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# DE
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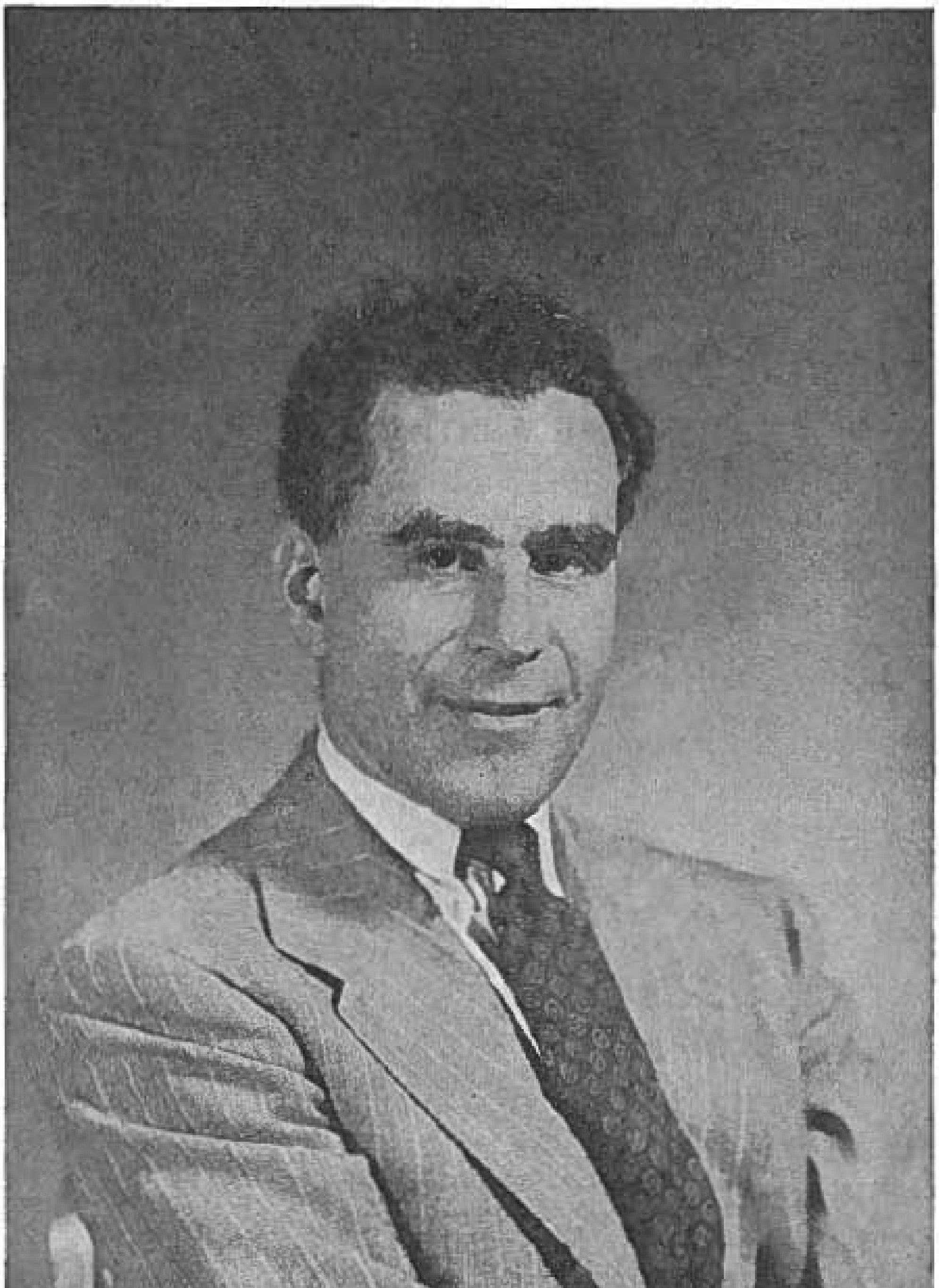

EN MEMORIA DE

WITOLD HUREWICZ

(1904–1956)

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La Comisión Organizadora.

## CONTENIDO

## ON THE STRUCTURE OF HIGHER TERMS OF THE SPECTRAL SEQUENCE OF A FIBRE SPACE

WITOLD HUREWICZ† AND EDWARD FADELL*

### 1. Introduction

Let $(E, B, p)$ denote a fibre space[1] with $B$ arcwise connected and $(E_r, d_r)$, $r = 1, 2, \ldots$, the associated spectral sequence. Furthermore, let $F = p^{-1}(b)$, $b \in B$ denote a fixed fibre. Then a well-known result of Leray-Serre states that $E_1 = C(B, H(F))$, the singular chains of $B$ with $H(F)$ as coefficients, and $d_1 : E_1 \to E_1$ is the boundary operator $\partial : C(B, H(F)) \to C(B, H(F))$ in the sense of local coefficients, where $\pi_1(B, b)$ operates on $H(F)$ in the usual manner. Hence, $E_2 = H(B, H(F))$ where the homology is in the sense of local coefficients. In case, $\pi_1(B, b) = 0$, therefore, $E_2 = H(B, H(F))$ where the coefficient group $H(F)$ is taken in the ordinary sense. In [2], the authors extended this latter result and showed that in case $B$ was $r$-connected that $E_i = H(B, H(F))$ for $i = 2, \cdots, r+1$ and $d_i = 0$ for $i = 2, \cdots r$. The purpose of this paper is to extend this result still further. An alternate way of stating the above Leray-Serre result is that if $\pi_0(B) = 0$, then $E_2$ depends only upon $B$ and the action of $\pi_1(B, b)$ on $H(F)$. Here we will show that if $B$ is $r-1$ connected then $E_i = H(B, H(F))$ for $2 \leq i \leq r$ and $E_{r+1}$ depends only upon $B$ and the action of $\pi_r(B, b)$ on $H(F)$. More precisely, we first show that $\pi_r(B, b)$ and $H(F)$ are *paired* to $H(F)$, $r > 1$. Then, in case $B$ is $r-1$ connected, $E_i = H(B, H(F))$ for $i = 2, \cdots, r$ $d_i = 0$ for $i = 2, \cdots, r-1$ and $d_r : E_r = H(B, H(F)) \to E_r$ is given by the cap product

$$d_r(h) = \gamma \cap h, \qquad h \in H(B, H(F))$$

where $\gamma$ is the characteristic cohomology class of $B$ and the cap product is defined in terms of the pairing of $\pi_r(B, b)$ and $H(F)$ to $H(F)$[2].

REMARK. The corresponding result for singular cohomology is also valid, where cup product replaces cap product and $\pi_r(B)$ and $H^*(F, G)$ (the cohomology group of $F$ with coefficients in $G$) are suitably paired.

### 2. Preliminaries

2.1. FIBRE SPACES. In this paper we employ the concept of fibre space as given in [1] and for the reader's convenience we recall the basic definitions. Given a triple $(E, B, p)$ where $p : E \to B$ is a map, let $\Omega_p$ denote the subset of $E \times B^I$ given by

$$\Omega_p = \{(e, \omega) \in E \times B^I : p(e) = \omega(0)\}.$$

† Due to the untimely death of Professor Hurewicz, the second-named author has prepared this joint account of their research and accepts full responsibility for its accuracy.

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[1] In this paper we used the term fibre space in the sense of [1], see §2.

[2] This simple description of $d_r$ was suggested by Norman Steenrod.

Then, we have a natural map $\tilde{p} : E^I \to \Omega_p$ given by

$$\tilde{p}(\alpha) = (\alpha(0),\ p(\alpha))$$

where $p(\alpha)(t) = p(\alpha(t))$, $0 \leq t \leq 1$. Finally, we say that $(E, B, p)$ is a *fibre space* provided $\tilde{p} : E^I \to \Omega_p$ admits a cross section, i.e. a map $\Lambda : \Omega_p \to E^I$ such that $\tilde{p} \circ \Lambda = 1$.[3] The map $\Lambda$ is referred to as a lifting function. It is easy to show that any two lifting functions are homotopic in the class of lifting functions, i.e. given any two lifting functions $\Lambda_0$, $\Lambda_1$, there exists a homotopy $H : \Omega_p \times I \to E^I$ such that

(i) $H_0 = \Lambda_0$, $H_1 = \Lambda_1$

(ii) $\tilde{p}H[(e, \omega), t] = (e, \omega)$, $t \in I$, $(e, \omega) \in \Omega_p$.

The following fact will also be used in the sequel. Let $\tilde{E}$ denote the space of paths in $E$ emanating from a fibre $F$, i.e.

$$\tilde{E} = \{\alpha \in E^I : \alpha(0) \in F\}$$

where $F = p^{-1}(b)$, $b \in B$. Then, if $\Lambda$ is a lifting function for the fibre space $(E, B, p)$, let $\tilde{\Lambda} : \tilde{E} \to \tilde{E}$ be given by

$$\tilde{\Lambda}(\alpha) = \Lambda(\alpha(0), p\alpha),\ \alpha \in \tilde{E}.$$

Then one shows easily that $\tilde{\Lambda}$ is homotopic to the identity map $1 : \tilde{E} \to \tilde{E}$.

2.2. Singular theory based on cubes. Let $X$ denote a topological space and, employing the notation in Serre [3], $Q_n(X)$ the free abelian group generated by singular $n$-cubes in $X$. Letting $D_n(X)$ denote the subgroup generated by degenerate $n$-cubes, we set $C_n(X) = Q_n(X)/D_n(X)$. Then $C(X) = \sum_n C_n(X)$ is called the group of singular chains in $X$ (based on cubes) with integral coefficients. For an arbitrary coefficient group $G$ we set $C_n(X, G) = C_n(X) \otimes G$ and $C(X, G) = \sum_n C_n(X, G)$. Also, we set $C^n(X, G) = \text{Hom}\ (C_n(X), G)$. $C^*(X, G) = \sum_n C^n(X, G)$ is then the group of singular cochains with coefficients in $G$.

The boundary operator $\partial$ in $C(X)$ is given by

$$\partial u = \textstyle\sum_{i=1}^n (-i)^i\,[\lambda_i^1 u - \lambda_i^0 u]^4$$

where $u$ is a singular $n$-cube and

$$(\lambda_i^\varepsilon u)(x_1, \cdots, x_{n-1}) = u(x_1, \cdots, x_{i-1}, \varepsilon, x_i, \cdots, x_{n-1})$$

for $\varepsilon = 0, 1$; $1 \leq i \leq n$. Employing $\partial$, $C(X, G)$ and $C^*(X, G)$ become chain and cochain complexes, respectively, and we have therefore the singular homology and cohomology groups of $X$, namely

$$H(X, G) = \textstyle\sum_n H_n(X, G),\ H^*(X, G) = \sum_n H^n(X, G).$$

[3] This definition is easily seen to be equivalent to assuming the validity of the Covering Homotopy Theorem for all spaces as applied to $(E, B, p)$ and hence is stronger than the definition of fibre space in the sense of Serre.

[4] This $\partial$ differs in sign from that employed in [3].

2.3. Cap products. Let $u$ denote a singular $n$-cube in $X$. Following Serre [3], we define certain faces of $u$ as follows: Let $H$ denote a subset of $p$ elements from the set of indices $\{1, \cdots, n\}$ and $K$ the complement of $H$, containing, therefore, $q$ elements where $p + q = n$. Let $\varphi_K : K \to \{1, \cdots, q\}$ denote a strictly monotone function. For $\varepsilon = 0$ or $1$ we let $\lambda_H^\varepsilon u$ denote the following $q$-face of $u$

$$(\lambda_H^\varepsilon u)(x_1, \cdots, x_q) = u(y_1, \cdots, y_n)$$

where

$$y_i = \varepsilon \qquad \text{for} \quad i \in H$$

$$y_i = x_{\varphi_K(i)} \quad \text{for} \quad i \in K.$$

Also, set $\operatorname{sgn} H = (-1)^\nu$ where $\nu$ is the number of pairs $(i, j)$, $i \in H, j \in K$ such that $i > j$.

Suppose now that the groups $G_1$ and $G_2$ are paired to $G$. For $g_1 \in G_1$, $g_2 \in G_2$, let $(g_1, g_2)$ denote the element of $G$ obtained from pairing $g_1$ and $g_2$. For $f^q \in C^q(X, G_1)$, $u_{p+q}$ a singular cube in $C_{p+q}(X)$, set

$$f^q \cap u_{p+q} \otimes g_2 = \textstyle\sum_H \operatorname{sgn} H \, \lambda_K^1 u \otimes (f^q(\lambda_H^0 u), g_2), \; g_2 \in G_2.$$

It is not difficult to show the usual cap product identity

$$\partial(f^q \cap u_{p+q} \otimes g_2) = (-1)^p \, \delta f^q \cap u_{p+q} \otimes g_2 + f^q \cap \partial u_{p+q} \otimes g_2$$

where $\delta$ is the differential operator in $C^*(X, G_1)$. Therefore, the pairing of $C^q(X, G_1)$ and $C_{p+q}(X, G_2)$ to $C_p(X, G)$ leads to a pairing of $H^q(X, G_1)$ and $H_{p+q}(X, G_2)$ to $H_p(X \; G)$.

Remark. The above cap product differs, at the homology level, from the definition (adapted to cubical theory) given in Eilenberg [4] by a factor of $(-1)^{pq}$ where $n$ is the dimension of the second factor. In comparison with the above definition of $f^q \cap u_{p+q} \otimes g_2$, the Eilenberg definition (adapted to cubical theory) would read

$$f^q \cap u_{p+q} \otimes g_2 = \textstyle\sum_H \operatorname{sgn} H \, \lambda_K^0 u \otimes (f^q(\lambda_H^1 u), g_2).$$

2.4. The spectral sequence. Let $(E, B, p)$ denote a fibre space. We filter $A = C(E)$ singular chains of $E$ (integral coefficients) just as in Serre [3]. If $u$ is a singular $n$-cube in $X$, a coordinate index $i$, $1 \leqq i \leqq n$ is called db (degenerate base) for $u$ if

$$pu(x_1, \cdots, x_i, \cdots, x_n) = pu(y_1, \cdots, y_i, \cdots, y_n)$$

for arbitrary $x_i$, $y_i$. Otherwise $i$ is called a pb (proper base) coordinate for $u$. Then, we set

$$\dim_r u = \max \text{pb coordinate index for } u.$$

The filtration

$$0 = A^{-1} \subseteqq A^0 \subseteqq \cdots \subseteqq A^p \subseteqq A^{p+1} \subseteqq \cdots$$

is obtained by letting $A^p$ denote the subgroup of $A$ generated by singular cubes $u$ such that $\dim_r u \leqq p$. The spectral sequence associated with this filtration can be obtained as follows: $E_r^{p,q}$, $1 \leqq r < \infty$, is the image of $i_*$, where

$$i_* : H_n(A^p, A^{p-r}) \to H_n(A^{p+r-1}, A^{p-1}), \qquad n = p + q$$

is the map induced by the natural injection $i$. Furthermore, the differential operator

$$d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$$

is just the boundary operator

$$\partial_* : H_n(A^{p+r-1}, A^{p-1}) \to H_{n-1}(A^{p-1}, A^{p-r-1})$$

for the triple $(A^{p+r-1}, A^{p-1}, A^{p-r-1})$ restricted to $E_r^{p,q}$, $\quad n = p + q$.

2.5. Extended $r$-skeleton of an $n$-cell. Let $I^n$ denote an $n$-cell $n \geqq 1$ and $r$ an integer $1 \leqq r \leqq n$. As in §2.3, let $H$ denote a subset of $r$ elements from $\{1, \cdots, n\}$ and $K$ the complement of $H$. Let $\alpha : K \to \{0, 1\}$ denote a function on $K$ taking the values 0 or 1. Then set

$$F_K^\alpha = \{x \in I^n : x_k = \alpha(k) \quad \text{for} \quad k \in K\}.$$

Then, $F_K^\alpha$ is an ordinary $r$-face of $I^n$. Now, let $\bar{H}$ denote a set of $r + 1$ elements from $\{1, \cdots, n\}$ and $\bar{K}$ its complement. Let $(i, j)$ denote a fixed pair of indices in $\bar{H}$ and $\bar{\alpha} : \bar{K} \to \{0, 1\}$ a function on $K$ with 0 or 1 as values. Set

$$F_{\bar{K}}^{\bar{\alpha}}(i, j) = \{x \in I^n : x_k = \bar{\alpha}(k) \quad \text{for} \quad k \in \bar{K} \text{ and } x_i = x_j\}.$$

We call $F_{\bar{K}}^{\bar{\alpha}}(i, j)$ a *diagonal* $r$-face of $I^n$. Let $F_r$ denote the set union of all the ordinary $r$-faces and $\bar{F}_r$ the set union of all the diagonal $r$-faces. Then $F_r^* = F_r \cup \bar{F}_r$ will be called the *extended* $r$-skeleton of $I^n$.

Now, let $X$ denote a topological space and $Q(X) = \sum_n Q_n(X)$ as in §2.2. Let $Q^r(X)$ denote the subgroup of $Q(X)$ generated by singular cubes

$$u : (I^n, F_r^*) \to (X, x_0)$$

wheie $x_0$ is a fixed element of $X$. Then, set

$$C^r(X) = (Q^r(X))/(Q^r(X) \cap D(X)).$$

The following lemmas will be used in the sequel and their proofs, follow standard lines.

Lemma 1. *If $X$ is r-connected,* i.e. $\pi_i(X, x_0) = 0$, $i \leqq r$, *then the chain complexes $C^r(X)$ and $C(X)$ are chain equivalent and hence the singular homology groups of $X$ may be based on singular cubes whose extended r-skeleton lies at the fixed point $x_0$.*

Lemma 2. *If $(E, B, p)$ is a fibre space with $B$ r-connected, then the singular homology groups of $E$ may be based on singular cubes whose extended r-skeleton lies in the fibre $F = p^{-1}(b)$, $b \in B$ a fixed base point.*

2.6. A homotopy addition lemma. Let $F$ denote the extended $n - 1$ skeleton of $I^n$, $n > 1$, and let

$$u : (I^n, F) \to (X, x_0)$$

$X$ a space, $x_0 \in X$, denote a map. Then $u$ represents an element $\alpha$ in $\pi_n(X, x_0)$. Furthermore, let

$$u_k : (I^n, \dot{I}^n) \to (X, x_0)$$

denote the map given by

$$u_k(x_1, \cdots, x_n) = u(y_1, \cdots, y_n)$$

where

$$y_i = x_n x_i \qquad i < k$$
$$y_k = x_n$$
$$y_i = x_n x_{i-1} \quad i > k.$$

Since $u$ maps the extended $n - 1$ skeleton of $I^n$ into $x_0$, $u_k$ maps $I^n$ into $x_0$ and hence $u_k$ represents an element $\alpha_k$ in $\pi_n(X, x_0)$. The following lemma is then valid. Its proof follows standard lines and is omitted.

LEMMA. $\sum_{k=1} (-1)^{n-k} \alpha_k = \alpha$.

### 3. Pairing $\pi_n(B)$ and $H(F, G)$ to $H(F, G)$, $n > 1$

Let $(E, B, p)$ denote a fibre space, $F = p^{-1}(b)$, $b \in B$, a fibre and $\pi_n(B, b) = \pi_n(B), n > 1$, a homotopy group of $B$. Let $\alpha \in \pi_n(B)$ with representative $f : (I^n, \dot{I}^n) \to (B, b)$. Also, let $v : I^q \to F$, $q \geq 0$, denote a singular $q$-cube in $F$. We define a singular $q + n - 1$ cube $(f, v)$ in $F$ as follows. For $x \in I^{q+n-1}$, set ${}^{n-1}x = (x_1, \cdots, x_{n-1})$, $x^q = (x_n, \cdots, x_{q+n-1})$. Then, if $\Lambda$ is any lifting function for $(E, B, p)$ set

$$(f, v)(x) = \Lambda[v(x^q), \omega({}^{n-1}x)](1)$$

where $\omega({}^{n-1}x)$ is the loop in $B$ given by

$$\omega({}^{n-1}x)(t) = f({}^{n-1}x, t).$$

If now, $h \in H_q(F, G)$ is a homology class with representative cycle $z = \sum_j v_j \otimes g_j$, we denote by $(\alpha, h) \in H_{q+n-1}(F, G)$ the homology class containing the cycle $(f, z) = \sum_j (f, v_j) \otimes g_j$. It is easy to show that $(f, z)$ is indeed a cycle and $(\alpha, h)$ is independent of the representatives $f$ and $z$ chosen, as well as the lifting function $\Lambda$ employed. Furthermore, bilinearity, namely

$$(\alpha + \alpha', h) = (\alpha, h) + (\alpha', h)$$
$$(\alpha, h + h') = (\alpha, h) + (\alpha, h')$$

follows easily and hence $\pi_n(B)$ and $H_q(F, G)$ are paired to $H_{n+q-1}(F, G)$.

An alternate description of this pairing may be given as follows. Let $\Omega$ denote the loop space of $B$ based at $b \in B$. Then, any lifting function $\Lambda$ gives rise to a

$$\bar{\Lambda} : \Omega \times F \to F$$

map as follows:

$$\bar{\Lambda}(\omega, x) = \Lambda(x, \omega)(1), \qquad \omega \in \Omega, \qquad x \in F.$$

Applying the Künneth Theorem, we obtain induced homomorphisms

$$\bar{\Lambda}_* : H_p(\Omega) \otimes H_q(F, G) \to H_{p+q}(F, G).$$

Then, the pairing of $\pi_n(B)$ and $H(F, G)$ to $H(F, G)$ is given by the composition homomorphism

$$\pi_n(B) \otimes H(F, G) \xrightarrow{i \otimes 1} \pi_{n-1}(\Omega) \otimes H(F, G) \xrightarrow{j \otimes 1}$$
$$H_{n-1}(\Omega) \otimes H(F, G) \xrightarrow{\bar{\Lambda}_*} H(F, G),$$

where $i : \pi_n(B) \to \pi_{n-1}(\Omega)$ is the standard natural isomorphism and $j : \pi_{n-1}(\Omega) \to H_{n-1}(\Omega)$ is the Hurewicz homomorphism. The two descriptions are easily seen to yield identical pairings. We shall, however, have need for the explicit form of the pairing given initially in terms of representatives of homotopy and homology classes.

## 4. The basic map and identity

4.1. THE BASIC MAP. We assume in this section that $B$ is a fixed arcwise connected topological space and $b \in B$ a fixed base point. Let $\tilde{B}$ be the space of paths in $B$ starting at $b$, i.e. $\tilde{B} = B^I(0, b)$. $\tilde{B}$ is a fibre space over $B$ with map $\xi : \tilde{B} \to B$ given by

$$\xi(\omega) = \omega(1), \qquad \omega \in \tilde{B}.$$

Let $C(B)$ and $C(\tilde{B})$ denote the singular chains of $B$ and $\tilde{B}$, respectively. We define a dimension preserving homomorphism (not a chain map)

$$\varphi : C_n(B) \to C_n(\tilde{B})$$

as follows. Let $u$ denote a singular $n$-cube in $B$, and let $p$ denote an index between 1 and $n$, i.e. $1 \leqq p \leqq n$ and $q = n - p$. As in §2.3 let $H$ denote a subset of $p$ elements from $\{1, \cdots, n\}$ and $K$ its complement. For such an $H$ we first define a homomorphism

$$\varphi_H^p : C_n(B) \to C_n(\tilde{B})$$

as follows. Let $\alpha : H \to \{1, \cdots, p\}$, $\beta : K \to \{p+1, \cdots, n\}$ denote increasing functions. For $x \in I^n$, set

$$y(x) = (y_1, \cdots, y_n), \qquad z(x) = (z_1, \cdots, z_n)$$

where

$$\begin{cases} y_i = x_{\beta(i)}, & z_i = 1 \text{ for } i \in K \\ y_i = 0, & z_i = x_{\alpha(i)} \text{ for } i \in H. \end{cases}$$

Then, let $\alpha_x$ denote the arc

$$\alpha_x(t) = \begin{cases} u[2ty(x)], \ t \leqq 1/2 \\ u[(2-2t)y(x) + (2t-1)z(x)], \ t \geqq 1/2 \end{cases}$$

where $u$ is a given $n$-cube. Then, set

$$\varphi_H^p u(x) = \alpha_x$$

and the homomorphism

$$\varphi_H^p : C(B) \to C(\tilde{B})$$

is defined. To obtain $\varphi$ set

$$\varphi^p = \sum_H \operatorname{sgn} H \, \varphi^p_H \quad \text{and} \quad \varphi = \sum_p \varphi^p.$$

Note that the definition of $\varphi$ depends on $n$ but is not displayed in the notation. Let

$$0 = A^{-1} \subseteq A^0 \subseteq \cdots \subseteq A^p \cdots$$

denote the filtration of $C(\tilde{B})$ relative to the fibering $(\tilde{B}, B, \xi)$. Then if $u$ is an $n$-cube in $B$, we make the following important observation

$$\varphi^p u \in A^p = A^{n-q} \qquad 1 \leqq p \leqq n.$$

4.2. THE BASIC IDENTITY. Let $u$ denote a fixed $n$-cube and $p$, $q$, $H$ and $K$ as above. Suppose also that the indices in $H$ and $K$ are denoted by

$$H : i_1 < \cdots < i_p, \qquad K : j_1 < \cdots < j_q.$$

Then the following lemmas can be verified easily from the definitions.

LEMMA 1. For $k \leqq p$,

$$\lambda^0_k \varphi^p_H u = \varphi^{p-1}_{H*} \, \lambda^0_{i_k} u$$

*and* $(-1)^k \operatorname{sgn} H = (-1)^{i_k} \operatorname{sgn} H^*$ *where*

$$H^* = \{i_1, \cdots, i_{k-1}, i_{k+1} - 1, \cdots, i_p - 1\}.$$

LEMMA 2. *For* $1 \leqq k \leqq q$,

$$\lambda^0_{p+k} \, \varphi^p_H u = \lambda^1_{j_k - k + 1} \, \varphi^{p+1}_{H*} u$$

*and* $(-1)^{p+k} \operatorname{sgn} H = (-1)^{j_k - k + 1} \operatorname{sgn} H^*$ *where* $H^* = H \cup j_k$.

LEMMA 3. *If the $q$-skeleton of $u$ is at a fixed point $b \in B$, then for* $1 \leqq k \leqq p$

$$\lambda^1_{p+k} \, \varphi^p_H u = \varphi^p_{H^*} \, \lambda^1_{j_k} u$$

*and* $(-1)^{p+k} \operatorname{sgn} H = (-1)^{j_k} \operatorname{sgn} H^*$ *where* $H^* = \{i^*_1, \cdots, i^*_p\}$ *and* $i^*_e = i_e$ *or* $i_e - 1$ *according as* $i_e < j_k$ *or* $i_e > j_k$.

Now, suppose that $u$ is an $n$-cube whose $r - 1$ skeleton lies at $b \in B$. Then for any $q \leqq r$, the following basic identity $I_\varphi$ is valid,

$$(I_\varphi) \quad \partial \varphi u - \varphi \partial u = S_q(u) + R_q(u) + \partial \sum_{j \leqq n-q-1} \varphi^j u - \sum_{j \leqq n-q-1} \varphi^j \, \partial u$$

where

$$S_q(u) = \sum_{k=1}^{n-q} \sum_H (-1)^k \operatorname{sgn} H \, [\lambda^1_k \, \varphi^{n-q}_H \, u - \lambda^0_k \, \varphi^{n-q}_H u]$$

$$R_q(u) = \sum_{k=1}^{q} \sum_H (-1)^{n-q+k} \operatorname{sgn} H \, [\lambda^1_{n-q+k} \varphi^{n-q}_H \, u - \varphi^{n-q}_{H^*} \, \lambda^1_{j_k} u]$$

where in the expression $R_q(u)$, $H^*$ and $j_k$ have the following meaning: If $H = \{i_1 < \cdots < i_{n-q}\}$, and $K = \{j_i < \cdots < j_q\}$ is its complement, $H^* = \{i^*_1, \cdots, i^*_{n-q}\}$ where $i^*_m = i_m$ or $i^*_m = i_m - 1$ according as $i_m < j_k$ or $i_m > j_k$. $j_k$ is, of course, already indicated as an element of $K$.

The proof of this identity is immediate by induction on $q$, making use of the previous lemmas. $I_\varphi$ immediately implies

LEMMA. *If the $r-1$ skeleton of a singular $n$-cube $u$ in $B$ lies at $b \in B$ then*

$$\partial\varphi u - \varphi\partial u \in A^{n-r}$$

*and*

$$\partial\varphi u - \varphi\partial u = R_r(u) \text{ modulo } A^{n-r-1}.$$

## 5. Application of the basic map to fibre spaces

5.1. THE INDUCED MAP $\psi$. Let $(E, B, p)$ denote a fibre space, $B$ arcwise connected, $b \in B$ a fixed base point and $p^{-1}(b) = F$ the fibre over $b$. The homomorphism

$$\varphi : C(B) \to C(\tilde{B})$$

of the preceding section induces a homomorphism

$$\psi : C(B) \otimes C(F) \to C(E)$$

as follows. Let $u$ denote a $p$-cube in $B$, $v$ a $q$-cube in $F$. For $(x, y) \in I^{p+q}$, $x \in I^p$, $y \in I^q$ set

$$\psi^i_H(u \otimes v)(x, y) = \Lambda\,[v(y), \varphi^i_H u(x)](1)$$

where $1 \leqq i \leqq p$ and $H$ is a subset of $i$ indices from $\{1, \cdots, p\}$ as in the previous section and $\Lambda$ is any lifting function for $(E, B, p)$. Then set

$$\psi^i = \textstyle\sum_H \psi^i_H, \qquad \psi = \sum_{i=1}^p \psi^i.$$

We note that

$$\psi : C_p(B) \otimes C_q(F) \to C_{p+q}(E)$$

depends on $p$ and $q$ but they will not be displayed in the notation.

Now, let

$$0 = A^{-1} \subseteqq A^0 \subseteqq \cdots \subseteqq A^p \subseteqq \cdots$$

denote the filtration of $C(E)$ as in §2.4. Then, we note that

$$\psi^i : C_p(B) \otimes C(F) \to A^i \quad 1 \leqq i \leqq p.$$

5.2. THE IDENTITY $I_\psi$. The basic identity $I_\phi$ implies easily a corresponding identity for $\psi$ which we state as the

FUNDAMENTAL LEMMA. *If $u$ is a $p$-cube in $B$ whose $r-1$ skeleton lies at a fixed point $b \in B$, then*

$$(I_\psi) \quad \partial\psi(u \otimes v) - \psi\partial(u \otimes v) = R_r(u \otimes v) \text{ modulo } A^{p-r-1}$$

*where $v$ is a singular cube in $F$, $R_r(u \otimes v) \in A^{p-r}$ and*

$$R_r(u \otimes v) = \textstyle\sum_{k=1}^r \sum_H (-1)^{p-r+k} \operatorname{sgn} H[\lambda^1_{p-r+k}\, \psi^{p-r}_H(u \otimes v) - \psi^{p-r}_{H^*}(\lambda^1_{j_k}\, u \otimes v)]$$

*where $H$ ranges over subsets of $\{1, \cdots, p)$ which contain $n-r$ elements.*

As in §4.2, $H^*$ and $j_k$ have the following meaning. Let $K$ denote the complement of a given $H$. We write the elements of $K$ in increasing order $j_1 < \cdots < j_r$, thus determining $j_k$. If $H = \{i_1, \cdots, i_{p-r}\}$, $H^* = \{i_i^*, \cdots, i_{p-r}^*\}$ where $i_m^* = i_m$ or $i_m^* = i_m - 1$ according as $j_k > i_m$ or $j_k < i_m$.

5.3. THE MAIN RESULT. Now, in the chain complex $C_p(B) \otimes C(F)$ introduce the boundary operator

$$\partial_F(b \otimes f) = (-1)^p b \otimes \partial f, \qquad b \in C_p(B), \qquad f \in C(F).$$

Furthermore, $\psi$ induces homomorphisms

$$\psi_0 : C_p(B) \otimes C(F) \to A^p/A^{p-1}.$$

Since $B$ is arcwise connected, we apply the Fundamental Lemma for $r = 1$ and see that

$$\psi_0\, \partial_F = \partial \psi_0$$

and hence $\psi_0$ induces homomorphisms

$$\psi_1 : C_p(B) \otimes H_q(F) \to H_{p+q}(A^p, A^{p-1}) = E_1^{p,q}.$$

Now, let $u$ denote a $p + q$ cube in $E$ such that $\dim_r u \leqq p$. As in Serre [3], $Bu$ will denote the $p$-cube in $B$, $Fu$ the $q$-cube in $F$ given by

$Bu(x_1, \cdots, x_p) = p \cdot u(x_1, \cdots, x_p, y_1, \cdots, y_q)$ for any choice of $(y_1, \cdots, y_q)$.

$Fu(x_1, \cdots, x_q) = u(0, \cdots, 0, x_1, \cdots, x_q)$.

Next, define

$$\theta : A^p \to C_p(B) \otimes C(F)$$

by setting

$$\theta(u) = Bu \otimes Fu$$

for $u$ a generator of $A^p$. Then $\theta$ induces

$$\theta_0 : A^p/A^{p-1} \to C_p(B) \otimes C(F).$$

It is easy to see that $\partial_F \theta_0 = \theta_0 \partial$ and hence $\theta_0$ induces

$$\theta_1 : H_{p+q}(A^p, A^{p-1}) = E_1^{p,q} \to C_p(B) \otimes H_q(F).$$

THEOREM *$\psi_0$ and $\theta_0$ form a chain equivalence and hence $\psi_1$ and $\theta_1$ are isomorphisms onto.*

PROOF. The proof is given in the Appendix, §1.

Now, let us assume that for the fibre space $(E, B, p)$, $B$ is $r - 1$ connected with $r > 1$. We may then assume that the singular chains of $E$ are generated by singular cubes whose extended $r - 1$ skeletons lie in $F$, and that the singular chains of $B$ are generated by singular cubes whose extended $r - 1$ skeletons lie at the fixed base point $b$. As a matter of interest, the case $r = 1$ which gives the Leray-Serre result is given in the Appendix, §2. Define

$$\partial_B : C_p(B) \otimes H(F) \to C_{p-1}(B) \otimes H(F)$$

by

$$\partial_B u \otimes h = (\partial u) \otimes h.$$

Then, applying the Fundamental Lemma, it is easy to see that the following diagram commutes

$$\begin{array}{ccc} C(B) \otimes H(F) & \xrightarrow{\psi_1} & E_1 \\ \partial_B \downarrow & & \downarrow d_1 \\ C(B) \otimes H(F) & \xrightarrow[\psi_1]{} & E_1 \end{array}$$

and, recalling that $H(E_j) = E_{j+1}$, $\psi_1$ induces an isomorphism

$$\psi_2 : H(B, H(F)) \rightarrow E_2$$

which is a special case of the Leray-Serre result. Now, again applying the Fundamental Lemma, we see, step by step, that $\psi_2$ induces isomorphisms

$$\psi_i : H(B, H(F)) \rightarrow E_i$$

for $2 \leqq i \leqq r$ and the composition maps

$$d_i \cdot \psi_i : H(B, H(F)) \rightarrow E_i \rightarrow E_i$$

are 0 for $2 \leqq i \leqq r-1$, and hence $d_i = 0$ for $2 \leqq i \leqq r-1$ (in case $r > 2$) Next, we investigate the structure of the differential operator $d_r$. It should be remarked, that $\theta_1 : E_1 \rightarrow C(B) \otimes H(F)$ also induces, step by step, isomorphisms

$$\theta_i : E_i \rightarrow H(B, H(F)) \qquad 2 \leqq i \leqq r.$$

Consider, the composition

$$H(B, H(F)) \xrightarrow{\psi_r} E_r \xrightarrow{d_r} E_r \xrightarrow{\theta_r} H(B, H(F)).$$

Take a homology class $h \in H_p(B, H_q(F))$ and let $z$ denote a representative cycle of $h$ such that $z = \sum_{i,j} \mu_{i,j} u_i \otimes v_{i,j}$, with $\mu_{ij}$ integers, $u_i$ $p$-cells in $B$, $v_{i,j}$ $q$-cells in $F$ and $\sum_j \mu_{i,j} v_{i,j}$ is a cycle in $F$ for each $i$, representing a homology class in $F$ which we denote by $h_i$. Then, employing the Fundamental Lemma, $d_r \psi_r(h) \in E_r^{p-r, q+r-1}$ is determined entirely by

$$\textstyle\sum_{i,j} \mu_{ij} R_r(u_i - v_{ij}) = R_r(z).$$

In order to determine the structure of $\theta_r d_r \psi_r(h)$ we look at the image of $R_r(u \otimes v)$ under $\theta : A^{p-r} \rightarrow C_{p-r}(B) \otimes C(F)$ where $u$ and $v$ are as in the Fundamental Lemma §5.2. Let $H$ denote a subset of $p - r$ indices from $\{1, \cdots, p\}$ and $K$ be its complement. Then $f_H = \lambda_H^0 u$ represents an element of $\pi_r(B, b)$, and $f_H$ maps the extended $r - 1$ skeleton of $I^r$ into $b$. Following §2.6 let

$$f_{H,k}(x_1, \cdots, x_r) = f_H(x_r x_1, \cdots, x_r x_{k-1}, x_r, x_r x_k, \cdots, x_r x_{r-1}).$$

Letting $e : I^r \rightarrow b$ denote the natural representative of $0 \in \pi_r(B, b)$, set

$$\bar{f}_{H,k} = f_{H,k} + e$$

where the addition occurs in the $r^{\text{th}}$ coordinate. Then, the following lemma is easy to verify directly from definitions.

