Lectures on Characteristic Classes

by John Milnor

Notes by James Stasheff
(Spring 1957)

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I. n-plane bundles:

In the study of characteristic classes, we will be concerned with n-dimensional vector space bundles or, briefly, n-plane bundles.

Definition: An n-plane bundle consists of a triple $\{E, B, \pi\}$ with π a map (i.e. continuous function) from a Hausdorff space E onto a Hausdorff space B, and the structure of an n-dimensional vector space over the reals R in the fibres $\pi^{-1}(b)$ for all $b \in B$, satisfying the further requirements that

1) there exist a distinguished class of open sets $\{U\}$ covering B and n maps $c_i : U \longrightarrow E$ for each U, such that

2) each c_i is a cross-section, that is $\pi c_i(b) = b$ for each $b \in U$, and

3) the map $U \times R^n \longrightarrow \pi^{-1}(U)$ defined by $(b, \lambda_1, \ldots, \lambda_n) \longrightarrow \Sigma \lambda_i c_i(b), \lambda_i \in R$, is a homeomorphism. (This is the local product structure on E.)

We call B the base space, E the total space, π the projection and denote the triple and structure by a Greek letter e.g. $\zeta = \{E, B, \pi\}$. A superscript on a bundle indicates the dimension of the fibre $\pi^{-1}(b)$, e.g. ζ^n.

These notes sponsored by Princeton University under Air Force Contract No. AF 18(600) - 1494

Remark: Although not necessary for what follows, it should be noted that an n-plane bundle is an example of a fibre bundle. (See Steenrod; Topology of Fibre Bundles, 1951.) In fact, an n-plane bundle is exactly a fibre bundle with real n-dimensional vector space as fibre and $GL(n,R)$ as structural group.

Examples of n-plane bundles:

1) The product bundle $B \times R^n$

2) The tangent bundle τ^n of a differentiable manifold M^n of class C^1 or more. Here $B = M^n$ and E is the set of all pairs (b, contravariant vector at b).

3) The normal k-plane bundle ν^k of a differentiable manifold $M^n \subset R^{n+k}$ (For a differentiable manifold "$\subset$" is always to be read "differentiably imbedded in".) Here the base space B is again M^n and E is the set of all pairs (b, normal vector at b).

4) The 1-plane bundle or line bundle ξ_n^1 over real projective n-space P^n defined as follows. Consider P^n as the set of all unordered pairs $[x,-x]$ where x ranges over all unit vectors in R^{n+1}. The total space E is to be the set of all pairs $([x,-x],\lambda x)$ with λ a real number.

Remark 1. Every cross-section of this bundle (i.e. a map $\phi: B \longrightarrow E$ such that $\pi\phi$ = identity on B) is somewhere zero, $\phi(b) = (b,0,0,\dots,0)$ for some b. We call a "non-zero cross-section" one which is never zero. Proof that the bundle ξ_n^1 of (4) has no non-zero cross-sections, $n \geq 1$: Given any cross-section $\phi: P^n \longrightarrow E$ we can define a map $\lambda: S^n \longrightarrow R$ by $\phi([x,-x]) = ([x,-x], \lambda(x)x)$. Since $\lambda(-x) = -\lambda(x)$ and S^n is connected, there is a point x for which $\lambda(x) = 0$ or $\phi([x,-x]) = ([x,-x], 0)$.

Remark 2. The following alternative description of ξ_n^1 will be useful later. As total space E_1 take $S^n \times R$ with the identification $(x,\lambda) = (-x,-\lambda)$. Evidently the element $[(x,\lambda),(-x,-\lambda)]$ of E_1 can be identified with $([x,-x], \lambda x) \in E$. Therefore this new bundle is equivalent (see the next paragraph) to the one defined above.

Bundle maps and induced bundles:

Definition: A bundle map $f: \zeta \longrightarrow \eta$, where $\zeta = (E,B,\pi)$ and $\eta = (E',B',\pi')$ are n-plane bundles, is a pair of maps (f_B, f_E) such that

1) the following diagram is commutative

$$\begin{tikzcd}
E \arrow[r, "f_E"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\
B \arrow[r, "f_B"] & B'
\end{tikzcd}$$

(i.e. $\pi' f_E = f_B \pi$) and

2) $f_E | \pi^{-1}(b)$ is linear and non-singular for each b in B.

Special Case: B = B'

Definition: Two n-plane bundles ζ, η over B are equivalent if there is a bundle map $f: \zeta \longrightarrow \eta$ with f_B = Identity on B. This is an equivalence relation and using it we define

Definition: An n-plane bundle is trivial if it is equivalent to the product bundle $B \times R^n$.

Remark: A bundle is trivial if and only if there exist n independent cross-sections. (We use them to define f_E.)
Using this concept, we have

Definition: A differentiable manifold M^n is parallelizable if the tangent bundle $\tau^n(M^n)$ is trivial.

Induced bundle: Given a bundle ζ with $\pi: E \longrightarrow B$, another space B' and a map $f_{B'}: B' \longrightarrow B$, there is a construction by which we get another bundle $\{E', B', \pi'\}$ and a bundle map $f = (f_{E'}, f_{B'})$. Let E' be the subset of $B' \times E$ consisting of all pairs (b', e) with $b' \in B'$, $e \in E$ such that $f_{B'}(b') = \pi(e)$. Define $\pi': E' \longrightarrow B'$ by $\pi'(b', e) = b'$. Each fibre $\pi'^{-1}(b')$ will have the structure of a vector space isomorphic to $\pi^{-1}(f_{B'}(b'))$. Thus we have constructed the induced bundle, the bundle induced by $f_{B'}$. It is easy to verify that the projection map $f_{E'}(b', e) = e$ gives a bundle map $(f_{B'}, f_{E'})$ of the induced bundle into the original bundle.

Remark: Given two bundles ζ and η and a map f_B of their base spaces as indicated in the diagram, it is often possible to define a map f_E so that the pair is a bundle map. This is possible, if and only if ζ is equivalent to the bundle induced by f_B from η. For example, $P^k \subset P^{k+1}$, $E(\xi^1_k) \subset E(\xi^1_{k+1})$ and there is the obvious bundle map $f = (i_E, i_B)$ where i_E, i_B are the indicated inclusion maps. Thus ξ^1_k is equivalent to the 1-plane bundle over P^k induced from ξ^1_{k+1} by i_B: $P^k \subset P^{k+1}$.

$$\begin{tikzcd}
E \arrow[r, dashed, "f_E"] \arrow[d, "\pi", "\zeta"'] & E' \arrow[d, "\pi'"', "\eta"] \\
B \arrow[r, "f_B"'] & B'
\end{tikzcd}$$

We need one more relation between n-plane bundles:

The Whitney Bundle Sum: Given an m-plane bundle $\zeta = (E,B,\pi)$ and an n-plane bundle $\eta = (E',B,\pi')$, let E'' be the subset of $E \times E'$ consisting of all pairs (e,e') such that $\pi(e) = \pi'(e')$.

Define $\rho: E'' \longrightarrow E$ by $\rho(e,e') = e$

$\rho': E'' \longrightarrow E'$ by $\rho'(e,e') = e'$

$\pi'': E'' \longrightarrow B$ by $\pi'' = \pi\rho = \pi'\rho'$

$$\begin{tikzcd}
 & E'' \arrow[dl, "\rho"'] \arrow[d, "\pi''"] \arrow[dr, "\rho'"] & \\
E \arrow[r, "\pi"'] & B & E' \arrow[l, "\pi'"] \\
\zeta & & \eta
\end{tikzcd}$$

Since π, π' are projections of m, n-plane bundles respectively, π'' is the projection of an (m+n)-plane bundle, the Whitney sum, $\zeta \oplus \eta = (E'',B,\pi'')$

Example 5 : For a differentiable manifold $M^n \subset R^{n+k}$ we have that $\tau^n \oplus \nu^k$ is trivial (equivalent to the product bundle $M^n \times R^{n+k}$).

II. Stiefel-Whitney classes:

We begin to look at the cohomology of n-plane bundles. Henceforth unless otherwise stated, we will use some cohomology theory with Z_2 as coefficients. $H^i(X)$ will mean $H^i(X; Z_2)$ and $H^*(X)$, the direct sum $H^0(X) \oplus H^1(X) \oplus \cdots$. We have the following, similar to the axioms for Chern classes given in Hirzebruch, Neue topologische Methoden in der Algebraischen Geometrie, Berlin, 1956 p. 60:

Axioms for Stiefel-Whitney Classes

1) To each n-plane bundle ζ over a paracompact base space B, there corresponds an element $W(\zeta) = 1 + W_1(\zeta) + \cdots + W_n(\zeta)$ of $H^*(B)$ where $W_i \in H^i(B)$, such that

2) For a bundle map $f = (f_E, f_B); \zeta \longrightarrow \eta$ we have $f_B^*(W(\eta)) = W(\zeta)$

3) The Whitney Product Theorem holds: $W(\zeta \oplus \eta) = W(\zeta)W(\eta)$

i.e. $W_k(\zeta \oplus \eta) = \sum_{i+j=k} W_i(\zeta) \smile W_j(\eta)$

[originally proved by Whitney "On the Theory of Sphere-Bundles" Proceedings Nat. Ac. Sci. 26 p. 148 (1940)].

4) For the non-trivial line bundle over S^1 (which can be represented as the open Moebius band or, since $S^1 = P^1$, as ξ_1^1 of Ex 4)

$$W_1(\xi_1^1) \neq 0.$$

We will call $W_i(\zeta)$ the Stiefel-Whitney classes and $W(\zeta)$ the total Stiefel-Whitney class.

Consequences and examples.

A. Axioms 2) and 4) imply

4') For the bundle ξ_n^1 of example 4, $W(\xi_n^1) = 1 + \alpha$ where α is the non-zero element of $H^1(P^n)$.

For we have $S^1 = P^1 \subset P^2 \subset \cdot\cdot \subset P^n$ and using the inclusion maps in the bundle spaces as well; we define bundle maps

$$\begin{tikzcd}
\xi^1 \arrow[r] \arrow[rrr, bend right, "f"'] & \xi_2^1 \arrow[r] & -- \arrow[r] & \xi_n^1
\end{tikzcd}$$

Call the composition $f = (f_E, f_B) : \xi_1^1 \longrightarrow \xi_n^1$.

Then for $f_B^* : H^*(P^n) \longrightarrow H^*(S^1)$ we have

$$f_B^* \, (W_1(\xi_n^1)) = W_1(\xi_1^1) \neq 0,$$

hence $W_1(\xi_n^1)$ is the non-zero element of $H^1(P^n)$. The Stiefel-Whitney classes $W_i(\xi_n^1), i > 1$, are zero by Axiom 1.

Axiom 4') may be used instead of 4) in which case it would not be necessary to specify in Axiom 1) that $W(\zeta^n)$ has at most n-dimensional classes.

B. Axiom 2) gives us that W is a function of equivalence classes. In particular, if ζ is trivial then $W(\zeta) = 1$ since $f_E: B \times R^n \longrightarrow R^n$; $f_B: B \longrightarrow$ (point), gives a bundle map and $H^i(\text{point}) = 0$ for $i > 0$.

C. $W(\tau^n(S^n)) = 1$ where $\tau^n(S^n)$ is the tangent bundle of the n-sphere.

Proof: Let $f_B: S^n \longrightarrow P^n$ be the natural map and define the bundle map $f: \tau^n(S^n) \longrightarrow \tau^n(P^n)$ in the obvious manner. Then

$$W_i(\tau^n(S^n)) = 0; \; 0 < i < n \quad \text{since } H^i(S^n) = 0; \; 0 < i < n,$$

and $f_B^*: H^n(P^n) \longrightarrow H^n(S^n)$ is zero so that $0 = f_B^*(W_n(\tau^n(P^n))) = W_n(\tau^n(S^n))$. Therefore $W(\tau^n(S^n)) = 1$.

This can also be found through Axiom 3) since

D. Axiom 3) is "solvable," that is, given any two of $W(\zeta)$, $W(\eta)$, or $W(\zeta \oplus \eta)$ we can solve for the third. For example, given $W(\zeta)$ and $W(\zeta \oplus \eta)$, let us write $W(\zeta) = W = 1 + W_1 + W_2 + \cdots + W_n$, $W(\eta) = W' = 1 + W_1' + \cdots + W_m'$, and $W(\zeta \oplus \eta) = W'' = 1 + W_1'' + \cdots + W_{n+m}''$. Expanding Axiom 3) we find

$$W_1'' = 1 \smile W_1' + W_1 \smile 1$$

$$W_2'' = 1 \smile W_2' + W_1 \smile W_1' + W_2 \smile 1$$

$$W_r'' = 1 \smile W_r' + W_1 \smile W_{r-1}' + \cdots + W_{r-1} \smile W_1' + W_r \smile 1$$

We can solve for W'_r in terms of $W'_i, i < r$ and the known W_j and W''_k:

$$W'_r = 1 \smile W'_r = W''_r - (W_1 \smile W'_{r-1} + \dots + W_{r-1} \smile W'_1 + W_r \smile 1).$$

This formula together with $W'_1 = W''_1 - W_1$ to start things off, gives a complete recursive solution for the W'_i (Note that this procedure depends on the fact that W_0 is always equal to 1.) This discussion can be simplified by noticing that the set of all infinite sequences: $1 + \alpha_1 + \alpha_2 + \cdots$ where $\alpha_i \in H^i(X)$ forms a group under $\smile$, which is abelian since we are working mod 2. For example, $(1 + \alpha_1)^{-1} = 1 + \alpha_1 + \alpha_1^2 + \cdots$. Thus we can write $W(\zeta) = W(\zeta \oplus \eta) W^{-1}(\eta)$ to indicate the solvability of Axiom 3). In particular, for $M^n \subset R^{n+k}$, we know that $\tau^n(M^n) \oplus \nu^k(M^n)$ is trivial and that therefore $W(\tau^n \oplus \nu^k) = 1$. Thus we have:

Theorem 1. The Whitney Duality Theorem [see Lectures in Topology Univ. of Michigan Press, 1941, p. 133, especially (21.9]: If we have a differentiable Manifold $M^n \subset R^{n+k}$ with tangent bundle τ^n and normal bundle ν^k, then

$$W(\tau^n) W(\nu^k) = 1 \quad \text{or} \quad W(\nu^k) = W^{-1}(\tau^n).$$

We write $W^{-1} = 1 + \overline{W}_1 + \overline{W}_2 + \cdots + \overline{W}_k$ with as usual $W = 1 + W_1 + W_2 + \cdots + W_n$. (Note that by the above theorem, $W^{-1}(\tau^n)$ has classes of at most dim k.) Solving as above

$$\overline{W}_1 = W_1$$

$$\overline{W}_2 = W_1^2 + W_2$$

$$\overline{W}_3 = W_1^3 + W_3$$

$$\overline{W}_4 = W_1^4 + W_1^2 W_2 + W_2^2 + W_4 \qquad \text{etc.}$$

In particular, we have another proof for assertion C. Taking the usual imbedding of S^n in R^{n+1} we have,

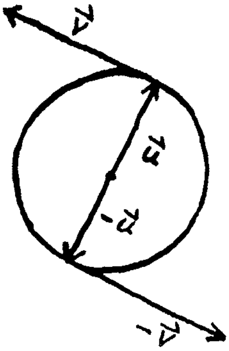
$W(\tau^n(S^n)) = W^{-1}(\nu^1(S^n))$. But $\nu^1(S^n)$ is trivial which implies $W(\tau^n(S^n)) = 1$.

III. Applications:

We will often be concerned with the situation of example 2, the tangent bundle τ^n to a differentiable manifold M^n. Though we have defined W only for bundles and not for manifolds, we will extend our use of classes by writing $W(M^n)$, defined as $W(\tau^n)$. For the sake of exposition, these are called the Stiefel class (of a manifold) and the Whitney class (of a tangent bundle) respectively.

Consider, for instance, $\tau^n(P^n)$. This can be represented in terms of the unit $S^n \subset R^{n+1}$ as follows.

Let $E(\tau^n(P^n))$ be the set of all unordered pairs $[(\vec{u},\vec{v}),(-\vec{u},-\vec{v})]$ where $(\vec{u},\vec{v})$ is an ordered pair with $\vec{v}$ perpendicular to the unit vector $\vec{u}$. In other words, $E(\tau^n(P^n))$ is the set of all pairs $(\vec{u},\vec{v})$ with $\vec{v}$ perpendicular to the unit vector $\vec{u}$ modulo the identification $(\vec{u},\vec{v})=(-\vec{u},-\vec{v})$.



On the other hand, consider the (n+1)-fold Whitney sum $\xi_n^1\oplus\cdots\oplus\xi_n^1$ where ξ_n^1 is the line bundle over P^n of example 4 . The bundle space $E(\xi_n^1\oplus\cdots\oplus\xi_n^1)$ consists of all (n+2)-tuples $(\vec{u};t_o,\cdots t_n)$ where

$$(u;t_o,\ldots t_n)\text{ is identified with }(-\vec{u},-t_o,\ldots,-t_n),$$

or $\quad(\vec{u};\vec{w})=(-\vec{u};-\vec{w}),\qquad|\vec{u}|=1.$ (See Remark 2 after Example 4 .) Notice $\vec{u}$ need not be perpendicular to $\vec{w}$, though at least

$$E(\tau^n)\subset E(\underbrace{\xi_n^1\oplus\cdots\oplus\xi_n^1}_{n+1}).$$

Let η^1 be the 1-plane bundle over P^n with $E(\eta^1)=\{(\vec{u},t\vec{u})\}$ modulo the identification $(\vec{u},t\vec{u})=(-\vec{u},-t\vec{u})$. Clearly η^1 is a trivial bundle.

As can readily be seen,

$$\tau^n\oplus\eta^1\text{ is equivalent to }\underbrace{\xi_n^1\oplus\cdots\oplus\xi_n^1}_{n+1}$$

Thus $W(\tau^n)W(\eta^1) = W(\xi_n^1)^{n+1}$ and since η^1 is trivial,

$$W(\tau^n) = W(\xi_n^1)^{n+1} = (1+\alpha)^{n+1}, \ \alpha \in H^1(P^n). \quad \text{Thus}$$

<u>Theorem 2</u>: The Stiefel class of projective n-space is given by $W(P^n) = (1+\alpha)^{n+1}$ where α is the non-zero element of $H^1(P^n)$. In other words, $W_i(P^n) = \binom{n+1}{i}\alpha^i ; \alpha \in H^1(P^n)$.

The following is a table of binomial coefficents mod 2.

We do not use the last coefficient since W has no n+1-dim component. e.g.

$$W(P^2) = 1 + \alpha + \alpha^2$$
$$W(P^3) = 1$$
$$W(P^4) = 1 + \alpha + \alpha^4 \quad \text{etc.}$$

$$\begin{array}{c}
\backslash 1 \\
1\ 0 \backslash 1 \\
1\ 1\ 1 \backslash 1 \\
1\ 0\ 0\ 0 \backslash 1 \\
1\ 1\ 0\ 0\ 1 \backslash 1 \\
1\ 0\ 1\ 0\ 1\ 0 \backslash 1
\end{array}$$

In making use of these formulas it is important to know that the powers $\alpha, \alpha^2, \ldots, \alpha^n$ are all non-zero. We will assume this to be known. (A proof based on the Gysin sequence will be given later in these notes.)

<u>Parallelizability</u>: Now we are ready to ask: which P^n are parallelizable? We have necessary conditions at hand since P^n parallelizable implies that $W(P^n) = 1$.

<u>Theorem 3</u>: $W(P^n) = 1$ if and only if $n+1$ is a power of two.

(From the above work, this reduces to an exercise in arithmetic mod 2)

Proof: Since $(\alpha+\beta)^2 = \alpha^2+\beta^2 \mod 2$

we have $(1+\alpha)^{2^r} = 1+\alpha^{2^r}$.

Therefore, for $n+1 = 2^r$, $W(P^n) = (1+\alpha)^{n+1} = 1+\alpha^{n+1} = 1$.

Conversely $n+1 = 2^r m$, m odd >1, implies that $W(P^n) = (1+\alpha)^{n+1} = (1+\alpha^{2^r})^m = 1+\binom{m}{1}\alpha^{2^r} + \cdots = 1+\alpha^{2^r}+\cdots \neq 1$ since $2^r < n+1$.

Thus the only P^n which can be parallelizable are $P^1, P^3, P^7, P^{15}, P^{31}, \cdots$. It is known that P^1, P^3, P^7 are in fact parallelizable and that P^{15} is not.

Immersion;

Definition: An immersion of M^n in R^{n+k} is a differentiable map $M^n \longrightarrow R^{n+k}$ such that the Jacobian is never singular (this means there is a well-defined tangent plane at every point). This differs from an imbedding in that "nice" self intersections are permitted.

Theorem 4: If the manifold M^n can be immersed in R^{n+k}, then the dual Stiefel-Whitney classes $\overline{W}_i(M^n)$ must be zero for $i > k$.

Proof: As in the case of an imbedding, $\tau^n \oplus \nu^k$ is trivial so that $W^{-1}(\tau^n) = W(\nu^k)$. But $W_i(\nu^k) = 0$ for $i > k$.

Applying this to P^n immersed in R^{n+1}, we have that

$$W(\nu^1) = 1 \text{ or } 1+\alpha \therefore W(P^n) = W(\tau^n) = W^{-1}(\nu^1) = 1 \text{ or } 1+\alpha+\alpha^2+\cdots+\alpha^n.$$

We have seen that $W(P^n) = 1$ if and only if $n+1 = 2^r$. On the other hand if $W(P^n) = (1+\alpha)^{n+1} = 1+\alpha+\cdots+\alpha^n$ then $(1+\alpha)^{n+2} = 1+\alpha^{n+1} \mod 2$ and again $H^{n+1}(P^n) = 0 \therefore \alpha^{n+1} = 0 \therefore (1+\alpha)^{n+2} = 1$.

As before this implies that $n+2 = 2^r$. Thus the only P^n which can be immersible in R^{n+1} are $P^1, P^2, P^3, P^6, P^7, P^{14}, P^{15}, \cdots$. It is known that P^1, P^2, P^3 are in fact immersible but that P^{15} is not. (See Milnor, "The immersion of n-manifolds in (n+1)-space", Comm. Math. Helv. 30 (1956), pp. 275-284.)

On the other hand, consider the case $n = 2^r$. Then

$$W(P^n) = (1+\alpha)^{n+1} = (1+\alpha)^{2^r}(1+\alpha) = (1+\alpha^{2^r})\,(1+\alpha) = 1+\alpha+\alpha^n,$$

and
$$\begin{aligned} W^{-1}(P^n) &= (1+\alpha^n)^{-1}(1+\alpha)^{-1} \\ &= (1+\alpha^n)\,(1+\alpha+\alpha^2+\cdots+\alpha^n) \\ &= (1+\alpha+\alpha^2+\cdots+\alpha^{n-1}) \qquad (\text{mod } 2). \end{aligned}$$

In other words
$$\begin{aligned} \overline{W}_i(P^n) &= 0, \; i = n \\ &\neq 0, \; i = 1,\cdots,n-1. \end{aligned}$$

Therefore by Theorem 2, for $n = 2^r$, P^n is not immersible in R^{2n-2}.

We would like to know how good an answer this is. I.e. for what dimensions $q > 2n-2$ can P^n be immersed in R^q? There is, in fact, the Theorem of Whitney: any M^n, $n > 1$, can be immersed in R^{2n-1}. [See Whitney, "Singularities of a Smooth n-Manifold in (2n-1) space", Ann. Math. 45 (1944) p. 247.] So for $n = 2^r$ we have the exact result: P^n is immersible in R^{2n-1}, but not immersible in R^{2n-2}.

Our results can be extended somewhat, as follows:
If P^9 is immersible in R^{14}, so is $P^8 \subset P^9$, but this we know is impossible, so P^9 is not immersible in R^{14}. Similarly, we have in general, if $n = 2^r + q$ where r is the largest power of 2 in n, then

P^n is not immersible in $R^{2^{r+1}-2}$.

Imbedding: Similar results can be obtained for imbedding. We will show later that if M^n is imbedded in R^{n+k}, then the highest Stiefel-Whitney class $W_k(\nu^k)$ is zero. Hence if M^n is imbeddable in R^{n+k} then $\overline{W}_i(M^n) = 0$ for $i \geq k$. In particular, for $n = 2^r + q$ as above, P^n is not imbeddable in $R^{2^{r+1}-1}$.

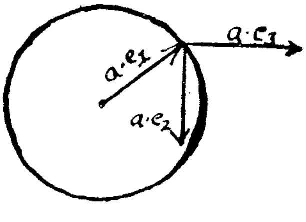
Division algebras: Another application of Stiefel-Whitney classes is in the question: for what n does there exist a division algebra of dim n over R? (Again we will get necessary but not sufficient conditions.) We are looking for a product operation in R^n which

1) is bilinear, and

2) has no zero-divisors.

Suppose such exists and choose a basis $\{e_1, \dots, e_n\}$ for R^n. Let a vary over R^n so that $a \cdot e_1$ varies over the unit S^{n-1} in R^n, $\{a \cdot e_1, \dots, a \cdot e_n\}$ are linearly independent by 1) and 2), and the projections of $a \cdot e_2, \dots, a \cdot e_n$ on the tangent plane to the unit sphere at $a \cdot e_1$ are still linearly independent. This in effect gives us n-1 linearly independent tangent vector fields on S^{n-1}, as $a \cdot e_1$ varies over S^{n-1}. If we identify a and -a we have that $(-a) \cdot e_1 = -(a \cdot e_1)$ is identified with $a \cdot e_1$ and thus we have n-1 linearly independent tangent vector fields on P^{n-1}.

$\therefore$ P^{n-1} is parallelizable or $n = 2^r$



In fact, we know that there are the following division algebras:

$n = 1$ R,	$n = 4$ Quaternions,
$n = 2$ Complex numbers,	$n = 8$ Cayley numbers.

IV. Stiefel-Whitney numbers.

We will now construct a tool which will allow us to compare cohomology classes of different manifolds. (So far we have only compared W's which could be represented in terms of the cohomology of a fixed manifold, P^n)

M^n will be a closed, possibly disconnected differentiable manifold.

Let μ be the fundamental class in $H_n(M^n, Z_2)$. (There is one since we use coefficient group Z_2)

For any $\gamma \in H^n(X, Z_2)$, there is defined the Kronecker index $<\gamma, \mu> \in Z_2$ [see Lefschetz Algebraic Topology, AMS, 1943, p. 118]
As usual, write $W(M^n) = 1 + W_1 + W_2 + \cdot\cdot + W_n$.
Now consider any monomial in $W_1, \ldots, W_n$ which is an element of $H^n(M^n, Z_2)$, that is, has total dimension n, i.e. $W_1^{r_1} \cdots W_n^{r_n}$ with $r_i \geq 0$, $r_1 + 2r_2 + \cdots + nr_n = n$. Each such monomial is of the proper dimension to obtain a Kronecker index; therefore we define

Definition: The Stiefel-Whitney number of the manifold M^n corresponding to the monomial $\gamma = W_1^{r_1} W_2^{r_2} \ldots W_n^{r_n}$ is the integer mod 2: $<\gamma, \mu>$.
In using Stiefel-Whitney numbers as a tool, we will usually be concerned with the complete set of numbers. When we compare the Stiefel-Whitney numbers of different manifolds, we naturally compare the numbers corresponding to the same monomial.

Let us apply this to projective spaces, about all we can work with at this point. For n even, $W_n(P^n) = (n+1)\alpha^n \neq 0$ so that $<W_n, \mu> \neq 0$. Similarly $W_1(P^n) = (n+1)\alpha \neq 0$ so that $<W_1^n, \mu> \neq 0$. (In the special case $n = 2^r$, we know that $W(P^n) = 1+\alpha+\alpha^n$, so that these are the only Stiefel-Whitney numbers different from zero.) For n odd, on the other hand, we can set $n+1 = 2m$ $W(P^n) = (1+\alpha)^{2m} = (1+\alpha^2)^m$. Therefore $W_i = 0$ for all odd i. Any monomial of dim n contains a factor of odd dimensional and therefore is zero. Thus all Stiefel-Whitney numbers are zero. This gives some indication of how much detail and structure this invariant overlooks. On the other hand, these numbers are very useful as is indicated by the following theorem and its converse.

Theorem of Pontrjagin: For B^{n+1}, a manifold with boundary M^n, the Stiefel-Whitney numbers of M^n are all zero. [see Pontrjagin, "Characteristic Cycles on Differentiable Manifolds," Math. Sbor. (NS) 21 (63), p. 233, AMS Translation 32].

In this case we represent the fundamental class of $H_n(M^n)$ not by μ but by $\partial\mu$ where μ is the fundamental class of $H_{n+1}(B^{n+1}, M^n)$.

Proof: By a standard result for arbitrary cohomology classes

$$< W_1^{r_1} \cdots W_n^{r_n}, \partial\mu > = < \delta(W_1^{r_1} \cdots W_n^{r_n}), \mu >$$

As usual, let τ^n be the tangent bundle to M^n. Let β^{n+1} be the tangent bundle to B^{n+1}, and let $\beta^{n+1}|M^n$ be the restriction of

β^{n+1} to M^n. (That is the bundle with total space $\pi^{-1}(M^n)$, base space M^n, and projection $\pi|M^n$ where π denotes the projection map of β^{n+1}). Choosing a Riemann metric on β^{n+1} (see the next section) there is a unique unit inward normal vector to M^n. This generates a trivial bundle θ^1. Clearly $\beta^{n+1}|M^n$ is equivalent to the bundle $\tau^n \oplus \theta^1$. In other words, $i: M^n \longrightarrow B^{n+1}$ is covered by a bundle map $f: \tau^n \oplus \theta^1 \longrightarrow \beta^{n+1}$, $f = (f_E, i)$.

Therefore $\quad i^*(W_i(\beta^{n+1})) = W_i(\tau^n)$.

But in general we have the exact sequence

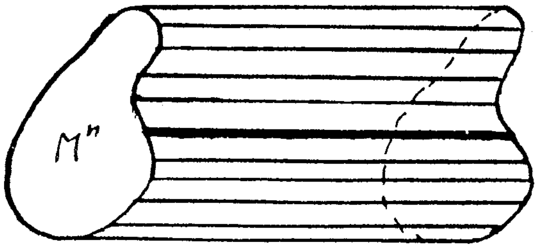
$$H^n(B^{n+1}) \xrightarrow{i^*} H^n(M^n) \xrightarrow{\delta} H^{n+1}(B^{n+1}, M^n).$$

Thus, by exactness, $\delta(W_1^{r_1} \cdots W_n^{r_n}) = 0$ and so all the Stiefel-Whitney numbers of M^n are 0.

The converse, due to Thom, is true, although much harder to prove:

Theorem of Thom: If all the Stiefel-Whitney numbers of M^n are zero, then M^n bounds. [see Thom "Quelques proprietres globale desvarietes differentiables", Comm. Math. Helv. 28 (1954) pp. 17-86, Thm IV.10].

For example, $M^n \cup M^n$, where we mean the union of disjoint copies, always bounds (This can be thoughtof as the two ends which bound a cylinder.)



More generally we define cobordism class:

Definition: M_1^n, M_2^n belong to the same cobordism class if there exists B^{n+1} with boundary $M_1^n \cup M_2^n$; and obtain the:

Theorem: M_1^n, M_2^n belong to the same cobordism class if and only if corresponding Stiefel-Whitney numbers are equal. [see Thom, op.cit. Cor IV.11].

V. Paracompactness:

We next give some basic tools necessary for the study of n-plane bundles. First let us define some of our terms:

Definition: A partition of Unity on X is an indexed collection $\{p_\alpha\}$ such that

1) each p_α is a map $X \longrightarrow [0,1]$,
2) each $x \in X$ has a neighborhood U_x such that $p_\alpha(U_x) = 0$ for all but a finite number of α's, and
3) $\sum_\alpha p_\alpha(x) = 1$, each $x \in X$

Definition: Given an indexed open covering $\{U_\alpha\}$ of a space X, an associated partition of unity is a partition of unity $\{p_\alpha\}$ with the same index set such that $p_\alpha = 0$ outside a closed subset V_α of U_α.

