\beta_i)$ intersects $K(\beta_{i+1})$ in a set of 0-simplices then $C_1(K(\beta);\mathbb{Z}) = \oplus_{i=1}^{\ell} C_1(K(\beta_i);\mathbb{Z})$. The proof follows induction on the number ℓ of crossings in the braid with the concatenation formula from Proposition 10.3.3. □

10.4 Comparison with other models

We now show that this model of a chain level Seifert pairing is chain equivalent to Banchoff's formula for the linking number of two space polygons and Ranicki's surgery-theoretic chain level linking formula.

Motivated by the Gauss map in Definition 9.1.3, Banchoff [Ban76] gave a combinatorial linking formula for two disjoint space polygons expressed in terms of the partial linking numbers of pairs of edges as follows.

Definition 10.4.1. Let $X = \{X_0, X_1, \ldots, X_{m-1}\}$ respectively $Y = \{Y_0, Y_1, \ldots, Y_{n-1}\}$ be a set of points in general position in $\mathbb{R}^3$.

(i) For a unit vector $\xi \in S^2$ let $p_\xi : \mathbb{R}^3 \to P$ denote the projection map from $\mathbb{R}^3$ onto the plane P orthogonal to ξ. A vector $\xi \in S^2$ in called *general* for X and Y if the projections $p_\xi(X), p_\xi(Y) \subset \mathbb{R}^2$ are in general position.

(ii) For a vector $\xi \in S^2$ which is general for X and Y, define $C_{i,j}(X,Y,\xi)$ to be the sign of $P_\xi(Y_{j+1} - Y_j) \times P_\xi(X_{i+1} - X_i).(\overline{X_i} - \overline{Y_j})$ if there are interior points $\overline{X_i}$ of the edge $X_i X_{i+1}$ and $\overline{Y_j}$ of the edge $Y_j Y_{j+1}$ such that $p_\xi(\overline{X_i}) = p_\xi(\overline{Y_j})$ and define $C_{i,j}(X,Y,\xi)$ to be zero otherwise

The linking number of two space polygons is then expressible as the sum of partial linking numbers of all edge pairs.

Theorem 10.4.2. [Ban76, p.1176-1177] For disjoint polygonal knots $X, Y \subset \mathbb{R}^3$ the value

$$C(X,Y,\xi) = \sum_{\substack{0 \leqslant i \leqslant m-1 \\ 0 \leqslant j \leqslant n-1}} C_{i,j}(X,Y,\xi) \in \mathbb{Z}$$

is independent of the choice of general vector $\xi \in S^2$. The linking number of the polygonal knots determined by X and Y is the average value of $C(X,Y,\xi)$, that is

$$\text{Lk}(X,Y) = \frac{1}{4\pi} \int_{\xi \in S^2} C(X,Y,\xi) d\omega = \frac{1}{4\pi} \sum_{\substack{0 \leqslant i \leqslant m-1 \\ 0 \leqslant j \leqslant n-1}} \int_{\xi \in S^2} C_{i,j}(X,Y,\xi) d\omega \in \mathbb{Z}$$

where ω is the volume form on S^2. Moreover this integral may be expressed in terms of dihedral angles of tetrahedra.

Ranicki gave an alternative chain level formula in terms of the Seifert graph. The Seifert graph of a braid records which strands of the braid cross but not whether the crossings and over-crossings or under-crossings.

Definition 10.4.3. The *Seifert graph* of a braid β is the 1-dimensional CW-complex $X(\beta)$ constructed from the canonical Seifert surface of β by collapsing each Seifert disc to a point and collapsing each twisted band to its core.

If β is an n-strand braid with ℓ-crossings then the Seifert graph $X(\beta')$ has ℓ 1-cells and n 0-cells and has a cellular chain complex of the form

$$d : C_1(X(\beta);\mathbb{Z}) \cong \mathbb{Z}^\ell \to C_0(X(\beta);\mathbb{Z}) \cong \mathbb{Z}^n.$$

If β' is another n-strand braid with ℓ' crossings then the Seifert graph $X(\beta')$ has a cellular chain complex of the form

$$d' : C_1(X(\beta');\mathbb{Z}) \cong \mathbb{Z}^{\ell'} \to C_0(X(\beta');\mathbb{Z}) \cong \mathbb{Z}^n.$$

The Seifert graph of the concatenated braid $\beta\beta'$ is a CW-complex which can be formed from the Seifert graphs of β and β' by identifying the 0-cells in pairs so that $X(\beta\beta')$ has $(\ell + \ell')$ 1-cells, n 0-cells and a cellular chain complex of the form

$$d'' = \begin{pmatrix} d & d' \end{pmatrix} : C_1(X(\beta\beta');\mathbb{Z}) \cong \mathbb{Z}^\ell \oplus \mathbb{Z}^{\ell'} \to C_0(X(\beta\beta');\mathbb{Z}) \cong \mathbb{Z}^n.$$

The closure of an n-strand geometric braid with ℓ-crossings arises as the trace of ℓ 0-surgeries on a disjoint union of n circles. Ranicki applied the algebraic theory of surgery to the geometric surgeries to obtain a formula which is defined inductively.

Definition 10.4.4.

(i) The *canonical generalised Seifert matrices* of the elementary regular n-strand braids σ_i, σ_i^{-1} are the 1×1 matrices

$$\psi_{\sigma_i} = \begin{pmatrix} 1 \end{pmatrix}, \quad \psi_{\sigma_i^{-1}} = \begin{pmatrix} -1 \end{pmatrix}.$$

(ii) Let β, β' be regular n-strand braids and let χ be the lower triangular $n \times n$ matrix with ones below the diagonal. The *generalised Seifert matrix* for the concatenated braid $\beta\beta'$ is the inductively defined matrix

$$\psi_{\beta\beta'} = \begin{pmatrix} \psi_\beta & -d^*\chi d' \\ 0 & \psi_{\beta'} \end{pmatrix} : C_1(X(\beta\beta');\mathbb{Z}) \times C_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}$$

Theorem 10.4.5. [Ran14, p.37-38] Let β, β' be regular n-strand braids. The generalised Seifert matrix

$$\psi_{\beta\beta'} : C_1(X(\beta\beta');\mathbb{Z}) \times C_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}$$

induces the Seifert form of $\beta\beta'$

$$\psi_{\beta\beta'} : H_1(X(\beta\beta');\mathbb{Z}) \times H_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}$$

on the homology level.

The equivalences of Banchoff's and Ranicki's models to the model we developed are both established via the following lemma.

Lemma 10.4.6. Let C and D be $\mathbb{Z}$-module chain complexes with C finitely generated free and concentrated in dimensions 0 and 1 and D concentrated in dimensions 1 and 2. If $H_0(C)$ is torsion free then the morphism

$$H_0(\mathrm{Hom}_{\mathbb{Z}}(C,D)) \to \mathrm{Hom}_{\mathbb{Z}}(H_1(C),H_1(D)); \quad f \mapsto f_*$$

is an isomorphism, that is any two chain maps $f,g : C \to D$ are chain homotopic if and only if $f_* = g_* : H_1(C) \to H_1(D)$.

