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Seminar notes on simply connected surgery

by Peter Orlik

During the fall of 1967 we studied simply connected surgery in Professor Montgomery's seminar by reading Milnor [15], Kervaire and Milnor [11], and parts of Novikov [17]. At Princeton University Professor Kervaire was lecturing on Browder-Novikov theory. Since at the time only the original papers were available for learning the subject, I decided to write some notes to serve until an introductory text appears in print and hopefully to unify the approach. As we proceeded to Wall [24] it became clear that the latter had been done there, but I was not convinced that his paper would be a suitable introduction to the topic. Therefore, I wrote these notes based on omitting enough in Wall [24] and adding a chapter on preliminary material and a chapter on applications.

Simply connected surgery offers all the geometric difficulty of the general case without the complicated algebra. We restrict ourselves to closed manifolds of dimension ≥ 5 or compact manifolds of dimension ≥ 6 for the usual reasons. For simplicity we treat only the smooth case. The PL case is entirely analogous and it is left to the reader to supply the necessary modifications.

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I. PRELIMINARIES.

1. Smooth category.

Throughout these notes we are working in the smooth (C^∞) category. Objects are compact smooth manifolds (closed if the boundary is empty) and maps are smooth maps.

The main reference for this section is Milnor [13].

Definition 1.1. A smooth m-manifold M^m is a (topological) m-manifold with a collection of C^∞ coordinate neighborhoods $\mathcal{D} = \{(U, h)\}$ satisfying

(i) $\mathcal{D}$ covers M.

(ii) $h_1 h_2^{-1} : h_2(U_1 \cap U_2) \longrightarrow R^m$ or R^m_+ is a smooth map.

(iii) $\mathcal{D}$ is maximal with respect to these properties.

Definition 1.2. A map $f : M^m \longrightarrow N^n$ between smooth manifolds is smooth if for every pair (U, h), (V, k) of coordinate systems of M and N the composite

$$kfh^{-1} : h(U) \longrightarrow R^n \text{ is smooth.}$$

Definition 1.3. The rank of f at $p \in M$ is the rank of the Jacobian $D(kfh^{-1})$ at p. This is independent of the choice of coordinate neighborhoods about p and $f(p)$.

Definition 1.4. $f : M^m \longrightarrow N^n$ is an immersion if rank $f = m$ at each point $p \in M$.

f is an imbedding if it is an immersion and a homeomorphism of M onto $f(M) \subset N$.

f is a diffeomorphism if it is an imbedding and $f(M) = N$ (hence $m = n$).

Definition 1.5. $Q^q \subset M^m$ is a submanifold if it is a subset, a topological manifold and the restriction of the coordinate neighborhoods gives a differentiable structure to Q^q.

Lemma 1.6. If $f : N^n \longrightarrow M^m$ is an imbedding then $f(N^n)$ is a submanifold of M^m.

The following result is proved in [13].

Theorem 1.7 (Whitney). M^m can be imbedded in R^{2m+1}.

Smoothing corners [11]. We shall often be in the following situation. Suppose N is a compact, smooth n-manifold with boundary and $S^{k-1} \times D^{n-k}$ is smoothly imbedded in the boundary. We wish to attach $D^k \times D^{n-k}$ along the imbedded $S^{k-1} \times D^{n-k}$, so that the resulting manifold N' has a differential structure compatible with the one given on N. This is easy except at the "corner" $S^{k-1} \times S^{n-k-1}$. A neighborhood of it looks like $S^{k-1} \times S^{n-k-1} \times Q$ where $Q \subset R^2$ denotes the three-quarter disc containing all $(r \cos \theta, r \sin \theta)$ with $0 \leq r < 1$, $0 \leq \theta \leq 3\pi/2$. To smooth this corner map Q onto the half disc H, consisting of all $(r \cos \theta', r \sin \theta')$ with $0 \leq r < 1$, $0 \leq \theta' \leq \pi$ by $\theta' = 2\theta/3$. Carry the differential structure of H back to Q making it a differentiable manifold. The same transformation in the neighborhood of $S^{k-1} \times S^{n-k-1}$ makes N' a differentiable manifold. We shall always assume that this has been done whenever needed.

2. Vector bundles.

The main references of this section are [8] and [13].

Definition 2.1. A k-dimensional real vector bundle ξ is a bundle (E, π, B) together with the structure of a k-dimensional real vector space R^k on each fiber $\pi^{-1}(b)$ such that each point $b \in B$ has an open neighborhood U and a trivialization $U \times R^k \longrightarrow \pi^{-1}(U)$ where the restriction $c \times R^k \longrightarrow \pi^{-1}(b)$ is a vector space isomorphism for each $c \in U$.

The structure group is $GL(n)$, the group of $n \times n$ real non-singular matrices. For paracompact B we may give ξ a riemannian metric [8, p. 36] and use it to reduce the structure group to $O(n)$, the group of $n \times n$ real, orthonormal matrices [8, p. 68].

Definition 2.2. The map $g : E(\xi) \longrightarrow E(\eta)$ is a bundle homomorphism of two bundles ξ, η over B if it is a vector space homomorphism in each fiber and the following diagram commutes

$$\begin{tikzcd} E(\xi) \arrow[r, "g"] \arrow[d, "\pi_\xi"] & E(\eta) \arrow[d, "\pi_\eta"] \\ B \arrow[r, "id"] & B \end{tikzcd}$$

If g is an isomorphism in each fiber it is called a <u>bundle isomorphism</u>, $\xi \approx \eta$. For paracompact B such an isomorphism has an inverse.

<u>Definition</u> 2.3. The <u>cartesian product</u> of two vector bundles ξ, η is defined as $\xi \times \eta$, where

$E(\xi \times \eta) = E(\xi) \times E(\eta)$

$B(\xi \times \eta) = B(\xi) \times B(\eta)$

$(\pi_\xi \times \pi_\eta)(x, y) = (\pi_\xi(x), \pi_\eta(y))$.

<u>Definition</u> 2.4. Given a diagram

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

where $\xi = (E, \pi, B)$ is a vector bundle and f a continuous map the <u>induced bundle</u> $f^*(\xi)$ is defined to have total space E' a subset of $X \times E$, $E' = \{(x, e) | f(x) = \pi(e)\}$. There is a bundle map g covering the natural projection $\lambda(x, e) = x$ defined by $g(x, e) = e$, making the following diagram commutative:

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, "\lambda"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

E' is unique up to isomorphism with this property. If ξ is trivial (isomorphic to $B \times F$) so is $f^*(\xi)$.

<u>Definition</u> 2.5. The <u>Whitney sum</u> $\xi \oplus \eta$ of the bundles ξ and η over B is the induced bundle from the diagram,

$$\begin{array}{ccc} E(\xi \oplus \eta) & \dashrightarrow & E(\xi) \times E(\eta) \\ \downarrow & & \downarrow \\ B & \xrightarrow{\Delta} & B \times B \end{array}$$

where Δ is the diagonal map $\Delta(x) = (x, x)$

$$\xi \oplus \eta = \Delta^*(\xi \times \eta) \ .$$

Note that the fiber of $\xi \oplus \eta$ is $F(\xi) \times F(\eta)$ and $\dim(\xi \oplus \eta) = \dim \xi + \dim \eta$.

<u>Lemma</u> 2.6. Whitney sum is commutative and associative.

<u>Definition</u> 2.7. Let ξ, η be two vector bundles with the same fiber dimension. A <u>bundle map</u> $f : \xi \longrightarrow \eta$ is a continuous map of the total spaces which induces isomorphism in the fibers.

$$\begin{array}{ccc} E(\xi) & \xrightarrow{f} & E(\eta) \\ {\scriptstyle \pi_\xi}\downarrow & & \downarrow{\scriptstyle \pi_\eta} \\ B(\xi) & \xrightarrow{g} & B(\eta) \end{array}$$

The induced map g is continuous. If $g = \text{id.}$ then f is a bundle isomorphism (see 2.2).

An <u>imbedding</u> is a bundle map which is an isomorphism <u>into</u>.

<u>Theorem</u> 2.8. If the continuous map $f : E(\xi) \longrightarrow E(\eta)$ is a vector space homomorphism in each fiber then f may be factored into a bundle homomorphism followed by a bundle map.

<u>Definition</u> 2.9. Let M^n be a smooth manifold, $x_0 \in M^n$. A <u>tangent vector at</u> x_0 is a presheaf map $X : \Gamma(U, x_0) \longrightarrow \mathbb{R} \times x_0$ from the sheaf of germs of smooth maps on M to the constant sheaf $\mathbb{R} \times M$. Thus

(i) X commutes with restrictions.

(ii) X is linear, i.e. $X(\alpha f + \beta g) = \alpha X(f) + \beta X(g)$.

Moreover we require that

(iii) $X(f \cdot g) = X(f) \cdot g(x_0) + f(x_0) \cdot X(g)$.

Notice that $X(1) = X(1 \cdot 1) = X(1) + X(1)$, hence $X(1) = 0$ and $X(c) = 0$.

Equivalently a tangent vector at x_0 is an assignment to every coordinate system $(u^1, \ldots, u^n)$ at x_0 an element $(\alpha^1, \ldots, \alpha^n) \in \mathbb{R}^n$ such that if $(\beta^1, \ldots, \beta^n)$ is assigned to $(v^1, \ldots, v^n)$ then $\alpha^i = \Sigma_j \dfrac{\partial u^i}{\partial u^j} \beta_j$. The map X is then just $X = \Sigma \alpha^i \dfrac{\partial}{\partial u^i}$.

Definition 2.10. For each x_0 the tangent vectors at x_0 form an n-dimensional vector space with basis $\frac{\partial}{\partial u^i}$. The totality of these is called the tangent bundle $E(\tau)$ of M. Define $\pi : E(\tau) \longrightarrow M$ to map the tangent vector X at x_0 to the point x_0.

Definition 2.11. For each $f : M_1 \longrightarrow M_2$ there is a bundle map $Df : E(\tau_1) \longrightarrow E(\tau_2)$ defined by $Df(X) = Y$ where $Y(g) = X(g \circ f)$, making the diagram commutative

$$\begin{tikzcd} E(\tau_1) \arrow[r, "Df"] \arrow[d, "\pi_1"'] & E(\tau_2) \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd}$$

D is called the derivative of the function f.

Definition 2.12. Let $f : M_1 \longrightarrow M_2$ be an immersion; M_1, M_2 smooth manifolds. The normal bundle ν_f is defined as follows:

Let τ_1, τ_2 be the tangent bundles of M_1, M_2. By (2.8) $Df : E(\tau_1) \longrightarrow E(\tau_2)$ may be factored into a bundle homomorphism $h : E(\tau_1) \longrightarrow E(f^*\tau_2)$ and a bundle map g.

$$\begin{tikzcd} E(\tau_1) \arrow[r, "h"] \arrow[d] & E(f^*\tau_2) \arrow[r, "g"] \arrow[d] & E(\tau_2) \arrow[d] \\ M_1 \arrow[r, "id"] & M_1 \arrow[r, "f"] & M_2 \end{tikzcd}$$

Since f is an immersion h is 1-1, hence an isomorphism into, hence an imbedding. Thus $f^*\tau_2/\text{image } h$ is a bundle over M_1. It is called the normal bundle ν_f.

Moreover

$$0 \longrightarrow \tau_1 \longrightarrow f^*\tau_2 \longrightarrow \nu_f \longrightarrow 0$$

is an exact sequence of bundle homomorphisms, thus $f^*\tau_2 \simeq \tau_1 \oplus \nu_f$.

(The splitting of an exact sequence of bundle homomorphisms requires only a paracompact base.)

Definition 2.13. ξ and η are stably equivalent, $\xi \underset{s}{\simeq} \eta$ if there exist trivial bundles ε_1 and ε_2 such that

$$\xi \oplus \varepsilon_1 \simeq \eta \oplus \varepsilon_2 \ .$$

Note that $\underset{s}{\simeq}$ is an equivalence relation.

Theorem 2.14. If M is a compact manifold the s-equivalence classes of vector bundles form an abelian group under $\oplus$.

(The above theorem holds for a much wider class of spaces.)

Let M^n be a smooth manifold. By (1.7) we can immerse M^n in a large euclidean space R^N with normal bundle ν. Since R^N has trivial tangent bundle (2.12) implies that

$$\tau_M \oplus \nu = \varepsilon^N \ .$$

and therefore the normal bundles of any two immersions are s-equivalent.

Definition 2.15. A smooth manifold M is parallelizable if it has trivial tangent bundle. It is s-parallelizable if it has stably trivial tangent bundle.

An s-parallelizable manifold is also called a π-manifold. Note that S^n with the standard differentiable structure is s-parallelizable, since adding the trivial line bundle gives the tangent bundle of R^{n+1}.

Lemma 2.16. M^n is a π-manifold if and only if it immerses in some R^N with trivial normal bundle.

Proof. Clear from $\tau_M \oplus \nu \simeq \varepsilon^N$.

The following three lemmas are proved in [11].

Lemma 2.17. Let ξ be a k-dimensional vector bundle over an n-dimensional complex, $k > n$. If ξ is stably trivial then it is trivial.

Lemma 2.18. If M^n is a submanifold of S^{n+N}, $N > n$, then M is s-parallelizable if and only if its normal bundle is trivial.

<u>Lemma</u> 2.19. A connected manifold with non-empty boundary is s-parallelizable if and only if it is parallelizable.

The proof of (2.17) uses a fact about classifying spaces, which we do not want to introduce here, (2.18) and (2.19) are immediate corollaries.

<u>Definition</u> 2.20. If $\xi = (E, \pi, B)$ is a trivial bundle of dimension n a <u>framing</u> of ξ is a given bundle isomorphism $\xi \approx B \times R^n$.

<u>Definition</u> 2.21. A <u>framed manifold</u> (M, F) is a π-manifold with a fixed trivialization F given for its stable tangent bundle, $\tau_M \oplus \varepsilon^k$. Note that if we frame S^{n+N}, $N > n$ (for example as a submanifold of R^{n+N+1}), then the framing of M gives an essentially unique framing of the (trivial) normal bundle of any imbedding $M^n \longrightarrow S^{n+N}$.

<u>Definition</u> 2.22. Let $f, g : X \longrightarrow Y$, where Y has a metric d, and let δ be a positive, continuous function defined on X. Then g is a δ-<u>approximation</u> to f if $d(f(x), g(x)) < \delta(x)$ for all $x \in X$.

<u>Theorem</u> 2.23. Given a smooth map $f : M^n \longrightarrow R^p$, $p \geq 2n$ and a continuous positive function δ on M^n there exists an immersion $g : M^n \longrightarrow R^p$ which is a δ-approximation to f.

Definition 2.24. Let $f : M^n \longrightarrow N^p$ be a map of smooth manifolds, W^{p-q} a submanifold of N^p. Call f <u>transverse regular</u> to W^{p-q} if for each $x \in f^{-1}(W)$ with $(u^1, \ldots, u^n)$ a coordinate system at x and $(v^1, \ldots, v^p)$ a coordinate system at $f(x)$ such that on W, $v^1 = v^2 = \ldots = v^q = 0$ the Jacobian of f has rank q and the induced map

$$(\tau_M)_x \xrightarrow{Df} (\tau_N)_{f(x)} \xrightarrow{pr} (\tau_N)_{f(x)}/(\tau_W)_{f(x)}$$

is an epimorphism for each $x \in M$. Here $(\tau_M)_x$ means the fiber of the tangent bundle at x and pr is projection. (For Df see (2.11)).

<u>Lemma</u> 2.25. If $f : M^n \longrightarrow N^p$ is transverse regular to W^{p-q}, then $V = f^{-1}(W)$ is a submanifold of dimension $n-q$ and the normal bundle of V in M, ν_V is isomorphic to $f^*(\nu_W)$, where ν_W is the normal bundle of W in N. Thus there is a bundle map g making the following

diagram commutative

$$\begin{array}{ccc} E(\nu_V) & \xrightarrow{g} & E(\nu_W) \\ {\scriptstyle \pi_V}\downarrow & & \downarrow{\scriptstyle \pi_W} \\ V & \xrightarrow{f} & W \end{array}$$

Theorem 2.26. Let $f : M^n \longrightarrow N^p$ be smooth, let W^{p-q} be a closed, smooth submanifold of N. Let A be a closed subset of M such that f is transverse regular to W at each $x \in A \cap f^{-1}(W)$. Let δ be a positive continuous function on N. Then there is a smooth map $g : M^n \longrightarrow N^p$ such that

(i) g is a δ-approximation to f.

(ii) g is transverse regular to W.

(iii) $g|A = f|A$.

Definition 2.27. Let ξ be a vector bundle with compact base. We may assume that it has a Riemannian metric and the structure group is reduced to $O(n)$. Consider the unit disc bundle $E_1(\xi)$ with boundary $\dot{E}_1(\xi)$, the associated unit sphere bundle. The Thom complex $T(\xi)$ is defined as the identification space $E_1(\xi)/\dot{E}_1(\xi)$.

An equivalent definition is the one point compactification of the total space of the bundle, $T(\xi) = E \cup e$. Note that $T(\xi)$ is a smooth manifold except at the identification point e, hence maps into it can be made transverse regular on manifolds missing e.

Lemma 2.28 (Thom isomorphism). Let M^n be an orientable smooth manifold and ν_M the normal bundle of M for some imbedding with codimension N. Then for all i we have an isomorphism

$$\Phi : H_i(M) \longrightarrow H_{i+N}(T(\nu_M)) .$$

Proof. Define Φ by the diagram

$$\begin{tikzcd}
H_i(M) \arrow[r, dashed, "\Phi"] \arrow[d, "\wr"] & H_{N+i}(T(\nu_M)) & H_{N+i}(E_1(\nu_M), \dot{E}_1(\nu_M)) \arrow[l, "\approx"'] \arrow[dl, "\approx"] \\
H^{n-i}(M) \arrow[r, "\approx"] & H^{n-i}(\nu_M) &
\end{tikzcd}$$

(Using a different proof this theorem applies for more general spaces.)

Definition 2.29. A regular homotopy of an immersion $F_0 : M \longrightarrow N$ is a homotopy $F : M \times I \longrightarrow N$ such that for each t, F_t is an immersion and the induced homotopy F^* of τ_M into τ_N is continuous.

The following theorem of Hirsch [6] will be needed as improved by Haefliger [5].

Theorem 2.30. Let V^v and M^m be smooth manifolds $v \leq m$, and $f : V \longrightarrow M$ a smooth map. Suppose V has a handle decomposition with no handle of dimension $> m-2$. Then regular homotopy classes of immersions homotopic to f correspond bijectively (by the tangent map) to stable homotopy classes of stable bundle monomorphisms $\tau_V \longrightarrow f^*\tau_M$.

For handle decomposition see Milnor [14].