Definition: X is paracompact if X is Hausdorff and given any indexed open covering of X there is an associated partition of unity.

Remark: The usual definition, which is equivalent, is: X is paracompact if X is Hausdorff and if every open covering has an open locally finite refinement. [For this definition and other properties of paracompactness, see Kelley, General Topology, VanNostrand, 1955, p. 156].

In particular, every metric space is paracompact as is every regular space which is a countable union of compact subsets. [see Morita, Math. Jap. vol 1 (1948) p. 60-68, Thm. 10]. These are all we will need. Note that separable manifolds are paracompact since they fall in both of these categories.

Illustrations of the Use of Paracompactness in Bundle Theory.

First: Definition. A Riemannian metric on an n-plane bundle is an inner product defined in each fibre [$e_1 \cdot e_2 = r \epsilon R$ for all $e_1, e_2 \epsilon E$ such that $\pi(e_1) = \pi(e_2)$] such that

$e_1 \cdot e_2$ is 1) symmetric: $e_1 \cdot e_2 = e_2 \cdot e_1$;

2) bilinear;

3) positive definite: $e_1 \cdot e_1 > 0$ except for $0 \cdot 0 = 0$; and

4) $e_1 \cdot e_2$ is a continuous function of two variables (Although $e_1 \cdot e_2$ is defined only for e_1, e_2 in the same fibre, we require continuity with respect to the topology of E, not just that of the fibre. I.e. if e_1', e_2' are

close to e_1, e_2 respectively and if $e_1 \cdot e_2$ and $e_1' \cdot e_2'$ are defined then $e_1' \cdot e_2'$ is close to $e_1 \cdot e_2$ in R.)

Remark: The term Riemannian metric is ordinarily used only in the tangent bundle, but this seems like a natural generalization.

Theorem 5: Every n-plane bundle ζ over a paracompact base X admits a Riemannian metric.

Proof: Case I: product bundle

We need only define the inner product on a basis of each fibre and extend by bilinearity. We can use as a basis for $\pi^{-1}(x), x \in X$ the $\{c_i(x)\}$ given by the cross-sections, which in the case of a product bundle can be taken to be global. We define $c_i(x) \cdot c_j(x) = \delta_{ij}$; the Kronecker $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

Case II: In general, let $\{U_\alpha\}$ be the distinguished class of open sets of X giving the local product structure for ζ. Let $\{p_\alpha\}$ be an associated partition of unity and $\{(e_1 \cdot e_2)_\alpha\}$ the associated Riemannian metric defined as in case I for each $\zeta | U_\alpha$

Define $e_1 \cdot e_2$ to be $\sum_\alpha p_\alpha(\pi(e_1))(e_1 \cdot e_2)_\alpha$ [with the convention $0 \cdot (\text{undefined}) = 0$ since $(e_1 \cdot e_2)_\alpha$ is defined only for $\pi(e_1) \in U_\alpha$].

It is easy to verify that this is

1) symmetric - since $\pi(e_1) = \pi(e_2)$,

2) bilinear - since it is a weighted sum of bilinear functions,

3) positive definite

and 4) continuous - since locally it is a finite sum of continuous functions. (For some neighborhood of x, all but a finite number of $p_\alpha = 0$.)

QED

Second Illustration: Grassman manifolds.

In classical differential geometry, there is encountered Gauss' construction of the spherical image of a manifold $M^n \subset R^{n+1}$.

This is a mapping of M^n into S^n given by mapping a point x of M^n into the unit vector at the origin of R^{n+1} with the same direction as the normal to M^n at x. More generally, for M^n immersed in R^{n+k} we associate with $x \in M^n$ the n-plane through the origin parallel to the tangent plane at x. [Tangent planes correspond 1-1 with the undirected normal in the case $k=1$]. This gives a map not of M^n into S^n but rather into $G_{n,k}$.

Definition: The Grassman manifold $G_{n,k}$ is the set of all n-dimensional subspaces in (n+k)-space (n-planes through the origin). This set has a natural structure as a differentiable manifold and is in fact compact. [see Steenrod, op.cit. p. 35] Note that there is no natural structure for the symbol for a Grassman manifold; there is no agreement in the literature.

By the usual duality between n-dimensional subspaces of R^{n+k} and (n+k)-n or k-dimensional subspaces, $G_{n,k} \approx G_{k,n}$. One example of $G_{n,k}$ is easy to picture: We obtain S^n as the set of unit vectors in R^{n+1} or, what is equivalent, the set of directed lines through the origin of R^{n+1}. Since the n-planes in $G_{n,k}$ are unoriented, we see that

$$G_{n,1} \approx G_{1,n} = P^n.$$

Now let γ_k^n be the n-plane bundle over $G_{n,k}$ with $E(\gamma_k^n)$ = set of all pairs (n-plane through origin, vector in that plane); e.g. $\gamma_k^1(G_{1,k}) = \xi_k^1$.

And we obtain

Theorem 6: For M^n immersed in R^{n+k}, there is an associated bundle map $f: \tau^n(M^n) \longrightarrow \gamma_k^n$ such that f_B is the generalized Gauss map:

$$M^n \longrightarrow G_{n,k}$$

This theorem is expressed by saying γ_k^n is "universal" for sufficiently large k: i.e. every tangent bundle maps into it.

The map f_E is defined in the obvious fasion and the verification that the pair is a bundle map is left to the reader.

In a still more general situation, we define

Definition: The infinite Grassman manifold G_n (i.e. $k = \infty$) is the set of all n-dimensional subspaces of R^∞, countably infinite dimensional Euclidean space, with the topology given as follows. Let $\{b_i\}$ be a basis for $R^\infty, i = 1, 2, \dots$ and let R^m be the subspace spanned by $b_1, \dots, b_m$. Then $R^1 \subset R^2 \subset \dots \subset R^\infty$, $G_{n,o} \subset G_{n,1} \subset \dots \subset G_n$, and this sequence of inclusions induces a topology on G_n by defining $H \subset G_n$

to be closed if and only if $H \cap G_{n,k}$ is closed for all k. Note: G_n is not metric, but is regular and a countable union of compact subsets $G_{n,k}$ and therefore is paracompact. (We will omit the proof that G_n is regular, since we will see presently that G_n is actually a CW-complex. Every CW-complex is known to be normal. [See J.H.C. Whitehead, "Combinatorial homotopy I", Bull. Amer.Math.Soc. 55 (1949), pp.213-245.]

As above we define γ^n, an n-plane bundle over G_n, with total space

$E(\gamma^n)$= set of all pairs (n-dimensional subspace of R^∞,vector in that subspace).

The following is a generalization of Theorem 6.

Theorem 7. For any n-plane bundle ζ^n over a paracompact base X there exists a bundle map $\zeta^n \longrightarrow \gamma^n$.

(Actually a somewhat stronger result holds. Any two such bundle maps $\zeta^n \longrightarrow \gamma^n$ are homotopic. Furthermore any two homotopic maps $X \longrightarrow G_n$ induce equivalent bundles. For this reason γ^n is called a universal bundle and G_n a classifying space for n-plane bundles.)

Proof: Case I: product bundle.

There exists a linear homeomorphism $h: E(\zeta) \longrightarrow X \times R^n$

Let ρ be the projection: $X \times R^n \longrightarrow R^n$. Then ρh is linear and non-singular in each fibre. Let $f: R^n \longrightarrow$ origin, $g: X \longrightarrow$ origin. Then $(\rho h, g)$ is a bundle map into the bundle $(R^n, 0, f)$, which maps into γ^n in the obvious fashion.

Case II: There is a countable distinguished covering $\{U_i\}$

Let $\{p_i\}$ be an associated partition of unity. R^∞ can be represented as $R^n \oplus R^n \oplus R^n \oplus \cdots$.

Map $E(\zeta) \xrightarrow{F} R^\infty$ by $F(e) = (p_1(\pi(e))f_1(e), p_2(\pi(e))f_2(e), \cdots)$

where $f_i : \pi^{-1}(U_i) \longrightarrow R^n$ as in case I.

F is continuous and is linear, non-singular on each fibre since each f_i is.

Let $g_B(x)$ = subspace $\subset R^\infty$ spanned by $\{F(e) | e \in \pi^{-1}(x)\}$.

therefore $g_B(x)$ is an element of G_n. That g_B is continuous can be checked easily since locally g_B lies in some finite $G_{n,k} \subset G_n$.

Define $g_E : E(\zeta) \longrightarrow E(\gamma^n)$ by

$$g_E(e) = (g_B(\pi(e)), F(e)).$$

Then (g_E, g_B) is the required bundle map.

Thus we will have proved the theorem as soon as we show

Lemma: Given an n-plane bundle ζ over a paracompact base space X, there is a countable covering $\{U_n\}$ of X such that the restrictions $\zeta | U_n$ are trivial.

Proof: Let $\{V_\alpha\}$ be the distinguished covering. Choose an associated partition of unity $\{p_\alpha\}$. Call the index set A and for each finite $S \subset A$

$$\text{let} \quad W_S = \{x | \underset{\alpha \in S}{\text{Min}}\, p_\alpha(x) > \underset{B \notin S}{\text{Max}}\, p_\beta(x)\}$$

$\{W_S\}$ is an open covering of X since

1) W_S is open by continuity of all p_α , and

2) $x \in W_{S_x}$ for $S_x = \{\alpha \in A \mid p_\alpha \ (x) > 0\}$ for each $x \in X$.

Let U_n be the union of W_S over all S with n elements. Again $\{U_n\}$ is an open covering.

Notice the W_S in U_n are disjoint since the S all have the same length and therefore for $S_1 \neq S_2$, there exist α, β such that $\alpha \in S_1, \alpha \notin S_2$; $\beta \in S_2, \beta \notin S_1$

Thus for $x \in W_{S_1}$, $P_\alpha(x) > p_\beta(x)$,
and for $x \in W_{S_2}$, $P_\alpha(x) < p_\beta(x)$. Therefore $W_{S_1} \cap W_{S_2} = 0$.

On the other hand for each $\alpha \in S$, $p_\alpha = 0$ outside V_α and therefore $W_S \subset V_\alpha$. Thus $\{U_N\}$ is a countable open covering giving the local product structure of ζ.

QED.

VI. The cohomology ring $H^*(G_n, Z_2)$.

In a little while, we will need to know something of the structure of G_n as a cell complex, and this we investigate by means of matrices over the reals. We need the following notions and theorems of matrix theory. [See, for example, Birkhoff and MacLane, A Survey of Modern Algebra, The Macmillan Co. 1946 p. $\overset{177}{271}$.]

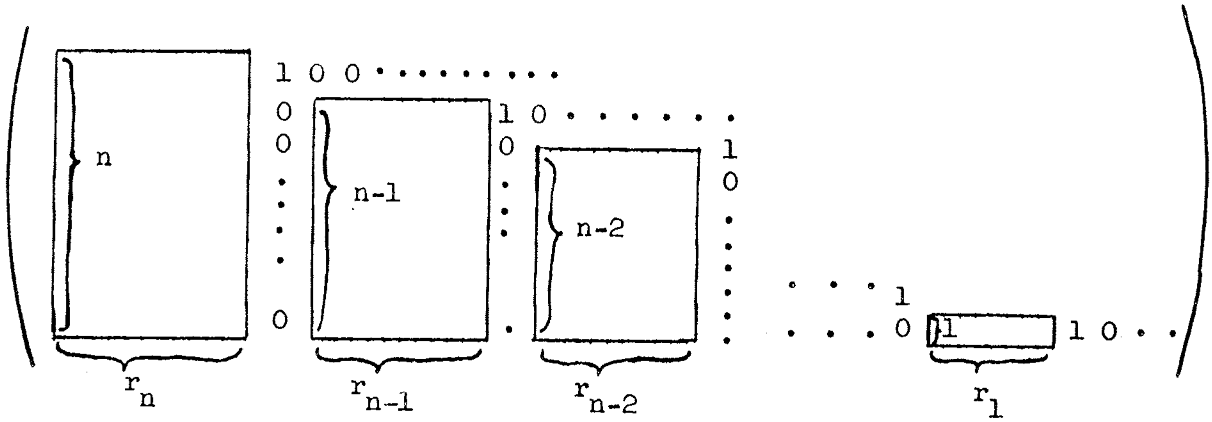
Definition: Two $n \times m$ matrices A, B are row equivalent if A can be obtained from B by a succession of elementary row operations i.e.

1) interchanging any two rows,
2) multiplication of a row by a non zero scalar,
3) addition of one row to another,

Definition: The row space of an $n \times (n+k)$ matrix is the subspace of R^{n+k} spanned by the n row vectors of the matrix (= "range" in Birkhoff and MacLane op. cit.)

Theorem: Two matrices are row equivalent if and only if they have the same row space.

Theorem: Every matrix is row equivalent to a matrix of canonical form, the reduced echelon matrix i.e.



where each $r_i \geq 0$, and the $r_i \times i$ blocks are arbitrary depending on the original matrix.

By these matrix tools, we have reduced the study of n-dimensional subspaces of R^{n+k} to the study of reduced echelon matrices. Since given a pattern as above, we can vary the entries in the blocks independently, each such pattern determines a vector space over R or a cell in $G_{n,k}$. Thus we have a cell subdivision of $G_{n,k}$. [For full details, see Ehresmann, Ann. Math. 35 (1934) p. 396.]

Example: $\begin{pmatrix} a & b & 1 & 0 & 0 & 0 & 0 & 0 \\ c & d & 0 & 1 & 0 & 0 & 0 & 0 \\ e & f & 0 & 0 & g & 1 & 0 & 0 \end{pmatrix}$ represents a 7-cell in $G_{3,5}$

Similarly for G_n, we look at such patterns in $n \times \infty$ matrices e.g.

$$\begin{pmatrix} a & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & 0 & d & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c & 0 & e & 0 & f & g & h & 1 & 0 & \cdot & \cdot & \end{pmatrix}$$ an 8-cell in G_3.

That is, for each sequence of n non-negative integers $r_1, \ldots, r_n$ we obtain a cell of dim $r_1 + 2\,r_2 + \cdots + n\,r_n$ in G_n. This gives G_n the structure of a CW-complex. Thus there is a unique 0-cell in each G_n: $\begin{pmatrix} 1 & & & \bigcirc \\ & 1 & & \\ & & 1 & \\ \bigcirc & & \ddots & 1 \end{pmatrix}$,

and a unique 1-cell: $\begin{pmatrix} 1 & & & & \bigcirc \\ & \ddots & & & \\ & & 1 & 0 & 0 \\ \bigcirc & & 0 & a & 1 \end{pmatrix}$.

For the special case $G_{1,k} = P^k$, we have one cell of each dimensional $0 \leq r_1 \leq k$.

Looking at the Stiefel-Whitney classes of γ^n, we see:

<u>Theorem 8</u>: The cohomology ring $H^*(G_n, Z_2)$ is a polynomial algebra over Z_2 generated by $W_1(\gamma^n), \cdots, W_n(\gamma^n)$.

<u>Proof:</u> First we show

<u>Lemma:</u> There are no relations among the $W_i(\gamma^n)$

<u>Proof</u>: if a polynomial $p(W_1(\gamma^n), \ldots, W_n(\gamma^n)) = 0$ then $p(W_1(\zeta^n), \ldots, W_n(\zeta^n) = 0$ for any ζ^n over a paracompact base X, since the bundle map given by Theorem 6 induces a homomorphism g^* such that $g^*(W_i(\gamma^n)) = W_i(\zeta^n)$ and thus

$$\begin{aligned} p(W_1(\zeta^n), \ldots, W_n(\zeta^n)) &= p(g^*(W_1(\gamma^n)), \ldots, g^*(W_n(\gamma^n))) \\ &= g^* p(W_1(\gamma^n), \ldots, W_n(\gamma^n)) \\ &= g^*(0) = 0. \end{aligned}$$

To prove the lemma, we need only find some ζ^n with no relations among the $W_i(\zeta^n)$.

Consider ξ^1_k: $W(\xi^1_k) = 1 + \alpha \quad \alpha \epsilon H^1(P^k; Z_2)$

Let $X = \underbrace{P^k \times \ . \ . \ . \times P^k}_{n}$ with projections π_i

into the n factors, P^k; $i = 1, \ldots, n$

It is known that

$H^*(P^\infty, Z_2)$ is the polynomial algebra generated by $\alpha \epsilon H^1(P^\infty; Z_2)$

and for $k = \infty$, $H^*(X)$ is the polynomial algebra generated by

$\alpha_1, \ldots, \alpha_n$ where $\alpha_i = \pi^*_i(\alpha)$

Let ζ^n be the bundle $\eta_1^1 \oplus \cdots \oplus \eta_n^1$ over X where η_i^1 is the bundle over X induced from ξ_k^1 by π_i.

Thus

$$\begin{aligned} W(\zeta^n) &= W(\eta_1^1) \ldots W(\eta_n^1) \\ &= \pi_1^* W(\xi_\infty^1) \ldots \pi_n^* W(\xi_\infty^1) \\ &= (1+\alpha_1)(1+\alpha_2) \ldots (1+\alpha_n) \end{aligned}$$

In other words

$$W_1(\zeta^n) = \alpha_1 + \ldots + \alpha_n$$

$$\vdots$$

$W_n(\zeta^n) = \alpha_1 \cdots \alpha_n$, where the polynomials which appear on the right are just the elementary symmetric functions σ_i in the α_i. From algebra we have [cf. Van der Waerden, Modern Algebra, Ungar, 1953 p. 79 or 176]

Theorem: For Λ a commutative ring with 1 and $x_1, \ldots, x_n$ indeterminate symbols, the symmetric elements of $\Lambda[x_1, \ldots, x_n]$ form a polynomial ring $\Lambda[\sigma_1, \ldots, \sigma_n]$.

This means in particular that if some polynomial p satisfies $p(\sigma_1, \ldots, \sigma_n) = 0$, then $p = 0$. Thus for ζ^n as above, $p(W_1(\gamma^n), \ldots, W_n(\gamma^n)) = 0$ implies $p(W_1(\zeta^n), \ldots, W_n(\zeta^n)) = 0$ and thus $p = 0$; or there are no (polynomial) relations among the $\{W_i(\gamma^n)\}$, which proves the lemma.

Thus we know that $H^*(G_n)$ contains the polynomial algebra generated by $\{W_i(\gamma^n)\}$.

Let $C^i(G_n)$ represent the i-cochains of G_n and $Z^i(G_n)$, the i-cocycles. The dimension of $C^i(G_n)$ as a vector space over Z_2 is the number of i-dimensional cells, which is finite since they correspond to sequences $r_1, \ldots, r_n$ with $r_1 + 2r_2 + \ldots + n r_n = i$. Moreover it is $\geq \dim Z^i(G_n) \geq i^{th}$ Betti number mod 2 $\geq$ number of monomials in $\{W_j(\gamma^n)\}$ of total dimension i, since $H^i(G_n) \supset$ i-dimensional part of the polynomial algebra generated by $\{W_j(\gamma^n)\}$. On the other hand, such monomials correspond to sequences $r_1, \ldots, r_n$ with $r_j \geq 0$, $r_1 + 2r_2 + \ldots + n r_n = i$. That is, there is a one to one correspondence between cells and monomials of the same dimension. Thus all the above inequalities are in fact equalities or:

$\dim C^i = \dim H^i(G_n) =$ number of monomials of dim i. Therefore $H^i(G_n)$ is the i-dimensional part of the polynomial algebra and

$H^*(G_n; Z_2)$ is the polynomial algebra over Z_2 generated by $W_1(\gamma^n), \ldots, W_n(\gamma^n)$. QED

Further for g^* as above $g^*: H^*(G_n) \longrightarrow H^*(\underbrace{P^\infty \times \ldots \times P^\infty}_{n})$ is an isomorphism onto the subalgebra consisting of all symmetric polynomials in $\alpha_1, \ldots, \alpha_n$.

Uniqueness of Stiefel-Whitney classes:

At this point, we still have not shown that there exists a collection of classes satisfying the given axioms, but before investigating that question we will prove

Theorem 9: There is at most one collection of classes compatible with the axioms.

Proof: Suppose we have two collections $\{W\}$ and $\{\widetilde{W}\}$ satisfying the axioms. As we showed in proving alternative Axiom 4'), $W(\xi_n^1)$ and $\widetilde{W}(\xi_n^1)$ must both equal $1+\alpha$, where α is the non zero element of $H^1(P^n)$. This still holds true for P^∞: $W(\xi_\infty^1) = 1+\alpha = \widetilde{W}(\xi_\infty^1); \alpha \in H^1(P^\infty)$. By naturality of W and $\widetilde{W}$ under mappings, in particular the projections $P^\infty \times P^\infty \times \cdots \times P^\infty \longrightarrow P^\infty$ of the previous section, $W(\eta_1^1) = \widetilde{W}(\eta_1^1)$. By Axiom 3) therefore, $W(\eta_1^1 \oplus \cdots \oplus \eta_n^1) = \widetilde{W}(\eta_1^1 \oplus \cdots \oplus \eta_n^1)$. For $g: P^\infty \times \cdots \times P^\infty \longrightarrow G_n$ as before

$$g^*(W(\gamma^n)) = W(\eta_1^1 \oplus \cdots \oplus \eta_n^1)$$

$$g^*(\widetilde{W}(\gamma^n)) = \widetilde{W}(\eta_1^1 \oplus \cdots \oplus \eta_n^1)$$

and g^* is a monomorphism so $W(\gamma^n) = \widetilde{W}(\gamma^n)$. But G_n is a classifying space; for any bundle ζ^n over a paracompact base X, there is a bundle map $f: \zeta^n \longrightarrow \gamma^n$, and so $f^*:(W(\gamma^n)) = W(\zeta^n)$, $f^*:(\widetilde{W}(\gamma^n)) = \widetilde{W}(\gamma^n)$. Thus for every ζ^n over a paracompact base space, $W(\zeta^n) = \widetilde{W}(\zeta^n)$. QED

Remark: It is possible to prove this for bundles restricted to manifolds for base, but not just for tangent bundles of manifolds.

VII. Existence of Stiefel-Whitney classes

We now proceed to prove the existence of Stiefel-Whitney classes by giving a construction in terms of known operations. For any n-plane bundle ζ with total space E, base space B and projection π, we denote by E_o the set of non-zero elements of E and by F_o, the set of all non-zero elements of $F = \pi^{-1}(b)$, a fibre. Clearly $F_o = F \cap E_o$.

Using singular theory and one of several techniques (e.g spectral sequences or that of the appendix) we have that

$$H^i(F,F_o;Z_2) = \begin{cases} 0 \text{ for } i \neq n \\ Z_2 \text{ for } i = n \end{cases} \quad \text{and} \quad H^i(E,E_o;Z_2) = \begin{cases} 0 \text{ for } i < n \\ H^{i-n}(B) \text{ for } i \geq n \end{cases}$$

(This can be seen intuitively, though not rigorously, without spectral sequences as follows: The unit n-cell is a deformation retract of R^n and the unit (n-1)-sphere is a deformation retract of (R^n-origin) = R^n_o. For B paracompact, we know that we can put a Riemannian metric on E. Looking at the cohomology of (E,E_o), we might just as well look at the cohomology of (E',E_o') where E' is the set of all elements of E with norm ≤ 1, E_o' is the set of all elements of E with norm 1, since as indicated above E and E_o have the same homotopy type as E' and E_o' respectively. Now assume that B is a cell complex. Take a fine enough cell subdivision of B so that we have a product bundle over each cell c^i. In (E', E_o') we are looking at $c^i \times$(n-cell) mod $c^i \times$(the boundary of that n-cell), thus we have a collection of cells covering E' and can extend it in a trivial fashion to give a cell subdivision of (E',E_o'). The relation between the cell structure of B and that of (E',E_o') indicates why the dimension of the cohomology gets shifted by n. As can be seen, there are no cells at all of dimension $< n$ which are not in E_o'.) Rigorously and more explicitly, it is possible to prove (see appendix):

Theorem 10: 1) $H^i(E,E_o) = 0$ for $i < n$

2) There exists a unique class U in $H^n(E,E_o)$ such that for each fibre $F = \pi^{-1}(b)$, we have $j_b^* U =$ the non-zero element U_b of $H^n(F,F_o)$ where j_b is the inclusion map $j_b: F,F_o \longrightarrow E,E_o$.

3) $H^i(E) \xrightarrow{\smile U} H^{i+n}(E,E_o)$ is an isomorphism for all i.

Now $\pi^*: H^*(B) \longrightarrow H^*(E)$ is an isomorphism, since there is the trivial zero cross-section of $B \longrightarrow E$ given by $b \longrightarrow (b,0)$ and the image of B under this cross-section is a deformation retract of E and homeomorphic to B. Following Thom, we combine these two isomorphisms in a new isomorphism

$$\phi = (\smile U)\circ \pi^*:\ \begin{tikzcd} H^j(B) \arrow[rr, "\phi"] \arrow[dr, "\pi^*"'] & & H^{j+n}(E,E_o) \\ & H^j(E) \arrow[ur, "\smile U"'] & \end{tikzcd}$$

and then define the Stiefel-Whitney classes as follows: $W_i(\zeta) = \phi^{-1} Sq^i \phi(1)$. To study this definition, we will assume as known the following properties of the Steenrod squares, Sq^i (read: square upper i):

1) For spaces X,Y with $X \supset Y$, Sq^i is an additive homomorphism

$$Sq^i: H^k(X,Y) \longrightarrow H^{k+i}(X,Y) \qquad \text{such that}$$

2) it is natural with respect to maps $f: X,Y \longrightarrow X',Y'$ i.e. $Sq^i f^* = f^* Sq^i$

3) $Sq^i(\alpha^k) = \begin{cases} 0 & \text{for } i > k \\ \alpha^k \smile \alpha^k & \text{for } i = k \end{cases}$ where k indicates the dimension of α

4) $Sq^0 =$ identity

5) (Cartan) $Sq^k(\alpha\smile\beta) = \sum_{i+j=k} Sq^i(\alpha)\smile Sq^j(\beta)$

Writing $Sq(\alpha)$ for $(Sq^0+ Sq^1+ \cdots + Sq^k+ \cdots)(\alpha)$, property 5) becomes $Sq(\alpha\smile\beta) = Sq(\alpha)\smile Sq(\beta)$. (Note that for dim $\alpha = k$, $Sq(\alpha)$ reduces to $(Sq^0+ Sq^1+ \cdots + Sq^k)(\alpha)$.) Thus Sq is a ring homomorphism

$$Sq:H^*(X,Y) \longrightarrow H^*(X,Y)$$

We can now write our construction of Stiefel-Whitney classes as

$$W(\zeta) = \phi^{-1}Sq\phi(1) = \phi^{-1}SqU .$$

Verification of the Axioms:

Axiom 1: Our construction gives elements of the proper dimension, i.e. $W_i(\zeta)\epsilon H^i(B)$ and by property 3) above $W_i = 0$ for $i > n$ and by property 4), $W_0 = 1$.

Axiom 2: Naturality under bundle maps: For $f = (f_E,f_B)$, f_E induces a map $g: E,E_0 \longrightarrow E',E'_0$ and by the definition of U, $g^*(U') = U$. Thus ϕ is natural and 2) above gives us that Sq is natural, and so W is natural.

Axiom 4: (We will return to Axiom 3) in a moment.) Let ξ_1^1 be as usual the twisted line bundle over $S^1 = P^1$, otherwise representable as the Moebius band. As can clearly be seen by homotopy type arguments similar to those above, we have $H^*(E,E_0) \approx H^*$ (Moebius band, Boundary of the Moebius band). Since we can obtain a Moebius band by removing a 2-cell from the projective plane, we have $H^*(E,E_0) \approx H^*(P^2$, 2-cell) which we know to be $H^0 = 0$, $H^1 = Z_2, H^2= Z_2$. Further it is known that for α the non-zero 1-dimensional class, $\alpha\smile\alpha$ is the non-zero 2-dimensional class. Therefore $Sq^1(\alpha) \neq 0$ and so $W_1(\xi^1) \neq 0$.

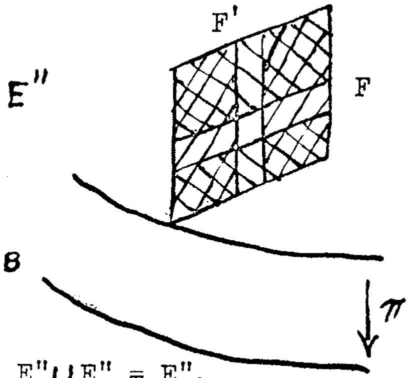
Axiom 3: We prove

<u>Theorem 11</u>: $W(\zeta\oplus\zeta') = W(\zeta)W(\zeta')$.

<u>Proof</u>: Let $\zeta'' = \zeta\oplus\zeta'$ and represent the total space of ζ by E, that of ζ' by E' and so on, with similar notation for the respective fibres F, F', F'', etc. From the structure of the Whitney bundle sum, we know

$$F \times F' = F''.$$

Let
$$E''_a = \bigcup_{\text{all fibres}} F_o \times F'$$
$$E''_c = \bigcup_{\text{all fibres}} F \times F'_o.$$



Obviously $E''_a \subset E''_o$, $E''_c \subset E''_o$ and it is clear that $E''_a \cup E''_c = E''_o$.

The following diagrams will be helpful in following the rest of the proof:

$$\begin{tikzcd}
E''_a \arrow[r, phantom, "\subset"] \arrow[d, "r_1"] & E'' \arrow[r, phantom, "\subset"] \arrow[d, "p_1"] & E'', E''_a \arrow[d, "q_1"] \\
E_o \arrow[r, phantom, "\subset"] & E \arrow[r, phantom, "\subset"'] \arrow[r, phantom, "i_1"', shift right=2] & E, E_o
\end{tikzcd}
\qquad
\begin{tikzcd}
E''_c \arrow[r, phantom, "\subset"] \arrow[d, "r_2"] & E'' \arrow[r, phantom, "\subset"] \arrow[d, "p_2"] & E'', E''_c \arrow[d, "q_2"] \\
E'_o \arrow[r, phantom, "\subset"] & E' \arrow[r, phantom, "\subset"'] \arrow[r, phantom, "i_2"', shift right=2] & E', E'_o
\end{tikzcd}$$

Diagram 1

Here p_1 and p_2 are the ρ and ρ' of the definition of the bundle sum (cf. Diagram 3 and page 5) and the restrictions of q_1 and q_2 just as r_1 and r_2 are the restrictions of p_1 and p_2.

Since the fibres are contractible, r_1 and r_2 are homotopy equivalences. Similarly for p_1 and p_2 and so on the cohomology level, we have

$$U \in H^*(E,E_o) \xrightarrow[\approx]{q_1^*} H^*(E'',E''_a)$$

$$U' \in H^*(E',E'_o) \xrightarrow[\approx]{q_2^*} H^*(E'',E''_c)$$

We assert that $q_1^*(U) \smile q_2^*(U') = U''$. By the uniqueness of U'' as given in Theorem 10, we need only show, for each $j_b : F''_b, F''_{b,o} \longrightarrow E'', E''_o$, that $j_b^*(q_1^* U \smile q_2^* U')$ is the non-zero element U''_b of $H^n(F''_b, F''_{b,o})$. Consider the following diagram

$$\begin{tikzcd}
U \otimes U' \in H^n(E,E_o) \otimes H^m(E',E'_o) \arrow[r, "q_1^* \otimes q_2^*"] \arrow[d, "j_3^* \otimes j_4^*"] & H^n(E'',E''_a) \otimes H^m(E'',E''_c) \arrow[r, "\smile"] \arrow[d, "j_1^* \otimes j_2^*"] & H^{n+m}(E'',E''_o) \arrow[d, "j^*"] \\
H^n(F,F_o) \otimes H^m(F',F'_o) \arrow[r, "q_3^* \otimes q_4^*"] & H^n(F'',F''_a) \otimes H^m(F'',F''_c) \arrow[r, "\smile"] & H^{n+m}(F'',F''_o)
\end{tikzcd}$$

Diagram 2

where we have written systematically F for the arbitrary fibre F_b, F_o for $F_{b,o}$ etc. and where $j_1 : F'', F''_a \longrightarrow E'', E''_a$, $j_2 : F'', F''_c \longrightarrow E'', E''_c$; $j_3 : F, F_o \longrightarrow E, E_o$; $j_4 : F', F'_o \longrightarrow E', E'_o$ are the inclusion maps.