Proof. Any $\mathbb{Z}$-module homomorphism $f : C_1 \to D_1$ fits into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & C_1 & \xrightarrow{d_C} & C_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\ 0 & \longrightarrow & D_2 & \xrightarrow[d_D]{} & D_1 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

so that there is an one-to-one correspondence between chain maps $f : C \to D$ and $\mathbb{Z}$-module homomorphisms $f : C_1 \to D_1$. It is then enough to show that if $f, f' : C \to D$ are chain maps satisfying $f_* = f'_* : H_*(C) \to H_*(D)$ then there is chain homotopy $\Delta : f \simeq f' : C \to D$. We can clearly choose $\Delta_r = 0 : C_r \to D_{r+1}$ if $r \neq 0, 1$ and it then suffices to construct $\mathbb{Z}$-module homomorphisms $\Delta_0 : C_0 \to D_1$ and $\Delta_1 : C_1 \to D_2$ such that $f - f' = \Delta_0 d_C + d_D \Delta_1 : C_1 \to D_1$.

The $\mathbb{Z}$-module $\mathrm{im}(d_C) \subset C_0$ is a submodule of a finitely generated free $\mathbb{Z}$-module and hence is also finitely generated free. Choose a basis $\{x_i\}_{i=1}^m$ of $\mathrm{im}(d_C)$ and for each x_i choose a point $z_i \in C_1$ such that $d_C(z_i) = x_i$. The short exact sequence

$$0 \to \mathrm{im}(d_C) \to C_0 \to H_0(C) \to 0$$

splits since $H_0(C)$ is f.g. free and hence there is an isomorphism $C_0 \cong \mathrm{im}(d_C) \oplus H_0(C)$. The $\mathbb{Z}$-module homomorphism $g : \mathrm{im}(d_C) \to D_1$ defined by $g(x_i) = (f - f')(z_i)$ induces a $\mathbb{Z}$-module homomorphism

$$\Delta_0 = (g, 0) : \mathrm{im}(d_C) \oplus H_0(C) \to D_1.$$

The $\mathbb{Z}$-module homomorphism $s : \mathrm{im}(d_C) \to C_1$ defined by $s(x_i) = z_i$ satisfies $d_C s = \mathrm{id}_{\mathrm{im}(d_C)}$ and hence provides a splitting of the short exact sequence

$$0 \to \ker(d_C) \to C_1 \to \mathrm{im}(d_C) \to 0$$

and induces an isomorphism $\mathrm{im}(d_C) \oplus \ker(d_C) \to C_1$. The $\mathbb{Z}$-module $\ker(d_C) \subset C_1$ is also finitely generated free and so choose a basis $\{y_j\}_{j=1}^n$ of $\ker(d_C)$. By assumption

$$(f - f')_* = 0 : H_1(C) = \ker(d_C) \to H_1(D) = \frac{D_1}{\mathrm{im}(d_D)}$$

and hence for each basis element y_j we may choose an element $w_j \in D_2$ such that $(f - f')(y_j) = d_D(w_j)$. The $\mathbb{Z}$-module homomorphism $f : \ker(d_C) \to D_2$ defined by $f(y_j) = w_j$

induces a $\mathbb{Z}$-module homomorphism $\Delta_1 = (0, f) : \mathrm{im}(d_C) \oplus \ker(d_C) \to D_2$. For element $c = (\sum_{i=1}^{m} \lambda_i z_i, \sum_{j=1}^{n} \mu_j y_j) \in C_1$ it follows that

$$\Delta_0 d_C(c) = \Delta_0(\sum_{i=1}^{m} \lambda_i x_i, 0) = \sum_{i=1}^{m} \lambda_i g(x_i) = \sum_{i=1}^{m}(f - f')(z_i) = (f - f')(\sum_{i=1}^{m} \lambda_i z_i)$$

and

$$\begin{aligned} d_D \Delta_1(c) = d_D f(\sum_{j=1}^{n} \mu_j y_j) = \sum_{j=1}^{n} \mu_j d_D f(y_j) = \sum_{j=1}^{n} \mu_j d_D(w_j) = \sum_{j=1}^{n} \mu_j (f - f')(y_j) \\ = (f - f')(\sum_{j=1}^{n} \mu_j y_j) \end{aligned}$$

and hence $(\Delta_0 d_C + d_D \Delta_1)(c) = (f - f')(c)$ as required. □

Proposition 10.4.7. Our model is chain homotopy equivalent to Banchoff's model.

Proof. Let $X = \{X_0, X_1, \ldots, X_{m-1}\}$ respectively $Y = \{Y_0, Y_1, \ldots, Y_{n-1}\}$ be a set of points in general position in $\mathbb{R}^3$. The set of vertices X respectively Y determines an oriented one-dimensional simplicial complex X respectively Y in $\mathbb{R}^3$ with positively oriented edges $\{e_i = X_i X_{i+1} | 0 \leqslant i \leqslant m-1\}$ respectively $\{f_j = Y_j Y_{j+1} | 0 \leqslant j \leqslant n-1\}$ where $X_m = X_0$ respectively $Y_n = Y_0$. By [Ban76, p.1176-1177] the linking number of the space polygons X and Y is given by

$$\mathrm{Lk}(X, Y) = \frac{1}{4\pi} \sum_{\substack{0 \leqslant i \leqslant m-1 \\ 0 \leqslant j \leqslant n-1}} \int_{\xi \in S^2} C_{i,j}(X, Y, \xi) d\omega \in \mathbb{Z}$$

where ω is the volume form on S^2. For basis elements e_i, f_j the associated integral $\frac{1}{4\pi} \int_{\xi \in S^2} C_{i,j}(X, Y, \xi) d\omega$ is in general a real number and not an integer. Banchoff's formula induces a bilinear pairing

$$\begin{aligned} &\mu : C_1(X; \mathbb{Z}) \times C_1(Y; \mathbb{Z}) \to \mathbb{R} \\ &\left(\sum_{i=0}^{m-1} a_i e_i, \sum_{j=0}^{n-1} b_j f_j \right) \mapsto \frac{1}{4\pi} \sum_{\substack{0 \leqslant i \leqslant m-1 \\ 0 \leqslant j \leqslant n-1}} a_i b_j \int_{\xi \in S^2} C_{i,j}(X, Y, \xi) d\omega. \end{aligned}$$

which has adjoint a $\mathbb{Z}$-module homomorphism

$$\mu : C_1(X; \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(C_1(Y; \mathbb{Z}), \mathbb{R}) = C^1(Y; \mathbb{R}).$$

Since X and Y are 1-dimensional simplicial complexes this is the same as a chain map $\mu : C_*(X; \mathbb{Z}) \to C^{2-*}(Y; \mathbb{R})$ by Lemma 10.4.6.