3. Poincaré complexes.

The main references for this section are Wall [22, §2] and Wall [24, §2].

We shall only give the simply connected definitions, hence assume that each component of every space in this section is simply connected.

Let X be a finite CW complex.

Definition 3.1. X is a Poincaré complex of dimension n if for some homology class $[X] \in H_n(X; Z)$

$$[X]\cap \; : H^r(X; Z) \longrightarrow H_{n-r}(X; Z)$$

is an isomorphism for each r. Call $[X]$ the fundamental class of X. It is determined up to sign in each component of X.

Clearly for any Z-module G we have isomorphisms

$$g\cap \; : H^r(X; G) \longrightarrow H_{n-r}(X; G) \; .$$

Similarly if (Y, X) is a finite CW pair then (Y, X) is called a <u>Poincaré pair</u> of dimension $(n+1)$ if there is a $[Y] \in H_{n+1}(Y, X; Z)$ such that

$$[Y] \cap : H^{r+1}(Y; Z) \longrightarrow H_{n-r}(Y, X; Z)$$

is an isomorphism for all r and each component of X is a Poincaré complex with fundamental class $\partial_*[Y]$.

Again for any Z-module G we have the isomorphisms

$$[Y] \cap : H^{r+1}(Y; G) \longrightarrow H_{n-r}(Y, X; G) \quad .$$

Moreover the diagram below commutes up to sign

$$\begin{tikzcd}
\arrow[r] & H^r(X; G) \arrow[r] \arrow[d, "{[X]\cap}"] & H^{r+1}(Y, X; G) \arrow[r] \arrow[d, "{[Y]\cap}"] & H^{r+1}(Y; G) \arrow[r] \arrow[d, "{[Y]\cap}"] & H^{r+1}(X; G) \arrow[r] \arrow[d, "{[X]\cap}"] & {} \\
& H_{n-r}(X, G) \arrow[r] & H_{n-r}(Y; G) \arrow[r] & H_{n-r}(Y, X; G) \arrow[r] & H_{n-r-1}(X; G) \arrow[r] & {}
\end{tikzcd}$$

hence $\cap [Y] : H^{r+1}(Y, X; G) \longrightarrow H_{n-r}(Y; G)$ is an isomorphism for each r.

A finite <u>Poincaré triad</u> is a finite CW triad $(Y; X_-, X_+)$ with $X_+ \cap X_- = W$ (possibly empty) such that each of the pairs $(Y, X_- \cup X_+)$, (X_-, W), (X_+, W) is a Poincaré pair with $j_* \partial_* [Y] = [X_+] - [X_-]$. Here ∂_* and j_* are given by

$$\begin{aligned} H_{r+1}(Y, X; Z) \xrightarrow{\partial_*} H_r(X; Z) \xrightarrow{j_*} H_r(X, W; Z) &\cong \\ H_r(X_+, W; Z) \oplus H_r(X_-, W; Z) & \end{aligned}$$

where the isomorphism at the right comes from the relative Mayer-Vietoris sequence of the triad $(Y; X_+, X_-)$ modulo W.

Here is a result of Wall [23] on the geometry of a Poincaré complex.

<u>Theorem</u> 3.2. Let X be a connected, simply connected Poincaré complex of dimension $n > 3$. Then X is homotopy equivalent to a complex

$K \cup_f e^n$ with $\dim K \leq n-1$. The pair (K, f) is unique up to homotopy type of K and homotopy and orientation of f.

Next we need the notions of homotopy groups and homology groups of a map, square, etc. For details see [16].

Definition 3.3. Let $\varphi : M \longrightarrow X$ be a map. The homotopy group $\pi_{k+1}(\varphi)$ is defined as homotopy classes of commutative diagrams preserving base point:

$$\begin{array}{ccc} S^k & \xrightarrow{i} & D^{k+1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & X \end{array}$$

We have the exact sequence

$$\cdots \longrightarrow \pi_{k+1}(\varphi) \longrightarrow \pi_k(M) \longrightarrow \pi_k(X) \longrightarrow \pi_k(\varphi) \longrightarrow \cdots .$$

Definition 3.4. Let φ denote the commutative square

$$\begin{array}{ccc} N & \xrightarrow{\varphi_2} & Y \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ M & \xrightarrow[\varphi_1]{} & X \end{array}$$

The group $\pi_{k+1}(\varphi)$ is defined by taking homotopy classes of the image of the model square

$$\begin{array}{ccc} D^k_+ & \longrightarrow & D^{k+1} \\ \uparrow & & \uparrow \\ S^{k-1} & \longrightarrow & D^k_- \end{array}$$

in φ. Again care must be taken to preserve base points to yield the exact sequences

$$\cdots \longrightarrow \pi_{k+1}(\varphi) \longrightarrow \pi_k(\varphi_1) \longrightarrow \pi_k(\varphi_2) \longrightarrow \pi_k(\varphi) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_{k+1}(\varphi) \longrightarrow \pi_k(i_1) \longrightarrow \pi_k(i_2) \longrightarrow \pi_k(\varphi) \longrightarrow \cdots$$

Similar definitions apply to cubes, etc. (see [24]).

Definition 3.5. The homology groups of the map $\varphi : M \longrightarrow X$ are defined as $H_{k+1}(M_\varphi, M)$ where M_φ is the mapping cylinder of φ. Thus we make φ an inclusion up to homotopy type and consider the relative group. We have the exact sequence

$$\cdots \longrightarrow H_{k+1}(\varphi) \longrightarrow H_k(M) \longrightarrow H_k(X) \longrightarrow H_k(\varphi) \longrightarrow \cdots$$

Definition 3.6. Consider the square φ

$$\begin{tikzcd} N \arrow[r, "\varphi_2"] & Y \\ M \arrow[u, "i_1"] \arrow[r, "\varphi_1"] & X \arrow[u, "i_2"'] \end{tikzcd}$$

Make all maps inclusions up to homotopy type. Define $H_{k+1}(\varphi) = H_{k+1}(Y, N \cup X)$. We have the exact sequences of a proper triad

$$\longrightarrow H_{k+1}(\varphi) \longrightarrow H_k(\varphi_1) \longrightarrow H_k(\varphi_2) \longrightarrow H_k(\varphi) \longrightarrow \cdots$$

$$\longrightarrow H_{k+1}(\varphi) \longrightarrow H_k(i_1) \longrightarrow H_k(i_2) \longrightarrow H_k(\varphi) \longrightarrow \cdots$$

Similarly for cubes, etc.

We shall also have opportunity to use the homology exact sequence connected with a triple of squares.

$$\begin{tikzcd} A \arrow[r] & B \arrow[r] & C \\ A' \arrow[u] \arrow[r] & B' \arrow[u] \arrow[r] & C' \arrow[u] \end{tikzcd}$$

Let φ_1 be the left square, φ_2 the right square and φ the outside square.

Then we have

$$\longrightarrow H_k(\varphi_1) \longrightarrow H_k(\varphi) \longrightarrow H_k(\varphi_2) \longrightarrow H_{k-1}(\varphi_1) \longrightarrow \dots$$

Definition 3.7. A map $\varphi : M \longrightarrow X$ of Poincaré complexes is of degree 1 if $\varphi_*[M] = [X]$.

Lemma 3.8. Let M^m, X^m be connected Poincaré complexes, $\varphi : M \longrightarrow X$ a map of degree 1, B a finitely generated Z-module. Then the diagram

$$\begin{tikzcd} H^r(M; B) \arrow[d, "{[M]\cap}"'] & H^r(X; B) \arrow[l, "\varphi^*"'] \arrow[d, "{[X]\cap}"] \\ H_{m-r}(M; B) \arrow[r, "\varphi_*"] & H_{m-r}(X; B) \end{tikzcd}$$

is commutative. Moreover $[M]\cap$ induces an isomorphism of the cokernel $K^r(M; B)$ of φ^* on the kernel $K_{m-r}(M; B)$ of φ_*. Thus if φ is k-connected, then φ_* and φ^* are isomorphisms for $r < k$ and $r > m-k$.

Similarly let $\varphi : (N, M) \longrightarrow (Y, X)$ be a map of degree 1 of Poincaré pairs. Then φ_* gives split surjections of homology groups for $M \longrightarrow X$, $N \longrightarrow Y$, $(N, M) \longrightarrow (Y, X)$ with kernels K_* and φ^* gives split injections of cohomology groups with cokernels K^*. The duality map $[N]\cap$ induces isomorphisms

$$K^*(N) \longrightarrow K_*(N, M), \; K^*(N, M) \longrightarrow K_*(N) \;.$$

The homology (cohomology) sequence of (N, M) is isomorphic to the direct sum of the sequence for (Y, X) and a sequence $K_*(K^*)$.

Proof. Commutativity of the first diagram follows from the naturality of cap products. The fact that each $H_r(M) \longrightarrow H_r(X)$ is onto implies that $K_r(M) \simeq H_{r+1}(\varphi)$. Hence the assertions of the first paragraph.

Now $\varphi_1 = ([M]\cap)\varphi^*([X]\cap)^{-1}$ is a right inverse for φ_*, so $H_{m-r}(M; B)$ splits into $K_{m-r}(M; B)$ and an isomorphic copy of $H_{m-r}(X; B)$. Similar considerations apply to cohomology and for pairs.

Suppose we have a Poincaré triad $(Y; X_+, X_-)$. Then the commutative diagram

$$\begin{tikzcd}
\longrightarrow & H^r(X_+; B) \arrow[r] \arrow[d, "{[X_+]\cap}"] & H^{r+1}(Y, X_+; B) \arrow[r] \arrow[d, "{[Y]\cap}"] & H^{r+1}(Y; B) \arrow[r] \arrow[d, "{[Y]\cap}"] & {} \\
\cdots \arrow[r] & H_{m-r}(X_+, W; B) \arrow[r] & H_{m-r}(Y, X_-; B) \arrow[r] & H_{m-r}(Y, X; B) \arrow[r] & {}
\end{tikzcd}$$

shows that the middle vertical map is an isomorphism.

Let $\psi : (N; M_+, M_-) \longrightarrow (Y; X_+, X_-)$ be a degree 1 map of Poincaré triads. In addition to $\psi_*[N] = [Y]$ it follows that $\psi_*[M_{\pm 1}] = [X_{\pm 1}]$. Also the diagram

$$\begin{tikzcd}
H^{r+1}(N, M_+; B) \arrow[d, "{[N]\cap}"'] & H^{r+1}(Y, X_+; B) \arrow[l, "\psi^*"'] \arrow[d, "{[Y]\cap}"] \\
H_{m-r}(N, M_-; B) \arrow[r, "\psi_*"] & H_{m-r}(Y, X_-; B)
\end{tikzcd}$$

is commutative. Using the argument of (3. 6) we have

$$[N]\cap \ : K^{r+1}(N, M_+; B) \cong K_{m-r}(N, M_-; B) \ .$$

Lemma 3.9. The direct sum splittings above are preserved in any of the homology or cohomology sequences of the triad.

This is immediate.

Assume now that (N, M_+, M_-) is a proper triad with $M_+ \cup M_- = M$ and $M_+ \cap M_- = L$. Combine the homology exact sequences of the triples $(N, M_{\pm 1}, L)$ and $(N, M, M_{\pm 1})$ into the diagram

$$\begin{tikzcd}[column sep=tiny]
H_{n+1}(M_+, L) \arrow[rr, bend left] \arrow[dr] & & H_{n+1}(N, M_-) \arrow[rr, bend left] \arrow[dr] & & H_n(M, M_+) \cong H_n(M_-, L) \arrow[rr, bend left] \arrow[dr] & & H_n(N, M_+) \\
& H_{n+1}(N, L) \arrow[ur] \arrow[dr] & & H_{n+1}(N, M) \arrow[ur] \arrow[dr] & & H_n(N, L) \arrow[ur] \arrow[dr] & \\
H_{n+1}(M_-, L) \arrow[rr, bend right] \arrow[ur] & & H_{n+1}(N, M_+) \arrow[rr, bend right] \arrow[ur] & & H_n(M, M_-) \cong H_n(M_+, L) \arrow[rr, bend right] \arrow[ur] & & H_n(N, M_-)
\end{tikzcd}$$

By a change of sign in the boundary map $H_{n+1}(N, M) \longrightarrow H_n(M_-, L)$ we can make the above diagram commutative. A further adjustment of signs shows that $\longrightarrow H_{n+1}(N, L) \longrightarrow H_{n+1}(N, M_-) \oplus H_{n+1}(N, M_+) \longrightarrow H_{n+1}(N, M) \longrightarrow H_n(N, L) \longrightarrow$ is exact. A sequence of this kind is just a relative Mayer-Vietoris sequence.

By (3.8) the kernels split off the diagram. In particular if $\psi : L \longrightarrow W$ is a homotopy equivalence, then all $K_r(L) = 0$ and the homology exact sequences of pairs involving L show that $K_n(M_-) \simeq K_n(M_-, L)$ and similarly for M_+ and N. Thus we have

$$(*) \qquad \begin{tikzcd}[column sep=tiny]
K_{n+1}(M_+) \arrow[rr, bend left] \arrow[dr] & & K_{n+1}(N, M_-) \arrow[rr, bend left] \arrow[dr] & & K_n(M_-) \arrow[rr, bend left] \arrow[dr] & & K_n(N, M_+) \\
& K_{n+1}(N) \arrow[ur] \arrow[dr] & & K_{n+1}(N, M) \arrow[ur] \arrow[dr] & & K_{n+1}(N) \arrow[ur] \arrow[dr] & \\
K_{n+1}(M_-) \arrow[rr, bend right] \arrow[ur] & & K_{n+1}(N, M_+) \arrow[rr, bend right] \arrow[ur] & & K_n(M_+) \arrow[rr, bend right] \arrow[ur] & & K_n(N, M_-)
\end{tikzcd}$$

Lemma 3.10. Let $\varphi : (N, M) \longrightarrow (Y, X)$ be a map of finite CW pairs. Let Y be connected and assume that $H_i(\varphi) = 0$ for $i < r$. If $H^{r+1}(\varphi) = 0$ then $H_r(\varphi)$ is free (with Z coefficients).

Proof. Replacing (Y, X) by the mapping cylinder of φ we may suppose that φ is an inclusion and $M = N \cap X$. Now all spaces are simply connected and $H_k(\varphi) = H_k(Y, N \cup X)$. The result follows from the universal coefficient theorem.

Note that the same applies to cohomology.

Corollary 3.11. Let $0 \longrightarrow C'_* \longrightarrow C_* \longrightarrow C''_* \longrightarrow 0$ be a short exact sequence of free chain complexes over Z, each with finite total rank. Assume that $H_i(C) = 0$ for $i \neq r$, $H_i(C'') = 0$ for $i \neq r+1$ and

$H^{r+1}(C) = H^{r+2}(C'') = 0$. Then $H_i(C') = 0$ for $i \neq 0$ and we have an exact sequence of free Z-modules

$$0 \longrightarrow H_{r+1}(C'') \longrightarrow H_r(C') \longrightarrow H_r(C) \longrightarrow 0 \ .$$

Corollary 3.12. Let $\varphi'' : (N, M) \longrightarrow (Y, X)$ be a map of finite CW complexes, Y connected. Denote the induced maps by $\varphi : N \longrightarrow Y$ and $\varphi' : M \longrightarrow X$. Assume that $H_i(\varphi'') = 0$, $i \neq r+1$, $H_i(\varphi) = 0$ $i \neq r$ and $H^{r+2}(\varphi'') = H^{r+1}(\varphi) = 0$. Then we have the short exact sequence of free Z-modules

$$0 \longrightarrow H_{r+1}(\varphi'') \longrightarrow H_r(\varphi') \longrightarrow H_r(\varphi) \longrightarrow 0 \ .$$

II. SURGERY ACCORDING TO WALL

1. Description

In this chapter we follow Wall [24] to describe surgery and define the obstruction groups. One basic difference is the assumption that we are working only in the simply connected case. This will be frequently omitted from statements although tacitly assumed throughout these notes.

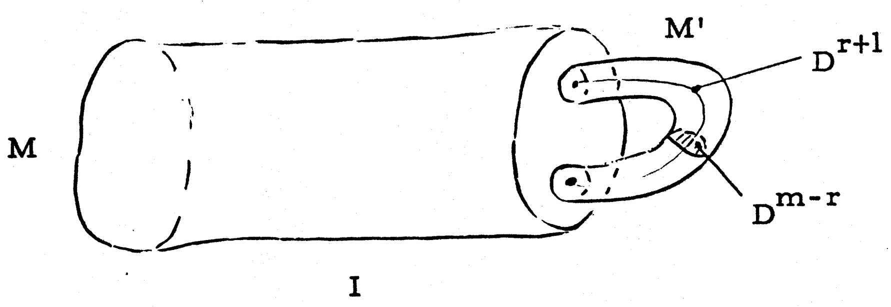
Let X be a simply connected topological space, M^n a closed smooth manifold and $\phi : M \longrightarrow X$ a map. The objective is to alter the map ϕ and the manifold M to make ϕ as near a homotopy equivalence as possible.

For the following description of surgery see also [15], [11] and [14].

Let $f : S^r \times D^{m-r} \longrightarrow M^m$ be an imbedding. The operation surgery replaces M by the manifold M' obtained by deleting the interior of $f(S^r \times D^{m-r})$ and replacing it by $D^{r+1} \times S^{m-r-1}$.

Now we want to define $\phi' : M' \longrightarrow X$.

In order to do this we shall look at the trace of the above surgery. Consider $M \times I$ and form a new manifold N by attaching $D^{r+1} \times D^{m-r}$ to $\{f(S^r \times D^{m-r}), 1\}$.



Call this attaching an $(r+1)$-handle to $M \times I$. Now we shall define a map $\psi : N \longrightarrow X$ such that $\psi | M = \phi$ and the required ϕ' is then defined as $\psi | M'$.

Defining ψ is a homotopy question. Up to homotopy N is just M with an $(r+1)$-cell attached to $\bar{f} = f | S^r \times 0$. Hence up to homotopy the construction is defined by

(i) the map $\bar{f} : S^r \longrightarrow M$

(ii) a nullhomotopy of $\phi \circ \bar{f}$

Let α denote the commutative diagram

$$\begin{array}{ccc} S^r & \longrightarrow & D^{r+1} \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{\phi} & X \end{array}$$

then equivalence classes of these diagrams define the relative homotopy group $\pi_{r+1}(\phi)$. Our surgery is therefore a surgery on the class $\alpha \in \pi_{r+1}(\phi)$.