The element $j^*(q_1^* U \smile q_2^* U')$ is obtained by following the outside edge of the diagram clockwise from $H^n(E,E_o) \otimes H^m(E', E'_o)$ to $H^{n+m}(F'',F''_o)$. By commutativity of the diagram, the same element is reached by the outside counter clockwise path. By the definition of U and U' we have $j_3^* U = U_b$, $j_4^* U' = U'_b$. Since the projections q_3, q_4 yield isomorphisms q_3^*, q_4^*, the element we reach in $H^{n+m}(F'',F''_o)$ is the non-zero element U''_b.

QED.

Since

1) $q_1^* U \smile q_2^* U' = U''$

we have $SqU'' = Sq(q_1^* U \smile q_2^* U')$.

By property of 5) of Sq,

$$SqU'' = Sq(q_1^* U) \smile Sq(q_2^* U').$$

Using the naturality of Sq this becomes

2) $SqU'' = (q_1^* \, Sq\, U) \smile (q_2^* \, SqU')$.

Our definition of W: $W(\zeta) = \phi^{-1} Sq \phi(1) = \phi^{-1} Sq\, U$ can be rewritten

3) $\pi^* W(\zeta) \smile U = SqU$

and similarly for ζ' and ζ''.

Combining 2) and 3) we have

4) $SqU'' = q_1^*(\pi^* W(\zeta) \smile U) \smile q_2^*(\pi'^* W(\zeta') \smile U')$.

We will make use of the relation $q_1^*(\gamma \smile \beta) = p_1^* \gamma \smile q_1^* \beta$ which holds for $\gamma \in H^*(E)$, $\beta \in H^*(E, E_0)$ (see Diagram 1). And the corresponding relation for q_2^* and p_2^*. Thus 4) becomes

$$SqU'' = (p_1^* \, \pi^* W(\zeta) \smile q_1^* U) \smile (p_2^* \, \pi'^* \, W(\zeta') \smile q_2^* \, U').$$

By commutativity of $\smile$ mod 2, we obtain

$$SqU'' = p_1^* \pi^* W(\zeta) \smile p_2^* \, \pi'^* W(\zeta') \smile q_1^* \, U \smile q_2^* U'.$$

Referring to diagram 3,

$\pi p_1 = \pi' p_2 = \pi''$

thus, using 1) we have

$$\begin{tikzcd}
 & E'' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\pi''"] & \\
E \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\
 & B &
\end{tikzcd}$$

Diagram 3

5) $SqU'' = \pi''(W(\zeta)\smile W(\zeta'))\smile U''$.

But the class $W(\zeta'')$ is uniquely defined by the equation

$$SqU'' = \pi''(W(\zeta''))\smile U''.$$

This completes the proof that $W(\zeta'') = W(\zeta)\smile W(\zeta')$.

VIII. Oriented Bundles:

Up to this point, we have been working strictly with Z_2 as coefficients for the cohomology we have used. This of necessity means that we overlook some detail in the structure; now we take a closer look using Z as coefficient group. Since part of the study we have conducted so far made strong use of the existence of a non-zero element of $H^n(E,E_o)$ which was guaranteed by using Z_2 as coefficients, we will have to limit ourselves when using Z as coefficients to oriented bundles, as will be seen in what follows. First, some preliminary definitions:

Definition: Two bases of a finite-dimensional vector space are equivalent if the determinant of the matrix expressing one in terms of the other is positive.

Definition: An orientation of a vector space V of dimension n is an equivalence class of bases.

This corresponds to choosing a generator (there are two) of $H_n(V,V_o;Z)$ (and incidently to the intuitive geometric idea of orientation). The correspondence can be given as follows: Let $v_1,\dots,v_n$ be a basis for V and Δ_n, the standard n-simplex with vertices $A_o,A_1,\dots,A_n$. The linear map $\Delta_n \longrightarrow V$ given by $A_o \longrightarrow -v_1-v_2\dots-v_n$, $A_i \longrightarrow v_i$ for $i = 1,2,\dots,n$ determines a generator of $H_n(V,V_o;Z)$ in the singular theory. Two bases will determine the same generator under this correspondence if and only if they are equivalent.

Definition: An oriented n-plane bundle is an n-plane bundle together with an orientation for each fibre such that these orientations are locally compatible, in the following sense. For each point b_o of the base space there should exist a neighborhood N and cross-sections

$$c_1,\dots,c_n: N \longrightarrow E$$

such that for each $b \in N$ the vectors $c_1(b),\dots,c_n(b)$ form a basis for the fibre F_b which is compatible with the given orientation of F_b.

In terms of cohomology this means that for each fibre F_b we have a distinguished generator $U_b \in H^n(F_b, F_{b,o}; Z)$. The local compatibility condition can then be put in the following form. For each $b_o \in B$ there should exist a neighborhood N and a cohomology class $u \in H^n(\pi^{-1}(N), \pi^{-1}(N)_o; Z)$ such that $j_b^*(u) = U_b$ for each $b \in N$, where $j_b: F_b, F_{b,o} \longrightarrow \pi^{-1}(N), \pi^{-1}(N)_o$ denotes the inclusion map. The proof that these two definitions of "oriented n-plane bundle" are equivalent is not difficult.

For an oriented bundle ζ with total space E, base B, and projection π, Theorem 10 can be generalized as follows:

Theorem 10': For ζ an oriented n-plane bundle as indicated,

1) $H^i(E, E_o; Z) = 0$ for $i < n$

2) There exists a unique class $U \in H^n(E, E_o; Z)$ such that $j_b^* U = U_b$ for all $b \in B$ where $j_b: F, F_o \longrightarrow E, E_o$ and $F = \pi^{-1}(b)$

3) $H^i(E; Z) \xrightarrow{\smile U} H^{i+n}(E, E_o; Z)$ is an isomorphism for all i.

(More generally, any commutative ring with unit may be used as coefficient group.) The proof will be given in the appendix.

Recall that before we had $\pi^*:H^i(B) \longrightarrow H^i(E)$ and we defined $\phi = (\smile U)\circ\pi^*:H^i(B) \longrightarrow H^{i+n}(E,E_o)$ and then working mod 2 we had $W(\zeta) = \phi^{-1}SqU$. In particular this meant $W_n = \phi^{-1}Sq^nU = \phi^{-1}(U\smile U)$ Now using our new U, this last construction can go through with coefficient group Z if we omit the reference to Sq^n.

Definition: The Euler class X of an n-plane bundle ζ is the class of $H^n(B;Z)$ defined by $X = \phi^{-1}(U\smile U)$ where U is as in Theorem ~~12~~ 10.

Remark 1: $X(\zeta)$ reduced mod 2 is $W_n(\zeta)$. $X(\zeta)$ is a strengthened Stiefel-Whitney class.

Remark 2: If n is odd, X is of order 2 since for U of odd dimension $U\smile U = -(U\smile U)$.

Theorem 12. $X = g^*i^*U$ where g is the homeomorphism of B into E given by any cross section, and $i:E \longrightarrow E,E_o$ is the injection.

Proof: Since ϕ is an isomorphism, we need only show that ϕg^*i^*U is equal to $\phi X = U\smile U$. But $\phi = (\smile U)\circ\pi^*$ so $\phi g^*i^*U = (\pi^*g^*i^*U)\smile U$ Since $g(B)$ is a deformation retract of E, $g\pi \simeq \text{identity}_E$ nd thus

$$\pi^*g^* = \text{identity}.$$

herefore we have

$$\phi g^*i^*U = (i^*U)\smile U.$$

ince i is the injection: $E \longrightarrow E,E_o$ we have $(i^*U)\smile U = U\smile U$ the up products being defined in the proper groups and therefore $g^*i^*U = U\smile U$. QED.

It is always possible to define X this way since there is always he zero cross section. If, however, g is a non-zero (never zero) cross section, $g^*i^* = 0$ and thus we have

Corollary: An oriented n-plane bundle ζ with $X(\zeta) \neq 0$ cannot have any non-zero cross section.

We won't attempt an axiomatization of Euler classes here, but note that Axiom 2), naturality under bundle maps, holds for Euler classes and Axiom 1) is satisfied except for the modification that we have an Euler class only in the dimension of the fibre. As for Axiom 3), let $V_1^m \times V_2^n$, where V_1^m and V_2^n have orientations $v_1, \ldots, v_m$ and $v_1', \ldots, v_n'$, be given the obvious orientation $v_1, \ldots, v_m, v_1', \ldots, v_n'$. By the way this means that the orientation of $V_2^n \times V_1^m$ is $(-1)^{mn}$ times the orientation of $V_1^m \times V_2^n$. Corresponding to Axiom 3) for Stiefel-Whitney classes, we have

Theorem 13: $X(\zeta \oplus \eta) = X(\zeta)X(\eta)$

The proof here is completely analogous to the proof of Theorem 11, using the same notation and the uniqueness of $U'' \in H^{m+n}(E'', E_0''; Z)$ as given by Theorem 10', to prove that $p_1^* U \smile p_2^* U' = U''$ and thus to show $X'' = X \smile X'$.

Note: Although the product formula looks completely analogous to the formula for Stiefel-Whitney classes, it works out rather differently in practice, since $W(\zeta)$ is a unit in the cohomology ring $\prod_i H^i(B; Z_2)$, the complete direct product, while $X(\zeta)$ is never a unit in $\prod_i H^i(B; Z)$. Given $X(\eta)$ and $X(\zeta \oplus \eta)$, this means it is not usually possible to solve for $X(\zeta)$.

Corollary: For ζ an oriented n-plane bundle, if $X(\zeta)$ is not of order 2, then ζ is not the sum of two odd dimensional bundles. In particular, this shows there does not exist a continuous field of oriented odd dimensional subspaces in the tangent bundle of a manifold with $X \neq 0$. (The hypothesis that the subspaces are oriented is not actually necessary.)

Corollary 2: ζ, an oriented n-plane bundle over a paracompact base B with $X(\zeta) \neq 0$, cannot have any non-zero cross section. (This gives an alternate proof for the corollary to the preceding Theorem 12 under the restricted condition that the base be paracompact.)

For if ζ has a non-zero cross section, let θ^1 be the line bundle spanned by the cross section and let η^{n-1} be the (n-1)-plane bundle orthogonal to θ^1 (in the Riemannian metric which we can assume since the base is paracompact). Since θ^1 is trivial, we obtain $X(\theta^1) = 0$ and hence the contradiction $0 \neq X(\zeta) = X(\theta^1)\, X(\eta^{n-1}) = 0$.

IX. Computations in a differentiable manifold.

1) The normal bundle

Using Theorem 12, we need knowledge of the maps

$$B \xrightarrow{g} E \xrightarrow{i} E, E_0 \quad (g = \text{zero cross section},\ i = \text{inclusion})$$

in order to study X, but this knowledge is available in a neighborhood of the zero cross section as will be seen in what follows. Let us first consider a simple case to illustrate the situation.

Let ν^k be the normal bundle to a closed differentiable manifold M^n imbedded in R^{n+k}. Instead of looking at the entire total space E, consider small vectors in each fibre, that is vectors of length $\leq \epsilon$ in the Riemannian metric which we know we can define. Denote this subset of E by $E(\epsilon)$. Similarly, the non-zero small vectors are to be denoted by $E_0(\epsilon)$. The inclusion map $E(\epsilon), E_0(\epsilon) \longrightarrow E, E_0$ is an excision so we have that $H^k(E, E_0) \approx H^k(E(\epsilon), E_0(\epsilon))$. Assuming the manifold M^n is differentiable of class C^2, since it is compact, we can pick an ϵ so that the map which assigns to each vector in $E(\epsilon)$ its endpoint in R^{n+k} is a 1-1 correspondence between $E(\epsilon)$ and a neighborhood N of M^n in R^{n+k}. Thus we have $H^k(E(\epsilon),E_0(\epsilon)) \approx H^k(N,N-M^n)$. Again by the excision axiom, we know that $H^k(N,N-M^n) \approx H^k(R^{n+k},R^{n+k}-M^n)$. Putting these three isomorphisms together we have an isomorphism

$$\psi: H^k(E,E_0) \longrightarrow H^k(R^{n+k},R^{n+k}-M^n)$$

Now assume that the normal bundle is oriented. (This is equivalent to the assumption that the tangent bundle is oriented.) Then the class $U \in H^k(E,E_0)$ is defined and determines $\psi U \in H^k(R^{n+k},R^{n+k}-M^n)$.

The inclusions

$$M^n \xhookrightarrow{i} R^{n+k} \xhookrightarrow{j} R^{n+k},R^{n+k}-M^n$$

gives maps of cohomology: $H^k(M^n) \xleftarrow{i^*} H^k(R^{n+k}) \xleftarrow{j^*} H^k(R^{n+k},R^{n+k}-M^n)$

under which, using the above isomorphisms, ψU goes into $X \in H^k(M^n)$. But $H^k(R^{n+k}) = 0$ so $X = i^*j^*\psi U = 0$. Thus we have proved

Theorem 14: If M^n is imbedded in R^{n+k} with an oriented normal bundle ν^k, then $X(\nu^k) = 0$. (Alternatively, without orientability, the same argument shows $W_k(\nu^k) = 0$, a fact we used on page 15.)

Remark: These results are true for imbedding but definitely do not carry over to immersions. For instance, consider the well known immersion of P^2 in R^3 (Boy's surface). According to the Whitney duality theorem, we have $W_1(\nu^1) \neq 0$. Recently, S. Smale has shown that S^2 can be immersed in R^4 so as to obtain any desired even multiple of the generator of $H^2(S^2;Z)$ for $X(\nu^2)$. Roughly, this multiple corresponds to the self-intersection number of S^2 as immersed. (See Bull. Amer. Math. Soc. 63, (1957), p. 196)

2) The tangent bundle of an oriented manifold.

Now let us turn our attention to the tangent bundle of a manifold M^n which is differentiable of class C^3. Such a manifold can be given* a Riemannian metric of class C^2. Let $F_b(\epsilon)$ denote the set of all tangent vectors at b of length $\leq \epsilon$. Then for ϵ sufficiently small a homeomorphism

$$F_b(\epsilon) \longrightarrow N \subset M^n$$

is defined by mapping each vector $\vec{v}$ into the endpoint of the geodesic which starts at b in the direction of $\vec{v}$ and has length $||\vec{v}||$. The image N is a neighborhood of b in M^n. Thus we have isomorphisms

$$\begin{tikzcd} H_n(F_b, F_{b,o}) \arrow[rrr, bend right, "\psi_b"'] & H_n(F_b(\epsilon), F_{b,o}(\epsilon)) \arrow[l, "\approx"'] \arrow[r, "\approx"] & H_n(N, N-b) \arrow[r, "\approx"] & H_n(M^n, M^n-b). \end{tikzcd}$$

Call the composite isomorphism, ψ_b.

We will say that M^n is oriented if its tangent bundle is oriented. If the orientation of each F_b is specified by a generator $\bar{U}_b \in H_n(F_b, F_{bo})$ then the corresponding generator $\psi_b(\bar{U}_b)$ of $H_n(M^n, M^n-b)$ will be denoted by $\bar{\mu}_b$. (Integer coefficients should be understood.)

Lemma 1. If M^n is a closed oriented differentiable manifold then there is a unique homology class $\bar{\mu} \in H_n(M^n)$ such that for each point b the inclusion homomorphism $H_n(M^n) \longrightarrow H_n(M^n, M^n-b)$ carries $\bar{\mu}$ into $\bar{\mu}_b$.

The class $\bar{\mu}$ is called the fundamental class of M^n. Proof of Lemma 1. A theorem of Cairns asserts that every differentiable manifold can be triangulated. For a recent proof see Whitney, Geometric integration theory, Princeton, 1957. However, under the hypothesis that M^n is triangulated, a proof of this Lemma has been given by Steenrod, Fibre

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* The proof is the same as our previous proof of the existence of a Riemannian metric, except that differentiable partitions of unity must be used.

Bundles, p. 200. (Steenrod works with the system of local coefficients $\pi_{n-1}(F_o)$. However the hypothesis that M^n is oriented implies that

$$\pi_{n-1}(F_o) \approx H_{n-1}(F_o) \approx H_n(F,F_o)$$

is canonically isomorphic to our coefficient group Z.) This completes the proof.

Lemma 2. If M^n is connected, as well as being closed, oriented, and differentiable, then the homology group $H_n(M^n)$ is infinite cyclic with generator $\bar{\mu}$. The cohomology group $H^n(M^n)$ is also infinite cyclic with a unique generator μ such that the Kronecker index $<\mu,\bar{\mu}>$ is + 1.

This is also proved by Steenrod (See the reference cited above. Compare Eilenberg and Steenrod, Algebraic Topology, p. 106.)

μ will be called the fundamental cohomology class of M^n. (It is definitely not defined unless the manifold is connected.)

Now consider the total space E of the tangent bundle. A map $E(\epsilon) \longrightarrow M^n \times M^n$ is defined by sending $(x,\vec{v})$ into (x,y) where y is the end point of a geodesic, as above. For ϵ sufficiently small this gives a homeomorphism of $E(\epsilon)$ onto a subset D of $M^n \times M^n$. Clearly D is a neighborhood of the diagonal Δ in $M^n \times M^n$. Thus $H^n(E(\epsilon),E_o(\epsilon))$ is isomorphic to $H^n(D,D-\Delta)$. Let ψ denote the composition of the following isomorphisms:

$$\begin{tikzcd}
H^n(E,E_o) \arrow[r, "\approx"] \arrow[rrr, bend right, "\psi"'] & H^n(E(\epsilon),E_o(\epsilon)) & H^n(D,D-\Delta) \arrow[l, "\approx"'] & H^n(M^n \times M^n, M^n \times M^n - \Delta) \arrow[l, "\approx"'] .
\end{tikzcd}$$

The class U in the first group corresponds to a class

$\psi U \epsilon H^n(M^n \times M^n, M^n \times M^n - \Delta)$. Finally define $\underline{U} = i^*\psi U$ where $i: M^n \times M^n \longrightarrow (M^n \times M^n, M^n \times M^n - \Delta)$ is the inclusion map. Thus we have

$$\begin{tikzcd} H^n(E,E_0) \arrow[r, "\psi"] & H^n(M^n \times M^n, M^n \times M^n - \Delta) \arrow[r, "i^*"] & H^n(M^n \times M^n) \\ U & & \underline{U} \end{tikzcd}$$

3) Computation of the class $\underline{U}$.

In the next sections, we will be engaged in investigating properties of Stiefel-Whitney classes and Euler classes through computation of the class $\underline{U}$. Our most important result will be Wu's formula for the Stiefel class of a manifold M^n: $W = SqV$ where V is characterized by the equation $\langle Sq\alpha, \bar{\mu} \rangle = \langle \alpha \smile V, \bar{\mu} \rangle$ for all $\alpha \epsilon H^*(M^n)$. This gives a direct computational construction for W which does not require knowledge of the tangent bundle. For Euler classes, we will elucidate a relation the reader has probably been suspecting, that of the Euler class to the Euler characteristic of a manifold. In the course of this development, we will obtain a proof of the Poincaré duality theorem.

Assume that the manifold M^n is connected. For the remainder of the section, we will consider two cases simultaneously.

Case 1: M^n is not necessarily oriented, but the coefficient group is Z_2.

Case 2: M^n is oriented and the coefficient group is a field Λ, usually the rational numbers, Q. The coefficient homomorphism $Z \longrightarrow \Lambda$ carries the fundamental class $\mu \epsilon H^n(M^n;Z)$ into a class in $H^n(M^n;\Lambda)$ which will also be denoted by μ.

In either case the group $H^n(M^n)$ is a one dimensional vector space over the coefficient field with generator μ. Let $\alpha_1, \ldots, \alpha_N$ be a basis for the cohomology of M^n. In particular, let $\alpha_1 = 1$. μ, as the generator of $H^n(M^n)$ will be some α_i. Using a field for coefficients, we know that $H^*(M) \otimes H^*(M) \longrightarrow H^*(M \times M)$ given by $a \otimes b \longrightarrow a \times b$ is an isomorphism. (This is a well-known result on the cohomology of products of finite complexes.) We can represent $\underline{U}$ consequently in terms of the generators

$\underline{U} = \sum_{i,j} c_{ij}\alpha_i \times \alpha_j$. (Since $\underline{U}$ is of dimension n, $c_{ij} = 0$ unless dim α_i + dim α_j = n.) Consider the map $f_b: M \longrightarrow M \times M$ defined by $f_b(y) = (b,y)$. As indicated in the diagram below, the compatibility condition on U reduces to $f_b^*(\underline{U}) = i_b^*\mu_b = \mu$

$$\begin{tikzcd}
U \arrow[rr] & & \underline{U} \\
H^*(E,E_o) \arrow[r, "\psi", "\approx"'] \arrow[d, "j_b^*"] & H^*(M^n \times M^n, M^n \times M^n - \Delta) \arrow[r, "i^*"] & H^*(M^n \times M^n) \arrow[d, "f_b^*"] \\
H^*(F_b, F_{b,o}) \arrow[r, "\psi_b", "\approx"'] & H^*(M^n, M^n - b) \arrow[r, "i_b^*"] & H^*(M^n) \\
U_b \arrow[r] & \mu_b \arrow[r] & \mu
\end{tikzcd}$$

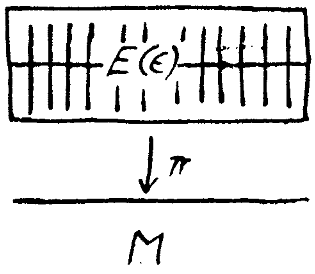
On the other hand, by the very definition of f_b it is clear that

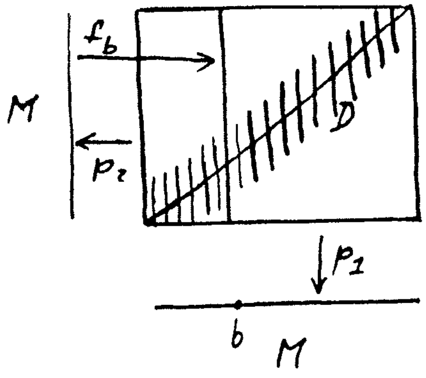
$$f_b^*(\alpha_i \times \alpha_j) = \begin{cases} 0 & \text{for dim } \alpha_i > 0 \\ \alpha_j & \text{for dim } \alpha_i = 0, \text{ that is for } \alpha_1, \text{ which is equal to 1.} \end{cases}$$

Thus the coefficient of the $1 \times \mu$ term must be 1 and we have

Formula 1: $\underline{U} = 1 \times \mu + \Sigma' c_{ij}\alpha_i \times \alpha_j$ where the summation Σ' extends over all i,j with dim $\alpha_i > 0$, dim $\alpha_j < n$.

To get more information about $\underline{U}$, consider the projections p_1, p_2 of $M^n \times M^n$ into its first and second factors respectively. Observe that $_1|D$ corresponds to $\pi|E(\epsilon)$ under the homeomorphism we have set up (see illustration). On the other hand, $p_1|\Delta = p_2|\Delta$ and since Δ is a deformation retract of D it follows that $p_1|D \simeq p_2|D$. ($\simeq$ is to be read "is homotopic to".)





Formula 2. $\underline{U} \smile (1 \times \alpha_k) = \underline{U} \smile (\alpha_k \times 1)$ for all α_k.

Proof: $1 \times \alpha_k = p_2^*(\alpha_k)$, $\alpha_k \times 1 = p_1^*(\alpha_k)$.

Consider the commutative diagram

$$\begin{tikzcd}
\overset{\psi U}{H^*(M\times M, M\times M - \Delta)} \arrow[r, "i^*"] \arrow[d, "e^*"', "\approx"] & \overset{\underline{U}}{H^*(M\times M)} \arrow[d, "d^*"] & H^*(M) \arrow[l, "p_1^*{,}p_2^*"'] \\
H^*(D, D-\Delta) \arrow[r, "j^*"] & H^*(D) &
\end{tikzcd}$$

We obtained $\underline{U}$ as the image under i^* of the class ψU in $H^n(M\times M, M\times M-\Delta)$ determined by U, so to compare $\underline{U}\smile(1\times\alpha_k)$ and $\underline{U}\smile(\alpha_k\times 1)$ we can first cup with ψU and then apply i^*. If the respective products with ψU are equal, then the products with $\underline{U}$ will also be equal. Further, since the excision homomorphism e^* is an isomorphism we can check the equality by taking d^* of $p_2^*(\alpha_k)$ and $p_1^*(\alpha_k)$ respectively and then cupping with $e^*\psi U$. Since $p_1|D \simeq p_2|D$ we know that $d^*p_1^*(\alpha_k) = d^*p_2^*(\alpha_k)$ hence the cup products with $e^*\psi U$ are equal and lifting back up into $H^*(M\times M, M\times M - \Delta)$ we have that $\psi U \smile (p_2^* \alpha_k) = \psi U \smile (p_1^* \alpha_k)$ QED.

Calling the coefficient field Λ, define a homomorphism $\gamma : H^*(M) \longrightarrow \Lambda$ by $\gamma(\alpha) = <\alpha, \overline{\mu}>$; $\overline{\mu}$, the fundamental class of $H_n(M)$. Using this homomorphism, define coefficients y_{jk} by $y_{jk} = \gamma(\alpha_j \smile \alpha_k)$. Extend γ to $(M)\otimes H^*(M)$ by $1\otimes\gamma$: $H^*(M) \otimes H^*(M) \longrightarrow H^*(M) \otimes \Lambda \approx H^*(M)$ and denote by h the corresponding homomorphism $h : H^{n+i}(M\times M) \longrightarrow H^i(M)$.

Now apply this homomorphism h to Formula 2.

On the left side we have

$$h(\underline{U} \smile (1\times\alpha_k)) = \sum_{i,j} c_{ij} h((\alpha_i \times \alpha_j) \smile (1\times\alpha_k))$$

$$= \sum_{i,j} c_{ij} h(\alpha_i \times \alpha_j \alpha_k) = \sum_{i,j} c_{ij} y_{jk} \alpha_i \ .$$

On the right side we have

$$h(\underline{U} \smile (\alpha_k \times 1)) = \sum_{i,j} (-1)^{\dim \alpha_j \dim \alpha_k} c_{ij} h(\alpha_i \alpha_k \times \alpha_j).$$

Formula 1 asserts that $c_{ij}\gamma(\alpha_j) = 0$ except for the single term

$c_{ij}\alpha_i \times \alpha_j = 1\times\mu$. Therefore

$$h(\underline{U} \smile (\alpha_k \times 1)) = (-1)^{n\dim \alpha_k} \alpha_k.$$

Comparing these two formulas we have

$$\sum_j c_{ij} y_{jk} = \begin{cases} 0 & \text{for } i \neq k \\ (-1)^{n\dim \alpha_k} & \text{for } i = k \end{cases}$$

Let C be the matrix (c_{ij}) and Y the matrix (y_{ij}). Then we have proved:

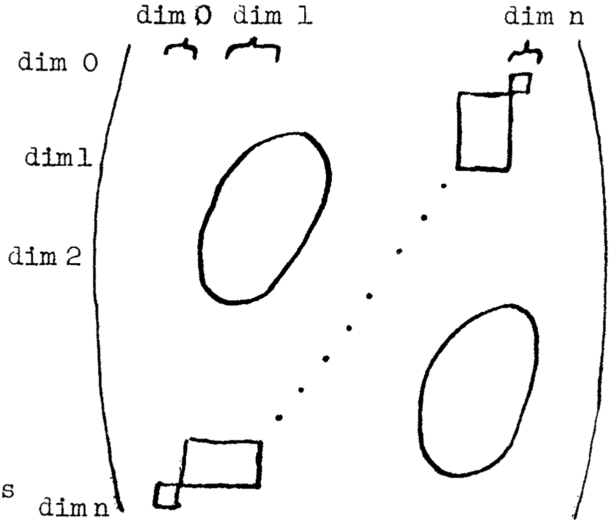
Theorem 15. The class $\underline{U} \in H^n(M^n \times M^n)$ is equal to $\Sigma c_{ij}\, \alpha_i \times \alpha_j$ where the matrix C of coefficients is, up to sign, the inverse of the matrix Y, where $y_{ik} = < \alpha_j \cup \alpha_k, \bar{\mu} >$. $CY = \begin{pmatrix} \pm 1 & & & & 0 \\ & \pm 1 & & & \\ & & \pm 1 & & \\ & & & \ddots & \\ 0 & & & & \pm 1 \end{pmatrix}$

If n is even, then C is actually the inverse of Y.

From Theorem 15, there follows as a corollary one of the classic results of combinatorial topology:

Corollary 1: (Poincaré Duality Theorem): For M^n a closed, connected manifold (oriented unless the coefficient field Λ is Z_2), the groups $H^i(M^n, \Lambda)$ and $H^{n-i}(M^n, \Lambda)$ have the same rank. Furthermore, these groups are dually paired to Λ by the correspondence $(\alpha, \beta) \longrightarrow < \alpha \cup \beta, \bar{\mu} >$.

Proof: Arrange the basis $\alpha_1, \ldots, \alpha_N$ in increasing order of dimension. Y will then have the form at the right. Since Y is a square matrix and the theorem shows it to be non-singular, each of the blocks in this form must be square, as can readily be seen from an elementary argument with matrices. Thus the ranks of the paired cohomology groups are equal and the pairing of the generators is given by $< \alpha_k \cup \alpha_j, \bar{\mu} >$



Note: We have given a proof only if M^n is differentiable of class C^3. The more general result can be obtained by somewhat finer reasoning.

4) The Euler Characteristic X.

We will now carry our computations over to the investigation of the Euler class X, and we will show

Theorem 16: If M^n is a closed connected oriented manifold, differentiable of class C^3, then the Euler class $X(\tau^n(M^n))$ is equal to the Euler characteristic X (i.e. the alternating sum of the Betti numbers) times the fundamental class $\mu \in H^n(M^n;Z)$. (This result is actually true for a C^1 manifold.)

Proof: For n odd, we have seen that X is of order 2. Since $H^n(M^n;Z)$ is infinite cyclic, this means that $X = 0$. By the Poincaré Duality Theorem, the Betti numbers in complementary dimensions all cancel out to give $X = 0$.

For n even, we will make a computational investigation of $X(\tau^n(M^n))$ using coefficients in a field, e.g. Q, the rationals. The theorem will follow for Z as coefficients since $H^n(M^n;Z) \longrightarrow H^n(M^n;Q)$ is an isomorphism into. Recall that according to Theorem 12, $X = g^*i^*U$ where $B \xrightarrow{g} E \xrightarrow{i} E,E_o$, g is the zero cross section. The following diagram relates these maps to our homeomorphism and ψU:

$$\begin{tikzcd}
U & & & \psi U & \\
H^*(E,E_0) \arrow[r, "\approx"] \arrow[d, "i^*"] & H^*(E(\epsilon),E_0(\epsilon)) \arrow[r, "\approx"] \arrow[dd] & H^*(D,D-\triangle) \arrow[ddl] & H^*(M\times M, M\times M-\triangle) \arrow[l, "\approx"'] \arrow[d, "i^*"] & \\
H^*(E) \arrow[dr, "g^*"'] & & & H^*(M\times M) \arrow[dll, "d^*"] & \underline{U} \\
 & H^*(M) & & & \\
 & X & & &
\end{tikzcd}$$

where $d: M \longrightarrow M\times M$ is the diagonal map which as can be seen corresponds to the zero cross section. Thus $X = d^*\underline{U}$.