Now consider the special case where $X = K$ and $Y = K^+$ where $K = K(\beta)$ is the fence for a braid β and K^+ is its push off in the positive normal direction. This yields a chain map $\mu : C_*(K; \mathbb{Z}) \to C^{2-*}(K^+; \mathbb{R})$. Recall that the simplicial complexes K^+ and K are simplicially isomorphic and the bilinear form $\lambda : C_1(K; \mathbb{Z}) \times C_1(K; \mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ may be considered as a bilinear form $\lambda : C_1(K; \mathbb{Z}) \times C_1(K^+; \mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}] \subset \mathbb{R}$. As above, this yields a chain map $\lambda : C_*(K; \mathbb{Z}) \to C^{2-*}(K^+; \mathbb{R})$. Both λ and μ compute linking numbers when we pass to homology, that is

$$[\lambda] = [\mu] : H_*(K;\mathbb{Z}) \to H^{2-*}(K^+;\mathbb{Z}) \to H^{2-*}(K^+;\mathbb{R}).$$

By the universal coefficients theorem there is an isomorphism

$$H^{2-*}(K^+;\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_{2-*}(K^+;\mathbb{Z}),\mathbb{R})$$

and the inclusion $\mathbb{Z} \subset \mathbb{R}$ induces a monomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(H_{2-*}(K^+;\mathbb{Z}),\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_{2-*}(K^+;\mathbb{Z}),\mathbb{R}).$$

It follows that there is a factorisation

$$[\lambda] = [\mu] : H_*(K;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_{2-*}(K^+;\mathbb{Z}),\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_{2-*}(K^+;\mathbb{Z}),\mathbb{R})$$

with both maps admitting the same factorisation through $\mathrm{Hom}_{\mathbb{Z}}(H^{2-*}(K^+;\mathbb{Z}),\mathbb{Z})$. By Lemma 10.4.6 there is a chain homotopy $\lambda \simeq \mu : C_*(K;\mathbb{Z}) \to C^{2-*}(K^+;\mathbb{R})$ so that the models are the same up to chain homotopy. ☐

Our model has the advantage over Banchoff's in that the averaged partial linking numbers are $\mathbb{Z}[\frac{1}{2}]$-valued and not $\mathbb{R}$-valued.

Proposition 10.4.8. Our model is chain homotopy equivalent to Ranicki's model.

Proof. Let β be a braid with Seifert graph X and fence K. We work with the opposite orientations to Ranicki, so the differential $d : C_1(X;\mathbb{Z}) \to C_0(X;\mathbb{Z})$ is the negative of the differential Ranicki uses. This does not effect the definition of generalised Seifert matrix [Ran14, p.37-38]. Ranicki also chooses the opposite positive normal direction when defining linking numbers. This implies that the canonical generalised Seifert 1×1 for the elementary n-strands braids σ_i and σ_i^{-1} are defined in our situation by $\psi_{\sigma_i} = \begin{pmatrix} -1 \end{pmatrix}$ and $\psi_{\sigma_i^{-1}} = \begin{pmatrix} 1 \end{pmatrix}$.

The Seifert graph $X = X(\beta)$ can be produced from the fence $K = K(\beta)$ by individually collapsing each horizontal row of simplices to a point so that the quotient map $q : K \to X$ is a homotopy equivalence. The chain map $q : C(K;\mathbb{Z}) \to C(X;\mathbb{Z})$ of cellular chain complexes is then a chain homotopy equivalence. The diagram

$$\begin{array}{ccc} H_1(K;\mathbb{Z}) \times H_1(K;\mathbb{Z}) & \xrightarrow{[\lambda]} & \mathbb{Z} \subset \mathbb{R} \\ {\scriptstyle q_* \times q_*}\big\downarrow{\scriptstyle \cong} & \nearrow_{[\psi]} & \\ H_1(X;\mathbb{Z}) \times H_1(X;\mathbb{Z}) & & \end{array}$$

is commutative since both $[\lambda]$ and $[\mu]$ compute the Seifert matrix of the Seifert surface of the link $\hat{\beta}$. This is implies that

$$[\lambda] = [q^{-1}\psi q] : H_*(K;\mathbb{Z}) \to H^{2-*}(K;\mathbb{Z}) \hookrightarrow H^{2-*}(K;\mathbb{R})$$

where as before the injection $H^{2-*}(K;\mathbb{Z}) \hookrightarrow H^{2-*}(K;\mathbb{R})$ is induced by the inclusion $\mathbb{Z} \subset \mathbb{R}$ and the universal coefficients theorem. By Lemma 10.4.6 there is a chain homotopy

$$\lambda \simeq q^{-1}\psi q : C_*(K;\mathbb{Z}) \to C^{2-*}(K;\mathbb{R})$$

giving a chain homotopy equivalence to Ranicki's model. □

Our model has the advantage over Ranicki's model in that the concatenation behaviour is additive and gives an instant chain level Seifert form whereas Ranicki's model is inductively defined.

Chapter 11

Applications to isotopy and signatures

We now define equivalence relations, called A- and $\widehat{A}$-equivalence, on the chain level Seifert pair (λ_β, d_β) of a braid β to produce a universal representation of the braid group and a representation of the braid group modulo conjugacy. We then construct a chain level formula for the ω-signature of a braid.

11.1 Isotopy of braids and their closures

We first examine the effect of isotopy on the chain level Seifert pair (λ_β, d_β), firstly by an isotopy of β and secondly by an isotopy of its closure $\widehat{\beta}$ in the solid torus $D^2 \times S^1$.

Definition 11.1.1. Two square matrices with entries in $\frac{1}{2}\mathbb{Z} \subset \mathbb{R}$ are *A-equivalent* if one can be transformed into the other by a finite sequence of *A-operations* defined as follows:

(i) $$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_j} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_j} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \text{ with } |i-j| \geqslant 2$$

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i^{-1}} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_j^{-1}} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_j^{-1}} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \text{ with } |i-j| \geqslant 2$$

(ii) $$\begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_j} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\sigma_i} & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_j} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\sigma_j} & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix} \text{ with } |i-j| = 1$$

$$\begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i^{-1}} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_j^{-1}} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_j^{-1}} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\sigma_j^{-1}} & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix} \quad \text{with } |i-j|=1$$

(iii) $$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 \\ 0 & 0 & 0 & B \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i^{-1}} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i} & 0 \\ 0 & 0 & 0 & B \end{pmatrix}$$

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i^{-1}} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

We now examine the effect of an A-operation on the chain level Seifert pair (λ_β, d_β) of a braid β. Once we write β as a concatenation of elementary braids then the effect of an A-operation on λ_β is clear from Theorem 10.3.4. It then remains examine the effect of an A-operation on the differential d_β. We first give a matrix representation for the differential d_{β_i} of an elementary n-strand braid and then examine the effect of concatenation on the differential.

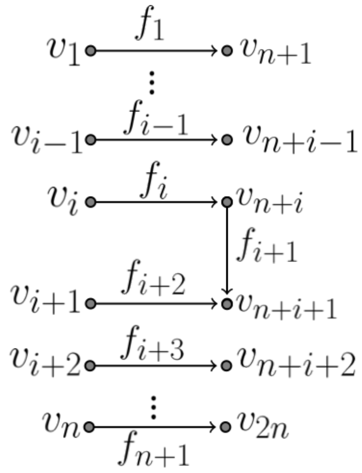
Lemma 11.1.2. The elementary n-strand braid β_i has a fence $K(\beta_i)$ with differential

$$d_{\beta_i} : C_1(K(\beta_i); \mathbb{Z}) \to C_0(K(\beta_i); \mathbb{Z})$$

represented by the $(n+1) \times 2n$ matrix

$$(d_{\beta_i})_{j,k} = \begin{cases} 1 & \text{if } 1 \leqslant k \leqslant i \text{ and } j = n+k \\ -1 & \text{if } 1 \leqslant k \leqslant i \text{ and } j = i \\ 1 & \text{if } k = i+1 \text{ and } j = n+i+1 \\ -1 & \text{if } k = i+1 \text{ and } j = n+i \\ 1 & \text{if } i+2 \leqslant k \leqslant n+1 \text{ and } j = n+k-1 \\ -1 & \text{if } i+2 \leqslant k \leqslant n+1 \text{ and } j = n+k-2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is the representation with respect to the ordered bases $(f_1, f_2, \ldots, f_{n+1})$ of $C_1(K(\beta_i); \mathbb{Z})$ and $(v_1, v_2, \ldots, v_{2n})$ of $C_0(K(\beta_i); \mathbb{Z})$ as shown below



□

Lemma 11.1.3. Let $\beta_i\beta_j$ be the concatenation of two elementary n-strand braids β_i and β_j with fences $K(\beta_i)$ and $K(\beta_j)$.

