*Topology* Vol. 26, No. 2, pp. 187–194, 1987
Printed in Great Britain.

0040-9383/87 \$3.00 + .00

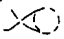

# JONES POLYNOMIALS AND CLASSICAL CONJECTURES IN KNOT THEORY

KUNIO MURASUGI

(*Received* 28 *January* 1986)

## §1. INTRODUCTION AND MAIN THEOREMS

LET $L$ be a tame link in $S^3$ and $V_L(t)$ the Jones polynomial of $L$ defined in [2]. For a projection $\tilde{L}$ of $L$, $c(\tilde{L})$ denotes the number of double points in $\tilde{L}$ and $c(L)$ the minimum number of double points among all projections of $L$.

A link projection $\tilde{L}$ is called *proper* if $\tilde{L}$ does not contain "removable" double points like  or  .

In this paper, we will prove some of the outstanding classical conjectures due to P.G. Tait [7].

THEOREM A. *(P. G. Tait Conjecture) Two (connected and proper) alternating projections of an alternating link have the same number of double points.*

THEOREM B. *The minimal projection of an alternating link is alternating. In other words, an alternating link always has an alternating projection that has the minimum number of double points among all projections. Moreover, a non-alternating projection of a prime alternating link cannot be minimal.*

The primeness is necessary in the last statement of Theorem B, since the connected sum of two figure eight knots is alternating, but it has a minimal non-alternating projection. Note that the figure eight knot is amphicheiral.

Theorems A and B follow easily from Theorems 1–4 (stated below) which show strong connections between $c(\tilde{L})$ and the Jones polynomial $V_L(t)$.

Let $\mathrm{d}_{\max} V_L(t)$ and $\mathrm{d}_{\min} V_L(t)$ denote the maximal and minimal degrees of $V_L(t)$, respectively, and $\operatorname{span} V_L(t) = \mathrm{d}_{\max} V_L(t) - \mathrm{d}_{\min} V_L(t)$.

THEOREM 1. *For any projection* $\tilde{\mathrm{L}}$ *of a link* L,

$$\operatorname{span} V_L(t) \le c(\tilde{L}) + \lambda - 1, \tag{1}$$

*where* $\lambda$ *is the number of connected components of* $\tilde{\mathrm{L}}$, *and therefore, if* L *has* $\lambda$ *split components, then*

$$\operatorname{span} V_L(t) \le c(L) + \lambda - 1. \tag{2}$$

If $L$ is an alternating link, then we are able to prove the following:

* The work was done while the author was visiting at the University of Geneva, Switzerland. This research was partially supported by NSERC-No. A4034.

THEOREM 2. *If* $\tilde{L}$ *is a connected proper alternating projection of an alternating link* L, *then*

$$\operatorname{span} V_L(t) = c(\tilde{L}). \tag{3}$$

(1) and (3) now yield that for any alternating link with $\lambda$ split components,

$$\operatorname{span} V_L(t) = c(L) + \lambda - 1. \tag{4}$$

If $L$ is prime, we can prove the converse of Theorem 2. In fact, we have

THEOREM 3. *Let* L *be a prime link. Then for any non-alternating projection* $\tilde{L}$ *of* L,

$$\operatorname{span} V_L(t) < c(\tilde{L}). \tag{5}$$

We should note that the primeness is necessary in Theorem 3, since the equality in (5) holds for a non-alternating projection of the square knot.

Using Theorems 2 and 3 we are able to give the complete characterization of links for which (4) holds.

THEOREM 4. *Let* L *be a non-split link. Then* (4) *holds for* L *if and only if* L *is the connected sum of alternating links.*

Besides Theorems A and B, these Theorems 1–4 yield several other consequences.

COROLLARY 5. *If a knot* K *is alternating and amphicheiral, then any proper alternating projection has an even number of double points.*

*Proof.* Let $K^*$ be the mirror image of $K$. Then $V_{K^*}(t) = V_K(t^{-1})$, [2]. Since $K = K^*$, $V_K(t)$ is symmetric and hence, span $V_K(t)$ is even and Corollary 5 follows from Theorem 2.

COROLLARY 6. *If* $L_1$ *and* $L_2$ *are alternating links, then*

$$c(L_1 \# L_2) = c(L_1) + c(L_2),$$

*where # means the connected sum.*

This solves Problem 1 in [1] for alternating links.