Now representing $\underline{U}$ again by $\sum_{i,j} c_{ij}\alpha_i \times \alpha_j$, we see that $X = \sum_{i,j} c_{ij}(\alpha_i \cup \alpha_j)$ which for the y_{ij} defined as before shows that $X = \Sigma c_{ij}y_{ij}\mu$ or in terms of matrices $X = \text{Trace } (CY^{\text{Transpose}})\mu$.

Since $\dim M^n$ is even, $X = \text{Trace } (Y^{-1}Y^T)\mu$.

Arrange the basis as follows(the ordering of the basis has not been used in our work so far except in corollary 1)

$$\underbrace{\alpha_1,\dots,\alpha_r}_{\text{even dim}},\ \underbrace{\alpha_{r+1},\dots,\alpha_N}_{\text{odd dim}}\ .$$

With respect to this basis, Y has the form

$$Y = \begin{pmatrix} Y_e & 0 \\ 0 & Y_o \end{pmatrix}$$ where Y_e refers to the even dimensional elements, Y_o the odd.

$$\text{Thus} \quad Y^{-1} = \begin{pmatrix} Y_e^{-1} & 0 \\ 0 & Y_o^{-1} \end{pmatrix}$$

$$\text{and} \quad Y^T = \begin{pmatrix} Y_e & 0 \\ 0 & -Y_o \end{pmatrix}$$

because of the anticommutativity of the cup product.

Therefore

$$\text{Trace } (Y^{-1}Y^{\text{Transpose}}) = \text{Trace} \begin{pmatrix} Y_e^{-1} & 0 \\ 0 & Y_o^{-1} \end{pmatrix} \begin{pmatrix} Y_e & 0 \\ 0 & -Y_o \end{pmatrix}$$

$$= \text{Trace} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \Sigma \text{ (even Betti numbers) } -\Sigma \text{ (odd Betti numbers) } = \chi.$$

QED.

5) Wu's Formula

Returning to Stiefel-Whitney classes, recall the definition according to Thom, $SqU = U \smile \pi^* W$. Under our canonical isomorphism $H^*(E,E_o) \to H^*(M\times M, M\times M-\Delta)$ and the inclusion homomorphism $i^*: H^*(M\times M, M\times M-\Delta) \to H^*(M\times M)$, U goes into $\underline{U}$ (see page 47) and the above relation becomes $Sq\underline{U} = \underline{U} \smile (W\times 1)$.

Again applying h (see page 50) after substituting $\underline{U} = \Sigma c_{ij} \alpha_i \times \alpha_j$, we have first

$$hSq(\Sigma c_{ij} \alpha_i \times \alpha_j) = W.$$

Using known properties of Sq, this gives $W = \Sigma c_{ij} h(Sq\alpha_i \times Sq\,\alpha_j)$.

Defining $s_j = \gamma(Sq\alpha_j) = < Sq\,\alpha_j, \bar{\mu} >$ we can rewrite our formula as

$$W = \Sigma c_{ij} s_j Sq\alpha_i,$$

or writing $V = \Sigma c_{ij} s_j \alpha_i$ we have $W = SqV$. Now, following Wu, observe that V is characterized by the equation $< Sq\,\alpha, \bar{\mu} > = < \alpha \smile V, \bar{\mu} >$.

In each dimension i the correspondence $\alpha \longrightarrow \langle Sq^{n-i}\alpha, \bar{\mu} \rangle$ defines an additive homomorphism of $H^i(M^n; Z_2)$ into Z_2. According to the Poincaré duality theorem there is a unique element $V_{n-i} \in H^{n-i}(M^n; Z_2)$ such that

$$\langle Sq^{n-i}\alpha, \bar{\mu} \rangle = \langle \alpha \smile V_{n-i}, \bar{\mu} \rangle$$

for each α. (Note that $V_o = 1$, $V_{n-i} = 0$ for $n-i > i$.) Defining $V = V_o + V_1 + \cdots + V_n = 1 + V_1 + \cdots + V_{[n/2]}$ this formula becomes $\langle Sq\,\alpha, \bar{\mu} \rangle = \langle \alpha \smile V, \bar{\mu} \rangle$ for all $\alpha \in H^*(M^n; Z_2)$. The element V defined in this way is equal to $\Sigma c_{ij}\, s_j \alpha_i$. Certainly V can be expressed in the form $\Sigma v_k \alpha_k$ for some coefficients v_k. Then the identity

$$\langle Sq\,\alpha_j, \bar{\mu} \rangle = \langle \alpha_j \smile V, \bar{\mu} \rangle$$

can be rewritten as

$$s_j = \sum_k y_{jk} v_k.$$

Now multiplying on the left by c_{ij} and summing over j we have

$$\sum_j c_{ij} s_j = \sum_k \delta_{ik} v_k = v_i. \qquad \text{QED}$$

Hence we have

Theorem 17 (Wu): $W(M) = SqV$ where V is characterized by the equation $\langle \alpha \smile V, \bar{\mu} \rangle = \langle Sq\,\alpha, \bar{\mu} \rangle$ for all $\alpha \in H^*(M)$. Since W is thus defined entirely in terms of cohomology and homology operations, we have:

Corollary: The Stiefel-Whitney classes of manifolds are invariants of the homotopy type.

Examples:

$P^n(C)$: For complex projective 4-space (eight real dimensions) we have the following system of generators: $1 \epsilon H^0$, $\alpha \epsilon H^2, \alpha^2 \epsilon H^4, \alpha^3 \epsilon H^6$ $\alpha^4 = \mu \epsilon H^8$, on which Sq operates as follows

$$Sq\,1 = 1,\ Sq\alpha = \alpha + \alpha^2,\ Sq\alpha^i = \alpha^i(1+\alpha)^i .$$

Thus

$$Sq^8 1 = 0 \text{ and } V_8 = 0,\ Sq^6\alpha = 0 \text{ and } V_6 = 0$$

$$Sq^4\alpha^2 = \alpha^4 \text{ and } V_4 = \alpha^2,\ Sq^2\alpha^3 = \alpha^4 \text{ and } V_2 = \alpha,$$

$$\text{or } V = 1 + \alpha + \alpha^2 .$$

Thus

$$W = SqV = 1 + (\alpha + \alpha^2) + \alpha^2(1 + 2\alpha + \alpha^2)$$

$$= 1 + \alpha + \alpha^4 .$$

In general, to calculate $W(P^n(C))$ we go through a procedure which is formally identical with the calculation for $W(P^n(R))$. But we already know the results in that case; thus we have:

<u>Theorem 8</u>: $W(P^n(C)) = (1 + \alpha)^{n+1}$ for α the non-zero class in H^2

Similarly $W(P^n(\text{Quaternions})) = (1+\alpha)^{n+1}$ for α the non-zero class in H^4

$W(\text{Cayley plane}) = 1 + \alpha + \alpha^2$ for α the non-zero class in H^8.

(These are the only known examples of differentiable manifolds M^n such that $H^*(M^n;Z_2)$ is a truncated polynomial ring. In fact, according to a theorem of Adem, if a complex K exists such that $H^*(K;Z_2)$ is generated by $\alpha \epsilon H^r$, $r \geq 1$, with relation $\alpha^{k+1} = 0$, $k \geq 2$, then r must be a power of 2. If $k > 2$, then r must be 1,2, or 4. Thus for $r < 16$ the above manifolds give the only possible truncated polynomial rings.)

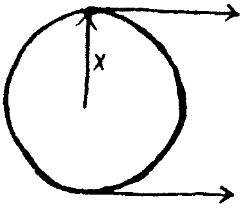
X. Obstructions:

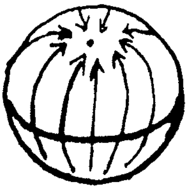
In the section which follows, we will assume familiarity with the definitions of obstruction and primary obstruction. (See, for example, Steenrod, Topology of Fibre Bundles §32,35). With terminology close to that of Steenrod, p. 190, given an n-plane bundle ζ^n we have for each $q < n$ the associated bundle $\mathcal{B}^q$ with base B and fibre $V'_{n,n-q}$, the Stiefel manifold of (n-q)-frames in n-space. By an (n-q)-frame, we mean just a set of n-q linearly independent vectors. (Note: Steenrod uses orthogonal unit (n-q)-frames in n-space; the modification does not affect the argument.) Explicitly, a point in the associated bundle fibre over $b \in B$ can be represented as (b, frame $(v_1,\ldots,v_{n-q})$ in the n-plane $\pi^{-1}(b)$). The primary obstruction to a cross section of $\mathcal{B}^q$ is an element o_{q+1} of $H^{q+1}(B;\pi_q(V'_{n,n-q}))$. This coefficient group is either Z or Z_2, depending on the dimensions. In general these are twisted coefficients, but this complication can be avoided by reducing mod 2; this we write as $(o_i)_2$. (In general, we lose nothing by this reduction since o_i can be recovered from $(o_i)_2$ but for the one dimension where we can calculate X. See Steenrod p. 195.) Now it is possible to interpret Stiefel-Whitney classes as follows:

Theorem 19: $o_i(\zeta)_2 = W_i(\zeta)$

Proof: Consider the bundle map $f = (f_B, f_E)$ mapping ζ into γ^n the canonical bundle over G_n. Since obstructions are natural with respect to bundle maps, we have $f_B^* \, o_i(\gamma^n)_2 = o_i(\zeta)_2$. Since $H^*(G_n;Z_2)$ is a polynomial algebra in the W_j, for each pair i,n we have that $o_i(\gamma^n)_2$ can be given as a polynomial $p_{i,n}$ in the Stiefel-Whitney classes $W_j(\gamma^n)$. The above relation shows that $o_i(\zeta)_2 = p_{i,n}(W_1(\zeta),\ldots,W_n(\zeta))$ and this formula is valid for all n-plane bundles, dependent only on i and n. We need to know the exact form of this polynomial, but this can be determined from a special case. For fixed i, let $B = G_{i-1}$ and $\zeta^n_i = \gamma^{i-1} \oplus \theta^{n-i+1}$ where θ^{n-i+1} is the trivial $(n-i+1)$-plane bundle. Now in general the associated bundle $\mathcal{B}^q$ has a cross section if and only if ζ can be split into a bundle sum with the trivial $n-q$ bundle θ^{n-q} as one summand. (Given the cross-section and using the usual Riemannian metric defined in an n-plane bundle over a paracompact base, we can split ζ by taking the orthogonal complement to the $n-q$ dimensional subspace spanned by the frame specified by the cross section. Conversely, the decomposition specifies a non-zero cross section of $\mathcal{B}^q$ by taking the $n-q$ frames which are the bases for the fibres of θ^{n-q}.) Thus we see that $o_i(\zeta^n_i) = 0$ and therefore $o_i(\zeta^n_i)_2 = 0$. On the other hand $W_j(\zeta^n_i) = W_j(\gamma^{i-1})$. Together this means that $0 = p_{i,n}(W_1(\gamma^{i-1}), \ldots, W_{i-1}(\gamma^{i-1}), 0, 0, \ldots, 0)$ where $W_1, \ldots, W_{i-1}$ generate a polynomial algebra. Since o_i is always of dimensional i, $p_{i,n}$ must have the form $p_{i,n}(x_1, \ldots, x_n) = \lambda x_i + p'_{i,n}(x_1, \ldots, x_{i-1})$. Now the equality $0 = p_{i,n}(x_1, \ldots, x_{i-1}, 0, \ldots, 0)$ implies that $p'_{i,n}$ must be identically zero. Thus we have proved: for each i,n there is a number $\lambda_{i,n}$ such that the identity $o_i(\zeta)_2 = \lambda_{i,n} W_i(\zeta)$ holds for all n-plane bundles.

A) Let $i = n$. We know that $\lambda_{i,n} = 1$ or 0. To prove the theorem in this case we need only show that for each i, there exists a bundle ζ^i with $o_i(\zeta^i)_2 \neq 0$. Let $B = P^i$ and let $\pi^{-1}(b)$ be the set of all vectors orthogonal to x in R^{i+1}, where P^i is considered as the unit S^i with antipodal points identified i.e. $b = [x, -x]$. We can start with a cross section on the $(i-1)$-skeleton as illustrated in the second figure. This extends without trouble until we reach a singularity at the poles which can be seen to correspond to a generator of the homotopy group. Thus $o_i(\zeta^i)_2 \neq 0$ for this particular ζ^i.





B) Suppose $n > i$. Repeat with $\zeta^n = \eta^i \oplus \theta^{n-i}$ where η^i is the bundle of A) . By the same reasoning, $o_i(\zeta^n)_2 \neq 0$ and so for all i,n we have shown $\lambda_{i,n} = 1$ or $o_i(\zeta)_2 = W_i(\zeta)$.

To follow the same procedure in order to relate the Euler class to an obstruction, we must work with the integers as coefficients and vill introduce the oriented analogue of γ^n. Let $\tilde{G}_n$ be the set of all oriented n-planes in R^∞ with topology defined to correspond to that of $_n$. As can easily be seen, the obvious map $\tilde{G}_n \longrightarrow G_n$ is a two-fold overing. Call $\tilde{\gamma}^n$, the bundle induced by this map from γ^n. Note that ^{n} is naturally an oriented bundle. For an oriented bundle ζ, we can ift the map $B \longrightarrow G_n$ into $\tilde{G}_n$ by using the orientation of the fibre $_b$ to determine which leaf of G_n to map b into (the local compatibility f orientations insures that this will be a continuous map). From this, t is easy to complete the diagram to et an oriented bundle map $\zeta \longrightarrow \tilde{\gamma}^n$.

$$\begin{tikzcd}
{} \arrow[dd, "\zeta"'] & {} \arrow[d, "\gamma^n"] \\
 & \tilde{G}_n \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & G_n
\end{tikzcd}$$

eorem 20: For ζ an oriented n-plane bundle over a paracompact base there is an orientation preserving bundle map f into $\tilde{\gamma}^n$ with f_B e canonical lifting of the map $B \longrightarrow G_n$.

Gysin Sequence:

Using Z as coefficients throughout this section and assuming ζ to be oriented, we have determined an element $U \in H^n(E,E_0;Z)$ and know that $\smile U: H^i(E) \longrightarrow H^{n+i}(E,E_0)$ is an isomorphism, as is $\phi: H^i(B) \approx H^{n+i}(E,E_0)$. From the exact sequence of the pair E,E_0:

$$\begin{tikzcd}
{} \arrow[r] & H^i(E,E_0) \arrow[r, "i^*"] & H^i(E) \arrow[r] & H^i(E_0) \arrow[r] & H^{i+1}(E,E_0) \arrow[r] & {} \\
{} \arrow[r] & H^{i-n}(B) \arrow[r, "\smile X"] \arrow[u, "\wr"', "\phi"] & H^i(B) \arrow[r, "\pi_0^*"] \arrow[u, "\wr"', "\pi^*"] & H^i(E_0) \arrow[r] \arrow[u, equal] & H^{i-n+1}(B) \arrow[r] \arrow[u, "\wr"', "\phi"] & {}
\end{tikzcd}$$

we get the lower exact sequence by the indicated isomorphisms.

The indicated map is $\smile X$ since

$$\pi^{*-1} i^* \phi\alpha = \pi^{*-1} i^*(\pi^*\alpha \smile U) = \pi^{*-1}(\pi^*\alpha \smile i^* U) = \alpha \smile \pi^{*-1} i^* U = \alpha \smile X \ .$$

That is,

Theorem 21 (Gysin): For an oriented n-plane bundle we have an exact sequence

$$\longrightarrow H^i(B) \xrightarrow{\smile X} H^{i+n}(B) \xrightarrow{\pi_0^*} H^{i+n}(E_0) \longrightarrow H^{i+1}(B) \longrightarrow$$

where π_0 is the restriction $\pi | E_0$.

Note: For unoriented bundles, we would get a corresponding exact sequence using Z_2 as coefficients and W_n in place of X.

The Euler class as an obstruction.

Now consider the top obstruction class $o_n(\zeta^n) \in H^n(B; \pi_{n-1}(V'_{n,1}))$. For an oriented bundle the coefficient group

$$\pi_{n-1}(V'_{n,1}) = \pi_{n-1}(F_o) \approx H_{n-1}(F_o) \approx H_n(F,F_o)$$

is canonically isomorphic to the integers Z. Hence the following statement makes sense.

Theorem 22: For an oriented n-plane bundle ζ^n, we have $o_n(\zeta^n) = X(\zeta^n)$.

Proof: Consider the Gysin sequence in the special case $B = \tilde{G}_n$.

$$\longrightarrow H^o(\tilde{G}_n) \xrightarrow{\smile X} H^n(\tilde{G}_n) \xrightarrow{\pi_o^*} H^n(E_o) \longrightarrow$$

We want to show the special case of the theorem, $X = o_n(\tilde{\gamma}^n) \in H^n(\tilde{G}_n; \pi_{n-1}(V'_{n,1}))$.

First we show $\pi_o^*(o_n(\tilde{\gamma}^n)) = 0$. Let η be the bundle over E_o induced by π_o from $\tilde{\gamma}^n$. By definition of the induced bundle, a point in $E(\eta)$ is a pair (e,e') where e is a point in $E_o(\tilde{\gamma}^n)$ and e' is any point in $E(\tilde{\gamma}^n)$ which belongs to the same fibre. The projection $E(\eta) \longrightarrow B(\eta) = E_o(\tilde{\gamma}^n)$ is given by $(e,e') \longrightarrow e$. Now the map $B(\eta) \longrightarrow E(\eta)$ given by $e \longrightarrow (e,e)$ is clearly a non-zero section of η. Therefore $o_n(\eta) = 0$, but by naturality with respect to bundle maps $o_n(\eta) = \pi_o^* o_n(\tilde{\gamma}^n)$. Hence $\pi_o^* o_n(\tilde{\gamma}^n) = 0$ as asserted.

By exactness of the above sequence, this implies that $o_n(\tilde{\gamma}^n) = \lambda \smile X$ for some $\lambda \in H^o(\tilde{G}_n)$. That is $o_n = \lambda_n X$ where λ_n is an integer since

$H^0(\tilde{G}_n) \approx Z$. We write λ_n to emphasize that the integer λ_n depends on the dimension of the bundle and not on the particular bundle, since the above formula relating o_n and X holds for all bundles by naturality (cf. the similar discussion for Stiefel-Whitney classes). Thus we can determine λ_n from special cases:

For n even, consider the tangent bundle $\tau^n(S^n)$. By Theorem 16, we know that X is $X(S^n)$ times the fundamental class, that is twice the fundamental class. On the other hand it is easy to verify that o_n is also twice the fundamental class in this case. Therefore

$$\lambda_n = +1.$$

For n odd, $\lambda_n = 0$ or 1 since X is already of order 2. To show $\lambda_n = 1$ we need only show that o_n is not zero for all n-plane bundles, but we have already done this while relating Stiefel-Whitney classes to obstruction. In fact, we even showed $(o_n)_2$ was not identically 0. Thus we have shown that the relation $o_n(\zeta^n) = X(\zeta^n)$ holds true for all n.

XI. Complex n-plane bundles

For many investigations in other branches of mathematics, e.g. the study of complex analytic manifolds, the structure of a real n-plane bundle is not a sufficient tool; it is therefore natural to give the following generalization of the definition of an n-plane bundle:

Definition: A complex n-plane bundle ω^n consists of a triple $\{E,B,\pi\}$ where π is a map from a Hausdorff space E onto a Hausdorff space B together with the structure of a complex n-dimensional vector space in each fibre $\pi^{-1}(b)$ satisfying the further conditions

1) there exist a distinguished class of open sets $\{U\}$ covering B and n maps $g_i : U \longrightarrow E$ for each U such that

2) each g_i is a cross section and

3) the map $U \times C^n \longrightarrow \pi^{-1}(U)$ defined by $(b, \lambda_1, \dots, \lambda_n) \longrightarrow \Sigma \lambda_i g_i(b)$, where $\lambda_i \in C$, is a homeomorphism.

Note: Throughout these notes we will represent the complex numbers by C.

Example: The tangent bundle τ^n of a complex analytic manifold M^n. A complex analytic n-manifold is defined analogously to a differentiable manifold except that we use n complex variables as local coordinates, and require that the functions relating the local coordinate systems must be analytic.

Remark: A complex n-plane bundle ω^n can be regarded as a real 2n-plane bundle ω^n_R by ignoring the multiplication by complex numbers.

Canonical orientation of ω^n_R

We can choose a basis $a_1, a_2, \dots, a_n$ over C for each fibre $\pi^{-1}(b)$. The real fibre, that is, the underlying real vector space of $\pi^{-1}(b)$, has a canonical orientation $a_1, ia_1, a_2, \dots, a_n, ia_n$. This orientation is independent of the choice of the complex basis $a_1, a_2, \dots, a_n$, since $GL(n,C)$ is connected and we can pass from this basis to any other continuously i.e. without change in sign.

Corollary: Every complex manifold has a standard orientation. As we have already seen in the real case, an orientation of the tangent bundle corresponds to an orientation of the manifold.

Corollary: For every complex n-plane bundle ω^n there is a well defined Euler class $X(\omega^n_R) \epsilon H^{2n}(B;Z)$.

If we take the bundle sum $\omega^n \oplus \phi^k$ of two complex plane bundles ω^n and ϕ^k, with bases $a_1,\ldots,a_n$ and $b_1,\ldots,b_k$ the vectors $a_1,\ldots,a_n,b_1,\ldots,b_k$ form a basis for $\omega^n \oplus \phi^k$. This means that the canonical orientation of $(\omega^n \oplus \phi^k)_R$ is $a_1, ia_1, \ldots, a_n, ia_n, b_1, ib_1, \ldots, b_k, ib_k$. Thus we see that in a natural way $(\omega^n \oplus \phi^k)_R \approx \omega^n_R \oplus \phi^k_R$ as oriented bundles. (This was one reason for defining the canonical orientation this way.) From this it follows that $X(\omega^n \oplus \phi^k)_R = X(\omega^n_R)X(\phi^k_R)$

Chern classes

We will now give an inductive definition of characteristic classes for a complex n-plane bundle, ω^n. We define a canonical complex (n-1)-plane bundle ω_0^{n-1} over $E_0(\omega^n)$. (As in the real case, $E_0(\omega^n)$ denotes the set of all non-zero vectors in $E(\omega^n) = E(\omega^n_R)$.)

A point in E_0 is specified by a fibre of ω^n and a non-zero vector in that fibre. We will obtain ω_0^{n-1} by considering the orthogonal (n-1)-space in that fibre. This can be done using the Hermitian metric, which can be defined in any complex n-plane bundle over a paracompact base B by a procedure analogous to that for real n-plane bundles (see Theorem 5). Alternatively it can be obtained algebraically by looking at the factor space. $E(\omega_0^{n-1})$ will consist of all pairs $(e_1, e_2 + Ce_1)$ where e_1 is the non-zero vector, e_2 is another vector in the same fibre and $e_2 + Ce_1$ is a coset. The projection $\pi' : E(\omega_0^{n-1}) \longrightarrow E_0(\omega^n)$ is defined by $\pi'(e_1, e_2 + Ce_1) = e_1$.

Recall that for real oriented 2n-plane bundles, we have a Gysin sequence

$$\longrightarrow H^{i-2n}(B) \xrightarrow{\smile X} H^i(B) \xrightarrow{\pi_0^*} H^i(E_0) \longrightarrow H^{i+1-2n}(B) \longrightarrow .$$

For $i \leq 2n-2$, $H^{i+1-2n}(B) = 0$ and so $\pi_o^*: H^i(B) \approx H^i(E_o)$.

__Definition__: The __Chern classes__ $c_i(\omega^n) \in H^{2i}(B,Z)$ are defined as follows, by induction on n.

$$c_i(\omega^n) = \begin{cases} 0 & \text{for} \quad i > n \\ X(\omega^n_R) & \text{for} \quad i = n \\ \pi_o^{*-1} c_i(\omega_o^{n-1}) & \text{for} \quad i < n. \end{cases}$$

The last expression is well defined since

$$\pi_o^* : H^{2i}(B) \longrightarrow H^{2i}(E_o)$$

is an isomorphism for $i < n$. The expression $c(\omega^n) = 1 + c_1(\omega^n) + \cdots + c_n(\omega^n)$ is called the total Chern class of ω^n.

__Lemma 1__: Chern classes are natural with respect to bundle maps i.e. for a bundle map $f = (f_E, f_B): \omega \longrightarrow \omega'$ we have $f_B^* \, c(\omega') = c(\omega)$.

__Proof__: 1) $f_B^* c_n(\omega') = c_n(\omega)$ since Euler classes are natural.

2) $f_E: E_o \longrightarrow E_o'$ can be covered by a bundle map $\omega_o \longrightarrow \omega_o'$, between the canonical (n-1)-plane bundles over E_o and E_o'. But $c_{n-1}(\omega^n) = \pi_o^{*-1} c_{n-1}(\omega_o^{n-1})$ and $c_{n-1}(\omega_o^{n-1}) = X(\omega^{n-1}_{o\,R})$ which is natural with respect to bundle maps. Since $f_B \pi_o = \pi_o' f_E$ we see that $c_{n-1}(\omega^n)$ is natural with respect to bundle maps. Descending this way, we show naturality of each $c_i(\omega^n)$ and so naturality of the total class $c(\omega^n)$.

$$\begin{tikzcd} E_o \arrow[r, "f_E"] \arrow[d, "\pi_o"] & E_o' \arrow[d, "\pi_o'"] \\ B \arrow[r, "f_B"'] & B' \end{tikzcd}$$

Lemma 2: Let θ^k be the trivial complex k-plane bundle, then $c(\omega^n \oplus \theta^k) = c(\omega^n)$.

Proof: It is sufficient to prove the assertion for θ^1 since the general case then follows by induction. Changing the notation for convenience, write $\omega^n = \phi^{n-1} \oplus \theta^1$. We want to show that $c(\omega^n) = c(\phi^{n-1})$. Since the bundle ω^n_R has a non-zero cross-section it is certainly true that $c_n(\omega^n) = X(\omega^n_R) = 0$ is equal to $c_n(\phi^{n-1})$. Let $f_B \colon B \longrightarrow E_0(\phi^{n-1} \oplus \theta^1)$ be the canonical cross-section. Then f_B is covered, in an obvious way, by a bundle map $\phi^{n-1} \longrightarrow \omega_0^{n-1}$. Thus by Lemma 1, $f_B^* c(\omega_0^{n-1}) = c(\phi^{n-1})$. But for $i < n$, $c_i(\omega_0^{n-1})$ is equal to $\pi_0^* c_i(\omega^n)$ by definition; so that

$$c_i(\omega^n) = (f_B^* \pi_0^*) c_i(\omega^n) = f_B^*(\pi_0^* c_i(\omega^n))$$
$$= f_B^*(c_i(\omega_0^{n-1})) = c_i(\phi^{n-1}).$$

This completes the proof.

We continue our complex analogy of real bundle theory with the following.
Definition: The complex Grassman manifold $G_{n,k}(C)$ is the set of all n-dimensional subspaces in C^{n+k} (When working with complex structures, dimensional notation will always refer to complex dimension unless otherwise stated.)

Just as in the real case, $G_{n,k}(C)$ has a natural structure as a differentiable manifold; in fact, $G_{n,k}(C)$ has a natural structure as a complex analytic manifold. For example, still paralleling the real case, $G_{1,k}(C) \approx P^k(C)$ is a complex projective space.

Similarly let $\gamma_k^n(C)$ be the n-plane bundle over $G_{n,k}(C)$, where

$E(\gamma^n_k(C))$ is the set of all pairs (n-dim subspace, vector in that subspace).

We investigate the structure of $H^*(P^k(C);Z)$. Applying the Gysin sequence to γ^1_k over $G_{1,k}(C) \approx P^k(C)$ and using the fact $X(\gamma^1_{k\,R}) = c_1(\gamma^1_k)$ we have

$$\longrightarrow H^{i+1}(E_0) \longrightarrow H^i(P^k(C)) \xrightarrow{\smile c_1} H^{i+2}(P^k(C)) \xrightarrow{\pi_0^*} H^{i+2}(E_0) \longrightarrow .$$

The space $E_0 = E_0(\gamma^1_k(C))$ is the set of all pairs (complex line through origin in C^{k+1}, non-zero vector in that line). Clearly, this is just the set C^{k+1}_0 of all non-zero vectors, which has the same homotopy type as S^{2k+1}. Hence E_0 has the same cohomology ring as S^{2k+1}. Thus the sequence becomes

$$0 \longrightarrow H^i(P^k(C)) \longrightarrow H^{i+2}(P^k(C)) \longrightarrow 0 \quad \text{for} \quad 0 \leq i \leq 2k-2.$$

That is, $H^0(P^k(C)) \approx H^2(P^k)) \approx \cdots \approx H^{2k}(P^k(C))$ and each group $H^{2i}(P^k(C))$ is infinite cyclic generated by $c_1(\gamma^1_k)^i$. For $i = -1$ and $k > 0$ the sequence becomes

$$\begin{tikzcd} {} \arrow[r] & H^{-1}(P^k(C)) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H^1(P^k(C)) \arrow[r] & H^1(E_0) \arrow[r] \arrow[d, phantom, "\shortparallel"] & {} \\ & 0 & & 0 & \end{tikzcd}$$

Combining this with the isomorphism

$$H^1(P^k(C)) \approx H^3(P^k(C)) \approx \cdots \approx H^{2k-1}(P^k(C))$$

we obtain $H^{2i+1}(P^k(C)) = 0$ for all i. That is:

<u>Theorem 23</u>: $H^*(G_{1,k}(C)) = H^*(P^k(C))$ is the truncated polynomial ring terminating in dimension $2k$ and generated by $c_1(\gamma^1_k(C))$.

A formally identical procedure can be carried through in the real case to show that $H^*(P^k;Z_2)$ is the truncated polynomial ring terminating in dimension k and generated by α, the non-zero element of $H^1(P^k;Z_2)$. In particular this means that $\alpha^2, \alpha^3, \cdots, \alpha^k$ are all different from zero, a fact of which we made extensive use in sections III and IV.

If we let $k \longrightarrow \infty$ we have shown explicitly that $H^*(G_1(C))$ is the polynomial ring generated by $c_1(\gamma^1(C))$. In general we will show

<u>Theorem 24</u>: $H^*(G_n(C))$ is the polynomial ring generated by $c_1(\gamma^n(C)), \cdots, c_n(\gamma^n(C))$.

<u>Proof</u> (by induction): We have already shown it to be true for $n = 1$. Using the Hermitian metric defined in C^{n+k}, i.e. $(\lambda_1, \ldots, \lambda_{n+k}) \cdot (\mu_1, \cdots, \mu_{n+k}) = \sum_{i=1}^{n+k} \lambda_i \bar{\mu}_i$, we know what is meant by orthogonality. For a point of $E_o(\gamma_k^n(C))$ given by an n-dimensional subspace of C^{n+k} and a non-zero vector therein, we take the complementary (orthogonal)(n-1)-dimensional subspace in the given subspace and thus obtain a map $\rho: E_o \longrightarrow G_{n-1,k+1}(C)$. On the other hand, given an (n-1)-plane in C^{n+k}, any orthogonal non-zero vector determines an n-plane and hence a point of E_o. In other words, ρ is a fibre map and the fibre is C_o^{k+1}. For $i \leq 2k$, the Gysin sequence of this bundle gives $\rho^*: H^i(G_{n-1,k+1}(C)) \approx H^i(E_o)$.