LEMMA 1.

$$\theta(R_r(u \otimes v)) = \sum_{k=1}^{r} \sum_H (-1)^{p-r+k} \operatorname{sgn} H\,[\lambda_K^1 u \otimes (\bar{f}_{H,k}, v) - \lambda_K^1 u \otimes v^*]$$

*where $v^*$ is a degenerate $q + r - 1$ cube in $F$ given by*

$$v^*(x_1, \cdots, x_{r-1}, y_1, \cdots, y_q) = \Lambda[v(y_1, \cdots, y_q), \tilde{b}](1)$$

*with $\tilde{b}$ the constant loop at $b$ and where $(\bar{f}_{H,k}, v)$ is the pairing of* §3.

We note the following fact which we shall use shortly. Using the notation of the above lemma, the homotopy addition theorem of §2.8 implies that $\sum_k (-1)^{r-k} \bar{f}_{H,k} \sim \lambda_H^0 u$.

Since

$$\partial\psi(z) - \psi\partial(z) = R_r(z) \text{ modulo } A^{p-r-1}$$

$R_r(z)$ represents a cycle in $A^{p-r}/A^{p-r-1}$ and hence determines an element $[R_r(z)]_1$ in $E_1^{p-r,q+r-1}$. Lemma 1, together with the fact mentioned above, immediately implies

LEMMA 2.

$$\theta_1[R_r(z)]_1 = (-1)^p \sum_i \sum_H \operatorname{sgn} H[\lambda_K^1 u_i \otimes (\gamma(\lambda_H^0 u_i), h_i)]$$

*where $\gamma : C_r(B) \to \pi_r(B, b)$ is the characteristic cocycle of $B$ and $(\gamma(\lambda_H^0 u_i), h_i)$ is the pairing of* §3.

Lemma 2 then gives our main result.

THEOREM. *$\theta_r d_r \psi_r(h) = (-1)^p = \bar{\gamma} \cap h$, $h \in H(B, H(F))$, where $\bar{\gamma}$ is characteristic cohomology class of $B$.*

Therefore, if we consider $d_r$ as acting on $H_p(B, H(F))$, we see that

$$d_r(h) = (-1)^p\, \bar{\gamma} \cap h,\ h \in H_p(B, H(F)).$$

We collect what we have shown in the following

MAIN THEOREM. *Let $(E, B, p)$ denote a fibre space with $B$ $r - 1$ connected. Then, in the spectral sequence $(E_j, d_j)$ associated with $(E, B, p)$,*

$$E_i = H(B, H(F)) \quad \text{for} \quad i = 2, \cdots, r$$

$$d_i = 0 \quad \text{for} \quad i = 2, \cdots, r - 1$$

*and*

$$d_r(h) = (-1)^p\, \bar{\gamma} \cap h, \quad h \in H_p(B, H(F))$$

*where $\bar{\gamma}$ is the characteristic cohomology class of $B$. Therefore, the first $r + 1$ terms of the spectral sequence depend only on $B$, $H(F)$ and the pairing of $\pi_r(B)$ and $H(F)$ to $H(F)$.*

REMARK. The main theorem remains valid if an arbitrary coefficient group $G$ is used for the singular chains of $E$. Naturally, one uses the pairing of $\pi_r(B)$ and $H(F, G)$ to $H(F, G)$ in this case.

APPENDIX

1. THEOREM. $\psi_0$ and $\theta_0$ form a chain equivalence (see §5.3).

PROOF. (a) We show first that $\theta_0\psi_0 \sim 1$. It should be remarked that if the fibre space $(E, B, p)$ were assumed regular [1], then it would follow that $\theta_0\psi_0 = 1$.

We proceed, however, without this assumption. Let $\tilde{b}$ denote the constant arc at the base point $b \in B$. For $x \in F$, set

$$\beta(x) = \Lambda[x, \tilde{b}](1)$$

where $\Lambda$ is a lifting function. Then we have the following

LEMMA. $\beta \sim 1 : F \to F$.

PROOF. The proof is immediate since

$$H(x, t) = \Lambda[x, \tilde{b}](t)$$

provides the connecting homotopy.

$\beta : F \to F$ induces a chain map $\beta_0 : C(F) \to C(F)$ and the above lemma implies that $\beta_0 \sim 1$, i.e. $\beta_0$ and 1 are chain homotopic.

Now consider

$$\theta_0 : A^p/A^{p-1} \leftrightarrows C_p(B) \otimes C(F) : \psi_0.$$

Since

$$\psi^i(u \otimes v) \in A^i \qquad 1 \leqq i \leqq p$$

where $\dim u = p$, $\dim v = q$, it follows that $\psi_0(u \otimes v) = [\psi^p(u \otimes v)]$, i.e. $\psi_0(u \otimes v)$ is determined entirely by $\psi^p(u \otimes v)$, and

$$\psi^p(u \otimes v)(x, y) = \Lambda[v(y), \omega_x](1)$$

where

$$\omega_x(t) = \begin{cases} u(0) & t \leqq 1/2. \\ u[(2t - 1)x] & t \leqq 1/2. \end{cases}$$

Therefore, it is easy to see that

$$\theta_0\psi_0(u \otimes v) = u \otimes \beta_0 v$$

and hence $\theta_0\psi_0 = 1 \otimes \beta_0 \sim 1$.

(b) Here we show that $\psi_0\theta_0 \sim 1$. First, employing the notation of §2.1 we define a map

$$\Gamma : \tilde{E} \to \tilde{E}$$

by setting

$$\Gamma(\alpha) = \Lambda[\alpha(0), \tilde{b} + p\alpha], \ \alpha \in \tilde{E}$$

where $\tilde{b}$ is the constant loop at $b$. Let $\bar{p} : \tilde{E} \to \tilde{B}$, where $\tilde{B}$ is the space of paths in $B$ emanating from $b$, be the natural map given by $\bar{p}(\alpha) = p\alpha$. Then it is easy to see that $\Gamma \sim \tilde{\Lambda} : \tilde{E} \to \tilde{E}$ with connecting homotopy $\Phi_0 : \tilde{E} \times I \to \tilde{E}$ such that

$$\bar{p}[\Phi_0(\alpha, s)](1) = \bar{p}\alpha(1), \qquad 0 \leqq s \leqq 1.$$

Furthermore, as noted in §2.1, $\tilde{\Lambda} \sim 1$, and in this case there is a connecting homotopy $\Phi_1 : \tilde{E} \times I \to \tilde{E}$ such that

$$\bar{p}[\Phi_1(\alpha, s)](t) = \bar{p}\alpha(t), \qquad 0 \leqq s, t \leqq 1.$$

Therefore, $\Gamma \sim 1$ and a connecting homotopy $\Phi : \tilde{E} \times I \to \tilde{E}$ may be chosen such that

$$\tilde{p}[\Phi(\alpha, s)](1) = \tilde{p}\alpha(1), \qquad 0 \leqq s \leqq 1.$$

Then if $w$ is a $p + q$-cube in $E$, with $\dim_\tau w = p$, $\tilde{w} : I^{p+q} \to \tilde{E}$ is given by

$$\tilde{w}(x, y)(t) = w(tx, y), \qquad x \in I^p, \quad y \in I^q.$$

Then, set

$$(Dw)(x, y, s) = \Phi(\tilde{w}(x, y), s)(1).$$

$D$ is then a homomorphism of degree $+1$.

$$D : C_{p+q}(E) \to C_{p+q+1}(E).$$

Furthermore, $D$ preserves filtration, i.e. $D(A^p) \subseteqq A^p$ and hence $D$ induces

$$D_0 : A^p/A^{p-1} \to A^p/A^{p-1}.$$

In addition we make the following observations. If $w$ is a $p + q$ cube in $E$, just as above,

(i) $Dw \in A^p$
(ii) $\lambda_i^\varepsilon Dw \in A^{p-1}$ for $1 \leqq i \leqq p \qquad \varepsilon = 0, 1$
(iii) $\lambda_{p+1}^\varepsilon Dw = D\lambda_i^\varepsilon w, \quad 1 \leqq i \leqq q, \qquad \varepsilon = 0, 1$
(iv) $\lambda_{p+q+1}^0 Dw = w$
(v) $\lambda_{p+q+1}^1 Dw = \psi^p\theta w.$

Therefore,

$$\partial Dw - (-1)^p D\partial w = (-1)^{p+q}[w - \psi^p\theta w] \text{ modulo } A^{p-1}$$

Since, as we observed in (a), $\psi_0 : C_p(B) \otimes C(F) \to A^p/A^{p-1}$ is determined entirely by $\psi^p$, it follows that for $c \in A^p/A^{p-1}$

$$\partial D_0 c - (-1)^p D\partial c = (-1)^{p+q}[c - \psi_0\theta_0 c]$$

and hence $\psi_0\theta_0 \sim 1$. The proof of our theorem is then complete.

2. The Leray-Serre result. Let $\bar{\partial} : C_p(B) \otimes H_q(F) \to C_{p-1}(B) \otimes H_q(F)$ denote the boundary operator for $C(B) \otimes H_q(F)$ in the sense of local coefficients, where the action of $\pi_1(B, b)$ on $H_q(F)$ can be obtained as follows. If $v$ is a $q$-cube in $F$ and $e$ is a loop in $B$, based at $b$, then set

$$(e, v)(x) = \Lambda[v(x), e](1), \qquad x \in I^q$$

where $\Lambda$ is a lifting function for $(E, B, p)$. $(e, v)$ is then a $q$-cube in $F$. Then, if $\alpha$ is an element of $\pi_1(B, b)$ with $e$ as a representative loop and $h$ is a $q$-homology class in $F$ with representative cycle $z = \sum_i \mu_i v_i$, $\mu_i$ integers, then

$$(\alpha, h) = [\textstyle\sum_i \mu_i(e, v_i)]$$

i.e. the $q$-homology class with representative cycle $\sum_i \mu_i(e, v_i)$, is the result of acting on $h$ with $\alpha$. In our notation, then,

$$\bar{\partial}(u \otimes h) = \textstyle\sum_{i=1}^p (-1)^i[\lambda_i^1 u \otimes (\alpha_i, h) - \lambda_i^0 u \otimes h]$$

where $\alpha_i$ has the $i^{\text{th}}$ edge of $u$, $e_i$ as representative, where, explicitly,

$$e_i(t) = u(0, \cdots, 0, \overset{i}{t}, 0, \cdots, 0).$$

$u$ is, of course, a $p$-cube in $B$ (whose vertices are assumed at $b$) and $h \in H_q(F)$. Our objective, then, is to show that the following diagram commutes.

$$\begin{array}{ccc} C_p(B) \otimes H_q(F) & \xrightarrow{\psi_1} & E_1^{p,q} = H_{p+q}(A^p, A^{p-1}) \\ \partial \downarrow & & \downarrow d_1 \\ C_{p-1}(B) \quad H_q(F) & \xrightarrow{\psi_1} & E_1^{p-1,q} = H_{p+q-1}(A^{p-1}, A^{p-2}). \end{array}$$

We again use the fact that the homomorphism $\psi_1$ is determined entirely by

$$\psi^p : C_p(B) \otimes C_q(F) \to A^p.$$

If $u$ is a $p$-cube in $B$ and $v$ a $q$-cube in $F$, one sees easily that

$$\partial \psi^p u \otimes v = \textstyle\sum_{i=1}^p (-1)^i[\lambda_i^1 \psi^p u \otimes v - \lambda_i^0 \psi^p u \otimes v] + (-1)^p \psi^p u \otimes \partial v.$$

Furthermore, one verifies directly from the definition of $\psi^p$ that

$$\theta \textstyle\sum_{i=1}^p (-1)^i[\lambda_i^1 \psi^p u \otimes v - \lambda_i^0 \psi^p u \otimes v] = \textstyle\sum_{i=1}^p (-1)^i[\lambda_i^1 u \otimes (\bar{e}_i, v) - \lambda_i^0 u \otimes (\tilde{b}, v)]$$

where $\theta$ operates on $A^{p-1}$, $\bar{e}_i$ is the product path $\tilde{b}e_i$, where $e_i$ is the $i^{\text{th}}$ edge of $u$ and $\tilde{b}$ is the constant arc at $b$. Therefore, if $z$ is a $q$-cycle in $F$

$$\theta \partial \psi^p(u \otimes z) = \textstyle\sum_{i=1}^p (-1)^i[\lambda_i^1 u \otimes (\bar{e}_i, z) - \lambda_i^0 u \otimes (\tilde{b}, z)]$$

and hence

$$\theta_1 d_1 \psi_1(u \otimes h) = \textstyle\sum_{i=1}^p (-1)^i[\lambda_i^1 u \otimes (\alpha_i, h) - \lambda_i^0 u \otimes h] = \bar{\partial}(u \otimes h)$$

where $h \in H_q(F)$ and $\alpha_i = [e_i] \in \pi_1(B)$, since $(1, h) = h$ where 1 is the identity in $\pi_1(B)$ and $\bar{e}_i \sim e_i$. Therefore, since $\psi_1\theta_1 = 1$, we have

$$d_1\psi_1 = \psi_1\bar{\partial}$$

and the above diagram commutes. Hence, we have the Leray-Serre result for singular homology, namely

$$E_2^{p,q} = H_p(B, H_q(F))$$

where the homology is in the sense of local coefficients.

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# FOUNDATIONS OF FIBRE BUNDLES

By Henri Cartan and Samuel Eilenberg[1]

The need of an abstract theory unifying the basic constructions of different types of fibre bundles has been felt for some time and several attempts along this line have already been made.

In this attempt, we adopt the point of view of categories and functors. We are interested in defining not only the bundles but also their maps, i.e., we are interested in defining a category $\mathscr{A}$ of bundles. Some bundles and some maps (i.e., a subcategory $\mathscr{M}$ of $\mathscr{A}$) are given to us in advance. Those are the *models*; every bundle restricted to sufficiently small open sets of the base should be isomorphic to one of the models. In this sense the bundles are "locally trivial", if we regard the models as being "trivial".

The problem of passing from $\mathscr{M}$ to $\mathscr{A}$ is thus a problem in extending a given category. An analysis shows that there are two distinct extension processes involved.

The first notion we introduce is that of a *faithful functor* $T: \mathscr{A} \to \mathscr{B}$ of one category into another (examples: the functor which to each differentiable manifold assigns the underlying topological space or to each topological group assigns the underlying discrete group). For such a functor the question of *transportability* may be raised, i.e., the question whether isomorphisms in $\mathscr{B}$ may be lifted to isomorphisms in $\mathscr{A}$. If $T$ is not transportable, the category $\mathscr{A}$ may be enlarged to a category $\mathscr{A}^\Gamma$ and the functor $T$ extended to a functor $T^\Gamma: \mathscr{A}^\Gamma \to \mathscr{B}$ which is transportable.

The second basic notion is that of a *local category*, that is a category $\mathscr{A}$ given with a functor $L$ into topological spaces. For each object $A$ of $\mathscr{A}$ and each open set $U$ of $L(A)$ the restriction $A|U$ is assumed to be given. For such a category one can ask the question whether a family of matching objects can be pieced together (collated) into larger objects. This leads to a second enlargement process for categories which is carried out by the use of sheafs.

The method of categories and functors is as usual accompanied by foundational difficulties. In the abstract theory, one can conveniently assume that the categories are sets and thus avoid all difficulty. In the applications, however, one wishes to think in terms of "the category of *all* topological spaces and all continuous maps" etc. This poses the usual danger of antinomies. There are various dodges that one could adopt, but a complete and honest solution of the difficulty is yet to be invented. In the present write-up the difficulty is totally bypassed; the categories are assumed to be sets.

## 1. Categories and functors

A *category* $\mathscr{A}$ is a composite object consisting of

(1) a set of elements $A$, $A'$ etc. called objects of $\mathscr{A}$,

[1] Work done while S. Eilenberg was engaged under contract AF 18 (603)–67.

(2) a function which to each pair $(A, A')$ of objects of $\mathscr{A}$ assigns a set $\mathrm{Hom}_{\mathscr{A}}(A, A')$,

(3) a function

$$\mathrm{Hom}_{\mathscr{A}}(A, A') \times \mathrm{Hom}_{\mathscr{A}}(A, A'') \to \mathrm{Hom}_{\mathscr{A}}(A, A'')$$

defined for any triple $(A, A', A'')$ of objects of $\mathscr{A}$.

Frequently instead of writing $f \in \mathrm{Hom}_{\mathscr{A}}(A, A')$ we shall write $f : A \to A'$. The elements of $\mathrm{Hom}_{\mathscr{A}}(A, A')$ will be referred to as homomorphisms, morphisms, mappings or transformations. The function given in (3) is called composition, and its effect on a pair $(f, g)$ is denoted by $gf$.

The above data are subject to the following axioms:

(c.1) If $f : A \to A'$, $g : A' \to A''$, $h : A'' \to A'''$ then $h(gf) = (hg)f$.

(c.2) For each object $A$ there exists $i_A : A \to A$ such that $fi = f$, $ig = g$ for each $f : A \to A'$ and $g : A'' \to A$.

It is easy to see that $i_A$ is unique; it will be called the *identity* of $A$. A morphism $f : A \to A'$ is called an *isomorphism* if there exists $g : A' \to A$ such that $gf = i_A$, $fg = i_A$. Such a $g$ is unique and we denote it by $f^{-1}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A (covariant) *functor* $T : \mathscr{A} \to \mathscr{B}$ consists of a pair of functions (both denoted by $T$). The first assigns to each object $A$ in $\mathscr{A}$ and object $T(A)$ of $\mathscr{B}$ and the second assigns to each morphism $f : A \to A'$ in $\mathscr{A}$ a morphism $T(f) = T(A) \to T(A')$ in $\mathscr{B}$. The following two conditions must be satisfied

(F.1) $T(gf) = T(g)T(f)$

(F.2) $T(i_A) = i_{T(A)}$.

Given functors $T : \mathscr{A} \to \mathscr{B}$, $S : \mathscr{B} \to \mathscr{C}$ the composition $ST : \mathscr{A} \to \mathscr{C}$ is defined in the evident fashion and is a functor.

Given functors $T, S : \mathscr{A} \to \mathscr{B}$, a *morphism* $\Phi : T \to S$ is a function which to each object $A$ of $\mathscr{A}$ assigns a morphism $\Phi(A) : T(A) \to S(A)$ in $\mathscr{B}$. The following condition must be fulfilled.

($M$) For each $f : A \to A'$ in $\mathscr{A}$ the diagram

$$\begin{array}{ccc} T(A) & \xrightarrow{T(f)} & T(A') \\ {\scriptstyle \Phi(A)}\downarrow & & \downarrow{\scriptstyle \Phi(A')} \\ S(A) & \xrightarrow[S(f)]{} & S(A') \end{array}$$

is commutative.

If each $\Phi(A)$ is an isomorphism, then $\Phi$ is said to be an isomorphism $T \approx S$.

## 2. Subcategories

Let $\mathscr{A}$ and $\mathscr{A}'$ be categories. We say that $\mathscr{A}'$ is a subcategory of $\mathscr{A}$ if (1) each object of $\mathscr{A}'$ is an object of $\mathscr{A}$, (2) each morphism in $\mathscr{A}'$ is a morphism in $\mathscr{A}$, and (3) the inclusion mapping $I : \mathscr{A}' \to \mathscr{A}$ is a functor.

A subcategory $\mathscr{A}'$ of $\mathscr{A}$ is said to be *full* if for any objects $A_1, A_2$ in $\mathscr{A}'$ we have $\mathrm{Hom}_{\mathscr{A}'}(A_1, A_2) = \mathrm{Hom}_{\mathscr{A}}(A_1, A_2)$.

Let $T : \mathscr{A} \to \mathscr{B}$ be a functor and let $\Gamma$ be a family of isomorphisms in $\mathscr{B}$ containing all identities. Let $\mathscr{A}'$ be a full subcategory of $\mathscr{A}$. The $\Gamma$-*closure* $\overline{\mathscr{A}'}$ of $\mathscr{A}'$ is the full subcategory determined by all objects $A$ such that there exists an isomorphism $f : A' \to A$ with $A'$ in $\mathscr{A}'$ and $T(f)$ in $\Gamma$. If $\overline{\mathscr{A}'} = \mathscr{A}$ then we say that $\mathscr{A}'$ is $\Gamma$-*dense* in $\mathscr{A}$.

## 3. Faithful functors

A functor $T : \mathscr{A} \to \mathscr{B}$ is called *faithful* if the following axioms hold

(Fid. 1) If $f, g \in \mathrm{Hom}_{\mathscr{A}}(A, A')$ and $T(f) = T(g)$ then $f = g$.

(Fid. 2) If $f : A \to A'$ is an isomorphism in $\mathscr{A}$ and $T(f) = i_{T(A)}$ then $A = A'$.

It follows from (Fid. 1) that, in (Fid. 2), $f = i_A$.

PROPOSITION 3.1. *Let $T : \mathscr{A} \to \mathscr{B}$ be a faithful functor and let $\phi : B \to B'$ be an isomorphism in $\mathscr{B}$. Then for each object $A$ in $\mathscr{A}$ such that $T(A) = B$ there exists at most one isomorphism $f : A \to A'$ in $\mathscr{A}$ such that $T(f) = \phi$.*

If an isomorphism $f$ with the above properties exists then we say that $f$ is a *lifting* of $\phi$ with origin $A$.

## 4. Transportability

Let $\Gamma$ be a family of isomorphisms in the category $\mathscr{B}$ satisfying the following properties:

($\Gamma$. 1) If $\gamma : B \to B'$ and $\gamma' : B' \to B''$ are in $\Gamma$ then $\gamma'\gamma$ and $\gamma^{-1}$ are in $\Gamma$,

($\Gamma$. 2) $i_B$ is in $\Gamma$ for each object $B$ in $\mathscr{B}$.

A faithful functor $T : \mathscr{A} \to \mathscr{B}$ is called $\Gamma$-*transportable* if for each $\gamma : B \to B'$ in $\Gamma$ and for each $A$ such that $T(A) = B$ there exists a lifting $f$ of $\gamma$ with origin $A$. If $\Gamma$ is the family of all isomorphisms in $\mathscr{B}$ then we say that $T$ is *transportable*.

THEOREM 4.1. *Every faithful functor $T : \mathscr{A} \to \mathscr{B}$ admits a factorization*

$$\mathscr{A} \xrightarrow{I} \mathscr{A}^{\Gamma} \xrightarrow{T^{\Gamma}} \mathscr{B}$$

*such that $\mathscr{A}$ is a $\Gamma$-dense subcategory of $\mathscr{A}^{\Gamma}$, $I$ is the inclusion functor and $T^{\Gamma}$ is faithful and $\Gamma$-transportable.*

The above factorization is essentially unique in the following sense: if

$$\mathscr{A} \xrightarrow{I'} \mathscr{A}' \xrightarrow{T'} \mathscr{B}$$

is another such factorization of $T$ then there exists a unique functor $S : \mathscr{A}^{\Gamma} \to \mathscr{A}'$ such that $SI = I'$, $T'S = T^{\Gamma}$. Further $S$ is an isomorphism of categories.

The proof is based on an explicit construction of a desired factorization of $T$. We begin by constructing a category $(\mathscr{A}, \Gamma)$ as follows: an object of $(\mathscr{A}, \Gamma)$ is a pair $(A, \gamma)$ where $A$ is an object in $\mathscr{A}$ and $\gamma : T(A) \to B$ is in $\Gamma$. A morphism $(A, \gamma) \to (A', \gamma')$ is a triple $(f, \gamma', \gamma)$ with $f : A \to A'$. Composition is defined by the rule

$$(f', \gamma'', \gamma')\,(f, \gamma', \gamma') = (f'f, \gamma''\gamma).$$

We obtain a factorization

$$\mathscr{A} \xrightarrow{J} (A, \Gamma) \xrightarrow{Q} \mathscr{B}$$

of $T$ by setting

$J(A) = (A, i_{T(A)}),$
$J(f) = (f, i_{T(A')}, i_{T(A)})$ for $f : A \to A'$,
$Q(A, \gamma) = B'$ for $\gamma : T(A) \to B'$,
$Q(f, \gamma', \gamma) = \gamma' T(f)\gamma^{-1}$ for $f : A \to A', \gamma : T(A) \to B, \gamma' : T(A') \to B'$.

The functor $J$ maps $\mathscr{A}$ isomorphically onto a full and dense subcategory of $(\mathscr{A}, \Gamma)$. The functor $Q$ satisfies (Fid. 1) but generally does not satisfy (Fid. 2). We therefore consider the class $\Omega$ of all isomorphisms in $(A, \Gamma)$ whose images under $Q$ are identities in $\mathscr{B}$. Thus $\Omega$ consists of all morphism $(f, \gamma', \gamma)$ such that $f$ is an $\mathscr{A}$-isomorphism and $\gamma' T(f) = \gamma$. The class $\Omega$ is closed under composition and inverse, contains all identities of $(A, \Gamma)$ but not other automorphisms. A class of isomorphisms with such properties leads quite generally to a construction of a quotient-category, in which the isomorphisms of $\Omega$ become identities. We define the category $\mathscr{A}^{\Gamma}$ to be the quotient-category $(A, \Gamma)/\Omega$. The functor $Q$ then factors as follow:

$$(A, \Gamma) \to A^{\Gamma} \xrightarrow{T^{\Gamma}} \mathscr{B},$$

while $I : \mathscr{A} \to \mathscr{A}^{\Gamma}$ is defined by composition

$$\mathscr{A} \xrightarrow{J} (\mathscr{A}, \Gamma) \to \mathscr{A}^{\Gamma}.$$

## 5. Local categories

A local category consists of

(i) a category $\mathscr{A}$,

(ii) a (covariant) functor $L : \mathscr{A} \to \mathscr{T}$ into a category $\mathscr{T}$ of topological spaces and continuous maps,

(iii) two functions which to each object $A$ of $\mathscr{A}$ and each open set $U \subset L(A)$ assign (1) an object $A|U$ of $\mathscr{A}$ (called the *restriction* of $A$ to $U$) and (2) an $\mathscr{A}$-morphism $i_A|U : A|U \to A$ (called the *inclusion*).

The composition

$$A|U \xrightarrow{i_A|U} A \xrightarrow{f} A'$$

is denoted by $f|U$ and is called the restriction of $f$ to $U$.

The above two functions are subject to the following set of axioms.

(Loc. 1) $L(A|U) = U$.

(Loc. 2) If $V \subset U$ are open subsets of $L(A)$, then $(A|U)|V = A|V$.

(Loc. 3) Let $A, A'$ be two objects in $\mathscr{A}$ with $L(A) = L(A')$. If there exists an open covering $\{U_i\}$, $i \in I$ of $L(A)$ such that $A|U_i = A'|U_i$ for every $i \in I$, then $A = A'$.

(Loc. 1′) $L(i_A|U$ is the inclusion map $U \to L(A)$.

(Loc. 2′) If $V \subset U$ are open sets of $L(A)$ then

$$(i_A|U)(i_{A|U}|V) = i_A|V.$$

(Loc. 3′) Given an open covering $\{U_i\}$, $i \in I$ of $L(A)$, $A \in \mathscr{A}$, and given morphisms $f_i : A|U_i \to A'$ in $\mathscr{A}$ such that

$$f_i|U \cap U_j = f_j|U_i \cap U_j$$

for each pair $(i, j)$, then there exists a unique $f : A \to A'$ in $\mathscr{A}$ such that

$$f|A_i = f_i \qquad \text{for all } i \in I.$$

(Loc. 4) Given any morphism $f : A \to A'$ in $\mathscr{A}$ and given an open $U'$ in $L(A')$ such that $L(f)$ maps $L(A)$ into $U'$, there exists a unique morphism $g : A \to A'|U'$ in $\mathscr{A}$ such that

$$A \longrightarrow A'|U' \xrightarrow{i_{A'}|U'} A'$$

is $f$.

The map $g$ of the last axiom is denoted by $U'|f$. The two-sided restriction $|U'|fU$ for $U$ open in $L(A)$ is defined as $U'|(f|U)$ and exists whenever $L(f)U \subset U'$.

## 6. Local functors

Let $\mathscr{A}$ and $\mathscr{B}$ be two local categories with functors $L_{\mathscr{A}} : \mathscr{A} \to \mathscr{T}$ and $L_{\mathscr{B}} : \mathscr{B} \to \mathscr{T}$ respectively. A (covariant) functor $S : \mathscr{A} \to \mathscr{B}$ is called *local* if the two following conditions are satisfied:

(LF. 1) For any object $A$ in $\mathscr{A}$ and any open subset $U$ of $L_{\mathscr{A}}(A)$ the topological space $V = L_{\mathscr{B}}S(A|U)$ is an open subset of $L_{\mathscr{B}}S(A)$ and

(i) $$S(A|U) = S(A)|V,$$

(ii) $$S(L_A|U) = i_{S(A)}|V.$$

(LF. 2) For every object $A$ of $\mathscr{A}$, the mapping $U \to V = L_{\mathscr{B}}S(A|U)$ which maps the open sets of $L_{\mathscr{A}}(A)$ into the open sets of $L_{\mathscr{B}}S(A)$ preserves finite intersections and arbitrary unions.

If further $S$ satisfies

(SLF) $$L_{\mathscr{A}} = L_{\mathscr{B}}S$$

then we say that $S$ is *strictly local*. In this case we have $V = U$ in (LF. 1) and condition (LF. 2) is redundant.

## 7. Subcategories of local categories

Let $\mathscr{A}$, $\mathscr{A}'$ be local categories and assume that $\mathscr{A}$ is a subcategory of $\mathscr{A}'$ (in the ordinary sense). If the inclusion functor $I : \mathscr{A} \to \mathscr{A}'$ is strictly local then we say that $\mathscr{A}$ is a *local subcategory* of $\mathscr{A}'$.

We say that $\mathscr{A}$ is *locally rich* in $\mathscr{A}'$ if $\mathscr{A}$ is a full local subcategory of $\mathscr{A}'$ and if for each object $A'$ of $\mathscr{A}'$ there exists an open covering $\{U_i\}$, $i \in I$ of $L(A')$ such that $A'|U_i$ is in $\mathscr{A}$ for every $i \in I$.

## 8. Collation

Let $S : \mathscr{A} \to \mathscr{B}$ be a strictly local functor. We shall say that $S$ is *collatable* if the following condition is satisfied

(Col.) Let $B$ be an object of $\mathscr{B}$ and $\{U_i\}$ $i \in I$ an open covering of $L_{\mathscr{B}}(B)$. Let $\{A_i\}$, $i \in I$ be a family of objects in $\mathscr{A}$ such that

$$S(A_i) = B|U_i \qquad \text{for all } i \in I$$

$$A_i|U_i \cap U_j = A_j|U_i \cap U_j \qquad \text{for all } i, j \in I.$$

Then there exists an object $A$ in $\mathscr{A}$ such that $A|U_i = A_i$ for all $i \in I$.

The uniqueness of $A$ and the fact that $S(A) = B$ follow from (Loc. 3). If $S$ is the functor $L_{\mathscr{A}} : \mathscr{A} \to \mathscr{T}$ connected with the local structure of $\mathscr{A}$ then we replace the phrase "$S$ is collatable" by "$\mathscr{A}$ is collatable".

THEOREM 8.1. *Every strictly local functor $S : \mathscr{A} \to \mathscr{B}$ admits a factorization*

$$\mathscr{A} \xrightarrow{I} \mathscr{A}^c \xrightarrow{S^c} \mathscr{B}$$

*such that $\mathscr{A}^c$ is a local category containing $\mathscr{A}$ as a local subcategory, $\mathscr{A}$ is locally rich in $\mathscr{A}^c$, $I$ is the inclusion map, $S^c$ is a strictly local functor which is collatable.*

The above factorization is essentially unique in the following sense. If

$$\mathscr{A} \xrightarrow{I'} \mathscr{A}' \xrightarrow{S} \mathscr{B}$$

is another such factorization of $S$ then there exists a unique functor $T : \mathscr{A}^c \to \mathscr{A}'$ such that $TI = I'$, $S'T = T^c$. Further $T$ is an isomorphism of local categories.

As in the case of Theorem 4.1, the proof is based on an explicit construction of a desired factorization of $S$.

Let $B$ be an object of $\mathscr{B}$ and let $S^{-1}(B)$ be the set of objects $A$ in $\mathscr{A}$ such that $S(A) = B$. Let $V \subset U$ be open sets in $L_{\mathscr{B}}(B)$; for each $A \in S^{-1}(B|U)$ we have $A|V \in S^{-1}(B|V)$. There results a mapping

$$\phi_{V,U} : S^{-1}(B|U) \to S^{-1}(B|V).$$

Clearly $\phi_{W,V} = \phi_{W,V} \circ \phi_{V,U}$ for $W \subset V \subset U \subset L_{\mathscr{B}}(B)$. The sets $S^{-1}(B|U)$ and the maps $\phi_{V,U}$ thus define a pre-sheaf on the space $L_{\mathscr{B}}(B)$. We denote by $F_B$ the resulting sheaf. By definition, an object of the category $\mathscr{A}^c$ will be a pair $(B, \sigma)$, where $B$ is an object in $\mathscr{B}$, and $\sigma$ is a cross-section of the sheaf $F_B$. We also define $S^c(B, \sigma) = B$.

If $U$ is an open set in $L_{\mathscr{B}}(B)$, the sheaf $F_{B|U}$ is then the restriction of the sheaf $F_B$ to the open set $U$. Thus each cross-section $\sigma$ of $F_B$ defines a cross-section $\sigma/U$ of $F_{B|U}$. We define $(B, \sigma)|U = (B|U, \sigma|U)$.

Let $A$ be an object of $\mathscr{A}$ and let $B = S(A)$. For each open set $U$ in $L_B(B) = L_{\mathscr{A}}(A)$ we have $A|U \in S^{-1}(B|U)$. If $V$ is an open subset of $U$ then $\phi_{V|U} A|U = A|V$. Thus we obtain a cross-section $\sigma_A$ of the sheaf $F_B$. We shall identify $A$ with $(B, \sigma_A)$. In this way the objects of $\mathscr{A}$ become a subset of the objects of $\mathscr{A}^c$. It is further easy to prove that for each $(B, \sigma)$ there exists an open covering $\{U_i\}$, $i \in I$, of $L_{\mathscr{B}}(B)$ such that $(B, \sigma)|U_i$ is an object of $\mathscr{A}$, for each $i \in I$.