Given a class α it gives an imbedding $\bar{f} : S^r \longrightarrow M$. In order to perform surgery, however, we need an imbedding $f : S^r \times D^{m-r} \longrightarrow M$. Now $S^r \times D^{m-r}$ is parallelizable. Hence if τ_M is the tangent bundle of M we need $\bar{f}^*\tau_M$ trivial (see I. 2.12). Thus we have the following requirement:

(i) There is an orientable vector bundle ν over X such that $\phi^*\nu$ is the stable normal bundle of M (in some S^{m+N}), or equivalently that there is a stable trivialization F of $\tau_M \oplus \phi^*\nu$. Since we want to preserve this property under surgery we require in addition that F extends to a stable trivialization of $\tau_N \oplus \psi^*\nu$. Thus we have the commutative diagram

$$\begin{array}{ccc} \tau_N & \longrightarrow & \nu \\ \downarrow & & \downarrow \\ N & \xrightarrow{\psi} & X \end{array}$$

where $\tau_{N|M} \simeq \tau_M \oplus \varepsilon^1$.

Assume we have X, ν, m satisfying (i). Consider triples (M, ϕ, F) where M is a smooth m-manifold $\phi : M \longrightarrow X$ a map and F a stable trivialization of $\tau_M \oplus \phi^*\nu$.

Disjoint union defines addition of triples. It is commutative, associative and has a zero $(M = \emptyset)$. Define an equivalence relation $(M_1, \phi_1, F_1) \sim (M_2, \phi_2, F_2)$ if there exists a compact $(m+1)$-manifold N such that $\partial N = M_1 \cup M_2$, a map $\psi : N \longrightarrow X$, $\psi|M_1 = \phi_1$, $\psi|M_2 = \phi_2$ and a stable trivialization of $\tau_N \oplus \psi^*\nu$ extending F_1 and F_2. (Use the inward normal along M_1 in N and the outward normal along M_2.)

Let $\Omega_m(X, \nu)$ denote the set of equivalence classes. It is an abelian group under the above addition. The inverse of (M, ϕ, F) is $(M, \phi, F \oplus (-\epsilon^1))$ in $M \times I$.

This equivalence is the same as equivalence by a sequence of surgeries [15].

Similarly we can define a relative version. Let (Y, X) be a pair of topological spaces, X, Y simply connected and (N, M) a pair of compact, smooth manifolds, $M = \partial N$. Let

$$\phi : (N, M) \longrightarrow (Y, X)$$

be a map of pairs. Assume that

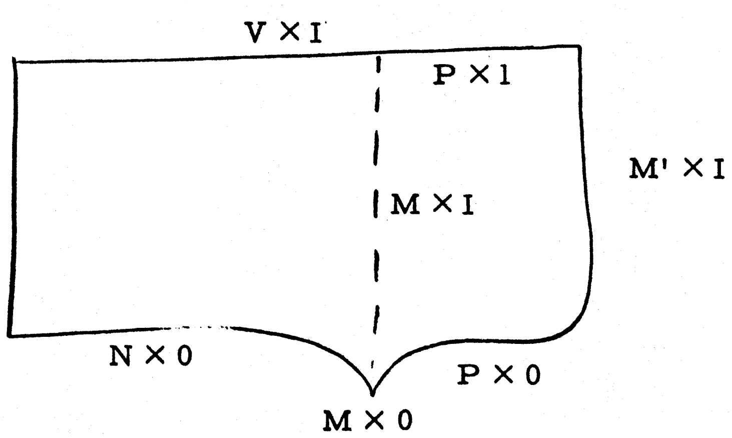
(i) there is a vector bundle ν over Y and a stable trivialization F of $\tau_N \oplus \phi^*\nu$ as before.

The cobordism group $\Omega_m(Y, X, \nu)$ is defined by the equivalence relation: $(N_1, \phi_1, F_1) \sim (N_2, \phi_2, F_2)$ if there is a manifold Q such that $\partial Q = N_1 \cup P \cup N_2$, $\partial P = M_1 \cup M_2$ and an extension of $\phi_1 \cup \phi_2$ to $\psi : (Q, P) \longrightarrow (Y, X)$ and an extension of F_1 and F_2 to a stable framing of $\tau_Q \oplus \psi^* \nu$.

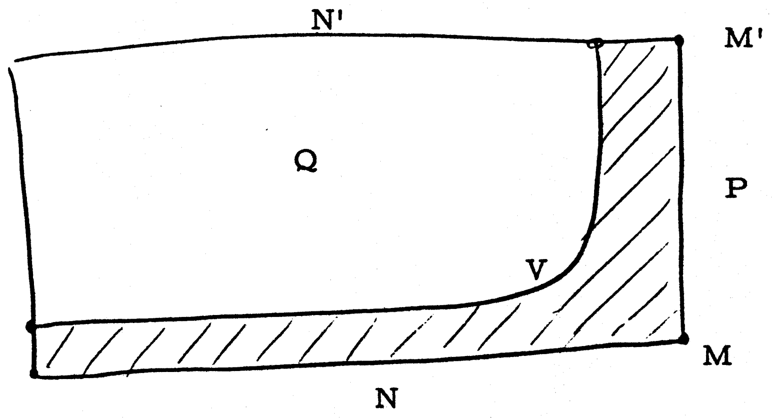
We can describe <u>bounded surgery</u> by first doing surgery on the boundary and then in the interior.

Doing surgery on ∂M we obtain a cobordism P, where $\partial P = M \cup M'$.

Now attach P to N along M to obtain a manifold V, where $\partial V = M'$. The cobordism of N is $V \times I$ with the corner along $M' \times 0$ rounded and a corner introduced along $M \times 0$.



To see that the cobordism Q of our equivalence relation is obtained this way, first construct V as above and the cobordism $V \times I$ of N to V, then find Q as a cobordism of V to N with boundary (M') fixed. (The shaded part is the cobordism of the figure above.)



The description of bounded cobordism has one disadvantage, it is not symmetrical with respect to N and N'. We shall return to this in the next section.

2. Surgery below the middle dimension

In this section we shall describe a necessary and sufficient condition for doing surgery on a class $\alpha \in \pi_{r+1}(\phi)$ and show that no difficulty is encountered below the middle dimension.

Theorem 2.1. Let $(M, \phi, F) \in \Omega_m(X, \nu)$. Any $\alpha \in \pi_{r+1}(\phi)$, $r \leq m-2$ determines a regular homotopy class of immersions $S^r \times D^{m-r} \longrightarrow M$. We can use the imbedding $f : S^r \times D^{m-r} \longrightarrow M$ to do surgery on α iff f is in this class.

Proof. Let

$$\begin{tikzcd} S^r \arrow[r, "i"] \arrow[d, "f_1"'] & D^{r+1} \arrow[d, "g_1"] \\ M \arrow[r, "\phi"] & X \end{tikzcd}$$

represent α. The stable trivialization F of $\tau_M \oplus \phi^*\nu$ pulled back by f, gives a stable trivialization of $f_1^*\tau_M \oplus f_1^*\phi^*\nu = f_1^*\tau_M \oplus i^*g_1^*\nu$. Since D^{r+1} is contractible we have a natural trivialization of $g_1^*\nu$, which induces a stable trivialization of $i^*g_1^*\nu$. Thus we obtain a stable trivialization of $f_1^*\tau_M$, which we view as a stable isomorphism with the trivial tangent bundle of $S^r \times D^{m-r}$.

By (I. 2.30) this determines a regular homotopy class of immersions if $r \leq m-2$.

Now let $f : S^r \times D^{m-r} \longrightarrow M$ be an imbedding. If it can be used for surgery on α, then its homotopy class must be that of $\partial_* \alpha$, $\partial_* : \pi_{r+1}(\phi) \longrightarrow \pi_r(M)$. Assuming this, we can take f_1 to be $f|S^r \times 0$. Construct N as described and extend ϕ to ψ using g_1. More precisely

$$\psi(M \times t) = \phi(M) \quad \text{and}$$
$$\psi(D^{r+1} \times D^{m-r}) = (g_1(D^{r+1}), \ \phi f(D^{m-r})).$$

Clearly ψ extends to a trivialization of $\tau_N \oplus \psi^*\nu$ on $M \times I$. The handle $D^{r+1} \times D^{m-r}$ is contractible and therefore it has a unique trivialization. This agrees with the trivialization f^*F on $S^r \times D^{m-r}$ because it is induced by the contraction of D^{r+1} and the stable isomorphism of $f^*\tau_M$ and $\tau_{S^r \times D^{m-r}}$. By our discussion these agree precisely when f lies in the regular homotopy class of α.

Corollary 2.2. If $m > 2r$, we can do surgery on α.

Proof. Since we have enough codimension general position gives an immersion $S^r \times D^{m-r} \longrightarrow M$ which defines an imbedding $f_0 : S^r \longrightarrow M$ representing α. Let τ_S denote the tangent bundle of S^r and ξ its normal bundle in M. By (I. 2.12)

$$f_0^*\tau_M \simeq \tau_S \oplus \xi$$

By the theorem we have a stable trivialization of $f_0^*\tau_M$ and τ_S is clearly stably trivial, hence ξ is stably trivial. By (I. 2.17) we have that ξ is trivial, since $m > 2r$. Thus we have an imbedding $f : S^r \times D^{m-r} \longrightarrow M$ representing α.

We use this to obtain

Theorem 2.3. Let $(M, \phi, F) \in \Omega_m(X, \nu)$ and assume that X is a finite simplicial complex. If $m \geq 2k$ then we can perform a finite number of surgeries on M with handles of dimension $\leq k$ to make ϕ k-connected.

Proof. As in (I. 3.5) replace X by the mapping cylinder of ϕ so that $\phi : M \longrightarrow X$ is an inclusion. Enumerate the s simplices of dimension $\leq k$ in $X - M$. Let $X_0 = M$ and let X_i be the result of attaching the first i simplices of $X - M$ to M. Let $N_0 = M \times I$. Now use induction on i. Suppose we already have a manifold N_{i-1}, $\partial N_{i-1} = M \cup M_{i-1}$, $\psi_{i-1} : N_{i-1} \longrightarrow X_{i-1}$ a homotopy equivalence and let $\psi_{i-1} | M_{i-1} = \phi_{i-1} : M_{i-1} \longrightarrow X$. Suppose the i-th simplex is of dimension $(r+1)$. It determines an element $\alpha \in \pi_{r+1}(X, X_{i-1})$. Since N_{i-1} is formed from M by attaching handles of dimension $\leq k$, starting from the other boundary it is formed from M_{i-1} by attaching handles of dimension $\geq (m+1-k) \geq k+1 \geq r+2$. Thus (N_{i-1}, M_{i-1}) is $(r+1)$-connected. Consider the homotopy exact sequence:

$$\begin{array}{ccccccccc} \longrightarrow \pi_{r+1}(X_{i-1}, M_{i-1}) & \longrightarrow & \pi_{r+1}(X, M_{i-1}) & \longrightarrow & \pi_{r+1}(X, X_{i-1}) & \longrightarrow & \pi_r(X_{i-1}, M_{i-1}) \longrightarrow \\ \| & & \| & & \| & & \| \\ 0 = \pi_{r+1}(N_{i-1}, M_{i-1}) & \longrightarrow & \pi_{r+1}(\phi_{i-1}) & \xrightarrow{\cong} & \pi_{r+1}(X, X_{i-1}) & \longrightarrow & \pi_r(N_{i-1}, M_{i-1}) = 0 \end{array}$$

Let α' map onto α and perform surgery on α'. This completes the induction.

We end up with $X_s = M \cup X^k$ and (X, X_s) is k-connected. By the above argument (N_s, M_s) is k-connected and N_s is homotopy equivalent to X_s. Thus $\phi_s : M_s \longrightarrow X$ is k-connected.

Now consider the relative versions. Our data consists of a pair of simply connected spaces (Y, X) with bundle ν and a smooth pair (N, M), $M = \partial N$ and a commutative diagram of maps

$$\phi : \begin{array}{ccc} N & \longrightarrow & Y \\ \uparrow & & \uparrow \\ M & \longrightarrow & X \end{array}$$

and a stable trivialization F of $\tau_N \oplus \phi^*\nu$.

Theorem 2.4. Let $(N, \phi, F) \in \Omega_m(Y, X, \nu)$. Any $\alpha \in \pi_{r+1}(\phi)$, $r \leq m-2$, determines a regular homotopy class of immersions

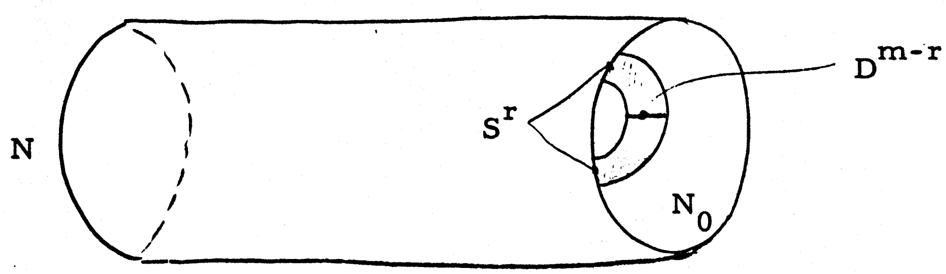
$$(D_-^r \times D^{m-r}, S^{r-1} \times D^{m-r}) \longrightarrow (N, M).$$

An imbedding f can be used to do surgery on α iff f is in this class.

Proof. The first part is similar to the proof of (2.1). If $f_1 : (D_-^r, S^{r-1}) \longrightarrow (N, M)$ represents the class $\partial_* \alpha$, then using f^*F and the contraction of $D_-^{r+1} \times D^{m-r}$ we can define a stable trivialization of $(f_1 | S^{r-1})^*(\tau_M)$ which extends to a stable trivialization of $f_1^* \tau_N$. A relative version of (I. 2.30) proves the first statement. Now suppose we have a nullhomotopy of $\phi \circ f$ given by α.

$$\begin{array}{ccc} (D_-^{r+1} \times D^{m-r}, S^r \times D^{m-r}) & \xrightarrow{i} & (D^{r+1} \times D^{m-r}, D_+^r \times D^{m-r}) \\ \downarrow f & & \downarrow \\ (N, M) & \xrightarrow{\phi} & (Y, X) \end{array}$$

We regard this as a nullhomotopy of $\phi | \mathrm{im} f$ and extend it to a homotopy of ϕ. Thus assume that $\phi(D_-^r \times D^{m-r}) = *$ and that the nullhomotopy is constant at the base point $*$ in X. Form N_0 by deleting the interior of $D_-^r \times D^{m-r}$ from N. Then ϕ induces a map $\phi_0 : (N_0, \partial N_0) \longrightarrow (Y, X)$ and N is obtained from N_0 by adding an $m-r$ handle. Obtaining N_0 from N this way will be called handle subtraction.



Thus $N \times I$ can be regarded as a cobordism between N and N_0 (after straightening corners). Finally the stable framing on $D_-^r \times D^{m-r}$ agrees with the one induced by contracting the handle iff the tangent bundle pulled back by f has the properties described, i.e. f is in the class of α.

Again this implies:

Corollary 2.5. If $m > 2r$, we can do surgery on α.

In order to obtain an analogue of (2.3) we have to assume that a k-connected map of the boundary $M \longrightarrow X$ induces a bijection $\pi_0 M \longrightarrow \pi_0 X$ and a k-connected map of each pair of components.

Theorem 2.6. Let (Y, X) be a finite simplicial pair $\pi_1(X) = \pi_1(Y) = 1$ and $(N, \phi, F) \in \Omega_n(Y, X, \nu)$. We can do surgery on ϕ to obtain

(i) if $n = 2k$, ϕ induces a k-connected map $N \longrightarrow Y$ and a $(k-1)$-connected map $M \longrightarrow X$, hence it is k-connected

(ii) if $n = 2k+1$, ϕ induces k-connected maps $N \longrightarrow Y$ and $M \longrightarrow X$, moreover ϕ is $(k+1)$-connected.

Proof. First restrict to the boundary, $(M, \phi|M, F|M)$. By (2.3) we can find a cobordism (P, ψ, F_0) to (M', ϕ', F') such that ϕ' is $(k-1)$-connected if $n = 2k$ and k-connected if $n = 2k+1$. By adding P to N we obtain a cobordism to (N', ϕ'', F'') where ϕ'' has the desired connectivity on $M' = \partial N'$. Now apply (2.3) to N' keeping M' fixed. This shows that we can make ϕ'' k-connected obtaining (N''', ϕ''', F'''). This proves the theorem for $n = 2k$ and it only remains to prove that for $n = 2k+1$ ϕ''' can be made $(k+1)$-connected.

The proof of (2.3) shows that if $\phi_1 = \phi|N : N \longrightarrow Y$, then $\pi_{k+1}(\phi_1)$ is represented by a finite number of cells. Choose a finite set of generators for $\pi_{k+1}(\phi_1)$. By (2.2) each can be represented by a framed imbedding of S^k. Connect each one by a tube to M so that we have framed embeddings of D_-^k. Now perform surgery as in (2.4).

Let H denote the union of the handles of this surgery and N_0 the constructed manifold, $\phi_0 : (N_0, M_0) \longrightarrow (Y, X)$ the resulting map. Note that

$(N_0, M_0) \longrightarrow (N, H \cup M)$ is an excision map. From the squares

$$\begin{array}{ccccc} N_0 & \longrightarrow & N & \longrightarrow & Y \quad \boxed{3} \\ \uparrow & \boxed{1} & \uparrow & \boxed{2} & \uparrow \\ M_0 & \longrightarrow & H \cup M & \longrightarrow & X \end{array} \qquad\qquad \begin{array}{ccccc} N & \longrightarrow & N & \longrightarrow & Y \quad \boxed{6} \\ \uparrow & \boxed{4} & \uparrow & \boxed{5} & \uparrow \\ M & \longrightarrow & H \cup M & \longrightarrow & X \end{array}$$

we have that $H_* \boxed{1} = 0$, hence $H_*(\phi_0) = H_* \boxed{3} \approx H_* \boxed{2}$. Moreover $\boxed{2} = \boxed{5}$ and

$$\begin{array}{ccccccccc} \cdots \longrightarrow H_{k+1}\boxed{4} & \longrightarrow & H_{k+1}\boxed{6} & \longrightarrow & H_{k+1}\boxed{5} & \longrightarrow & H_k\boxed{4} & \longrightarrow & \cdots \\ & & \| & & \| & & & & \\ \cdots \longrightarrow H_{k+1}\boxed{4} & \longrightarrow & H_{k+1}(\phi) & \longrightarrow & H_{k+1}(\phi_0) & \longrightarrow & H_k\boxed{4} & \longrightarrow & \cdots \end{array}$$

From the exact sequence of $\boxed{4}$ we have that $H_n(N, N) = 0$, hence $H_n\boxed{4} \simeq H_{n-1}(H \cup M, M)$ thus the sequence reduces to

$$\longrightarrow H_k(H \cup M, M) \longrightarrow H_{k+1}(\phi) \longrightarrow H_{k+1}(\phi_0) \longrightarrow H_{k-1}(H \cup M, M) \longrightarrow$$

Note that ϕ_0 and ϕ are k-connected, $(H, H \cap M) \longrightarrow (H \cup M, M)$ is an excision map and $(H, H \cap M)$ is just a collection of copies of $(D^k \times D^{k+1}, S^{k-1} \times D^{k+1})$. Thus $H_{k-1}(H \cup M, M) = 0$. In dimension k the original k-discs D^k_- represent images in $H_{k+1}(\phi)$ of generators of $\pi_{k+1}(\phi_1) = H_{k+1}(\phi_1)$. Now $H_{k+1}(\phi_1) \longrightarrow H_{k+1}(\phi)$ is onto because the map $M \longrightarrow X$ is k-connected and therefore $H_k(H \cup M, M) \longrightarrow H_{k+1}(\phi)$ is onto. This proves $H_{k+1}(\phi_0) = 0$.