Letting $k \longrightarrow \infty$ as usual, we have

$$\longrightarrow H^i(G_n(C)) \xrightarrow{\smile c_n} H^{i+2n}(G_n(C)) \xrightarrow{\rho^{*-1}\pi_o^*} H^{i+2n}(G_{n-1}(C)) \longrightarrow H^{i+1}(G_n(C)) \rightarrow$$

Referring to diagram 4, we see that by naturality of Chern classes under bundle maps, $\rho^{*-1}\pi_o^*$ takes the Chern classes of $\gamma^n(C)$ into those of $\gamma^{n-1}(C)$ which by the induction hypothesis, are the generators of $H^*(G_{n-1}(C))$. This means that $\rho^{*-1}\pi_o^*$ is an epimorphsim [onto $H^*(G_{n-1}(C))$]. In other words the exact sequence becomes:

$$\xrightarrow{0} H^i(G_n) \xrightarrow{\smile c_n} H^{i+2n}(G_n) \longrightarrow H^{i+2n}(G_{n-1}) \xrightarrow{0} .$$

$$\begin{tikzcd}
{} \arrow[d, "\gamma_o^{n-1}"'] & & {} \arrow[d, "\gamma^{n-1}"] \\
E_o(\gamma^n) \arrow[rr, "\rho"'] \arrow[d, "\pi_o"] & & G_{n-1}(C) \\
G_n(C) & &
\end{tikzcd}$$

Diagram 4

We want to show 1) that every element a of $H^*(G_n(C))$ is a polynomial in $c_1,\ldots,c_n$ and 2) that no non-trivial polynomial is zero. We will prove both assertions by induction; 1) will be proved by induction on the dimension of a. [At the same time, we have the induction hypothesis on the structure of $H^*(G_{n-1}(C))$].

Certainly the assertion is true for dim a = -1. Since $\rho^{*-1}\pi_o^*(a)\epsilon H^*(G_{n-1}(C))$, it is a polynomial in $c_1(\gamma^{n-1}(C)),\ldots,c_{n-1}(\gamma^{n-1}(C))$ i.e. $\rho^{*-1}\pi^*(a) = p(c(\gamma^{n-1}(C)),\ldots,c(\gamma^{n-1}(C)))$. To simplify notation we will write c_i for $c_i(\gamma^n(C))$ and c_i' for $c_i(\gamma^{n-1}(C))$, and λ for $\rho^{*-1}\pi_0^*$. Thus we have shown $\lambda(a)$ can be written as some polynomial $p(c_1',c_2',\ldots,c_{n-1}')$. Consider $a' = a-p(c_1,\ldots,c_{n-1})\epsilon H^*(G_n(C))$. We see that $\lambda(a')= o$ which by exactness of the above sequence means there is some $a''\epsilon H^*(G_n(C))$ such that $a' = a''\smile c_n$. Now a'' has a smaller dimension than a and hence by our special induction for 1) can be written as a polynomial in $c_1,\ldots,c_n$. Therefore $a' = a''\smile c_n$ is a polynomial in $c_1,\ldots,c_n$. But this implies $a = a' + p(c_1,\ldots,c_{n-1})$ is a polynomial in $c_1,\ldots,c_n$. QED

As for 2), suppose $p(c_1,\ldots,c_n) = 0$. Then $\lambda[p(c_1,\ldots,c_n)] = p(c_1',\ldots,c_{n-1}',0) = 0$. This means that $p(*,\ldots,*,0)$ must be identically zero as a polynomial. In other words, $p(x_1,\ldots,x_n)$ has x_n as a factor; $p = x_np'$. Again we use a subsidiary induction, this time on the dimension of p. Certainly 2) holds for dim $p = -1$. Having $p = x_np'$, we know $p(c_1,\ldots,c_n) = p'(c_1,\ldots,c_n)\smile c_n = 0$. Since $\smile c_n$ is a monomorphism, this means $p'(c_1,\ldots,c_n) = 0$. By the induction hypothesis, $p'\equiv 0$ thus $p = p'\smile c_n \equiv 0$. QED

Just as for real n-plane bundles we prove:

Theorem 25: Every complex n-plane bundle over a paracompact base has a bundle map into $\gamma^n(C)$ covering the generalized Gauss map into $G_n(C)$. (As in the real case, $G_n(C)$ is assumed to have the weak topology.)

Product theorem for Chern classes

We will use this universal bundle construction to prove the product theorem for Chern classes. Let ω^m and υ^n be complex plane bundles over the same paracompact base B. Then there exist bundle maps $\omega^m \longrightarrow \gamma^m$ and $\upsilon^n \longrightarrow \gamma^n$. (The C's will be omitted whenever they are clear from the context.) The corresponding maps $B \longrightarrow G_m$, $B \longrightarrow G_n$ of the base space combine to give a map

$$f_B\colon B \longrightarrow G_m \times G_n.$$

Let γ_1^m and γ_2^n be the bundles over $G_m \times G_n$ induced by the projection maps $p_1\colon G_m \times G_n \longrightarrow G_m$, $p_2\colon G_m \times G_n \longrightarrow G_n$ respectively. Then we have a bundle map $\omega^m \longrightarrow \gamma_1^m$, where the dotted arrow in the following diagram is defined so that the diagram is commutative.

$$\begin{tikzcd}
E(\omega^m) \arrow[rr] \arrow[dr, dashed] \arrow[dd, dash] & & E(\gamma^m) \arrow[dd, dash] \\
& E(\gamma_1^m) \arrow[ur] \arrow[dd] & \\
B \arrow[rr] \arrow[dr, "f_B"'] & & G_m \\
& G_m \times G_n \arrow[ur, "p_1"'] &
\end{tikzcd}$$

Similarly we have a bundle map $\upsilon^n \longrightarrow \gamma_2^n$, and hence a bundle map $\omega^m \oplus \upsilon^n \longrightarrow \gamma_1^m \oplus \gamma_2^n$.

Thus we have proved the following: The bundles γ_1^m and γ_2^n over $G_m \times G_n$ are universal for pairs of bundles, in the sense that given any two bundles ω^m and υ^n of the same dimensions over a paracompact space

B, there exist bundle maps $\omega^m \longrightarrow \gamma_1^m$, $\upsilon^n \longrightarrow \gamma_2^n$ and $\omega^m \oplus \upsilon^n \longrightarrow \gamma_2^m \oplus \gamma_2^n$ covering the same map f_B of the base space.

The existence of these bundle maps, together with Lemma 1, gives us

$$(1) \quad c(\omega^m) = f_B^* \, c(\gamma_1^m), \; c(\upsilon^n) = f_B^* c(\gamma_2^n), \; c(\omega^m \oplus \upsilon^n) = f_B^* \, c(\gamma_1^m \oplus \gamma_2^n).$$

Since $H^*(G_m)$ and $H^*(G_n)$ are both polynomial rings generated by the respective Chern classes, it follows that $H^*(G_m \times G_n)$ is a polynomial ring generated by the Chern classes

$$c_1(\gamma_1^m), \ldots, c_m(\gamma_1^m), \; c_1(\gamma_2^n), \ldots, \; c_n(\gamma_2^n).$$

(This is a consequence of the Künneth sequence

$$\begin{tikzcd} 0 \arrow[r] & H^*(G_m) \otimes H^*(G_n) \arrow[r, "\times"] & H^*(G_m \times G_n) \arrow[r] & \operatorname{Tor}(H^*(G_m), H^*(G_n)) \arrow[r] \arrow[d, equal] & 0 \\ & & & 0 & \end{tikzcd}$$

which is known to be exact for any pair of finite complexes. This sequence is exact in this case also since each subcomplex $G_{n,k}$ is finite.)

Now consider the Chern class $c(\gamma_1^m \oplus \gamma_2^n)$. Since it belongs to this polynomial ring $H^*(G_m \times G_n)$, there must be a unique polynomial $p_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ with integer coefficients such that

$$c(\gamma_1^m \oplus \gamma_2^n) = p_{m,n}(c_1(\gamma_1^m), \ldots, c_m(\gamma_1^m), \ldots, c_n(\gamma_2^n)).$$

Applying f_B^* to both sides of this equation, and using the naturality conditions (1) we have:

Lemma: For any pair of complex plane bundles ω^m, υ^n over a paracompact base B, the formula

$$c(\omega^m \oplus \upsilon^n) = p_{m,n}(c_1(\omega^m), \ldots, c_m(\omega^m), c_1(\upsilon^n), \ldots, c_n(\upsilon^n))$$

Recall the definition of Chern classes. In particular $c_n(\omega^n) = X(\omega^n_R)$ and so we have the product theorem for the top Chern class from that for Euler classes:

1) $c_m(\omega^m)c_n(\upsilon^n) = c_{m+n}(\omega^n \oplus \upsilon^n)$.

Recall further that we have already proved a special case of the product theorem

2) $c(\omega^n \oplus \theta^1) = c(\omega^n)$ (see Lemma 2, of this section).

Now we are ready to prove in general:

Theorem 26: $c(\omega^m \oplus \upsilon^n) = c(\omega^m)\, c\,(\upsilon^n)$.

In other words, the polynomial $p_{m,n}(x_1,\ldots,x_m,y_1,\ldots,y_n)$ of Lemma 3 is in fact $(1 + x_1 + \cdots + x_m)(1 + y_1 + \cdots + y_n)$.

Proof: By induction on $m+n$. Certainly the assertion is true if $m+n = 0$ or 1 or if either m or n is zero. By induction, assume the theorem true for $m+n-1$. Look at $\omega^m \oplus \upsilon^{n-1} \oplus \theta^1$. Grouping it one way, $c(\omega^m \oplus \upsilon^{n-1} \oplus \theta^1) = c((\omega^m \oplus \upsilon^{n-1}) \oplus \theta^1)$.

By 2) we have $= c(\omega^m \oplus \upsilon^{n-1})$.

By the induction hypothesis, $= c(\omega^m)\, c\,(\upsilon^{n-1})$.

On the other hand, associating the other way:

$$c(\omega^m \oplus \upsilon^{n-1} \oplus \theta^1) = c(\omega^m \oplus (\upsilon^{n-1} \oplus \theta^1)).$$

By the lemma and 2) this is $= p_{m,n}(c_1(\omega^m),\ldots,c_m(\omega^m), c_1(\upsilon^{n-1}),\ldots,c_{n-1}(\upsilon^{n-1}), 0)$.

This is true for all complex bundle pairs; in particular, considering the bundle $\gamma_1^m \oplus \gamma_2^{n-1}$ where there are no polynomial relations, this must be a polynomial identity

$$(1+x_1+\cdots+x_m)(1+y_1+\cdots+y_{n-1}) = p_{m,n}(x_1,\ldots,x_m,y_1,\ldots,y_{n-1},0) .$$

In other words

$$p_{m,n}(x_1,\ldots,x_m,y_1,\ldots,y_n) \equiv (1+x_1+\cdots+x_m)(1+y_1+\cdots+y_n) \bmod (y_n)$$

where (y_n) is the ideal generated by y_n.

If we repeat the same procedure with $\theta^1 \oplus \omega^{m-1} \oplus \upsilon^n$, we find

$$p_{m,n}(x_1,\ldots,x_m,y_1,\ldots,y_n) \equiv (1+x_1+\cdots+x_m)(1+y_1+\cdots+y_n) \bmod (x_m).$$

It is a simple algebraic conseauence that

$$p_{m,n}(x_1,\ldots,x_m,y_1,\ldots,y_n) \equiv (1+x_1+\cdots+x_m)(1+\cdots+y_n) \bmod (x_m)\cap(y_n) = (x_my_n).$$

That is, $p_{m,n} = (1+\cdots+x_m)(1+\cdots+y_n) + zx_my_n$ where z belongs to the polynomial ring concerned. By 1), the only term of dimension $\geqq 2m+2n$ (the dimension of the top class is twice that of the bundle over C) is x_my_n; that is, $z = 0$. QED

Application (again analogous to the real case):

Theorem 27: $c(\tau^n(P^n(C))) = (1+\alpha)^{n+1}$ where α is the standard generator of $H^2(P^n(C);Z)$ (i.e. the one corresponding to the standard generator of $H^2(S^2)$ under the inclusion $S^2 = P^1(C) \subset P^n(C)$. As a complex manifold, S^2 has a uniquely distinguished generator of $H^2(S^2;Z)$.)

Proof: Complex projective n-space $P^n(C)$ can be represented as the unit $S^{2n+1} \subset C^{n+1}$ under the identification $\vec{u} = \lambda\vec{u}$ for all $\lambda \in C, |\lambda| = 1$. Then $E(\tau^n(P^n(C)))$ can be represented as the set of all pairs $(\vec{u},\vec{v})$, with $\|\vec{u}\| = 1$ and $\vec{u}\cdot\vec{v} = 0$ in the Hermitian metric, under the identification $(\vec{u},\vec{v}) = (\lambda\vec{u},\lambda\vec{v})$ for all $\lambda \in C, |\lambda| = 1$. Consider the complex line bundle

ξ_n^1 over $P^n(C)$ obtained from $S^{2n+1}\times C$ by the identification $(\vec{u},\rho) = (\lambda\vec{u},\lambda\rho)$ where $\rho\epsilon C$ and λ is as above and, as in the real case, take the (n+1)-fold bundle sum $\xi_n^1\oplus\cdots\oplus\xi_n^1$. Then $E(\xi_n^1\oplus\cdots\oplus\xi_n^1)$ can be represented as the set of pairs $(\vec{u},\vec{v})\epsilon S^{2n+1}\times C^{n+1}$ with the identification $(\vec{u},\vec{v}) = (\lambda\vec{u},\lambda\vec{v})$ where λ is as above. Comparing this with $E(\tau^n(P^n(C))$ we see $E(\xi_n^1\oplus\cdots\oplus\xi_n^1)\supset E(\tau^n(P^n(C)))$. On the other hand $\underbrace{\xi_n^1\oplus\cdots\oplus\xi_n^1}_{n+1}$ has a cross section (taking $u\epsilon S^{2n+1}$ into $(\vec{u},\vec{u})$). By taking the orthogonal compliment to this cross section, using the Hermitian metric, $\underbrace{\xi_n^1\oplus\cdots\oplus\xi_n^1}_{n+1}$ splits into $\tau^n(P^n(C))\oplus\theta^1$. By the product theorem, we have $c(\tau^n(P^n(C))) = c(\xi_n^1)^{n+1} = (1+c_1(\xi_n^1))^{n+1}$. The inclusions $S^2 = P^1(C)\subset P^2(C)\subset\cdots$ are covered by bundle maps $\xi_1^1\to\xi_2^1\to\cdots$ and by naturality $c_1(\xi_n^1)$ goes into $c_1(\xi_1^1)$. In fact, the homomorphism $H^2(P^n(C))\to H^2(P^1(C))$ is an isomorphism. Considering $S^2 = P^1(C)$ as a complex manifold, there is a distinguished generator α of $H^2(S^2;Z)$ and $c_1(\xi_1^1)$ must be some multiple of this standard generator. We have shown that $c(\tau^1(P^1(C))) = (1+c(\xi_1^1))^2$ or $c_1(\tau^1(P^1(C))) = 2c_1(\xi_1^1)$. On the other hand, by definition, $c_1(\tau^1(P^1(C))) = \chi(P^1(C)) = X(S^2)$. As is known, $X(S^2) = 2\alpha$ which shows that $c_1(\xi_1^1) = \alpha$ or $c(\tau^n(P^n(C))) = (1+\alpha)^{n+1}$. QED

<u>Corollary</u>: α^n is the fundamental class of $H^{2n}(P^n(C))$ since $(n+1)\alpha^n = c_n(\tau^n) = X(P^n(C)) = \chi\mu$ and it is known that the Euler characteristic is $n+1$. (Since $H^*(P^n(C))$ is the truncated polynomial ring, α^n is a generator. Here we have settled the ambiguity as to whether it was + or - the fundamental cohomology class.)

Conjugate bundle.

In order to gain more information about the characteristic classes of complex n-plane bundles, we introduce a new tool.

Definition: Two complex n-plane bundles ω and υ are conjugate equivalent if there is a map $f\colon E(\omega) \longrightarrow E(\upsilon)$ such that 1) ω_R and υ_R are equivalent under f and 2) $f(\lambda e) = \overline{\lambda} f(e)$ for all $e \in E(\omega)$. We will denote υ by $\overline{\omega}$.

Note: Conjugate equivalence is not an equivalence relation since in general ω is not conjugate equivalent to itself. For example, consider $\tau^1(P^1(C))$. (Ignoring the complex structure, this is just the tangent bundle of the 2-sphere.) If this bundle were self-conjugate, there would be defined a map of the tangent plane at each point into itself so that the complex structure (rotation by i) was reversed. The only such maps are obtained by reflection in some line of the plane. We would thus have a continuous field of lines in the tangent bundle of the 2-sphere, but this is impossible according to the Corollary to Theorem 13. Hence $\tau^1(P^1(C))$ is not self-conjugate. An alternative proof of this will be given below using Chern classes. Conjugate equivalence is however an involutive relation, like the relation between two oriented bundles which are equivalent except that their orientations are opposed, in that the conjugate equivalent to the conjugate equivalent of a bundle is equivalent to the original bundle. There is a canonical representative of $\overline{\omega}$; namely, the bundle with the identical total space and conjugate structure in each fibre.

Example: Over $P^n(C) = G_{1,n}(C)$ we have made use of two line bundles, $\xi_n^1(C)$ and $\gamma_n^1(C)$. They are in fact conjugate equivalent.

Looking at the Chern classes of the conjugate bundle we see:

Theorem 28. $c(\overline{\omega}) = 1 - c_1(\omega) + c_2(\omega) - c_3(\omega) + \cdots$

Proof: Let $v_1, \ldots, v_n$ be a basis for the complex fibre $F = \pi^{-1}(b)$ of ω for some arbitrary $b \in B$. This gives $v_1, iv_1, v_2, iv_2, \ldots, v_n, iv_n$ as the orientation of the real fibre. Applying f which gives the conjugate equivalence, $f(v_1), f(v_2), \ldots, f(v_n)$ gives $f(v_1), if(v_1), \ldots, f(v_n), if(v_n)$ as the orientation of $(\overline{\omega})_R$. On the other hand applying f to the orientation of the real fibre we get $f(v_1), -if(v_1), \ldots, f(v_n), -if(v_n)$ which is $(-1)^n$ times the orientation of $\overline{\omega}_R$. Thus we see $X(\overline{\omega}_R) = (-1)^n X(\omega_R)$ and so $c_n(\overline{\omega}) = (-1)^n c_n(\omega)$ which checks with the formula.

To check the formula for the lower dimensional classes, recall the definition $c_{n-1}(\omega^n) = \pi_o^{*-1} c_{n-1}(\omega_o^{n-1})$. But $c_{n-1}(\omega_o^{n-1}) = X(\omega_{o,R}^{n-1})$; by the above argument therefore, $c_{n-1}(\omega_o^{n-1}) = (-1)^{n-1} c_{n-1}(\overline{\omega}_o^{n-1})$. Descending in this way, we obtain the above formula for the total Chern class.

Note: This gives us a new proof of our earlier assertion that $\tau^1(P^1(C))$ is not self-conjugate for $c_1(\tau^1(P^1(C))) = 2\mu$ (see proof of Theorem 21) which is not of order 2.

XII. Pontrjagin Classes

To complete our study of characteristic classes of n-plane bundles, we need one new tool: the construction of the complex n-plane bundle induced by a real bundle. There are two ways of looking at the new structure although the structure itself is the same.

Definition: Given a real n-plane bundle ζ the induced complex n-plane bundle ζ_C with the same base B is obtained by considering as fibre over b the set of all formal sums $x + iy$ where $x, y \in F_b$, the fibre of ζ.

(Each fibre of ζ_C is an n-dimensional vector space over C as desired.)

Alternative definition: Given a real n-plane bundle ζ, the induced complex n-plane bundle ζ_C with the same base space B is defined as follows: $E(\zeta_C) = E(\zeta \oplus \zeta)$ and multiplication over C is defined in each fibre by $i \cdot (x,y) = (-y,x)$.

Using this second definition, it is easy to see that

Lemma: ζ_C is equivalent to its conjugate $\overline{\zeta_C}$, which is the same as saying ζ_C is conjugate equivalent to itself.

Proof: Let $f:E(\zeta_C) \longrightarrow E(\zeta_C)$ be defined by $f(x,y) = (x,-y)$. Clearly f gives the equivalence of the real bundle structures. Further $f[i \cdot (x,y)] = f[(-y,x)] = (-y,-x) = -i \cdot (x,-y)$. QED

If we look at the Chern classes of ζ_C we see $c(\zeta_C) = c(\overline{\zeta_C})$ which, by our result on the Chern classes of the conjugate bundle, gives us

$$c(\zeta_C) = 1 + c_1(\zeta_C) + c_2(\zeta_C) + \cdots$$

$$= c(\overline{\zeta_C}) = 1 - c_1(\zeta_C) + c_2(\zeta_C) - c_3(\zeta_C) + \cdots$$

Thus we have $\quad 2c_1(\zeta_C) = 2c_3(\zeta_C) = \cdots = 0$

This means that these odd classes carry a limited amount of information; we therefore confine our attention to the even classes.

Definition: For a real n-plane bundle ζ, the i-th Pontrjagin class $p_i(\zeta)$ is defined to be $(-1)^i c_{2i}(\zeta_C) \in H^{4i}(B)$. (The reason for $(-1)^i$, such as it is, will appear below.)

The total Pontrjagin class $p(\zeta)$ is defined to be

$1+p_1(\zeta) + p_2(\zeta) + \cdots + p_{[n/2]}(\zeta)$. (The highest Chern class is c_n since ζ_C is a complex n-plane bundle and thus the highest Pontrjagin class corresponds to $[n/2]$, the integral part of $n/2$.)

As for the other classes we have studied, we would like the Pontrjagin classes to satisfy the product formula, but we are likely to run into trouble because we have thrown away the odd dimensional Chern classes of ζ_C. The factors $(-1)^i$ we have introduced will cause no trouble since if $(1+c_2+c_4+\cdots)(1+c_2'+c_4'+\cdots) = (1+c_2''+c_4''+\cdots)$ then $(1-c_2+c_4-\cdots)(1-c_2'+c_4'-\cdots) = (1-c_2''+c_4''-\cdots)$. In fact, throwing away the odd dimensional classes forces a revision of the product theorem as follows:

Theorem 29: $p(\zeta \oplus \eta)-p(\zeta)p(\eta)$ is a sum of elements of order 2

Proof: $\zeta_C \oplus \eta_C = (\zeta \oplus \eta)_C$ and by the product theorem for Chern classes

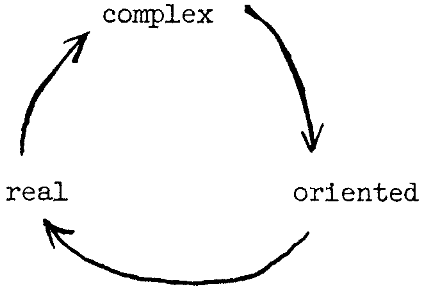
$$c(\zeta \oplus \eta)_C = c(\zeta_C)\,c(\eta_C)$$

We know the odd dimensional classes not included in the Pontrjagin classes are all of order 2. QED

Example: If we look at $p(\tau^n(S^n))$ we see that it is trivially 1 unless $n = 4k$, in which case $p(\tau^n) = 1 + p_k$. However, $\tau^n \oplus \nu^1$ is trivial as is ν^1 so $p(\tau^n \oplus \nu^1)-p(\tau^n)p(\nu^1) = 1-(1+p_k) = -p_k$ must be of order 2. But $H^n(S^n) = Z$ has no element of order 2 other than zero. That is, $p_k(\tau^{4k}) = 0$ and so $p(\tau^n(S^n)) = 1$ for all n.

We see that the Pontrjagin classes of spheres are uninteresting; it turns out that the things to look at are complex projective spaces;

but first let us consolidate our gains. At this point, we have a situation which is represented symbolicly at the right. Given a real n-plane bundle we now can obtain the induced complex n-plane bundle. Given a complex n-plane bundle we can look at its underlying real structure to obtain a real oriented 2n-plane bundle. Given a real oriented 2n-plane bundle, we can always ignore the orientation to get a real 2n-plane bundle. In otherwords, we can start at any point on the circle above and traverse it in the clockwise direction; notice that when we return to the original point we do not have the original bundle but rather one of twice the dimension. We would next like to investigate the behavior of characteristic clases under this sequence of operations. In particular, we have



<u>Theorem 30:</u> For ω^n a complex n-plane bundle,
$(-1)^i p_i(\omega^n_R) = \sum_{k+j=2i} (-1)^j c_k(\omega^n) c_j(\omega^n)$

<u>Proof:</u> By definition

$$p_i(\omega^n_R) = (-1)^i c_{2i}(\omega^n_{RC})$$

where ω^n_{RC} is obtained by neglecting the complex structure to get an oriented real 2n-plane bundle, ignoring the orientation, and then complexifying to get a complex 2n-plane bundle. By definition, $E(\omega^n) = E(\omega^n_R)$ and $E(\omega^n_{RC}) = E(\omega^n_R \oplus \omega^n_R)$. Therefore $E(\omega^n_{RC}) = E(\omega^n \oplus \omega^n)$. Our problem is to compare the complex structure of $E(\omega^n_{RC})$ with that of $E(\omega^n \oplus \omega^n)$. A point in $E(\omega^n_{RC})$ is given by a pair (x,y) where x and y

belong to the same fibre of ω^n. The multiplication by i defined in ω^n will always appear inside the pair; the multiplication by i for ω^n_{RC} will appear outside the pair and will be written with $\cdot$ as, for example, $i\cdot(x,y)$. Let $E(\omega_1)$ be the subspace of $E(\omega^n_{RC})$ consisting of all pairs $(x,-ix)$. This space $E(\omega_1)$ is invariant under $i\cdot$ as we have defined it, for $i\cdot(x,-ix) = (ix,x)$ which is of the required form. Similarly $E(\omega_2)$, defined as the subspace of all pairs of the form (x,ix), is invariant under $i\cdot$ since $i\cdot(x,ix) = (-ix,x)$. Now $\omega^n_{RC} = \omega_1 \oplus \omega_2$ since any point (x,y) of $E(\omega^n_{RC})$ can be written as $(\frac{x+iy}{2}, \frac{y-ix}{2}) + (\frac{x-iy}{2}, \frac{y+ix}{2})$. Moreover ω_1 is equivalent to ω^n. Consider the map $f:(x,-ix) \longrightarrow x$ taking $E(\omega_1)$ into $E(\omega)$. Since $i\circ(x,-ix) = (ix,x)$, we have $f(i\cdot(x,-ix)) = ix = if(x,-ix)$ and f gives the equivalence of the complex bundles. Similarly, ω_2 is equivalent to $\overline{\omega}^n$. Let $g(x,ix) = x$ take $E(\omega_2)$ into $E(\omega^n)$, then $g(i\circ(x,ix)) = g(-ix,x) = -ix = -ig(x,ix)$ as required. Thus we have shown:

<u>Lemma</u>: ω^n_{RC} is equivalent to $\omega^n \oplus \overline{\omega}^n$.

By the product theorem for Chern classes

$$c(\omega^n_{RC}) = c(\omega)c(\overline{\omega}) = (1+c_1(\omega^n) + c_2(\omega^n)+\cdots)(1-c_1(\omega^n)+c_2(\omega^n)-\cdots)$$

Observe that the minus signs in this formula coooperate to cancel out all the odd dimensional classes in the product. The result can be stated

$$1-p_1(\omega^n_R)+p_2(\omega^n_R)-\cdots = (1+c_1(\omega^n)+\cdots)(1-c_1(\omega^n)+c_2(\omega^n)-\cdots)$$

$$= \sum_{k,j}(-1)^j c_k(\omega^n)c_j(\omega^n)$$

Broken down this is

$$(-1)^i p_i(\omega_R^n) = \sum_{k+j=2i} (-1)^j c_k(\omega^n) c_j(\omega^n), \quad \text{QED.}$$

These formulas can be written as follows

$$p_1(\omega_R^n) = c_1^2(\omega^n) - 2c_2(\omega^n)$$

$$p_2(\omega_R^n) = c_2^2(\omega^n) - 2c_1(\omega^n)c_3(\omega^n) + 2c_4(\omega^n)$$

$$p_3(\omega_R^n) = c_3^2(\omega^n) - 2c_2(\omega^n)c_4(\omega^n) + 2c_1(\omega^n)c_5(\omega^n) - 2c_6(\omega^n)$$

etc.

Example: We already know $c(\tau^n(P^n(C))) = (1+\alpha)^{n+1}$ where $\alpha \in H^2(P^n(C))$. It is clear that $c(\bar{\tau}^n) = (1-\alpha)^{n+1}$ and by the above formula

$$1 - p_1(\tau_R^n) + p_2(\tau_R^n) - p_3(\tau_R^n) + \cdots = c(\tau^n)\,c(\bar{\tau}^n) = (1-\alpha^2)^{n+1}$$

Therefore $p(\tau_R^n(P^n(C))) = (1+\alpha^2)^{n+1}$.

Since there will be no ambiguity we will write $p(M^n)$ for $p(\tau_R^n(M^n))$ where M^n is a complex manifold. In particular $p(P^1(C)) = (1+\alpha^2)^2 = 1$ where $\alpha \in H^2(P^1(C))$, since $H^i(P^1(C)) = 0$ for $i > 2$. (This checks with our previous result since $P^1(C) = S^2$.)

Further

$$p(P^2(C)) = (1+\alpha^2)^3 = 1+3\alpha^2,$$

$$p(P^3(C)) = (1+\alpha^2)^4 = 1+4\alpha^2,$$

$$p(P^4(C)) = (1+\alpha^2)^5 = 1+5\alpha^2+10\alpha^4, \quad \text{etc.}$$

These last results were obtained from the sequence

complex ⟶ oriented real ⟶ real ⟶ complex

(see diagram above.) If we follow the sequence

oriented real ⟶ real ⟶ complex ⟶ oriented real

instead we find that starting with an oriented n-plane bundle ζ^n we have

$$\zeta^n_{CR} = \pm(\zeta^n \oplus \zeta^n)$$

the only question being the agreement of the orientations.

The orientation in each fibre is given by a basis $v_1,\ldots,v_n$ for that fibre. The corresponding orientation for $\zeta^n \oplus \zeta^n$ is given by 1) $(v_1,o)\cdots(v_n,o),(o,v_1),\ldots,(o,v_n)$. On the other hand using $v_1,\ldots,v_n$ as a basis for the complex fibre of ζ^n_C, the corresponding real basis for ζ^n_{CR} is given by $(v_1,o),i\cdot(v_1,o),(v_2,o),i\cdot(v_2,o),\ldots,(v_n,o),i\cdot(v_n,o)$ or 2) $(v_1,o),(o,v_1),(v_2,o),(o,v_2),\cdots,(v_n,o),(o,v_n)$. It is easy to determine the sign of the permutation relating these two bases (and therefore relating the corresponding orientations of ζ_{CR} and $\zeta^n \oplus \zeta^n$). The permutation can be effected by moving each (o,v_i) to the left in 1) until it is in the proper place for 2), and the sign can thus be seen to be

$$(-1)^{(n-1)+(n-2)+\cdots+2+1} = (-1)^{\frac{1}{2}n(n-1)}$$

If we confine our attention to even dimensional bundles (where the Euler class is not necessarily of order 2) we have

Lemma: $\zeta^{2n}_{CR} = (-1)^n(\zeta^{2n} \oplus \zeta^{2n})$ for any oriented 2n-plane bundle ζ^{2n}.

Therefore, looking at the Euler classes we have,

Theorem 31. For any oriented 2n-plane bundle $p_n(\zeta^{2n}) = (X(\zeta^{2n}))^2$.

Proof: $p_n(\zeta^{2n}) = (-1)^n \; c_{2n}(\zeta^{2n}_C) = (-1)^n X(\zeta^{2n}_{CR})$

$$= (-1)^n X(\,(-1)^n(\zeta^{2n} \oplus \zeta^{2n})) = X(\zeta^{2n} \oplus \zeta^{2n}).$$

Thus by the product theorem for Euler classes, $p_n(\zeta^{2n}) = (X(\zeta^{2n}))^2$ QED.

(This is the one place where we find it convenient to have defined $p_i(\zeta^{2n})$ with the factor $(-1)^i$.)

Structure of $H^*(\tilde{G}_n(R);\Lambda)$

For Λ a coefficient ring which contains 1/2 (so that we need not worry about elements of order 2, e.g. the rationals Q), we can now give the structure of the cohomology ring $H^*(\tilde{G}_n(R);\Lambda)$. (See pg. 59 for definition.)

The result will be only slightly more complicated than the cases $H^*(G_n(R);Z_2)$ and $H^*(G_n(C);Z)$ which we have already computed.