(i) The decompositions

$$K(\beta_i) = (K(\beta_i) \smallsetminus K(\beta_j)) \sqcup (K(\beta_i) \cap K(\beta_j))$$
$$K(\beta_j) = (K(\beta_i) \cap K(\beta_j)) \sqcup (K(\beta_j) \smallsetminus K(\beta_i))$$

imply that the differentials

$$d_\beta : C_1(K(\beta_i);\mathbb{Z}) \to C_0(K(\beta_i);\mathbb{Z})$$
$$d_{\beta'} : C_1(K(\beta_j);\mathbb{Z}) \to C_0(K(\beta_j);\mathbb{Z})$$

may be written as

$$\begin{pmatrix} d'_{\beta_i} \\ d''_{\beta_i} \end{pmatrix} : C_1(K(\beta_i);\mathbb{Z}) \to C_0(K(\beta_i) \smallsetminus K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z})$$
$$\begin{pmatrix} d'_{\beta_j} \\ d''_{\beta_j} \end{pmatrix} : C_1(K(\beta_j);\mathbb{Z}) \to C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_j) \smallsetminus K(\beta_i);\mathbb{Z})$$

where $(n+1)\times 2n$-matrix representation of d_{β_i} from Lemma 11.1.2 induces $(n+1)\times n$-matrix representations of d'_{β_i} and d''_{β_i} with

$$(d'_{\beta_i})_{k,l} = (d_{\beta_i})_{k,l}, \quad (d''_{\beta_i})_{k,l} = (d_{\beta_i})_{n+k,l} \quad (1 \leqslant k \leqslant n, 1 \leqslant l \leqslant n+1)$$

and similarly for d_{β_j} and $d'_{\beta_j}, d''_{\beta_j}$.

(ii) The decomposition

$$K(\beta_i) \cup K(\beta_j) = (K(\beta_i) \smallsetminus K(\beta_j)) \sqcup (K(\beta_i) \cap K(\beta_j)) \sqcup (K(\beta_j) \smallsetminus K(\beta_i))$$

implies that the regular n-strand braid with two crossings $\beta_i\beta_j$ has a fence $K(\beta_i\beta_j)$ with differential

$$d_{\beta_i\beta_j} : C_1(K(\beta_i\beta_j);\mathbb{Z}) \to C_0(K(\beta_i\beta_j);\mathbb{Z})$$

has a block decomposition

$$\begin{pmatrix} d'_{\beta_i} & 0 \\ d''_{\beta_i} & d'_{\beta_j} \\ 0 & d''_{\beta_i} \end{pmatrix} : C_1(K(\beta_i);\mathbb{Z}) \oplus C_1(K(\beta'_j);\mathbb{Z}) \to$$
$$C_0(K(\beta_i) \smallsetminus K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_j) \smallsetminus K(\beta_i);\mathbb{Z}).$$

Proof. The simplicial complexes $K(\beta_i), K(\beta_j) \subset K(\beta_i\beta_j)$ intersect in a 0-dimensional simplicial complex so that

$$\begin{aligned} C_0(K(\beta_i);\mathbb{Z}) &= C_0(K(\beta_i) \smallsetminus K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z}) \\ C_0(K(\beta_j);\mathbb{Z}) &= C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_j) \smallsetminus K(\beta_i);\mathbb{Z}) \\ C_0(K(\beta_i\beta_j);\mathbb{Z}) &= C_0(K(\beta_i) \smallsetminus K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_j) \smallsetminus K(\beta_i);\mathbb{Z}) \\ C_1(K(\beta_i\beta_j);\mathbb{Z}) &= C_1(K(\beta_i);\mathbb{Z}) \oplus C_1(K(\beta_j);\mathbb{Z}) \end{aligned}$$

from which the decomposition of the differentials is clear. □

This decomposition may be extended to a concatenation of elementary braids.

Proposition 11.1.4. Let $\beta = \beta_1\beta_2 \dots \beta_\ell$ be a regular n-strand braid with ℓ crossings where each β_i is an elementary n-strand braid. The decompositions

$$\begin{aligned} K(\beta_i) &= (K(\beta_i) \smallsetminus K(\beta_{i+1})) \sqcup (K(\beta_i) \cap K(\beta_{i+1})) \\ K(\beta_{i+1}) &= (K(\beta_i) \cap K(\beta_{i+1})) \sqcup (K(\beta_{i+1}) \smallsetminus K(\beta_i)) \end{aligned}$$

imply that the differentials

$$\begin{aligned} d_{\beta_i} &: C_1(K(\beta_i);\mathbb{Z}) \to C_0(K(\beta_i);\mathbb{Z}) \\ d_{\beta_{i+1}} &: C_1(K(\beta_{i+1});\mathbb{Z}) \to C_0(K(\beta_{i+1});\mathbb{Z}) \end{aligned}$$

may be written as

$$\begin{pmatrix} d'_{\beta_i} \\ d''_{\beta_i} \end{pmatrix} : C_1(K(\beta_i);\mathbb{Z}) \to C_0(K(\beta_i) \smallsetminus K(\beta_{i+1});\mathbb{Z}) \oplus C_0(K(\beta_i) \cap K(\beta_{i+1});\mathbb{Z})$$
$$\begin{pmatrix} d'_{\beta_{i+1}} \\ d''_{\beta_{i+1}} \end{pmatrix} : C_1(K(\beta_{i+1});\mathbb{Z}) \to C_0(K(\beta_i) \cap K(\beta_{i+1});\mathbb{Z}) \oplus C_0(K(\beta_{i+1}) \smallsetminus K(\beta_i);\mathbb{Z}).$$