Now the morphisms in $\mathscr{A}^c$ can be defined. Let $(B, \sigma)$, $(B', \sigma)$ be objects in $\mathscr{A}^c$ and let $f : B \to B$: be a morphism in $\mathscr{B}$. Consider the pairs $(U, U')$ where $U$ is an open set of $L_{\mathscr{B}}(B)$, $U'$ is an open set in $L_{\mathscr{B}}(B)'$ and $L_{\mathscr{B}}(f)U \subset U'$ (i.e.,$U'|f|U$ is defined), and further such that $(B, \sigma)|U$ and $(B', \sigma')|U'$ are objects of $\mathscr{A}$. A morphism $g : (B, \sigma) \to (B', \sigma')$ with $S^c(g) = f$ is defined as a family of $\mathscr{A}$-morphisms

$$g_{(U,U')} : (B, \sigma)|U \to (B', \sigma')|U',$$

defined for each pair $(U, U')$ as above as satisfying the following conditions

$$S(g_{(U,U')}) = U'|f|U$$

$$V'|g_{(U,U')}|V = g_{(V,V')}$$

where $(V, V')$ is another pair as above with $V \subset U$, $V' \subset U'$.

The inclusion maps $i_{(B,\sigma)}|U : (B, \sigma)|U \to (B, \sigma)$ can now be defined in an evident fashion. The remaining verifications are fairly lengthy but straightforward.

## 9. Local categories and transportability

In §4 we considered a category $\mathscr{B}$ and a family $\Gamma$ of isomorphisms in $\mathscr{B}$ satisfying conditions $(\Gamma.1)$ and $(\Gamma.2)$. Here we shall assume that $\mathscr{B}$ is a local category and that $\Gamma$ satisfies the following additional property:

$(\Gamma.3)$ Let $\gamma : B \to B'$ be an isomorphism in $\mathscr{B}$ and let $\{U_i\}$ and $\{U'_i\}$, $i \in I$, be open coverings of $L_{\mathscr{B}}(B)$ and $L_{\mathscr{B}}(B')$ respectively, such that $L_{\mathscr{B}}(\gamma)U_i = U'_i$ for each $i \in I$. Then $\gamma \in \Gamma$ if and only if each $U'_i|\gamma|U_i$ is in $\Gamma$.

Assuming this additional property of $\Gamma$ we have the following results.

THEOREM 9.1. *Every strictly local faithful functor $S : \mathscr{A} \to \mathscr{B}$ admits a factorization*

$$\mathscr{A} \xrightarrow{I} \mathscr{A}^\Gamma \xrightarrow{S^\Gamma} \mathscr{B}$$

*such that $\mathscr{A}$ is a local and $\Gamma$-dense subcategory of the local category $\mathscr{A}^\Gamma$, $I$ is the inclusion functor and $S^\Gamma$ is strictly local, faithful and $\Gamma$-transportable.*

The factorization is unique in the same sense as that of Theorem 4.1. The essential part of the proof consists in the introduction of the structure of a local category in the category $\mathscr{A}^\Gamma$ given in Theorem 4.1.

THEOREM 9.2. *Let $S : \mathscr{A} \to \mathscr{B}$ be a strictly local and faithful functor. Then the functor $S^c : \mathscr{A}^c \to \mathscr{B}$ of Theorem* 8.1. *is faithful. If $S$ is $\Gamma$-transportable then so is $S^c$.*

## 10. Perfect functors

Let $S : \mathscr{A} \to \mathscr{B}$ be a strictly local functor, let $\mathscr{A}'$ be a full local category of $\mathscr{A}$, and let $\Gamma$ be a family of isomorphism in $\mathscr{B}$ satisfying $(\Gamma.1)$–$(\Gamma.3)$. We shall say that $\mathscr{A}'$ is *locally $\Gamma$-dense* in $\mathscr{A}$ if the $\Gamma$-closure $\mathscr{A}'$ of $\overline{\mathscr{A}'}$ (cf. §2) is a rich subcategory of $\mathscr{A}$.

A strictly local functor $S : \mathscr{A} \to \mathscr{B}$ is called *$\Gamma$-perfect* if it is faithful, $\Gamma$-transportable and collatable.

THEOREM 10.1. *Every strictly local faithful functor $S : \mathscr{A} \to \mathscr{B}$ admits a factorization*

$$\mathscr{A} \xrightarrow{I} \mathscr{A}^* \xrightarrow{S^*} \mathscr{B}$$

*where $\mathscr{A}^*$ is a local category in which $\mathscr{A}$ is a locally $\Gamma$-dense subcategory, $I$ is the inclusion and $S^*$ is $\Gamma$-perfect.*

This factorization is essentially unique in the same sense as in the earlier theorems.

For the proof, we first apply Theorem 9.1 and obtain a factorization

$$\mathscr{A} \xrightarrow{I_1} \mathscr{A}^\Gamma \xrightarrow{S^\Gamma} \mathscr{B},$$

then we apply Theorem 8.1 to $S^\Gamma$ and obtain the factorization

$$\mathscr{A}^\Gamma \xrightarrow{I_2} \mathscr{A}^{\Gamma_c} \xrightarrow{S^{\Gamma_c}} \mathscr{B}.$$

Then $\mathscr{A}$ is locally $\Gamma$-dense in $\mathscr{A}^* = \mathscr{A}^{\Gamma_c}$ and the strictly local functor $S^* = S^{\Gamma_c}$ is faithful and $\Gamma$-transportable by Theorem 9.2 and is collatable. Thus $S^*$ is $\Gamma$-perfect.

Let $S : \mathscr{A} \to \mathscr{B}$ be $\Gamma$-perfect and let $\mathscr{M}$ be a locally $\Gamma$-dense subcategory of $\mathscr{A}$. The uniqueness feature of Theorem 10.1 implies that $\mathscr{A}$ may be identified with $\mathscr{M}^* = \mathscr{M}^{\Gamma_c}$.

## 11. Fibre bundles

Let $\mathscr{B}$ be a local category and $\mathscr{F}$ any category. The category $(\mathscr{B}, \mathscr{F})$ of *forma bundles* with bases in $\mathscr{B}$ and fibres in $\mathscr{F}$ is defined as follows: An object of $(\mathscr{B}, \mathscr{F})$ is a pair $(B, \phi)$ consisting of an object $B$ in $\mathscr{B}$ and a function $\phi$ which to each point $x$ of $L_{\mathscr{B}}(B)$ assigns an object $\phi x$ of $\mathscr{F}$. A morphism $(B, \phi) \to (B', \phi')$ is a pair $(f, \psi)$ consisting of a morphism $f : B \to B'$ in $\mathscr{B}$ and a function $\psi$ which to each $x \in L_{\mathscr{B}}(B)$ assigns a morphism $\psi x : \psi x \to \phi' x'$ in $\mathscr{B}$, where $x' = L_{\mathscr{B}}(f)x$. Composition is defined in the obvious manner. We have a natural functor $Q : (\mathscr{B}, \mathscr{F}) \to \mathscr{B}$ given by $Q(B, \phi) = B$, $Q(f, \psi) = f$. Composing $Q$ with $L_{\mathscr{B}} : \mathscr{B} \to \mathscr{T}$ we obtain a functor $L_{(\mathscr{B}, \mathscr{F})} : (\mathscr{B}, \mathscr{F}) \to \mathscr{T}$. The structure of $(\mathscr{B}, \mathscr{F})$ as a local category (relative to the functor $L_{(\mathscr{B}, \mathscr{F})}$ is defined by setting $(B, \phi)|U = (B|U, \phi|U)$, with the inclusion maps $i_{(B, \phi)}|U$ defined in the obvious manner. The functor $Q$ is then strictly local. In $(\mathscr{B}, \mathscr{F})$ we consider the family $\Gamma$ of all isomorphisms $(f, \psi)$ for which $f$ is an identity in $\mathscr{B}$. This family $\Gamma$ satisfies properties $(\Gamma.1)$–$(\Gamma.3)$.

A category of bundles with bases $\mathscr{B}$ and fibres $\mathscr{F}$ is by definition a local category $\mathscr{A}$ together with a $\Gamma$-perfect functor $S : \mathscr{A} \to (\mathscr{B}, \mathscr{F})$. The category $\mathscr{A}$ is completely determined by any locally $\Gamma$-dense subcategory $\mathscr{M}$. Conversely, given a local category $\mathscr{M}$ together with a strictly local and faithful functor $S : \mathscr{M} \to (\mathscr{B}, \mathscr{F})$, there is an essentially unique category of bundles $\mathscr{A}$ (with base $\mathscr{B}$ and fibres $\mathscr{F}$) in which $\mathscr{M}$ is locally $\Gamma$-dense.

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# SUR LA TOPOLOGIE DES VARIÉTÉS ALGÉBRIQUES EN CARACTÉRISTIQUE $p$.

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## Introduction

Comme l'a signalé A. Weil, l'un des problèmes les plus intéressants de la géométrie algébrique sur un corps de caractéristique $p > 0$ est de donner une définition satisfaisante des "nombres de Betti" et des "groupes d'homologie" d'une variété algébrique $X$ (supposée projective et non singulière).

En ce qui concerne les nombres de Betti, j'avais proposé dans [13] de les définir par la formule suivante (imitée du cas classique):

$$B_n = \sum_{r+s=n} h^{r,s} \quad \text{où} \quad h^{r,s} = \dim H^s(X, \Omega^r),$$

$\Omega^r$ désignant le faisceau des germes de formes différentielles régulières de degré $r$ sur $X$.

Les $B_n$ ainsi définis ont certaines des propriétés que l'on est en droit d'attendre de "nombres de Betti" : par exemple, ils vérifient la "dualité de Poincaré" $B_n = B_{2d-n}$, si $d = \dim X$, cf. [13]. Cependant des résultats récents ont montré qu'ils peuvent posséder des propriétés pathologiques : c'est ainsi que, si $g$ désigne la dimension de la variété d'Albanese de $X$, on peut avoir $g < h^{0,1}$ (Igusa [6]), et aussi $h^{0,1} \neq h^{1,0}$ (cf. nº 20). Ces faits montrent que les $B_n$ ne fournissent, tout au plus, qu'une *majoration* des nombres de Betti cherchés.

D'ailleurs, si l'on se place au point de vue "groupes d'homologie", l'insuffisance des $H^s(X, \Omega^r)$ est claire : ce sont des espaces vectoriels de caractéristique $p$, alors que, comme l'a mis en évidence Weil, on a besoin de groupes *de caractéristique zéro*, de façon à pouvoir y définir des traces et démontrer une *formule de Lefschetz* (donnant le nombre de points fixes d'une application régulière de $X$ dans lui-même).

Dans le présent mémoire nous indiquons comment l'on peut effectivement attacher à $X$ des groupes $H^q$ qui soient des modules sur un anneau $\Lambda$ de caractéristique zéro, analogue à l'anneau des entiers $p$-adiques; ces groupes sont définis comme les limites projectives des groupes de cohomologie de $X$ à valeurs dans des *faisceaux de vecteurs de Witt*. Ces groupes de cohomologie sont étudiés dans le §1; on y verra notamment comment on peut définir la *torsion* de $X$, au moyen d'opérations semblables à celles de Bockstein; il semble bien que ce soit cette torsion qui soit responsable des phénomènes pathologiques cités plus haut. Nous avons dû laisser sans réponse une question importante : les $H^q$ sont-ils des $\Lambda$-modules de type fini? (c'est vrai si $q = 0$ ou 1). De plus, les $H^q$ ne constituent certainement qu'une partie de la cohomologie de $X$, celle qui correspond aux $h^{0,q}$ du cas classique: c'est dire que nous n'avons encore aucune définition raisonnable des "nombres de Betti" à proposer.

Le cas des *courbes*, auquel est consacré le §2, est cependant encourageant. Le $\Lambda$-module $H^1$ est alors un $\Lambda$-module libre de rang égal à $2g - \sigma$, $g$ désignant le genre de $X$ et $\sigma$ le rang du groupe des éléments d'ordre $p$ de la jacobienne de $X$; l'entier $\sigma$ peut être déterminé au moyen de la *matrice de Hasse-Witt* de $X$. Dans les démonstrations, un rôle décisif est joué par une opération sur les formes différentielles qui vient d'être introduite par P. Cartier; comme les résultats de Cartier sur ce sujet n'ont pas encore été publiés, nous avons reproduit la définition et les principales propriétés de cette opération.

Enfin le §3 montre comment la cohomologie à valeurs dans les vecteurs de Witt permet de classifier les *revêtements cycliques d'ordre $p^n$*, étendant ainsi aux variétés de dimension quelconque des résultats connus pour les courbes ([5], [12]).

## §I. Cohomologie à valeurs dans les vecteurs de Witt

### 1. Vecteurs de Witt

Soit $p$ un nombre premier qui restera fixé dans toute la suite. Si $A$ est un anneau commutatif, à élément unité, de caractéristique $p$, nous désignerons par $W_n(A)$ l'anneau des vecteurs de Witt de longueur $n$ à coefficients dans $A$ (cf. [21], §3). Rappelons qu'un élément de $W_n(A)$ est un système $\alpha = (a_0, \cdots, a_{n-1})$ avec $a_i \in A$; si $\beta = (b_0, \cdots, b_{n-1})$ est un autre vecteur, la somme:

$$\alpha + \beta = (c_0, \cdots, c_{n-1})$$

est donnée par des formules:

$$\begin{aligned} c_0 &= a_0 + b_0 \\ c_1 &= a_1 + b_1 - \textstyle\sum_{m=1}^{m=p-1} \frac{1}{p}\binom{p}{m} a_0^m b_0^{p-m} \\ &\cdots \\ c_i &= a_i + b_i + f_i(a_0, b_0, \cdots, a_{i-1}, b_{i-1}) \\ &\cdots \end{aligned}$$

où les $f_i$ sont des polynômes à coefficients entiers dont on trouvera le procédé de formation dans [21]. De même, la différence et le produit de deux vecteurs sont donnés par des opérations polynomiales.

Les anneaux $W_n(A)$ sont reliés par les opérations suivantes :

(a) L'endomorphisme de Frobenius $F : W_n(A) \to W_n(A)$ qui applique le vecteur $(a_0, \cdots, a_{n-1})$ sur le vecteur $(a_0^p, \cdots, a_{n-1}^p)$.

(b) L'opération de décalage $V : W_n(A) \to W_{n+1}(A)$ qui applique le vecteur $(a_0, \cdots, a_{n-1})$ sur le vecteur $(0, a_0, \cdots, a_{n-1})$.

(c) L'opération de restriction $R : W_{n+1}(A) \to W_n(A)$ qui applique le vecteur $(a_0, \cdots, a_n)$ sur le vecteur $(a_0, \cdots, a_{n-1})$.

Les opérations $F$ et $R$ sont des homomorphismes d'anneaux; elles commutent entre elles. L'opération $V$ est additive, et vérifie l'identité $(Vx) \cdot y = V(x \cdot FRy)$ pour $x \in W_n(A)$, $y \in W_{n+1}(A)$. On a en outre $RVF = FRV = RFV = p$ (multiplication par $p$).

Nous noterons $W(A)$ l'anneau des vecteurs de Witt $(a_0, \cdots, a_n, \cdots)$ de longueur infinie; c'est la limite projective, pour $n$ infini, du système formé par les $W_n(A)$ et les homomorphismes $R$. Les opérations $V$ et $F$ sont définies sur $W(A)$ et vérifient la relation $VF = FV = p$; comme $V$ et $F$ sont injectives, on en conclut que l'anneau $W(A)$ est un anneau de caractéristique 0.

Exemple. Prenons pour $A$ le corps $\boldsymbol{F}_p = \boldsymbol{Z}/p\boldsymbol{Z}$; l'anneau $W_n(\boldsymbol{F}_p)$ est alors canoniquement isomorphe à $\boldsymbol{Z}/p^n\boldsymbol{Z}$, et l'anneau $W(\boldsymbol{F}_p)$ est canoniquement isomorphe à l'anneau $\boldsymbol{Z}_p$ des entiers $p$-adiques; dans ce cas, l'opération $F$ est l'identité.

Plus généralement, si $k$ est un corps parfait de caractéristique $p$, l'anneau $W(k)$ est un anneau de valuation discrète, non ramifié, complet, ayant $k$ pour corps des restes (cf. [21], §3); en particulier, $W(k)$ est un anneau principal, d'unique idéal maximal $pW(k)$ vérifiant $W(k)/pW(k) = k$.

## 2. Faisceaux de vecteurs de Witt sur une variété algébrique

Soit $X$ une variété algébrique définie sur un corps algébriquement clos $k$ de caractéristique $p$, et soit $\mathcal{O}$ le faisceau de ses anneaux locaux (cf. [14], n° 34). Pour tout $x \in X$, l'anneau $\mathcal{O}_x$ est un anneau de caractéristique $p$, et, si $n$ est un entier $\geqq 1$, on peut former l'anneau $W_n(\mathcal{O}_x)$; lorsque $x$ varie, les $W_n(\mathcal{O}_x)$ forment de façon naturelle un faisceau d'anneaux, que nous noterons $\mathcal{W}_n$. En tant que faisceau *d'ensembles*, $\mathcal{W}_n$ est isomorphe à $\mathcal{O}^n$; mais, bien entendu, les lois de composition de ces deux faisceaux sont différentes si $n \geqq 2$.

Les opérations $F$, $V$ et $R$ du n° 1 définissent des opérations sur les faisceaux $\mathcal{W}_n$ que nous noterons par les mêmes symboles. On a la suite exacte, valable si $n \geqq m$:

$$(1) \qquad 0 \to \mathcal{W}_m \xrightarrow{V^{n-m}} \mathcal{W}_n \xrightarrow{R^m} \mathcal{W}_{n-m} \to 0.$$

Pour $m = 1$ on a $\mathcal{W}_m = \mathcal{O}$, d'où la suite exacte:

$$(2) \qquad 0 \to \mathcal{O} \xrightarrow{V^{n-1}} \mathcal{W}_n \xrightarrow{R} \mathcal{W}_{n-1} \to 0.$$

On voit ainsi que $\mathcal{W}_n$ est extension multiple de $n$ faisceaux isomorphes à $\mathcal{O}$; cela permet d'étendre aux $\mathcal{W}_n$ un grand nombre de resultats connus pour le faisceau $\mathcal{O}$; par exemple, on peut facilement montrer (en utilisant [14], n$^{\text{os}}$ 13 et 16) que les $\mathcal{W}_n$ sont des faisceaux cohérents d'anneaux, au sens de [14], n° 15.

Puisque les $\mathcal{W}_n$ sont des faisceaux de groupes abéliens, les groupes de cohomologie $H^q(X, \mathcal{W}_n)$ sont définis pour tout entier $q \geqq 0$. Si l'on note $\Lambda$ l'anneau $W(k)$, les $\mathcal{W}_n$ sont des $\Lambda$-modules, annulés par $p^n\Lambda$, et il en est donc de même des $H^q(X, \mathcal{W}_n)$. Les opérations induites par $F$, $V$ et $R$ sur les $H^q(X, \mathcal{W}_n)$ sont semilinéaires: on a les formules

$$(3) \qquad F(\lambda w) = \lambda^p F(w), \qquad V(\lambda w) = \lambda^{p^{-1}} V(w), \qquad R(\lambda w) = \lambda R(w), \qquad \lambda \in \Lambda.$$

La proposition suivante donne les principales propriétés élémentaires des $H^q(X, \mathcal{W}_n)$:

PROPOSITION 1. (a) *On a* $H^q(X, \mathscr{W}_n) = 0$ *pour* $q > \dim X$.

(b) *Si* $X$ *est une variété affine, on a* $H^q(X, \mathscr{W}_n) = 0$ *pour* $q > 0$.

(c) *Si* $X$ *est une variété projective, les* $\Lambda$*-modules* $H^q(X, \mathscr{W}_n)$ *sont des modules de longueur finie.*

(d) *Si* $\mathfrak{U}$ *est un recouvrement fini de* $X$ *par des ouverts affines, on a* $H^q(\mathfrak{U}, \mathscr{W}_n) = H^q(X, \mathscr{W}_n)$ *pour tout* $q \geqq 0$.

(e) *A toute suite exacte* $0 \to \mathscr{W}_n \to \mathscr{B} \to \mathscr{C} \to 0$, *où* $\mathscr{B}$ *et* $\mathscr{C}$ *sont des faisceaux quelconques, est associée une suite exacte de cohomologie:*

$$\cdots \to H^q(X, \mathscr{W}_n) \to H^q(X, \mathscr{B}) \to H^q(X, \mathscr{C}) \to H^{q+1}(X, \mathscr{W}_n) \to \cdots .$$

Puisque le faisceau $\mathcal{O}$ est un faisceau algébrique cohérent, la suite exacte (2) vérifie les hypothèses du Théorème 5 de [14], n° 47, et l'on obtient une suite exacte de cohomologie:

$$(4) \qquad \cdots \to H^q(X, \mathcal{O}) \to H^q(X, \mathscr{W}_n) \to H^q(X, \mathscr{W}_{n-1}) \to \cdots .$$

En utilisant (4), on ramène immédiatement les assertions (a), (b), (c) de la Proposition 1 au cas particulier $n = 1$, où elles sont connues ([15], th. 2-[14], n° 46-[14], n° 66). Les assertions (d) et (e) résultent de (b) en appliquant les raisonnements de [14], n° 47.

REMARQUE. En utilisant (b), on peut montrer que les groupes de cohomologie $H^q(X, \mathscr{W}_n)$, définis ici par la méthode des recouvrements, coincident avec ceux définis par Grothendieck comme les $\mathrm{Ext}^n$ du foncteur $\Gamma(X, \mathscr{F})$.

### 3. Opérations de Bockstein

La construction des faisceaux $\mathscr{W}_n$ n'est pas spéciale aux variétés algébriques et aux faisceaux de leurs anneaux locaux. Nous aurions pu l'appliquer à un complexe simplicial $K$, en remplaçant le faisceau $\mathcal{O}$ par le faisceau constant $\mathbf{Z}/p\mathbf{Z}$; à la place de $\mathscr{W}_n$, nous aurions obtenu le faisceau constant $\mathbf{Z}/p^n\mathbf{Z}$. Ainsi, les groupes $H^q(X, \mathscr{W}_n)$ apparaissent comme les analogues des groupes de cohomologie de $K$ mod $p^n$; nous allons poursuivre cette analogie en définissant des "opérations de Bockstein" jouissant de propriétés semblables à celles du cas classique.

D'après la Proposition 1, (e), la suite exacte (1) donne naissance à une suite exacte de cohomologie, et, en particulier, à un opérateur de cobord

$$\delta^q_{n,m} : H^q(X, \mathscr{W}_{n-m}) \to H^{q+1}(X, \mathscr{W}_m), \qquad n \geqq m.$$

Le cobord $\delta^q_{n,m}$ sera appelé une *opération de Bockstein* en dimension $q$. Par définition, on a donc la suite exacte:

$$\cdots \to H^q(X, \mathscr{W}_m) \xrightarrow{V^{n-m}} H^q(X, \mathscr{W}_n) \xrightarrow{R^m} H^q(X, \mathscr{W}_{n-m}) \xrightarrow{\delta^q_{n,m}} H^{q+1}(X, \mathscr{W}_m) \to \cdots$$

Les opérations de Bockstein sont semi-linéaires (de façon précise, $\delta^q_{n,m}$ est $p^{n-m}$-linéaire) et commutent avec $F$; elles vérifient avec $V$ et $R$ des relations de commutation que nous laissons au lecteur le soin d'expliciter.

Lorsque $n \geqq 2m$, l'idéal $V^{n-m}(\mathscr{W}_m)$ de $\mathscr{W}_n$ est un idéal *de carré nul*; cela permet de calculer l'effet de $\delta^q_{n,m}$ sur un cup-produit. On trouve:

$$\delta^q_{n,m}(x \cdot y) = \delta^r_{n,m}(x) \cdot F^{n-m}R^{n-2m}y + (-1)^r F^{n-m}R^{n-2m}x \cdot \delta^s_{n,m}(y), \tag{5}$$

où $x \in H^r(X, \mathscr{W}_{n-m})$ et $y \in H^s(X, \mathscr{W}_{n-m})$, avec $r + s = q$.

Par analogie avec le cas classique, nous dirons que $X$ *n'a pas de torsion* (homologique) *en dimension* $q$ si les $\delta^q_{n,m}$ sont nuls pour tous les couples $(n,m)$, avec $n \geqq m$. En vertu de la suite exacte écrite plus haut, cela signifie que les homomorphismes

$$R^m : H^q(X, \mathscr{W}_n) \to H^q(X, \mathscr{W}_{n-m})$$

sont surjectifs; on vérifie d'ailleurs facilement qu'il suffit que les homomorphismes $H^q(X, \mathscr{W}_n) \to H^q(X, \mathscr{O})$ le soient.

EXEMPLES. Une variété algébrique $X$ de dimension $r$ n'a de torsion ni en dimension $r$ (puisque $H^{r+1}(X, \mathscr{W}_m) = 0$ d'après la Proposition 1), ni en dimension 0 (car toute section $f$ du faisceau $\mathscr{O}$ se remonte en une section $(f, 0, \cdots, 0)$ du faisceau $\mathscr{W}_n$). Ainsi, une *courbe algébrique* est sans torsion. Par contre, les surfaces construites par Igusa dans [6] ont de la torsion en dimension 1; nous verrons au n° 20 un exemple analogue.

*Les opérations* $\beta_n$.

A côté des opérations de Bockstein que nous venons de définir, et qui opèrent sur les divers groupes $H^q(X, \mathscr{W}_n)$, il y a intérêt à introduire des opérations $\beta_n$, non partout définies, opérant sur

$$H^*(X, \mathscr{O}) = \textstyle\sum_{q=0}^{q=\infty} H^q(X, \mathscr{O}).$$

La première de ces opérations

$$\beta^q_1 : H^q(X, \mathscr{O}) \to H^{q+1}(X, \mathscr{O})$$

n'est autre que l'opération de Bockstein $\delta^q_{2,1}$ associée à la suite exacte:

$$0 \to \mathscr{O} \to \mathscr{W}_2 \to \mathscr{O} \to 0.$$

On a $\beta^q_1 \circ \beta^{q-1}_1 = 0$, ce qui permet de poser $H^q(X, \mathscr{O})_2 = \mathrm{Ker}\,(\beta^q_1)/\mathrm{Im}\,(\beta^{q-1}_1)$; l'opération $\beta^q_2$ appliquera alors $H^q(X, \mathscr{O})_2$ dans $H^{q+1}(X, \mathscr{O})_2$, et ainsi de suite.

De façon précise, posons:

$$Z^q_n = \begin{cases} \mathrm{Im}\,[H^q(X, \mathscr{W}_n) \xrightarrow{R^{n-1}} H^q(X, \mathscr{O})] \\ \mathrm{Ker}\,[H^q(X, \mathscr{O}) \xrightarrow{\delta^q_{n,n-1}} H^{q+1}(X, \mathscr{W}_{n-1})] \end{cases} \tag{6}$$

et

$$B^q_n = \begin{cases} \mathrm{Ker}\,[H^q(X, \mathscr{O}) \xrightarrow{V^{n-1}} H^q(X, \mathscr{W}_n)] \\ \mathrm{Im}\,[H^{q-1}(X, \mathscr{W}_{n-1}) \xrightarrow{\delta^{q-1}_{n,1}} H^q(X, \mathscr{O})]. \end{cases} \tag{7}$$

Les $Z_n^q$ (resp. les $B_n^q$) vont en décroissant (resp. en croissant) avec l'entier $n$, et les $Z_n^q$ contiennent les $B_m^q$; pour $n = 1$, on a $B_1^q = 0$ et $Z_1^q = H^q(X, \mathcal{O})$; pour $n = 2$, on a $B_2^q = \operatorname{Im}(\beta_1^{q-1})$ et $Z_2^q = \operatorname{Ker}(\beta_1^q)$, de telle sorte que $Z_2^q/B_2^q = H^q(X, \mathcal{O})_2$. De façon générale, on posera $H^q(X, \mathcal{O})_n = Z_n^q/B_n^q$; si $x \in Z_n^q$, choisissons un $y \in H^q(X, \mathscr{W}_n)$ tel que $R^{n-1}y = x$, et posons $z = \delta_{n+1,1}^q(y)$, qui est un élément de $H^{q+1}(X, \mathcal{O})$; on vérifie tout de suite que l'application $x \to z$ définit par passage au quotient un homomorphisme

$$\beta_n^q : H^q(X, \mathcal{O})_n \to H^{q+1}(X, \mathcal{O})_n,$$

et que l'on a $\operatorname{Ker}(\beta_n^q) = Z_{n+1}^q/B_n^q$ et $\operatorname{Im}(\beta_n^{q-1}) = B_{n+1}^q/B_n^q$. Les $\beta_n^q$ sont les opérations cherchées. Pour qu'elles soient identiquement nulles, il faut et il suffit que $X$ n'ait pas de torsion: cela résulte immédiatement de l'expression (6). On notera la formule suivante, conséquence de la formule (5):

$$\beta_n(x \cdot y) = \beta_n(x) \cdot F^n(y) + (-1)^{\deg(x)} F^n(x) \cdot \beta_n(y). \tag{8}$$

En particulier, en prenant pour $y$ un élément de degré 0, on voit que $\beta_n$ est une opération $p^n$-linéaire.

Remarques. (1) Nous aurions également pu définir les $\beta_n$ comme les différentielles successives de la suite spectrale définie par la filtration $\{V^k\mathscr{W}_{N-k}\}$ de $\mathscr{W}_N$ ($N$ étant pris suffisamment grand).

(2) Il y a tout lieu de penser que l'on peut définir des puissances réduites de Steenrod dans $H^*(X, \mathcal{O})$ et que $\beta_1$ coïncide avec l'une de ces puissances. En tout cas, lorsque $p = 2$, un calcul direct montre que l'opération

$$\beta_1^1 : H^1(X, \mathcal{O}) \to H^2(X, \mathcal{O})$$

coïncide bien avec le cup-carré.

## 4. Un lemme sur les limites projectives

Nous aurons besoin au n° 5 du résultat suivant (bien connu dans le cas des espaces vectoriels):

Lemme 1. *La limite projective d'une suite exacte de modules de longueur finie est une suite exacte.*

Rappelons brièvement la démonstration. Soit $I$ un ensemble ordonné filtrant pour une relation d'ordre notée $\geq$, et soient $(A_i, f_{ij})$, $(A'_i, f'_{ij})$ et $(A''_i, f''_{ij})$ trois systèmes projectifs, indexés par $I$, formés de modules de longueur finie sur un anneau $\Lambda$; supposons donnée, pour tout $i \in I$, une suite exacte:

$$A_i \xrightarrow{g_i} A'_i \xrightarrow{h_i} A''_i,$$

avec $f'_{ij}g_i = g_j f_{ij}$, $f''_{ij}h_i = h_j f''_{ij}$ si $i \geq j$ (les applications $f_{ij}, \ldots, h_i$ étant semi-linéaires). Dans ces conditions, il nous faut démontrer que la suite:

$$\lim(A_i, f_{ij}) \xrightarrow{g} \lim(A'_i, f'_{ij}) \xrightarrow{h} \lim(A''_i, f''_{ij})$$

est une suite exacte.

Soit donc $(a'_i) \in \lim(A'_i, f'_{ij})$ un élément du noyau de $h$; cela signifie que $h_i(a'_i) = 0$ pour tout $i \in I$, et si l'on pose $B_i = g_i^{-1}(a'_i)$, les $B_i$ sont des sous-modules affines non vides des $A_i$, avec $f_{ij}(B_i) \subset B_j$. Soit $\mathfrak{S}$ l'ensemble des systèmes $\{C_i\}$ où les $C_i$ sont des sous-modules affines non vides des $B_i$, vérifiant $f_{ij}(C_i) \subset C_j$. L'ensemble $\mathfrak{S}$, ordonné par inclusion descendante, est un ensemble inductif; cela résulte immédiatement du fait que les sous-modules affines d'un module de longueur finie vérifient la condition minimale. D'après le théorème de Zorn, $\mathfrak{S}$ possède un élément minimal, soit $\{C_i\}$. Si $i_0 \in I$, les $f_{ii_0}(C_i)$, $i \geqq i_0$, sont des sous-modules affines de $C_{i_0}$, d'intersections finies non vides; en appliquant à nouveau la condition minimale aux sous-modules affines de $A_i$, on voit que l'intersection des $f_{ii_0}(C_i)$ est non vide; soit $a_{i_0}$ un élément de cette intersection. Posons maintenant $C'_i = f_{ii_0}^{-1}(a_{i_0}) \cap C_i$ si $i \geqq i_0$, et $C'_i = C_i$ sinon. On a $\{C'_i\} \in \mathfrak{S}$, comme on le voit tout de suite, d'où $C'_i = C_i$ en vertu du caractère minimal de $\{C_i\}$. En particulier, on a $C'_{i_0} = C_{i_0}$, ce qui signifie que $C_{i_0}$ est réduit à $\{a_{i_0}\}$. Ceci s'applique à tout indice $i \in I$, et montre que $C_i = \{a_i\}$; on a $f_{ij}(a_i) = a_j$, et $g_i(a_i) = a'_i$, ce qui montre bien que $\{a_i\}$ est un élément de $\lim(A_i, f_{ij})$ ayant $\{a'_i\}$ pour image, cqfd.

## 5. Cas des variétés projectives

Nous supposerons à partir de maintenant que $X$ est une *variété projective*. Les $H^q(X, \mathcal{O})$ sont alors des $k$-espaces vectoriels de dimension finie, ce qui entraîne diverses simplifications; par exemple, les $Z_n^q$ et les $B_n^q$ définis au n° 3 forment des suites stationnaires, et les homomorphismes $\beta_n^q$ sont nuls pour $n$ assez grand: nous noterons $Z_\infty^q$ (resp. $B_\infty^q$) la valeur limite de $Z_n^q$ (resp. de $B_n^q$) pour $n \to +\infty$.

Pour tout entier $q \geqq 0$, les $\Lambda$-modules $H^q(X, \mathcal{W}_n)$ et les homomorphismes $R^{n-m}: H^q(X, \mathcal{W}_n) \to H^q(X, \mathcal{W}_{n-m})$ forment un *système projectif*. La limite projective de ce système sera notée $H^q(X, \mathcal{W})$, ou simplement $H^q$; c'est l'analogue, dans le cas classique, de la cohomologie à coefficients entiers $p$-adiques; on notera toutefois que nous n'avons pas défini les $H^q$ comme des groupes de cohomologie de $X$ à valeurs dans un certain faisceau, mais simplement comme des limites projectives de tels groupes.

Les $H^q$ sont des $\Lambda$-modules, de façon évidente; de plus, ils peuvent être munis, par passage à la limite, des opérations $V$ et $F$; comme d'ordinaire, $V$ est $p^{-1}$-linéaire, $F$ est $p$-linéaire, et l'on a $VF = FV = p$. Du fait que les $H^q(X, \mathcal{W}_n)$ sont des $\Lambda$-modules de longueur finie, on peut appliquer le Lemme 1 aux suites exactes:

$$\ldots \to H^q(X, \mathcal{W}_N) \xrightarrow{V^n} H^q(X, \mathcal{W}_{N+n}) \to H^q(X, \mathcal{W}_n) \to \ldots,$$

et l'on obtient les suites exactes:

$$\ldots \to H^q \xrightarrow{V^n} H^q \to H^q(X, \mathcal{W}_n) \xrightarrow{\delta_n^q} H^{q+1} \to \ldots \tag{9}$$

Pour $n = 1$, l'image de $H^q$ dans $H^q(X, \mathcal{O})$ n'est autre que $Z_\infty^q$: cela résulte du Lemme 1. Ainsi, *pour que $X$ n'ait pas de torsion en dimension $q$, il faut et il suffit que $\delta_1^q$ soit nul*, et les autres $\delta_n^q$ sont alors automatiquement nuls.