Theorem 32. If Λ is an integral domain containing $\frac{1}{2}$ then the cohomology ring $H^*(\tilde{G}_{2m+1};\Lambda)$ is a polynomial ring generated by

$$p_1(\gamma^{2m+1}),\cdots,p_m(\gamma^{2m+1}).$$

The cohomology ring $H^*(\tilde{G}_{2m};\Lambda)$ is a polynomial ring generated by

$$p_1(\gamma^{2m}),\;\cdots,p_{m-1}(\gamma^{2m}),\;\text{and}\;\; X(\gamma^{2m}).$$

This can be summarized by saying that $H^*(\tilde{G}_n;\Lambda)$ is the polynomial ring generated by $p_1,\ldots,p_{[n/2]}$, and X, modulo the relation

$$X = 0 \qquad \text{for } n \text{ odd}$$

$$X^2 = p_{n/2} \qquad \text{for } n \text{ even.}$$

Proof by induction on n. Since $\tilde{G}_o$ is a point, we can clearly start the induction. Just as in the complex case we have an exact sequence

$$\longrightarrow H^i(\tilde{G}_n) \xrightarrow{X} H^{i+n}(\tilde{G}_n) \xrightarrow{\lambda} H^{i+n}(\tilde{G}_{n-1}) \longrightarrow H^{i+1}(\tilde{G}_n) \longrightarrow$$

where $\lambda = \rho^{*-1}\pi^*_o$ carries the Pontrjagin classes of $\tilde{G}_n$ into those of $\tilde{G}_{n-1}$.

Case 1. Assume that the theorem is true for $\tilde{G}_{2m-1}$. That is $H^*(\tilde{G}_{2m-1})$ is a polynomial ring generated by $p_1,\ldots,p_{m-1}$. Now the argument used in the proof of Theorem 24 shows that $H^*(\tilde{G}_{2m})$ is a polynomial ring generated by $p_1,\ldots,p_{m-1}$ and X.

Case 2. Assume that $H^*(\tilde{G}_{2m})$ has this form. Since $X(\tilde{\gamma}^{2m+1}) = 0$ (with coefficient group Λ) the above sequence, for $n = 2m+1$, $j = i+2m+1$, becomes

$$\xrightarrow{0} H^j(\tilde{G}_{2m+1}) \xrightarrow{\lambda} H^j(\tilde{G}_{2m}) \longrightarrow H^{j-2m}(\tilde{G}_{2m+1}) \xrightarrow{0} .$$

Thus $H^*(\tilde{G}_{2m+1})$ can be considered as a subring of $H^*(\tilde{G}_{2m})$. This subring is known to contain the elements $p_1,\ldots,p_{m-1}$, and $p_m = X^2$. Thus, if R^* denotes the subring generated by $p_1, \ldots,p_m$, we have

$$R^* \subset \lambda H^*(\tilde{G}_{2m+1}) \subset H^*(\tilde{G}_{2m})$$

which implies that

a) rank $R^j \leq$ rank $H^j(\tilde{G}_{2m+1})$.

(For the concept of rank, see for example Eilenberg and Steenrod, p. 52.)

From the exact sequence above we see that

$$\text{rank } H^j(\tilde{G}_{2m+1}) + \text{rank } H^{j-2m}(\tilde{G}_{2m+1}) = \text{rank } H^j(\tilde{G}_{2m}).$$

But the equality

$$\text{rank } R^j + \text{rank } R^{j-2m} = \text{rank } H^j(\tilde{G}_{2m})$$

is easily verified. (In fact $H^j(\tilde{G}_{2m}) = R^j \oplus X^{2m} \smile R^{j-2m}$.)

Therefore

$$\text{rank } H^j(\tilde{G}_{2m+1}) + \text{rank } H^{j-2m}(\tilde{G}_{2m+1}) = \text{rank } R^j + \text{rank } R^{j-2m}$$

Using a) for both j and $j-2m$, we have rank R^j = rank $H^j(\tilde{G}_{2m+1})$. From this it follows easily that $R^j = H^j(\tilde{G}_{2m+1})$ which completes the proof.

XIII. Pontrjagin numbers

1. Partitions

A partition ω of an integer k is an unordered sequence $i_1 \cdots i_r$ of positive integers with sum k. The set of all such partitions will be denoted by $\Pi(k)$ and the number of partitions by $\pi(k)$.

[For $k = 0,1,2,3,4$ the number $\pi(k)$ is equal to 1,1,2,3,5 respectively. As k tends to infinity, a theorem of Hardy and Ramanujan asserts that

$$\pi(k) \sim \frac{1}{4k\sqrt{3}} e^{\pi\sqrt{\frac{2k}{3}}}$$

For further information see Ostmann [14].

The natural composition operation $\Pi(k) \times \Pi(\ell) \longrightarrow \Pi(k+\ell)$ will be denoted by juxtaposition:

if $\omega = i_1 \cdots i_r$, $\omega' = j_1 \cdots j_s$, then $\omega\omega' = i_1 \dots i_r \, j_1 \dots j_s$.

This composition operation is associative, commutative and has an identity element, which is denoted by   . It is also possible to define a partial ordering relation among partitions. A refinement of $i_1 \cdots i_r$ will mean any partition which can be written in the form $\omega_1 \cdots \omega_r$ with $\omega_1 \in \Pi(i_1), \dots, \omega_r \in \Pi(i_r)$.

2. Pontrjagin numbers

Let M^n be a compact, oriented, differentiable manifold with tangent bundle τ^n and fundamental homology class μ_n. Given any partition $i_1 \dots i_r \in \Pi(k)$, define the $(i_1 \dots i_r)$-th Pontrjagin number

$p_{i_1} \cdots p_{i_r}[M^n]$ of M^n to be the integer $< p_{i_1}(\tau^n) \cdots p_{i_r}(\tau^n), \mu_n >$.

Note that this is zero unless $n = 4k$.

(Compare Stiefel-Whitney numbers page 16.)

As an example consider the complex projective space $P^{2n}(C)$. Recall (pg. 82) that

$$p_i(P^{2n}(C) = \binom{2n+1}{i} \alpha^{2i}$$

where $\alpha \in H^2(P^{2n}(C);Z)$ and $<\alpha^{2n}, \mu_{4n}> = 1$.

Hence

$$p_{i_1} \cdots p_{i_r}[P^{2n}(C)] = \binom{2n+1}{i_1} \cdots \binom{2n+1}{i_r}$$

for any $i_1 \cdots i_r \in \Pi(n)$.

It is frequently useful to consider various linear combinations of the Pontrjagin numbers of a manifold. The rest of this chapter will be concerned with one such set of linear combinations. Others will occur in Chapter XV.

3. Symmetric functions; the polynomials s_ω.

Consider a polynomial ring in n variables over the integers: $Z[t_1, \ldots, t_n]$. This is made into a graded ring by defining the degree of each t_i to be 1. The elementary symmetric functions $\sigma_1, \ldots, \sigma_n$ are defined by

1) degree $\sigma_i = i$, and

2) $1 + \sigma_1 + \cdots + \sigma_n = (1 + t_1) \cdots (1 + t_n)$.

[There is an important connection between symmetric functions and Pontrjagin classes due to Borel. For our purposes this can be motivated as follows. Suppose that a bundle ζ^{2n} splits into a sum $\zeta_1^2 \oplus \cdots \oplus \zeta_n^2$ of 2-plane bundles. Then the identity

$$1 + p_1(\zeta^{2n}) + \cdots + p_n(\zeta^{2n}) = (1 + p_1(\zeta_1^2)) \cdots (1 + p_1(\zeta_n^2))$$

shows that $p_i(\zeta^{2n})$ is the i-th elementary symmetric function of $p_1(\zeta_1^2), \ldots, p_1(\zeta_n^2)$.]

Let S denote the graded subalgebra of $Z[t_1, \ldots, t_n]$ consisting of the polynomials which are left fixed by all permutations of $t_1, \ldots, t_n$. A standard theorem asserts that $S = Z[\sigma_1, \ldots, \sigma_n]$, where $\sigma_1, \ldots, \sigma_n$ are algebraically independent.

An alternative description of S is the following: Define two monomials in $t_1, \ldots, t_n$ to be equivalent if some permutation of $t_1, \ldots, t_n$ carries one into the other. Define

$$\Sigma\, t_1^{i_1} \ldots t_r^{i_r}$$

to be the summation of all monomials equivalent to $t_1^{i_1} \ldots t_r^{i_r}$. (For example $\sigma_i = \Sigma\, t_1 \ldots t_i$)

Lemma. An additive basis for S^k = subspace of S of dimension k, $k \leq n$, is given by the set of polynomials

$$\Sigma\, t_1^{i_1} \ldots t_r^{i_r}$$

where $i_1 \ldots i_r$ ranges over all partitions of k.

The proof is not difficult.

Now define a polynomial in k variables $s_{i_1 \ldots i_r}$ belonging to S^k, where $i_1 \ldots i_r \in \Pi(k)$, by the identity

$$s_{i_1 \ldots i_r}(\sigma_1, \ldots, \sigma_k) = \Sigma\, t_1^{i_1} \ldots t_r^{i_r} .$$

(This polynomial does not depend on n, as long as the condition $k \leq n$ is satisfied.)

The first twelve such polynomials are

$$s(\ ) = 1 \ ;$$

$$s_1(\sigma_1) = \sigma_1 \quad ;$$

$$\begin{cases} s_2(\sigma_1, \sigma_2) = \sigma_1^2 - 2\sigma_2 \\ s_{11}(\sigma_1, \sigma_2) = \sigma_2 \ ; \end{cases}$$

$$\begin{cases} s_3(\sigma_1, \sigma_2, \sigma_3) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\ s_{12}(\sigma_1, \sigma_2, \sigma_3) = \sigma_1\sigma_2 - 3\sigma_3 \\ s_{111}(\sigma_1, \sigma_2, \sigma_3) = \sigma_3 \ ; \text{ and} \end{cases}$$

$$\begin{cases} s_4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4 \\ s_{13} = \sigma_1^2\sigma_2 - 2\sigma_2^2 - \sigma_1\sigma_3 + 4\sigma_4 \\ s_{22} = \sigma_2^2 - 2\sigma_1\sigma_3 + 2\sigma_4 \\ s_{112} = \sigma_1\sigma_3 - 4\sigma_4 \\ s_{1111} = \sigma_4 \ . \end{cases}$$

(For more information see van der Waerden [26] Chapter 26; in particular the exercises.

4. A product formula; the group $\Gamma_1 A^*$

It will be convenient to introduce the following concept. Given any graded ring A^* with unit, define a ring ΓA^* as the cartesian product

$$A^0 \times A^1 \times A^2 \times \quad \dots$$

with composition operations

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (a_0 + b_0, a_1 + b_1, \dots) \quad \text{and}$$

$$(a_0, a_1, \dots) \cdot (b_0, b_1, \dots) = (a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, \dots) .$$

Each element $a_i \in A^i$ will be identified with the sequence $(0, \dots, 0, a_i, 0, \dots)$ in ΓA^*. Whenever no confusion is possible, the sequence $(a_0, a_1, \dots)$ will be written as a formal sum $a_0 + a_1 + \cdots$.

Let $\Gamma_1 A^* \subset \Gamma A^*$ denote the subset consisting of sequences $(1, a_1, a_2, \dots)$ with leading term 1. Then $\Gamma_1 A^*$ is a multiplicative group.

As an example, given any commutative ring with unit Λ and given a space X, the groups $A^i = H^{4i}(X;\Lambda)$ give rise to a commutative graded Λ-algebra, which will be denoted by $H^{4*}(X;\Lambda)$. The total Pontrjagin class

$$p = 1 + p_1 + \cdots + p_n = (1, p_1, \dots, p_n, 0, \dots)$$

with coefficients in Λ of a bundle over X is an element of $\Gamma_1 H^{4*}(X;\Lambda)$. Recall (p. 79) the identity

$$p(\zeta \oplus \eta) = p(\zeta) p(\eta)$$

holds whenever $H^{4*}(X;\Lambda)$ has no 2-torsion.

Given any partition $\omega \in \Pi(k)$ and given $a \in \Gamma_1 A^*$, where A^* is commutative, define $s_\omega(a) \in A^k$ to be $s_\omega(a_1, a_2, \dots, a_k)$.

Theorem 33. The polynomials s_ω satisfy the identity

$$s_\omega(a\cdot b) = \Sigma_{\omega_1\omega_2 = \omega}\ s_{\omega_1}(a)\cdot s_{\omega_2}(b)\ ,$$

to be summed over all pairs ω_1, ω_2 such that $\omega_1\omega_2 = \omega$.

As an example, for $\omega = k$, this formula takes the particularly simple form:

Corollary 1. $s_k(a\cdot b) = s_k(a) + s_k(b)$.

Given any n-plane bundle ζ, the elements $s_\omega(p(\zeta)) \in H^{4k}(X;\Lambda)$ can be considered as new characteristic classes of ζ.

Corollary 2. The identity

$$s_\omega(p(\zeta \oplus \eta)) = \Sigma_{\omega_1\omega_2 = \omega}\ s_{\omega_1}(p(\zeta))s_{\omega_2}(p(\eta))$$

holds modulo 2-torsion.

Now consider a compact, oriented, differentiable manifold M^n. For each $\omega \in \pi(k)$, define a new characteristic number by the formulas

$$s_\omega[M^n] = 0 \quad \text{if} \quad n \neq 4k$$

$$s_\omega[M^{4k}] = 0 \quad < s_\omega(p(\tau^{4k})), \mu_{4k} >$$

These numbers are linear combinations of the Pontrjagin numbers, and conversely the Pontrjagin numbers can be expressed as linear combinations of these. However the new numbers satisfy a very simple product formula.

Corollary 3 (Thom).

$$s_\omega[M_1 \times M_2] = \Sigma_{\omega_1\omega_2 = \omega}\ s_{\omega_1}[M_1]\, s_{\omega_2}[M_2]\ .$$

Note that most of the terms on the right drop out for dimensional reasons. For example:

Corollary 4. If M_1 and M_2 both have positive dimension then

$$s_k[M_1 \times M_2] = 0.$$

The characteristic numbers $s_k[M^n]$ will turn out to be particularly important.

Example. For the manifold $P^{2n}(C)$, since $p = (1+\alpha^2)^{2n+1}$, the class p_i can be considered as the i-th elementary symmetric function in $\alpha^2, \dots, \alpha^2$. Hence $s_k(p)$ is equal to

$$\Sigma(\alpha^2)^k = (2n+1)\,\alpha^{2k} .$$

In particular

$$s_n[P^{2n}(C)] = 2n+1 \neq 0.$$

It follows from Corollary 4 that $P^{2n}(C)$ cannot be expressed as a product of positive dimensional manifolds.

Proof of Theorem 33. Consider the special case $A^* = Z[t_1, \dots, t_{2k}]$,

$$a = (1+t_1)\cdots(1+t_k), \qquad b = (1+t_{k+1})\cdots(1+t_{2k}),$$

where the t_i are algebraically independent of degree 1. Then the elements $a_1, \dots, a_k$ and $b_1, \dots, b_k$ are algebraically independent. Hence if Theorem 33 is true in this special case, it will be true universally.

Let $\omega = i_1 \dots i_r$. By definition $s_\omega(a \cdot b)$ is equal to $\Sigma t_1^{i_1} \dots t_r^{i_r}$. Each term of this sum has the form $t_{\alpha_1}^{i_1} \cdots t_{\alpha_r}^{i_r}$ where $\alpha_1, \dots, \alpha_r$ are distinct numbers between 1 and 2k. Let ω_1 be the partition formed by those exponents i_q such that $1 \leq \alpha_q \leq k$, and let ω_2 be the partition formed by the remaining i_q. The sum of all the terms corresponding to a given decomposition $\omega = \omega_1 \omega_2$ is clearly just

$$s_{\omega_1}(a)\, s_{\omega_2}(b).$$

Since every such decomposition occurs, this completes the proof.

Corollaries 1, 2 and 4 are clear.

Proof of Corollary 3. For $i = 1, 2$ the tangent bundle τ_i of M_i, together with the projection $M_1 \times M_2 \longrightarrow M_i$, induces a bundle ζ_i over M_i. The tangent bundle τ of $M_1 \times M_2$ may be identified with the sum $\zeta_1 \oplus \zeta_2$. Hence Corollary 2 takes the form

$$s_\omega(p(\tau)) \equiv \Sigma_{\omega_1\omega_2=\omega} \; s_{\omega_1}(p(\tau_1))s_{\omega_2}(p(\tau_2)) \qquad \text{(mod 2 torsion).}$$

The fact that the Kronecker index with integral coefficients ignores torsion, together with the identities $\mu = \mu_1 \times \mu_2$ and

$$<\alpha \times \beta, \mu_1 \times \mu_2> \; = \; <\alpha, \mu_1><\beta, \mu_2> ,$$

completes the proof.

5. Linear independence of Pontrjagin numbers.

The object of this section will be to prove the following theorem, which shows that the $\pi(n)$ Pontrjagin numbers of a general 4n-manifold satisfy no linear relations.

Theorem 34. (Thom) The $\pi(n) \times \pi(n)$ matrix

$$\| \, p_{i_1} \cdots p_{i_r} [\, P^{2j_1}(C) \times \cdots \times P^{2j_s}(C) \,] \, \| ,$$

where $i_1 \dots i_r$ and $j_1 \dots j_s$ range over $\Pi(n)$, is non-singular.

Remark. In place of the manifolds $P^2(C)$, $P^4(C)$, ... one could substitute any sequence $M^4, M^8, \dots$ of manifolds which satisfy the conditions $s_k[M^{4k}] \neq 0$.

Example. For $n = 2$,

$$p_1^2[P^2(C) \times P^2(C)] = 18 \qquad p_1^2[P^4(C)] = 25$$

$$p_2[P^2(C) \times P^2(C)] = 9 \qquad p_2[P^4(C)] = 10,$$

so that the determinant is $-45 \neq 0$. It is evident that the direct approach of simply computing the matrix will not help much in the general case.

Proof of Theorem 34. In place of the Pontrjagin numbers themselves we will use the linear combinations $s_\omega[M]$. The following formula is a direct generalization of Theorem 33 Corollary 3.

$$(1) \quad s_\omega[M_1 \times \cdots \times M_r] = \Sigma_{\omega_1 \cdots \omega_r = \omega}\, s_{\omega_1}[M_1] \cdots s_{\omega_r}[M_r].$$

Suppose that the manifolds $M_1, \ldots, M_r$ have dimensions $4i_1, \ldots, 4i_r$ respectively. Then the term

$$s_{\omega_1}[M_1] \cdots s_{\omega_r}[M_r]$$

is zero unless $\omega_1 \in \Pi(i_1), \ldots, \omega_r \in \Pi(i_r)$. This proves:

(2) $s_\omega[M_1 \times \cdots \times M_r] = 0$ unless ω is a refinement of $i_1 \ldots i_r$.

For the special case $\omega = i_1 \ldots i_r$ the formula becomes

$$(3) \quad s_{i_1 \ldots i_r}[M_1 \times \cdots \times M_r] = s_{i_1}[M_1] \cdots s_{i_r}[M_r],$$

since all the other terms are necessarily zero.

Now choose some sequence $M^4, M^8, \ldots, M^{4n}$ of manifolds such that $s_i[M^{4i}] \neq 0$ for $i = 1, 2, \ldots, n$. Let $M_{i_1 \ldots i_r}$ denote the product manifold $M^{4i_1} \times \cdots \times M^{4i_r}$.

Then we will prove:

(4) the matrix $\| s_\omega[M_{\omega'}] \|_{\omega, \omega' \in \Pi(n)}$ is non-singular.

In fact let $\omega_1, \ldots, \omega_{\pi(n)}$ denote the partitions of n, numbered so that, if ω_j is a refinement of ω_k, then $j \geq k$. Assertion (2) implies that

$$s_{\omega_j}[M_{\omega_k}] = 0 \quad \text{for} \quad j < k,$$

while (3) implies that

$$s_{\omega_j}[M_{\omega_j}] \neq 0 .$$

Thus the matrix is triangular and nonsingular. This completes the proof of (4), and therefore of Theorem 34, for the $p_{i_1} \ldots p_{i_r}[M_{\omega'}]$ are linear combinations of the $s_\omega[M_{\omega'}]$ so that dependence of the former would imply dependence of the latter.

XIV. Cobordism

This chapter will give a presentation of the cobordism theory of Thom [23].

1. The ring Ω^*.

All manifolds considered are to be compact, oriented and differentiable unless otherwise stated. The word "differentiable" will always mean "differentiable of class C^∞". We construct an operation of addition among manifolds of the same dimension:

Definition. $M_1^n + M_2^n$ will represent the disjoint union $M_1^n \cup M_2^n$. It is natural therefore to write kM^n for the union of k disjoint copies of M^n, $k \geq 0$. Further define $-M^n$ to be the same manifold but with the opposite orientation.

This sum operation has a zero element: namely the vacuous manifold. Note however that $M^n - M^n$ is not equal to zero.

An equivalence relationship between manifolds of the same dimension is defined as follows (as was indicated briefly on p. 19):

Definition: M^n is a boundary if there exists a compact, oriented, differentiable bounded-manifold B^{n+1} whose boundary is M^n. The induced differentiable structure on the boundary M^n should coincide with the differentiable structure originally given. [The differentiable structure on B^{n+1} may be specified by a coordinate system $\{(U_\alpha, f_\alpha)\}$ where

1) the U_α are open sets covering B^{n+1};

2) each $f_\alpha : U_\alpha \longrightarrow R^{n+1}$ is a homeomorphism, either onto R^{n+1} or onto a closed half-space; and

3) for each α, β the composition

$$f_\alpha f_\beta^{-1} : f_\beta(U_\alpha \cap U_\beta) \longrightarrow R^{n+1}$$

is differentiable (i.e. can be extended to a differentiable map defined on a neighborhood of $f_\beta(U_\alpha \cap U_\beta)$). For further details see [12] Appendix 1.]

Definition: $M_1^n \sim M_2^n$ (read: M_1^n and M_2^n belong to the same cobordism class) if $M_1^n - M_2^n$ is a boundary.

It is clear that this relation is reflexive and symmetric; that it is transitive can be seen using the obvious construction. If B_{12}^{n+1} has boundary $M_1^n - M_2^n$ and B_{23}^{n+1} has boundary $M_2^n - M_3^n$, then B_{12}^{n+1} and B_{23}^{n+1} are identified along the common boundary M_2^n. The resulting structure can be smoothed out to give a C^∞-manifold whose boundary is $M_1^n - M_3^n$. (See [12] Appendix I, Lemma 4.) If we denote by + the operation on equivalence classes induced by the operation + on the manifolds, the classes form an abelian group Ω^n under +. Ω^n is the cobordism group in dimension n.

A bilinear pairing from Ω^m and Ω^n to Ω^{m+n} is defined by the correspondence $M_1^m, M_2^n \longrightarrow M_1^m \times M_2^n$. Thus the sequence $\Omega^* = (\Omega^0, \Omega^1, \Omega^2, \ldots)$ of cobordism groups has the structure of a graded ring. It is easily verified that $M_1^m \times M_2^n$ is isomorphic (as an oriented manifild) to $(-1)^{mn} M_2^n \times M_1^m$. Thus the cobordism ring is anticommutative.

The Pontrjagin numbers provide a basic tool for studying this cobordism ring.

Theorem 35 (Pontrjagin). If M^n is a boundary, then every Pontrjagin number $p_{i_1} \ldots p_{i_r}[M^n]$ is zero.

Proof: The argument is completely analogous to that on page 17.

Since the identity

$$p_{i_1} \ldots p_{i_r}[M_1 + M_2] = p_{i_1} \ldots p_{i_r}[M_1] + p_{i_1} \ldots p_{i_r}[M_2]$$

is clearly satisfied we have:

Corollary 1. For each $i_1 \ldots i_r \in \Pi(k)$ the correspondence

$$M^{4k} \longrightarrow p_{i_1} \ldots p_{i_r}[M^{4k}]$$

defines a homomorphism of Ω^{4k} into Z.

Comparing Theorems 34 and 35 we have:

Corollary 2: The manifolds

$$P^{2i_1}(C) \times \cdots \times P^{2i_r}(C)$$

with $i_1 \ldots i_r \in \Pi(k)$ represent linearly independent elements of the cobordism group Ω^{4k}. Hence the group Ω^{4k} has rank $\geqq \pi(k)$.

The principal object of Chapter XIV will be to show that this is a best possible result. (That is Ω^{4k} has rank exactly $\pi(k)$; while Ω^n has rank zero for $n \neq 0 \pmod 4$.)

[Remark. The actual structure of the first few groups is the following:

$\Omega^0 = Z$, since the only 0-manifolds are finite sets of points, and the algebraic number of points determines the cobordism class uniquely.

$\Omega^1 = \Omega^2 = \Omega^3 = 0$. It is well known that every 1-manifold or (orientable!) 2-manifold bounds. The corresponding assertion for 3-manifolds is non-trivial.

$\Omega^4 = Z$ generated by the complex projective plane $P^2(C)$.

$\Omega^5 = Z_2$, $\Omega^6 = \Omega^7 = 0$, $\Omega^8 = Z + Z$ generated by $P^2(C) \times P^2(C)$ and $P^4(C)$.

For further information see Dold [1] and Milnor [13].]

2. The Thom space of a bundle.

Let ζ be an n-plane bundle over a compact base space $B(\zeta)$. By the *Thom space* $T(\zeta)$ will be meant the one point compactification of the total space $E(\zeta)$. The base space will be identified with the subset of $E(\zeta)$ corresponding to the zero cross-section. Thus we have

$$B(\zeta) \subset E(\zeta) \subset T(\zeta).$$

The point at infinity will be denoted by t_0.

Remark 1. The following alternative definition is sometimes more convenient. Choosing a Riemannian metric, let E' denote the subspace of E consisting of vectors of length ≤ 1, and let E'_0 denote the subspace of unit vectors. Define T' as the identification space E'/E_0'. Then the correspondence

$$\vec{v} \longrightarrow \vec{v}/(1 - \|\vec{v}\|)$$

gives rise to a homeomorphism

$$h: T' \rightarrow T.$$

Remark 2. Thom's notation is as follows. Let G be a closed subgroup of the orthogonal group O_n, and let ζ be the n-plane bundle associated with a universal bundle for G. Then Thom denotes $T(\zeta)$ by $M(G)$.

The following two lemmas describe the structure of the Thom space.

Lemma 1. If $B(\zeta)$ is a finite cell complex then $T(\zeta)$ is an (n-1)-connected finite cell complex.

Proof. For each open q-cell e of B, the inverse image $\pi^{1}(e) \subset E \subset T$ is an open (n+q)-cell. If the cells $\{e_i\}$ cover B then the cells $\{\pi^{-1}(e_i)\}$, together with the point t_o, cover T. Note that there are no cells in dimensions 1 through n-1.

Let D^q denote the unit ball in R^q, and let $f: D^q \longrightarrow B$ be a characteristic map for the cell e. The induced bundle η over D^q is necessarily a product bundle $D^q \times R^n \supset D^q \times D^n$. Hence the composition of the natural maps

$$D^q \times D^n = E'(\eta) \longrightarrow E'(\zeta) \longrightarrow T'(\zeta) \xrightarrow{h} T(\zeta)$$

gives the required characteristic map for $\pi^{-1}(e)$. This completes the proof.

Lemma 2. If ζ is an oriented n-plane bundle, then each cohomology group $H^{n+k}(T(\zeta), t_o)$ is isomorphic to $H^k(B(\zeta))$, $h \geq 0$.

Proof. There are natural isomorphisms

$$H^k(B) \xrightarrow{\phi} H^{n+k}(E, E_o) \xleftarrow{\text{excision}} H^{n+k}(T, T\text{-}B).$$

(See the appendix for the details of ϕ.)

Since the space T-B is contractible to the point t_o, this last group can be replaced by $H^{n+k}(T, t_o)$, which completes the proof.

3. Regular values of differentiable maps.

Let W be an open subset of euclidean space R^n, and let $f: W \longrightarrow R^k$ be a differentiable map.

Definition: A point $y \in R^k$ is a regular value of f if, for each $x \in f^{-1}(y)$, the Jacobian matrix

$$\| \partial f_i(x) / \partial x_j \|$$

has rank k. (The case $f^{-1}(y)$ vacuous is not excluded. For example if $n < k$ then y is a regular value only if $f^{-1}(y)$ is vacuous.) More generally, for any subset C of W, we will say that y is a regular value of $f|C$ if the Jacobian matrix has rank k for all $x \in f^{-1}(y) \cap C$.

Motivation for this definition is provided by

Lemma 3. If y is a regular value of f then $f^{-1}(y)$ is a differentiable submanifold of W, with dimension $n-k$.

Proof. This follows immediately from the implicit function theorem. (See for example, Graves [6] p. 138.)

The following extremely delicate theorem shows that regular values exist.

Theorem of Sard. If $f:W \longrightarrow R^k$ is differentiable (of class C^∞) then the set of all $y \in R^k$ which are not regular values has measure zero.

For the proof, see Sard [15].

The following lemma is based on this theorem. Let C be a compact subset of W, and V a neighborhood of C, with $\bar{V}$ compact $\subset W$.

Lemma 4. Given any differentiable map $f:W \longrightarrow R^k$, and given $\varepsilon > 0$, there exists a differentiable map $g:W \longrightarrow R^k$ such that

(1) $g|C$ has the origin 0 as regular value;

(2) g coincides with f outside of V; and

(3) $|g_i(x) - f_i(x)| < \varepsilon$, $|\partial g_i(x)/\partial x_j - \partial f_i(x)/\partial x_j| < \varepsilon$

for all x in W, all $1 \leq i \leq k$, and all $1 \leq j \leq n$.

Proof. Let $\lambda : W \longrightarrow R$ be a differentiable function which takes the value 1 on C and the value 0 on $W - V$. (See Steenrod [20] p. 26.) If y is any regular value of f, then the function g defined by

$$g(x) = f(x) - \lambda(x)y$$

will certainly satisfy conditions (1) and (2). But, according to the theorem of Sard, the vector y can be chosen arbitrarily close to the origin. Hence condition (3) can also be satisfied.

Finally, the following will be needed:

Lemma 5. Let C again denote a compact subset of W, and $g : W \longrightarrow R^k$ a map such that $g|C$ has 0 as regular value. Then there exists $\varepsilon > 0$ such that, if $h : W \longrightarrow R^k$ satisfies

$$|h_i(x) - g_i(x)| < \varepsilon, \quad |\partial h_i(x)/\partial x_j - \partial g_i(x)/\partial x_j| < \varepsilon$$

for all $x \in C$, then $h|C$ also has 0 as regular value.

The proof is straightforward.

4. Transverse regularity.

Let $f : M \longrightarrow M'$ be a differentiable map, and M'' a submanifold of M'.

Definition: f is transverse regular on M'' if, for each $y \in M''$ and each $x \in f^{-1}(y)$, the induced map from the tangent vector space at x to the normal vector space at y

$$F_x \longrightarrow F'_y \longrightarrow F'_y / F''_y$$

is onto. (Notice in particular that dim $M \geq$ dim M' - dim M" if $f(M)$ intersects M", and if $n = n' - n''$ then $f(M)$ must be normal to M" at the intersections.)

Using Lemma 3 it is not hard to see that the inverse image $f^{-1}(M'')$ is a differentiable manifold of dimension $n - n' + n''$, providing that f is transverse regular on M''.

Consider the following situation: Let M^n and B be compact differentiable manifolds, and let ζ be a differentiable k-plane bundle over B. That is we assume that the total space E has a differentiable structure compatible with the bundle structure. Then B is a differentiable submanifold of E with normal bundle equivalent to ζ.

<u>Theorem</u> 36. Every map $f:M^n \longrightarrow T(\zeta)$ is homotopic to a map h which

(I) is differentiable on $h^{-1}(E)$ (i.e. where ever differentiability makes sense); and

(II) is transverse regular on B.