The decomposition

$$K(\beta) = \cup_{i=1}^{\ell} K(\beta_i) = (K(\beta_1) \smallsetminus K(\beta_2)) \sqcup (\sqcup_{i=1}^{\ell-1}(K(\beta_i) \cap K(\beta_{i+1}))) \sqcup (K(\beta_\ell) \smallsetminus K(\beta_{\ell-1}))$$

implies β has a fence $K(\beta)$ with differential

$$d_\beta : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z})$$

which has a block decomposition

$$\begin{pmatrix} d'_{\beta_1} & 0 & \dots & 0 & 0 \\ d''_{\beta_1} & d'_{\beta_2} & \dots & 0 & 0 \\ 0 & d''_{\beta_2} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & d'_{\beta_{\ell-1}} & 0 \\ 0 & 0 & \dots & d''_{\beta_{\ell-1}} & d'_{\beta_\ell} \\ 0 & 0 & \dots & 0 & d''_{\beta_\ell} \end{pmatrix}:$$

$$\oplus_{i=1}^n C_1(K(\beta_i);\mathbb{Z}) \to C_0(K(\beta_1)\smallsetminus K(\beta_2)) \oplus (\oplus_{i=1}^{\ell-1} C_0(K(\beta_i)\cap K(\beta_{i+1});\mathbb{Z})) \oplus C_0(K(\beta_\ell)\smallsetminus K(\beta_{\ell-1})).$$

Proof. Follows by induction on ℓ with the base case $\ell = 2$ given by Lemma 11.1.3 and the equality

$$C_0(K(\beta_i)\smallsetminus K(\beta_{i+1});\mathbb{Z}) = C_0(K(\beta_{i-1})\cap K(\beta_i);\mathbb{Z}) \quad (2 \leqslant i \leqslant \ell - 1).$$

□

Corollary 11.1.5. The elementary n-strand braid relations

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geqslant 2$

(ii) $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i-j| = 1$

(iii) $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$

have the effect of replacing the differentials

(i) $\begin{pmatrix} d'_{\sigma_i} & 0 \\ d''_{\sigma_i} & d'_{\sigma_j} \\ 0 & d''_{\sigma_j} \end{pmatrix} : C_1(K(\sigma_i\sigma_j);\mathbb{Z}) \to C_0(K(\sigma_i\sigma_j);\mathbb{Z})$

respectively

$$\begin{pmatrix} d'_{\sigma_i^{-1}} & 0 \\ d''_{\sigma_i^{-1}} & d'_{\sigma_j^{-1}} \\ 0 & d''_{\sigma_j^{-1}} \end{pmatrix} : C_1(K(\sigma_i^{-1}\sigma_j^{-1});\mathbb{Z}) \to C_0(K(\sigma_i^{-1}\sigma_j^{-1});\mathbb{Z})$$

(ii) $\begin{pmatrix} d'_{\sigma_i} & 0 & 0 \\ d''_{\sigma_i} & d'_{\sigma_j} & 0 \\ 0 & d''_{\sigma_j} & d'_{\sigma_i} \\ 0 & 0 & d''_{\sigma_i} \end{pmatrix} : C_1(K(\sigma_i\sigma_j\sigma_i);\mathbb{Z}) \to C_0(K(\sigma_i\sigma_j\sigma_i);\mathbb{Z})$

respectively

$$\begin{pmatrix} d'_{\sigma_i^{-1}} & 0 & 0 \\ d''_{\sigma_i^{-1}} & d'_{\sigma_j^{-1}} & 0 \\ 0 & d''_{\sigma_j^{-1}} & d'_{\sigma_i^{-1}} \\ 0 & 0 & d''_{\sigma_i^{-1}} \end{pmatrix} : C_1(K(\sigma_i^{-1}\sigma_j^{-1}\sigma_i^{-1});\mathbb{Z}) \to C_0(K(\sigma_i^{-1}\sigma_j^{-1}\sigma_i^{-1});\mathbb{Z})$$

(iii) $\begin{pmatrix} d'_{\sigma_i} & 0 \\ d''_{\sigma_i} & d'_{\sigma_i^{-1}} \\ 0 & d''_{\sigma_i^{-1}} \end{pmatrix} : C_1(K(\sigma_i\sigma_i^{-1});\mathbb{Z}) \to C_0(K(\sigma_i\sigma_i^{-1});\mathbb{Z})$

and

$$\begin{pmatrix} d'_{\sigma_i^{-1}} & 0 \\ d''_{\sigma_i^{-1}} & d'_{\sigma_i} \\ 0 & d''_{\sigma_i} \end{pmatrix} : C_1(K(\sigma_i^{-1}\sigma_i);\mathbb{Z}) \to C_0(K(\sigma_i^{-1}\sigma_i);\mathbb{Z})$$

by the differentials

(i) $\begin{pmatrix} d'_{\sigma_j} & 0 \\ d''_{\sigma_j} & d'_{\sigma_i} \\ 0 & d''_{\sigma_i} \end{pmatrix} : C_1(K(\sigma_j\sigma_i);\mathbb{Z}) \to C_0(K(\sigma_j\sigma_i);\mathbb{Z})$

respectively

$$\begin{pmatrix} d'_{\sigma_j^{-1}} & 0 \\ d''_{\sigma_j^{-1}} & d'_{\sigma_i^{-1}} \\ 0 & d''_{\sigma_i^{-1}} \end{pmatrix} : C_1(K(\sigma_j^{-1}\sigma_i^{-1});\mathbb{Z}) \to C_0(K(\sigma_j^{-1}\sigma_i^{-1});\mathbb{Z})$$

(ii) $\begin{pmatrix} d'_{\sigma_j} & 0 & 0 \\ d''_{\sigma_j} & d'_{\sigma_i} & 0 \\ 0 & d''_{\sigma_i} & d'_{\sigma_j} \\ 0 & 0 & d''_{\sigma_j} \end{pmatrix} : C_1(K(\sigma_j\sigma_i\sigma_j);\mathbb{Z}) \to C_0(K(\sigma_j\sigma_i\sigma_j);\mathbb{Z})$

respectively

$$\begin{pmatrix} d'_{\sigma_j^{-1}} & 0 & 0 \\ d''_{\sigma_j^{-1}} & d'_{\sigma_i^{-1}} & 0 \\ 0 & d''_{\sigma_i^{-1}} & d'_{\sigma_j^{-1}} \\ 0 & 0 & d''_{\sigma_j^{-1}} \end{pmatrix} : C_1(K(\sigma_j^{-1}\sigma_i^{-1}\sigma_j^{-1});\mathbb{Z}) \to C_0(K(\sigma_j^{-1}\sigma_i^{-1}\sigma_j^{-1});\mathbb{Z})$$

(iii) $0 : C_1(K(1);\mathbb{Z}) = 0 \to C_0(K(1);\mathbb{Z})$

Definition 11.1.6. Let β and β' be regular n-strand braids. The chain level Seifert pairs (λ_β, d_β) and $(\lambda_{\beta'}, d_{\beta'})$ are *A-equivalent* if there exists a finite sequence of A-operations which transforms both λ_β to $\lambda_{\beta'}$ and d_β to $d_{\beta'}$.

Proposition 11.1.7. The A-equivalence class of the chain level Seifert pair of an n-strand geometric braid β is an isotopy invariant.

Proof. Two geometric n-strand braids β, β' are isotopic if and only if they are isotopic to regular n-strand braids determined by braid words β, β' from the alphabet $\{\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$ such that β' can be obtained from β by applying finitely many of the relations

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geqslant 2$

(ii) $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i-j| = 1$

(iii) $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$

and their inverses. By Theorem 10.3.4 and Proposition 11.1.4, these relations and their inverses correspond to transformations (i)-(iii) in the definition of A-equivalence of a chain level Seifert pair. □

The isotopy invariance of the A-equivalence class of the chain level Seifert pair of a braid yields a universal representation of the braid group.