Pour $n$ quelconque, la suite exacte (9) montre que l'image de $H^q$ dans $H^q(X, \mathcal{W}_n)$

s'identifie à $H^q/V^nH^q$; il en résulte que $H^q$ est limite projective des $H^q/V^nH^q$, ce qui signifie:

(a) *que* $\bigcap V^nH^q = 0$,

(b) *que* $H^q$ *est complet pour la topologie définie par les sous-groupes* $V^nH^q$.

Posons $T^q_n = \operatorname{Ker}(V^n : H^q \to H^q)$; d'après (9), c'est aussi l'image de l'homomorphisme $\delta^{q-1}_n$, ce qui montre que c'est un sous-module de longueur finie de $H^q$. On a évidemment $T^q_n \subset T^q_{n+1}$, et les suites exactes:

$$\begin{array}{ccccc} & & H^{q-1}(X, \mathscr{W}_n) & & \\ & & V \downarrow & & \\ H^{q-1} & \longrightarrow & H^{q-1}(X, \mathscr{W}_{n+1}) & \xrightarrow{\delta^{q-1}_{n+1}} & H^q \\ & & R^n \downarrow & & \\ & & H^{q-1}(X, \mathcal{O}) & & \end{array} \tag{10}$$

montrent que $T^q_{n+1}/T^q_n$ est isomorphe à $Z^{q-1}_{n+1}/Z^{q-1}_\infty$. Il en résulte que la suite des $T^q_n$ est stationnaire; nous désignerons par $T^q$ sa limite, et nous l'appellerons la *composante de torsion* de $H^q$; la relation $T^q = 0$ signifie, en vertu de ce qui précède, que $X$ n'a pas de torsion en dimension $q - 1$. Il est facile de calculer la longueur $l(T^q)$ du $\Lambda$-module $T^q$; on trouve:

$$l(T^q) = \sum_{n=1}^{n=\infty} l(Z^{q-1}_n/Z^{q-1}_\infty) = \sum_{n=1}^{n=\infty} n \cdot l(\operatorname{Im}(\beta^{q-1}_n)). \tag{11}$$

Remarque. Jusqu'à présent, les $\Lambda$-modules $H^q$ se comportent exactement comme les groupes de cohomologie d'un complexe fini $K$ à coefficients dans $\mathbf{Z}_p$, les $T^q$ jouant le rôle des composantes de torsion. Mais, alors qu'il est évident que les $H^q(K, \mathbf{Z}_p)$ sont des $\mathbf{Z}_p$-modules de type fini (i.e. engendrés par un nombre fini d'éléments), *il n'est nullement évident que les* $H^q$ *soient des* $\Lambda$*-modules de type fini.* En fait, c'est le cas pour $H^0$ qui est isomorphe à $\Lambda^r$ ($r$ désignant le nombre de composantes connexes de $X$), et c'est aussi le cas pour $H^1$ si $X$ est normale (cf. Proposition 4); par contre, ce n'est *pas* le cas pour le groupe $H^1$ d'une courbe de genre 0 ayant un point de rebroussement ordinaire (cf. n° 6). De façon générale, je conjecture que tous les $H^q$ d'une variété projective *non singulière* sont des $\Lambda$-modules de type fini.

Proposition 2. *Supposons que* $H^q$ *soit un* $\Lambda$*-module de type fini. Alors son module de torsion est* $T^q$ *et, si l'on pose* $L^q = H^q/T^q$, *le* $\Lambda$*-module* $L^q$ *est un* $\Lambda$*-module libre, de rang égal à* $l(L^q/VL^q) + l(L^q/FL^q)$.

Tout d'abord, on sait qu'il existe un entier $n$ tel que $T^q = T^q_n$, d'où le fait que $V^n$ est identiquement nul sur $T^q$; comme $p = FV$, on en conclut que tout élément de $T^q$ est annulé par $p^n$, ce qui montre que $T^q$ est contenu dans le sous-module de torsion $T'$ de $H^q$. Soit maintenant $V' : T'/T^q \to T'/T^q$ l'application déduite de $V$ par passage au quotient; vu la définition de $T^q$, l'application $V'$ est injective; mais, puisque $H^q$ est supposé être un module de type fini sur l'anneau principal $\Lambda$, le module $T'$ est un module de longueur finie, et l'application $V'$ est alors bijective. D'où:

$$T' = VT' + T^q,$$

et, en appliquant $V^n$,

$$V^nT' = V^{n+1}T' = \cdots$$

Puisque $\bigcap V^nH^q = 0$, on en déduit $V^nT' = 0$, d'où $T' \subset T^q$ et $T' = T^q$, ce qui démontre la première partie de la proposition.

Il est alors évident que $L^q = H^q/T^q$ est un $\Lambda$-module libre, de rang égal à la dimension du $k$-espace vectoriel $L^q/pL^q = L^q/FVL^q$. On a:

$$\dim_k(L^q/FVL^q) = l(L^q/VL^q) + l(VL^q/FVL^q) = l(L^q/VL^q) + l(L^q/FL^q),$$

puisque $V$ est un semi-isomorphisme de $L^q$ sur $VL^q$; ceci achève de démontrer la proposition.

COROLLAIRE. *Si $H^1$ est un module de type fini, c'est un module libre.*

En effet, $T^1$ est réduit à 0, puisqu'une variété n'a pas de torsion en dimension 0.

La Proposition 2 montre que, si $H^q$ est un $\Lambda$-module de type fini, $L^q/FL^q$ est un module de longueur finie, et il en est de même de $H^q/FH^q$, puisque $H^q$ ne diffère de $L^q$ que par le module de longueur finie $T^q$. Inversement:

PROPOSITION 3. *Si $H^q/FH^q$ est un module de longueur finie, alors $H^q$ est un module de type fini.*

L'hypothèse entraîne que $l(VH^q/VFH^q) < +\infty$, d'où:

$$l(H^q/pH^q) = l(H^q/VFH^q) < +\infty.$$

Il est donc possible de choisir dans $H^q$ des éléments $x_1, \cdots, x_k$ en nombre fini, dont les images dans $H^q/pH^q$ engendrent ce module; si $H'$ désigne le module engendré par les $x_i$ dans $H^q$, on a donc:

$$H^q = pH^q + H'. \tag{12}$$

Prouvons que $H' = H^q$. Montrons d'abord que $H'$ est *dense* dans $H^q$, muni de la topologie définie par les $V^nH^q$. Posons $M_n = H^q/(H' + V^nH^q)$; la relation (12) montre que $M_n = p \cdot M_n$, et, comme $M_n$ est un module de longueur finie (puisque quotient de $H^q/V^nH^q$), ceci entraîne $M_n = 0$, d'où $H^q = H' + V^nH^q$ pour tout $n$, ce qui signifie bien que $H'$ est dense dans $H^q$. Montrons maintenant que $H'$ est *complet* pour la topologie induite par celle de $H^q$, ce qui entraînera qu'il est *fermé*, donc égal à $H^q$. Posons $H'_n = H' \cap V^nH^q$; les $H'_n$ sont des sous-modules de $H'$ formant une base de voisinages de 0 pour la topologie induite sur $H'$ par $H^q$; on a $\bigcap H'_n = 0$ et les quotients $H'/H'_n$ sont de longueur finie; comme $H'$ est un module de type fini sur l'anneau local complet $\Lambda$, il en résulte que la topologie définie par les $H'_n$ est identique à la topologie $p$-adique de $H'$, définie par les sous-modules $p^kH'$ (cf. [11], p. 9, prop. 2, qui s'étend immédiatement aux modules de type fini sur un anneau semi-local complet); comme $H'$ est complet pour la topologie $p$-adique, ceci achève la démonstration.

COROLLAIRE 1. *Pour que tous les $H^q$, $q \geqq 0$, soient des modules de type fini, il faut et il suffit que les limites projectives des modules $H^q(X, \mathscr{W}_n/F\mathscr{W}_n)$ soient des modules de longueur finie.*

Soit $S^q = \lim H^q(X, \mathscr{W}_n/F\mathscr{W}_n)$. Par passage à la limite à partir des suites exactes:

$$(13) \quad \cdots \to H^q(X, \mathscr{W}_n) \xrightarrow{F} H^q(X, \mathscr{W}_n) \to H^q(X, \mathscr{W}_n/F\mathscr{W}_n) \to H^{q+1}(X, \mathscr{W}_n) \to \cdots$$

on obtient la suite exacte:

$$(14) \quad \cdots \to H^q \xrightarrow{F} H^q \to S^q \to H^{q+1} \to \cdots$$

Si les $H^q$ sont des modules de type fini, on a vu que le conoyau de $F$ est un module de longueur finie, donc aussi son noyau; la suite exacte (14) montre alors bien que $S^q$ est de longueur finie. Inversement, si $S^q$ est de longueur finie, il en est de même du conoyau de $F$, et l'on peut appliquer la Proposition 3.

(Il est facile de voir que les $H^q(X, \mathscr{W}_n/F\mathscr{W}_n)$ et les $S^q$ sont des $\Lambda$-modules annulés par $p$, autrement dit sont des *espaces vectoriels sur $k$*.)

COROLLAIRE 2. *Soit $q$ un entier $\geqq 0$; supposons que $X$ n'ait de torsion ni en dimension $q-1$ ni en dimension $q$, et que l'homomorphisme*

$$F : H^q(X, \mathcal{O}) \to H^q(X, \mathcal{O})$$

*soit surjectif. Alors $H^q$ est un $\Lambda$-module libre de rang égal à* $\dim H^q(X, \mathcal{O})$.

Puisque $X$ n'a pas de torsion en dimension $q$, on a $Z^q_\infty = H^q(X, \mathcal{O})$ et l'hypothèse faite sur $F$ signifie que $F : H^q/VH^q \to H^q/VH^q$ est surjectif. On en déduit aussitôt, par récurrence sur $n$, qu'il en est de même de $F : H^q/V^nH^q \to H^q/V^nH^q$, et, en appliquant le Lemme 1, on voit que $FH^q = H^q$. Comme $X$ n'a pas de torsion en dimension $q-1$, on a $T^q = 0$ et $H^q = L^q$. Le corollaire s'ensuit, en appliquant les Propositions 2 et 3.

(Nous laissons au lecteur le soin d'énoncer un résultat plus général, sous la seule hypothèse que $F : Z^q_\infty \to Z^q_\infty$ soit surjectif.)

## 6. Un contre-exemple

Soit $X$ une courbe de genre zéro, présentant un point de rebroussement ordinaire $P$; nous allons voir que *$H^1(X, \mathscr{W})$ n'est pas un $\Lambda$-module de type fini.*

Si $X'$ désigne la courbe déduite de $X$ par normalisation, l'application canonique $X' \to X$ est un homéomorphisme, ce qui nous permet d'identifier les espaces topologiques $X$ et $X'$. Si $\mathcal{O}$ et $\mathcal{O}'$ désignent respectivement les faisceaux des anneaux locaux de $X$ et de $X'$, on a $\mathcal{O}_x \subset \mathcal{O}'_x$ et $\mathcal{O}_x = \mathcal{O}'_x$ pour $x \neq P$; quant à $\mathcal{O}_P$, c'est le sous-anneau de $\mathcal{O}'_P$ formé des fonctions $f$ dont la différentielle $df$ s'annule en $P$ (une telle fonction s'écrit donc

$$f = a_0 + a_2t^2 + a_3t^3 + \cdots;$$

c'est la définition même d'un point de rebroussement ordinaire).

On obtient ainsi une suite exacte:

$$0 \to \mathcal{O} \to \mathcal{O}' \to \mathcal{Q} \to 0, \tag{15}$$

où $\mathcal{Q}$ est un faisceau concentré en $P$, et tel que $\mathcal{Q}_P = k$. D'où une suite exacte de cohomologie:

$$0 \to H^0(X, \mathcal{Q}) \xrightarrow{\delta} H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}'). \tag{16}$$

On a $H^0(X, \mathcal{Q}) = \mathcal{Q}_P = k$, et $H^1(X, \mathcal{O}') = H^1(X', \mathcal{O}') = 0$ (puisque $X'$ est une courbe non singulière de genre 0). Il en résulte que dim $H^1(X, \mathcal{O}) = 1$, d'où, par récurrence sur $n$, $l(H^1(X, \mathcal{W}_n)) = n$; on a d'ailleurs, pour tout entier $n$, une suite exacte analogue à (15):

$$0 \to \mathcal{W}_n \to \mathcal{W}'_n \to \mathcal{Q}_n \to 0, \tag{17}$$

et l'homomorphisme cobord $\delta : H^0(X, \mathcal{Q}_n) \to H^1(X, \mathcal{W}_n)$ est bijectif. L'opération $F : \mathcal{W}'_n \to \mathcal{W}'_n$ applique évidemment $\mathcal{W}_n$ dans lui-même, donc définit un homomorphisme de la suite exacte (17) dans elle-même; de plus, si $f \in \mathcal{O}'_P$, la fonction $Ff = f^p$ a une différentielle identiquement nulle, donc appartient à $\mathcal{O}_P$; ainsi, $F$ applique le faisceau $\mathcal{W}'_n$ dans $\mathcal{W}_n$, et le faisceau quotient $\mathcal{Q}_n$ dans 0. Si l'on considère alors le diagramme commutatif:

$$\begin{array}{ccc} H^0(X, \mathcal{Q}_n) & \xrightarrow{\delta} & H^1(X, \mathcal{W}_n) \\ F \downarrow & & F \downarrow \\ H^0(X, \mathcal{Q}_n) & \xrightarrow{\delta} & H^1(X, \mathcal{W}_n), \end{array}$$

on voit que $F : H^1(X, \mathcal{W}_n) \to H^1(X, \mathcal{W}_n)$ est identiquement nul. Il s'ensuit que $p$ annule $H^1(X, \mathcal{W}_n)$ qui est donc un *espace vectoriel sur* $k$, de dimension égale à $n$, d'après ce qui a été dit plus haut. Quant à $H^1$, limite projective des $H^1(X, \mathcal{W}_n)$, *c'est un espace vectoriel sur* $k$ *de dimension infinie* (il est topologiquement isomorphe à l'espace produit $k^{\mathbf{N}}$, $\mathbf{N}$ désignant l'ensemble des entiers $\geqq 0$); ce n'est donc pas un $\Lambda$-module de type fini.

Remarque. La suite exacte (15) s'applique plus généralement à toute courbe $X$ et à sa normalisée $X'$; la suite exacte (16) montre alors que dim $H^1(X, \mathcal{O})$ n'est pas autre chose que le "genre" $\pi$ de $X$, au sens défini par Rosenlicht dans [9]; en appliquant [14], n° 80, on voit donc que *le genre arithmétique de la courbe* (*à singularités*) $X$ *est égal à* $1 - \pi$, si $X$ est connexe.

## 7. Le premier groupe de cohomologie d'une variété projective normale

Soit tout d'abord $A$ un anneau commutatif quelconque, et soit

$$\alpha = (a_0, \cdots, a_{n-1})$$

un élément de $W_n(A)$. Nous associerons à $\alpha$ la forme différentielle de degré 1 donnée par la formule suivante:

$$D_n(\alpha) = da_{n-1} + a_{n-2}^{p-1}\, da_{n-2} + \cdots + a_0^{p^{n-1}-1}\, da_0. \tag{18}$$

Lorsque $A$ est un anneau de caractéristique 0, les composantes $a^{(0)}, a^{(1)}, \cdots, a^{(n-1)}$ de $\alpha$ sont définies (cf. [21], §1), et l'on a évidemment:

$$D_n(\alpha) = \frac{1}{p^{n-1}} da^{(n-1)}. \tag{19}$$

De la formule (19) on déduit aussitôt:

$$D_n(\alpha + \beta) = D_n(\alpha) + D_n(\beta) \tag{20}$$

et

$$D_n(\alpha \cdot \beta) \equiv D_n(\alpha) \cdot b_0^{p^{n-1}} + a_0^{p^{n-1}} \cdot D_n(\beta) \bmod p. \tag{21}$$

En vertu du principe de prolongement des identités, la formule (20) reste valable lorsque $A$ est un anneau de caractéristique $p$, alors que la formule (21) est remplacée par la suivante:

$$D_n(\alpha \cdot \beta) = D_n(\alpha) \cdot F^{n-1}R^{n-1}\beta + F^{n-1}R^{n-1}\alpha \cdot D_n(\beta). \tag{22}$$

Ceci s'applique notamment à l'anneau local $A = \mathcal{O}_x$ d'un point $x$ sur une variété normale $X$, et l'on obtient ainsi un homomorphisme

$$D_n : W_n(\mathcal{O}_x) \to \Omega_x^1,$$

en désignant par $\Omega_x^1$ le $\mathcal{O}_x$-module des germes de formes différentielles de degré 1 sur $X$ qui n'ont pas de pôle en $x$ (i.e. dont le diviseur polaire ne passe pas par $x$).

Si l'on a $\alpha \in FW_n(\mathcal{O}_x)$, c'est-à-dire si les $a_0, \cdots, a_{n-1}$ sont des puissances $p$-èmes, on a évidemment $D_n(\alpha) = 0$; inversement, il est classique que la relation $D_1(a) = da = 0$ entraîne que $a$ est une puissance $p$-ème dans le corps $k(X)$ des fonctions rationnelles sur $X$; plus généralement, il n'est pas difficile de montrer (par exemple en utilisant l'opération $C$ de Cartier, cf. n° 10) que la relation $D_n(\alpha) = 0$ entraîne que chacun des $a_i$ est une puissance $p$-ème $b_i^p$, avec $b_i \in k(X)$; mais la relation $b_i^p = a_i$ montre que $b_i$ est *entier* sur $\mathcal{O}_x$, donc appartient à $\mathcal{O}_x$, vu l'hypothèse de normalité faite sur $X$. Ainsi, le noyau de $D_n$ est exactement $FW_n(\mathcal{O}_x)$ et, en passant aux faisceaux, on obtient:

LEMME 2. *L'application $D_n$ définit par passage au quotient une injection du faisceau $\mathscr{W}_n/F\mathscr{W}_n$ dans le faisceau $\Omega^1$ des germes de formes différentielles dépourvues de pôles.*

Supposons maintenant que $X$ soit une variété *projective* et *normale*. D'après le Lemme 2, $H^0(X, \mathscr{W}_n/F\mathscr{W}_n)$ est un sous-espace vectoriel de $H^0(X, \Omega^1)$, qui est un espace vectoriel de dimension finie ($\Omega^1$ étant un faisceau algébrique cohérent); on en déduit que $\dim H^0(X, \mathscr{W}_n/F\mathscr{W}_n)$ est *bornée* pour $n \to +\infty$; soit $\nu$ cette borne, et posons:

$$g = \dim Z_\infty^1 = \dim [\mathrm{Im} : H^1 \to H^1(X, \mathcal{O})].$$

PROPOSITION 4. *Les hypothèses et notations étant comme ci-dessus, le $\Lambda$-module $H^1 = H^1(X, \mathscr{W})$ est un module libre de rang $\leq g + \nu$, l'égalité ayant lieu si $X$ n'a pas de torsion en dimension* 1.

Les $H^0(X, \mathscr{W}_n/F\mathscr{W}_n)$ forment une suite croissante de sous-espaces de $H^0(X, \Omega^1)$, et il existe donc un entier $m$ tel que l'on ait dim $H^0(X, \mathscr{W}_n/F\mathscr{W}_n) = \nu$ pour $n \geqq m$. De la suite exacte de faisceaux:

$$0 \to \mathscr{W}_n \xrightarrow{F} \mathscr{W}_n \to \mathscr{W}_n/F\mathscr{W}_n \to 0,$$

on déduit la suite exacte suivante (qui n'est qu'un cas particulier de (13)):

$$0 \to H^0(X, \mathscr{W}_n/F\mathscr{W}_n) \to H^1(X, \mathscr{W}_n) \xrightarrow{F} H^1(X, \mathscr{W}_n).$$

Comme $H^1(X, \mathscr{W}_n)$ est un $\Lambda$-module de longueur finie, on tire de là:

$$l(H^1(X, \mathscr{W}_n)/FH^1(X, \mathscr{W}_n)) = \nu \qquad \text{pour } n \geqq m. \tag{23}$$

Puisque $H^1/FH^1$ est limite projective des $H^1(X, \mathscr{W}_n)/FH^1(X, \mathscr{W}_n)$, on a aussi $l(H^1/FH^1) \leqq \nu$, ce qui, d'après la Proposition 3, entraîne que $H^1$ est un $\Lambda$-module de type fini. De plus, on sait que $T^1 = 0$, d'où $L^1 = H^1$, avec les notations du n$^o$ 5, et $H^1/VH^1 = Z^1_\infty$; en appliquant la Proposition 2, on en déduit que $H^1$ est un $\Lambda$-module libre de rang égal à dim $Z^1_\infty + l(H^1/FH^1) \leqq g + \nu$, ce qui démontre la première partie de la proposition.

Supposons maintenant $X$ sans torsion en dimension 1. Les homomorphismes

$$R : H^1(X, \mathscr{W}_{n+1}) \to H^1(X, \mathscr{W}_n)$$

sont surjectifs, donc aussi les homomorphismes obtenus par passage au quotient

$$R : H^1(X, \mathscr{W}_{n+1})/FH^1(X, \mathscr{W}_{n+1}) \to H^1(X, \mathscr{W}_n)/FH^1(X, \mathscr{W}_n).$$

Mais, si $n \geqq m$, ces deux modules ont même longueur $\nu$, et il s'ensuit que $R$ est bijectif; en passant à la limite, il en est donc de même de l'homomorphisme $H^1/FH^1 \to H^1(X, \mathscr{W}_n)/FH^1(X, \mathscr{W}_n)$, et l'on a $l(H^1/FH^1) = \nu$; en appliquant à nouveau la Proposition 2 on en conclut bien que le rang de $H^1$ est égal à $g + \nu$, cqfd.

Remarques. (1) Même lorsque $X$ a de la torsion en dimension 1, on peut calculer le rang de $H^1$. On trouve: $\mathrm{rg}(H^1) = g + \nu - l(T^2/FT^2)$.

(2) La Proposition 4 est encore valable si l'on ne suppose plus que $X$ est normale mais seulement que les relations $a \in k(X)$, $a^p \in \mathcal{O}_x$ entraînent $a \in \mathcal{O}_x$; cela suffit en effet à assurer que $\mathscr{W}_n/F\mathscr{W}_n$ est un sous-faisceau de $\Omega^1$.

## §2. Cas des courbes algébriques

Dans tout ce §4, $X$ désignera une *courbe algébrique irréductible, complète* (donc projective), *sans singularités*, définie sur le corps algébriquement clos $k$, de caractéristique $p > 0$.

### 8. Rappel

Montrons d'abord comment les groupes de cohomologie $H^1(X, \mathcal{O})$ et $H^1(X, \Omega^1)$ s'interprètent en termes classiques (cf. [1]):

Soit $K = k(X)$ le corps des fonctions rationnelles sur $X$; nous considérerons

$K$ comme un faisceau constant sur $X$ (cf. [14], n° 36), contenant $\mathcal{O}$ comme sous-faisceau. On a donc la suite exacte:

$$0 \to \mathcal{O} \to K \to K/\mathcal{O} \to 0. \tag{24}$$

Puisque $K$ est un faisceau constant, et que $X$ est irréductible, on a $H^1(X, K) = 0$; la suite exacte de cohomologie associée à (24) donne donc naissance à la suite exacte:

$$K \to H^0(X, K/\mathcal{O}) \to H^1(X, \mathcal{O}) \to 0. \tag{25}$$

Cette dernière suite exacte est facile à interpréter. Soit $R$ l'algèbre des *répartitions* sur $X$ (cf. [1], p. 25); rappelons qu'un élément $r \in R$ est une famille $\{r_x\}_{x \in X}$ où les $r_x$ sont des éléments de $K$ appartenant à $\mathcal{O}_x$ pour presque tout $x$ (i.e. sauf pour un nombre fini). Les répartitions $r = \{r_x\}$ telles que $r_x \in \mathcal{O}_x$ pour tout $x$ forment un sous-anneau $R(0)$ de $R$; celles qui sont telles que tous les $r_x$ soient égaux à un même élément de $K$ forment un sous-anneau de $R$ que l'on peut identifier à $K$. On voit tout de suite que $R/R(0)$ est canoniquement isomorphe à $H^0(X, K/\mathcal{O})$, et la suite exacte (25) donne donc en définitive un isomorphisme:

$$R/(R(0) + K) \approx H^1(X, \mathcal{O}). \tag{26}$$

Nous identifierons en général $H^1(X, \mathcal{O})$ et $R/(R(0) + K)$ au moyen de l'isomorphisme précédent. On sait ([1], chaps. II et VI) que l'espace vectoriel $R/(R(0) + K)$ est dual de l'espace $H^0(X, \Omega^1)$ des formes différentielles de lère espèce, la dualité se faisant au moyen de la forme bilinéaire:

$$\langle r, \omega \rangle = \textstyle\sum_{x \in X} \operatorname{res}_x (r_x \omega). \tag{27}$$

En particulier, on a dim $H^1(X, \mathcal{O}) = g$, *genre* de la courbe $X$.

La forme bilinéaire (27) peut aussi être considérée comme le cup-produit de $r \in H^1(X, \mathcal{O})$ et de $\omega \in H^0(X, \Omega^1)$, à valeurs dans $H^1(X, \Omega^1)$ qui est canoniquement isomorphe à $k$ (ce dernier isomorphisme s'obtient de la façon suivante: à une classe de cohomologie on associe, comme dans (26), une classe de "répartition-différentielles" $\{\omega_x\}_{x \in X}$ et, à une telle répartition, on fait correspondre l'élément $\sum_{x \in X} \operatorname{res} (\omega_x)$ qui appartient à $k$). C'est là un cas particulier du "théorème de dualité", dont on trouvera l'énoncé général dans [13], th. 4.

Remarque. Une formule analogue à (26) vaut pour $H^1(X, \mathcal{W}_n)$, ainsi que pour $H^1(X, \mathcal{L}(D))$, $D$ désignant un diviseur de $X$.

## 9. La matrice de Hasse-Witt

Nous allons chercher la matrice de l'opération semi-linéaire

$$F : H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O})$$

par rapport à une base convenable de $H^1(X, \mathcal{O})$.

Remarquons d'abord que l'identification (26) transforme $F$ en l'élévation à la puissance $p$-ème dans $R$. D'autre part, en utilisant la dualité entre $R/(R(0) + K)$

et $H^0(X, \Omega^1)$, on voit qu'il existe $g$ points $P_1, \cdots, P_g$ appartenant à $X$ tels que, si $t_1, \cdots, t_g$ sont des paramètres uniformisants en ces points, les répartitions:

$$r_i = \{r_{i,x}\} \quad \text{où} \quad r_{i,x} = \begin{cases} 0 & \text{si } x \neq P_i \\ 1/t_i & \text{si } x = P_i \end{cases}, \qquad 1 \leqq i \leqq g,$$

forment une *base* du $k$-espace vectoriel $R/(R(0) + K)$. (Un tel système de $g$ points est parfois appelé "non-spécial", cf. [1], p. 129.)

Soit $A = (a_{ij})$ la matrice de $F$ par rapport à la base des $r_i$. Par définition, on a donc:

$$r_i^p \equiv \textstyle\sum_{j=1}^{j=g} a_{ij} r_j \bmod (R(0) + K),\ 1 \leqq i \leqq g.$$

Ces congruences signifient qu'il existe des fonctions $g_i \in K$ telles que:

$$g_i \equiv r_i^p - \textstyle\sum_{j=1}^{j=g} a_{ij} r_j \bmod R(0).$$

En d'autres termes, chaque $g_i$ est régulière en dehors des points $P_1, \cdots, P_g$ et admet $\delta_{ij}/t_j^p - a_{ij}/t_j$ pour partie polaire au point $P_j$ ($\delta_{ij}$ désignant comme à l'ordinaire le symbole de Kronecker). On reconnait là la définition de la *matrice de Hasse-Witt* de $X$ (cf. [5]). Nous avons donc démontré:

PROPOSITION 5. *La matrice de* $F : H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O})$ *par rapport à la base des* $r_i$ ($1 \leqq i \leqq g$) *n'est autre que la matrice de Hasse-Witt de* $X$.

Nous aurons besoin par la suite d'utiliser la *réduction de Jordan* de $F$ (cf. [5] ainsi que [3], n° 10). Rappelons brièvement en quoi elle consiste:

De façon générale, soit $F$ un endomorphisme $p$-linéaire d'un espace vectoriel $V$, de dimension finie, sur un corps algébriquement clos $k$ de caractéristique $p$. L'espace $V$ se décompose canoniquement en somme directe

$$V = V_s \oplus V_n, \tag{28}$$

où $V_s$ et $V_n$ sont stables par $F$, l'endomorphisme $F$ étant *nilpotent* sur $V_n$ et *bijectif* sur $V_s$; les dimensions de $V_s$ et $V_n$ seront notées respectivement $\sigma(V)$ et $\nu(V)$. On montre en outre que $V_s$ possède une base $e_1, \cdots, e_\sigma$ telle que $F(e_i) = e_i$ pour tout $i$; les $v \in V$ tels que $F(v) = v$ sont les combinaisons linéaires à coefficients entiers mod $p$ des $e_i$, et forment donc un groupe fini $V^F$ d'ordre $p^\sigma$ et de type $(p, \cdots, p)$; l'existence de la base $e_i$ fournit également le résultat suivant, qui nous sera utile plus loin: l'application $1 - F : V \rightarrow V$ est *surjective*.

Soit $V'$ l'espace vectoriel dual de $V$. Le *transposé* $F'$ de $F$ est un endomorphisme $p^{-1}$-linéaire de $V'$ défini par la formule:

$$\langle Fv, v' \rangle = \langle v, F'v' \rangle^p \quad \text{pour} \quad v \in V \quad \text{et} \quad v' \in V'. \tag{29}$$

A la décomposition (28) correspond la décomposition duale:

$$V' = V'_s \oplus V'_n. \tag{30}$$

Si $e'_i$ désigne la base de $V'_s$ duale de $e_i$, on a encore $F'e'_i = e'_i$ pour $1 \leqq i \leqq \sigma$, et les $v' \in V'$ tels que $F'v' = v'$ sont les combinaisons linéaires à coefficients entiers des

$e'_i$; ces $v'$ forment donc un groupe *dual* du groupe $V^F$. (On observera que la décomposition (30) vaut pour *tout* endomorphisme $p^{-1}$-linéaire d'un $k$-espace vectoriel de dimension finie, puisqu'un tel endomorphisme peut toujours être considéré comme le transposé d'un endomorphisme $p$-linéaire.)

Ce qui précède s'applique notamment au cas où $V = H^1(X, \mathcal{O})$ et $V' = H^0(X, \Omega^1)$. On écrira alors simplement $\sigma$ et $\nu$ à la place de $\sigma(V)$ et de $\nu(V)$; on a $g = \sigma + \nu$. Avec les notations de [5], l'entier $\sigma$ n'est pas autre chose que le *rang* de la matrice $AA^p \cdots A^{p^{g-1}}$.

Le résultat suivant, dû à P. Cartier (non publié), sera démontré au n° 10:

PROPOSITION 6. *Pour tout entier $m \geqq 1$, l'image de l'homomorphisme*

$$D_m : H^0(X, \mathscr{W}_m/F\mathscr{W}_m) \to H^0(X, \Omega^1) \quad (\text{cf. n}^\circ\ 7)$$

*est égale au noyau de la $m$-ème itérée $F'^m$ de $F'$.*

(Pour $m = 1$, ce résultat est facile à démontrer directement, et était d'ailleurs déjà connu, cf. [12], n° 6).

Il résulte de la prop. 6 que, pour $m$ assez grand, l'image de $D_m$ est égale à la "composante nilpotente" $H^0(X, \Omega^1)_n$ de $H^0(X, \Omega^1)$ et a donc pour dimension $\nu$. Ainsi, l'entier $\nu$ défini ci-dessus coïncide avec celui défini au n° 7 comme Sup. dim $H^0(X, \mathscr{W}_m/F\mathscr{W}_m)$; en appliquant la Proposition 4, et tenant compte du fait qu'une courbe n'a pas de torsion, on obtient finalement:

PROPOSITION 7. *Le $\Lambda$-module $H^1(X, \mathscr{W})$ est un module libre de rang égal à $g + \nu = 2g - \sigma$.*

En particulier, ce rang *ne dépend que de la matrice de Hasse-Witt* de la courbe $X$, ce qui n'était nullement évident *a priori*.

## 10. Une nouvelle opération sur les formes différentielles

Pour démontrer la Proposition 6, nous aurons besoin d'une opération sur les formes différentielles qui a été définie par P. Cartier dans le cas des variétés de dimension quelconque. Dans le cas particulier des courbes, auquel nous nous limiterons, cette opération avait déjà été envisagée par J. Tate [17].

Soit $x$ un point de $X$, et soit $t$ un élément de $\mathcal{O}_x$ dont la différentielle $dt$ ne s'annule pas en $x$. On vérifie alors immédiatement que les $p$ fonctions $1, t, \cdots, t^{p-1}$ forment une *base* de $\mathcal{O}_x$ considéré comme module sur $\mathcal{O}_x^p$; en d'autres termes, toute fonction $f \in \mathcal{O}_x$ s'écrit d'une manière et d'une seule sous la forme:

$$f = f_0^p + f_1^p t + \cdots + f_{p-1}^p t^{p-1}, \qquad \text{avec } f_i \in \mathcal{O}_x. \tag{31}$$

Les $f_i^p$ sont des combinaisons linéaires des dérivées successives

$$d^k f/dt^k, \qquad 0 \leqq k \leqq p-1;$$

en particulier, on a $f_{p-1}^p = -d^{p-1}f/dt^{p-1}$.

Soit $\omega = f\,dt$ un élément de $\Omega_x^1$, et posons:

$$C(\omega) = f_{p-1}dt; \tag{32}$$

l'opération $C : \Omega^1_x \to \Omega^1_x$ ainsi définie est *l'opération de Cartier et Tate*. On montre (cf. [17], th. 1) qu'*elle ne dépend pas* de l'élément $t$ choisi; de plus, en prenant $f$ dans $K$ et non plus dans $\mathcal{O}_x$, on prolonge $C$ en une opération définie sur *toutes* les différentielles (régulières ou non) de $X$.

Les deux propositions suivantes sont dues à Cartier:

Proposition 8. (i) $C(\omega_1 + \omega_2) = C(\omega_1) + C(\omega_2)$.

(ii) $C(f^p\omega) = fC(\omega)$.

(iii) $C(df) = 0$.

(iv) $C(f^{p-1}df) = df$.

(v) *La suite* $0 \to \mathscr{W}_m/F\mathscr{W}_m \xrightarrow{D_m} \Omega_1 \xrightarrow{C^m} \Omega_1 \to 0$ *est une suite exacte* $(m \geqq 1)$.