<u>Proof</u>: First choose a map $f_o:M^n \longrightarrow T(\zeta)$ which coincides with f on $f^{-1}(t_o)$, and which is differentiable on $f^{-1}(E)$. (Compare Steenrod [20] §6.7.) Let $\{B_j\}$ be a covering of B by coordinate neighborhoods. Thus the bundle ζ restricted to B_j is equivalent to $B_j \times R^k$, and the projections of $B_j \times R^k$ into the two factors correspond to maps

$$\pi:\pi^{-1}(B_j) \longrightarrow B_j \ , \qquad \rho_j:\pi^{-1}(B_j) \longrightarrow R^k \ .$$

Choose a covering of $f_o^{-1} B$ by open sets $W_1, \ldots, W_m \subset f_o^{-1}(E)$. These sets should be small enough so that

1) each W_i is diffeomorphic to an open subset of R^n, and

2) each $f_o(W_i)$ is contained in $\pi^{-1}B_j$ for some $j = j(i)$.

Choose smaller open sets U_i, V_i with

$$\overline{U}_i \subset V_i, \quad \overline{V}_i \subset W_i$$

so that the union $U = U_i \cup \ldots \cup U_m$ still contains $f_o^{-1} B$.

Now Lemma 4 will be used to construct a series of modifications $f_1, \ldots, f_m$ of f_o. Each f_i will coincide with f_{i-1} except on V_i. Each projection $\pi f_i : f^{-1}(E) \longrightarrow B$ will coincide with $\pi f_o : f^{-1}E \longrightarrow B$. Thus to construct these modifications, it is only necessary to construct maps $W_i \longrightarrow R^k$ which coincide with the composition g_i of

$$W_i \xrightarrow{f_{i-1}|W_i} \pi^{-1}(B_j) \xrightarrow{\rho_j} R^k$$

outside of V_i; where $j = j(i)$.

Assume by induction that $f_{i-1} : M^n \longrightarrow T(\zeta)$ has been defined, as above; so that

1) $f_{i-1} \mid \overline{U}_1 \cup \ldots \cup \overline{U}_{i-1}$ is transverse regular on B, and

2) $f_{i-1}^{-1} B \subset U$.

For the case $i = 1$, both conditions are certainly satisfied. Consider the composition g_i above, carrying W_i into R^k. Choose an approximation $g_i' : W_i \longrightarrow R^k$, as in Lemma 4, so that

(a) $g_i' \mid \overline{U}_i$ has the origin as a regular value,

(b) g_i' coincides with g_i outside of V_i, and

(c) the approximation is sufficiently close so that $g_i' \mid (\overline{U}_1 \cup \ldots \cup \overline{U}_{i-1}) \cap W_i$ has the origin as regular value (making use of Lemma 5); and so that $g_i'(W_i - U)$ does not contain the origin.

Now define f_i by the conditions

$$\pi f_i(x) = \pi f_o(x) \qquad \text{for all} \quad x \in f_o^{-1}(E)$$

$$\rho_j f_i(x) = g_i'(x) \qquad \text{for all} \quad x \in W_i$$

$$f_i(x) = f_{i-1}(x) \qquad \text{for all} \quad x \notin V_i .$$

Conditions 1) and 2) above are clearly satisfied, since regularity of the g_i corresponds to transverse regularity of the f_i along B. The required map $h:M^n \longrightarrow T(\zeta)$ is now given by $h = f_m$. The conditions that:

1) $f_m | \overline{U}$ is transverse regular on B, and

2) $f_m^{-1} B \subset U$;

guarantee that f_m is transverse regular on B.

Remark: Suppose that M^n is an oriented manifold and that ζ is an oriented bundle. Then the manifold $h^{-1}(B) \subset M^n$ has a standard orientation induced as follows:

(1) The map h induces a bundle map of the normal bundle ν^k of $h^{-1}(B)$ in M^n into the normal bundle of B in E, which is equivalent to ζ. Hence ν^k is oriented.

(2) For any submanifold there is a bundle map

$$\tau^{n-k} \oplus \nu^k \longrightarrow \tau^n .$$

Hence if the tangent bundle τ^n and the normal bundle ν^k are oriented, there is an induced orientation for τ^{n-k}.

Lemma 6: Let f and g be homotopic maps of M^n into $T(\zeta)$ which are both differentiable wherever possible and both transverse regular on B. Then the oriented manifold $f^{-1}(B)$ and $g^{-1}(B)$ belong to the same cobordism class.

Proof: The homotopy will give the bounding manifold. That is, choose a homotopy

$$h_o\colon M\times[0,5] \longrightarrow T(\zeta)$$

so that $h_o(x,t) = f(x)$ for $t \leq 2$, $h_o(x,t) = g(x)$ for $t \geq 3$, and so that h_o is differentiable on $h_o^{-1}(E)$.

Then, just as in the proof of Theorem 36, h_o can be approximated by a map h_m which is transverse regular on B. Furthermore this approximation can be chosen so that $h_m(x,t) = h_o(x,t)$ for $t \leq 1$ or $t \geq 4$.

[Choose the open sets W_i in $M^n\times(1,4)$ so as to cover the compact set $h_o^{-1}(B) \times [2,3]$. Then the argument of Theorem 36 shows that h_n will be transverse regular over $M^n\times[2,3]$. It is only necessary to choose all of the approximations close enough so that transverse regularity is not lost on the remainder of $M^n\times[0,5]$]
The inverse image $h_m^{-1}(B)$ will then be the required bounded-manifold with boundary diffeomorphic to $g^{-1}(B) - f^{-1}(B)$.

5. The main theorem.

In the place of the manifold M^n of the previous section, subsubstitute the $(n+k)$-sphere.

Lemma 7. Let ζ^k be an oriented differentiable k-plane bundle. The correspondence which assigns to each transverse regular map $f\colon S^{n+k} \longrightarrow T(\zeta^k)$ the manifold $f^{-1}(B(\zeta^k))$ gives rise to a homomorphism λ of the homotopy group $\pi_{n+k}(T(\zeta^k))$ into the cobordism group Ω^n.

Proof: Theorem 36 and Lemma 6 imply that every element of the homotopy group corresponds to a unique element of the cobordism group. It is clear that this correspondence is a homomorphism.

Now consider the universal bundle $\tilde{\gamma}^k$ of oriented k-planes through the origin in k+h-dimensional Euclidean space (see p. 59). The main result of cobordism theorem is the following:

Theorem of Thom. If k and h are sufficiently large, then the homomorphism

$$\lambda:\pi_{n+k}(T(\tilde{\gamma}_h^k)) \longrightarrow \Omega^n$$

is an isomorphism onto.

(Thom's notation for $T(\tilde{\gamma}_h^k)$ is $M(SO(k))$.), Thus the computation of the cobordism group is reduced to a problem in homotopy theory. For our purposes it will be sufficient to prove half of this theorem.

Lemma 8. For $k,h \geq n$ the homomorphism

$$\lambda:\pi_{n+k}(T(\tilde{\gamma}_h^k)) \longrightarrow \Omega^n$$

is onto.

Proof. Start with any manifold M^n. According to Whitney [28], M^n can be imbedded in R^{n+k} providing that $k \geq n$. Let ν^k denote the normal bundle and $E_\varepsilon(\nu^k)$ the subset of the total space consisting of normal vectors of length $< \varepsilon$. Here ε should be small enough so that the correspondence

$$\text{normal vector} \xrightarrow{e} \text{end point}$$

defines a diffeomorphism e of $E_\varepsilon(\nu^k)$ onto a neighborhood U of M^n. Define a map

$$f:R^{n+k} \longrightarrow T(\nu^k)$$

transverse regular along M^n by

$$f(x) = t_o \quad \text{for} \quad x \notin U$$

$$f(e(\vec{v})) = \vec{v}/(\varepsilon - \|\vec{v}\|) \quad \text{for} \quad e(\vec{v}) \in U .$$

Let $g:\nu^k \longrightarrow \tilde{\gamma}_n^k$ be the generalized Gauss map, defined by the correspondence

normal plane $\longrightarrow$ parallel plane through origin

(see p. 22-23) and let

$$g_T:T(\nu^k) \longrightarrow T(\tilde{\gamma}_n^k)$$

denote the induced map of the Thom space. Then the composition

$$g_T f:R^{n+k} \longrightarrow T(\tilde{\gamma}_n^k)$$

is clearly transverse regular on $B(\tilde{\gamma}_n^k) = \tilde{G}_{k,n}$. Furthermore the inverse image is

$$f^{-1}g_T^{-1}(\tilde{G}_{k,n}) = M^n .$$

Now replacing euclidean space by its one-point compactification S^{n+k}, this completes the proof that

$$\lambda:\pi_{n+k}(T(\tilde{\gamma}_n^k)) \longrightarrow \Omega^n$$

is onto. The more general case $h \geq n$ is easily handled by the same method.

6. Homotopy and cohomotopy groups modulo $\mathcal{C}$.

Let $\mathcal{C}$ denote the class of all finite abelian groups. A homomorphism

$$h:A \longrightarrow B$$

between abelian groups is called a $\mathcal{C}$-isomorphism if the kernel and cokernel ($= B/hA$) belong to $\mathcal{C}$. (This concept is due to Serre [16].)

Lemma 9. Let X be a finite complex which is $(k-1)$-connected. Then the Hurewicz homomorphism

$$\phi_*:\pi_r(X) \longrightarrow H_r(X;Z)$$

is a $\mathcal{C}$-isomorphism for $r \leq 2k-2$.

Instead of giving a detailed proof, it is sufficient to observe that this Lemma is dual, in the sense of the Spanier-Whitehead duality [19], to Lemma 10 below.

If X is a finite complex of dimension $\leq 2n-2$ then the set of all homotopy classes of maps

$$f:X \longrightarrow S^n$$

form a group $\pi^n(X)$, called the n-th cohomotopy group. The "co-Hurewicz homomorphism"

$$\phi^*:\pi^n(X) \longrightarrow H^n(X;Z)$$

is defined by

$$(f) \longrightarrow f^*(\sigma^n)$$

where σ^n generates $H^n(S^n;Z)$. (For further details see [18].)

Lemma 10 (Serre): The homomorphism

$$\phi^*:\pi^n(X) \longrightarrow H^n(X;Z)$$

is a $\mathcal{C}$-isomorphism (for dim $X \leq 2n-2$).

For the proof see Serre [16].

Applying Spanier-Whitehead duality, Lemma 9 follows.

7. The structure of Ω^* modulo $\mathcal{C}$

By the rank of an abelian group is meant the maximal number of elements which are linearly independent.

Theorem 37. The cobordism group Ω^n is finitely generated and has rank

$$\begin{aligned} &\pi(s) \text{ for } n = 4s \\ &\quad 0 \text{ for } n \not\equiv 0 \pmod 4). \end{aligned}$$

The proof will be based on

Lemma 11: Assume that k and h are sufficiently large. Each of the following groups is finitely generated, and has rank $\pi(s)$ or 0 according as $n = 4s$ or $n \not\equiv 0 \pmod 4$:

(1) the cohomology group $H^n(\tilde{G}_{k,h};Z)$;

(2) the cohomology group $H^{n+k}(T(\tilde{\gamma}_h^{\,k});Z)$;

(3) the homology group $H_{n+k}(T(\tilde{\gamma}_h^{\,k});Z)$; and

(4) the homotopy group $\pi_{n+k}(T(\tilde{\gamma}_h^{\,k}))$.

Proof. Assertion (1) follows from Theorem 32. According to Lemma 2 the cohomology groups of the Thom space are isomorphic to those of the base space, with a dimension shift. This proves (2). Assertion (3) now follows from the universal coefficient theorem, together with the fact that $T(\tilde{\gamma}_h^{\,k})$ is a finite complex (Lemma 1). Assertion (4) now follows since the Hurewicz homomorphism is a $\mathcal{C}$-isomorphism (Lemma 9).

Proof of Theorem 37. According to Lemma 8, Ω^n is a homomorphic image of $\pi_{n+k}(T(\tilde{\gamma}_h^{\,k}))$. Therefore Ω^n is finitely generated and

$$\operatorname{rank} \Omega^{4s} \le \pi(s), \quad \operatorname{rank} \Omega^n = 0 \text{ for } n \not\equiv 0 \pmod 4).$$

But according to Theorem 35 Corollary 2:

$$\operatorname{rank} \Omega^{4s} \ge \pi(s).$$

This completes the proof.

Now consider the tensor product of Ω^* with the rational numbers. The argument shows that the vector space $\Omega^{4s} \otimes Q$ has rank $\pi(s)$, and also gives an explicit basis: namely the set of products

$$P^{2i_1}(C) \times \cdots \times P^{2i_r}(C), \quad i_1 \ldots i_r \in \Pi(s) .$$

This proves:

Corollary 1. The algebra $\Omega^* \otimes Q$ has the structure of a polynomial algebra generated by the complex projective spaces $P^{2i}(C)$, $i = 1,2,\ldots$.

Corollary 2. If all of the Pontrjagin numbers of M^n are zero, then some multiple kM^n, $k > 0$, is a boundary.

For otherwise there would be too many linearly independent elements of Ω^n.

Remark Thom has made the following conjecture (unpublished). If all the Pontrjagin numbers and the Stiefel-Whitney numbers of M^n are zero, then M^n is a boundary. This conjecture is supported by the fact that Ω^n has no odd torsion (Milnor [13]), that is, if M^n is not a boundary, no odd multiple bounds. (Note that Thom has proved a weaker statement if we ignore questions of orientation (p. 18): if all the Stiefel-Whitney numbers are zero, then M^n is an (unoriented) boundary.)

XV The index theorem

The material in this chapter is due to Hirzebruch [7], [9].

1. Multiplicative sequences.

Let Λ be a fixed commutative ring with unit. (In the main application Λ will be the rational numbers.)

Review of Chapter XIII. The symbol A^* will stand for a graded commutative Λ-algebra with unit. To each such A^* corresponds a group $\Gamma_1 A^*$ with elements

$$a = (1, a_1, a_2, \ldots) = 1 + a_1 + a_2 + \cdots .$$

For each $\omega \in \Pi(n)$ there is a polynomial

$$s_\omega : \Gamma_1 A^* \longrightarrow A^n ,$$

which satisfies the product formula of Theorem 33.

Consider a sequence of polynomials

$$K_1(x_1), \ K_2(x_1,x_2), \ K_3(x_1,x_2,x_3), \ \ldots$$

with coefficients in Λ such that, if the variable x_i is assigned degree i:

(1) Each K_n is homogeneous of degree n.

Then given A^* as above and given $a \in \Gamma_1 A^*$, define a new element $K(a) \in \Gamma_1 A^*$ by the formula

$$K(a) = (1, K_1(a_1), \ K_2(a_1,a_2), \ldots)$$

<u>Definition:</u> $\{K_n\}$ is a <u>multiplicative sequence</u> of polynomials if the identity

(2) $\quad K(a \cdot b) = K(a) \cdot K(b)$

is satisfied for all A^* and all $a,b \in \Gamma_1 A^*$.

[Examples.(I) Given any constant $\lambda \in \Lambda$ the polynomials

$$K_n(x_1,\ldots,x_n) = \lambda^n x_n$$

form a multiplicative sequence. The cases $\lambda = +1$ (identity map) and $\lambda = -1$ (compare Theorem 28 p. 77) are of particular interest.

(II) The identity $K(a) = a^{-1}$ defines a multiplicative sequence with

$$K_1(x_1) = -x_1 , \quad K_2(x_1,x_2) = x_1^2 - x_2 ,$$

$$K_3(x_1 x_2, x_3) = -x_1^3 + 2x_1 x_2 - x_3, \quad \text{etc.}$$

These polynomials describe the relations between the Pontrjagin classes of the tangent bundle and normal bundle of a manifold in Euclidean space.

(III) The polynomials $K_{2n+1} = 0$,

$$K_{2n}(x_1,\dots,x_{2n}) = x_n^2 - 2x_{n-1}x_{n+1} + \cdots \mp 2x_1x_{2n-1} \pm 2x_{2n}$$

form a multiplicative sequence. (Compare p. 82).]

The following theorem gives a description of the set of all possible multiplicative sequences. Consider the polynomial ring $\Lambda[t]$, with degree $t = 1$. Then $\Gamma_1 \Lambda[t]$ is the set of all formal power series

$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$$

with coefficients in Λ. In particular $1 + t$ is an element of $\Gamma_1 \Lambda[t]$.

Theorem 38 (Hirzebruch). Given a formal power series $f(t) \in \Gamma_1 \Lambda[t]$, there is one and only one multiplicative sequence $\{K_n\}$ satisfying the condition

$$K(1+t) = f(t).$$

(The condition $K(1+t) = f(t)$ is equivalent to the condition that the coefficient of x_1^n in each $K_n(x_1,\dots,x_n)$ be λ_n).

Definition. $\{K_n\}$ will be called the multiplicative sequence belonging to the power series $f(t)$.

[Examples. The three multiplicative sequences mentioned above belong to the power series $1 + \lambda t$, $1 - t + t^2 - + \cdots$, and $1 + t^2$ respectively.]

Remark. Suppose that $\{K_n\}$ belongs to $f(t)$. Then the identity

$$K(1 + a_1) = f(a_1)$$

holds for any A^* and any $a_1 \in A^1$. However this identity is no longer true if something of degree $\neq 1$ is substituted for a_1.

Proof of existence. Given

$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$$

define

$$K_n(x_1,\ldots,x_n) = \Sigma_{\omega\,\epsilon\Pi(n)} \lambda_{i_1}\cdots\lambda_{i_r} \; s_{i_1\ldots i_r}(x_1,\ldots,x_n),$$ where $\omega = i_1\ldots i_r \epsilon \Pi(n)$. Introducing the abbreviation $\lambda_{i_1\ldots i_r} = \lambda_{i_1}\cdots\lambda_{i_r}$, this means that

$$K(a) = \Sigma_n \Sigma_{\omega\,\epsilon\Pi(n)} \lambda_\omega s_\omega(a)$$

or

$$= \Sigma_\omega \lambda_\omega s_\omega(a)$$

where the summation is over all partitions of all the integers. Now

$$K(a\cdot b) = \Sigma_\omega \lambda_\omega s_\omega(a\cdot b) = \Sigma_\omega \lambda_\omega \Sigma_{\omega_1\omega_2=\omega} \; s_{\omega_1}(a)\cdot s_{\omega_2}(b)$$

$$= \Sigma_{\omega_1\omega_2=\omega} \lambda_{\omega_1} s_{\omega_1}(a)\cdot \lambda_{\omega_2} s_{\omega_2}(b)$$

$$= \Sigma_{\omega_1} \lambda_{\omega_1} s_{\omega_1}(a)\cdot \Sigma_{\omega_2} \lambda_{\omega_2} s_{\omega_2}(b)$$

where again the summations are over all partitions of all the integers. Hence

$$K(a\cdot b) = K(a)\cdot K(b).$$

For the special case $a = 1+t$, note that

$$s_\omega(1+t) = \begin{cases} t^k & \text{if } \omega = k \\ 0 & \text{if } \omega = i_1\ldots i_r,\ r > 1. \end{cases}$$

Hence $K(1+t) = 1 + \Sigma \lambda_k t^k = f(t)$, as required.

Proof of uniqueness. Consider the special case $A^* = \Lambda[t_1,\ldots,t_n]$ where the t_i are algebraically independent of degree 1, and

$$\sigma = (1+t_1)\cdots(1+t_n).$$

Then

$$K(\sigma) = K(1+t_1) \cdots K(1+t_n) = f(t_1) \cdots f(t_n).$$

Taking the homogeneous part of degree n, it follows that $K_n(\sigma_1,\ldots,\sigma_n)$ is completely determined by the power series $f(t)$. Since $\sigma_1,\ldots,\sigma_n$ are algebraically independent, this completes the proof.

Remark. Hirzebruch has given the following, more convenient, description of $\{K_n\}$ in terms of $f(t)$:

Assertion: The coefficient of $x_{i_1} \ldots x_{i_r}$ in $K_n(x_1,\ldots,x_n)$ is equal to $s_{i_1 \ldots i_r}(\lambda_1,\ldots,\lambda_n)$.

Comparing this with the uniqueness proof above, the following identity is obtained

$$K_n(x_1,\ldots,x_n) = \Sigma \lambda_{i_1} \ldots \lambda_{i_r} s_{i_1 \ldots i_r}(x_1,\ldots,x_n)$$

$$= \Sigma s_{j_1 \ldots j_k}(\lambda_1,\ldots,\lambda_n) x_{j_1} \ldots x_{j_k} .$$

This evidently expresses a symmetry property of the collection of polynomials s_ω.

Definition: Given any multiplicative sequence $\{K_i(x_1,\ldots,x_i)\}$ with rational coefficients define the K-genus $K[M^n]$ of a (compact, oriented, differentiable) manifold to be zero if n is not devisible by 4 and

$$K_s[M^{4s}] = < K_s(p_1,\ldots,p_s), \mu_{4s} >$$

for $n = 4s$, where the p_i denote the Pontrjagin classes of the tangent bundle.

Lemma 1. The correspondence $M \longrightarrow K[M]$ defines a ring homomorphism from the cobordism ring Ω^* to the rational numbers. (Or an algebra homomorphism from $\Omega^* \otimes Q$ to Q.)

Proof. It is clear that this correspondence is additive, and that the K-genus of a boundary is zero. For a product manifold $M \times M'$ with Pontrjagin class $p \times p'$, we have $K(p \times p') = K(p) \times K(p')$, hence

$$< K(p \times p'), \mu \times \mu' > = < K(p), \mu > < K(p'), \mu' > ,$$

which completes the proof.

Remark: The converse is not hard to prove: Any ring homomorphism $\Omega^* \longrightarrow Q$ is given by the K-genus for some uniquely determined K.

2. The index theorem

The index I of a manifold M^n is defined to be zero if n is not a multiple of 4, and as follows for $n = 4s$. Choose a basis $\alpha_1, \ldots, \alpha_r$ for $H^{2s}(M^{4s}; Q)$ so that the symmetric matrix

$$\| < \alpha_i \smile \alpha_j, \mu_{4s} > \|$$

is diagonal. Then $I(M^{4s})$ is the number of positive diagonal entries minus the number of negative ones (i.e. the signature of the quadratic form in the usual terminology). The following three properties will be needed:

(1) $I(M_1 + M_2) = I(M_1) + I(M_2)$,

(2) $I(M_1 \times M_2) = I(M_1) \cdot I(M_2)$

(for the proof see Hirzebruch [9]), and

(3) if M is a boundary then $I(M) = 0$.

(The proof, due to Thom [22], is based on the Poincaré duality theorem). In other words I gives rise to a ring homomorphism from Ω^* to the integers.

Remark. Although these properties will be needed only for differentiable manifolds, they are true for much more general (compact, oriented) manifolds.

Theorem 39 (Hirzebruch). Let $\{L_k(p_1,\ldots,p_k)\}$ be the multiplicative sequence of polynomials belong to the power series

$$\sqrt{t}/\tanh\sqrt{t} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + - \cdots + (-1)^{k-1}\frac{2^{2k}}{(2k)!}B_k t^k + \cdots .$$

Then the index I of any (compact, oriented, differentiable) manifold M^{4k} is equal to the L-genus $L_k[M^{4k}]$.

(Here B_k denotes the k-th Bernoulli number: $B_1 = 1/6$, $B_2 = 1/30$, ... The first three polynomials are

$$L_1 = \frac{1}{3}p_1, \quad L_2 = \frac{1}{45}(7p_2 - p_1^2), \quad L_3 = \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3), \ \ldots)$$

Proof. Since both I and $L[\ \ ]$ define algebra homomorphisms

$$\Omega^*\otimes Q \longrightarrow Q,$$

it is sufficient to check this assertion on a set of generators for the algebra $\Omega^*\otimes Q$. According to Theorem 37 Corollary 1, such a set of generators is provided by the complex projective spaces $P^{2k}(C)$. Since $H^{2k}(P^{2k}(C);Q)$ is generated by α^k, with $<\alpha^k \smile \alpha^k, \mu_{4k}> = +1$, it follows that the index of $P^{2k}(C)$ is $+1$.

Recall that the Pontrjagin class p of $P^{2k}(C)$ is $(1+\alpha^2)^{2k+1}$. (See p. 82.) We have

$$L(1+\alpha^2+0+\ldots) = \alpha/\tanh\alpha,$$

and hence

$$L(p) = (\alpha/\tanh\alpha)^{2k+1} .$$

The Kronecker index $<L(p), \mu_{4k}>$ is equal to the coefficient of α^{2k} in this power series. Replacing α by the complex variable z, this coefficient can be evaluated by (1) dividing by $2\pi i z^{2k+1}$ and (2) integrating around the origin. But the substitution $u = \tanh z$ shows that

$$\oint \frac{dz}{(\tanh z)^{2k+1}} = \oint \frac{du}{u^{2k+1}(1-u^2)} = \oint \frac{1}{u^{2k+1}}(1+u^2+\dots)du = 2\pi i .$$

Therefore $< L(p), \mu_{4k} > = +1$, which completes the proof.

Corollary 1. The L-genus of any manifold is an integer.

The index I is an integer by definition. In other words the Pontrjagin numbers of any manifold satisfy congruences:

$$p_1[M^4] \equiv 0 \pmod 3,$$

$$7p_2[M^8] - p_1{}^2[M^8] \equiv 0 \pmod{45}, \text{ etc.}$$

Corollary 2. The L-genus of a manifold is a homotopy type invariant of the oriented manifold, since I is a homotopy type invariant by definition. It is likely that the Pontrjagin numbers themselves are not homotopy type invariants. (The Pontrjagin classes are definitely not homotopy type invariants: see Chapter XVI § 5.)

Unsolved Problem: Is the L-genus the only linear combination of the Pontrjagin numbers which is a homotopy type invariant over the integers?

3. An axiomatic description of characteristic classes.

This section will sketch another application of multiplicative sequences, without giving detailed proofs. For further information see Wu[29] as well as [7].

Lemma 2. Let Λ be an integral domain containing $\frac{1}{2}$. Then the cohomology algebra

$$H^*(G_n; \Lambda)$$

of the real Grassmann space is a polynomial algebra generated by

$p_1(\gamma^n),\ldots,p_{[n/2]}(\gamma^n)$.

Proof. This follows from Theorem 32, together with the fact that $\tilde{G}_n$ is a 2-fold covering of G_n, and the fact that the Euler class changes sign when the orientation of a bundle is reversed.

It follows that characteristic classes with coefficients in Λ can only be defined in dimensions divisible by 4.

Let $\{K_n\}$ be any multiplicative sequence with rational coefficients. Then the formulas

$$k_n(\zeta) = K_n(p_1(\zeta),\ldots,p_n(\zeta)),$$

clearly define "characteristic classes" of ζ with the following properties:

(1) For each real n-plane bundle ζ over a paracompact base B, the classes $k_n(\zeta) \in H^{4n}(B;\Lambda)$ are defined.

(2) the operation $\zeta \longrightarrow k_n(\zeta)$ is natural with respect to bundle maps; and

(3) the sum $k = 1 + k_1 + \cdots \in \Gamma_1 H^*(B;\Lambda)$ satisfies

$$k(\zeta \oplus \eta) = k(\zeta)k(\eta).$$

The following converse is easy to prove. Consider the 2-plane bundle $\xi_R^{\ 1}$ over $P^\infty(C)$. (Compare pages 63, 75. The total Pontrjagin class of ξ_R^1 is $1+\alpha^2$, where α^2 has dimension 4.)

Lemma 3. Suppose that an operation k satisfying (1), (2), (3) is given. Define a formal power series $f(t)$ by the condition

$$f(\alpha^2) = k(\xi_R^{\ 1})$$

and let $\{K_n\}$ be the corresponding multiplicative sequence. Then

$$k_n(\zeta) = K_n(p_1(\zeta),\ldots,p_n(\zeta))$$

for all ζ.

As an example for any odd prime q, consider the reduced q-th power operation

$$\mathcal{P}^i : H^r(X;Z_q) \longrightarrow H^{r+2i(q-1)}(X;Z_q).$$

In analogy with Thom's definition of Stiefel-Whitney classes (p. 35) it is natural to define a characteristic class

$$Q_i(\zeta) \in H^{2i(q-1)}(B(\zeta);Z_q)$$

by the identity $Q_i(\zeta) = \phi^{-1}\mathcal{P}^i\phi(1)$.

Theorem (Wu) This characteristic class $Q_i(\zeta)$ is equal to $K_{\frac{1}{2}i(q-1)}(p(\zeta))$ mod q where $\{K_j\}$ is the multiplicative sequence over Z_q corresponding to the power series

$$f(t) = 1 + t^{\frac{1}{2}(q-1)}$$

[Thus for $q = 3$, $Q_i(\zeta) = p_i(\zeta)$; and

for $q = 5$, $Q_i = p_i^2 - 2p_{i-1}p_{i+1} + - \cdots \pm 2p_{2i}$]

The proof is not difficult.

Remark Just as in the mod 2 case it can be shown that $Q_i(\tau^n)$, for the tangent bundle τ^n of a manifold, is a homotopy type invariant. (Compare Theorem 17 p. 55.) In fact

$$Q_i = V_i + \mathcal{P}^1 V_{i-1} + \mathcal{P}^2 V_{i-2} + \cdots$$

where V_j is characterized by the identity $<\mathcal{P}^j\alpha, \mu_n> = <\alpha \cup V_j, \mu_n>$ for all $\alpha \in H^{n-2j(q-1)}(M^n;Z_q)$. As indicated earlier, less is known about the existence of linear combinations of the Pontrjagin numbers which are homotopy type invariants with integral coefficients.

XVI Combinatorial Pontrjagin classes.

For any triangulated manifold K^n Thom [25] has defined classes $\ell_i \in H^{4i}(K^n;Q)$ which are combinatorial invariants. In the case of a differentiable manifold, suitably triangulated, these coincide with the Hirzebruch classes $L_i(p_1,\ldots,p_i)$ of the tangent bundle τ^n. Since the equations $\ell_i = L_i(p_1,\ldots,p_i)$ can be uniquely solved for the Pontrjagin classes:

$$p_1 = 3\ell_1, \qquad p_2 = \frac{1}{7}(45\ell_2 - 9\ell^2)$$

it follows that the rational Pontrjagin classes $p_i(\tau^n) \in H^{4i}(M^n;Q)$ are combinatorial invariants. (This remark depends on the fact that the coefficient of p_k in L_k is never zero. See Hirzebruch [9].)

Sections 1 to 4 will give a new version of Thom's construction. Section 5 will give two applications.

(1) An example is given of two simply connected manifolds which belong to the same homotopy type, but are not combinatorially equivalent.

(2) An example is given of a combinatorial manifold which does not possess any differentiable structure compatible with the given combinatorial structure.

1. The differentiable case.

In order to motivate the definition we will first give a new interpretation for the classes $L_i(p_1,\ldots,p_i)$ in a differentiable n-manifold. The restriction $n \geq 8i+2$ will be needed at first.

Consider a differentiable map $f\colon M^n \longrightarrow S^{n-4i}$ of class C^∞.

Lemma 1. For almost every point $y \in S^{n-4i}$ the inverse image $f^{-1}(y)$ is a differentiable manifold $M^{4i} \subset M^n$ having trivial normal bundle.

Here "almost every" means "except on a set of measure zero".

Proof: (Compare Chapter XIV §3.) It follows from the Theorem of Sard that almost every $y \in S^{n-4i}$ is a regular value of f. But if y is a regular value then $f^{-1}(y)$ is a manifold (Chapter XIV Lemma 3); and the normal bundle of $f^{-1}(y)$ maps into the tangent bundle of S^{n-4i} at y, hence is a product bundle.

Let σ^k, μ_n denote the standard generators of $H^k(S^k;Z)$, $H_n(M^n;Z)$ respectively. The class $L_i(p_1(\tau^n),\ldots,p_i(\tau^n)) \in H^{4i}(M^n;Q)$ will be written as $L_i(\tau^n)$.