Corollary 2.7. If $n = 2k+1$ and $M \longrightarrow X$ is already k-connected, then all further surgery can be performed in a prescribed (non-empty) open subset. The effect on M is just that of surgery on spheres which have trivial framed imbeddings.

3. The bounded case

Up to this point we had a map $\phi : M^m \longrightarrow X$ and a bundle ν over X together with a stable trivialization F of $\tau_M \oplus \phi^* \nu$. This enabled us to simplify ϕ considerably. If we are to make ϕ a homotopy equivalence, however, we need additional assumptions.

Since M^m satisfies Poincaré duality, a homotopy equivalent space will do the same, thus we must assume

(ii) X is a finite Poincaré complex of virtual dimension m and the Thom isomorphism corresponding to ν takes $[X]$ to a spherical class.

Moreover if $\phi : M \longrightarrow X$ is a homotopy equivalence then the proper choice of sign for $[X]$ makes it a degree 1 map. This is preserved under cobordism, hence we may assume

(iii) $\phi : M \longrightarrow X$ is a degree 1 map.

Similar considerations apply to pairs (Y, X).

Definition. A map $\phi : M \longrightarrow X$ [resp. $\phi : (N, M) \longrightarrow (Y, X)$] satisfying (i), (ii), (iii) will be called a surgery map.

In this section we shall give a complete solution for the case when $\partial N = M \neq \emptyset$. We restate our data as follows:

(Y, X) is a connected finite Poincaré pair of virtual dimension $n \geq 6$, $\pi_1(Y) = 1$, $\pi_1(X_i) = 1$ for each component X_i of X; there is a smooth manifold pair (N, M), $M = \partial N$ with a degree 1 map $\phi : (N, M) \longrightarrow (Y, X)$ and there is a vector bundle ν over Y with spherical Thom class and a stable trivialization F of $\tau_N \oplus \phi^*\nu$ which reduces to the boundary.

Theorem 3.1. In the above situation we can perform surgery on (N, M) to make ϕ a homotopy equivalence; moreover, the resulting manifold pair (N_0, M_0) is unique up to diffeomorphism in the bordism class $(N, \phi, F) \in \Omega_m(Y, X,$

Proof. Uniqueness follows immediately from existence applied to the cobordism between two solutions mapped into $(Y \times I, Y \times \partial I)$. Since the end result is a simply connected h-cobordism, the assertion is proved.

Now we proceed with the construction. The proof naturally breaks up into two cases according to the parity of n.

The case $n = 2k$.

By (2.6) we can perform surgery on ϕ to make the induced map $\phi_2 : M \longrightarrow X$ $(k-1)$-connected and $\phi_1 : N \longrightarrow Y$ k-connected. By (I. 3.10) $K_k(N, M)$ is free. In fact we have the isomorphism

$$\pi_{k+1}(\phi) \simeq H_{k+1}(\phi) = K_k(N, M) .$$

If we choose generators e_i for $K_k(N, M)$, they determine classes $\alpha_i \in \pi_{k+1}(\phi)$ and by (2.4) these in turn determine regular homotopy classes of immersions

$$f_i : (D^k \times D^k, \partial D^k \times D^k) \longrightarrow (N, M) .$$

We claim that the f_i are regularly homotopic to disjoint imbeddings, and hence we can perform surgery to kill $\pi_{k+1}(\phi)$. It is enough to show this for the "cores", $\bar{f}_i : (D^k, \partial D^k) \longrightarrow (N, M)$ and use "small" neighborhoods of these discs.

We shall use a technique called "piping". Put the $\bar{f}_i$ in mutual general position. The only intersections (and self-intersections) are isolated points B in the interior of N. At each point the intersecting sheets meet transversely. Choose arcs β, β' from B along the sheets to M, meeting no other singularities. Then $\beta \cup \beta'$ is an arc in N with both ends in M. Since $\pi_1(N, M) = 0$ we can find a singular disc Δ with boundary β, β' and an arc β'' connecting the endpoints inside M. Put Δ in general position. Since $k \geq 3$ it is then imbedded disjointly from the discs $\bar{f}_i(D^k)$, except along β and β'. Now construct a regular homotopy of $\bar{f}_i$ leaving everything fixed except a neighborhood of β. It pulls β across Δ past β' eliminating the intersection B and introducing no new intersections.

By induction each $\bar{f}_i$ (and f_i) is converted to disjoint imbeddings.

Now perform handle subtraction as in (2.4) to complete the argument.

<u>The case</u> $n = 2k+1$.

By (2.6) we may assume that $\phi_1 : N \longrightarrow Y$ and $\phi_2 : M \longrightarrow X$ are k-connected and $\pi_{k+1}(\phi) = K_k(N, M) = 0$. We have the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{k+2}(\phi) & \longrightarrow & H_{k+1}(\phi_2) & \longrightarrow & H_{k+1}(\phi_1) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & K_{k+1}(N, M) & \longrightarrow & K_k(M) & \longrightarrow & K_k(N) & \longrightarrow & 0 \end{array}$$

where each group is free (see I. 3.12).

By a theorem of Namioka [16] since ϕ, ϕ_1, ϕ_2 are k-connected $\pi_{k+2}(\phi) \longrightarrow H_{k+2}(\phi) \cong K_{k+1}(N, M)$ is an isomorphism, hence we can apply (2.4) to obtain framed immersions

$$\bar{f}_i : (D^{k+1}, S^k) \longrightarrow (N, M)$$

to represent basis elements of $K_{k+1}(N, M)$.

Next we shall modify $\bar{f}_i$ by regular homotopy to obtain disjoint imbeddings of the boundaries $S^k \longrightarrow M$. Put the $\bar{f}_i$ in general position and consider the intersections and self-intersections. These are 1-dimensional and along any two sheets meet transversely. They form certain circles (of no interest) and arcs β with both ends of M. Find in each sheet at β a disc Δ_i whose other side, β_i, lies in M. The loop $\beta_1 \cup \beta_2$ is in M. It spans the disc $\Delta_1 \cup \Delta_2$ in N, hence it is null-homotopic in N and $(\pi_1 M = 1)$ also in M. Span $\beta_1 \cup \beta_2$ by a disc Δ in M. Since $k \geq 3$ we may suppose that Δ is imbedded meeting the images of the $\bar{f}_i$ only in $\beta_1 \cup \beta_2$. Deform a neighborhood of β_1 across Δ to eliminate the intersections at the ends. By induction all intersections and self-intersections of $\bar{f}_i(S^k)$ on M are eliminated.

Recall (I. 3.8) that we have the isomorphism

$$[N] \cap : K^k(N) \longrightarrow K_{k+1}(N, M)$$

where $K^k(N)$ is dual to $K_k(N)$ with dual base.

Since $K_{k+1}(N, M)$ injects into $K_k(M)$ we have represented a base of $K_{k+1}(N, M)$ by imbedded framed spheres $\bar{f}_i(S^k)$ in M. Attach corresponding $(k+1)$-handles to N. Let U be the union of these handles and the resulting pair (N', M'). Since the $\bar{f}_i(S^k)$ are nullhomotopic in N, $K_k(N)$ is unchanged. In fact up to homotopy N' is just $N \cup$ a bouquet of $(k+1)$-spheres, thus $K_{k+1}(N')$ is again free.

The exact sequence of the triple $M' \subset M' \cup U \subset N'$ is

$$\ldots \longrightarrow K_{k+2}(N', M' \cup U) \longrightarrow K_{k+1}(M' \cup U, M') \longrightarrow K_{k+1}(N', M') \longrightarrow$$

$$\longrightarrow K_{k+1}(N', M' \cup U) \longrightarrow K_k(M' \cup U, M') \longrightarrow K_k(N', M') \longrightarrow K_k(N', M' \cup U) \longrightarrow \ldots$$

Here by excision $K_r(N', M' \cup U) \simeq K_r(N, M) = 0 \quad r \neq k+1$ and $K_r(M' \cup U, M') \simeq K_r(U, U \cap M') = 0 \quad r \neq k$ thus the above sequence reduces to

$$0 \longrightarrow K_{k+1}(N', M') \longrightarrow K_{k+1}(N, M) \longrightarrow K_k(U, U \cap M') \longrightarrow K_k(N', M') \longrightarrow 0$$

Moreover $K_k(U, U \cap M')$ is free with one basis element corresponding to each handle (represented by the fiber of the normal disc to $\bar{f}_i(S^k)$ or equivalently by the core of the dual handle). The map

$$K_{k+1}(N, M) \longrightarrow K_k(U, U \cap M')$$

is dual to

$$K_{k+1}(U, U \cap M) \longrightarrow K_k(N)$$

representing the attaching maps, and hence zero. Thus $K_{k+1}(N', M') \simeq K_{k+1}(N, M)$ and we have a new free kernel $K_k(N', M')$ dual to $K_{k+1}(N')$. The attached handles correspond to a basis of $K_{k+1}(N, M)$, hence we have an isomorphism

$$K_{k+1}(N') \longrightarrow K_{k+1}(N', M')$$

The map of the duals

$$K_k(N') \longrightarrow K_k(N', M')$$

is also an isomorphism, hence the exact sequence

$$0 \longrightarrow K_{k+1}(N') \longrightarrow K_{k+1}(N', M') \longrightarrow K_k(M') \longrightarrow K_k(N') \longrightarrow K_k(N', M') \longrightarrow 0$$

yields $K_k(M') = 0$, and $\phi_2' : M' \longrightarrow X$ is a homotopy equivalence.

Now choose a basis for $K_k(N')$. Using interior surgery on the elements of $\pi_{k+1}(\phi_1') \simeq K_k(N')$ we obtain a cobordism P from N' to N''. Consider the induced map of Poincaré triads

$$(P; N' \cup M' \times I, N'') \longrightarrow (Y \times I, Y \times 0 \cup X \times I, Y \times 1)$$

Identify N' with $N' \cup M' \times I$. In the exact sequence

$$0 \longrightarrow K_{k+1}(N') \longrightarrow K_{k+1}(P) \longrightarrow K_{k+1}(P, N') \xrightarrow{d} K_k(N') \longrightarrow K_k(P) \longrightarrow 0$$

the map d is an isomorphism by construction, so $K_k(P) = 0$ and $K_{k+1}(N') \longrightarrow K_{k+1}(P$ is an isomorphism. On the other hand in

$$0 \longrightarrow K_{k+1}(N'') \longrightarrow K_{k+1}(P) \xrightarrow{\bar{d}} K_{k+1}(P, N'') \longrightarrow K_k(N'') \longrightarrow 0$$

$\bar{d}$ is dual to d, hence $K_k(N'') = K_{k+1}(N'') = 0$.

Thus $\phi_1'' : N'' \longrightarrow Y$ is a homotopy equivalence, which together with the fact that $\phi_1'' | M' = \phi_2'$ implies the homotopy equivalence

$$\phi'' : (N'', M') \longrightarrow (Y, X).$$

completing the proof.

At this stage we could deduce the results for closed manifolds by removing a disc (care must be taken how to alter the Poincaré space), performing bounded surgery and looking at the obstruction to putting the disc back. Thus we would only need to compute the groups P_n as for example in Kervaire and Milnor [11]. On the other hand more insight is gained by doing the closed case as a genuine surgery problem. Moreover it resembles the non-simply connected case instead of emphasizing the advantages of simple connectivity. We shall return to the computation of P_n in chapter III.

4. The closed case, $m = 2k$.

Here we shall consider the case when X is a Poincaré complex of dimension $2k \geq 6$, M a closed, smooth manifold and $\varphi : M \longrightarrow X$ a surgery map. We could assume that M has boundary which is fixed throughout and all results would be valid but the statements and proofs more complicated.

By (2.3) we may assume that φ is k-connected.

By (I.3.10) we have $G = K_k(M) = \pi_{k+1}(\varphi)$ free and Poincaré duality induces an isomorphism of G and $\hat{G} = K^k(M)$. Since φ is k-connected $\hat{G} \simeq \mathrm{Hom}_Z(G, Z)$.

An element of $G = \pi_{k+1}(\varphi)$ is represented by a well defined regular homotopy class of immersions of $S^k \times D^k$ in M. Through the isomorphism $\pi_{k+1}(\varphi) \simeq K_k(M)$ we may identify it with the homology class of the core and use homology intersections to define a bilinear pairing $\lambda : G \times G \longrightarrow Z$. This is clearly well defined.

Represent elements of G by immersions $f : S^k \longrightarrow M$ (which may be "fattened" when needed). Such an immersion will not necessarily preserve the base point but we can run an arc from the base point of M to the base point of the sphere, $f(1)$. Addition is represented by joining by an arc thickened to a copy of $D^k \times I$ with ends $D^k \times \partial I$ on the two spheres and using $\partial D^k \times I$ for piping (i.e. based connected sum).

Theorem 4.1. Intersections define a map $\lambda : G \times G \longrightarrow Z$ such that if $x, y \in G$, $a \in Z$ we have

(i) λ is bilinear

(ii) $\lambda(y, x) = (-1)^k \lambda(x, y)$.

Let $V_k = Z/\{1-(-1)^k\}Z = \begin{cases} Z & k \text{ even} \\ Z_2 & k \text{ odd} \end{cases}$. Then self-intersections define a map $\mu : G \longrightarrow V_k$ such that

(iii) $\lambda(x, x) = \mu(x) + (-1)^k \mu(x)$

(iv) $\mu(x+y) - \mu(x) - \mu(y) = \lambda(x, y)$

(v) $\mu(xa) = a^2 \mu(x)$.

Finally x is represented by an embedding iff $\mu(x) = 0$.

Note that in (iii) although $\mu(x) \in V_k$, $\mu(x) + (-1)^k\mu(x)$ is well defined in Z. In fact $\mu(x)$ is half the self-intersection number for k even and the Arf-Kervaire cohomology operation for k odd (see [11]). In (iv) $\lambda(x, y)$ is taken mod 2 if k is odd. (v) shows that μ is a quadratic form, again take mod 2 for k odd.

Proof. Let S_1 and S_2 be two immersed k-spheres in M put in general position. They intersect transversely in a finite number of points P. To each P assign a sign $\varepsilon_P = \pm 1$ as follows. Orient M at the base point $*$ and transport the orientation to P by the path chosen above to $f_1(1) \in S_1$. Since $\pi_1(M) = 1$ the choice of this path is immaterial. Define ε_P to be the sign of intersection of S_1 and S_2 with respect to this orientation at P.

Define $\lambda(S_1, S_2) = \Sigma_P \varepsilon_P$ over all intersection points P. Clearly λ is well defined for elements of G and it is bilinear. To compute $\lambda(y, x)$ note that the sign of the intersection changes by $(-1)^k$ by interchanging the order, $\varepsilon'_P = (-1)^k \varepsilon_P$.

Now let S_1 be an immersed sphere in general position, so it has only a finite set of transverse self-intersections. At each P two branches of S_1 cross. By using the above procedure and specifying on ordering of these branches we can compute ε_P. If we interchange the order $\varepsilon'_P = (-1)^k \varepsilon_P$. Consider the sum $\Sigma_P \varepsilon_P$ over all self-intersection points with an arbitrary ordering at each. Define $\mu(S_1)$ to be the element of V_k defined by $\Sigma_P \varepsilon_P$. Note that $\mu(S_1)$ is unchanged by a change of any of the above choices. Changing S_1 by a regular homotopy can be done so that the self-intersections, hence μ, vary continuously except at a finite set of points where two self-intersections appear or disappear together. At such an occurrence the two self-intersections determine opposite ε_P, thus μ is constant. Hence $\mu : G \longrightarrow V_k$ is well defined.

Since the self-intersections of the connected sum of S_1 and S_2 consist of the self-intersections of S_1, those of S_2 and the intersections of S_1 with S_2 (iv) follows. For (iii) note that $\lambda(x, x)$ is the intersection of two different spheres S_1 and S'_1 representing x. In particular choose S_1 as above, take a tubular neighborhood and let S'_1 be a cross-section of the normal bundle of

S_1 (fiber D^k). This does not intersect the zero section (S_1) since our immersions are framed. Each self-intersection of S_1 gives rise to two intersections of S_1 and S_1' with opposite order. (v) follows from (iii) and (iv).

Finally note that $\mu(x) = 0$ is clearly a necessary condition for finding an imbedded representative for x. We want to show it is also sufficient for $k \geq 3$.

Let $\mu(x) = 0$ and let S represent x such that its self-intersections are in pairs (P_i, Q_i) with $\varepsilon(P_i) = -\varepsilon(Q_i) = 1$ (with appropriate choices of order of the two branches at each intersection). Join P_i to Q_i by an arc β_i along one branch and an arc β_i' along the other. The loop defined by $\beta_i \cup \beta_i'$ is null-homotopic and P_i and Q_i have opposite signs on it. Such singularities are removable, see [14].

Definition. A free Z-module G together with maps λ and μ as above will be called a special Hermitian form (G, λ, μ).

If the special Hermitian form G is generated by two elements $\{e, e^*\}$ with $\mu(e) = \mu(e^*) = 0$, $\lambda(e, e^*) = 1$ it is called a standard plane.

Define the direct sum of special Hermitian forms by $(G_1, \lambda_1, \mu_1) \oplus (G_2, \lambda_2, \mu_2) =$
$= (G_1 \oplus G_2, \lambda_1 \oplus \lambda_2, \mu_1 + \mu_2)$.

A direct sum of standard planes is called a kernel.

Lemma 4.2. A special Hermitian form (G, λ, μ) is a kernel if and only if G has a free submodule H with a base extending to G and hence defining a basis for G/H, such that $\lambda(H \times H) = 0$, $\mu(H) = 0$ and the map $G/H \longrightarrow \mathrm{Hom}^{\cdot}_Z(H, Z)$ induced by λ is an isomorphism. Such a submodule H is called a subkernel.