Les formules (i), (ii) et (iii) résultent immédiatement de (31) et (32); pour la formule (iv), voir [17], Lemme 1. Il est clair que $C$ est surjectif, et (v) se réduit donc à montrer que $\mathrm{Ker}\,(C^m) = \mathrm{Im}\,(D_m)$. Pour $m = 1$, cela signifie que $C(\omega) = 0 \Rightarrow \omega = df$, ce qui est immédiat sur les formules (31) et (32); à partir de là, on va raisonner par récurrence sur $m$, en utilisant la formule (déduite des formules (i) à (iv)) :

(vi) $$CD_m\alpha = D_{m-1}R\alpha \quad \text{pour} \quad \alpha \in \mathscr{W}_m.$$

Il est clair que (vi) entraîne que $\mathrm{Im}\,(D_m) \subset \mathrm{Ker}\,(C^m)$; inversement, soit $\omega \in \Omega^1_x$ tel que $C^m(\omega) = 0$; vu l'hypothèse de récurrence, il existe $\beta \in W_{m-1}(\mathcal{O}_x)$ tel que $D_{m-1}\beta = C(\omega)$; si l'on choisit un $\alpha \in W_m(\mathcal{O}_x)$ tel que $R\alpha = \beta$, on aura, d'après (vi), $C(\omega - D_m\alpha) = 0$, d'où, d'après ce qu'on a vu plus haut, $\omega - D_m\alpha = df$; en posant alors $\alpha' = \alpha + V^{m-1}f$, on aura bien $\omega = D_m\alpha'$, cqfd.

Proposition 9. *L'homomorphisme* $C : H^0(X, \Omega^1) \to H^0(X, \Omega^1)$ *coïncide avec la transposée* $F'$ *de l'opération* $F$.

Il nous faut montrer que, si $\omega$ est une forme différentielle, et $r$ une répartition, on a:

$$\langle r^p, \omega\rangle = \langle r, C\omega\rangle^p.$$

Ceci s'écrit, en vertu de (27):

$$\textstyle\sum_{x\in X} \mathrm{res}_x\,(r^p_x\omega) = \sum_{x\in X} \mathrm{res}_x\,(r_xC\omega)^p,$$

ce qui résulte de la formule suivante, facile à vérifier:

(33) $\mathrm{res}_x\,(\pi)^p = \mathrm{res}_x\,(C\pi)$, $\pi$ étant une forme différentielle quelconque.

La Proposition 6 est maintenant une conséquence évidente de la Proposition 8, (v) et de la Proposition 9.

Remarque. Comme l'a montré Cartier, l'opération $C$ peut être définie sur les formes différentielles *fermées* d'une variété algébrique de dimension quelconque; pour les formes de degré 1, les formules (i) à (iv) de la Proposition 8 subsistent sans changement alors que (v) doit être formulée de façon légèrement différente (il faut tenir compte du fait que $C$ et ses itérées ne sont pas partout définies).

## 11. Classes de diviseurs d'ordre $p$

Soit $G$ le groupe des classes de diviseurs de $X$, au sens de l'équivalence linéaire; soit $G_p$ le sous-groupe des éléments $d \in G$ tels que $pd = 0$.

PROPOSITION 10. *Le groupe $G_p$ est canoniquement isomorphe au groupe additif des différentielles $\omega \in H^0(X, \Omega^1)$ qui vérifient $C(\omega) = \omega$. En particulier, c'est un groupe fini d'ordre $p^\sigma$.*

(Pour la définition de l'entier $\sigma$, voir n° 9.)

Nous allons tout d'abord définir une application $\theta : G_p \to H^0(X, \Omega^1)$.

Soit $d \in G_p$, et soit $D$ un diviseur appartenant à la classe $d$; puisque $pd = 0$, il existe une fonction $f \neq 0$ telle que $pD = (f)$; posons $\omega = df/f$, différentielle "logarithmique" de $f$. Si l'on change $D$ en un diviseur équivalent $D + (g)$, ceci a pour effet de multiplier $f$ par $g^p$, ce qui ne change pas $df/f$; donc $\omega$ ne dépend que de $d$, et peut être notée $\theta(d)$. Enfin, si $x \in X$, l'équation $pD = (f)$ montre que l'on peut écrire $f = t^p u$, où $u$ est une unité de $\mathcal{O}_x$, d'où $df/f = du/u$ ce qui montre que $df/f$ n'a pas de pôle en $x$; ainsi $\theta(d)$ est bien une différentielle de 1ère espèce.

On vérifie tout de suite que l'application $\theta$ est un homomorphisme injectif de $G_p$ dans $H^0(X, \Omega^1)$. On a de plus $\theta(d) = df/f$, et les formules (ii) et (iv) de la Proposition 8 montrent que:

$$C(df/f) = C(f^{p-1}\, df/f^p) = C(f^{p-1}\, df)/f = df/f.$$

Inversement, si une forme différentielle $\omega$ vérifie l'équation $C(\omega) = \omega$, elle est de la forme $df/f$ d'après un théorème de Jacobson ([7], th. 15); si de plus $\omega$ est une forme de première espèce, l'ordre de la fonction $f$ en un point quelconque de $X$ est divisible par $p$, ce qui signifie que $(f) = pD$, d'où $\omega = \theta(d)$, en désignant par $d$ la classe du diviseur $D$. Ainsi $\theta$ est bien un isomorphisme de $G_p$ sur l'ensemble des points fixes de $C$ (ou de $F'$, cela revient au même d'après la Proposition 9), cqfd.

REMARQUES. (1) La Proposition 10 était connue ([12], Satz II) dans le cas particulier où le corps de base $k$ est la clôture algébrique de $F_p$, cette hypothèse permettant d'utiliser la théorie du corps de classes.

(2) La Proposition 10 a été étendue aux variétés normales de dimension quelconque par Cartier (le seul point non évident étant de montrer que l'équation $C(\omega) = \omega$ caractérise encore les différentielles logarithmiques).

(3) On peut donner de la Proposition 10 une démonstration toute différente, basée sur la théorie de la jacobienne (cf. n° 19).

### 12. Exemple: courbes elliptiques

On a alors $\dim H^1(X, \mathcal{O}) = \dim H^0(X, \Omega^1) = g = 1$, et la matrice de Hasse-Witt de $X$ se réduit à un scalaire $A$, l'*invariant de Hasse* de la courbe (cf. [4]); il n'est déterminé de façon unique qu'une fois choisi un élément de base dans $H^1(X, \mathcal{O})$ ou $H^0(X, \Omega^1)$. Si, en caractéristique $p \neq 2$, on suppose $X$ donnée sous la forme de Legendre:

$$y^2 = x(x-1)(x-\lambda),$$

on peut prendre pour élément de base de $H^0(X, \Omega^1)$ la forme différentielle $dx/y$, et l'invariant $A$ est une fonction $P(\lambda)$ de $\lambda$. M. Deuring [2] a montré que $P(\lambda)$ est un polynôme de degré $(p-1)/2$ en $\lambda$ qui n'est identiquement nul pour aucune valeur de $p$; il n'y a donc qu'un *nombre fini* de courbes elliptiques telles que $A = 0$, pour une caractéristique donnée.

Résumons les propriétés de $X$ suivant que $A$ est nul ou non:

(i) $A \neq 0$ (cas "général"). On a $\nu = 0$, $\sigma = 1$. Le groupe des éléments de $X$ d'ordre $p$ a $p$ éléments; il existe $\omega \in H^0(X, \Omega^1)$, $\omega \neq 0$, avec $\omega = df/f$, $f \in k(X)$. Le $\Lambda$-module $H^1(X, \mathscr{W})$ est un module libre de rang 1.

(ii) $A = 0$ (cas "exceptionnel"). On a $\nu = 1$, $\sigma = 0$. Le groupe des éléments de $X$ d'ordre $p$ a un seul élément; toute forme $\omega \in H^0(X, \Omega^1)$ s'écrit $\omega = df$, avec $f \in k(X)$. Le $\Lambda$-module $H^1(X, \mathscr{W})$ est un module libre de rang 2.

Signalons également que, d'après Deuring [2] (resp. Dieudonné [3]), la condition $A \neq 0$ est nécessaire et suffisante pour que l'anneau des endomorphismes de $X$ soit commutatif (resp. pour que le groupe algébrique $X$ soit "analytiquement isomorphe" au groupe multiplicatif $G_m$).

## §3. Revêtements cycliques d'ordre $p^n$ d'une variété algèbrique

Les n$^{os}$ 13, 14, 15 ci-dessous sont consacrés à diverses propriétés élémentaires des revêtements; dans ces n$^{os}$, la caractéristique du corps de base $k$ est quelconque.

### 13. Quotient d'une variété algébrique par un groupe fini d'automorphismes

Soit $Y$ une variété algébrique, sur laquelle opère (à droite) un groupe fini $G$; dans tout ce qui suit, nous supposerons vérifiée la condition:

(A) *Toute orbite de $G$ est contenue dans un ouvert affine de $Y$.*

Puisqu'une orbite est un ensemble *fini*, la condition précédente est vérifiée si $Y$ est une sous-variété localement fermée d'un espace projectif: on le voit en appliquant les Lemmes 1 et 2 de [14], n° 52.

Soit $X$ l'ensemble quotient $Y/G$, que nous munirons de la topologie quotient de la topologie de Zariski de $Y$; nous noterons $\pi$ la projection canonique: $Y \to X$. Si $f$ est une fonction définie au voisinage d'un point $x \in \mathrm{M}$, nous dirons que $f$ est *régulière en $x$* si $f \circ \pi$ est régulière au voisinage de $\pi^{-1}(x)$; on définit ainsi un sous-faisceau $\mathcal{O}_X$ du faisceau $\mathscr{F}(X)$ des germes de fonctions sur $X$.

Lemme 3. *La topologie et le faisceau précédent définissent sur $X$ une structure de variété algébrique.*

Supposons d'abord que $Y$ soit une variété affine, d'anneau de coordonnées $A$, et soit $A^G$ l'ensemble des éléments de $A$ laissés fixes par $G$. On vérifie tout de suite que $A^G$ est une $k$-algèbre de type fini, sans éléments nilpotents, donc est l'anneau de coordonnées d'une variété affine $Z$; on montre ensuite, par des raisonnements élémentaires, que $Z$, munie de sa topologie de Zariski et de son faisceau d'anneaux locaux, est isomorphe à $Y/G$, muni de la topologie et du faisceau définis ci-dessus; ceci démontre le Lemme 3 lorsque $Y$ est affine.

Dans le cas général, l'hypothèse (A) montre que l'on peut recouvrir $Y$ au moyen d'un nombre fini d'ouverts affines $V_i$, stables par $G$. D'après ce qui précède, $X$ est donc recouvert par les ouverts affines $U_i = V_i/G$, ce qui montre que $X$ vérifie l'axiome ($VA_{\text{I}}$) de [14], n° 34. Quant à ($VA_{\text{II}}$), il résulte de ce que $X \times X$ est isomorphe à $(Y \times Y)/(G \times G)$.

Nous ne poursuivrons pas l'étude de $Y/G$ dans le cas général. Signalons seulement que $Y/G$ est une variété *affine* (resp. *complète*) si et seulement si $Y$ a la même propriété (pour les variétés affines, cela résulte de la démonstration du Lemme 3 et du Théorème 1 de [15]—pour les variétés complètes, cela résulte directement de la définition donnée dans [15], §4).

NOTE. Dans la littérature, on trouvera surtout discuté le cas particulier (qui est le plus important pour les applications) où $Y$ est une variété *irréductible* et *normale*; il en est alors de même de $X$ qui peut être identifiée à la normalisée de la variété des "points de Chow" des orbites de $G$; inversement, $Y$ est la normalisée de $X$ dans l'extension des corps de fonctions rationnelles $k(Y)/k(M)$. Pour une discussion de ce point de vue, cf. [8], §1.

## 14. Revêtements

Les notations étant celles du n° précédent, nous dirons que $Y$ est un *G-revêtement de X* (ou encore un revêtement de groupe de Galois $G$), si le groupe $G$ opère *sans points fixes* sur $Y$, i.e. si:

$$y \cdot g = y, \quad y \in Y, g \in G \text{ entraînent } g = e.$$

Bien entendu, si $X'$ est isomorphe à $X$, on dira encore que $Y$ est un revêtement de $X'$.

L'ensemble des classes de $G$-revêtements de $X$ sera noté $\pi^1(X, G)$. Comme dans le cas topologique, c'est un foncteur covariant en $G$ et contravariant en $X$:

(a) Si $Y$ est un $G$-revêtement de $X$, et si $f : X' \to X$ est une application régulière, on a un revêtement induit $Y'$ de $X'$ ($Y'$ est l'image réciproque de $\Delta$ par

$$f \times \pi : X' \times Y \to X \times X).$$

D'où une application $f^1 : \pi^1(X, G) \to \pi^1(X', G)$.

(b) Si $f$ est un homomorphisme de $G$ dans un groupe fini $G'$, on fait opérer $G$ sur $Y \times G'$ par la formule usuelle:

$$(y, g') \cdot g = (y \cdot g, f(g^{-1}) \cdot g'); \tag{34}$$

en posant $Y \times_G G' = (Y \times G')/G$, on vérifie (en se ramenant au cas des variétés affines, comme dans la démonstration du Lemme 3) que $Y \times_G G'$ est un $G'$-revêtement de $X$. D'où une application $f_1 : \pi^1(X, G) \to \pi^1(X, G')$.

Lorsque $G$ est *abélien*, on peut appliquer (b) à l'homomorphisme canonique $G \times G \to G$, d'où une application de $\pi^1(X, G \times G)$ dans $\pi^1(X, G)$. En utilisant la formule (facile à vérifier):

$$\pi^1(X, G \times H) = \pi^1(X, G) \times \pi^1(X, H), \tag{35}$$

on voit que l'on a défini une loi de composition sur $\pi^1(X, G)$; des raisonnements classiques montrent que cette loi de composition fait de $\pi^1(X, G)$ un *groupe abélien*.

Remarque. Supposons que $k = C$ et que $X$ soit une variété projective connexe. En utilisant les résultats de [16], on peut montrer que les revêtements de $X$ (au sens ci-dessus) sont en correspondance bijective avec les revêtements topologiques de l'espace $X^h$ que l'on obtient en munissant $X$ de la topologie "usuelle" (cf. [16], n° 5). Si $G$ est un groupe fini, les éléments de $\pi^1(X, G)$ correspondent donc aux classes d'homomorphismes de $\pi_1(X^h)$ dans $G$, modulo l'équivalence définie par les automorphismes intérieurs de $G$; si $G$ est abélien, on a ainsi:

$$\pi^1(X, G) = \text{Hom}\,(\pi_1(X^h), G),$$

ce qui justifie dans ce cas la notation $\pi^1(X,G)$.

## 15. Espaces fibrés associés à un revêtement

Soit $Y$ un $G$-revêtement de $X$, et supposons d'abord que $k = C$. On peut considérer $Y$ comme un espace fibré analytique principal, de base $X$, et de groupe structural le groupe discret $G$; si $f$ est un homomorphisme de $G$ dans un groupe algébrique $H$, on déduit de $Y$, par extension du groupe structural, un espace fibré analytique principal $Y \times_G H$, de groupe structural $H$; cet espace fibré peut être plus simple à étudier que le revêtement $Y$. C'est la méthode introduite par Weil ([18], Chap. III) lorsque $X$ est une courbe, $H$ étant un groupe linéaire $GL_n(C)$.

Essayons d'imiter cette construction dans le cas général. Il est toujours possible de définir $Y \times_G H$ comme la variété quotient $(Y \times H)/G$, le groupe $G$ opérant par la formule (34). Le groupe $H$ opère à droite sur $Y \times_G H$, et l'ensemble quotient $(Y \times_G H)/H$ s'identifie à $X$. Mais $Y \times_G H$ n'est pas toujours un *espace fibré algébrique* (au sens de Weil [20], c'est-à-dire localement trivial): le lemme suivant fournit un critère pour que ce soit le cas:

Lemme 4. *Supposons que, pour tout $x \in X$, il existe un voisinage saturé $U$ de $\pi^{-1}(x)$, et une application régulière $\theta : U \to H$ telle que:*

$$\theta(y \cdot g) = \theta(y) \cdot f(g) \quad \textit{pour} \quad y \in U \quad \textit{et} \quad g \in G. \tag{36}$$

*Alors $Y \times_G H$ est un espace fibré algébrique principal, de base $X$, et de groupe structural $H$.*

La question étant locale, on peut supposer que $U = Y$. Soit alors

$$\alpha : Y \times H \to Y \times H$$

l'application définie par la formule:

$$\alpha(y, h) = (y, \theta(y) \cdot h). \tag{37}$$

Il est clair que $\alpha$ est birégulière. De plus, en combinant (36) et (37), on voit que $\alpha$ commute aux opérations de $G$ (en faisant opérer $G$ sur le second $Y \times H$ par les opérations de $G$ sur $Y$ seulement). Par passage au quotient, $\alpha$ définit donc une application birégulière $\bar{\alpha} : Y \times_G H \to X \times H$, commutant avec les opérations de H. Ceci montre bien que $Y \times_G H$ est un espace fibré algébrique, cqfd.

PROPOSITION 11. *L'hypothèse du Lemme 4 est vérifiée lorsque $H$ est un sous-groupe algébrique du groupe linéaire $GL_n(k)$ vérifiant la condition*:

$$\text{(R)} - \textit{Il existe une section rationnelle } GL_n(k)/H \to GL_n(k).$$

(Cf. [16], n° 20, pour une discussion de la condition (R)).

Soit $x \in X$, et soient $y_1, \cdots, y_r$ les éléments de $\pi^{-1}(x)$; d'après la condition (A) du n° 13 on peut trouver des fonctions régulières au voisinage de $\pi^{-1}(x)$ et prenant aux $y_i$ des valeurs données. Si l'on désigne par $M_n(k)$ l'algèbre des matrices carrées d'ordre $n$ sur $k$, il existe donc un voisinage ouvert saturé $V'$ de $\pi^{-1}(x)$, et une application régulière $a : V' \to M_n(k)$ telle que $a(y_1)$ (resp. $a(y_i)$, $2 \leqq i \leqq r$) soit la matrice unité (resp. la matrice 0). Posons alors:

$$\theta'(y) = \textstyle\sum_{h \in G} a(y \cdot h) \cdot f(h^{-1}) \quad \text{pour} \quad y \in V.$$

Un calcul immédiat montre que $\theta' : V' \to M_n(k)$ vérifie (36); de plus, on a $\theta'(y_1) = 1 \in GL_n(k)$; il existe donc un voisinage ouvert saturé $V$ de $\pi^{-1}(x)$ que $\theta'$ applique dans $GL_n(k)$.

Mais l'hypothèse (R) signifie qu'il existe un voisinage ouvert $W$ de l'élément neutre de $GL_n(k)$, saturé pour les translations à droite de $H$, et une "rétraction" $r : W \to H$ telle que:

$$r(w \cdot h) = r(w) \cdot h \quad \text{si} \quad w \in W \quad \text{et} \quad h \in H.$$

Si l'on pose alors $U = \theta'^{-1}(W)$ et $\theta = r \circ \theta'$, l'application $\theta$ est bien une application régulière de $U$ dans $H$ vérifiant (36), cqfd.

COROLLAIRE. *Supposons que $H$ soit l'un des groupes $GL_n(k)$, $SL_n(k)$, $Sp_n(k)$, ou un groupe linéaire résoluble (par exemple le groupe additif $G_a$). Alors $Y \times_G H$ est un espace fibré principal algébrique.*

Il faut vérifier la condition (R) dans chaque cas. C'est trivial pour $GL_n(k)$ et $SL_n(k)$, facile pour $Sp_n(k)$ (cf. [16], n° 20); dans le cas d'un groupe linéaire résoluble, c'est un théorème de Rosenlicht ([10], th. 10).

EXEMPLE. *Revêtements cycliques d'ordre premier à $p$.*

Prenons pour $G$ le groupe cyclique $\boldsymbol{Z}/n\boldsymbol{Z}$, avec $(n, p) = 1$. A toute racine primitive $n$-ème de l'unité est associé un isomorphisme $f$ de $G$ dans $k^* = GL_1(k)$. En appliquant le corollaire à la Proposition 11, on associe à tout revêtement $Y \in \pi^1(X, G)$ un espace fibré à groupe $k^*$, c'est-à-dire un élément $f(Y)$ de $H^1(X, \mathcal{O}^*)$ (en désignant par $\mathcal{O}^*$ le faisceau des $\mathcal{O}_x^*$, groupes multiplicatifs des éléments inversibles des $\mathcal{O}_x$). Si l'on suppose $X$ projective, un raisonnement semblable à celui de la Proposition 12 ci-après montre que $f$ est *un isomorphisme de $\pi^1(X, \boldsymbol{Z}/n\boldsymbol{Z})$ sur le sous-groupe des éléments $d \in H^1(X, \mathcal{O}^*)$ vérifiant $nd = 0$.* Lorsque $X$ est non singulière, le groupe $H^1(X, \mathcal{O}^*)$ n'est autre que le groupe des *classes de diviseurs* de $X$ (cf. [20], §3), et le résultat précédent est bien connu (cf. [8], où il est déduit de la théorie de Kummer).

## 16. Revêtements cycliques d'ordre $p$

Soit $G = \boldsymbol{Z}/p\boldsymbol{Z}$. Si l'on identifie $G$ au corps premier $\boldsymbol{F}_p$, on obtient un plongement $f$ de $G$ dans le groupe additif $G_a$ du corps de base $k$. En appliquant le corollaire à la Proposition 11 à $f$ on fait correspondre à tout $G$-revêtement $Y$ de $X$ un espace fibré algébrique principal de base $X$ et de groupe structural $G_a$, autrement dit un élément de $H^1(X, \mathcal{O})$. On a donc obtenu une application canonique

(38) $$f_1 : \pi^1(X, \mathbf{Z}/p\mathbf{Z}) \to H^1(X, \mathcal{O}).$$

Proposition 12. *Si $X$ est une variété projective, l'application $f_1$ est un isomorphisme du groupe $\pi^1(X, \mathbf{Z}/p\mathbf{Z})$ sur $H^1(X, \mathcal{O})^F$, sous-groupe de $H^1(X, \mathcal{O})$ formé des éléments $\xi$ vérifiant:*

(39) $$F\xi = \xi.$$

Le fait que $f_1$ soit un homomorphisme est facile à vérifier. Cherchons l'image de cette homomorphisme. Si l'on note $\wp : G_a \to G_a$ l'application $\wp(\lambda) = \lambda^p - \lambda$ (i.e. $\wp = F - 1$), on a une suite exacte:

(40) $$0 \to G \xrightarrow{f} G_a \xrightarrow{\wp} G_a \to 0.$$

Le fait que $\wp \circ f = 0$ montre que l'homomorphisme composé:

$$\pi(X, G) \xrightarrow{f_1} H^1(X, \mathcal{O}) \xrightarrow{\wp} H^1(X, \mathcal{O})$$

est identiquement nul, ce qui signifie que l'image de $f_1$ est contenue dans $H^1(X,\mathcal{O})^F$. Inversement, soit $Z$ un espace fibré correspondant à un élément de $H^1(X, \mathcal{O})^F$, c'est-à-dire tel que l'espace fibré $\wp(Z)$ (déduit de $Z$ par $\wp : G_a \to G_a$) soit trivial. Le groupe $G$ opére sur $Z$, et la suite exacte (40), jointe à un raisonnement local évident, montre que $Z/G$ s'identifie à $\wp(Z)$; si donc l'on a une section $s : X \to \wp(Z)$ qui identifie $X$ à une sous-variété $s(X)$ de $\wp(Z)$, l'image réciproque $Y$ de $s(X)$ dans $Z$ sera un $G$-revêtement de $X$, donc un élément de $\pi^1(X, G)$. De plus, on vérifie facilement que $f(Y) = Y \times_G G_a$ s'identifie canoniquement à $Z$, ce qui montre bien que $Z \in \mathrm{Im}\,(f_1)$.

Reste à montrer que le noyau de $f_1$ est réduit à 0 (c'est le seul point qui fasse intervenir l'hypothèse que $X$ est projective). Soit donc $Y$ un $G$-revêtement tel que $f(Y)$ soit isomorphe à $X \times G_a$; l'injection $f : G \to G_a$ définit une injection de $Y$ dans $X \times G_a$; mais $X$ est une variété complète, donc aussi $Y$, et l'image de $Y$ dans le facteur $G_a$ ne peut consister qu'en un nombre fini de points (cf. [15], §4, par exemple). Il en résulte tout de suite que $Y$ est trivial sur chaque composante connexe de $X$, donc aussi sur $X$ tout entier, ce qui achève la démonstration.

Corollaire 1. *Soit $\sigma$ la dimension de la "composante semi-simple" $H^1(X, \mathcal{O})_s$ de $H^1(X, \mathcal{O})$* (cf. n° 9). *Le groupe $\pi^1(X, \mathbf{Z}/p\mathbf{Z})$ est un groupe fini d'ordre $p^\sigma$.*

Cela résulte de ce qui a été dit au n° 9.

Corollaire 2. *Une variété de dimension $\geqq 2$ qui est une intersection complète n'a aucun revêtement cyclique de degré $p$ non-trivial.*

En effet, si $X$ est une telle variété, on sait que $H^1(X, \mathcal{O}) = 0$, cf. [14], n° 78.

Remarque. Si X n'est pas irréductible, $X$ peut posséder des revêtements localement triviaux; ils correspondent au sous-groupe $H^1(X, \mathbf{Z}/p\mathbf{Z})$ de $H^1(X, \mathcal{O})^F$.

## 17. Variante

On peut obtenir les résultats du n° précédent par une autre méthode, reposant sur le lemme suivant:

LEMME 5. *Soient $X$ une variété algébrique, $G$ un groupe fini, $Y$ un $G$-revêtement de $X$, et $x$ un point de $X$. Désignons par $\mathcal{O}'_x$ l'anneau des germes de fonctions régulières au voisinage de $\pi^{-1}(x) \subset Y$. L'anneau $\mathcal{O}'_x$ est un anneau semi-local sur lequel opère $G$, et l'on a:*

$$(41)\qquad H^0(G, \mathcal{O}'_x) = \mathcal{O}_x \quad et \quad H^q(G, \mathcal{O}'_x) = 0 \quad pour \quad q > 0.$$

Le fait que $H^0(G, \mathcal{O}'_x) = \mathcal{O}_x$ résulte de la définition d'une variété quotient donnée au n° 13. D'autre part l'anneau semi-local $\mathcal{O}'_x$ a pour anneaux locaux les $\mathcal{O}_y$, $y \in \pi^{-1}(x)$; il s'ensuit (cf. [11], p. 15) que le complété $\hat{\mathcal{O}}'_x$ de $\mathcal{O}'_x$ est isomorphe au produit des $\hat{\mathcal{O}}_y$; comme le groupe $G$ opère sans point fixe sur $\pi^{-1}(x)$, on en déduit, en appliquant un résultat classique de cohomologie des groupes:

$$(42)\qquad H^q(G, \hat{\mathcal{O}}'_x) = 0 \quad \text{pour} \quad q > 0.$$

Mais $\mathcal{O}'_x$ est un module de type fini sur $\mathcal{O}_x$ (pour le voir, prendre pour $Y$ une variété affine, et expliciter $\mathcal{O}_x$ et $\mathcal{O}'_x$ en fonction de l'anneau de coordonnées de $Y$); il s'ensuit (cf. [16], Annexe, par exemple) que l'on a:

$$(43)\qquad \hat{\mathcal{O}}'_x = \mathcal{O}'_x \otimes \hat{\mathcal{O}}_x, \text{ le produit tensoriel étant pris sur } \mathcal{O}_x.$$

Comme $\hat{\mathcal{O}}_x$ est un $\mathcal{O}_x$-module plat ([16], *loc.cit.*), on déduit de (43):

$$(44)\qquad H^q(G, \hat{\mathcal{O}}'_x) = H^q(G, \mathcal{O}'_x) \otimes \hat{\mathcal{O}}_x.$$

Du fait que le couple $(\mathcal{O}_x, \hat{\mathcal{O}}_y)$ est plat ([16], prop. 27), les relations (42) et (44) entraînent $H^q(G, \mathcal{O}'_x) = 0$ pour $q > 0$, cqfd.

REMARQUES. (1) La démonstration de (42) montre en outre que l'on a:

$$\hat{\mathcal{O}}_y = \hat{\mathcal{O}}_x \quad \text{si} \quad \pi(y) = x.$$

Autrement dit, la projection $\pi$ est un isomorphisme "analytique".

(2) En utilisant le Lemme 5, on peut démontrer l'existence d'une *suite spectrale* analogue à celle de Cartan-Leray; cette suite aboutit à $H^*(X, \mathcal{O})$ et a pour terme $E_2$ le groupe bigradué $H^*(G, H^*(Y, \mathcal{O}_Y))$. Cf. un mémoire de A. Grothendieck à paraître prochainement.

Revenons maintenant au cas $G = Z/pZ$. Comme la fonction $1 \in \mathcal{O}'_x$ a une trace nulle, la relation $H^1(G, \mathcal{O}'_x) = 0$ entraîne l'existence d'une fonction $\theta \in \mathcal{O}'_x$ vérifiant:

$$(45)\qquad \theta^\sigma = \theta + 1 \; (\sigma \text{ désignant le genérateur de } Z/pZ).$$

A l'écriture près, c'est l'équation (36). On remarquera que, si $Y$ est irréductible, $\theta$ est un *générateur d'Artin-Schreier de l'extension* $k(Y)/k(X)$.

Une fois démontrée l'existence des fonctions $\theta$, la construction de la classe de cohomologie $\xi$ associée à $Y$ ne présente plus de difficultés: on commence par construire un recouvrement ouvert $\{U_i\}$ de $X$, et des fonctions $\theta_i$, régulières sur

$V_i = \pi^{-1}(U_i)$, et vérifiant (45). Si l'on pose $f_{ij} = \theta_i - \theta_j$ dans $V_i \cap V_j$, les $f_{ij}$ sont invariants par $G$, et constituent un 1-*cocycle de* $\{U_i\}$ *à valeurs dans le faisceau* $\mathcal{O}$, dont la classe de cohomologie n'est autre que l'élément $\xi$ cherché. Les autres résultats de la Proposition 12 ne présentent pas davantage de difficultés. Par exemple, le fait que $F\xi = \xi$ se démontre en remarquant que les $g_i = \theta_i^p - \theta_i$ sont invariants par $G$, donc forment une 0-cochaîne de $\{U_i\}$ à valeurs dans $\mathcal{O}$, dont le cobord est $f_{ij}^p - f_{ij}$.

## 18. Revêtements cycliques d'ordre $p^n$

Soit $n$ un entier $\geqq 1$, et soit $G = Z/p^nZ$; on peut identifier canoniquement $G$ au groupe $W_n(F_p)$, cf. n° 1. Comme $F_p$ se plonge dans $k$, on a ainsi défini un isomorphisme $f$ de $G$ dans le groupe $W_n = W_n(k)$. Ce dernier groupe est un groupe *algébrique*, en correspondance birégulière avec $k^n$; c'est de plus un groupe *linéaire*: cela se voit, soit directement, soit en invoquant [10], th. 16, cor. 4. On peut donc appliquer à $W_n$ le corollaire à la Proposition 11: si $Y$ est un $G$-revêtement de $X$, l'espace $Y \times_G W_n$ est un espace fibré principal de groupe structural $W_n$, c'est-à-dire un élément de $H^1(X, \mathcal{W}_n)$. Comme au n° 16, on a donc obtenu une application

$$f_1 : \pi^1(X, Z/p^nZ) \to H^1(X, \mathcal{W}_n). \tag{46}$$

Proposition 13. *Si $X$ est une variété projective, l'application $f_1$ est un isomorphisme de $\pi^1(X, Z/p^nZ)$ sur $H^1(X, \mathcal{W}_n)^F$.*

La démonstration étant identique à celle de la Proposition 12, nous ne la répèterons pas; indiquons simplement que, ici encore, elle repose essentiellement sur le fait que l'homomorphisme $\wp = F - 1$ définit par passage au quotient un isomorphisme de $W_n/G$ sur $W_n$.

Soient maintenant $n$ et $m$ deux entiers, avec $n \geqq m$; on a un homomorphisme canonique de $Z/p^nZ$ sur $Z/p^mZ$, d'où, d'après le n° 14, b) un homomorphisme $\pi^1(X, Z/p^nZ) \to \pi^1(X, Z/p^mZ)$; cherchons l'image de cet homomorphisme:

Proposition 14. *Soit $\alpha$ un élément de $\pi^1(X, Z/p^mZ)$ et soit $\xi = f_1(\alpha)$ la classe de cohomologie qui lui est associée. Pour que $\alpha$ appartienne à l'image de $\pi^1(X, Z/p^nZ)$, il faut et il suffit que $\delta^1_{n,n-m}(\xi) = 0$.*

(Pour la définition de l'opération de Bockstein $\delta^1_{n,n-m}$, voir n° 3.)

Nous aurons besoin du lemme suivant:

Lemme 6. *Soit $H$ un $\Lambda$-module de longueur finie, et soit $F$ un endomorphisme $p$-linéaire de $H$. L'application $\wp = F - 1 : H \to H$ est alors une surjection.*

Il existe un entier $n$ tel que l'on ait $p^nH = 0$; nous raisonnerons par récurrence sur $n$. Lorsque $n = 1$, $H$ est un $k$-espace vectoriel de dimension finie, et le fait que $\wp$ est surjectif est connu (cf. n° 9); le cas général résulte de l'hypothèse de récurrence, appliquée à $pH$ et à $H/pH$.

Nous pouvons maintenant démontrer la Proposition 14:

Si $\alpha$ est image d'un élément $\beta \in \pi^1(X, Z/p^nZ)$, correspondant à une classe de cohomologie $\eta \in H^1(X, \mathcal{W}_n)^F$, on voit tout de suite que $\xi = R^{n-m}\eta$, d'où évidemment $\delta^1_{n,n-m}(\xi) = 0$.

Réciproquement, soit $\xi \in H^1(X, \mathscr{W}_m)^F$ vérifiant l'équation précédente; il nous faut montrer que $\xi$ s'écrit $\xi = R^{n-m}(\eta)$, avec $\eta \in H^1(X, \mathscr{W}_n)^F$, c'est-à-dire $F\eta = \eta$. Or, par définition même des opérations de Bockstein, la relation $\delta^1_{n,n-m}(\xi) = 0$ signifie que $\xi = R^{n-m}(\eta')$, avec $\eta' \in H^1(X, \mathscr{W}_n)$. De plus, la relation $F\xi = \xi$ montre que $R^{n-m}(F\eta' - \eta') = 0$, i.e. $F\eta' - \eta' = V^m\theta$ avec $\theta \in H^1(X, \mathscr{W}_{n-m})$. En appliquant le Lemme 6 à $H^1(X, \mathscr{W}_{n-m})$, on peut écrire $\theta = F\theta' - \theta'$, et, en posant $\eta = \eta' - V^m\theta'$, on obtient un élément vérifiant les propriétés requises, cqfd.

COROLLAIRE. *Si $X$ n'a pas de torsion en dimension* 1, *le groupe* $\pi^1(X, \boldsymbol{Z}/p^n\boldsymbol{Z})$ *est somme directe de $\sigma$ groupes isomorphes à* $\boldsymbol{Z}/p^n\boldsymbol{Z}$.

(Pour la définition de $\sigma$, voir Proposition 12, Corollaire 1.)

Désignons par $H_n$ le groupe $\pi^1(X, \boldsymbol{Z}/p^n\boldsymbol{Z})$, considéré comme sous-groupe de $H^1(X, \mathscr{W}_n)$; d'après la Proposition 14, l'homomorphisme canonique

$$R^{n-1} : H_n \to H_1$$

est surjectif. De plus il est clair que son noyau est $VH_{n-1}$. On déduit de là, par récurrence sur $n$, que $H_n$ est un groupe fini d'ordre $p^{n\sigma}$; comme il est plongé dans $H^1(X, \mathscr{W}_n)$, on a $p^nH_n = 0$. De plus, le composé:

$$H_n \xrightarrow{R} H_{n-1} \xrightarrow{V} H_n$$

est la multiplication par $p$ (en vertu de la formule $FVR = p$ et du fait que $F$ est l'identité sur $H_n$); puisque $R$ est surjectif (d'après la Proposition 14), ceci entraîne que $H_{n-1} = pH_n$, et l'on voit donc que $H_n/pH_n = H_1$ a $p^\sigma$ éléments. Ceci suffit à prouver que $H_n$ est somme directe de $\sigma$ groupes cycliques d'ordre $p^n$, cqfd.