Corollary 6 follows from (4) and Theorem 4 since

$$\operatorname{span} V_{L_1 \# L_2}(t) = \operatorname{span} V_{L_1}(t) + \operatorname{span} V_{L_2}(t).$$

To state the final corollary, we define the twist number $w$ at each double point $v$ in a projection $\tilde{L}$ as indicated in Fig. 1.

Define $w(\tilde{L}) = \sum_{v \in \tilde{L}} w(v)$, where the summation is taken over all double points in $\tilde{L}$.

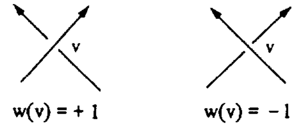



Fig. 1.

COROLLARY 7. *Let* $\tilde{L}_1$ *and* $\tilde{L}_2$ *be proper alternating projections of a special alternating link* L. *Then* $w(\tilde{L}_1) = w(\tilde{L}_2)$.

*Proof.* Since $L$ is special alternating, any proper alternating projection must be proper special alternating [5]. Therefore, $w(\tilde{L}_i) = c(\tilde{L}_i)$ or $-c(\tilde{L}_i)$, $i = 1, 2$, and hence, $w(\tilde{L}_1) = w(\tilde{L}_2)$ or $-w(\tilde{L}_2)$. However, the sign of the signature of $L$ is determined by the sign of $w(\tilde{L})$ of any special alternating projection $\tilde{L}$ of $L$, [5]. Therefore, $w(\tilde{L}_1) = w(\tilde{L}_2)$, q.e.d.

Corollary 7 shows that $w(\tilde{K})$ is a knot type invariant as long as we consider a special alternating knot $K$ and its proper alternating projections. It is still not known whether or not $w(\tilde{K})$ is a knot type invariant when $K$ is an alternating knot and $\tilde{K}$ a proper alternating projection.

Proofs of Theorems 1–4 will be given in the next two sections.

This work is inspired by the work of L. Kauffman [4] and conversations with C. Weber of the University of Geneva, to whom I would like to express my gratitude.

After submitting the paper, I learned that M. B. Thistlethwaite also obtained the same results using a completely different method [8].

Finally, I would like to thank the referee for his invaluable suggestions.

## §2. PROOF OF THEOREM 1

We use the bracket polynomial $P_{\tilde{L}}(A)$ defined by L. Kauffman [4] rather than the original Jones polynomial.

For a projection $\tilde{L}$ of a link $L$, $P_{\tilde{L}}(A)$ is defined recursively by using the following fundamental identities (6)–(8):

(6) If $\tilde{L} = 0$, then $P_{\tilde{L}}(A) = 1$.

(7) If $\tilde{L}$ is the disjoint union of two link projections $\tilde{L}_1$ and $\tilde{L}_2$ then

$$P_{\tilde{L}}(A) = -(A^2 + A^{-2})P_{\tilde{L}_1}(A)P_{\tilde{L}_2}(A).$$

(8) If $\tilde{L}$, $\tilde{L}_0$ and $\tilde{L}_\infty$ are completely identical projections except at a small neighborhood of one crossing in $\tilde{L}$, where they are related by the diagrams below, then

$$P_{\tilde{L}}(A) = AP_{\tilde{L}_0}(A) + A^{-1}P_{\tilde{L}_\infty}(A).$$

It is shown that $P_{\tilde{L}}(A)$ is uniquely determined by these identities. However, $P_{\tilde{L}}(A)$ may be different from $P_{\tilde{L}'}(A)$ for another projection $\tilde{L}'$ of the same link. Nevertheless, $P_{\tilde{L}}(A)$ and the Jones polynomial $V_L(t)$ are related by the following formula [4]:

$$V_L(t) = (-t^{3/4})^{w(\tilde{L})}P_{\tilde{L}}(t^{-1/4}), \tag{9}$$

where $w(\tilde{L})$ is the integer defined in Section 1.

Since span $V_L(t) =$ span $P_{\tilde{L}}(t^{-1/4}) =$ span $P_{\tilde{L}}(t^{1/4}) = 1/4$ span $P_{\tilde{L}}(t) = 1/4$ span $P_{\tilde{L}}(A)$, to prove Theorem 1 it is sufficient to show the following:

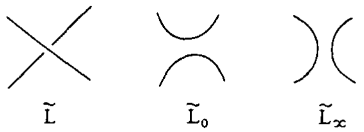



Fig. 2.