Proof. If (G, λ, μ) is a kernel then the conditions are satisfied. Conversely let $\{e_i\}$ be a base for H. There is a dual base of $\mathrm{Hom}_Z(H, Z)$, which induces by the above isomorphism a base of G/H. Choose representative elements $\{e_i^{**}\}$ in G. By hypothesis $\{e_i, e_i^{**}\}$ is a symplectic base of G and we have

$$\mu(e_i) = 0, \ \lambda(e_i, e_j) = 0, \ \lambda(e_i, e_j^{**}) = \delta_{ij} \ .$$

Let $\mu_i = \mu(e_i^{**})$ and

$$e_j^* = e_j^{**} + (-1)^{k-1}[e_j\mu_j + \sum_{i<j} e_i\lambda(e_i^{**}, e_j^{**})] \;.$$

It gives

$$\mu(e_i^*) = 0, \;\; \lambda(e_i, e_j^*) = \delta_{ij} \;\text{ and }\; \lambda(e_i^*, e_j^*) = 0$$

hence G is a kernel. The base $\{e_i, e_i^*\}$ provides an isomorphism of G with a direct sum of standard planes. This also proves that if $H_1 \longrightarrow H_2$ is an isomorphism of subkernels of G_1, G_2, then it extends to an isomorphism $G_1 \longrightarrow G_2$.

Call two subkernels H_1, H_2 of (G, λ, μ) complementary if $H_1 \cap H_2 = 0$, $H_1 \oplus H_2 = G$. Then there is an obvious isomorphism of H_2 with G/H_1. By the above argument we can lift a base of G/H_1 to lie in H_2. Hence any two complementary subkernels are isomorphic to the pair described above.

Lemma 4.3. If (G, λ, μ) is a special Hermitian form, then $(G, \lambda, \mu) \oplus (G, -\lambda, -\mu)$ is a kernel.

Proof. Let $\{e_i\}$ be a base of G. Write e_i', e_i'' for the corresponding elements of the two summands. Then

$$\lambda(e_i' + e_i'', e_j' + e_j'') = \lambda(e_i', e_j') + \lambda(e_i'', e_j'') = \lambda(e_i, e_j) - \lambda(e_i, e_j) = 0$$

$$\mu(e_i' + e_i'') = \mu(e_i') + \mu(e_i'') = \mu(e_i) - \mu(e_i) = 0 \;.$$

The submodule H of $G \oplus G$ freely generated by $e_i' + e_i''$ is a subkernel since λ induces an isomorphism of $(G \oplus G)/H$ and $\mathrm{Hom}_Z(H, Z)$. Let the e_i' give a basis for $(G \oplus G)/H$ and the dual of $\{e_j' + e_j''\}$ for $\mathrm{Hom}_Z(H, Z)$. The matrix of the map is

$$a_{ij} = \lambda(e_i', e_j' + e_j'') = \lambda(e_i, e_j)$$

which is also the matrix of $G \longrightarrow \mathrm{Hom}_Z(G; Z)$. This map is an isomorphism by hypothesis.

Definition 4.4. The groups P_{2k} are defined as follows. Consider the semi-group of special Hermitian forms under $\oplus$. Write $X \sim X'$ if there are kernels K, K' such that $X \oplus K$ and $X' \oplus K'$ are isomorphic. Since the sum of kernels is a kernel this is an equivalence relation. Divide out and consider the quotient semi-group. By (4.3) (G, λ, μ) has an inverse in the quotient, hence it is a group. Call it P_{2k}.

The addition of kernels is described geometrically as follows.

Lemma 4.5. If $\varphi : M \longrightarrow X$ is a k-connected surgery map, then performing surgery on a (k-1)-sphere corresponds to adding a standard plane to (G, λ, μ).

Proof. $\pi_k(\varphi) = 0$, hence we are doing surgery on the zero element. Thus the (k-1)-sphere is regularly homotopic and by general position isotopic to an unknotted one inside a disc $D^{2k} \subset M$ with standard framing. Surgery replaces M by the connected sum $M \# S^k \times S^k$. G is replaced by the orthogonal direct sum of G with a standard plane whose basis elements $\{e, e^*\}$ correspond to $S^k \times 1$ and $1 \times S^k$.

Now suppose that $\varphi : M \longrightarrow X$ is a surgery map. Its surgery obstruction $\theta(M, \varphi, F)$ is defined as follows. Use (2.3) to make φ k-connected. Let $\varphi' : M' \longrightarrow X$ be the resulting k-connected surgery map. Define $\theta \in P_{2k}$ to be the equivalence class of $K_k(M') = G$ in P_{2k}. Naturally, we have to prove that it is well defined, i.e. independent of the surgeries employed to obtain (M', φ', F').

Theorem 4.6. The surgery obstruction $\theta(M, \varphi, F)$ depends only on the bordism class of (M, φ, F). This class has a representative with φ a homotopy equivalence if and only if $\theta = 0$.

Proof. Suppose we have bordant triples (M_-, φ_-, F_-) and (M_+, φ_+, F_+) where φ_+ and φ_- are k-connected and (N, ψ, F) is the cobordism. Regard ψ as a map of triads, $Y = X \times I$

$$\psi : (N; M_-, M_+) \longrightarrow (Y; X \times 0, X \times 1) .$$

By (2.6) we can do surgery on N relative to ∂N to make $N \longrightarrow Y$ k-connected. We can obtain $K_k(\psi) = 0$ by using handle subtraction (2.4) keeping for example M_- fixed. The only effect on $K_k(M_+)$ is adding standard planes, leaving θ unchanged. Thus all K_i of M_- and M_+ vanish except for $i = k$ and the same is true for $\partial N = M_+ \cup (-M_-)$. In fact the only nonzero groups are

$$0 \longrightarrow K_{k+1}(N, \partial N) \longrightarrow K_k(\partial N) \longrightarrow K_k(N) \longrightarrow 0 \ .$$

Moreover $K_k(M_+)$ and $K_k(M_-)$ are free, hence $K_k(\partial N) = K_k(M_+) \oplus K_k(M_-)$ is free and therefore its subgroup $K_{k+1}(N, \partial N)$ is free. Since $j_* \partial_* [N] = [M_+] - [M_-]$ the special Hermitian form defined on $K_k(\partial N)$ by using immersed spheres in each component and connected sums of spheres in the same component is the sum of a form representing the surgery obstruction for M_+ and the negative of a corresponding form for M_-. To prove that these are equal we show that $K_k(\partial N)$ is a kernel. This is clear if we can show that $K_{k+1}(N, \partial N)$ is a subkernel. This we shall prove in (4.7). Assume therefore that θ only depends on the bordism class.

Clearly if φ is a homotopy equivalence, then $\theta = 0$. Conversely assume $\theta = 0$. This means that we may assume that $K_k(M)$ is a kernel with standard base $\{e_i, e_i^*, 1 \leq i \leq r\}$. Since $\mu(e_r) = 0$ the class $e_r \in K_k(M) = \pi_{k+1}(\varphi)$ is represented by a framed embedded sphere $S_r \subset M$ by (4.1). By (2.1) we can do surgery on M using this sphere. Let N be the trace of the surgery. Up to homotopy $N \sim M \cup e^{k+1}$ and if M_+ is the resulting manifold then $N \sim M_+ \cup e^k$. The homomorphism η

$$K_k(M) \longrightarrow H_k(N) \xrightarrow{\eta} H_k(N, M_+) \simeq \mathbb{Z}$$

has an immediate geometric interpretation by intersection numbers with e_r. Since $\lambda(e_r, e_r^*) = 1$, η is surjective. So φ_+ is still k-connected, the surgery from M_+ to M is on a trivial (k-1)-sphere and $M \approx M_+ \# (S^k \times S^k)$.

Now $K_k(M_+)$ may be identified with the kernel with base $\{e_i, e_i^*; 1 \leq i \leq r-1\}$ by the diagram (see I.3.*)

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr, bend left] \arrow[dr] & & K_{k+1}(N,M) \arrow[rr, bend left, "e_r"] \arrow[dr] & & K_k(M) \arrow[rr, bend left, "\cap e_r"] \arrow[dr] & & K_k(N,M_+) \arrow[rr, bend left] \arrow[dr] & & 0 \\
& 0 \arrow[ur] \arrow[dr] & & K_{k+1}(N,\partial N) \arrow[ur] \arrow[dr] & & K_k(N) \arrow[ur] \arrow[dr] & & 0 \arrow[ur] & \\
& & 0 \arrow[ur] \arrow[rr, bend right] & & K_k(M_+) \arrow[ur] \arrow[rr, bend right] & & 0 \arrow[ur] & &
\end{tikzcd}$$

Here $K_{k+1}(N,M)$ and $K_k(N,M_+)$ can be identified with Z. We can identify $K_{k+1}(N,\partial N)$ with the submodule of $K_k(M)$ generated by all e_i and e_i^* except e_r^* and $K_k(M_+)$ with $K_k(M)/\{e_r\}$. Moreover the geometric description is mirrored in the algebra. In fact this construction is the reverse of taking connected sum with $S^r \times S^r$. The result now follows from induction on the rank of $K_k(M)$.

Let us return to invariance under cobordism.

Lemma 4.7. Let $\varphi : (N,M) \longrightarrow (Y,X)$ be a surgery map, $M \neq \phi$, $\dim N = 2k+1 \geq 5$. Suppose φ induces k-connected maps $\varphi_1 : M \longrightarrow X$, $\varphi_2 : N \longrightarrow Y$ and that $K_k(N,M) = 0$. Then $K_{k+1}(N,M)$ is a subkernel in $K_k(M)$.

Proof. By assumption the only non-zero groups are

$$(1) \qquad 0 \longrightarrow K_{k+1}(N,M) \longrightarrow K_k(M) \longrightarrow K_k(N) \longrightarrow 0 .$$

By duality $K_k(N) \approx K^{k+1}(N,M)$. Since $K_k(N,M) = 0$ by assumption, $K^{k+1}(N,M) \approx \mathrm{Hom}(K_{k+1}(N,M); Z)$ and hence $K_k(N)$ is free. Therefore (1) splits.

We want to apply (4.2). In addition to the isomorphism

$K_k(N) \approx \mathrm{Hom}(K_{k+1}(N, M); Z)$ we need to know that λ and μ vanish identically on $K_{k+1}(N, M)$.

Since $\varphi_1 : M \longrightarrow X$ is k-connected, $K_k(M) \approx \pi_{k+1}(\varphi_1)$ is generated by classes represented by maps of spheres. These were used to define the special Hermitian form of $K_k(M)$. Let $x \in K_{k+1}(N, M)$. Represent $\partial x \in K_k(M)$ as a sum of maps of spheres, each being a framed immersion. These spheres have classes in $\pi_k(N)$ and we claim that the sum of these classes is zero. By the exactness of (1) it is zero in $K_k(N)$. Now consider the square of φ

$$\begin{tikzcd} N \arrow[r, "\varphi_2"] & Y \\ M \arrow[u] \arrow[r, "\varphi_1"] & X \arrow[u] \end{tikzcd} .$$

The homotopy exact sequences of φ and φ_2 and Hurewicz homomorphisms

$$\begin{tikzcd} & K_k(M) \arrow[r] \arrow[d, equal] & K_k(N) \arrow[r] \arrow[d, equal] & 0 & \\ \cdots \arrow[r] & \pi_{k+1}(\varphi_1) \arrow[r] & \pi_{k+1}(\varphi_2) \arrow[r] \arrow[d, equal] & \pi_{k+1}(\varphi) \arrow[r] \arrow[u] & 0 \\ \cdots \arrow[r] & \pi_{k+1}(Y) \arrow[r] & \pi_{k+1}(\varphi_2) \arrow[r] & \pi_k(N) \arrow[r] & \pi_k(Y) \arrow[r] & 0 \end{tikzcd}$$

give us a map $K_k(N) \longrightarrow \pi_k(N)$ proving the claim. (Note that φ_2 is k-connected, hence $\pi_{k+1}(\varphi_2) \approx H_{k+1}(\varphi_2) \approx K_k(N)$.)

We now have a map into N of a $(k+1)$-sphere with discs removed such that the boundary spheres are mapped by the above framed immersions. Thus the framed immersions of the S^k in M extend to a framed immersion of a punctured S^{k+1} in N. Let T and T' be representatives for x and x' obtained this way and moved into general position. They meet in a finite set of circles and arcs with both ends representing intersections of ∂x and $\partial x'$. Homologically all such intersections cancel in pairs, thus $\lambda(\partial x, \partial x') = 0$.

The self-intersections of T can be computed as follows. Along each arc choose an order of the two branches of T meeting there. Note that the self-intersections of ∂T at the two ends have opposite signs, so they cancel in pairs and $\mu(\partial x) = 0$. This completes the proof.

We have proved (4.7) under weaker assumptions then needed for (4.6), but the full strength will be utilized in the next section.

5. The closed case, $m = 2k+1$.

Assume X is a finite Poincaré complex of formal dimension $m = 2k+1 \geq 5$ and $\varphi : M \longrightarrow X$ a surgery map, where $(M, \varphi, F) \in \Omega_m(X, \nu)$. By (2.3) we may assume that φ is k-connected.

Choose a set of generators for $\pi_{k+1}(\varphi) = K_k(M)$. By general position they can be represented by disjoint framed imbeddings $f_i : S^k \times D^{k+1} \longrightarrow M$, each connected to the base point by a path. Let $U = \bigcup_i f_i(S^k \times D^{k+1})$, $M_0 = M - \operatorname{Int} U$. Since the f_i are trivial in X with given null-homotopies, we can replace φ by a homotopic map such that $\varphi(U) = *$ and the null-homotopies of f_i are constant.

The troube is that performing surgery may not reduce $K_k(M)$. We need to study the effect of surgery on $K_k(M)$.

By (I.3.2) we may suppose that $\dim X = m$ and X has only one m-cell, so we have a finite Poincaré pair (X_0, S^{m-1}) and $X = X_0 \cup D^m$. Using a cellular approximation of $\varphi | M_0$ we may suppose after a further homotopy that φ is a map of degree 1 of the Poincaré triads

$$\varphi : (M; M_0, U) \longrightarrow (X; X_0, D^m) \ .$$

Combine the exact sequences of groups K_i for the pairs (M, M_0), (M, U), $(M_0, \partial U)$ and $(U, \partial U)$ with excisions to the diagram

$$
(1) \qquad
\begin{tikzcd}
0 \arrow[dr] \arrow[rr, bend left] & & K_{k+1}(M, M_0) = K_{k+1}(U, \partial U) \arrow[dr] \arrow[rr, bend left] & & K_k(M_0) \arrow[dr] \arrow[rr, bend left] & & 0 \\
& K_{k+1}(M) \arrow[ur] \arrow[dr] & & K_k(\partial U) \arrow[ur] \arrow[dr] & & K_k(M) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] \arrow[rr, bend right] & & K_{k+1}(M, U) = K_{k+1}(M_0, \partial U) \arrow[ur] \arrow[rr, bend right] & & K_k(U) \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}
$$

Now φ maps $(U, \partial U)$ to (D^m, S^{m-1}) and the latter has trivial absolute and relative middle homology. Thus we can replace $K_{k+1}(U, \partial U)$, $K_k(\partial U)$, $K_k(U)$ by $H_{k+1}(U, \partial U)$, $H_k(\partial U)$, $H_k(U)$ respectively.

By (I. 3.10) the groups $K_{k+1}(M_0, \partial U)$ and $K_k(M_0)$ are free. Let $H = H_k(\partial U)$. By (4.7) H is a kernel and $S = K_{k+1}(U, \partial U)$ and $A = K_{k+1}(M_0, \partial U)$ are subkernels. In fact we have an explicit representation of the former since U is a disjoint union of copies of $S^k \times D^{k+1}$ and we can take the classes of $S^k \times 1$ and $1 \times \partial D^{k+1}$ as basis for H to identify it with the standard kernel.

Let $\mathcal{H}_r$ denote the standard kernel with basis $\{E_i, E_i^*; 1 \leq i \leq r\}$ and $\mu(E_i) = \mu(E_i^*) = 0$, $\lambda(E_i, E_j) = \lambda(E_i^*, E_j^*) = 0$, $\lambda(E_i, E_j^*) = (-1)^k \lambda(E_j^*, E_i) = \delta_{ij}$. Let $\mathcal{S}_r$ denote the subkernel generated by $\{E_i\}$.

Let H_r be given the basis obtained from the imbeddings $f_i(S^k \times D^{k+1})$, where $e_i = f_i(1 \times \partial D^{k+1})$, $e_i^* = f_i(S^k \times 1)$. If we identify $H_r \longrightarrow \mathcal{H}_r$ by $e_i \longrightarrow E_i$, $e_i^* \longrightarrow E_i^*$, then it sends $S_r \longrightarrow \mathcal{S}_r$ isomorphically. Thus S_r is generated by $\{e_i\}$.

Our aim is to kill $K_k(M)$. One way to do this is to show that $\partial : K_{k+1}(M, M_0) \longrightarrow K_k(M_0)$ is an isomorphism. Equivalently we could show that S_r and A_r are complementary subkernels of H_r.

By the remark following (4.2) any isomorphism of S_r onto A_r extends to an automorphism $H_r \longrightarrow H_r$ and through the above identification to $\mathcal{H}_r \xrightarrow{a} \mathcal{H}_r$. We need to study the effect on a of admissible change of choices made so far.

First we introduce some additional notation.

Definition 5.1. In the following

SU_r denotes the group of automorphisms of $\mathcal{H}_r$, (i.e., Z-module automorphisms preserving λ and μ).

TU_r denotes the subgroup leaving $\mathcal{S}_r$ setwise invariant (and induces an automorphism on it).

UU_r denotes the subgroup leaving $\mathcal{S}_r$ pointwise invariant (i.e. induces the identity automorphism on it).

SL_r denotes the group of automorphisms of $\mathcal{S}_r$.

The following sequence is split exact:

$$1 \longrightarrow UU_r \longrightarrow TU_r \longrightarrow SL_r \longrightarrow 1 .$$

The last map is surjective since we can define a splitting homomorphism $h : SL_r \longrightarrow TU_r$. Let $V = (v_{ij})$*) be a matrix representing an element of SL_r and $V^t = (v_{ji})$ its transpose, $W = (V^t)^{-1}$. Let

$$h(V) : e_i \longrightarrow e_j v_{ji},\ e_i^* \longrightarrow e_j^* w_{ji}$$

and note that t is an involutory anti-automorphism of SL_r. A matrix representation of UU_r is given by maps of the form

$$e_i \longrightarrow e_i,\ e_i^* \longrightarrow e_i^* + e_j c_{ji}$$

since an element of UU_r induces the identity on $\mathcal{S}_r$ hence also on its dual, which is the same as the quotient module. Now such a transformation preserves λ only if $C + (-1)^k C^* = 0$. In order to preserve μ we need

*) matrices act on the right.

in addition that $C = D - (-1)^k D^*$ for some D. The last restriction affects only the diagonal elements. Thus for k even C is anti-symmetric with zeros on the main diagonal and for k odd C is symmetric with even entries on the main diagonal.