REMARQUE. Lorsque $X$ a de la torsion en dimension 1 la détermination explicite de $\pi^1(X, \boldsymbol{Z}/p^n\boldsymbol{Z})$ peut encore se faire de manière analogue, mais plus compliquée. Nous nous bornerons à donner le résultat:

Soient $Z^1_m$, $m = 1, 2, \cdots$, les sous-espaces vectoriels de $H^1(X, \mathscr{O})$ définis par les formules (6) du n° 3; soit $\sigma_m$ la dimension de la "composante semi-simple" de $Z^1_m/Z^1_{m+1}$, et soit $\tau$ la dimension de la composante semi-simple de $Z^1$. Soit $H$ le groupe abélien de type fini défini par la formule:

$$H = \textstyle\sum_{m=1}^{\infty} (\boldsymbol{Z}/p^m\boldsymbol{Z})^{\sigma_m} + \boldsymbol{Z}^\tau. \tag{47}$$

Le groupe $\pi^1(X, \boldsymbol{Z}/p^n\boldsymbol{Z})$ est alors *isomorphe à* $\mathrm{Hom}(H, \boldsymbol{Z}/p^n\boldsymbol{Z})$; autrement dit, tout se passe (au point de vue des revêtements cycliques d'ordre une puissance de $p$) comme si la variété $X$ avait un "groupe fondamental" isomorphe à $H$.

## 19. Courbes algébriques et jacobiennes

Soit $X$ une courbe algébrique irréductible, complète, et non singulière. Du fait que $X$ n'a pas de torsion, le corollaire à la Proposition 14 montre que $\pi^1(X, \boldsymbol{Z}/p^n\boldsymbol{Z})$ est somme directe de $\sigma$ groupes cycliques d'ordre $p^n$; de plus, d'après le n° 9, l'entier $\sigma$ est égal au rang de la "composante semi-simple" de la matrice de Hasse-Witt $A$ de $X$, c'est-à-dire au rang de $A \cdot A^p \cdots A^{p^{g-1}}$. On retrouve ainsi les résultats de Hasse-Witt [5] et de Schmid-Witt [12].

Soit en outre $\phi : X \to J$ l'application canonique de $X$ dans sa jacobienne (pour tout ce qui concerne jacobiennes et variétés abéliennes, cf. [19]). D'après un résultat (inédit) de Rosenlicht, l'homomorphisme

$$\phi^* : H^1(J, \mathcal{O}_J) \to H^1(X, \mathcal{O}_X)$$

est bijectif. D'après la Proposition 12 il en est donc de même de

$$\phi^1 : \pi^1(J, \mathbf{Z}/p\mathbf{Z}) \to \pi^1(X, \mathbf{Z}/p\mathbf{Z}).$$

Dans le langage de [8], cela signifie que tout revêtement cyclique d'ordre $p$ de $X$ est "du type d'Albanese".

Nous montrerons ailleurs que tout revêtement d'une variété abélienne $A$ est donné par une isogénie $B \to A$; ce point étant admis, des raisonnements classiques montrent que $\pi^1(A, \mathbf{Z}/n\mathbf{Z})$ s'identifie au dual du groupe $A_n$ des points $a \in A$ vérifiant $na = 0$. Appliquant ceci à $J$ et utilisant l'isomorphisme $\phi^1$, on en conclut que $\pi^1(X, \mathbf{Z}/p\mathbf{Z})$ est dual du groupe $J_p$, lui-même isomorphe au groupe $G_p$ du n° 11; on retrouve ainsi le fait que le groupe $G_p$ est d'ordre $p^\sigma$ (Proposition 10).

## 20. Un exemple

Proposition 15. *Soient $G$ un groupe fini, et $r$ un entier $\geqq 1$. Il existe une variété algébrique $Y$ de dimension $r$, non singulière, qui est une intersection complète dans un espace projectif convenable, et sur laquelle le groupe $G$ opère sans points fixes.*

*De plus, dans le cas où $r = 2$ et $G = \mathbf{Z}/p\mathbf{Z}$ (avec $p \geqq 5$), on peut imposer à $Y$ d'être une surface dans $\mathbf{P}_3(k)$.*

Nous allons construire $Y$ en suivant une méthode due à Godeaux dans le cas classique:

Considérons une représentation linéaire $R$ du groupe $G$, et notons $n + 1$ son degré. Puisque $G$ opère linéairement sur $k^{n+1}$, il opère par passage au quotient sur $P = \mathbf{P}_n(k)$, et la variété quotient $P/G$ est bien définie (cf. n° 13). Nous allons tout d'abord montrer comment l'on peut définir un plongement projectif de cette variété:

Soit $S$ la sous-algèbre de $k[X_0, \cdots, X_n]$ formée des polynômes invariants par $G$; c'est une algèbre graduée, qui est de type fini sur $k$. Nous noterons $S_d$ la composante homogène de degré $d$ de $S$, et nous poserons:

$$S(d) = \Sigma_{m=0}^{\infty} S_{md}. \tag{48}$$

Un raisonnement élémentaire (analogue à celui utilisé dans la "normalisation projective" des variétés) montre que l'on peut choisir $d$ de telle sorte que tous les éléments de $S(d)$ soient des polynômes en ceux de $S_d$. Si l'on gradue $S(d)$ en considérant $S_{md}$ comme de degré $m$, ceci signifie que $S(d)$ est une algèbre graduée *engendrée par ses éléments de degré* 1, donc peut être considérée comme l'anneau de coordonnées projectives d'une sous-variété $Z$ de l'espace projectif $\mathbf{P}_s(k)$, avec

$s+1=\dim S_d$. Si l'on choisit une base $f_0, \cdots, f_s$ de $S_d$, les $f_i$ définissent par passage au quotient une application régulière $f: P \rightarrow Z$.

L'application $f$ est invariante par $G$ et définit par passage au quotient *un isomorphisme birégulier de $P/G$ sur $Z$* (nous omettons la vérification de ce fait, qui est pénible, mais ne présente pas de difficulté essentielle). C'est donc le plongement projectif cherché.

Soit maintenant $Q$ l'ensemble des points $y \in P$ tels qu'il existe $g \in G$, avec $g \neq e$ et $y \cdot g = y$; soit $Q' = f(Q)$. Les ensembles $Q$ et $Q'$ sont des sous-variétés fermées de $P$ et de $Z$, de même dimension. Nous supposerons vérifiée la condition suivante:

$$r < n - \dim (Q). \tag{49}$$

On observera que $P - Q$ est un $G$-revêtement de $Z - Q'$, ce qui montre (en utilisant la Remarque 1 du n° 17) que $Z - Q'$ est une variété non singulière.

Soit alors $L$ une sous-variété linéaire de $P_s(k)$, de dimension égale à $s-n+r$; si $L$ est choisie "en position générale", l'inégalité (49) montre qu'elle ne rencontre pas $Q'$; de plus, elle rencontre transversalement $Z$, ce qui entraîne que l'intersection $X = Z \cap L$ soit une variété *non singulière, de dimension $r$, et ne rencontrant pas $Q'$*. Posons $Y = f^{-1}(X)$; il est clair que $Y$ est un $G$-revêtement de $X$. De plus, si $L$ est définie par l'annulation de $g_1, \cdots, g_{n-r}$, combinaisons linéaires des $f_i$, la variété $Y$ sera définie par l'annulation de ces mêmes $g_i$, considérés comme éléments de $k[X_0, \cdots, X_n]$; soit $\mathfrak{a}$ l'idéal engendré par les $g_i$; si l'on montre que $\mathfrak{a}$ n'est autre que l'idéal défini par $Y$, il en résultera bien que $Y$ est une intersection complète. Or, d'après le théorème de Macaulay, $\mathfrak{a}$ est intersection d'idéaux primaires $\mathfrak{q}_\alpha$ correspondant aux idéaux premiers $\mathfrak{p}_\alpha$ associés aux composantes irréductibles $Y_\alpha$ de $Y$. Soit alors $y \in Y_\alpha$, et soit $x = f(y) \in Z$; comme, par hypothèse, $L$ est transversale à $Z$ en $x$, les $g_i$ définissent dans $\mathcal{O}_x$ des éléments faisant partie d'un système régulier de paramètres (au sens de [11], p. 29); la relation $\hat{\mathcal{O}}_y = \hat{\mathcal{O}}_x$ montre qu'il en est de même dans $\mathcal{O}_y$, et l'idéal local $\mathfrak{a}\mathcal{O}_y$ est donc un idéal premier, d'où $\mathfrak{q}_\alpha = \mathfrak{p}_\alpha$ pour tout $\alpha$, ce qui montre bien que $Y$ est une intersection complète. Comme $Y$ est non singulière et connexe (comme toute intersection complète, cf. [14], n° 78, Proposition 5, par exemple), on voit en outre que $Y$ est *irréductible*.

La Proposition 15 sera donc démontrée si nous prouvons que l'on peut toujours choisir une représentation $R$ de $G$ vérifiant (49), et de dimension 4 dans le cas $r = 2$, $G = Z/pZ$, $p \geq 5$. Or c'est immédiat:

(a) Dans le cas général, on prend la somme directe d'un nombre suffisant ($r$ par exemple) de copies de la représentation régulière.

(b) Dans le cas particulier $r = 2$, $G = Z/pZ, p \geq 5$, on fait correspondre au générateur de $G$ l'endomorphisme $1 + N$ de $k^4$, où $N$ est défini par les formules $N(e_i) = e_{i+1}$, $0 \leq i \leq 2$, et $N(e_3) = 0$. L'ensemble $Q$ est réduit au point de coordonnées homogènes $(0, 0, 0, 1)$, et l'on a $r = 2, n = 3$, $\dim(Q) = 0$, ce qui vérifie bien l'inégalité (49), cqfd.

Remarque. La méthode suivie plus haut pour définir un plongement projectif de $P/G$ a une portée plus générale; en l'utilisant, on peut démontrer que, si $Y$

est une variété projective sur laquelle opère $G$, la variété $Y/G$ est aussi une variété projective.

PROPOSITION 16. *Soit $G = \mathbf{Z}/p\mathbf{Z}$, avec $p \geqq 5$. Soit $Y$ la surface de $\mathbf{P}_3(k)$ dont l'existence est affirmée par la Proposition 15, et posons $X = Y/G$. La surface $X$ est une surface projective, non singulière, vérifiant:*

$$H^0(X, \Omega^1) = 0 \quad et \quad H^1(X, \mathcal{O}) \neq 0. \tag{50}$$

Puisque $Y$ est connexe, c'est un revêtement non trivial de $X$, donc qui correspond à un élément $\xi \neq 0$ dans $H^1(X, \mathcal{O})^F$. D'autre part, on montre (par le même raisonnement que dans le cas classique) qu'il n'y a pas de forme différentielle de première espèce $\neq 0$ sur une surface non singulière de $\mathbf{P}_3(k)$; donc $H^0(Y, \Omega^1) = 0$, et comme $H^0(X, \Omega^1)$ est un sous-espace de $H^0(Y, \Omega^1)$, il est aussi réduit à 0, cqfd.

REMARQUES. (1) En utilisant la suite spectrale du revêtement $Y \to X$ (cf. n° 17), on peut préciser (50) et montrer que $h^{0,1} = \dim H^1(X, \mathcal{O})$ est égal à 1; de plus, l'operation de Bockstein

$$\beta_1 : H^1(X, \mathcal{O}) \to H^2(X, \mathcal{O})$$

n'est pas nulle.

Ceci montre que le "groupe fondamental" $H$ de $X$, au sens du n° 18, est isomorphe à $\mathbf{Z}/p\mathbf{Z}$.

On observera par ailleurs que le groupe des classes de diviseurs d'ordre $p$ de $X$ est réduit à 0, puisque ce groupe est isomorphe à un sous-groupe de $H^0(X, \Omega^1)$ (cf. n° 11).

(2) Plus généralement, on peut appliquer la Proposition 15 à un $p$-groupe abélien $G$ quelconque. On obtient ainsi une variété $X = Y/G$ dont le "groupe fondamental" $H$ est isomorphe à $G$ (cela se voit en remarquant que $Y$ joue le rôle d'un "revêtement universel" de $X$, en vertu du Corollaire 2 à la Proposition 12).

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# LES CLASSES CARACTÉRISTIQUES DE PONTRJAGIN DES VARIÉTÉS TRIANGULÉES

PAR R. THOM

Avant de nous occuper des variétés polyèdrales, il nous sera utile de généraliser quelque peu la notion de polyèdre. Introduisons dans ce but la notion d'"espace différentiable par morceaux."

## Structure différentiable par morceaux

Un espace $E$ est dit "différentiable par morceaux" de classe $C^m$ s'il peut être défini ainsi qu'il suit: on se donne des ouverts $U^n$ d'espace euclidien $R^n$, $n$ variable, et une famille d'applications différentiables de classe $C^m$ $g_{\alpha\beta}: U^n_\alpha \rightarrow U^p_\beta$; dans ces conditions, $E$ est le quotient de la réunion $U = \bigcup_\alpha U_\alpha$ des $U_\alpha$ par les relations d'identification définies par les applications $g_{\alpha\beta}$ (ou un sous-espace de ce quotient).

Sur un tel espace différentiable par morceaux, on a la notion de fonction réelle de classe $C^m$ (et même la notion de forme différentielle). La notion de dérivée partielle d'une fonction peut être définie dans chaque ouvert $U_\alpha$; il en résulte qu'on peut parler, sur $E$, de l'espace $D^m$ des fonctions de classe $C^m$, muni de la topologie définie par l'écart sur les dérivées partielles d'ordre $r$ ($r \leq m$). De même pour les applications de $E$ dans un espace euclidien. Tous ces espaces sont métriques complets, donc de Baire.

Il importe de dire tout de suite que, si l'on n'impose aucune condition aux relations d'identification $g_{\alpha\beta}$, l'espace $E$ a toutes chances de se trouver muni d'une topologie fortement dégénérée, en général non séparée. De ce fait, il n'est pas impossible qu'il n'existe sur $E$ d'autres fonctions différentiables que les constantes. D'ailleurs, certains espaces de ce type s'introduisent dans l'étude des feuilletages de variétés, comme quotients des structures feuilletées.

Nous ferons sur les espaces $E$ des hypothèses assez restrictives:

(1) Les applications $g_{\alpha\beta}$ sont des *injections*, partout de rang maximum; il suffira de se donner des applications $g_{\alpha\beta}: U^{n-K}_\alpha \rightarrow U^n_\beta$, les autres s'en déduisant par transitivité.

(2) On suppose que l'intersection de deux $U^n_\alpha$ est contenue toujours dans une réunion de $U^{n-1}$ de dimension inférieure.

(3) *Hypothèse de position générale.* Soient $g: U^p_\alpha \rightarrow U^n_\gamma$, $U^{p'}_\beta \rightarrow U^n_\gamma$ deux injections; si $U^p_\alpha$ n'est pas contenu dans $U^{p'}_\beta$, alors les images $g(U^p_\alpha)$, $g(U^{p'}_\beta)$ sont des sous-variétés de $U^n_\gamma$ qui se coupent en tout point commun *en position générale*: En un point $x$ commun à $g(U^p_\alpha)$, $g(U^{p'}_\beta)$ les plans tangent à ces sous-variétés se coupent suivant un $q$-plan dont la codimension (par rapport à $U^n_\gamma$) est somme des codimensions de $g(U^p_\alpha)$, $g(U^{p'}_\beta)$. Ce $q$-plan est d'ailleurs le $q$-plan tangent au $U$ d'intersection auquel appartient $x$. De même pour l'intersection de plusieurs $g(U^p_\alpha)$ incidents à un même $U^n_\gamma$.

Il est clair qu'un complexe simplicial $K$ est susceptible d'une définition de cette espèce: il suffit d'attacher à tout $p$-simplexe $s_p$ une boule ouverte $B_p$ le contenant rectilinéairement; on prend pour $K$ la portion de l'espace $E$ obtenue à partir des $B_p$ par identification en ne conservant, pour chaque $B_p$, que la portion d'espace intérieure à $s_p$. Finalement, les espaces que nous considérons "généralisent" les polyèdres au sens suivant: les "cellules" $U$ ne sont pas nécessairement homologiquement triviales; ce sont seulement des variétés à bord (connexes paracompactes) dont le bord peut présenter des singularités du type suivant: les cellules qui composent le bord se coupent dans $U$ en position générale. Bien entendu, les espaces ainsi obtenus sont encore des polyèdres, en vertu des théorèmes généraux de triangulation des variétés différentiables (J. H. C. Whitehead [6]).

On va étendre aux applications différentiables de polyèdres les théorèmes de régularisation connus pour les variétés. Enonçons dans ce but un Lemme.

Lemme de position générale. *Soit* $(G)$ *un ensemble de plans de* $R^n$ *(considéré comme espace affine); on suppose que les plans de* $(G)$ *se coupent en position générale. Soit* $F$ *une fonction réelle sur* $G$*, de classe* $C^m$ *sur chacun des plans de* $(G)$*; alors* $F$ *est la restriction à* $(G)$ *d'une fonction* $F_1$ *de classe* $C^m$ *sur* $R^n$.

L'extension de la fonction $F$ en $F_1$ va se démontrer par induction sur le nombre $r$ des plans qui composent $(G)$. Pour $r = 1$, la propriété est presque évidente: on forme un voisinage tubulaire $T$ du plan $X$, de rayon $a$, et on définit une rétraction différentiable $p : T \to X$ sur le plan $X$; désignons par $g(u)$ une fonction de classe $C^\infty$, décroissant de 1 à 0 lorsque $u$ croit de 0 à $a$. Pour tout point $x$ extérieur à $T$, on posera $F_1(x) = 0$, pour un point $x$ de $T$ situé à la distance $u$ de $X$, on posera $F_1(x) = g(u)F(p(x))$. Si les dérivées $d^kg/(du)^k$ ont été supposées nulles pour $u = 0$ et $u = a$, alors la fonction $F_1$ est de classe $C^m$ et répond à la question.

Supposons le lemme établi pour un système $(G_1)$ de $(r - 1)$ plans, et soit $(X)$ un plan qu'on ajoute à $(G_1)$. Le plan $(X)$ coupe par hypothèse tout plan de $(G_1)$ en position générale; on pourra par suite définir sur un voisinage de $(X)$ une métrique riemannienne pour laquelle tout plan de $(G_1)$ coupe $(X)$ orthogonalement; grâce à cette métrique, on définira un voisinage tubulaire $T$ de $(X)$, de rayon géodésique $a$, et une rétraction différentiable normale $p : T \to X$, telle que $p(T \cap Y) = Y \cap X$ pour tout plan $Y$ de $(G_1)$. Soit $f$ la fonction donnée sur $(G) = (G_1) \cup X$; par induction il existe une fonction $F$, de classe $C^m$ sur $R^n$, qui coincide avec $f$ sur $(G_1)$; formons sur $(X)$ la fonction $v = f - F$; $v$, de classe $C^m$, est nulle sur l'intersection $G_1 \cap X$. On applique alors la construction précédente à la fonction $v$ pour $X$ seul; comme la rétraction $p$ conserve $(G_1)$, on obtient une fonction $w = g(u(x)) \cdot v(p(x))$ de classe $C^m$, qui est nulle sur $(G_1)$. Par suite la somme

$$F_1 = F + w$$

répond bien à la question.

Remarque. L'hypothèse que les plans de $(G)$ se coupent en position générale joue un rôle absolument essentiel dans cette démonstration. On obtiendra un contre-exemple très simple en prenant pour $(G)$ trois droites concourantes du plan, par exemple les axes $Ox$, $Oy$ et la première bissectrice $x = y$. Une fonction $f$

dérivable sur ces trois droites n'est la restriction d'une fonction différentiable du plan que si les dérivées de $f$ le long de ces trois droites en 0 satisfont à une relation linéaire évidente.

### Point régulier; valeur régulière d'une application

Soit $f$ une application de classe $C$ d'une polyèdre $K$ dans l'espace euclidien $R^k$; on dira qu'un point $x$ est régulier pour l'application $f$ si, pour toute cellule $U$ contenant $x$, l'application $f$, restreinte à $U$, admet en $x$ un point régulier, i.e., un point où le rang de $f$ est *égal* à $k$.

Il résulte de cette définition que tous les points du squelette de dimension $(k - 1)$ sont nécessairement non réguliers.

Une valeur $y \in R^k$ de $f$ sera dite *régulière* si l'image réciproque $f^{-1}(y)$ ne contient que des points réguliers (ou aucun point!).

Théorème de Sard. *Si un complexe $K$ est de dimension $n$, et si $K$ ne comporte qu'une infinité dénombrable au plus de cellules, toute application $f$ de $K$ dans $R^k$ n'admet de valeurs non régulières que sur un ensemble de mesure nulle de $R^k$, dés que sa classe $m$ est $\geq n - k + 1$.*

Généralisation immédiate du théorème classique [3].

Théorème de la fibration. *Soit $f$ une application de $K$ dans $R^k$, régulière sur un point $O$ de $R^k$ ainsi que sur tout point d'une boule ouverte $U$ de centre $O$, de rayon $r$. Si l'application $f$ est de plus propre sur $U$ (i.e., l'image réciproque de tout compact est un compact), alors l'application $f$ définit une fibration de $f^{-1}(U)$ sur $U$.*

On construit dans $f^{-1}(U)$ un champ de $k$-plans transverses aux images réciproques $f^{-1}(y)$, et ceci dans chaque cellule $Z_j$ de $K$. Une telle construction est évidemment possible pour toute cellule $Z_k$ du $k$-squelette; en effet $Z_k \cap f^{-1}(U)$ est appliqué par $f$ avec rang maximum sur $U$, et le $k$-plan transverse est évidement, en tout point de $Z_k \cap f^{-1}(U)$, le plan des vecteurs tangents à $Z_k$.

Supposons construit le champ $H$ de $k$-plans transverses sur le $(r - 1)$-squelette $K^{(r-1)}$; il faut montrer que la prolongation de $H$ est possible de façon différentiable sur le $r$-squelette. Soit $Z_r$ une $r$-cellule; sur $Z_r \cap f^{-1}(U)$ les coordonnées $u_1, u_2, \cdots, u_k$ de $U$ peuvent être prises comme fonction coordonnées; par suite, en tout point $x$ de $Z_r \cap f^{-1}(U)$ l'ensemble des $k$-plans transverses est représenté, dans une carte locale $(u_1, \cdots, u_k, v_1, \cdots, v_{r-k})$ par tous les systèmes linéaires de la forme:

$$v_j = \sum a_j^i u_i.$$

C'est donc un espace vectoriel de coordonnées $a_j^i$, de dimension $k \cdot (r - k)$. L'ensemble des $k$-plans transverses aux images réciproques constitue donc un fibré sur $Z_r \cap f^{-1}(U)$, à fibre vectorielle. Une section de ce fibré nous est déjà donnée sur le bord $Z_r$ par le champ $H$; la fibre étant contractile, la section peut se prolonger de façon continue sur tout $Z_r \cap f^{-1}(U)$. Par ailleurs, en vertu du lemme de position générale, ce prolongement pourra s'effectuer de façon différentiable sur un voisinage du bord, donc partout.

On a ainsi établi l'existence, dans $f^{-1}(U)$, d'un champ $H$ de $k$-plans transverses aux images réciproques $f^{-1}(y)$. Ce champ n'est pas, en général, intégrable dans chaque

$Z_r$; il peut néanmoins servir à définir des transversales par le procédé suivant: A tout point y de $U$ associons la demi-droite $Oy$; dans toute cellule $Z_r$ l'image réciproque $f^{-1}(Oy)$ est une sous-variété de dimension $r - k + 1$; dans cette sous-variété, le champ $H$ définit un système de trajectoires différentiables $H_y$; ce système de trajectoires transversales $H_y$ permet de définir un homéomorphisme de $f^{-1}(O)$ sur $f^{-1}(y)$ (car, puisque $f$ est propre, $f^{-1}(Oy)$ est compact, et toute trajectoire peut être prolongée de $f^{-1}(O)$ a $f^{-1}(y)$). L'homéomorphisme $h_y$ ainsi défini dépend continuement de y et permet de définir un homéomorphisme global $h$ de $f^{-1}(U)$ sur $U \times f^{-1}(O)$, ce qui démontre le théorème.

Etant donné un polyèdre $K$ on montrera, comme dans [5], que l'ensemble des applications $f: K \to R^k$ qui n'admettent pas un point donné comme valeur régulière forme un sous-ensemble rare de $L(K;\ R,\ m)$ pourvu que $m$ soit assez grand (maigre, si $K$ est infini paracompact.)

## Applications $t$-régulières sur une sous-variété

Soit $N$ une sous-variété diff. plongée d'une variété $M^p$; si $N$ est de codimension $q$, on peut supposer $N$ définie par un système de cartes locales du type $U \to R^q$; comme dans [5], un point $x \in K$ est $t$-régulier sur $N$ si l'application composée $g \circ f(x) \to R^q$ admet $x$ pour point régulier; l'application $f$ de $K$ dans $M$ est $t$-régulière sur la sous-variété $N$ si tout point de l'image réciproque $f^{-1}(N)$ est $t$-régulier sur $N$.

Comme dans [5] on montera que l'ensemble des applications $f$ de $K$ dans $M$ non $t$-régulières sur $N$ est un fermé rare de $L(K,\ V;\ m)$ si $K$ est compact et $m$ assez grand (maigre si $K$ est dénombrable..).

Remarque. On sait que si $f$ est une application simpliciale de $K$ sur $K'$, l'application $f$ définit une fibration locale sur l'intérieur de tout simplexe de dimension maximum de $K'$. Le théorème de fibration est donc connu pour les applications simpliciales mais il parait difficile d'adapter la notion de "$t$-régularité" au cadre des applications simpliciales. C'est ce qui justifie l'introduction des structures différentiables par morceaux.

## Variétés triangulées

Par variété triangulée, on entend un polyèdre qui est homologiquement une variété de dimension $n$: i.e., les nombres de Betti locaux autour de chaque point sont ceux de la $n$-boule ouverte. Il n'est donc pas nécessaire que le voisinage de tout point soit une boule topologique.

*Image réciproque d'une valeur régulière.* Soit $V^n$ une variété triangulée de dimension $n$, $f: V \to R^k$ une application régulière sur 0. L'image réciproque $f^{-1}(0)$ est dans ces conditions une variété triangulée de dimension $(n - k)$.

Ceci résulte du fait qu'on peut trouver pour tout point $x$ un système fondamental de voisinages qui soient la fois saturés pour les images réciproques $f^{-1}(y)$, et un système de transversales $(H)$. En vertu des théorèmes classiques sur les fibrations d'espaces euclidiens, la fibre est une sous-variété homologique de dimension $(n - k)$. (Il importe de remarquer qu'on peut définir en $x$ une famille fondamentale de voisinages du type ci-dessus qui sont homéomorphes: ceci résulte du fait que $f^{-1}(y)$ est un polyèdre, donc tout point $x$ a un voisinage conique dans $f^{-1}(y)$; ce qui a pour effet que ces voisinages ont même cohomologie que la limite inductive, donc celle de l'espace euclidien $R^n$).

Ce résultat se généralise immédiatement aux images réciproques par des applications $t$-régulières. On a:

*Si une application $f : V^n \to M^p$ est t-régulière sur la sous-variété $N$ de codimension $q$, l'image réciproque $f^{-1}(N)$ est une sous-variété homologique de codimension $q$.*

Ce résultat s'étend également aux variétés à bord: si $Q^n$ est une variété à bord triangulée, de bord $V$, et si $f : Q^n$ est $t$-régulière sur $N \subset M$, l'image réciproque par $f$ de $N$ est une sous-variété à bord homologique $X$, de codimension $q$, dont le bord $Y$ est une sous-variété $Y$ de $V$, de codimension $q$ (également homologique).

Nous allons maintenant définir entre variétés triangulées une relation d'équivalence qui généralise quelque peu la notion d'équivalence combinatoire.

Définition. *Variétés J-équivalentes.* Deux variétés triangulées $V$, $V'$ seront dites $J$-équivalentes si: (1) Elles ont toutes deux même type d'homotopie $(T)$; (2) Elles sont cobordantes et il existe une variété à bord triangulée $Q$ admettant pour bord $V \cup V'$ ($V - V'$ si $V$ et $V'$ sont orientées), dont le type d'homotopie est $(T)$, et telle que les injections $i : V \to Q$, $i' : V \to Q$ soient des homotopies-équivalences.

Dans ces conditions, chacune des variétés $V$, $V'$ est rétracte par déformation de $Q$; la construction usuelle faite sur les variétés cobordantes montre qu'on a bien là une relation d'équivalence; il y a transitivité. On ne connait pas d'exemple de variétés ayant même type d'homotopie qui ne soient pas cobordantes; par contre, il existe des variétés (de dimension 7) qui ont même type d'homotopie, mais ne satisfont pas à la condition (2), et ne sont donc pas $J$-équivalentes.

Définition. *Sous-variétés à structure orthogonale.* On dira qu'une sous-variété $W$ de $V^n$, de codimension $q$, est *à* structure orthogonale normale, s'il existe une application $f$ de $V$ dans le complexe $M(SO(q))$, $t$-régulière sur la grassmannienne $G_q$, telle que $W$ soit l'image réciproque par $f$ de la grassmannienne $G_q$.

Une sous-variété à structure normale orthogonale admet des classes caractéristiques normales (de Stiefel-Whitney et de Pontrjagin), images par $f^*$ des classes corresopndantes de la grassmannienne $G_q$. Dans le cas ou $V^n$ est différentiable, ainsi que $f$, ces classes sont les classes caractéristiques du fibré des vecteurs normaux. Dans le cas généralisé des variétés triangulées, il n'y a plus de fibré des vecteurs normaux au sens strict. Le plongement à structure normale orthogonale jouit de plus de la propriété de transitivité énoncée dans le lemme:

Lemme 1. *Soit $P^{n-q}$ une sous-variété à structure normale orthogonale de la variété $V^n$, $W^{n-q-r}$ une sous-variété à structure orthogonale normale de $P^{n-q}$; dans ces conditions, $W^{n-q-r}$ est une sous-variété à structure orthogonale normale de $V^n$, et cette structure a mêmes classes caractéristiques que le "joint" (au sens de Whitney) de la structure normale de $W$ dans $P$ par la restriction à $W$ de la structure normale de $P$ dans $V$.*

Soient $f : P \to M(SO(r))$, $g : V \to M(SO(q))$ les applications qui définissent $W = f^{-1}(G_r)$, $P = g^{-1}(G_q)$. On peut supposer que l'application $g$ plonge biunivoquement $P$ dans $G_q$, de telle façon que des plans tangents à des cellules incidentes se coupent en position générale. L'application donnée $f : P \to M(SO(r))$ se prolonge différentiablement à un voisinage $U$ de $g(P)$ dans $G_q$; soit $f_1$ ce prolongement $f_1 : U \to M(SO(r)$; on peut supposer $f_1$ $t$-régulière sur $G_r$, puisque $f_1$ l'est, restreinte à $P$; alors l'image réciproque par $f_1$ de $G_r$ dans $U$ est une sous-variété $Z$ de codimension $r$ dans $U$, donc de codimension $(q + r)$ dans $M(SO(q))$. La structure normale de $Z$ dans $M(SO(q))$ est le joint de la structure normale induite de $G_r$ par $f_1$, et de la restriction à $Z$ de la structure normale à $G_q$ dans $M(SO(q))$; l'application $g : V \to M(SO(q))$ est $t$-régulière sur $Z$, et l'on a $W = g^{-1}(Z)$. Ceci démontre la propriété énoncée.

## La fonction $\tau$

Deux applications $f$, $g$ de $V^n$ dans $M(SO(q))$, $t$-régulières sur $G_q$, *homotopes*, définissent des sous-variétés $W$, $W'$ images réciproques de $G_q$ par $f$ et $g$ qui sont $L$-équivalentes au sens de [5] (il ne s'agit ici, toutefois, que de variétés homologiques). Par suite, les index $\tau(W)$, $\tau(W')$ sont égaux. Ainsi: à toute classe d'homotopie d'applications de $V^n$ dans $M(SO(q))$ est attachée de façon invariante l'index des sous-variétés réalisantes. C'est cette fonction $\tau$ qui va nous permettre de définir des classes de Pontrjagin rationnelles dans la cohomologie de $V$. Le résultat précédent se généralise légérement comme suit: si $V$ et $V'$ sont deux variétés $J$-équivalentes (donc de même type d'homotopie), les ensembles de $L$-classes de $V$ et $V'$ sont isomorphes, l'isomorphisme de $L_k(V)$ sur $L_k(V')$ étant induit par une homotopie-équivalence de $V'$ sur $V$. Dans ces conditions, deux $L$-classes homologues de $V$, $V'$ sont réalisées par des sous-variétés $W$, $W'$ qui sont cobordantes dans la variété $X$ de bord $V' - V$; donc $\tau(W) = \tau(W')$. La fonction $\tau$ est ainsi un invariant pour la classe de $J$-équivalence de $V$. En particulier, c'est un *invariant combinatoire* pour toute variété triangulée $V$: en effet, deux variétés triangulées $V$, $V'$ qui admettent des subdivisions isomorphes sont de ce fait $J$-équivalentes.

Dans [1], F. Hirzebruch a introduit une fonction $\tau(u, v \cdots w)$ qui attache à tout système de classes de cohomologie de dimension 2 de $V$ l'index de la sous-variété intersection complète des hypersurfaces duales aux classes $u$, $v \cdots w$. Ce système de classes définit évidemment une $L$-classe, et la fonction index virtuel de Hirzebruch rentre ainsi dans le cas général de la fonction $\tau(L_k)$. Par suite, cette fonction est un invariant combinatoire de $V^n$. On verra plus tard que le symbolisme de Hirzebruch se généralise au cas combinatoire; on pourra démontrer, en conséquence, la relation fonctionnelle

$$\tau(u + v, w) = \tau(u, w) + \tau(v, w) - \tau(u, v, u + v, w)$$

comme dans le cas différentiable considéré par Hirzebruch.

Ceci nécessitera une définition généralisée des classes de Pontrjagin à l'aide de la fonction $\tau$. Nous aurons besoin, dans ce but, d'un lemme de pure homotopie, énoncé ci-dessous:

LEMME 2. *Soit $K$ un complexe fini, de dimension $n$, $B$ un complexe fini tel que $\pi_j(B) = 0$ pour $j < m$. Si $n < 2m - 2$, l'ensemble des applications de $K$ dans $B$*

*est muni d'une loi d'addition (cohomotopie), et l'ensemble des classes d'applications de $K$ dans $B$ forme un groupe abélien $G(K; B)$. Si on désigne par $\mathscr{C}$ la classe des groupes finis (au sens de* J. P. Serre [4]), *alors $G(K; B)$ est isomorphe* mod $\mathscr{C}$ *au groupe* Hom $(H_*(K; Z), H_*(B; Z))$.

*Ou encore: $G(K; B)\otimes Q$ ($Q$ corps des rationnels) est isomorphe au groupe* Hom $(H^*(B; Q), H^*(K; Q)$.

Il suffit de démontrer la propriété suivante: Si deux applications $f$, $g$ de $K$ dans $B$ induisent le même homomorphisme $f_*, g_*$; $H_*(K; Z)\to H_*(B; Z)$, alors il existe un entier non nul $N$ tel que les multiples $N\cdot f$ et $N\cdot g$ soient homotopes.

En appliquant la propriété à la différence $f-g$, on se ramène à démontrer. Si une application $f$ de $K$ dans $B$ induit un homomorphisme $f_*: H_*(K; Q)\to H_*(B; Q)$ qui est nul, il existe un entier $\neq 0$ $N$ tel que $N\cdot f$ soit homotope à zéro.