For any connected projection $\tilde{L}$ of a link $L$,

$$\operatorname{span} P_{\tilde{L}}(A) \leq 4c(\tilde{L}). \tag{10}$$

*Proof of* (10). Let $\tilde{L}$ be a proper connected projection of $L$ in $S^2$. $\tilde{L}$ divides $S^2$ into finitely many domains, which we will classify as shaded or unshaded. Let $\Gamma$ be the graph of a link projection $\tilde{L}$ such that each vertex of $\Gamma$ corresponds to an unshaded domain and each edge of $\Gamma$ corresponds to a double point of $\tilde{L}$. We call an edge $e$ of $\Gamma$ positive or negative according to the diagrams shown in Fig. 3.

We should note that $\tilde{L}$ is alternating if and only if either all the edges of $\Gamma$ are positive or all the edges are negative. A graph $\Gamma$ is called *oriented* if every edge is either positive or negative.

Let $p$ and $n$ be the number of positive and negative edges in $\Gamma$, respectively, and hence $p+n = c(\tilde{L})$.

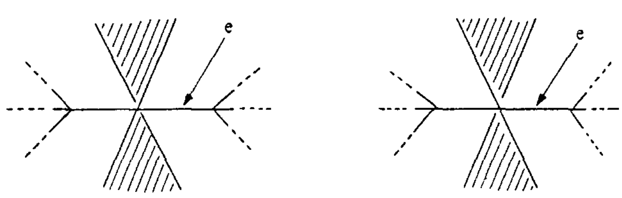
Now to evaluate $P_{\tilde{L}}(A)$, we have to *smooth* a double point on an edge $e$. For convenience, we call these smoothings *parallel* or *transverse* according to the diagrams shown in Fig. 4.

When we apply either a parallel or a transverse smoothing on every edge in $\Gamma$, we obtain a trivial link of several components, and $P_{\tilde{L}}(A)$ is the sum of the bracket polynomials of these links multiplied by $A^k$ with some integer $k$. In order to obtain a more precise formula of $P_{\tilde{L}}(A)$, we will use the folllowing notation in the rest of the paper.

For an oriented graph $\Gamma$, we denote by $\Gamma^*$ the dual graph of $\Gamma$ which is oriented in such a way that an edge $e^*$ in $\Gamma^*$ is positive (or negative) if and only if $e^*$ intersects a positive (or negative) edge in $\Gamma$.

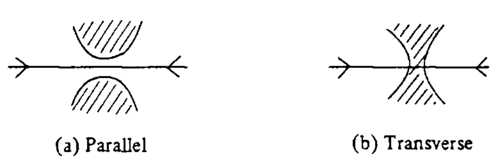
$\Gamma_+$ and $\Gamma_-$ denote, respectively, the subgraphs of $\Gamma$ that consist of all positive edges and their end vertices, and of all negative edges and their end vertices.

For integers $a$ and $b, 0 \leq a \leq p$ and $0 \leq b \leq n$, $\tilde{\mathscr{L}}_\Gamma(a, b)$ denotes the link projection obtained from the link projection with $\Gamma$ as its oriented graph, by applying parallel smoothings on exactly $a$ positive edges and $b$ negative edges, and by applying transverse smoothings on $p-a$ positive edges and $n-b$ negative edges in $\Gamma$. ($\tilde{\mathscr{L}}_\Gamma(a, b)$ is called a *state* in [4].) $\tilde{\mathscr{L}}_\Gamma(a, b)$ is a trivial link. For each pair $(a, b)$, there exist $\binom{p}{a}\binom{n}{b}$ such trivial links. Let $S(a, b)$ denote the collection of these links.




(a) Positive (b) Negative

Fig. 3.



(a) Parallel (b) Transverse

Fig. 4.

Let $\mu_\Gamma(a, b)$ (or $\mu(a, b)$ when no confusion may occur) be the maximal number of components a link in $S(a, b)$ can have. For each of the following pairs $(a, b) = (0, 0)$, $(p, 0)$, $(0, n)$ and $(p, n)$, $S(a, b)$ consists of only one link and we know that $\mu(0, 0)$ and $\mu(p, n)$ are, respectively, the number of vertices of $\Gamma$ and that of $\Gamma^*$, and hence $\mu(0, 0) + \mu(p, n) - 2$ is the number of edges in $\Gamma$.