Given the imbeddings f_i the subkernels are already determined. The particular identification $H_r \longrightarrow \mathcal{H}_r$ determines α. If a different one gives β then $\beta\alpha^{-1}$ preserves the subkernel $\mathcal{S}_r$ corresponding to $K_{k+1}(U, \partial U)$, hence $\beta\alpha^{-1}$ lies in TU_r. Thus the set $\{f_i\}$ determines uniquely a coset $TU_r \cdot \alpha$ in SU_r.

Next consider the change induced by a regular homotopy of f_i. A regular homotopy of S^k in M^{2k+1} is an immersion of $S^k \times I$ in $M^{2k+1} \times I$. Since the ends are imbedded, we can calculate the self-intersection in $V_{k+1}(\cong Z$ or $Z_2)$ of a regular homotopy or the mutual intersection (in Z) of two such as in §4. Denote the regular homotopy by $\{\eta_i\}$ and the end results by $\bar{\ }$. Let the self-intersection of η_i be ν_i and the intersection of η_i with η_j be ρ_{ij}. Then $\rho_{ii} = \nu_i + (-1)^{k+1}\nu_i$ and $\rho_{ji} = (-1)^{k+1}\rho_{ij}$ for $1 \leq i, j \leq r$. Thus $P = (\rho_{ij})$ is a matrix of the form $D - (-1)^k D^*$ and hence P determines an element γ of UU_r. We claim that the result of the regular homotopy replaces α by $\alpha\gamma$.

Consider the new diagram (1) with $U, M_0, \partial U$ replaced by $\overline{U}, \overline{M}_0, \partial\overline{U}$. We wish to show that the result of the regular homotopy corresponds to the change of cosets from $TU_r \cdot \alpha$ to $TU_r \cdot \alpha\gamma$, where

$$\gamma : e_i \longrightarrow \bar{e}_i,\ e_i^* \longrightarrow \bar{e}_i^* - \sum_j \bar{e}_j \rho_{ji} \ .$$

Clearly $\gamma : H_r \longrightarrow \overline{H}_r$ is an isomorphism since P is invertible, moreover $\gamma : S_r \longrightarrow \overline{S}_r$ is naturally an isomorphism. Thus we investigate $\gamma : A_r \longrightarrow \overline{A}_r$. In particular we wish to show that $\gamma(A_r) \subset \overline{A}_r$.

If $x \in A_r$ then x is represented by a chain in ∂U such that $x = \partial c$, c chain in M_0. Let

$$x = \sum_i e_i u_i + \sum_i e_i^* v_i \ .$$

We need to show that $\bar{x} = \gamma(x)$ can be represented as a chain in $\partial\bar{U}$, $\bar{x} = \partial\bar{c}$, $\bar{c}$ chain in $\bar{M}_0$.

$$\bar{x} = \gamma(x) = \sum_i \bar{e}_i u_i + \sum_i (\bar{e}_i^{\,*} - \sum_j \bar{e}_j \rho_{ji}) v_i =$$

$$= \sum_i \bar{e}_i u_i + \sum_i \bar{e}_i^{\,*} v_i - \sum_{i,j} \bar{e}_j \rho_{ji} v_i \ .$$

Now $\bar{x} + \sum_{i,j} \bar{e}_j \rho_{ji} v_i = \partial r$ bounds in M (namely the regular homotopies capped off with c). Let s denote the chain $\sum_{i,j} \bar{f}_j (1 \times D^{k+1}) \rho_{ji} v_i$. Clearly $\bar{x} = \partial(r-s)$. We need to show that $r-s$ is represented by a chain in $\bar{M}_0$. Now $r-s$ is a chain in M and it is a cycle in $\bar{M}_0$, let it be represented by $w \in H_{k+1}(M, \bar{M}_0)$. Similarly $\bar{x}$ is a chain in $\bar{M}_0$ and $\partial w = \bar{x}$ regarded in $H_k(\bar{M}_0)$. We also have the intersection pairing

$$\begin{array}{c} H_{k+1}(M, \bar{M}_0) \\ \| \\ H_k(\bar{U}) \otimes H_{k+1}(\bar{U}, \partial\bar{U}) \longrightarrow Z \ . \end{array}$$

Let us calculate $\bar{e}_t^{\,*} \cdot w$. If this is zero for each t, then we can pull w off interior $\bar{U}$ and represent $r-s$ in $\bar{M}_0$.

Recall that

$$r-s = c + \sum_i \xi_i u_i + \sum_i \eta_i v_i - \sum_{i,j} \bar{f}_j (1 \times D^{k+1}) \rho_{ji} v_i$$

where ξ_i and η_i are the obvious chains in M such that $\partial\xi_i = \bar{e}_i - e_i$, $\partial\eta_i = \bar{e}_i^{\,*} - e_i^{\,*}$ (η_i is the regular homotopy, ξ_i moving the framing).

In order to compute the intersections we need to consider U and $\bar{U}$ in M simultaneously.

Let $M \times I \longrightarrow M$ be projection onto the first factor. By an isotopy we may assume $U \cap \bar{U} = \varphi$. Next we claim that the intersection of η_t and η_i in $M \times I$ is the same as the intersection of $\bar{e}_t^{\,*}$ and η_i in M. This follows by moving all intersection points of η_i and η_t to a small straight collar near $\bar{e}_t^{\,*}$ in M. Hence

$$\bar{e}_t^* \eta_i - \bar{e}_t^* \cdot \sum_j \bar{f}_j (1 \times D^{k+1}) \rho_{ji} = \rho_{ti} - \rho_{ti} = 0 \ .$$

Thus we have proved that a sequence of r elements generating $G = K_r(M)$ determines a double coset $TU_r \cdot \alpha \cdot UU_r \subset SU_r$ and that α can be replaced by any element of the double coset.

Now assume that both $\{x_1, \ldots, x_r\}$ and $\{y_1, \ldots, y_s\}$ generate G. We can pass from the first to the second by a sequence of operations. Write $y_i = \sum_j x_j \lambda_{ji}$. Then

$$\begin{aligned} \{x_1, \ldots, x_r\} &\longrightarrow \{x_1, \ldots, x_r, 0\} \longrightarrow \{x_1, \ldots, x_r, y_1\} \longrightarrow \\ &\longrightarrow \{x_1, \ldots, x_r, y_1, 0\} \longrightarrow \{x_1, \ldots, x_r, y_1, y_2\} \longrightarrow \ldots \\ &\longrightarrow \{x_1, \ldots, x_r, y_1, \ldots, y_s\} \longrightarrow \{y_1, \ldots, y_s, x_1, \ldots, x_r\} \longrightarrow \\ &\longrightarrow \ldots \longrightarrow \{y_1, \ldots, y_s\} \end{aligned}$$

Each operation is one of:

(T1) Adjoin or delete a zero.

(T2) Permute the elements.

(T3') Add to the last element a linear combination of the others.

This can be reduced to a combination of:

(T3) Replace the first element by $\pm$ itself.

(T4) Replace the first element by the sum of the first two.

Consider the effect of (T1) - (T4) on α.

(T1) This adds an imbedding of $S^k \times D^{k+1}$ whose image lies in a $D^{2k} \subset M_0$. It takes the direct sum of (1) with the diagram (with all the natural maps)

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr, bend left] \arrow[dr] & & Z \arrow[rr, bend left] \arrow[dr] & & Z \arrow[rr, bend left] \arrow[dr] & & 0 \\
& 0 \arrow[ur] \arrow[dr] & & Z + Z \arrow[ur] \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & \\
0 \arrow[rr, bend right] \arrow[ur] & & Z \arrow[rr, bend right] \arrow[ur] & & Z \arrow[rr, bend right] \arrow[ur] & & 0
\end{tikzcd}$$

Thus the effect of (T1) is to form the direct sum of α with

$$\sigma = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$$

(T2) has the effect of altering the defining basis of H_r, i.e. conjugating α by a permutation.

(T3) only alters the basis of H_r, it conjugates α by the direct sum of $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ with the identity.

(T4) join the two copies of $S^k \times D^{k+1}$ by a thickened arc to obtain an imbedding of a trivial handlebody $\mathcal{H}$ (i.e. handles unlinked). The effect of the change of basis is performing a diffeomorphism of $\mathcal{H}$. Consider $\mathcal{H}$ as the thickening of the join of S^{k-1} in R^k with the vertices of an equilateral triangle in R^2 (the join is taken in a smooth neighborhood of R^m). The required diffeomorphism is rotation in R^2 by $2\pi/3$. Adjoining the thickened arc to U does not change (1). The above diffeomorphism changes the preferred basis of H_r and hence conjugates α by the automorphism

$$\begin{aligned} e_1' &= e_1 + e_2 \qquad & e_1^{*\prime} &= e_1^* \\ e_2' &= e_2 \qquad & e_2^{*\prime} &= e_2^* - e_1^* . \end{aligned}$$

The effects of (T2), (T3) and (T4) combine to an elementary basis change of rZ and hence generate E_r, the group of $r \times r$ matrices with 1 on the main diagonal and one other non-zero off-diagonal entry. Since $E_r \subset SL_r$ for any $\xi \in E_r$ we can conjugate α by $h(\xi) \in TU_r$. From above α is equivalent to any member of the double coset $TU_r \cdot \alpha \cdot UU_r$. Provided $E_r = SL_r$ the effect of (T2) - (T4) is to replace α by an arbitrary element of the double coset $TU_r \cdot \alpha \cdot TU_r \subset SU_r$. If we let r go to infinity in the limit we certainly have $E = SL$.

The effect of (T1) is to stabilize, but rather than $\alpha \longrightarrow \alpha \oplus 1$ by $\alpha \longrightarrow \alpha \oplus \sigma$.

The natural inclusions $\alpha \longrightarrow \alpha \oplus 1$ give

$$SU_r \subset SU_{r+1} \subset \cdots$$

where the limit is SU.

The inclusion $\alpha \longrightarrow \alpha \oplus \sigma$ gives rise to

$$SU_r \longrightarrow SU_{r+1} \longrightarrow \cdots$$

where the inclusions are <u>not</u> group homomorphisms but compatible with the natural left and right actions of SU_r. Taking limit we obtain SU' admitting left and right actions of SU. It has a natural base point, Σ, the direct sum of copies of σ.

Let RU be the subgroup of SU generated by TU and $\sigma \in SU_1$.

<u>Theorem</u> 5.2. Surgery can be completed to a homotopy equivalence iff α is equivalent to Σ under the two-sided action of RU.

<u>Proof.</u> Let $f_1(S^k \times D^{k+1})$ be one of the framed imbeddings of U. Performing surgery leaves the groups of the exact sequence

$$0 \longrightarrow K_{k+1}(M_0, \partial U) \longrightarrow K_k(\partial U) \longrightarrow K_k(M_0) \longrightarrow 0$$

unchanged, but the basis of $K_k(\partial U)$ is altered, since $S^k \times 1$ and $1 \times S^k$ are interchanged. Hence α is replaced by $\alpha\sigma$. Thus the class of α in RU is invariant under surgeries on k-spheres. Now suppose $\varphi : M \longrightarrow X$ and $\varphi' : M' \longrightarrow X$ are k-connected maps which are cobordant. In order to show α a surgery invariant we need to go from φ to φ' by surgeries on k-spheres alone. Let (N, ψ, G) be any cobordism between (M, φ, F) and (M', φ', F'). By (2.3) we can assume ψ is (k+1)-connected, thus (N, M) and (N, M') are k-connected pairs. Now by a relative handle decomposition theorem N is built up from M with no handles of dimension $\leq k$ and another with no handles of dimension $> k+1$. These two can be satsified simultaneously since $K_{k+1}(N, M)$ is free. Attach handles representing basis elements of $K_{k+1}(N, M)$ to M, then the rest of N is an h-cobordism of the resulting handlebody to M'. (See [14].)

Thus the class of α is a surgery invariant.

If φ is a homotopy equivalence, then take $U = \phi$, α is a zero matrix, hence stably $\alpha = \Sigma$.

Conversely suppose that for $\xi, \eta \in RU$, $\alpha = \xi\Sigma\eta$. Choose r so large that $\xi, \eta \in RU_r$. Then $\alpha = \xi\Sigma\eta = \Sigma(\Sigma^{-1}\xi\Sigma)\eta$. Note that Σ operates on $2rZ$ as a finite product of conjugates by permutation of summands (which belong to TU_r) composed with copies of σ, hence $\Sigma \in RU_r$. Similarly $\Sigma^{-1}\xi\Sigma \in RU_r$. Thus $\alpha = \Sigma\beta$, $\beta \in RU_r$. Again choosing r large we may assume that β is a product of elements of the form $\sigma, \nu, h(\varepsilon)$, where $\nu \in UU_r$, $\varepsilon \in E_r$. Multiplying α on the right with σ corresponds to a surgery; by the other elements just a change of basis. Thus by induction on the length of β we may assume $\alpha = \Sigma$. This, however, implies that: $\partial : K_{k+1}(M, U) \longrightarrow K_k(U)$ is an isomorphism, hence $K_k(M) = 0$ and φ is a homotopy equivalence.

It can be shown [24] that RU is a normal subgroup of SU containing the commutator and hence $P_{2k+1} = SU/RU$ is the obstruction group. We shall see in the next section that it turns out to be zero.

III. APPLICATIONS.

1. The groups P_m.

In this section we shall compute the simply connected surgery obstruction groups from the algebraic definitions given in Chapter II. Since again the results are well known and available in the literature, only an outline is given for the difficult case, P_{2k+1}. We shall remedy this by giving an alternative proof, which is more geometric, in the following section.

1.1. $P_{4n} \simeq Z$.

Proof. Recall the data from (II.4). We have (G, λ, μ) a special Hermitian form consisting in our case of a free Z-module G, a bilinear pairing $\lambda : G \times G \longrightarrow Z$ which is symmetric and a map $\mu : G \longrightarrow Z$ such that $\lambda(x, x) = 2\mu(x)$, hence λ is even. Since λ is just the intersection pairing its matrix is unimodular. Let Λ denote the matrix (unimodular, even, symmetric) of λ and $\sigma(\Lambda)$ its signature. If $\sigma(\Lambda) = 0$ it is possible to choose a basis for G such that it becomes a sum of standard planes (Milnor [15]). On the other hand the signature is an invariant of (G, λ, μ) under stabilization. Since there exists a special Hermitian form with $\sigma(\Lambda) = 8$ and since every unimodular even, symmetric bilinear form has signature divisible by 8 we conclude that $\sigma/8 : P_{4n} \longrightarrow \mathbb{Z}$ is an isomorphism.

For completeness let $G = Z + Z + Z + Z + Z + Z + Z + Z$. Define λ by the matrix

$$\Lambda = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

and $\mu(x) = \frac{1}{2}\lambda(x, x)$. Then $\sigma(\Lambda) = 8$.

1.2. $P_{4n+2} \simeq \mathbb{Z}_2$.

This time $V_{2n+1} \simeq \mathbb{Z}_2$, thus our data give (G, λ, μ), $\lambda : G \times G \longrightarrow \mathbb{Z}$

a bilinear, skew-symmetric pairing with $\lambda(x,x) = 0$ for all x and $\mu : G \longrightarrow \mathbb{Z}_2$ a quadratic form with

$$\mu(x+y) = \mu(x) + \mu(y) + \lambda(x,y) \bmod 2 \ .$$

Every such form is the direct sum of only two types (see [11])

$$H^0 = \langle e, e^*; \mu(e) = \mu(e^*) = 0, \ \lambda(e,e^*) = 1\rangle$$

$$H^1 = \langle e, e^*; \mu(e) = \mu(e^*) = 1, \ \lambda(e,e^*) = 1\rangle \ .$$

Here H^0 is the standard plane. One can prove that $H^1 \oplus H^1 \simeq H^0 \oplus H^0$, hence only the parity of the number of H^1-s in a decomposition of G matters. The latter is a stable invariant and is called the Arf-invariant $c(G, \lambda, \mu)$.

Let $\{e_i, e_i^*\}$ be a symplectic basis for (G,λ,μ), i.e. $\lambda(e_i, e_j) = 0$, $\lambda(e_i^*, e_j^*) = 0$, $\lambda(e_i, e_j^*) = \delta_{ij}$ and define

$$c(G,\lambda,\mu) = \sum_i \mu(e_i)\cdot\mu(e_i^*) \qquad \bmod 2 \ .$$

Clearly c is zero on a kernel, on the other hand $c(H^1) = 1$ and the geometric interpretation of μ shows that we cannot complete the surgery.

Thus $c : P_{4n+2} \longrightarrow Z_2$ is an isomorphism.

1.3. $P_{2k+1} = 0$.

Since $P_{2k+1} = SU/RU$ it would be desirable to be able to show that $SU \approx RU$. To my best knowledge there is no direct proof of this fact in the literature. Rather, the proof proceeds as follows. Recall that we have a diagram

$$\begin{tikzcd}
0 \arrow[dr] \arrow[rr, bend left] & & S \arrow[dr] \arrow[rr, bend left] & & H/A \arrow[dr] \arrow[rr, bend left] & & 0 \\
& G^0 \arrow[ur] \arrow[dr] & & H \arrow[ur] \arrow[dr] & & G \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] \arrow[rr, bend right] & & A \arrow[ur] \arrow[rr, bend right] & & H/S \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}$$

where H is a kernel with subkernels S and A and G^0 is the free part of G by duality.

Clearly $G = H/S+A$. Wall [21] defines a complete set of invariants for (S, A) in this situation by studying a bilinear form on $G^* = G/G^0$,

$$b : G^* \times G^* \longrightarrow \mathbb{Q}/\mathbb{Z}$$

induced by λ.

It is a matter of direct computation (see Wall [20]) to show that the allowable changes (T1) - (T4) enable us first to make $G^0 = 0$ and then reduce the order of G^* to 1. We shall do this in the next section in a more geometric way.

2. The Kervaire-Milnor proof for $P_{2k+1} = 0$.

In this section we are going to interpret P_m as the group of framed cobordism classes of framed m-manifolds with boundary a homotopy sphere. The cobordism is interior, i.e. it leaves the boundary fixed. The group structure is given by connected sum along the boundary (see [11]).

First recall the closing remark of (II.3). Suppose we are given a surgery map $\varphi_1 : N^m \longrightarrow X$, $m \geq 6$. Choose a small m-disc D in the top dimensional cell of X (see I.3.2) and approximate φ_1 by a map which is smooth in a neighborhood of D. Thus $R = \varphi_1^{-1}(D)$ is a compact manifold with boundary. Perform bounded surgery (as in II.3) on R to make it homotopy equivalent to D so that each time we subtract a handle from R we add it to the complement. Call the new map $\varphi_2 : N^m \longrightarrow X$ and $\varphi_2^{-1}(D) = B$.