Il est clair qu'on peut tout d'abord trouver un multiple $g = N_1\cdot f$ tel que l'homomorphisme $g$ induit par $g$ sur les $x$ homologies *entières* $g_*: H_*(K; Z)\to H_*(B; Z)$ soit nul. Soit $T$ le cone sur $K$. On se propose d'étendre l'application $g$ de $K$ à $T$; on se heurte à des obstructions qui sont des classes de $H^{q+1}(T, K; \pi_q(B))\cong H_q(K; \pi_q(B))$. Or, d'après un résultat classique en $\mathscr{C}$-théorie, l'homomorphisme de Hurewicz $\pi_i(B)\to H_i(B; Z)$ est un $\mathscr{C}$-isomorphisme pour $i<2m$. Si $\pi_q(B)$ est d'ordre fini $m$, toute classe obstruction $w\in H^q(K; \pi_q(B))$ peut être annulée en remplaçant l'application $g$ par sa multiple $m\cdot g$. Si $\pi_q(B)$ n'est pas fini, on lui substitue le groupe $\mathscr{C}$-isomorphe $H_q(B; Z)$; la classe $\bar{w}$, image de $w\in H^q(K; \pi_q(B))$, n'est autre que l'image par $g^*$ de la classe fondamentale Hom $H_q(B); H_q(B))\cong H^q(B; H_q(B; Z))$. Par suite, cette image $\bar{w}$ est nulle mod $\mathscr{C}$, et l'extension est possible après une éventuelle multiplication.

Nous sommes maintenant en mesure de définir les

Classes de Pontrjagin rationnelles. *On attache à toute variété triangulée orientée $V$ (ainsi qu'a toute variété à bord orientée) un système de classes $p_i\in H^{4i}(V; Q)$, univoquement définies par les axiomes suivants:*

(1) $p^0 = 1$.

(2) Théorème de "dualité". *Si $W$ désigne une sous-variété de $V^n$ à structure normale orthogonale (de classes normales $n_i$, de classes "tangentes" $q_j$), et si $i$ désigne l'injection de $W$ dans $V$; on a:*

$$(1)\ i^*(1+p_1+p_2+\cdots p_r) = (1+n_1+n_2+\cdots+n_q)\cup(1+q_1+q_2\cdots q_s).$$

(3) Formule de l'index. *Pour toute variété triangulée $V^{4j}$, on a $\tau(V^{4j}) = l_j(p_i)$, $l_j$ polynome ($L_j$) de Hirzebruch* (cf. [1]).

Pour démontrer que ces axiomes définissent effectivement des classes $p_i$, on procède par induction sur l'indice $i$. Supposons qu'on veuille définir la classe $p_1$. Sur les variétés de dimension 4, la valeur de $p_1$ est donnée par l'Axiome $3 = p_1\in H^4(M^4) = 3\,\tau(M^4)$. On va définir $p_1$ dans les variétés de dimension $>10$; on peut toujours se ramener à ce cas; en vertu de l'Axiome 2, en effet les classes $p_i$ d'une variété $V$ et celles du produit de $V$ par une sphère $S^q$ de grande dimension sont les mêmes. Si la dimension $n$ de la variété $V$ est $>10$, on peut affirmer que l'ensemble

$L_4$ des classes d'homotopie d'applications de $V^n$ dans $M(SO(n-4))$ est muni d'une structure de groupe abélien, l'addition de deux classes correspondant à la réunion des variétés représentatives. Il existe un homomorphisme canonique $h$ de $L_4$ sur $H_4(M; Z)$, à cause du fait, démontré en [5], que toute classe d'homologie de dimension 4 d'une variété peut être réalisée par une sous-variété à structure orthogonale normale. Soit $z \in H_4(M; Z)$; on réalise la classe $z$ par une sous-variété à structure orthogonale normale $W^4$, de nombre de Pontrjagin normal $q_1$. L'axiome 2 donne alors la valeur de $i^*(p_1)$ sur $W$:

$$i^*(p_1) = 3\tau(W) + q_1. \tag{2}$$

Le second membre de (2) est évidemment un invariant de la L-classe de $W$, et définit par suite un homomorphisme $g$ de $L_4$ dans $Z$. Pour démontrer que cet homomorphisme $g$ définit un homomorphisme du groupe $H_4(M; Z)$ dans $Z$ il suffit de montrer que $g$ s'annule sur le noyau $Y^4$ de l'homomorphisme $h$: $L_4 \to H_4(M; Z)$. La cohomologie de $M(SO(n-4))$ comprend: un générateur $U$ en dimension $n-4$, et un générateur $X$ en dimension $n$, en coefficients rationnels. Il résulte du Lemme 2 que $L_4 \otimes Q$ est isomorphe au produit $H^{n-4}(V^n; Q) \otimes H^n(V^n; Q)$; toute classe de $L_4 \otimes Q$ est entièrement déterminée par la donnée des deux classes images $f^*(U)$ et $f^*(X)$. La première de ces classes, $f^*(U)$ est duale de la classe d'homologie de la sous-variété correspondante; pour une classe de $Y^4 \otimes Q$, on a $f^*(U) = 0$; par suite $Y^4 \otimes Q$ est isomorphe à $H^n(V^n; Q)$. Si $V^n$ est connexe, $Y \otimes Q$ n'a qu'un générateur, qu'on peut aisément expliciter. Dans un simplexe de dimension maximum de $V^n$, plongeons le plan projectif complexe $PC(2)$; une telle sous-variété est définie par une application $f$: $V^n \to M(SO(n-4))$, dont voici le type d'homologie: l'application $f$ se factorise en $V^n \underset{h}{\to} S^n \underset{v}{\to} M(SO(n-4))$, où $h$ est de degré 1, et où l'application $v$ a une type d'homologie aisé à calculer. La classe normale $q_1$ de $PC(2)$ plongé dans $S^n$ est donnée par la formule classique de dualité soit $0 = 3\tau + q_1$ et l'on a $v^*(X) = \phi^*(q_1) = -3\ S$ ($S$ classe fondamentale de $H^n(S^n)$); donc $f^*(X) = -3\ V^n$. Sur $PC(2)$, le second membre de (2) donne par suite:

$$3\tau + q_1(PC(2)) = 0.$$

Le second membre de (2) est donc nul sur le noyau $Y \otimes Q$, et la classe $p_1$ est ainsi déterminée (c'est, en ce cas, un élément de Hom $(H_4(V^n), Z)$).

Reste à montrer que les classes $p_1$ ainsi définies vérifient les axiomes 1–2–3. Pour 1 et 3, c'est évident. Pour vérifier l'axiome (2), considérons une sous-variété $P$ de $V^n$ à structure orthogonale normale, et soit $j$ l'injection de $P$ dans $V$, $q_1$ la classe $(p_1)$ normale de $P$ dans $V$. Il faut montrer que, si on désigne par $p^P, p^V$, les classes $p_1$ de $P$ et $V$ resp., on a :

$$j^*(p^V) = p^P + q_1. \tag{3}$$

On prend la valeur des deux membres de (3) sur une sous-variété $Z^4$ de $P$ à structure orthogonale normale (de classe normale $c_1$). On obtient:

$$3\tau(Z) + (c_1 + q_1) = 3\tau(Z) + c_1 + q_1$$

en remarquant que la structure normale de $Z$ dans $V$ est le joint de la structure normale à $Z$ dans $P$ par la restriction à $Z$ de la structure normale de $P$ dans $V$ (Lemme 1).

L'existence des classes $p_1$ étant ainsi établie, on montre par induction sur l'indice $i$ l'existence et l'unicité des classes $p_i$. Supposons donc établie, pour toute variété triangulée, l'existence des classes $p_j$, $j < i$. On définira $p_i$ tout d'abord sur les variétés $V^{4i}$ de dimension $4i$ grâce à la formule de l'index (Axiome 3) $\tau(V^{4i}) = \langle l_j(p_r), V^{4i}\rangle$. Observons, de façon essentielle, que le coefficient (rationnel) de $p_i$ dans le polynôme $L_i$ de Hirzebruch n'est jamais nul (cf. [1]).

On détermine ensuite les classes $p_i$ dans les variétés de grande dimension $n > 4i + 2$; on peut toujours se ramener à ce cas en faisant le produit de la variété par une sphère de dimension assez grande. Comme tout-à-l'heure, on détermine $p_i$ d'une variété $M$ en calculant la valeur prise par $p_i$ sur une sous-variété $X^{4i}$ de dimension $4i$, à structure normale orthogonale; on sait en effet qu'il existe une base de l'homologie rationnelle $H_{4i}(M; Q)$ constituée de sous-variétés à structure normale orthogonale. L'axiome (2) permettra dés lors d'évaluer $\langle i^*(p_i), X^{4i}\rangle$ en fonction de nombres caractéristiques tangents $p_{j_1} \cdots p_{j_2}(X^{4i})$, de nombres normaux $q_{j_1} \cdots q_{j_s}(X^{4i})$, et de nombres mixtes $p_{i_1} \cdots q_{j_1}(X)$. Il résulte immédiatement de cette expression que la valeur prise par $i^*(p_i)$ sur $X^{4i}$ ne dépend que de la $L$-classe de la sous-variété $X^{4i}$, c'est-à-dire de la classe d'applications $g: M \to M(SO(n-4i))$ qui définit $X^{4i}$. De plus, cette expression, définit, par réunion des sous-variétés réalisantes, un homorphisme du groupe $L_{4i}$ des $L$-classes dans le groupe des rationnels $Q$.

Pour démontrer que cet homomorphisme définit une classe de cohomologie $p_i$, il suffit de vérifier qu'il s'annule sur le noyau $Y^{4i}$ de l'homomorphisme canonique $L_{4i} \to H_{4i}(M; Q)$. Avant de vérifier ce fait, il nous faut déterminer $Y^{4i} \otimes Q$.

D'après le Lemme 2, le groupe $L_{4i} \otimes Q$ est isomorphe au groupe Hom $(H^*(M(SO(n-4i)), H^*(M; Q))$. Or $H^*(M(SO(n-4i)))$ est l'image par l'isomorphisme $\phi^*$ (qui augmente la dimension de $n - 4i$) de la cohomologie de la grassmannienne $G_{n-4i}$, i.e. une algèbre de polynomes $S\ (q_1, q_2, \cdots q_i)$ engendrée par les classes de Pontrjagin $q_j \in H^{4j}(G_{n-4i}; Q)$. Un élément de $L_{4i} \otimes Q$ est donc déterminé, dés qu'on s'est donné les images, par l'homomorphisme $f^*$ induit par $f: M \to M(SO\,(n-4i))$, de toutes les classes $\phi^*(q_{j_1} \cdots q_{j_2})$, où $(q_{j_1} \cdots q_{j_2})$ parcourt tous les monomes en $q_j$ de poids total $\leq i$. En particulier, l'image $f^*\phi^*(1)$ donne precisément la classe $x \in H^{n-4i}(M; Q)$, duale, par la dualité de Poincaré, à la sous-variété $X^{4i}$ définie par $f$. Les éléments du noyau $Y^{4i} \otimes Q$ sont donc caractérisés par le fait que $f^*\phi^*(1) = 0$, les $f^*\phi^*(q_{j_1} \cdots q_{j_2})$ pouvant par ailleurs être des classes arbitraires de $H^*(M)$. On va exhiber un système de générateurs pour le noyau $Y^{4i} \otimes Q$; pour tout indice $r \leq i$, on formera un sous-groupe $Y_i^r$ de $L$-classes, tel que, si $q_{j_1}, \cdots, q_{j_2}$ constitue une base des polynomes de poids $r$ (base de $H^{4r}(G)$), on ait:

$$f^*(\phi^*(q_{j_1}, \cdots, q_{j_s}) = y^r_{j_1 j_2 \cdots j_s} \in H^{n-4i+4r}(M; Q)$$

avec
$$f^*(\phi^*(q_{j_1}, \cdots, q_{j_m}) = 0$$

si $$\sum j_m > r$$

et $$f^* (\phi^*(q_{j_1}, \cdots, q_{j_t}) = z_{j_1 \cdots j_t},$$

si $\sum j_s < r$, les $z_j$ étant des classes de $M$ fonctions des $y^r_{j_1 \cdots j_s}$ supposées donnés à l'avance dont nous n'aurons pas à nous préoccuper. Il est clair, dans ces conditions, que tout élément de $Y^{4i} \otimes Q$ est combinaison linéaire d'éléments de $Y^r_i$, où $r$ varie de 1 à $i$. On va réaliser les $Y^r_i$ par des applications $f$ de $M$ dans $M(SO(n-4i))$ d'un type spécial.

## Applications de type zéro

Rappelons le théorème de J.-P. Serre [9]: Etant donnée une classe de cohomologie $y \in H^{n-4i+4r}(M; Q)$ $(n > 8i+2)$, il existe un entier non nul $N$ tel que, pour une application $h$ de $M$ dans la sphère $S^{n-4i+4r}$ de classe fondamentale $s$, on ait $h^*(s) = N \cdot y$. On supposera cette application $h$ régulière sur l'hémisphère Nord $N$ de la sphère $S^{n-4i+4r}$. On se donne alors une sous-variété $W^{4r}$ de $S^{n-4i}$, supposée plongée dans l'hémisphère $N$. L'image réciproque $Z^{4i} = h^{-1}(W^{4r})$ est alors le produit topologique de l'image réciproque $h^{-1}(0) = V^{4i-4r}$ par $W^{4r}$. La projection $r$ de $Z^{4i}$ sur $V^{4i-4r}$ peut être définie par un système de trajectoires orthogonales à la fibration définie par l'application $h$, et, dans la décomposition de Künneth de la cohomologie de $Z^{4i}$, $H^*(Z^{4i}) = H^*(V^{4i-4r}) \otimes H^*(W^{4r})$, on peut écrire, pour toute classe $y$ de $H^*(V^{4i-4r})$, $r^*(y) = y \otimes 1$. Désignons alors par $i$ et $j$ les injections de $V^{4i-4r}$ $Z^{4i}$, dans $M$. L'injection $j$ peut se factoriser en:

$$Z^{4i} \xrightarrow{k} h_j^{-1}(N) \xrightarrow{u} M,$$

et l'image réciproque $h^{-1}(N)$ est homéomorphe au produit $h^{-1}(0) \times N$; l'application $r: Z^{4i} \to V$ se prolonge en une rétraction par déformation $r: h^{-1}(N) \to V^{4i-4r}$. Il en résulte que, pour toute classe $x \in H^*(M)$, on a $j^*(x) = k^*(u^*(x))$, avec $u^*(x) = (r^*)(i^*(x))$. Comme $k^*(r^*) = (r^*)$ il vient $j^*(x) = i^*(x) \otimes 1$.

La sous-variété $W^{4r}$ étant différentiablement plongée dans $S^{n-4i+4r}$, il existe une application $g: S^{n-4i+4r} \to M(SO(n-4i))$, telle que l'image réciproque par $g$ de la grassmannienne $G_{n-4i}$ plongée dans $M(SO(n-4i))$ soit la sous-variété $W^{4r}$. D'où résulte, par composition:

$$M \xrightarrow{h} S^{n-4i+4r} \xrightarrow{g} M(SO(n-4i))$$

une application $f$ de $M$ dans $M(SO(n-4i))$; la $L$-classe associée sera dite $L$-classe de *type zéro*. On va montrer que sur la sous-variété $Z^{4i} = f^{-1}(G_{n-4i})$ associée à une $L$-classe de type zéro, on a

$$\langle j^*(p_i), Z^{4i} \rangle = 0.$$

Désignons par $\sum c_j$ la classe de Pontrjagin tangente de $W^{4r}$. La classe de Pontrjagin du fibré des vecteurs normaux est alors donnée par:

$$\textstyle\sum_j \bar{c}_j = 1 / \sum_j c_j.$$

Désignons par $\sum q_z$ la classe de Pontrjagin de $Z_{4i}$, par $\sum p_i$ celle de $M$. La sous-variété $Z^{4i}$ de $M$ est homologue à zéro dans $M$, car elle est homologue à zéro dans $h^{-1}(N) = N \times V^{4(i-r)}$. Il faudra donc montrer que la valeur de la classe $j^*(p_i)$ sur le cycle fondamental de $Z^{4i}$ est nulle.

Or $j^*(p_i)$ est donnée par l'Axiome (2):

$$\text{(II)} \qquad \sum j^*(p_k) = \sum_j q_j / \sum_m c_m.$$

Pour $k < i$, on sait que les classes $p_k$ existent, et, par suite, $j^*(p_k) = i^*(p_k) \otimes 1$. On aura donc $j^*(p_k) = 0$ pour $i - r < k < i$, (car $i^*(x)$ est une classe de $H^*(V^{4i-4r})$). Après avoir chassé le dénominateur, et supprimé les termes de degré $>4i$, (II) donne:

$$(4) \qquad j^*(p_i) = \sum q_j - (1 + j(p_1) + j(p_2) + \cdots + j(p_{i-r}) \cdot h^* \textstyle\sum_m c_m.$$

La relation (4) est supposée par induction valable pour les termes de degré $<4i$; il suffit donc de montrer que, si l'on porte dans (4) la valeur de $q_i$ donnée par la formule de l'index, alors la composante de degré $4i$ du second membre de (4) s'annule sur le cycle fondamental de $Z^{4i}$. On suppose dans ce but que $W^{4r}$ et $V^{4i-4r}$ sont connexes; ce n'est pas une restriction, car on peut toujours—dans ce calcul—se limiter à une composante connexe de $V^{4i-4r}$, et $W^{4r}$ sera toujours prise connexe.

On supposera nul le premier membre de (4), et on en déduira la valeur de $q_i$; on obtient:

$$\sum q_j = \sum j^*(p_k) \cdot h^* \sum c_m.$$

Cette relation est équivalente à celle obtenue en appliquant le "foncteur multiplicatif" $l$ (au sens de F. Hirzebruch):

$$\sum l_j(q_j) = \sum j^*(l_k(p_k)) \cdot h^* \sum l_m(c_m).$$

En appliquant au cycle fondamental de $Z^{4i} = V^{4(i-r)} \times W^{4r}$, et en remarquant que $j^*(p_j) = i^*(p_j) \otimes 1$, et que les classes $i^*(p_j)$, sont, pour $j \leq i - r$, les classes de Pontrjagin de la variété $V^{4(i-r)}$,

$$\begin{aligned} \langle l_i(q_j), Z^{4i}\rangle &= \langle j^* l_{i-r}(p_j) \cdot h^* l_r(c_m), Z^{4i}\rangle \\ &= \langle l_{i-r}(p_j^V) \cdot h^* l_r(c_m), Z^{4i}\rangle \\ &= \langle l_{i-r}(p_j^V) \cdot V^{4(i-r)}\rangle \langle l_r(c_m)\, W^{4r}\rangle \end{aligned}$$

soit

$$\tau(V^{4(i-r)}) \cdot \tau(W^{4r}) = \tau(Z^{4i}).$$

La valeur de $l(q_j)$ est donc précisément celle que donne la formule de l'index, puisque $Z^{4i}$ est homéomorphe au produit $V^{4(i-r)} \times W^{4r}$. Nous avons donc bien vérifié que $\langle j^*(p_i), Z^{4i}\rangle$ est nul, propriété que nous devions démontrer pour toute $L$-classe de type zéro.

Il reste à vérifier qu'on peut engendrer tout le noyau $Y_i \otimes Q$ avec des $L$-classes de type zéro. On le voit comme suit: Rappelons d'abord que, si l'on associe à tout monôme de poids $r$ $p_{j_1} \cdots p_{j_s}$ un rationnel $m_{j_1 \ldots j_s}$, il existe toujours un entier non nul $N$ tel que les entiers $N \cdot m_{j_1 \ldots j_s}$ soient les nombres caractéristiques (tangents ou normaux) d'une variété $W^{4r}$; on peut réaliser $W^{4r}$, par exemple, comme une somme de produits d'espaces projectifs complexes de dimension (complexe) paire (cf. [1]). Cela étant, supposons qu'on veuille construire explicitement une $L$-classe de $M$ vérifiant les relations: $g^*\phi^*(p_{j_1} \cdots p_{j_s}) = m_{j_1 \ldots j_s} \cdot y$, où les images $g^*\phi^*(p_{j_1} \cdots p_{j_s})$ sont toutes multiples d'une même classe $y \in H^{n-4i+4r}$ de $M$. On réalisera la classe $y$ (ou une classe multiple $N_1 \cdot y$) comme image de la classe fondamentale d'une sphère $S^{n-4i+4r}$, puis on plongera dans $S^{n-4i+4r}$ une sous-variété $W^{4r}$ dont les nombres caractéristiques normaux $p_{j_1} \cdots p_{j_s}$ sont proportionnels aux rationnels $m_{j_1 \ldots j_s}$. On aura ainsi défini une $L$-classe répondant—à un facteur entier près—aux conditions demandées. Cette $L$-classe est de type zéro, et il est clair que tout élément de $Y_i^r$ est combinaison linéaire à coefficients rationnels de classes de cette forme. Nous établissons ainsi que les classes de type zéro engendrent tout le noyau $Y^{4i}$. Ceci démontre donc l'existence des classes $p_i$. Il reste à démontrer que ces classes $p_i$ satisfont aux Axiomes 1, 2, 3; la seule démonstration non triviale est relative à l'Axiome 2; elle est entiérement analoque à celle donnée pour la classe $p_1$, aussi nous ne la répéterons pas ici.

Nous avons explicité la démonstration précédente dans le seul cas, où $M$ est une variété compacte; si $M$ est une variété à bord, de bord $V$, on doit remarquer que la cohomologie $H^{4i}(M)$ et l'homologie $H_{4i}(M)$ (homologie singulière des chaines finies) sont des espaces vectoriels duaux sur les rationnels. Il suffit donc de réaliser les classes de $H_{4i}(M; Q)$ par des sous-variétés à structure normale orthogonale; on est ainsi ramené à étendre la théorie de la réalisation des classes au cas des variétes à bord, ce qui ne fait aucune difficulté: les $L$-classes d'une variété à bord $M$, de bord $V$, correspondent biunivoquement aux classes d'homotopie des applications de $M$ dans le complexe $M(SO(n-4i))$ qui envoient le bord $V$ sur le point "$a$" compactifiant de $M(SO(n-4i))$.

Il résulte de la démonstration précédente que les classes $p_i$ sont déterminées exclusivement et univoquement à partir de la fonction $\tau$ associée à toute $L$-classe. Il en résulte que les classes $p_i$ (en coefficients rationnels) de deux variétés $J$-équivalentes $V$ et $V'$ se correspondent par la $J$-équivalence. En particulier, si $V$ et $V'$ sont deux complexes simpliciaux isomorphes, leurs classes $p_i$ se correspondent par cet isomorphisme (invariance combinatoire des classes de Pontrjagin).

Soit $V$ une variété (topologique) triangulée; elle admet de ce fait des classes $p_i$; si $V$ admet une structure différentiable ($S$) pour laquelle la triangulation donnée est différentiable (i.e., les simplexes sont des sous-variétés différentiablement plongées), alors les classes de Pontrjagin de la structure (S) sont les classes $p_i$ de la structure "différentiable par morceaux" associée à la triangulation; ces classes sont alors des classes entières. Nous obtenons donc ainsi:

THÉORÈME. *Pour qu'une variété $V$, différentiable par morceaux, puisse être munie d'une structure différentiable globale qui induise la structure diff. par morceaux donnée, il faut que les classes $p_i$ associées soient des classes entières.*

## Relations avec la Hauptvermutung

Soient $K$, $K'$ deux complexes simpliciaux, $f: K \to K'$ un homéomorphisme de $K$ sur $K'$. On peut alors donner à la Hauptvermutung de la topologie les deux formes suivantes:

*Forme faible.* Les complexes $K$ et $K'$ présentent des subdivisions simpliciales isomorphes.

*Forme forte.* Les complexes $K$ et $K'$ présentent des subdivisions simpliciales isomorphes, et l'homéomorphisme $g$ défini par cet isomorphisme est arbitrairement voisin de $f$.

Si l'on admet la forme forte de la Hauptvermutung, il en résulte que deux variétés $V$, $V'$ différentiables et homéomorphes admettent des subdivisions simpliciales diff plongées isomorphes, et l'isomorphisme entre polyèdres est arbitrairement voisin de l'homéomorphisme donné; dans ces conditions, les classes de Pontrjagin de $V$ et $V'$ se correspondent par cet homéomorphisme. En admettant donc la Hauptvermutung sous sa forme forte, on en conclut que les classes de Pontrjagin, prises à coefficients rationnels, sont des invariants topologiques de la variété.

Si on admet seulement la forme faible de la Hauptvermutung, on peut seulement affirmer l'invariance des classes $p_i \in H^{4i}(V; Q)$ modulo un automorphisme de la variété $V$.

Soit $V$ une variété polyèdrale; et soient $p_i \in H^{4i}(V; Q)$ ses classes de Pontrjagin; supposons que $V$ admette par ailleurs une structure différentiable, même sans aucun rapport avec la triangulation initiale. La Hauptvermutung (forte ou faible) permet alors d'affirmer que les classes de Pontrjagin (rationnelles) de ces deux structures se correspondent; donc les classes $p_i$ de la structure triangulée sont des classes entières.

Exemples. Désignons par $B(h, p)$ le fibré de base $S^4$, fibre $S^3$ admettant pour invariant de Hopf $h$, pour nombre de Pontrjagin $p_1(S^4) = p$. Si $h = 0$, le fibré admet une section, et $B(0, p)$ a même cohomologie que le produit $S^4 \times S^3$; la classe $p_1$ de cette variété est alors $p \cdot S^4$; on peut par suite affirmer que les fibrés $B(0, p)$ et $B(0, p')$, où $p$ diffère de $p'$ sont combinatoirement distincts.

J. Milnor a montré récemment [2] que les fibrés $B(1, p)$ ($p = 1 \bmod 2$) sont tous isomorphes à la sphère $S^7$, l'homéomorphisme ne pouvant toutefois pas, dans certains cas, être rendu différentiable. En ajoutant au "mapping cylinder" du fibré $B(1, p)$ une boule de dimension 8 dont le bord $S^7$ s'identifie à $B(1, p)$, J. Milnor obtient une variété triangulée $M^8$ dont la seconde classe de Pontrjagin $p_2$ n'est pas, en général, une classe entière. Si l'on admet la Hauptvermutung, cette variété ne peut par suite être munie d'aucune structure différentiable globale. On voit ainsi que le conjecture—communément admise—: "Toute variété polyèdrale peut être munie d'une structure différentiable globale" est contradictoire avec la Hauptvermutung, même sous forme faible.

Généralisation aux fibrés. Soit $E$ un polyèdre, $p$ une application de $E$ sur un polyèdre $B$; on suppose l'application $p$ homologiquement localement triviale, la fibre $p^{-1}(x)$ ayant la cohomologie de l'espace euclidien $R^K$ (l'image réciproque

$p^{-1}(U)$ de tout ouvert $U$ assez petit a même cohomologie que le produit $U \times R^K$). Supposons $B$ plongée dans une variété à bord $Y$ dont $B$ est rétracte par déformation; alors la rétraction $r : Y \to B$ définit un fibré induit de $E$, $Q$ sur $Y$; $Q$ est également une variété à bord contenant $Y$ comme sous-variété; on a dans $Q$ et $Y$ des classes de Pontrjagin; la formule de "dualité" de l'Axiome 2 permet alors de définir dans la cohomologie de $Y$ des classes de Pontrjagin normales $q_j$. On peut voir aisément que les restrictions à $H^*(B)$ de ces classes $q_j$ sont indépendantes de la variété à bord $Y$ choisie; on pourrait prendre en particulier un voisinage de $B$ pour un plonguement rectilinéaire de $B$ dans un espace euclidien de dimension assez grande. Les classes $q_j$ ainsi définies (à coefficients rationnels) sont des invariants combinatoires de l'application fibrée $p : E \to B$. Si cette fibration admet $SO(k)$ pour groupe de structure, on retrouve les classes de Pontrjagin de la fibration au sens usuel.

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# A SPECTRAL RESOLUTION OF COMPLEX STRUCTURE

D. C. SPENCER

In order to motivate the main result of this paper, we consider the simple special case of complex euclidean $m$-space $\mathbf{C}^m$ with coordinates $z_1, \cdots, z_m$ and with the usual euclidean hermitian metric (which is kählerian). A harmonic of degree $n$ is a homogeneous polynomial of degree $n$ in the coordinates $z_1, \cdots, z_m$ and their conjugates $\bar{z}_1, \cdots, \bar{z}_m$ which is annihilated by the laplacian $\Delta = 4 \sum_{\alpha=1}^{m} \partial^2/\partial z_\alpha \partial \bar{z}_\alpha$. We shall say that a harmonic of degree $n$ is of type $(p, q)$, $p + q = n$, if it is homogeneous of degree $p$ in the coordinates $z_\alpha$, homogeneous of degree $q$ in the $\bar{z}_\alpha$. Given an arbitrary homogeneous polynomial $u$ of degree $n$, we have the unique decomposition $u = \sum_{p+q=n} u_{pq}$ where $u_{pq}$ is homogeneous of degree $p$ in the $z_\alpha$, homogeneous of degree $q$ in the $\bar{z}_\alpha$. Since $\Delta u_{pq}$ is homogeneous of degree $p - 1$ in the $z_\alpha$, homogeneous of degree $q - 1$ in the $\bar{z}_\alpha$, we conclude from the independence of the $\Delta u_{pq}$ that $\Delta u = 0$ if and only if $\Delta u_{pq} = 0$ for each $u_{pq}$. Hence the complex vector space $\mathbf{H}^n$ of harmonics of degree $n$ has the direct-sum decomposition $\mathbf{H}^n = \sum_{p+q=n} \mathbf{H}^{p,q}$ where $\mathbf{H}^{p,q}$ is the subspace of harmonics of type $(p, q)$. We remark that this decomposition of harmonics is analogous to the decomposition of harmonic differential forms according to type, but differs from the latter in that the integers $p, q$ are not restricted to be less than or equal to the complex dimension $m$.

If we consider the complex euclidean unit ball $V = \{z | \sum_{\alpha=1}^{m} |z_\alpha|^2 < 1\}$, the harmonics of type $(p, q)$ may be characterized by the property that their boundary values are eigen-functions belonging to the eigen-value $\frac{1}{2}q/(p + q + m - 1)^2$ of a certain operator acting on the Hilbert space of norm-finite functions in the boundary of $V$ where the scalar product is defined in terms of the metric induced in the boundary. The vanishing eigen-values characterize holomorphy since the harmonics with $q = 0$ are precisely the holomorphic ones.

The purpose of this paper is to generalize the preceding considerations to an arbitrary compact submanifold $V$ of an almost-complex manifold $X$ with hermitian metric whose almost-complex structure is integrable in a neighbourhood of the boundary of $V$. It is necessary to assume that the boundary of $V$ is smooth, more precisely that the boundary of $V$ is a differentiable submanifold of $X$ which is regularly imbedded. We establish the existence of a basis for a class of sufficiently smooth harmonic functions in $V$ which is bigraded according to type in the same manner as the harmonic polynomials of complex euclidean space. The functions of the basis are determined by a generalization of the eigen-value property described above, and the vanishing eigen-values characterize holomorphy. Thus we obtain generalized harmonics of type $(p, q)$ which we call Stekloff functions since they generalize classical real eigen-functions studied by Stekloff, a fact which was called to the author's attention by M. Schiffer.

For the sake of completeness we begin in Section 1 with a general discussion of structure, in particular almost-complex structure, along lines formulated by H. K. Nickerson and the author. We assume the definitions of local category and faithful functor as given by H. Cartan and S. Eilenberg in another part of this volume. This section (Section 1) is independent of the remainder and may be omitted by the reader.

## 1. Structure on a space, almost-complex structure

Let $\mathcal{M}$ be a subcategory of the category of topological spaces and continuous maps which satisfies the conditions that any open subset of an object of $\mathcal{M}$ is an object of $\mathcal{M}$ and that the restriction of any map of $\mathcal{M}$ to an open subset of its domain is a map of $\mathcal{M}$. The maps of $\mathcal{M}$ will be called regular maps. We remark that $\mathcal{M}$ is a local category in the sense of Cartan and Eilenberg.

Let $X$ be a topological space. For each open $U \subset X$ let $\mathcal{M}(U)$ be the subcategory of $\mathcal{M}$ consisting of objects homeomorphic to $U$ together with the maps connecting objects of $\mathcal{M}(U)$. Let

$$S(U) = \{\varphi | \varphi \text{ a homeomorphism from an object of } \mathcal{M}(U) \text{ onto } U\}.$$

If $V \subset U$, we have $S(U) \to S(V)$ where $\varphi \in S(U)$ goes into $(\varphi^{-1}|V)^{-1}$. Also let

$$G(U) = \{(\varphi, \psi) | \varphi, \psi \in S(U), \quad g = \varphi^{-1} \circ \psi, g, g^{-1} \in \mathcal{M}(U)\}.$$

Then we have the restriction map $G(U) \to G(V)$ where $(\varphi, \psi)$ goes into $((\varphi^{-1}|V)^{-1}, (\psi^{-1}|V)^{-1})$. An element of $G(U)$ is an identity if it is of the form $(\varphi, \varphi)$, $\varphi \in S(U)$. The product $(\varphi_1, \psi_1) \circ (\varphi_2, \psi_2)$ is defined in $G(U)$ if and only if $\psi_1 = \varphi_2$ in which case $(\varphi_1, \psi_1) \circ (\varphi_2, \psi_2) = (\varphi_1, \psi_2)$. It is clear that the product commutes with restriction so the restriction is a homomorphism in the sense of groupoids. Finally we denote a general element of $G(U)$ by $g$.

The assignments $U \to G(U)$, $U \to S(U)$ define preshaves on $X$ inducing sheaves which will be denoted by $G$, $S$ respectively.

If $g \in G(U)$, $g = (\varphi, \psi)$, $\varphi, \psi \in S(U)$, we shall say that $\varphi$ and $\psi$ are equivalent, and we shall denote the equivalence classes of $S(U)$ modulo $G(U)$ by $S(U)/G(U)$. Then $U \to S(U)/G(U)$ is a presheaf inducing a sheaf on $X$ which will be denoted by $S/G$. An $\mathcal{M}$-structure on $X$ is an element of $H^0(X, S/G)$.

A 1-cochain with values in $G$ defined on the nerve of an open covering $\mathfrak{U} = \{U_i\}$ of $X$ is an assignment $U_{ij} \to g_{ij}$ where $g_{ij} \in G(U_{ij})$, $U_{ij} = U_i \cap U_j$, and (i) $g_{ji} = g_{ji}^{-1}$; (ii) $g_{ij|k} \circ g_{jk|i}$ is defined where $g_{ij|k}$ denotes the image of $g_{ij}$ under $G(U_{ij}) \to G(U_{ijk})$, $U_{ijk} = U_i \cap U_j \cap U_k$. The coboundary $\delta : \{g_{ij}\} \to \{g_{ijk}\}$ is defined by $g_{ijk} = g_{ij|k} \circ g_{jk|i} \circ g_{ki|j}$, $g_{ijk} \in G(U_{ijk})$, and a 1-cochain $\{g_{ij}\}$ is a cocycle if $g_{ijk} = 1_{ijk}$ where $1_{ijk}$ is an identity of $G(U_{ijk})$. Two 1-cocycles $\{g_{ij}\}$, $\{\hat{g}_{ij}\}$ are cohomologous if $\hat{g}_{ij} = g_{i|j} \circ g_{ij} \circ g_{j|i}^{-1}$ where $g_i \in G(U_i)$. Then $H^1(X, G)$ is defined as the direct limit of the $H^1(\mathfrak{U}, G) : H^1(X, G) = \varinjlim H^1(\mathfrak{U}, G)$.