First we prove the following:

LEMMA 1. *For any integers $a$ and $b$, $0 \le a \le p$ and $0 \le b \le n$, $\mu(a, b) + a + b$ is an increasing function of $a$ and $b$, and $\mu(a, b) - a - b$ is a decreasing function of $a$ and $b$.*

*Proof.* If we change a transverse smoothing to a parallel smoothing, or vice versa, the number of components of the resulting trivial link increases or decreases by one. Therefore,

$$\begin{aligned} &|\mu(a+1, b) - \mu(a, b)| \le 1, \quad \text{and} \\ &|\mu(a, b+1) - \mu(a, b)| \le 1, \end{aligned} \tag{11}$$

and hence, we have

$$\begin{aligned} &(1)\ \mu(a+1, b) + 1 \ge \mu(a, b) \ge \mu(a+1, b) - 1, \\ &(2)\ \mu(a, b+1) + 1 \ge \mu(a, b) \ge \mu(a, b+1) - 1. \end{aligned} \tag{12}$$

An easy induction now gives us a proof of Lemma 1.

Using Lemma 1, we can see that

$$\begin{aligned} &(1)\ \mu(0, n) + n - b \ge \mu(0, b) \ge \mu(a, b) - a, \\ &(2)\ \mu(p, 0) + p - a \ge \mu(a, 0) \ge \mu(a, b) - b. \end{aligned} \tag{13}$$

Now let $b_i(\Gamma)$ denote the $i$th Betti number of a graph $\Gamma$ as a 1-complex.

LEMMA 2. *If $\Gamma$ is the graph of $\tilde{L}$, then*

$$\begin{aligned} &(1)\ \mu(p, 0) = b_1(\Gamma_+) + b_1(\Gamma^*_-) + 1, \\ &(2)\ \mu(0, n) = b_1(\Gamma_-) + b_1(\Gamma^*_+) + 1, \end{aligned} \tag{14}$$

*and hence*

$$(3)\ \mu(p, 0) + \mu(0, n) \le c(\tilde{L}) + 2.$$

*Proof.* Since $\mu(p, 0)$ and $\mu(0, n)$ are dual, it may suffice to prove (14) (1).

To compute $\mu(p, 0)$, we first remove all negative edges (but no vertices) from $\Gamma$, and denote by $\Gamma_0$ the resulting (possibly disconnected) planar subgraph of $\Gamma$. Then the boundary of a regular neighborhood of $\Gamma_0$ in $S^2$ is exactly the trivial link $\tilde{\mathscr{L}}_\Gamma(p, 0)$. Therefore, $\mu(p, 0)$, the number of components of $\tilde{\mathscr{L}}_\Gamma(p, 0)$ is given by $b_1(\Gamma_0) + b_0(\Gamma_0)$. Since $b_i(\Gamma_0) = \text{rank } H_i(\Gamma_0)$, $i = 0,1$, we have $b_1(\Gamma_0) = b_1(\Gamma_+)$, and $b_0(\Gamma_0) = b_1(S^2 - \Gamma_0) + 1$ yields $b_0(\Gamma_0) = b_1(\Gamma^*_-) + 1$. This proves (14) (1).

To show (14) (3), note that $\mu(p, n)$ is the number of boundary components of a regular neighborhood of $\Gamma (= \Gamma_+ \cup \Gamma_-)$ in $S^2$. Since $b_1(\Gamma_+ \cap \Gamma_-) = 0$, we have $b_1(\Gamma_+) + b_1(\Gamma_-) \le b_1(\Gamma)$. Furthermore, since $\Gamma$ is connected and $\mu(p, n) = b_1(\Gamma) + b_0(\Gamma)$, it follows that $b_1(\Gamma_+) + b_1(\Gamma_-) + 1 \le \mu(p, n)$. Similarly, $b_1(\Gamma^*_+) + b_1(\Gamma^*_-) + 1 \le \mu(0, 0)$, and therefore, $\mu(p, 0) + \mu(0, n) \le \mu(p, n) + \mu(0, 0) = c(\tilde{L}) + 2$. This proves Lemma 2.