Theorem 11.1.8. Let $n \geqslant 2$ and denote by F_n the free group on the set of elementary n-strand braids $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ and denote by B_n denote the braid group. The map

$$(\lambda, d) : F_n \to \{\text{chain level Seifert pairs}\}, \quad \beta \mapsto (\lambda_\beta, d_\beta)$$

is a bijection which respects the concatenation of braid words such that words $\beta, \beta' \in F_n$ differ by the relations in the braid group if and only if the chain level Seifert pairs $(\lambda_\beta, d_\beta), (\lambda_{\beta'}, d_\beta)$ are A-equivalent. This induces a well defined bijection

$$(\lambda, d) : B_n \to \frac{\{\text{chain level Seifert pairs}\}}{A-\text{equivalence}}, \quad [\beta] \mapsto [(\lambda_\beta, d_\beta)]$$

which is group homomorphism and which determines a commutative diagram

$$\begin{array}{ccc} F_n & \xrightarrow[\cong]{(\lambda,d)} & \{\text{chain level Seifert pairs}\} \\ \downarrow & & \downarrow \\ B_n & \xrightarrow[\cong]{(\lambda,d)} & \frac{\{\text{chain level Seifert pairs}\}}{A\text{–equivalence}} \end{array}$$

where the vertical maps are quotient maps.

Proof. This follows from Corollary 11.1.5 and Proposition 11.1.7. □

Example 11.1.9. Let β be the regular 4-strand braid with 8-crossings represented by the braid word $\beta = \sigma_1\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_1$. The sequence of isotopies

$$\begin{aligned} \sigma_1\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_1 &= \sigma_3\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_1 \\ &= \sigma_3\sigma_2\sigma_1\sigma_2\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_1 \\ &= \sigma_3\sigma_2\sigma_3^{-1}\sigma_1 \\ &= \sigma_2\sigma_3\sigma_3^{-1}\sigma_1 \\ &= \sigma_2\sigma_1 \end{aligned}$$

arising from applying the relations of the braid group B_4, implies that the chain level Seifert pairing

$$\lambda_\beta : C_1(K(\beta); \mathbb{Z}) \times C_1(K(\beta); \mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$$

and differential

$$d_\beta : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z})$$

is A-equivalent to the chain level pairing

$$\lambda_{\sigma_2\sigma_1} = \begin{pmatrix} \lambda_{\sigma_2} & 0 \\ 0 & \lambda_{\sigma_1} \end{pmatrix}$$

$$:(C_1(K(\sigma_2);\mathbb{Z}) \oplus C_1(K(\sigma_1);\mathbb{Z})) \times (C_1(K(\sigma_2);\mathbb{Z}) \oplus C_1(K(\sigma_1);\mathbb{Z}));\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$$

and differential

$$d_{\sigma_2\sigma_1} : C_1(K(\sigma_2\sigma_1);\mathbb{Z}) \to C_0(K(\sigma_2\sigma_1);\mathbb{Z})$$

We now construct a second equivalence relation which corresponds to isotopy of the closure of a braid inside in the solid torus.

Definition 11.1.10. Two square real matrices with entries in $\frac{1}{2}\mathbb{Z} \subset \mathbb{R}$ are $\widehat{A}$-*equivalent* if one can be transformed into the other by a finite sequence of $\widehat{A}$-*operations* defined as follows:

(i) A-operations

(ii) $A \mapsto \begin{pmatrix} \lambda_\alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \lambda_{\alpha^{-1}} \end{pmatrix}$ for α an elementary n-strand braid

(iii) $\begin{pmatrix} \lambda_\alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \lambda_{\alpha^{-1}} \end{pmatrix} \mapsto A$ for α an elementary n-strand braid

The $\widehat{A}$-operations have the following effect on the differential of a fence.

Proposition 11.1.11. Let $\beta = \beta_1\beta_2\ldots\beta_\ell$ be a regular n-strand braid with ℓ crossings where each β_i is an elementary n-strand braid and let α be an elementary n-strand braid. The conjugacy transformation $\beta \in B_n \mapsto \alpha\beta\alpha^{-1}$ is such that if the fence $K(\beta)$ has differential represented by the block matrix as in Proposition 11.1.4

$$d_\beta = \begin{pmatrix} d'_{\beta_1} & 0 & \ldots & 0 & 0 \\ d''_{\beta_1} & d'_{\beta_2} & \ldots & 0 & 0 \\ 0 & d''_{\beta_2} & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & d'_{\beta_{\ell-1}} & 0 \\ 0 & 0 & \ldots & d''_{\beta_{\ell-1}} & d'_{\beta_\ell} \\ 0 & 0 & \ldots & 0 & d''_{\beta_\ell} \end{pmatrix} : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z})$$

then the fence $K(\alpha\beta\alpha^{-1})$ has differential represented by the block matrix

$$d_{\alpha\beta\alpha^{-1}} = \begin{pmatrix} d'_{\alpha} & 0 & 0 & \dots & 0 & 0 \\ d''_{\alpha} & d'_{\beta_1} & 0 & \dots & 0 & 0 \\ 0 & d''_{\beta_1} & d'_{\beta_2} & \dots & 0 & 0 \\ 0 & 0 & d''_{\beta_2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & d'_{\beta_\ell} & 0 \\ 0 & 0 & 0 & \dots & d''_{\beta_\ell} & d'_{\alpha^{-1}} \\ 0 & 0 & 0 & \dots & 0 & d''_{\alpha^{-1}} \end{pmatrix} : C_1(K(\alpha\beta\alpha^{-1});\mathbb{Z}) \to C_0(K(\alpha\beta\alpha^{-1});\mathbb{Z})$$

Proof. Follows from Proposition 11.1.4. □

Definition 11.1.12. Let β, β' be regular n-strand braids. The chain level Seifert pairs (λ_β, d_β) and $(\lambda_{\beta'}, d_{\beta'})$ are $\widehat{A}$-*equivalent* if there exists a finite sequence of $\widehat{A}$-operations which transforms both λ_β to $\lambda_{\beta'}$ and d_β to $d_{\beta'}$.

Proposition 11.1.13. The $\widehat{A}$-equivalence class of the chain level Seifert pair of an n-strand geometric braid β is an isotopy invariant of the closure $\widehat{\beta}$ inside the solid torus.

Proof. By [KT08, Theorem 2.1] for any regular n-strand braids $\beta, \beta' \in B_n$, the closed braids $\widehat{\beta}, \widehat{\beta'}$ are isotopic in the solid torus if and only if β and β' are conjugate in B_n. The proof is then similar to the proof of Proposition 11.1.7 but now with the conjugacy of elements in the braid group corresponding to operations (ii) and (iii) in the definition of $\widehat{A}$-equivalence. □

The isotopy invariance of the $\widehat{A}$-equivalence class of the chain level Seifert pair of a braid yields a representation of the quotient of the braid group by the conjugacy relation.