Now $\varphi_2 : (B, \partial B) \longrightarrow (D^m, S^{m-1})$ is a homotopy equivalence (and since $m \geq 6$ we have Smale's Poincaré theorem). Let $N_0 = N - \text{int } B$, $X_0 = X - \text{int } D$. Then $\partial N_0 = \partial B$, $\partial X_0 = S^{m-1}$ and $\varphi_2 | = \varphi_0 : (N_0, \partial N_0) \longrightarrow (X_0, \partial X_0)$ is a surgery map, where $(X_0, \partial X_0)$ is a Poincaré pair of dimension m. Notice, moreover, that ν is trivial over D and hence over the sphere ∂X_0, thus $\varphi_0^* \nu$ and F give a stable framing of $\tau_{\partial N_0}$.

Use theorem (II.3.1) to obtain a homotopy equivalence

$$\varphi_0' : (N_0', \partial N_0') \longrightarrow (X_0, \partial X_0) \ .$$

Consider the trace of the boundary surgery, V: $\partial V = \partial N_0 \cup \partial N_0'$. We have a map

$$\psi : V \longrightarrow \partial X_0$$

covered by the stable framing we carried along and ψ is a homotopy equivalence restricted to either end since $\psi | \partial N_0 = \varphi_0 | \partial N_0$ and $\psi | \partial N_0' = \varphi_0' | \partial N_0'$. Attach B to ∂N_0 and D to ∂X_0, call $V \underset{\partial N_0}{\cup} B = M$, $\partial M = \partial N_0'$ and define a new map

$$\varphi : (M, \partial M) \longrightarrow (D^m, S^{m-1})$$

using ψ and $\varphi_0 | \partial N_0 = \varphi_2 | \partial B$. Clearly φ is a surgery map where the bundle over D^m is trivial giving a stable framing of τ_M. By (I.2.19) this shows that M is in fact parallelizable. Notice moreover that $\varphi | \partial M = \varphi_0' | \partial N_0'$ is already a homotopy equivalence.

Thus N is homotopy equivalent to X if and only if M is (interior) framed cobordant to a contractible manifold.

Boundary connected sum gives a group structure to framed manifolds with boundary a homotopy sphere and the differentiable pair (D^m, S^{m-1})

represents the zero element, hence we have

Theorem 2.1. P_m can be identified with the group of (interior) framed cobordism classes of framed m-manifolds with boundary a homotopy sphere.

In view of this $P_{2k+1} = 0$ is established by showing ([11])

Theorem 2.2. If m is odd we can use surgery in the interior of M to make φ a homotopy equivalence.

Proof. By (II. 2.5) we can perform interior framed surgery to make φ k-connected. We want to kill $K_k(M, \partial M) = K_k M = H_k M$. Let $f : S^k \times D^{k+1} \longrightarrow M$ be an imbedding and M' the result of surgery on f. Let $M_0 = M - \operatorname{int} f(S^k \times D^{k+}$ Then there is a commutative diagram

$$\begin{tikzcd}
 & & H_{k+1}M' \arrow[d, "\cdot\rho'"] & & \\
 & & Z \arrow[d, "\varepsilon"] \arrow[dr, "\rho"] & & \\
H_{k+1}M \arrow[r, "\cdot\rho"] & Z \arrow[r, "\varepsilon'"] \arrow[dr, "\rho'"'] & H_k M_0 \arrow[r, "i"] \arrow[d, "i'"] & H_k M \arrow[r] & 0 \\
 & & H_k M' \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

such that the horizontal and vertical sequences are exact. It follows that the quotient group $H_k M/\rho(Z)$ is isomorphic to $H_k M'/\rho'(Z)$.

Here ρ is the class of $f(S^k \times 0)$ in $H_k M$ and $\rho : Z \longrightarrow H_k M$ is $\rho(1) = \rho$. The map $\cdot\rho : H_{k+1}M \longrightarrow Z$ carries $\tau \in H_{k+1}M$ into the intersection number $\tau\cdot\rho$. (Similarly for ρ', the class of $f(0 \times S^k)$ in $H_k M'$.)

The horizontal sequence comes from the exact sequence of the pair (M, M_0). By excision

$$H_j(M, M_0) \approx H_j(S^k \times D^{k+1}, S^k \times S^k) \approx \begin{cases} Z & j = k+1 \\ 0 & j < k+1 \end{cases} .$$

It corresponds to $G^0 \longrightarrow S \longrightarrow H/A \longrightarrow G \longrightarrow 0$ in section 1. Since the generator of $H_{k+1}(M, M_0)$ has intersection number ± 1 with the cycle $f(S^k \times 0)$ which represents ρ, the homomorphism $H_{k+1}M \longrightarrow Z$ may be described as $\tau \longrightarrow \tau \cdot \rho$. The element $\varepsilon' = \varepsilon'(1) \in H_k M_0$ corresponds to $f(* \times S^k)$. Similar description yields the vertical sequence. Thus $\varepsilon = \varepsilon(1)$ is the class of $f(S^k \times *)$. Also $i(\varepsilon) = \rho$ and $i'(\varepsilon') = \rho'$. The isomorphisms

$$H_k M/\rho(Z) \simeq H_k M_0/\varepsilon(Z) + \varepsilon'(Z) \simeq H_k M'/\rho'(Z)$$

follow from the diagram.

If we define a primitive element $\rho \in H_k M$ to be one for which there is a $\tau \in H_{k+1}M$ such that $\tau \cdot \rho = 1$, then $i : H_k M_0 \longrightarrow H_k M$ is an isomorphism and hence $H_k M' \simeq H_k M/\rho(Z)$. Thus any primitive element can be killed by surgery. This implies that $H_k M$ may be reduced to its torsion subgroup (i.e. G^0 may be killed and only G^* remains). For suppose ρ generates an infinite cyclic summand of $H_k M$. By Poincaré duality there is a $\tau_1 \in H_{k+1}(M,\partial M)$ such that $\tau_1 \cdot \rho = 1$. But $H_{k+1}M \longrightarrow H_{k+1}(M,\partial M) \longrightarrow H_{k+1}(\partial M) = 0$ shows that τ_1 can be lifted back to $H_{k+1}M$.

Let W be an orientable homology manifold of dimension $2r$ and K a field. Define the semi-characteristic $\chi^*(\partial W; K)$ to be

$$\chi^*(\partial W; K) \equiv \sum_{i=1}^{r-1} \operatorname{rank} H_i(\partial W; K) \mod 2 .$$

Lemma 2.3. The rank of the bilinear pairing

$$H_r(W; K) \otimes H_r(W; K) \longrightarrow K ,$$

given by intersection numbers, is congruent modulo 2 to $\chi^*(\partial W; K)$ plus the Euler characteristic $\chi(W)$.

Proof. Consider the exact sequence (all coefficients in K)

$$H_r W \xrightarrow{h} H_r(W,\partial W) \longrightarrow H_{r-1}(\partial W) \longrightarrow \dots \longrightarrow H_0(W,\partial W) \longrightarrow 0 \ .$$

Counting shows that the rank of h is equal to the alternating sum of the ranks of the vector spaces to the right of h. Reducing modulo 2 and using $\operatorname{rank} H_i(W,\partial W) = \operatorname{rank} H_{2r-i}W$ we have

$$\begin{aligned}\operatorname{rank} h &\equiv \sum_{i=0}^{r-1} \operatorname{rank} H_i(\partial W) + \sum_{i=0}^{2r} \operatorname{rank} H_i W \\ &\equiv \chi^*(\partial W; K) + \chi(W) \bmod 2 \ .\end{aligned}$$

On the other hand the rank of

$$h : H_r W \longrightarrow H_r(W,\partial W) \simeq \operatorname{Hom}_k(H_r W; K)$$

is just the rank of the intersection pairing.

<u>Assume</u> $k = 2n$, $m = 4n+1$.

<u>Lemma</u> 2.4. If k is even surgery on f changes the k-th Betti number of M.

<u>Proof</u>. Put a cone over ∂M to obtain a closed manifold $\overline{M}$. Similarly let $\overline{M}'$ be the result of the surgery and W the trace, $\dim W = 2k+2$, $\partial W = \overline{M} \times 0 \cup \overline{M}' \times 1$. W has the homotopy type of $\overline{M}$ with a $(k+1)$-cell attached. Since $\dim \overline{M} = 2k+1$, $\chi(\overline{M}) = 0$, hence $\chi(W) = \chi(\overline{M}) + (-1)^{k+1} = (-1)^{k+1}$.

Since k is even the intersection pairing

$$H_{k+1}(W; Q) \otimes H_{k+1}(W; Q) \longrightarrow Q$$

is skew-symmetric, hence it has even rank. Setting $K = Q$ in Lemma 2.3 we obtain

$$\chi^*(\overline{M} \cup \overline{M}'; Q) + (-1)^{k+1} \equiv 0 \mod 2$$

hence $\chi^*(\overline{M}; Q) \not\equiv \chi^*(\overline{M}'; Q)$. Since $H_i\overline{M} \simeq H_i\overline{M}' = 0$ for $0 < i < k$ we have

$$\operatorname{rank} H_k(\overline{M}; Q) \not\equiv \operatorname{rank} H_k(\overline{M}'; Q)$$

but then also

$$\operatorname{rank} H_k(M; Q) \neq \operatorname{rank} H_k(M'; Q) \quad .$$

This lemma suffices to finish the argument for k even. By above we may assume that $H_k M$ is a torsion group. Let $f : S^k \times D^{k+1} \longrightarrow M$ represent a non-trivial $\rho \in H_k M$. We have

$$H_k M/\rho(Z) \simeq H_k M'/\rho'(Z) \quad .$$

Since the group $\rho(Z)$ is finite it follows from the above lemma that $\rho'(Z)$ is infinite. Now

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\rho'} H_k M' \longrightarrow H_k M'/\rho'(Z) \longrightarrow 0$$

is exact, hence the torsion subgroup of $H_k M'$ injects into $H_k M'/\rho'(Z)$ and therefore it is strictly smaller than $H_k M$. By further surgery we can kill the infinite part of $H_k M'$ to obtain M'' with

$$H_k M'' \simeq \text{torsion subgroup of } H_k M' < H_k M \quad .$$

Induction on the order of $H_k M$ completes the argument.

<u>Assume</u> $k = 2n+1$, $m = 4n+3$.

We shall now use the move of (II.5) we have not employed yet -- changing f by a regular homotopy. Suppose $f : S^k \times D^{k+1} \longrightarrow M$ is an imbedding. If $\beta : S^k \longrightarrow SO_{k+1}$ is a smooth map then we can define a new imbedding

$$f_\beta : S^k \times D^{k+1} \longrightarrow M$$

by $f_\beta(u, v) = f(u, v \cdot \beta(u))$ where $\cdot$ denotes the usual action of SO_{k+1} on D^{k+1}. Clearly f_β represents the same homotopy class as f. They extend to the same stable framing (given by F) of τ_M if and only if β is in the kernel of

$$s_* : \pi_k(SO_{k+1}) \longrightarrow \pi_k(SO_{2k+2}) \; .$$

This kernel is infinite cyclic for k odd.

Choose β in the kernel of s_* and denote by M'_β the result of surgery on f_β. Note that M_0 is independent of β. So is the class of $f_\beta(* \times S^k)$, i.e. $\varepsilon \in H_k M_0$. On the other hand the class $f_\beta(S^k \times *)$ depends on β, in particular its class $\varepsilon_\beta \in H_k M_0$ is given by

$$\varepsilon_\beta = \varepsilon + j(\beta)\varepsilon'$$

where the homomorphism

$$j_* : \pi_k(SO_{k+1}) \longrightarrow Z \simeq \pi_k(S^k)$$

is induced by the canonical map $j : SO_{k+1} \longrightarrow S^k$ defined by $j(\xi) = x \cdot \xi$, $x \in S^k$.

We may identify the stable group $\pi_k(SO_{2k+1})$ with the stable group $\pi_k(SO_{k+2})$. The exact sequence

$$\pi_{k+1}(S^{k+1}) \xrightarrow{\partial} \pi_k(SO_{k+1}) \xrightarrow{s_*} \pi_k(SO_{k+2})$$

arises from the fibration $SO_{k+1} \longrightarrow SO_{k+2} \longrightarrow S^{k+1}$. Recall that β is in the kernel of s_*.

The composition

$$\pi_{k+1}(S^{k+1}) \xrightarrow{\partial} \pi_k(SO_{k+1}) \xrightarrow{j_*} \pi_k(S^k)$$

is known to carry a generator of $\pi_{k+1}(S^{k+1})$ onto twice a generator of $\pi_k(S^k)$ for odd k. Thus the integer $j_*(\beta)$ can be any multiple of 2.

Next we shall consider the effect of replacing ε by $\varepsilon_\beta + j(\beta)\varepsilon'$ on the homology of M'_β. We have

$$0 \longrightarrow Z \xrightarrow{\varepsilon'} H_k M_0 \xrightarrow{i} H_k M \longrightarrow 0$$

from the beginning of the proof, where i carries $\varepsilon \in H_k M_0$ into an element ρ of order $r > 1$. Clearly $r\varepsilon$ is a multiple of ε', say

$r\varepsilon + r'\varepsilon' = 0$. Since ε' is not a torsion element, these two elements can satisfy no other relation. Now $\varepsilon_\beta = \varepsilon + j(\beta)\varepsilon'$, hence

$$r\varepsilon_\beta + (r'-r_j(\beta))\varepsilon' = 0 \quad .$$

Using the exact sequence

$$Z \xrightarrow{\varepsilon_\beta} H_kM_0 \xrightarrow{i'_\beta} H_kM' \longrightarrow 0$$

we see that i'_β carries ε' into some $\rho'_\beta \in H_kM'_\beta$ of order $|r' - rj(\beta)|$.

Since $H_kM'_\beta/\rho'_\beta(Z) \cong H_kM/\rho(Z)$ we see that $H_kM'_\beta$ is smaller than H_kM_β iff

$$0 < |r' - rj(\beta)| < r \quad .$$

From above $j(\beta)$ can be any even integer, thus $j(\beta)$ can be chosen so that

$$- r < r' - rj(\beta) \leq r \quad .$$

This choice of $j(\beta)$ will give $H_kM'_\beta < H_kM_\beta$ unless r' is divisible by r.

We need to study the residue class of r' modulo r.

Recall the definition of linking numbers ([18, §77]). Let $\rho \in H_pM$, $\tau \in H_qM$ be homology classes of finite order, with $\dim M = p+q+1$. Consider the homology exact sequence

$$\cdots \longrightarrow H_{p+1}(M; Q/Z) \xrightarrow{\alpha} H_pM \xrightarrow{i_*} H_p(M; Q) \longrightarrow \cdots$$

associated with the coefficient sequence

$$0 \longrightarrow Z \xrightarrow{i} Q \longrightarrow Q/Z \longrightarrow 0 \quad .$$

Since ρ is of finite order, $i_*\rho = 0$ and $\rho = \alpha(\nu)$ for some $\nu \in H_{p+1}(M; Q/Z)$. The pairing

$$Q/Z \otimes Z \longrightarrow Q/Z$$

defined by multiplication induces a pairing

$$H_{p+1}(M; Q/Z) \otimes H_q M \longrightarrow Q/Z$$

defined by intersection of homology classes. Denote it by a dot.

The <u>linking number</u> $L(\rho, \tau)$ is the rational number modulo 1 defined by

$$L(\rho, \tau) = \nu \cdot \tau \quad .$$

This linking number is well defined, and satisfies the symmetry relation

$$L(\rho, \tau) + (-1)^{pq} L(\tau, \rho) = 0 \quad .$$

In fact for $p = q = k$ this is just the bilinear form b derived from λ we referred to at the closing of section 1.

<u>Lemma</u> 2.5. The ratio r'/r modulo 1 is, up to sign, equal to the self-linking number $L(\rho, \rho)$.

<u>Proof</u>. Since $r\varepsilon + r'\varepsilon' = 0$ in $H_k M_0$, the cycle $r\varepsilon + r'\varepsilon'$ on ∂M_0 bounds a chain c in M_0. Let $c_1 = f(* \times D^{k+1})$ denote the cycle in $f(S^k \times D^{k+1}) \subset M$ with boundary ε'. The chain $c - r'c_1$ has boundary $r\varepsilon$, hence $(c - r'c_1)/r$ has boundary ε, representing the homology class ρ in $H_k M$. Taking the intersection of this chain with $f(S^k \times 0)$, which represents ρ we obtain $\pm r'/r$, since c is disjoint from $f(S^k \times 0)$ and c_1 has intersection number ± 1 with it. Thus $L(\rho, \rho) = \pm r'/r \bmod 1$.

Now if $L(\rho, \rho) \neq 0$, then $r' \not\equiv 0 \bmod r$, hence ρ can be replaced by a class of smaller order via surgery. Hence, unless $L(\rho, \rho) = 0$ for all $\rho \in H_k M$, $H_k M$ can be simplified. Recall that we assume that k is odd.

<u>Lemma</u> 2.6. If $H_k M$ is a torsion group, with $L(\rho, \rho) = 0$ for every $\rho \in H_k M$, then $H_k M$ is a direct sum of Z_2 - s.

<u>Proof</u>. The relation $L(\xi, \eta) + (-1)^{k^2} L(\eta, \xi) = 0$ and $k^2 \equiv 1 \bmod 2$ show that L is symmetric. If all self-linking numbers are zero, the identity

$$L(\xi+\eta, \xi+\eta) = L(\xi, \xi) + L(\eta, \eta) + L(\xi, \eta) + L(\eta, \xi)$$

implies that

$$2L(\xi, \eta) = 0$$

for all ξ and η. But, according to the Poincaré duality theorem for torsion groups [18, p. 245], L defines a completely orthogonal pairing

$$T_k M \otimes T_k M \longrightarrow Q/Z \ .$$

Hence the identity $2L(\xi, \eta) = L(2\xi, \eta) = 0$ for all η implies $2\xi = 0$.

Thus a sequence of surgeries reduces $H_k M$ to a group of the form $Z_2 \oplus Z_2 \oplus \ldots \oplus Z_2 = sZ_2$.

Note that M and M'_β are parallelizable and so is the trace of the surgery, W. It follows from the formulas of Wu that the Steenrod operation

$$Sq^{k+1} : H^{k+1}(W, \partial W; Z_2) \longrightarrow H^{2k+2}(W, \partial W; Z_2)$$

is zero (see [10, Lemma 7.9]). Hence every $\xi \in H_{k+1}(W; Z_2)$ has self-intersection number $\xi \cdot \xi = 0$.

Lemma 2.7. Suppose that every mod 2 homology class $\xi \in H_{k+1}(W; Z_2)$ has self-intersection number $\xi \cdot \xi = 0$. Then surgery necessarily changes the rank of the mod 2 homology group $H_k(M; Z_2)$.