We now return to the proof of (10). Repeated applications of (6)–(8) give us the following:

$$P_{\tilde{L}}(A) = \sum_{\tilde{\mathscr{L}}} A^{-a+(p-a)} A^{b-(n-b)} \{-(A^2+A^{-2})\}^{|\tilde{\mathscr{L}}|-1}, \tag{15}$$

where the summation runs over all trivial link diagrams $\tilde{\mathscr{L}}$ in $S(a, b)$, and $0 \leq a \leq p$ and $0 \leq b \leq n$, and $|\tilde{\mathscr{L}}|$ denotes the number of components of $\tilde{\mathscr{L}}$. Since $|\tilde{\mathscr{L}}| \leq \mu(a, b)$, by definition, in (15), we see that

$$d_{\max} P_{\tilde{L}}(A) \leq \max_{a,b} \{p - n - 2a + 2b + 2\mu(a, b) - 2\} \quad \text{and}$$

$$d_{\min} P_{\tilde{L}}(A) \geq \min_{a,b} \{p - n - 2a + 2b - 2\mu(a, b) + 2\}. \tag{16}$$

However, (13) shows that

$$\begin{aligned} 2\mu(a, b) - 2a + 2b - 2n &\leq 2\mu(0, n), \quad \text{and} \\ -2\mu(a, b) - 2a + 2b &\geq -2\mu(p, 0) - 2p, \end{aligned}$$

and hence

$$\begin{aligned} d_{\max} P_{\tilde{L}}(A) &\leq p + n + 2\mu(0, n) - 2, \quad \text{and} \\ d_{\min} P_{\tilde{L}}(A) &\geq -p - n - 2\mu(p, 0) + 2. \end{aligned} \tag{17}$$

Using (14) (3), we obtain finally

$$\begin{aligned} \operatorname{span} P_{\tilde{L}}(A) &\leq 2(p+n) + 2\mu(0, n) + 2\mu(p, 0) - 4 \\ &\leq 2c(\tilde{L}) + 2\{c(\tilde{L}) + 2\} - 4 \\ &= 4c(\tilde{L}). \end{aligned} \tag{18}$$

This proves (10) and the proof of Theorem 1 is now complete.

## §3. PROOFS OF THEOREMS 2–4

We will use the same notation used in Section 2.

*Proof of Theorem* 2. Let $L$ be a non-split alternating link and $\tilde{L}$ a proper connected alternating projection. We may assume without loss of generality that all edges of the graph $\Gamma$ of $\tilde{L}$ are positive. Therefore, $\Gamma_+ = \Gamma$ and $\Gamma_- = \phi$, and hence, $p = c(\tilde{L})$ and $n = 0$ (and $b = 0$). Now to prove Theorem 2 we must show that

$$\begin{aligned} &(1)\ d_{\max} P_{\tilde{L}}(A) = p + 2\mu(0, 0) - 2, \quad \text{and} \\ &(2)\ d_{\min} P_{\tilde{L}}(A) = -p - 2\mu(p, 0) + 2. \end{aligned} \tag{19}$$

However, to prove (19), it suffices to show that each of $A^{p+2\mu(0,0)-2}$ and $A^{-p-2\mu(p,0)+2}$ appears exactly once in the summation (15). [See (17).] In other words, it is enough to show that

$$\begin{aligned} &(1)\ \mu(0, 0) > \mu(a, 0) - a, \qquad \text{for} \quad 0 < a \leq p, \quad \text{and} \\ &(2)\ \mu(p, 0) + p > \mu(a, 0) + a, \quad \text{for} \quad 0 \leq a < p. \end{aligned} \tag{20}$$

Now we know that $\mu(0, 0) = v(\Gamma)$, the number of vertices in $\Gamma$, and $\mu(p, 0) = v(\Gamma^*)$ the number of vertices in $\Gamma^*$, and further since $\tilde{L}$ is proper, we see easily from the definition that $\mu(1, 0) = v(\Gamma) - 1$ and hence trivially $\mu(0, 0) > \mu(1, 0) - 1$. Since $\mu(a, 0) - a$ is a decreasing function of $a$ (Lemma 1), it follows that $\mu(0,0) > \mu(1,0) - 1 \geq \mu(a,0) - a$, for $0 < a \leq p$. This proves (20) (1).

Similarly, since $\tilde{L}$ is proper, $\mu(p,0) = \mu(p-1,0) + 1$ and trivially $\mu(p,0) + 1 > \mu(p-1,0)$. Since $\mu(a,0) + a$ is an increasing function of $a$, it follows that $\mu(p,0) + p > \mu(p-1,0) + p - 1 \geqq \mu(a,0) + a$, for $0 \leq a < p$. This proves (20) (2), and the proof of Theorem 2 is complete.