Theorem 11.1.14. Let $n \geqslant 2$ and denote by F_n the free group on the set of elementary n-strand braids $\{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$ and by let B_n denote the braid group. The map

$$(\lambda, d) : F_n \to \{\text{chain level Seifert pairs}\}, \quad \beta \mapsto (\lambda_\beta, d_\beta)$$

is a bijection such that conjugate words $\beta, \beta' \in B_n$ have chain level Seifert pairs $(\lambda_\beta, d_\beta), (\lambda_{\beta'}, d_\beta)$ which are $\widehat{A}$-equivalent. This induces a well-defined bijection

$$(\lambda, d) : \frac{B_n}{\text{conjugacy}} \to \frac{\{\text{chain level Seifert pairs}\}}{\widehat{A} - \text{equivalence}}, \quad [\beta] \mapsto [(\lambda_\beta, d_\beta)]$$

and determines a commutative diagram

$$\begin{array}{ccc} B_n & \xrightarrow[\cong]{(\lambda,d)} & \frac{\{\text{chain level Seifert pairs}\}}{A\text{-equivalence}} \\ \downarrow & & \downarrow \\ \frac{B_n}{\text{conjugacy}} & \xrightarrow[\cong]{(\lambda,d)} & \frac{\{\text{chain level Seifert pairs}\}}{\widehat{A}\text{-equivalence}} \end{array}$$

Moreover, words $\beta, \beta' \in F_n$ differ by the relations in the braid group plus conjugacy if and only if the chain level Seifert pairs $(\lambda_\beta, d_\beta), (\lambda_{\beta'}, d_\beta)$ are $\widehat{A}$-equivalent so that there is commutative diagram

$$\begin{array}{ccc} F_n & \xrightarrow[\cong]{(\lambda,d)} & \{\text{chain level Seifert pairs}\} \\ \downarrow & & \downarrow \\ \dfrac{B_n}{\text{conjugacy}} & \xrightarrow[\cong]{(\lambda,d)} & \dfrac{\{\text{chain level Seifert pairs}\}}{\widehat{A}\text{-equivalence}} \end{array}$$

which factors as

$$\begin{array}{ccc} F_n & \xrightarrow[\cong]{(\lambda,d)} & \{\text{chain level Seifert pairs}\} \\ \downarrow & & \downarrow \\ B_n & \xrightarrow[\cong]{(\lambda,d)} & \dfrac{\{\text{chain level Seifert pairs}\}}{A\text{-equivalence}} \\ \downarrow & & \downarrow \\ \dfrac{B_n}{\text{conjugacy}} & \xrightarrow[\cong]{(\lambda,d)} & \dfrac{\{\text{chain level Seifert pairs}\}}{\widehat{A}\text{-equivalence}} \end{array} .$$

Proof. Follows from Theorem 11.1.8 and Proposition 11.1.13. □

11.2 Signatures of braids

We now use the chain level Seifert pair (λ_β, d_β) of a braid β to give a chain level combinatorial formula for the ω-signature of a braid.

Definition 11.2.1. If L is an oriented link with Seifert matrix V then the *signature* of L is the signature $\sigma(L)$ of the symmetric form $(H_1(\Sigma;\mathbb{Z}), V+V^t)$. For a unit complex number $\omega \neq 1$ the *ω-signature* of L is the signature $\sigma_\omega(L)$ of the hermitian form $(H_1(\Sigma;\mathbb{C}), (1-\omega)V+(1-\overline{\omega})V^t)$.

The -1-signature of an oriented link is the same as its signature.

Proposition 11.2.2. [Rol90, p.219] For an oriented link L and a unit complex number $\omega \neq 1$ the value $\sigma_\omega(L)$ does not depend on the choice of Seifert surface for L.

The signature of a link may also be interpreted as the signature of a 4-manifold with boundary.

Proposition 11.2.3. [KT76] Let $L \subset S^3$ be a link with Seifert surface $\Sigma \subset S^3 = \partial D^4$. Keeping the boundary of Σ fixed in S^3, push Σ inside D^4 to form a new surface Σ' with boundary L. If W is the two-fold branched cover of D^4 branched along Σ' then W is an oriented 4-manifold with boundary such that ∂W is a 2-fold cover of S^3 branched over L. Moreover, there exists a choice of basis such that the intersection form on $H_2(W)$ is represented by the matrix $V + V^t$ so that $\sigma(L) = \sigma(W)$.

Definition 11.2.4. If β is braid and if $\omega \neq 1$ is a unit complex number then the *ω-signature* of β is the ω-signature $\sigma_\omega(\beta)$ of the oriented link $\widehat{\beta}$.

Example 11.2.5. From Example 10.2.4 the 2-strand braid $\beta = \sigma_1\sigma_1\sigma_1$ with closure $\widehat{\beta}$ the trefoil knot has Seifert matrix V and symmetrisation $V + V^t$ given by

$$V = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad V + V^t = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

so that $\sigma(\beta) = -2$.

Theorem 11.2.6. Let β be a braid with chain level Seifert pair (λ_β, d_β) and let $\omega \neq 1$ be a unit complex number. The ω-signature of β may be expressed on the chain level as the signature of the hermitian form

$$\left(C_1(K(\beta);\mathbb{C}) \oplus C^0(K(\beta);\mathbb{C}), \begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix}\right)$$

so that

$$\sigma_\omega(\beta) = \sigma\left(\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix}\right).$$

Proof. The $\mathbb{C}$-coefficients chain level Seifert pair $\lambda_\beta : C_1(K(\beta);\mathbb{C}) \times C_1(K(\beta);\mathbb{C}) \to \mathbb{C}$ determines a commutative diagram

$$\begin{array}{ccc} 0 & \xrightarrow{\quad 0 \quad} & C^0(K(\beta);\mathbb{C}) \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d_\beta^*} \\ C_1(K(\beta);\mathbb{C}) & \xrightarrow{(1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t} & C^1(K(\beta);\mathbb{C}) \\ {\scriptstyle d_\beta}\downarrow & & \downarrow \\ C_0(K(\beta);\mathbb{C}) & \xrightarrow{\quad 0 \quad} & 0 \end{array} .$$

The algebraic lemma of [RS76, p.26] implies that the signature of the hermitian form

$$\left(H_1(K(\beta);\mathbb{C}), (1-\omega)V + (1-\overline{\omega})V^t\right)$$

is equal to the signature of the hermitian form

$$\left(C_1(K(\beta);\mathbb{C}) \oplus C^0(K(\beta);\mathbb{C}), \begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix}\right)$$

and hence

$$\sigma_\omega(\beta) = \sigma\left(\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix}\right).$$

□

This chain level formula shows that the signature of a braid is not additive under the concatenation of braids.

Corollary 11.2.7. Let $\omega \neq 1$ be a unit complex number. The ω-signature concatenation defect

$$\sigma_\omega(\beta\beta') - \sigma_\omega(\beta) - \sigma_\omega(\beta')$$

is equal to the difference in signature between the block matrix

$$\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta'^t & d_\beta''^t & 0 & 0 \\ d_\beta' & 0 & 0 & 0 & 0 \\ d_\beta'' & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & 0 & d_{\beta'}'' \\ 0 & 0 & d_{\beta'}'^t & d_{\beta'}''^t & (1-\omega)\lambda_{\beta'} + (1-\overline{\omega})\lambda_{\beta'}^t \end{pmatrix}$$

and the block matrix

$$\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta'^t & d_\beta''^t & 0 & 0 & 0 \\ d_\beta' & 0 & 0 & 0 & 0 & 0 \\ d_\beta'' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & 0 & 0 & d_{\beta'}'' \\ 0 & 0 & 0 & d_{\beta'}'^t & d_{\beta'}''^t & \lambda_{\beta'} + \lambda_{\beta'}^t \end{pmatrix}$$

where we have decomposed the differential of $K(\beta\beta')$ in terms of the differentials of $K(\beta)$ and $K(\beta')$ as in Proposition 11.1.4.