Theorem 40: For every differentiable map $f:M^n \longrightarrow S^{n-4i}$ and almost every $y \in S^{n-4i}$ the Kronecker index

$$< L_i(\tau^n) \smile f^*(\sigma^{n-4i}), \mu_n >$$

is equal to the index I of the manifold $f^{-1}(y) = M^{4i}$. In the case $n \geq 8i+2$ the class $L_i(\tau^n)$ is completely characterized by this identity.

Proof: Let τ^{4i} be the tangent bundle of M^{4i}, and

$$j:M^{4i} \longrightarrow M^n$$

the inclusion map. Then j is covered by a bundle map $\tau^{4i} \oplus \nu^{n-4i} \longrightarrow \tau^n$. Since the normal bundle ν^{n-4i} is trivial, this means that $L_i(\tau^{4i})$ is equal to $j^*L_i(\tau^n)$. Hence the index

$$I(M^{4i}) = < L_i(\tau^{4i}), \mu_{4i}>$$

is equal to $< L_i(\tau^n), j_*\mu_{4i} >$.

The Poincaré dual of the homology class

$$j_* \mu_{4i} \in H_{4i}(M^n;Z)$$

is clearly the cohomology class

$$f^*(\sigma^{n-4i}) \in H^{n-4i}(M^n;Z)$$

Therefore

$$< L_i(\tau^n),\ j_* \mu_{4i} > \ = \ < L_i(\tau^n),\ f^*(\sigma^{n-4i}) \cap \mu_n >$$

$$= \ < L_i(\tau^n) \cup f^*(\sigma^{n-4i}),\ \mu_n > \ .$$

(For a discussion of cap products, see the appendix.) This proves the first assertion of Theorem 39.

To prove the second recall that, for $n \leq 2k-2$, the group $H^k(M^n;Z)$ is generated, modulo $\mathcal{C}$, by those cohomology classes of the form $f^*(\sigma^k)$, where $\mathcal{C}$ is the class of finite abelian groups. (This is a result of Serre [16]. See Chapter XIV Lemma 10.) Now substituting $k = n - 4i$, the restriction $n \leq 2(n-4i)-2$ becomes $n \geq 8i+2$.

[Remark. As a method for computing $L_i(\tau^n)$, Theorem ~~39~~ C is probably hopeless. However the following consequence might be useful in studying cohomotopy groups of manifolds.

Corollary 1. For any element $\phi^*(f)$ in the image of

$$\phi^* : \pi^{n-4i}(M^n) \longrightarrow H^{n-4i}(M^n;Z) \ ,$$

the Kronecker index

$$< L_i(\tau^n) \cup \phi^*(f),\ \mu_n >$$

is an integer.

This is non-trivial since the class $L_i(\tau^n)$ is usually not an integral class. For example, for the complex projective space $P^m(C)$, the class $1 + L_1 + L_2 + \cdots$ is given by

$$L(\tau^{2m}) = (\alpha/\tanh \alpha)^{m+1} = 1 + \frac{m+1}{3}\alpha^2 + \frac{5m^2+3m-2}{90}\alpha^4 + \cdots$$

A typical consequence is:

Corollary 2. If $m \equiv 0 \pmod 3$ then every element of the image $\phi^* \pi^{m-8}(P^m(C)) \subset H^{m-8}(P^m(C);Z)$ is divisible by 9.]

2. The combinatorial case

The following will be a convenient class of objects to work with. Let K be a finite simplicial complex.

Definition: K is a (compact, simplicial) (rational) homology n-manifold if the star boundary of each simplex has the rational homology groups of an (n-1)-sphere. [This condition can also be put in the following topologically invariant form: for each point $x \in K$ the local homology groups

$$H_i(K, K-x; Q)$$

should be zero for $i \neq n$ and isomorphic to Q for $i = n$.]

Each component of such a complex K is clearly a simple n-circuit. (See Eilenberg and Steenrod [4] p. 106.) Hence it makes sense to require that K be oriented. The orientation may be specified by an element

$$\mu \in H_n(K;Z).$$

Similarly one can define the concept of a "bounded homology n-manifold" K. In this case the boundary $\dot{K}$ is a homology (n-1)-manifold, and the orientation is specified by $\mu \in H_n(K,\dot{K};Z)$.

Let Σ^r denote the boundary of an (r+1)-simplex. The key lemma will be the following

Lemma 2. Let f be a piecewise linear map from a homology n-manifold K to Σ^r, $r = n-4j$. Then:

(1) For almost every $y \in \Sigma^r$ the inverse image $f^{-1}(y)$ is a homology $4j$-manifold. Given orientations for K and Σ^r, there is an induced orientation for $f^{-1}(y)$.

(2) The index $I f^{-1}(y)$ is independent of y for almost all y. Denote this constant value by $I(f)$. Finally:

(3) The integer $I(f)$ depends only on the homotopy class of f.

Here "almost every" can be taken to mean "except for y belonging to some lower dimensional complex".

(Remark: There is some analogy between this definition of $I(f)$, and the definition of the Hopf invariant.)

The proof will be based on:

<u>Lemma</u> 3: Let $f:K \longrightarrow L$ be a simplicial map and let y belong to the <u>interior</u> Δ of a simplex of L. Then $f^{-1}(\Delta)$ is homeomorphic to $\Delta \times f^{-1}(y)$.

(This assertion would be false for a closed simplex.)

<u>Proof</u>. Let $A_o, \ldots, A_r$ denote the vertices of Δ and let

$$y = t_o A_o + \cdots + t_r A_r$$

(where $t_i > 0$, $\Sigma t_i = 1$). Then any $x \in f^{-1}(\Delta)$ can be expressed uniquely as

$$x = t_o{}' A_o{}' + \cdots + t_r{}' A_r{}'$$

with $A_i{}' \in f^{-1}(A_i)$ being points of the boundary of the simplex of K to which x belongs. The required homeomorphism

$$f^{-1}(\Delta) \longleftrightarrow \Delta \times f^{-1}(y)$$

is now defined by

$$x \longleftrightarrow (t_o{}' A_o + \cdots + t_r{}' A_r, t_o A_o{}' + \cdots + t_r A_r{}').$$

Remark: Note that the composition

$$f^{-1}(\Delta) \longrightarrow \Delta \times f^{-1}(y) \longrightarrow \Delta$$

is just the original map f (restricted to $f^{-1}(\Delta)$). Hence $f^{-1}(y')$ is homeomorphic to $f^{-1}(y)$ for all $y' \in \Delta$.

Proof of Lemma 2: (1). Subdivide K and Σ^r so that f is simplicial. Assume that y belongs to the interior Δ of a top dimensional simplex. Then by Lemma 3, $\Delta \times f^{-1}(y)$ has the local homology groups of an n-manifold. Since Δ has the local homology groups of an $(n-4i)$-manifold, it follows easily that $f^{-1}(y)$ is a homology $4i$-manifold. Furthermore, given orientations for Δ and $\Delta \times f^{-1}(y)$, there is clearly an induced orientation for $f^{-1}(y)$.

The remark above stated that $f^{-1}(y')$ is homeomorphic to $f^{-1}(y)$ for all $y' \in \Delta$. Therefore:

(4) If y is chosen as above, then the index $I f^{-1}(y')$ is independent of y' at least for y' in a neighborhood of y.

Now suppose that f and g are homotopic piecewise linear maps $K \longrightarrow \Sigma'$. Then they are related by a piecewise linear homotopy

$$h:K \times [0,1] \longrightarrow \Sigma^r.$$

Subdividing and choosing $y \in \Delta$ as above, a similar argument shows that $h^{-1}(y)$ is a bounded homology manifold with boundary $f^{-1}(y) - g^{-1}(y)$. Since the index of a boundary is zero, this implies that

(5) If f is homotopic to g, then $I f^{-1}(y) = I g^{-1}(y)$ for almost all y.

Given $f:K^n \longrightarrow \Sigma^r$ let y_1 and y_2 be any two points satisfying (4) above. Let

$$u:\Sigma^r \longrightarrow \Sigma^r$$

be a piecewise linear homeomorphism of degree +1 carrying y_1 into y_2. Then uf is homotopic to f, hence

$$I(f^{-1}u^{-1}z) = I(f^{-1}z)$$

for almost all z. Choosing z close to y_2, so that $u^{-1}z$ is close to y_1, it follows from (4) that

$$I f^{-1}(y_1) = I f^{-1}(y_2).$$

This proves assertion (2). Since (3) now follows from (5), this completes the proof.

Lemma 4. If $n \geq 8i+2$ then the correspondence $f \longrightarrow I(f)$ defines a homomorphism from the cohomotopy group $\pi^{n-4i}(K^n)$ to the integers.

Proof. It follows from the definition of addition in the cohomotopy group that $(f+g)^{-1}(y)$ is the disjoint union of $f^{-1}(y)$ and $g^{-1}(y)$, providing f and g are chosen carefully within their homotopy chasses. Hence $I(f+g) = I(f) + I(g)$.

The main theorem will now follow easily:

Theorem 41. For $n \geq 8i+2$ there exists a unique cohomology class

$$\ell_i \in H^{4i}(K^n;Q)$$

such that the identity

$$< \ell_i \cup f^*(\sigma), \mu > = I(f)$$

is satisfied for every map $f: K^n \longrightarrow \Sigma^{n-4i}$.

(Here σ denotes the standard generator of $H^{n-4i}(\Sigma^{n-4i};Z)$.)

Proof: Consider the diagram

$$\begin{tikzcd}
\pi^{n-4i}(K^n) \arrow[r, "I"] \arrow[d, "\emptyset^*"] & Z \arrow[d, "\text{inclusion}"] \\
H^{n-4i}(K^n;Z) \arrow[r, dashed, "I'"] & Q .
\end{tikzcd}$$

Since $\emptyset^*$ is a $\mathcal{C}$-isomorphism (Chapter XIV Lemma 10), the bottom arrow can be filled in uniquely. By the Poincaré duality theorem, the resulting homomorphism I' is given by the formula

$$\beta \longrightarrow < \ell_i \smile \beta, \mu_n >$$

for some unique $\ell_i \in H^{4i}(K^n;Q)$. This completes the proof.

3. The compatibility theorem $\ell_i(M^n) = L_i(\tau^n)$.

Now it is necessary to compare the combinatorial and differentiable situations.

By a triangulation of a space is meant a homeomorphism of a simplical complex onto the space. J. H. C. Whitehead has shown that a differentiable manifold M^n has a preferred class of triangulations

$$t:K \longrightarrow M^n$$

which are called C^1-triangulations. (See [27], [12].) The complex K which occurs is uniquely determined, up to combinatorial equivalence (= piecewise linear homeomorphism). Hence the class $(t^*)^{-1} \ell_i \in H^{4i}(M^n;Q)$ does not dpend on the particular C^1-triangulation t which is chosen. This class will be denoted by $\ell_i(M^n)$. (It is of course defined only for $n \geq 8i + 2$.)

Theorem 42. The class $\ell_1(M^n)$ (defined for a differentiable manifold by a combinatorial procedure) is equal to the Hirzebruch class L_1 of the tangent bundle.

Proof: Let $f:M^n \longrightarrow S^r$ be a differentiable map. We will construct a diagram

$$\begin{tikzcd}
K \arrow[r, "t"] \arrow[d, "f_1"] & M^n \arrow[d, "f"] \\
K_1 \arrow[r, "t_1"] & S^r
\end{tikzcd} ,$$

commutative up to homotopy, where t and t_1 are C^1-triangulations, so that

$$I\, f_1^{-1}(y_1) = I\, f^{-1}(y)$$

for almost all $y_1 \in K_1$, $y \in S^r$. Together with Theorems 40 and 41, this will complete the proof.

Let $y \in S^r$ be any regular value of f. If B is a sufficiently small ball around y, then the inverse image $f^{-1}(B)$ can be considered as the product $B \times f^{-1}(y)$. Choose a C^1-triangulation

$$t_1 : K_1 \longrightarrow S^r$$

so that some subcomplex K_2 of K_1 triangulates B; and choose a C^1-triangulation $K_3 \longrightarrow f^{-1}(y)$. Then the product triangulation

$$K_2 \times K_3 \longrightarrow B \times f^{-1}(y)$$

can be extended to a triangulation $t:K \longrightarrow M^n$.

The composition $t_1^{-1} f\, t : K \longrightarrow K_1$ will not, in general, be piecewise linear. However its restriction to $K_2 \times K_3$ is just the projection map into $K_2 \subset K_1$, hence is piecewise linear. Choose a piecewise linear map $f_1 : K \longrightarrow K_1$ which agrees with $t_1^{-1} f\, t$ on $K_2 \times K_3$, and approximates it elsewhere. Let y_1 denote $f_1^{-1}(y)$. Then $f_1^{-1}(y_1)$ will be homeomorphic to $f(y)$. Hence

$$I f_1^{-1}(y_1{}') = I f_1^{-1}(y_1) = I f^{-1}(y) = I f^{-1}(y')$$

for all $y_1{}'$, y' close to y_1 and y respectively.

This completes the proof.

4. The unrestricted case:

So far the condition $n \geq 8i + 2$ has been imposed. However given K^n one can always form the product space $K^n \times \Sigma^m$ with m large. The class $\ell_i(K^n)$ can then be defined as the class induced from $\ell_i(K^n \times \Sigma^m)$ be the natural inclusion map. It is not hard to show that this new class is well defined, and has the expected properties (For example $< \ell_i(K^{4i}), \mu_{4i} > = I(K^{4i})$.)

Another extension which can easily be made is to bounded homology manifolds. It is only necessary to substitute the relative cohomotopy groups

$$\pi^{n-4i}(K^n, \dot{K}^n)$$

and the Lefschetz duality theorem in the above discussion.

5. Applications

The first example which will be discussed was discovered independently by Thom [24] p. 81, Tamura [21], and Shimada [17].

Lemma 5: Given integers m and n with $n \geq 4$, there exists an n-plane bundle ζ over S^4 with $p_1(\zeta) = 2m\sigma$.

(σ = standard generator of $H^4(S^4;Z)$.)

Remark: A corresponding assertion for any sphere S^{4k} has recently been proved by Borel, Hirzebruch and Bott. The integer 2 must be replaced by

$$(2k-1)!\ \text{G.C.D.}(k+1\ ,2).$$

This is a best possible result.

Proof of Lemma 5: First assume that $n \geq 8$. Let τ^8 denote the tangent bundle of the Quaternion projective plane $P^2(K)$, and let u generate $H^4(P^2(K);Z) \approx Z$. According to Hirzebruch [8], the Pontrjagin class of τ^8 is $1 + 2u + 7u^2$. Since this space is 3-connected, there exists a map

$$f:S^4 \longrightarrow P^2(K)$$

satisfying $f^*(u) = m\sigma$. The induced bundle ζ^8 over S^4 will now satisfy Lemma 5 for $n = 8$. For $n > 8$m the Whitney sum

$$\zeta^8 \oplus \text{trivial } (n-8)\text{-plane bundle}$$

will satisfy the Lemma.

Next consider the case $n = 7$. Using obstruction theory, it is seen that ζ^8 has a non-zero cross-section. In other words ζ^8 is the Whitney sum of the required 7-plane bundle ζ^7 and a trivial line bundle. This construction can be iterated until the case $n = 4$ is reached, which completes the proof.

[The obstruction to further iteration is $X(\zeta^4) \in H^4(S^4;Z)$. This is definitely non-zero since $w_4(\tau^8) \neq 0$. See Theorem 18, p. 56.]

Lemma 6: Let ζ^n be a differentiable n-plane bundle over B and let τ^r be the tangent bundle of B. Then the Pontrjagin class of the total space E of ζ^n is given by

$$p(E) = \pi^*(p(\zeta^n)p(\tau^r)).$$

Similarly, if E_o' is the set of unit vectors in E, then

$$p(E_o') = \pi_o'^*(p(\zeta^n)p(\tau^r)).$$

Proof: The tangent bundle of E is the Whitney sum of

(1) the bundle of vectors tangent to the fibre, and

(2) the bundle of vectors normal to the fibre.

Since (1) maps into ζ^n and (2) maps into τ^r, the first assertion is clear. Since the normal bundle of E_o' in E is trivial, the second assertion follows.

Example 1: Consider an n-plane bundle ζ over S^4 where (for convenience) $n \geq 6$. Then it follows from the Gysin sequence that the homomorphism

$$\pi^*(H^4(S^4;Z) \longrightarrow H^4(E_o';Z)$$

is an isomorphism. If $p_1(\zeta) = 2m\sigma$, it follows that

$$p_1(E_o') = 2m\pi_o'^*(\sigma)$$

since $p(\tau^n(S^n)) = 1$ (p. 79.).

Since the Pontrjagin class of E_o' is a combinatorial invariant, it follows that the integer $|m|$ is a combinatorial invariant of E_o'. Thus as m varies we obtain infinitely many manifolds which are combinatorially distinct.

On the other hand, according to James and Whitehead [10], these manifolds E_o', for fixed n, fall into a finite number of distinct homotopy types (namely 13). This proves:

Assertion: There exist two differentiable simply conected 9-manifolds which have the same homotopy type, but are not combinatorially equivalent.

(The dimension 9 can easily be improved to 7.) It is not known whether these manifolds are homeomorphic. In dimension 3 there do exist manifolds which have the same homotopy type but are not homeomorphic, although the proof in that case hinges on the fundamental group

The next example is due to Thom [25]. (See also Milnor [11, [12] and Shimada [17].)

Lemma 7: Given integers i, j satisfying $i \equiv 2j \pmod 4$, there exists an oriented 4 plane bundle η over S^4 having characteristic classes

$$p_1(\eta) = i\,\sigma\,, \quad X(\eta) = j\,\sigma\,.$$

Remark: The integers i, j actually determine the equivalence class of the bundle. This is due to the fact that the homotopy group $\pi_4(\tilde{G}_4) \approx \pi_3(SO_4)$ is $Z + Z$.

Proof of Lemma 7: First let η range over all oriented 4-plane bundles over S^4. Observe that the corresponding set of pairs $p_1(\eta), X(\eta)$ forms a subgroup of the direct sum

$$H^4(S^4;Z) \oplus H^4(S^4;Z) \approx Z \oplus Z.$$

In fact each such bundle η_1 is induced by some map $f_1 : S^4 \longrightarrow \tilde{G}_4$. Given two such maps f_1, f_2 form the sum or difference $f_1 \pm f_2$ as defined in homotopy theory, and let η be the bundle induced by $f_1 \pm f_2$. Then it can be seen that the characteristic classes of η are:

$$p_1(\eta) = p_1(\eta_1) \pm p_1(\eta_2), \quad X(\eta) = X(\eta_1) \pm X(\eta_2).$$

Now consider the following two bundles: (1) The tangent bundle τ^4 of S^4, which satisfies

$$p_1(\tau^4) = 0 \qquad X(\tau^4) = 2\,\sigma\,. \text{ (see p. 79.)}$$

(2) In Lemma 5 take $n = 4$, $m = 1$. The resulting bundle ζ^4 will satisfy

$$p_1(\zeta^4) = 2\sigma , \qquad w_4(\zeta^4) \neq 0 ,$$

since $w_4(P^2(K)) \neq 0$ [See Theorem 18, p. 56] and hence

$$X(\zeta^4) = \text{ some odd multiple of } \sigma .$$

But starting with the pairs (0,2), (2,2r+1) and forming sums and differences, one can obtain any pair (i,j) which satisfies $i \equiv 2j \pmod 4$. This completes the proof.

Example 2: In particular consider the bundles for which $j = 1$ (that is $X = \sigma$). Then i can be any integer congruent to 2 modulo 4. Using the Gysin sequence, it is seen that the corresponding total spaces E_o' all have the homotopy type of the 7-sphere. Actually each such E_o' is homeomorphic to S^7(see [11]); and even combinatorially equivalent to S^7 (see [12]). Now consider the Thom space T of such a bundle. T can be formed from the space E' of vectors of length ≤ 1 by attaching a cone over the boundary E_o'. Since E_o' is a 7-sphere, it follows that E' is a compact 8-manifold. Furthermore any C^1-triangulation (in the sense of Whitehead) of E' gives rise to a triangulation of T.

It follows from Lemma 2 of Chapter XIV that

$$H^k(T;Z) = \begin{cases} Z & \text{for } k = 0,4,8 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore it is easily verified that the homomorphisms

$$H^4(S^4;Z) \xrightarrow{\pi'^*} H^4(E';Z) \longleftarrow H^4(T;Z)$$

are isomorphisms. Let σ', σ'' be the elements in the second two groups corresponding to the standard generator σ. By the Poincaré duality theorem, $< \sigma'' \cup \sigma'', \mu_8 >$ must be ± 1. Hence, choosing the orientation μ_8 properly, the index $I(T)$ is $+1$.

According to Lemma 6, the Pontrjagin class p_1 of the bounded manifold E' is $i\sigma'$. Hence the combinatorial Pontrjagin classes $p_1(E')$ and $p_1(T)$ are the rational classes corresponding to $i\sigma'$ and $i\sigma''$ respectively. Therefore the Pontrjagin number $p_1{}^2[T]$ is equal to i^2.

Using the index theorem

$$I(T) = \frac{7}{45}\, p_2[T] - \frac{1}{45}\, p_1{}^2[T]$$

it follows that the other Pontrjagin number is given by

$$p_2[T] = \frac{45 + i^2}{7} \quad .$$

But in general this is not an integer. (e.g. for $i = 6$.) Since a Pontrjagin number of a differentiable manifold must be an integer, this implies

Assertion: For $i \not\equiv \pm 2 \pmod 7$ the triangulated 8-manifold T possesses no differentiable structure which is compatible with the given triangulation.

As a corollary, it follows that the differentiable 7-manifold E_0' is not diffeomorphic to S^7. For otherwise T could be given a differentiable structure which was compatible.

In conclusion, here is an

Unsolved Problem. Let $\Omega_H{}^n$ [or $\Omega_C{}^n$] denote the analogue of the cobordism group in which homology n-manifolds [or combinatorial (= formal) n-manifolds] are used in place of differentiable n-manifolds. What is the structure of these groups, and what can be said about the natural homomorphism $\Omega^n \longrightarrow \Omega_C{}^N \longrightarrow \Omega_H{}^n$? It should be noted that in the combinatorial case, Pontrjagin numbers are not invariants of homotopy type (cf. p. 118) for it can be shown that among the 8-manifolds above for which $p2[T] = \frac{45 + i^2}{7}$ (for any $i \equiv 2(4)$), there are only finitely many homotopy types.

Appendix: The Thom isomorphism ϕ

Let ζ be an oriented n-plane bundle with projection $\pi:E \longrightarrow B$. This Appendix will give a proof that the cohomology group $H^{n+i}(E,E_o;\Lambda)$ is isomorphic to $H^i(B;\Lambda)$. (See Theorem 10' p. 40.) A corresponding theorem for homology groups is included as part of the proof. The corresponding proof for the unoriented case, with coefficient group Z_2, is left to the reader.

1. Construction of the cohomology class u

Let SE denote the total singular complex of E, and define the relative Eilenberg subcomplex $S_n(E;E_o)$ as the set of all singular simplexes

$$f:\Delta^r \longrightarrow E$$

such that f maps the (n-1)-skeleton of Δ^r into E_o. Then the following assertion will be proved.

Lemma 1: The inclusion $S_n(E;E_o) \longrightarrow SE$ induces isomorphisms of homology groups.

Next a canonical cocycle $d \in Z^n(S_n(E;E_o), SE_o)$ will be defined. Intuitively speaking, d(f) can be considered as the intersection number of the image $f(\Delta^n)$ with $B = E - E_o$. (Note that every n-simplex f in $S_n(E;E_o)$ maps the boundary of Δ^n into E_o.)

Finally $u \in H^n(E,E_o)$ will be defined as the cohomology class determined by d.

<u>Lemma</u> 2. The correspondence $\zeta \longrightarrow u(\zeta)$ is natural with respect to bundle maps. For the special case

$$B = \text{point}, \quad E = R^n$$

u is the standard generator of $H^n(R^n,R_o^n)$.

The proofs will be based on the following construction. Given any map

$$f:\Delta^r \longrightarrow E$$

let η denote the bundle over Δ^r induced from ζ by the composition πf, and let f' be the unique cross-section of η such that the diagram

$$\begin{tikzcd}
E(\eta) \arrow[r] & E \\
\Delta^r \arrow[u, "f'"] \arrow[ur, "f"] &
\end{tikzcd}$$

is commutative. Note that η is necessarily a product bundle. (See Steenrod [20] 11.6.) Hence the pair $E(\eta),E_o(\eta)$ is (n-1)-connected and $H_n(E(\eta),E_o(\eta))$ is infinite cyclic with a preferred generator.

<u>Proof of Lemma 1</u>: First observe that the pair E,E_o is (n-1)-connected, since in fact the above argument shows that any map $f_1:\Delta^r,\dot{\Delta}^r \longrightarrow E,E_o$ can be factored through a pair $E(\eta),E_o(\eta)$ which is (n-1)-connected.

But now Lemma 1 can be proved by an argument completely analogous to that given by Eilenberg [2] Chapter VI.

Definition of $d(f)$: Any n-simplex f of $S_n(E;E_o)$ gives rise to a map

$$f_1:\Delta^n,\dot{\Delta}^n \longrightarrow E,E_o.$$

Define $d(f)$ as the degree of the associated map

$$f_1':\Delta^n,\dot{\Delta}^n \longrightarrow E(\eta),E_o(\eta).$$

This defines a cochain $d \in C^n(S_n(E;E_o),SE_o)$.

It is easily verified that the coboundary of d is zero.

The proof of Lemma 2 will be left as an exercise.

2. The homology isomorphism.

Recall that the cap product of a singular n-cochain c with a singular $(n+i)$-simplex is defined to be the product of the "front i-face" of the simplex with the integer obtained by evaluating c on the "back n-face". The following properties will be needed.

(1) $\partial(c \frown a) = c \frown \partial a + (-1)^i(\delta c) \quad a$,

(2) $< c_1 \smile c_2, a > = < c_1, c_2 \frown a >$, and

(3) the cap product gives rise to a bilinear pairing

$$H^n(X,Y) \otimes H_{n+i}(X,Y) \longrightarrow H_i(X).$$

Lemma 3. The correspondence

$$a \longrightarrow \pi_*(u \frown a)$$

defines an isomorphism $\emptyset$ of $H_{n+i}(E,E_o;Z)$ onto $H_i(B;Z)$.

The proof will be divided into four cases.

Case 1: ζ is a product bundle so that $(E,E_o) = B \times (R^n,R_o^n)$. Let μ denote the standard generator of $H_n(R^n,R_o^n)$. It follows from the Künneth Theorem that the correspondence

$$a \longrightarrow a \times \mu$$

defines an isomorphism of $H_i(B)$ onto $H_{n+1}(E,E_o)$. (See Eilenberg and Zilber [5] together with Eilberg and Cartan [3] p. 113.) But it follows from Lemma 2, together with a short computation, that $\phi(a \times \mu) = \pi_*(u \cap (a \times \mu))$ is equal to a.

Case 2: B is the union of open subsets B', B'' with intersection B''', where the Lemma is known to be true for ζ restricted to B', B'' and B'''. It will be shown that the following diagram of Mayer-Vietoris sequences is commutative:

$$\begin{tikzcd}
{} \arrow[r] & H_{n+1}(E''', E_o''') \arrow[r] \arrow[d, "\phi'''"] & H_{n+1}(E', E_o') \oplus H_{n+1}(E'', E_o'') \arrow[r] \arrow[d, "\phi' + \phi''"] & H_{n+1}(E, E_o) \arrow[r, "\partial"] \arrow[d, "\phi"] & {} \\
{} \arrow[r] & H_i(B''') \arrow[r] & (H_i(B') \oplus H_i(B'') \arrow[r] & H_i(B) \arrow[r, "\partial"] & {}
\end{tikzcd}$$

Since ϕ', ϕ'' and ϕ''' are known to be isomorphisms, it will follow from the Five Lemma ([4] p. 16) that ϕ is an isomorphism.

The Mayer-Vietoris sequence can be derived as follows: The natural homomorphism of singular chain groups

$$C_*(B') \oplus C_*(B'') \longrightarrow C_*(B)$$

has kernel isomorphic to $C_*(B''')$ and an image $C_*(B, \{B', B''\})$ which is chain equivalent to $C_*(B)$. (See [4] p. 197.) Now the short exact sequence

$$(1) \qquad 0 \longrightarrow C_*(B''') \longrightarrow C_*(B') \oplus C_*(B'') \longrightarrow C_*(B; \{B', B''\}) \longrightarrow 0$$

gives rise to the required sequence of homology groups. Similarly the short exact sequence

(2) $0 \to C_*(E''', E_o''') \to C_*(E', E_o') + C_*(E'', E_o'') \to C_*(E, E_o; \{E', E''\}) \to 0$

gives rise to the relative Mayer-Vietoris sequence.

Choose a representative cocycle $z \in Z^n(E, E_o; \{E', E''\})$ for the class u, and let z', z'', z''' be the appropriate restrictions. Then a chain mapping from the sequence (2) to the sequence (1) is defined by the formulas

$$a''' \longrightarrow \pi_\#(z''' \frown a'''), \ \ldots, a \longrightarrow \pi_\#(z \frown a).$$

This chain mapping induces the required homomorphism between the Mayer-Vietoris sequences.

Case 3. B is the union of finitely many distinguished open sets $V_1, \ldots, V_k$. For $k = 1$, the assertion follows from Case 1. For $k \geq 1$ it follows by induction , applying Case 2 to the pair

$$B' = V_1 \cup \ldots \cup V_{k-1}, \quad B'' = V_k.$$

Note in particular that this argument applies whenver B is compact.

General case: Let B^* range over all compact subsets of B. Then $H_i(B)$ is the direct limit of the groups $H_i(B^*)$; and $H_{n+i}(E, E_o)$ is the direct limit of the corresponding groups $H_{n+i}(E^*, E_o^*)$. Since the assertion is true for each B^*, this completes the proof.

Remark: The arguments given for cases 2 and 3 would apply equally well to cohomology. However the limiting argument does not apply to cohomology.

3. The cohomology isomorphism.

Consider the homomorphism of Lemma 1 on the chain level. That is choose a cocycle $z \in Z^n(E,E_o)$ which represents u, and define

$$\phi_\#: C_{n+i}(E,E_o) \longrightarrow C_i(B)$$

by $\phi_\#(a) = \pi_\#(z \cap a)$. It is easily verified that $\phi_\#$ is onto. Let $C_i(K)$ denote the kernel of $\phi_\#$. Then there is a exact sequence

$$\cdots \longrightarrow H_i(K) \longrightarrow H_{n+i}(E,E_o) \longrightarrow H_i(B) \xrightarrow{\partial} \cdots$$

It follows from Lemma 3 that the chain complex K has trivial homology.

Now for any coefficient group Λ consider the corresponding cohomology sequence

$$\cdots \longrightarrow H^i(B;\Lambda) \xrightarrow{\phi^*} H^{n+i}(E_o;\Lambda) \longrightarrow H^i(K;\Lambda) \xrightarrow{\delta} \cdots$$

It follows from the universal coefficient theorem that K has trivial cohomology, so that ϕ^* is an isomorphism.

The identities

$$< \phi^\# c,a > = < c,\phi_\# a > = < c,\pi_\#(z \cap a) > = < \pi^\# c \cup z ,a >$$

show that ϕ^* is just the correspondence

$$\gamma \longrightarrow \pi^* \gamma \cup u.$$

Thus we have proved

<u>Lemma</u> 4. An isomorphism

$$\phi: H^i(B;\Lambda) \longrightarrow H^{n+i}(E,E_o;\Lambda)$$

is given by the correspondence

$$\gamma \longrightarrow \pi^*(\gamma) \cup u.$$

Now to complete the <u>proof of Theorem</u> 10' (p. 40), it is only necessary to show that u is characterized by the condition

$$j_b^* u = u_b \in H^n(\pi^{-1}b, \pi^{-1}b \cap E_o)$$

for each $b \in B$. But applying the isomorphism ϕ^{-1}, this is equivalent to showing that the element

$$1 \in H^0(B)$$

is characterized by the fact that its Kronecker index with each point b is 1. Since this is clear, this completes the proof.

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