*Remark.* We actually proved that for an alternating link $L$, the coefficients of the terms of $V_L(t)$ of maximal and minimal degrees are $+1$ or $-1$. (See [8].)

Another proof of Theorem 2 is also given in [5].

*Proof of Theorem* 3. We may assume that $\tilde{L}$ is connected and proper. Let $\Gamma$ be the graph of $\tilde{L}$. If $\Gamma$ has a cut vertex, then $\Gamma$ is the one-point union of two subgraphs $\Gamma_1$ and $\Gamma_2$. Let $\tilde{L}_1$ and $\tilde{L}_2$ be the link projections whose graphs are $\Gamma_1$ and $\Gamma_2$, respectively. Then $L$ is the connected sum of two links $L_1$ and $L_2$ whose projections are $\tilde{L}_1$ and $\tilde{L}_2$, respectively. Since $L$ is prime, one of $L_i$, say $L_1$, is unknotted, and hence, $V_L(t) = V_{L_2}(t)$. Therefore, Theorem 1 yields that span $V_L(t) =$ span $V_{L_2}(t) \leq c(\tilde{L}_2) < c(\tilde{L}_1) + c(\tilde{L}_2) = c(\tilde{L})$, since $c(\tilde{L}_1) \neq 0$. Therefore, we may assume that $\Gamma$ has no cut vertices.

Now suppose span $V_L(t) = c(\tilde{L})$. Then span $P_{\tilde{L}}(A) = 4c(\tilde{L})$, and hence, as we have seen in the proof of Theorem 1, we must have the equality in (14) (3). Therefore, $b_1(\Gamma_+) + b_1(\Gamma_-) + 1 = \mu(p,n)$ and $b_1(\Gamma_+^*) + b_1(\Gamma_-^*) + 1 = \mu(0,0)$. However, the first equality (and hence the second equality as well) holds if and only if $b_1(\Gamma_+) + b_1(\Gamma_-) = b_1(\Gamma)$. This is possible only if $\Gamma$ is a positive or negative graph. Therefore, $\tilde{L}$ must be an alternating projection. This proves Theorem 3.

*Proof of Theorem* 4. If a non-split link $L$ is the connected sum of alternating links $L_i$, $i = 1, 2, \ldots, k$, then $L$ has a connected proper projection $\tilde{L}$ and each $L_i$ has a connected proper alternating projection $\tilde{L}_i$ such that

$$c(\tilde{L}) = \sum_{i=1}^{k} c(\tilde{L}_i).$$

Since

$$\text{span } P_{\tilde{L}}(A) = \sum_{i=1}^{k} \text{span } P_{\tilde{L}_i}(A),$$

it follows from (3) that

$$\text{span } V_L(t) = \sum_{i=1}^{k} \text{span } V_{L_i}(t) = \sum_{i=1}^{k} c(\tilde{L}_i) = c(\tilde{L}).$$

Conversely, if span $P_{\tilde{L}}(A) = 4c(\tilde{L})$ for some connected proper projection $\tilde{L}$ of a link $L$, then, as we have seen in the proof of Theorem 3, the equality in (14) (3) must hold; therefore, $\Gamma$ is either a positive or negative graph, or $\Gamma$ has cut vertices $v_1, \ldots, v_r$ which separate $\Gamma$ into positive and/or negative graphs. Therefore, $L$ is the connected sum of (positive or negative) alternating links. This proves Theorem 4.

## REFERENCES

1. J. C. Hausmann (ed.): Knot theory, *Proceedings Plans-sur-Bex*, Switzerland, *Springer Lecture Notes in Mathematics* **685** (1977).
2. V. F. R. Jones: A polynomial invariant for knots via von Neumann algebras, *Bull. AMS.* **89** (1985), 103–111.

3. L. H. KAUFFMAN: Formal knot theory, Lecture Notes **30**, Princeton University Press, Princeton, 1983.
4. L. H. KAUFFMAN: State models for knot polynomials, to appear.
5. K. MURASUGI: On a certain numerical invariant of link types, *Trans. AMS.* **117** (1965), 387–422.
6. K. MURASUGI: Jones polynomials of alternating links, *Trans. AMS.* **295** (1986), 147–174.
7. P. G. TAIT: On knots, *Scientific paper I*, 273–347, Cambridge University Press, London, 1898.
8. M. B. THISTLETHWAITE: A spanning tree expansion of the Jones polynomial, to appear.

*University of Toronto,*
*Toronto, Canada,*
*M5S1A1.*