Proof. By Proposition 10.3.3 the chain level Seifert pairing for the concatenation $\beta\beta'$ is represented by the block diagonal matrix

$$\lambda_{\beta\beta'} = \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \lambda_{\beta'} \end{pmatrix}$$

so that there is an equality of block matrices

$$\begin{pmatrix} (1-\omega)\lambda_{\beta\beta'} + (1-\overline{\omega})\lambda_{\beta\beta'}^t & d_{\beta\beta'}^t \\ d_{\beta\beta'} & 0 \end{pmatrix} = \begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & 0 & d_\beta'^t & d_\beta''^t & 0 \\ 0 & \lambda_{\beta'} + {\lambda_{\beta'}}^t & 0 & d_{\beta'}'^t & d_{\beta'}''^t \\ d_\beta' & 0 & 0 & 0 & 0 \\ d_\beta'' & d_{\beta'}' & 0 & 0 & 0 \\ 0 & d_{\beta'}'' & 0 & 0 & 0 \end{pmatrix}.$$

One can then perform identical row and column exchanges to find a congruence

$$\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & 0 & d_\beta'^t & d_\beta''^t & 0 \\ 0 & (1-\omega)\lambda_{\beta'} + (1-\overline{\omega}){\lambda_{\beta'}}^t & 0 & d_{\beta'}'^t & d_{\beta'}''^t \\ d_\beta' & 0 & 0 & 0 & 0 \\ d_\beta'' & d_{\beta'}' & 0 & 0 & 0 \\ 0 & d_{\beta'}'' & 0 & 0 & 0 \end{pmatrix}$$

$$\simeq \begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta'^t & d_\beta''^t & 0 & 0 \\ d_\beta' & 0 & 0 & 0 & \\ d_\beta'' & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & d_{\beta'}'^t & d_{\beta'}''^t & (1-\omega)\lambda_{\beta'} + (1-\overline{\omega})\lambda_{\beta'}^t \end{pmatrix}$$

so that by Theorem 11.2.6

$$\sigma_\omega(\beta\beta') = \sigma\begin{pmatrix} \lambda_{\beta\beta'} + \lambda_{\beta\beta'}^t & d_{\beta\beta'}^t \\ d_{\beta\beta'} & 0 \end{pmatrix} = \sigma\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta'^t & d_\beta''^t & 0 & 0 & 0 \\ d_\beta' & 0 & 0 & 0 & 0 & 0 \\ d_\beta'' & 0 & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & d_{\beta'}'^t & d_{\beta'}''^t & \lambda_{\beta'} + \lambda_{\beta'}^t \end{pmatrix}.$$

On the other hand, there is an equality and congruence of block matrices

$$\begin{pmatrix} (1-\omega)\lambda_{\beta'} + (1-\overline{\omega})\lambda^t_{\beta'} & d^t_{\beta'} \\ d_{\beta'} & 0 \end{pmatrix} = \begin{pmatrix} (1-\omega)\lambda_{\beta'} + (1-\overline{\omega})\lambda^t_{\beta'} & d'^t_{\beta'} & d''^t_{\beta'} \\ d'_{\beta'} & 0 & 0 \\ d''_{\beta'} & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & d'_{\beta'} \\ 0 & 0 & d'_{\beta'} \\ d'^t_{\beta'} & d''^t_{\beta'} & \lambda_{\beta'} + \lambda^t_{\beta'} \end{pmatrix}$$

so that

$$\begin{aligned} \sigma_\omega(\beta) + \sigma_\omega(\beta') &= \sigma_\omega \begin{pmatrix} \lambda_\beta + \lambda^t_\beta & d^t_\beta \\ d_\beta & 0 \end{pmatrix} + \sigma \begin{pmatrix} \lambda_{\beta'} + \lambda^t_{\beta'} & d^t_{\beta'} \\ d_{\beta'} & 0 \end{pmatrix} \\ &= \sigma \begin{pmatrix} \lambda_\beta + \lambda^t_\beta & d^t_\beta & 0 & 0 \\ d_\beta & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\beta'} + \lambda^t_{\beta'} & d^t_{\beta'} \\ 0 & 0 & d_{\beta'} & 0 \end{pmatrix} \\ &= \sigma \begin{pmatrix} \lambda_\beta + \lambda^t_\beta & d'^t_\beta & d''^t_\beta & 0 & 0 & 0 \\ d'_\beta & 0 & 0 & 0 & 0 & 0 \\ d''_\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d'_{\beta'} \\ 0 & 0 & 0 & 0 & 0 & d'_{\beta'} \\ 0 & 0 & 0 & d'^t_{\beta'} & d''^t_{\beta'} & \lambda_{\beta'} + \lambda^t_{\beta'} \end{pmatrix}. \end{aligned}$$

□

11.3 An open question

One would wish to find an elementary closed form expression for the ω-signature concatenation defect, but this is not possible in general. In the spirit of [KT76] Gambaudo and Ghys [GG05] constructed from n-strand braids β, β' an oriented, compact, connected 4-manifold $M(\beta, \beta')$ of signature zero in such that way that $M(\beta, \beta')$ can be obtained by glueing three oriented 4-manifold manifolds $C(\beta), C(\beta'), C(\beta\beta')$ with signatures which satisfy

$$\sigma(C(\beta)) = \sigma(\beta), \quad \sigma(C(\beta')) = \sigma(\beta'), \quad \sigma(C(\beta\beta')) = \sigma(\beta\beta').$$

They extended this to an equivariant version for branched cyclic covers where there is an action of $\mathbb{Z}_k$ on $M(\beta, \beta'), C(\beta), C(\beta')$, $C(\beta\beta')$ which respects the decomposition of $M(\beta\beta')$ and used an equivariant version of Wall's non-additivity theorem for the signature [Wal69] to express the ω-signature concatenation defect in in terms of the Meyer cocycle and the Burau-Squier hermitian representation of the braid group $\mathcal{B}_\omega : B_\infty \to \mathbf{Sp}(\infty, \mathbb{R})$. Bourrigan [Bou13] gave a different proof using infinite cyclic covers.

Theorem 11.3.1. ([GG05, Theorem A], [Bou13, Chapter V]). Let $\omega \neq 1$ be a root of unity. The ω-signature of the concatenated braid $\beta\beta'$ is related to the ω-signature of the braids β, β' by

$$\sigma_\omega(\beta\beta') = \sigma_\omega(\beta) - \sigma_\omega(\beta') - Meyer(\mathcal{B}_\omega(\beta), \mathcal{B}_\omega(\beta')).$$

This suggests the following:

Open question: Is it possible to use the chain level Seifert pair (λ_β, d_β) of a braid and the L-theory techniques of [Ran98] to express the ω-signature concatenation defect in terms of an L-theoretic analogue of the Meyer cocycle?

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