Proof. Analogous to (2.4). The hypothesis $\xi \cdot \xi = 0$ for all ξ, guarantees that the intersection pairing

$$H_{k+1}(W; Z_2) \otimes H_{k+1}(W; Z_2) \longrightarrow Z_2$$

has even rank.

By (2.7) surgery on f_β changes the rank of $H_k(M; Z_2)$. The effect of this surgery on $H_k(M; Z)$, provided β is chosen properly, will be to replace ρ of order $r = 2$ by an element ρ'_β of order r'_β, where

$$-2 < r'_\beta \leq 2, \qquad r'_\beta \equiv 0 \pmod 2) \ .$$

Thus r'_β is 0 or 2. Now using the sequence

$$0 \longrightarrow Z_{r'_\beta} \longrightarrow H_k M'_\beta \longrightarrow H_k M_\beta / \rho_\beta(Z) \longrightarrow 0$$

where the right hand group is isomorphic to $(s-1)Z_2$, we see that $H_k M'_\beta$ is one of the following extensions:

$$H_k M'_\beta = \begin{cases} Z + (s-1)Z_2 \\ Z_2 + (s-1)Z_2 \\ Z + (s-2)Z_2 \\ Z_4 + (s-2)Z_2 \end{cases} .$$

The first two possibilities are excluded by (2.7). In the last two further surgery will replace $H_k M'_\beta$ by a group which is definitely smaller than $H_k M$.

This completes the case k odd and proves the theorem.

3. A braid.

In [12] Levine constructs a sequence of braids. We shall describe the stable version originally due to Kervaire and Milnor. For proofs and details the reader should consult [12].

Consider the following groups, whose elements are the objects under the given equivalence relation. Let $n \geq 5$.

P_n: Objects: framed n-manifolds with boundary a homotopy sphere. Equivalence relation: framed (interior) cobordism. Group operation: bounded framed connected sum. These groups were computed as

$n \equiv 0$	1	2	3	mod 4
$P_n = Z$	0	Z_2	0	

θ_n: Objects: homotopy n-spheres. Equivalence relation: h-cobordism. Group operation: connected sum. Kervaire and Milnor [11] showed that the θ_n are finite and computed them for $n \leq 18$. Only partial results are available for $n > 18$.

$F\theta_n$: Objects: framed homotopy n-spheres. Equivalence relation:

framed h-cobordism. Group operation: framed connected sum. For $n \leq 14$ these groups are listed in Novikov [17, p. 386]. They are finite except for $n \equiv 3, 7 \bmod 8$ when $F\theta_n \simeq Z +$ finite group.

A_n: Objects: closed n-manifolds, framed in the complement of a finite set of points. Equivalence relation: cobordism framed in the complement of the trace of these points. Group operation: connected sum framed in the complement of the above points. Again the structure is unknown except that the A_n are finite for $n \not\equiv 0 \bmod 4$ and $A_{4k} \simeq Z +$ finite group.

$\pi_n(O)$: Homotopy groups of the infinite orthogonal group O. Let O(k) denote the group of $k \times k$ real orthogonal matrices and define the inclusion $O(k) \longrightarrow O(k+1)$ by $\gamma \longrightarrow \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}$. The direct limit of O(k) is O. Its homotopy groups were computed by Bott [1]:

$n \equiv$	0	1	2	3	4	5	6	7	mod 8
$\pi_n(O) =$	Z_2	Z_2	0	Z	0	0	0	Z	

$\prod_n$: Objects: framed closed n-manifolds. Equivalence relation: framed cobordism. Group operation: framed connected sum. We can identity these groups with the stable homotopy groups of spheres as follows.

<u>Thom-Pontrjagin construction</u>:

Let M^n be a closed n-manifold with τ_M stably trivial. For sufficiently large $N > n$ we can imbed M in S^{n+N}:

$$\bar{f} : M^n \longrightarrow S^{N+n}$$

and the normal bundle of $f(M^n)$ is trivial. ($\tau_M \oplus \nu \simeq \bar{f}^* \tau_S$, $(\tau_M \oplus \varepsilon_1) \oplus \nu \simeq \bar{f}^* \tau_S \oplus \varepsilon_1$, thus ν is stably trivial. By (I.2.17) ν is trivial.) Thus we have an imbedding

$$f : M^n \times D^N \longrightarrow S^{N+n} \quad .$$

Define a map

$$a : S^{N+n} \longrightarrow S^N$$

by sending S^{N+n} - int $f(M^n \times D^N)$ to the base point.

Conversely let

$$\alpha : S^{N+n} \longrightarrow S^N$$

represent an element of the stable n-stem. We can approximate α by a smooth map transverse regular at, say x. Then $\alpha^{-1}(x) = M$ is a smooth n-manifold in S^{N+n} and if F_x is a framing of the tangent plane at x then the pullback gives a framing of M.

Finally we need to know that the correspondence is well defined. This follows, since two closed framed n-manifolds are framed cobordant if and only if the corresponding maps α_i are homotopic. (See [9].)

Now consider the exact sequences:

$$(1) \qquad \cdots \longrightarrow \pi_n(O) \xrightarrow{w_1} F\theta_n \xrightarrow{\varphi_1} \theta_n \xrightarrow{\partial_1} \pi_{n-1}(O) \longrightarrow \cdots$$

w_1 sends an element of $\pi_n(O)$ into the standard sphere whose usual framing is twisted by this element

φ_1 forgets the framing

∂_1 is the obstruction to framing θ_n. By Kervaire and Milnor [11, 3.1] every homotopy sphere is s-parallelizable, hence ∂_1 is zero.

Thus (1) splits into short exact sequences

$$0 \longrightarrow \pi_n(O) \longrightarrow F\theta_n \longrightarrow \theta_n \longrightarrow 0$$

$$(2) \qquad \cdots \longrightarrow F\theta_n \xrightarrow{w_2} \prod_n \xrightarrow{\varphi_2} P_n \xrightarrow{\varphi_2} F\theta_{n-1} \longrightarrow \cdots$$

w_2 the natural inclusion

φ_2 cut out a disc. Clearly φ_2 is zero for $n \neq 4k+2$. Recent results of Browder [3] show that φ_2 is zero except possibly when $n = 2^j-2$.

∂_2 restrict to the boundary with framing

$$(3) \qquad \cdots \longrightarrow \theta_n \xrightarrow{w_3} A_n \xrightarrow{\varphi_3} P_n \xrightarrow{\partial_3} \theta_{n-1} \longrightarrow \cdots$$

w_3 give the complement of a point the natural framing

φ_3 cut out a disc containing all points without framing

∂_3 restrict to boundary (no framing preserved)

$$(4) \qquad \cdots \longrightarrow \pi_n(O) \xrightarrow{w_4} \prod_n \xrightarrow{\varphi_4} A_n \xrightarrow{\partial_4} \pi_{n-1}(O) \longrightarrow \cdots$$

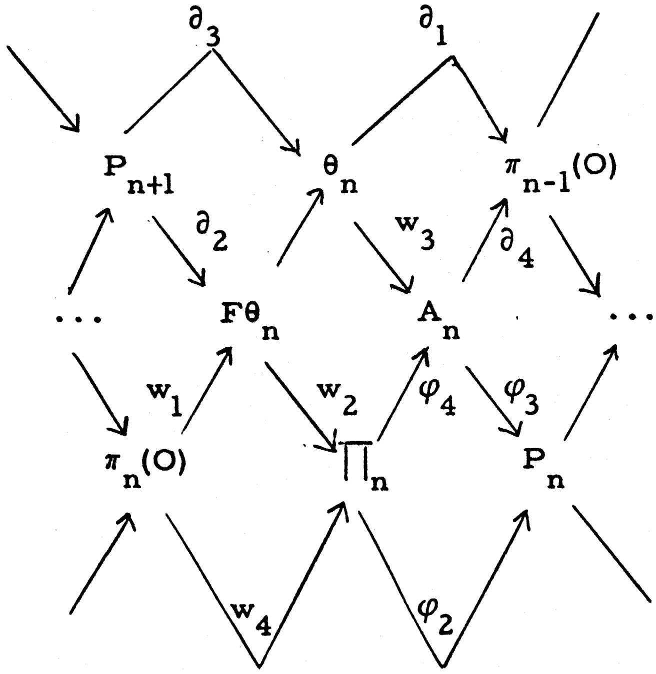
w_4 is the Hopf-Whitehead J homomorphism defined as follows. Let $N > n$. For $\beta \in \pi_n(O)$ represent β by a smooth map $\beta : S^n \longrightarrow SO_{N+n}$. The standard subsphere $S^n \subset S^{n+N}$ has trivial normal bundle, thus we have an imbedding $f : S^n \times D^N \longrightarrow S^{n+N}$. Use β to twist the standard framing (as in section 2) and the Thom-Pontrjagin construction to obtain a map

$$\alpha : S^{n+N} \longrightarrow S^N .$$

φ_4 natural inclusion

∂_4 obstruction to completing the framing. We may think of an element of A_n as framed on the complement of a disc (containing all the unframed points) and then the obstruction is clearly in $\pi_{n-1}(O)$.

The four exact sequences can be collected in the diagram; which is commutative up to sign.



The reader familiar with G and PL will recognize the braid as the homotopy exact sequence of fibrations of $O \subset PL \subset G$.

4. Browder-Novikov theory.

The idea of surgery on a map was invented by Browder [2] and Novikov [17] in order to solve the problems below. The present description is based, in addition to these papers, on a talk by Browder at the Tulane Conference on Transformation Groups and the earlier mentioned lectures by Kervaire.

1. Problem: Suppose we have a topological space X. Is X of the homotopy type of a closed smooth manifold?

Certainly, we need some assumptions.

(i) X has the homotopy type of a finite complex,

(ii) X is a Poincaré complex for some n.

Now recall the situation of a surgery map, $\varphi : M \longrightarrow X$. In addition to (i) and (ii) we also had a bundle ν over X and a stable trivialization F of $\tau_M \oplus \varphi^*\nu$. Since M^n imbeds in some large sphere S^{n+N} with normal bundle ν_M and $\tau_M \oplus \nu_M$ is stably trivial, $\varphi^*\nu$ and ν_M are stably equivalent. Let $E_1(\nu_M)$ denote the unit disc bundle of ν_M and $\dot{E}_1(\nu_M)$ its boundary. The Thom complex of ν_M, $T(\nu_M)$ is defined by (I.2.27)

$$T(\nu_M) = E_1(\nu_M)/\dot{E}_1(\nu_M) \ .$$

Lemma 4.1. The Hurewicz homomorphism

$$h : \pi_{n+N}(T(\nu_M)) \longrightarrow H_{n+N}(T(\nu_M))$$

is onto (spherical).

Proof. By the Thom isomorphism (I.2.28)

$$\Phi : H_i(M^n) \longrightarrow H_{N+i}(T(\nu_M))$$

we need only to show that a generator, $\pm\Phi[M]$, is in the image of h. The map $S^{N+n} \longrightarrow T(\nu_M)$ collapsing $[S^{N+n} \setminus E(\nu_M)] \cup \dot{E}(\nu_M)$ to a point certainly represents a class in $\pi_{n+N}(T(\nu_M))$ whose image is $\pm\Phi[M]$.

In view of (4.1) we must require that

(iii) there is an oriented vector bundle ν over X (of fiber dimension $N >> n$) such that $\pi_{n+N}(T\nu) \longrightarrow H_{n+N}(T\nu)$ is onto. We say that ν has a spherical Thom class.

Theorem 4.2. Suppose X is simply connected and satisfies (i), (ii) and (iii) above. Then there is a smooth manifold M^n and a surgery map $\varphi : M^n \longrightarrow X$. The obstruction to making φ a homotopy equivalence is an element of P_n.

Proof. Imbed X in a suitably large euclidean space R^k. Let U be a smooth neighborhood of X in R^k such that $i : X \longrightarrow U$ is a homotopy equivalence with inverse $r : U \longrightarrow X$. We have the homotopy equivalence of Thom complexes $Tr : T(r^*\nu) \longrightarrow T(\nu)$ with inverse s. Let $\alpha : S^{n+N} \longrightarrow T\nu$ represent a spherical class. Consider

$$S^{n+N} \xrightarrow{\alpha} T\nu \xrightarrow{s} T(r^*\nu) \quad .$$

Approximate this map by a map g transverse regular at U, the zero section of $T(r^*\nu)$. Now $g^{-1}(U) = M$ is a submanifold of S^{n+N}. Since U is closed in $T(r^*\nu)$, $g^{-1}(U)$ is closed in S^{n+N}, hence compact. Since $\partial U = \phi$, $\partial M = \phi$. Finally codim M = codim $U = N$, hence $\dim M = n$. Let $\varphi = r \circ g|M : M \longrightarrow U \longrightarrow X$. Clearly $\varphi^*\nu = \nu_M$ and φ is a (degree 1) surgery map. Application of Chapter II yields the desired result.

Thus the above theorem answers the 1 Problem in the affirmative when n is odd. For even n the situation is the following. If $n = 4s$ we have an additional cobordism invariant of M, its signature $\sigma(M)$. Recall the definition: we have a bilinear form

$$\begin{array}{c} H^{2s}(M; \mathbb{R}) \otimes H^{2s}(M; \mathbb{R}) \longrightarrow \mathbb{R} \\ u \otimes v \longrightarrow (u \cup v)[M] \end{array}$$

defined by cup product evaluated on the orientation class. It is a well-known theorem of Hirzebruch [7] that there is a universal polynomial L_s in the Pontrjagin classes $p_i \in H^{4i}(M; Z)$ with rational coefficients such that

$$L_s(p_1, \ldots, p_s)[M] = \sigma(M) \quad .$$

Recall [7, p. 65-66] that the Pontrjagin classes of M are just the Pontrjagin classes of the tangent bundle τ_M.

Now by assumption $\tau_M \oplus \varphi^* \nu$ is stably trivial, hence $\varphi^* \nu$ is the stable inverse of τ_M. Let $-\nu$ denote the stable inverse of ν. Define the dual Pontrjagin classes $\overline{p}_i(\nu) = p_i(-\nu) \in H^{4i}(X; Z)$. Define the signature of X, $\sigma(X)$ by the bilinear pairing in the middle cohomology as above. Let $[X]$ denote the fundamental class of the Poincaré complex X.

Theorem 4.2 (continued). Suppose X is simply connected and satisfies (i), (ii), and (iii). Then
(a) if n is odd X has the homotopy type of a smooth manifold,
(b) if $n = 4s$ then X has the homotopy type of a smooth manifold if and only if

$$L_s(\overline{p}_1(\nu), \ldots, \overline{p}_s(\nu))[X] = \sigma(X) \quad .$$

The problem is undecided for $n = 4k+2$. For any given bordism class the obstruction lies in Z_2, but it is not clear how it depends on the choice of α. This naturally raises the question:

2. Problem: How unique is the above construction?

Theorem 4.3. Suppose α and α' are homotopic and $\varphi : M \longrightarrow X$ and $\varphi' : M' \longrightarrow X$ are homotopy equivalences. Then for

n even $\quad$ M and M' are diffeomorphic

n odd $\quad$ there exists a homotopy sphere Σ^n bounding a parallelizable manifold such that M and $M' \# \Sigma$ are diffeomorphic.

Proof. Let Λ be the homotopy between α and α', $\Lambda : S^{n+N} \times I \longrightarrow T(\nu)$. Define the map

$$\Lambda' : S^{n+N} \times I \longrightarrow T(\nu) \times I$$

by $\Lambda'(x,t) = (\Lambda_t(x), t)$. Now use the above construction and a relative transverse regularity theorem to obtain a map

$$\psi : N \longrightarrow X \times I$$

where N is a compact $(n+1)$-manifold, $\partial N = M \cup (-M')$ and $\psi | M = \varphi$, $\psi | M' = \varphi'$ are homotopy equivalences. Moreover ψ is a surgery map.

It follows that the obstruction to making ψ a homotopy equivalence keeping the ends fixed lies in P_{n+1}, hence the claim when n is even.

For n odd there may be an obstruction. But this can be eliminated if we allow changes in the boundary (see theorem II.3.1). It is quite clear from section 2 that the obstruction may be killed by adding a π-manifold with boundary a homotopy sphere. The h-cobordism theorem provides the transition from the homotopy statement to diffeomorphism

5. Further topics.

There is a great wealth of applications of the theory. For example:

Homotopy smoothings of manifolds.

The object is to study the collection of smooth manifolds homotopy equivalent to a given manifold. The idea is present in Browder-Novikov theory and roughly one has the following:

Let $hS(M)$ denote the set of manifolds homotopy equivalent to M under the equivalence relation

$$\begin{tikzcd} M_1 \arrow[d, "f"'] \arrow[dr, "g_1"] & \\ M_2 \arrow[r, "g_2"'] & M \end{tikzcd}$$

g_1 and g_2 are the given homotopy equivalences, f is a diffeomorphism making the diagram homotopy commutative.

Let $nm(M)$ denote the set of normal maps of M. An element is a vector bundle ν over M of fiber dimension $N >> n$ and a homotopy class of maps $\alpha : S^{n+N} \longrightarrow T(\nu)$ whose image generates $H_{n+N}T(\nu)$. From (4.2) this corresponds to a surgery map

$$\begin{array}{ccc} \varphi^*\nu \big\downarrow & & \big\downarrow \nu \\ \varphi : M' & \longrightarrow & M \end{array}$$

where M' is smooth, φ of degree 1 and $\tau_{M'} \oplus \varphi^*\nu$ stably trivial.

For M simply connected there is the exact sequence

$$P_{n+1} \xrightarrow{\#} hS(M) \xrightarrow{\eta} nm(M) \xrightarrow{s} P_n \ .$$

The map s sends the surgery setup into the obstruction to making φ a homotopy equivalence. The map $\#$ just adds a homotopy sphere which bounds a π-manifold to a manifold M'. (See 4.3.) The map η associates with the homotopy equivalence $\varphi : M' \longrightarrow M$ a vector bundle $\nu = \bar{\varphi}^*\nu_{M'}$, where $\bar{\varphi}$ is a homotopy inverse to φ and $\nu_{M'}$ is the stable normal bundle of M'. We shall not prove that the maps are well defined and exact. The reader should consult Sullivan [19] for details.

<u>Transformation groups.</u>

The study of free involutions on homotopy spheres using surgery was initiated by Browder and Livesay [4]. Several other papers have appeared along those lines.

The free actions of S^1 and S^3 on homotopy spheres can also be investigated by surgery methods. One has to classify homotopy smoothings of complex and quaternionic projective spaces.

<u>Hauptvermutung.</u>

An outstanding application of the PL-theory is the result that if two simply connected PL-manifolds of dimension ≥ 5 with no 2-torsion in 3-dimensional homology are topologically homeomorphic, then they are PL-isomorphic (Sullivan [19]).

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