The coboundary map $\delta^* : H^0(X, S/G) \to H^1(X, G)$ is defined as follows: let $c \in H^0(X, S/G)$ be represented by $\{U_i, \varphi_i, M_i\}$, $\varphi_i : M_i \to U_i$, $\varphi_i \in S(U_i)$, $M_i \in \mathcal{M}(U_i)$; then $\delta^* c \in H^1(X, G)$ is represented by $g_{ij} = (\varphi_{i|j}, \varphi_{j|i})$ where $\varphi_{i|j} = (\varphi_i^{-1}|U_{ij})^{-1}$.

Let $c, \hat{c} \in H^0(X, S/G)$ be two $\mathscr{M}$-structures. Then $c, \hat{c}$ are said to be equivalent if there exists a homeomorphism $f : X \to X$ and representations $\{U_i, \varphi_i, M_i\}$, $\{V_i, \psi_i, \hat{N}_i\}$ of $c, \hat{c}$ respectively such that $g_i = \psi_i^{-1} \circ f_i \circ \varphi_i \in \mathscr{M}(U_i)$ where $f_i = f|U_i$.

Given an arbitrary homeomorphism (continuous) $f : X \to X$, then

$$f^*: H^0(X, S/G) \to H^0(X, S/G)$$

(transport of structure) is bijective. Similarly $f^*: H^1(X, G) \to H^1(X, G)$ is bijective.

Let $F$ be the group of homeomorphisms (continuous) of $X$ onto $X$; then $F$ operates on $H^0(X, S/G)$, $H^1(X, G)$ and we say that two elements of either set are equivalent if one is transformed into the other by $f^*$, $f \in F$. We denote by $H^0(X, S/G)/F$, $H^1(X, G)/F$ respectively the sets of equivalence classes of $H^0(X, S/G), H^1(X, G)$. Since it is clear that $\delta^*(f^* c^0) = f^*(\delta^* c^0)$, $c^0 \in H^0(X, S/G)$, there is an induced map $\eta^*: H^0(X, S/G)/F \to H^1(X, G)/F$ such that the following diagram is commutative:

$$\begin{array}{ccc} H^0(X, S/G) & \longrightarrow & H^0(X, S/G)/F \\ \downarrow \delta^* & & \downarrow \eta^* \\ H^1(X, G) & \longrightarrow & H^1(X, G)/F \end{array}$$

THEOREM 1.1. *The maps $\delta^*$ and $\eta^*$ are injective.*

The proof is straightforward and will be omitted.

If $\mathscr{N}$ is a local category whose objects (models) are the same as those of $\mathscr{M}$ but whose maps define a sheaf of groupoids $G_{\mathscr{N}}$ containing $G_{\mathscr{M}}$, there is a faithful functor $T : \mathscr{M} \to \mathscr{N}$ which induces maps such that the following diagram is commutative:

$$\begin{array}{ccc} H^0(X, (S/G)_{\mathscr{M}}) & \longrightarrow & H^1(X, G_{\mathscr{M}}) \\ \downarrow & & \downarrow \\ H^0(X, (S/G)_{\mathscr{N}}) & \longrightarrow & H^1(X, G_{\mathscr{N}}). \end{array}$$

The local category $\mathscr{M}$ for almost-complex structure may be defined as follows. The objects of $\mathscr{M}$ are pairs $(M, s)$ where $M$ is a subdomain of real euclidean $2m$-space $\mathbf{R}^{2m}$ and where $s : M \to P \times_H Q$ is a differentiable cross-section of the bundle $P \times_H Q$ associated to the principal tangent bundle $P$ of $M$, $H = GL(2m, \mathbf{R})$ (group of $P$), $Q = GL(2m, \mathbf{R})/GL(m, \mathbf{C})$. By differentiable we shall always mean differentiable of class $C^\infty$. The map $s$ defines a (differentiable) reduction of the structure group $GL(2m, \mathbf{R})$ to $GL(m, \mathbf{C})$ (regarded as a real subgroup of $GL(2m, \mathbf{R})$), and this reduction in turn defines a direct-sum decomposition of the complexified tangent bundle $CT$ of $M$, $CT = T \otimes_{\mathbf{R}} C$, $C$ the trivial bundle $M \times \mathbf{C}$, namely: $CT = P(CT) \oplus Q(CT)$ where $Q(CT)$ is isomorphic to $P(CT)$ under conjugation, $Q(CT) = \overline{P(CT)}$. This splitting induces a direct-sum decomposition of the complex vector space of differential forms on $M$ according to type; projection onto the space of forms of type $(p, q)$ will be denoted by $\prod_{p,q}$. The exterior differential $d$ operating on the differential forms splits: $d = \partial + \bar{\partial}$ where $\partial$ is the anti-derivation of degree 1 of the exterior algebra of differential forms which is characterized by the following two properties: (i) for a function $f$, $\partial f = P^*(df) = \prod_{1,0}(df)$; (ii) $\partial d + d\partial = 0$. The conjugate $\bar\partial$ of $\partial$ is defined similarly with $Q^* = \bar{P}^* = \prod_{0,1}$ replacing $P^* = \prod_{1,0}$. The maps of the local category $\mathscr{M}$ are those differentiable maps $f$ whose induced homomorphisms $f^*$ of the exterior algebra of differential forms commute with $\bar\partial$. An element $c \in H^0(X, (S/G)_{\mathscr{M}})$ defines an almost-complex structure on $X$ and, if $X$ is paracompact (which includes Hausdorff) we shall say that $X$ (with structure $c$) is an almost-complex manifold of complex dimension $m$. If $\mathscr{N}$ is the local category for differentiable structure whose objects are subdomains of $\mathbf{R}^{2m}$ and whose maps are all differentiable maps, there is a faithful functor $T: \mathscr{M} \to \mathscr{N}$ which induces on $X$ the structure of differentiable manifold and it is well-known (and easy to prove) that $X$ with the induced structure of differentiable manifold is orientable and has a natural orientation. The compatible collation of the structure of the model objects of $\mathscr{M}$ induces a decomposition of the space of differential forms on $X$; projection onto the subspace of forms of type $(p, q)$ and the splitting of the operator $d$ will be denoted by the same letters as for the model objects. A function which is annihilated by $\bar\partial$ will be called holomorphic; the holomorphic functions obviously form a ring. Finally let $J = \sqrt{-1}(\prod_{1,0} - \prod_{0,1})$; then $J$ is an operator on the space of 1-forms satisfying $J^2 = -1$.

On the other hand, given a differentiable manifold ($X$ with an $\mathscr{N}$-structure), an almost-complex structure (if it exists) is defined by a differentiable cross-section $s: X \to P \times_H Q$ of the bundle $P \times_H Q$ associated to the principal tangent bundle $P$ of $X$ where $H$ and $Q$ are as defined above. Two almost-complex structures are homotopic if the corresponding maps $s_0, s_1: X \to P \times_H Q$ are homotopic in the usual sense, that is if there exists a differentiable map $s: X \times I \to P \times_H Q$, $I = \{t|0 \leqq t \leqq 1\}$ which satisfies $s|X \times 0 = s_0$, $s|X \times 1 = s_1$. We denote by $c_t \in H^0(X, S/G)$ the structure defined by $s_t = s|X \times t$.

An almost-complex structure is integrable if and only if $\bar\partial^2 = 0$ (which is equivalent to the condition $\partial^2 = 0$).

If $X$ has an almost-complex structure, the sheaf $S/G$ induced by the presheaf $U \to H^0(U, (S/G))$ is the sheaf of germs of almost-complex structures. A germ of $(S/G)_x$, $x \in X$, is represented by an almost-complex neighborhood $U$ whose structure, if $U$ is a cell, is defined by a differentiable map $f: U \to Q = GL(2m, \mathbf{R})/GL(m, \mathbf{C})$. Two germs of $(S/G)_x$ will be called equivalent if they are represented by almost-complex neighborhoods $U, V$ of $X$ whose structures, defined by $(U, \varphi, M)$, $(V, \psi, N)$, are connected by a homeomorphism $f: U \to V$ carrying $x$ into $x$ such that $g = \varphi^{-1} \circ f \circ \psi$ is a map of $\mathscr{M}$.

Denote by $(S/G)_i$ the subsheaf of $S/G$ composed of germs of integrable almost-complex structures on $X$. A germ of $(S/G)_x$ will be called kählerian if it is represented by a kählerian almost-complex neighborhood of $x$.

THEOREM 1.2 (E. Calabi). *$((S/G)_i)_x$ is the subset of $(S/G)_x$ composed of kählerian germs.*

The construction of a local Kähler metric is achieved by choosing a real differentiable function $\varphi$ in a neighborhood of $x$ which vanishes at $x$ together with its gradient $d\varphi$ and which satisfies the further condition that its matrix of second partial derivatives is positive-definite at $x$. Let $\omega = -\frac{1}{2}\prod_{1,1} dJd\varphi$; then $\omega$ is the fundamental 2-form associated with a positive-definite hermitian metric in a neighborhood of $x$ and it can be verified that $\omega = -\frac{1}{2}\, dJd\varphi + \tau(d\varphi)$ where $\tau$ is the torsion tensor of the almost-complex structure: $\tau = 0$ if and only if $\bar{\partial}^2 = 0$. Thus $d\omega = 0$, which is equivalent to the metric being kählerian, if and only if the structure is integrable.

## 2. Spectral resolution of the structure

A finite sub-manifold $V$ of a differentiable manifold $X$ is a subdomain of $X$ whose closure is compact and whose boundary is a differentiable submanifold of $X$ which is regularly imbedded. It follows, in particular, that the boundary of a finite submanifold $V$ consists of finitely many connected components.

Now let $V$ be a finite submanifold of an hermitian almost-complex manifold $X$ of complex dimension $m$ with the induced almost-complex structure and metric and assume that the structure is integrable in some neighborhood in $X$ of the boundary of $V$. We shall denote the boundary of $V$ by $bV$; $bV$ is a real differentiable manifold of dimension $2m - 1$ with a riemannian metric induced from the hermitian metric of $X$.

The hermitian metric of $X$ induces a splitting of the tangent bundle $T$ of $X$ restricted to a sufficiently small neighborhood $U$ in $X$ of $bV$, namely $T = T_t \oplus T_n$, where $T_n$ is the bundle of tangent vectors along the geodesics orthogonal to $bV$ and where $T_t$ is the bundle of tangent vectors along the $(2m - 1)$-dimensional hypersurfaces orthogonal to the geodesics. This splitting defines a local-product structure on $U$, and we say that a 1-form in $U$ is of type $(1, 0)$ if it vanishes on the tangents to the geodesics, of type $(0, 1)$ *if it annihilates the tangent spaces of the hypersurfaces* orthogonal to the geodesics. A differential form of degree $p$ in $U$ then splits uniquely into the sum of a form of type $(p, 0)$ (its tangential component) and a form of type $(p - 1, 1)$ (its normal component). Given a differential form $\varphi$ in $V$ we shall denote by $t\varphi$, $n\varphi$ respectively the restrictions to $bV$ of its tangential and normal components.

Denote by $\mathbf{H}$ the Hilbert space of norm-finite functions in $bV$, by $\mathbf{H}_V$ the Hilbert space of norm-finite functions in $V$, and by $\mathbf{P}_V$ the subspace of $\mathbf{H}_V$ composed of harmonic functions in $V$ which belong to the closure of the domain of the operator $d$. Let $W$ be a finite submanifold of $X$ which contains the adherence of $V$ in its interior, and let $G_W$ be the Green's operator (see [7]) of $W: (G_W\, \varphi)(y) = (\varphi(x), g(x, y))_W$ for any function $\varphi$ in the Hilbert space $\mathbf{H}_W$ of norm-finite functions in $W$ where $(\varphi, \psi)_W$ denotes the scalar product on $W$. Introduce the operator

$$(2.1) \qquad (G\,\varphi)(y) = \int_{bV} \overline{g(x, y)}\, \varphi\,(x) dS_x, \quad \varphi \in \mathbf{H},$$

where $dS_x$ is the volume element of $bV$. Then $G$ may be regarded either as a map $G: \mathbf{H} \rightarrow \mathbf{P}_V$ or as a symmetric completely continuous transformation $G: \mathbf{H} \rightarrow \mathbf{H}$.

LEMMA 2.1. *The map* $G: \mathbf{H} \rightarrow \mathbf{H}$ *is injective* (*kernel zero*).

PROOF. Suppose that $G\varphi = 0$ in $\mathbf{H}$, $\varphi \in \mathbf{H}$. Then $G\varphi = 0$ in $\mathbf{P}_V$ and in $\mathbf{P}_{W-V}$ since $G\varphi$ defines harmonic functions in $V$, $W-V$ each of which has vanishing boundary values (the boundary values of $G$ on $bW$ vanish by definition of the operator $G_W$). On the other hand, $n\bar{\partial}G\varphi$ decreases by $\varphi$ as the boundary of $V$ is crossed from $V$ into $W-V$ (Lemma 2.3.8 of [5]); since $n\,\bar{\partial}G\varphi$ has vanishing boundary values on either side of $bV$, we conclude that $\varphi = 0$ in $\mathbf{H}$. We remark that the hypothesis of the integrability of the structure near the boundary enters into the proof of Lemma 2.3.8 of [5].

Let $T : \mathbf{H} \to \mathbf{H}$ be defined as $T = n\bar{\partial}G$; then $T$ is a singular operator which has been investigated in detail in the paper [5]. We denote the adjoint operator of $T$ by $T^*$. It was shown in [5] that the addition of bounded operators to $T$, $T^*$ makes both operators regularizable in the sense of Giraud-Mihlin ([4, 6]) and this result implies the following theorem:

THEOREM 2.1. *The operators $T$, $T^*$ are bounded.*

Next we show:

LEMMA 2.2. *The operator $G \circ T = G \circ n\bar{\partial}G$ is self-adjoint in $\mathbf{H}$; that is $G \circ T = T^* \circ G$.*

PROOF. We apply Green's formula to $G\varphi \in \mathbf{P}_V$, $\varphi \in \mathbf{H}$, and we obtain

$$G\varphi + \int_{bV} \overline{(n\bar{\partial}g)} \cdot G\varphi \cdot dS_x = G(n\,\bar{\partial}\,G\,\varphi).$$

By passage to $bV$:

$$G(n\,\bar{\partial}\,G\,\varphi) \to (G \circ T)\varphi, \qquad G\varphi + \int_{bV} \overline{(n\bar{\partial}g)} \cdot G\varphi \cdot dS_x \to (T^* \circ G)\varphi.$$

Thus $G \circ T = T^* \circ G$ as stated.

Let $B : \mathbf{H} \to \mathbf{H}$ be defined by

$$B = G \circ T = T^* \circ G. \tag{2.2}$$

Since $G$ is completely continuous and $T$ bounded (Theorem 2.1), we see that $B$ is a completely continuous, self-adjoint transformation, $B : \mathbf{H} \to \mathbf{H}$, $B^* = B$.

Now let $\{\lambda_i\}$, $\{\beta_i\}$ be the eigen-values and eigen-functions defined by

$$B\beta_i = \lambda_i\beta_i \tag{2.3}$$

where $\lambda_i$ tends to zero as $i$ approaches infinity. We write

$$\mathbf{B} = \{\beta_i \mid B\beta_i = \lambda_i\beta_i\} \tag{2.4}$$

and adjoin the space

$$\mathbf{A} = \{\alpha_i \mid \alpha_i \in \mathbf{H},\ B\alpha_i = 0\} \tag{2.5}$$

to obtain a complete orthonormal base $\{\beta_i, \alpha_i\}$ for $\mathbf{H}$, namely:

$$\mathbf{H} = \mathbf{B} \oplus \mathbf{A}. \tag{2.6}$$

Finally, let $\mathbf{S} = G(\mathbf{H}) \subset \mathbf{P}_V$,

$$\mathbf{S}_1 = G(\mathbf{B}) \subset \mathbf{P}_V, \qquad \mathbf{S}_0 = G(\mathbf{A}) \subset \mathbf{P}_V. \tag{2.7}$$

We call $\mathbf{S} = \mathbf{S}_1 \oplus \mathbf{S}_0$ the set of Stekloff functions, and $\mathbf{S}_0$ is the subset of $\mathbf{S}$ consisting of those functions which are holomorphic in $V$.

LEMMA 2.3. *The eigen-values $\lambda_i$ occurring in* (2.3) *are positive numbers.*

PROOF. We distinguish scalar products over $V$ by affixing a subscript $V$; scalar products in $bV$ will carry no subscripts. Then if $\gamma_i \in \mathbf{S}_1$, $\gamma_i = G\beta_i \in \mathbf{P}_V$, $\beta_i \in \mathbf{B}$, we have by Green's formula:

$$(\bar\partial\gamma_i, \bar\partial\gamma_i)_V = \int_{bV} \gamma_i \cdot \overline{(n\,\bar\partial\,\gamma_i)} \cdot dS_x = (G\beta_i, n\,\bar\partial\, G\beta_i)$$
$$= (\beta_i, Gn\,\bar\partial\, G\beta_i) = \bar\lambda_i(\beta_i, \beta_i) = \bar\lambda_i.$$

Thus $\bar\lambda_i > 0$ which shows that $\lambda_i > 0$ as stated.

We assign to each Stekloff function of $\mathbf{S}_1$ its corresponding eigen-value $\lambda$ where, if $\gamma = G\beta$, $\beta \in B$, $\lambda$ is defined by (2.3) and we assign to each function of $\mathbf{S}_0$ the eigen-value $\lambda = 0$. The non-negative numbers $\lambda$ will be called Stekloff numbers; their vanishing characterizes holomorphy in a Stekloff function: a Stekloff function is holomorphic if and only if its Stekloff number is zero. In the case of the complex euclidean ball $V = \{z \mid \sum |z_\alpha|^2 < 1\}$, we may choose $W = \mathbf{C}^m$, $g(z, \zeta) = 1/|z - \zeta|^{2m-2}$ where $|z - \zeta|$ is the ordinary euclidean distance of the points $z$, $\zeta \in \mathbf{C}^m$, and we may verify that a harmonic of type $(p, q)$ has the Stekloff number $\lambda_{pq} = \frac{1}{2}q/(p + q + m - 1)^2$. The bigradation of the Stekloff functions in the general case will be carried through in §3.

Now let

$$\mathbf{O} = \{\varphi | \varphi \in \mathbf{H},\ T^* \varphi = 0\}. \tag{2.8}$$

Thus $\mathbf{O}$ spans the subspace of $\mathbf{H}$ which corresponds to boundary values of holomorphic functions in $V$ (see [5]). The following result is obvious:

THEOREM 2.2. *We have the orthogonal decomposition*

$$\mathbf{H} = [T(\mathbf{H})] \oplus \mathbf{O}, \tag{2.9}$$

*where $[T(\mathbf{H})]$ denotes the closure in $\mathbf{H}$ of the space of functions $T(\varphi)$, $\varphi \in \mathbf{H}$.*

## 3. Bigradation of the Stekloff functions

We continue to assume that $V$ is a finite submanifold of an hermitian almost-complex manifold $X$ of complex dimension $m$ whose structure is integrable in a neighborhood of the boundary of $V$.

Let $\tau$ be a positive number and define

$$d_\tau = \tau d + \bar\partial. \tag{3.1}$$

All the considerations of §2 remain valid with $d_\tau$ replacing $\bar\partial$. We shall distinguish the operators of §2 defined in terms of $d_\tau$ by affixing a subscript $\tau$.

The operator $T_\tau : \mathbf{H} \to \mathbf{H}$ is defined to be $T = nd_\tau G_\tau/(2\tau + 1)$, and we denote the adjoint operator of $T_\tau$ by $T^*_\tau$. The operators $T_\tau$, $T^*_\tau$ have been investigated in detail in the paper [5] where it is shown that, for $\tau > 0$, these operators are regularizable in the sense of Giraud-Mihlin ([4, 6]). It is important to remark that, for $\tau > 0$, the space $\mathbf{S}_{0\tau}$ is isomorphic to $\mathbf{C}$ (complex numbers) and that $\mathbf{O}_\tau$, the space spanned by the boundary values of holomorphic functions, is also isomorphic to $\mathbf{C}$. We consider now the Stekloff numbers $\lambda_\tau$, $\lambda_\tau \geqq 0$, associated with the functions of $\mathbf{S}_\tau$. As $\tau$ tends to zero, each $\lambda_\tau$ converges to a Stekloff number of $\mathbf{S}$ and the vector subspace of $\mathbf{S}_\tau$ corresponding to $\lambda_\tau$ converges to a subspace of the space spanned by the functions of $\mathbf{S}$ belonging to $\lambda = \lim_{\tau\to 0} \lambda_\tau$. If $\lambda = \lim \lambda_\tau > 0$, the space spanned by the functions of $\mathbf{S}$ belonging to $\lambda$ is finite-dimensional.

We have the decomposition

$$\mathbf{S}_\tau = \sum \mathbf{S}_{\lambda_\tau} \tag{3.2}$$

where $\mathbf{S}_{\lambda_\tau}$ is the finite-dimensional vector space spanned by the functions which belong to $\lambda_\tau$, $\tau > 0$, and similarly, for $\tau = 0$,

$$\mathbf{S} = \sum \mathbf{S}_\lambda, \tag{3.3}$$

where $\mathbf{S}_\lambda$ is finite-dimensional for $\lambda > 0$.

We shall say that two functions $f, g$ of $\mathbf{S}$ are equivalent if, for each sufficiently small $\tau > 0$, there exist functions $f_\tau$, $g_\tau$ belonging to the same $\mathbf{S}_{\lambda_\tau}$ which converge to $f, g$ respectively. It is clear that this is a relation of equivalence, and to each equivalence class of $\mathbf{S}_\lambda$ there is associated a unique $\lambda_\tau$, $\lim_{\tau\to 0} \lambda_\tau = \lambda$. Writing $\eta_\tau = \lambda_\tau - \lambda$, $\lambda = \lim \lambda_\tau$, we have

$$\eta_\tau = c_1\tau + c_2\tau^2 + \cdots + c_k\tau^k + O(\tau^{k+1}) \tag{3.4}$$

where $k$ may be any positive integer, and we associate to each equivalence class of $\mathbf{S}_\lambda$ the first non-zero coefficient $c_i$ in the development (3.4), provided that such a non-zero coefficient exists for some arbitrary large $k$. If no non-zero coefficient exists, we assign the number 0 to the equivalence class. The real number determined in this way minus $\lambda$ will be denoted by $r$, and the Stekloff number $\lambda$ will be denoted by $s$. We have then the decomposition.

$$\mathbf{S} = \sum\nolimits_{r,s} \mathbf{S}_{r,s} \tag{3.5}$$

and we say that the functions of $\mathbf{S}_{r,s}$ are of type $(r, s)$. The sum of the numbers $r$ and $s$ is a generalization of the total degree of a harmonic and thus reflects the structure of a Stekloff function.

In the case of the complex euclidean ball $V = \{z \mid \sum |z_\alpha|^2 < 1\}$, the harmonics of type $(p, q)$ span $\mathbf{S}_{r,s}$ where

$$\begin{cases} r = \dfrac{1}{2}\dfrac{p}{(p+q+m-1)^2} \\ s = \dfrac{1}{2}\dfrac{q}{(p+q+m-1)^2}\,. \end{cases} \tag{3.6}$$

## 4. Homotopy and the principle of upper semi-continuity

Let $X$ be a real differentiable manifold of dimension $2m$ with the two hermitian almost-complex structures $c_0$, $c_1$ connected by a differentiable homotopy of structure where $c_t$ is an hermitian almost-complex structure for each $t$, $0 \leqq t \leqq 1$, whose hermitian metric $\omega_t$ also depends differentiably on $t$. Let $V$ be a finite submanifold of $X$, and let $V_t$ be $V$ with structure and metric induced from $c_t$, $\omega_t$. We assume that there is a fixed neighborhood (independent of $t$) of the boundary of $V$ in $X$ throughout which the structure $c_t$ is integrable for each $t$, $0 \leqq t \leqq 1$. If $W$ is a finite submanifold of $V$ containing the adherence of $V$ as compact subset, the Green's operator $G_{W_t}$, the corresponding operator $G_t$, and the various other operators of Section 2 will vary continuously, and even differentiably, with $t$. We denote dependence on $t$ by attaching a subscript.

For each $t$, $0 \leqq t \leqq 1$, we have the decomposition

$$\mathbf{S}_t = \mathbf{S}_{0,t} \oplus \mathbf{S}_{1,t} \tag{4.1}$$

where $\mathbf{S}_{0,t}$ denotes the subspace of $\mathbf{S}_t$ spanned by the Stekloff functions of type $(r, 0)$ (holomorphic functions), $\mathbf{S}_{1,t}$ the subspace of $\mathbf{S}_t$ spanned by functions of type $(r, s)$, $s > 0$. We say that the variation of complex structure is continuous at $t_0$ if and only if the decomposition (4.1) is continuous at the parameter point $t_0$ in the sense that $\lim \mathbf{S}_{0,t} = \mathbf{S}_{0,t_0}$, $\lim \mathbf{S}_{1,t} = \mathbf{S}_{1,t_0}$ as $t$ approaches $t_0$. We may also define left and right continuity. The following result is immediate:

THEOREM 4.1 (*principle of upper semi-continuity*). *As $t$ approaches $t_0$*, $\lim \mathbf{S}_{0,t} \subset \mathbf{S}_{0,t_0}$.

The following example shows that $\lim \mathbf{S}_{0,t}$ may be a proper subset of $\mathbf{S}_{0,t_0}$. Let $V_0$ be the euclidean ball $V = \{z \mid \sum |z_\alpha|^2 < 1\}$ with the structure $c_0$ of $\mathbf{C}^m$ and with the euclidean metric $\omega_0$, and let $V_t$ be $V$ with the structure $c_t$ defined by a differentiable homotopy such that $c_t$ coincides with $c_0$ outside the euclidean ball of radius $1/2$. Define $\omega_t = \prod_{1,1}(t)\omega_0$. Then, for all sufficiently small $t > 0$, $\omega_t$ is a positive-definite hermitian metric. If we make the deformation in such a way that $c_t$ is non-integrable for $t > 0$, then clearly $\mathbf{S}_{0,t}$ is isomorphic to $\mathbf{C}$ for $t > 0$ while $\mathbf{S}_{0,0}$ is the infinite-dimensional space spanned by the harmonics of type $(r, 0)$; hence $\lim \mathbf{S}_{0,t} = \mathbf{C}$ is a proper subset of $\mathbf{S}_{0,0}$.

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# COMPLEX ANALYTIC CONNECTIONS IN FIBRE BUNDLES[1]

M. F. Atiyah

## Introduction

In the theory of differentiable fibre bundles the notion of a connection plays an important role. It is therefore natural that we should investigate the corresponding situation for complex analytic fibre bundles. It turns out that complex analytic connections do not in general exist, and in this way we obtain an obstruction element which corresponds in a certain sense to the curvature of a differentiable connection. Under suitable circumstances this obstruction element generates the characteristic cohomology ring of the bundle in a manner analogous to the differentiable case.

The preceding ideas are applied in particular to a problem of Weil [12]. We show how Weil's main result fits into the general theory and we discuss various aspects of the problem.

## §1. Complex analytic connections

Let $P$ be a principal complex analytic fibre bundle with a complex Lie group $G$ as structure group and a complex manifold $X$ as base. Let $\mathcal{T}$ denote the tangent bundle of $P$. Since $G$ operates on $P$ it also operates on $\mathcal{T}$ and we put $Q = \mathcal{T}/G$. An element of $Q$ is therefore an invariant tangent vector field of $P$ defined along one of its fibres. $Q$ has a natural structure of (complex analytic) vector bundle over $X$, and there is a homomorphism of $Q$ onto the tangent bundle $T$ of $X$ induced by the projection of $P$ onto $X$. The kernel of this homomorphism is $L(P)$, the vector bundle over $X$ (with fibre the Lie algebra $L(G)$ of $G$) associated to $P$ by the adjoint representation. This can be seen as follows. The sub-bundle $\mathcal{F}$ of $\mathcal{T}$ consisting of vectors tangential to the fibres of $P$ is canonically isomorphic with the product bundle $P \times L(G)$. The action of $g \in G$ is then given by $(p, l)g = (pg, ad(g)^{-1}l)$, so that $\mathcal{F}/G = P \times_G L(G) = L(P)$.

Thus to every principal bundle $P$ we can associate the exact sequence of vector bundles over $X$:

$$\mathscr{A}(P) : 0 \to L(P) \to Q \to T \to 0.$$

Definition. A *connection* in $P$ is a splitting homomorphism $T \to Q$ of the exact sequence $\mathscr{A}(P)$.

If $f : T \to Q$ is a connection in $P$, then $Q \cong L(P) \oplus T$. In general however the exact sequence $\mathscr{A}(P)$ may not split, and we are led therefore to the study of extensions of vector bundles.

[1] Full details and proofs of the results stated here will be found in a paper of the same title appearing in the Transactions of the American Mathematical Society.

## §2. Extensions of vector bundles

Let $E'$, $E''$ be two vector bundles over $X$. Then an extension of $E''$ by $E'$ is an exact sequence of vector bundles over $X$

$$A : O \to E' \to E \to E'' \to 0.$$

Two extensions $A_1$, $A_2$ of $E''$ by $E'$ are equivalent if there is an isomorphism $g : A_1 \to A_2$ which is the identity on $E'$ and $E''$. The classification problem is then solved by the following result:

PROPOSITION 1. *The equivalence classes of extensions of $E''$ by $E'$ are in one-one correspondence with the elements of* $H^1(X, \text{Hom}\ (\mathbf{E}'', \mathbf{E}'))$.

Here $\mathbf{E}''$, $\mathbf{E}'$ denote the sheaves of germs of holomorphic sections of $E''$ and $E'$ respectively. Hom $(\mathbf{E}'', \mathbf{E}')$ denotes the sheaf of germs of $\mathbf{O}$-homomorphisms of $\mathbf{E}''$ into $\mathbf{E}'$, where $\mathbf{O}$ is the sheaf of germs of holomorphic functions (cf. Serre [9]).

Proposition 1 follows from the general theory of fibre bundles (cf. Grothendieck [5]). We remark only that, if $O \to E' \to E \to E'' \to 0$ is a given extension, $E$ is locally isomorphic with $E' \oplus E''$, and two different local isomorphisms differ by a local automorphism of $E' \oplus E''$ of the form $I + \phi$, where $I$ is the identity and $\phi : E'' \to E'$ locally.

## §3. The obstruction element

By Proposition 1 the extension $\mathscr{A}(P)$ corresponds to an element $a(P) \in H^1(X, \text{Hom}\ (\mathbf{T}, \mathbf{L(P)}))$. If we identify Hom $(\mathbf{T}, \mathbf{L(P)})$ with $\mathbf{L(P)} \otimes \mathbf{\Omega}^1$ under the canonical isomorphism[2] we may regard $a(P)$ as an element of $H^1(X, \mathbf{L(P)} \otimes \mathbf{\Omega}^1)$. A connection exists in $P$ if and only if $a(P) = 0$. If $X$ is a Stein manifold $H^1(X, \mathbf{L(P)} \otimes \mathbf{\Omega}^1) = 0$, and so every principal bundle has a connection. The same result holds in the differentiable case since the relevant sheaves are then fine. If $X$ is a compact complex manifold we shall see that $a(P)$ is in general non-zero.

A case of special interest arises when $G = GL_r(C)$. In this case $G$ is the group of automorphisms of a vector space $V$, and $L(G) = \text{End}\ V$ is the vector space of all endomorphisms of $V$. If $E = P \times_G V$ is the associated vector bundle it follows that $L(P) = \text{End}\ E = E \otimes E^*$, where $E^*$ is the dual of $E$. Hence, identifying the corresponding sheaves, $a(P) \in H^1(X, \text{End}\ \mathbf{E} \otimes \mathbf{\Omega}^1)$.

## §4. The characteristic ring

If $P$ is a differentiable principal bundle, and if $\Theta$ is the curvature of a connection we can construct the characteristic ring of $P$ as follows. Let $F$ be an invariant polynomial of $G$ (which we assume to be compact), that is $F : L(G) \otimes_R \cdots \otimes_R L(G) \to R$ is an $R$-homomorphism symmetric and invariant under $ad(G)$. Then $F(\Theta, \cdots, \Theta)$ is a closed differential form on $X$ and so defines an element of $H^*(X, R)$. By taking all the invariant polynomials $F$ we obtain all the characteristic ring (with real coefficients) (cf. Chern [4]).

We return now to the complex analytic situation. Let $F$ be an invariant polynomial of $G$, that is $F : L(G) \otimes_C \cdots \otimes_C L(G) \to C$ is a $C$-homomorphism, symmetric

[2] As usual $\mathbf{\Omega}^k$ denotes the sheaf of germs of holomorphic differential forms of degree $k$.

and invariant under $ad(G)$. Then $F$ induces a vector bundle homomorphism $F : L(P) \otimes \cdots \otimes L(P) \to 1$, where 1 denotes the trivial line-bundle. This in turn gives rise to a sheaf homomorphism $\mathbf{F} : \mathbf{L}(\mathbf{P}) \otimes \cdots \otimes \mathbf{L}(\mathbf{P}) \to \mathbf{O}$. Using $\mathbf{F}$ we can define a cup-product multiplication.

$$H^1(X, \mathbf{L}(\mathbf{P}) \otimes \mathbf{\Omega}^1) \otimes \cdots \otimes H^1(X, \mathbf{L}(\mathbf{P}) \otimes \mathbf{\Omega}^1) \to H^k(X, \mathbf{\Omega}^k),$$

where $k$ is the degree of $F$. We consider in particular the image of $a(P) \otimes \cdots \otimes a(P)$ under this homomorphism, and we denote the resulting element of $H^k(X, \mathbf{\Omega}^k)$ by $F(a(P))$. Then our main result is the following:

THEOREM 1. *Let $X$ be a compact Kähler manifold, and $G$ a semi-simple complex Lie group or $GL_r(C)$. Let $P$ be a principal $G$-bundle over $X$, and let $a(P)$ be the obstruction element defined by $\mathscr{A}(P)$. Then the set of $F(a(P))$ where $F$ runs through the invariant polynomials of $G$ is identical with the set of elements in the characteristic cohomology ring of $P$.*

We remark first that, since $X$ is a compact Kähler manifold, $H^k(X, \mathbf{\Omega}^k)$ is canonically isomorphic with a subgroup of $H^{2k}(X, C)$. Next, let $G_1$ be a maximal compact subgroup of $G$. Then the structure group of $P$ may be reduced (differentiably) to $G_1$. Let $\mathscr{C}_1 \subset H^*(X, R)$ be the characteristic cohomology ring of the $G_1$-bundle obtained from $P$. Then, by the characteristic cohomology ring of $P$ we mean the ring $\mathscr{C}_1 \otimes C \subset H^*(X, C)$.

Two different proofs of Theorem 1 may be given, and we give brief indications of both.

(1) Let $P_1$ be a $G_1$-bundle which is $G$-equivalent (differentiably) to $P$. Then it has been shown by Nakano [8] and Singer [11] that there exists a differentiable connection in $P_1$ such that the induced connection in $P$ has a curvature $\Theta$ of type (1, 1). Moreover $\Theta$ corresponds to $a(P)$ under the Dolbeault isomorphism associated with the vector bundle $L(P)$ (cf. [10]) Using this fact, the natural relation between invariant polynomials of $G$ and those of $G_1$, and the multiplicative properties of the Dolbeault isomorphism we obtain the Theorem.

(2) We first prove Theorem 1 for $GL_r(C)$. This is done, following Hirzebruch [6], in three stages: (i) for line-bundles, (ii) for split bundles (i.e., bundles with the triangular group as structure group), (iii) for general bundles. Cases (i) and (ii) are easily proved, and (iii) reduces to (ii) by lifting to a bigger space (also compact Kähler). The Theorem for a semi-simple group $G$ now follows from the Theorem for $GL_r(C)$ by considering all complex analytic representations $G \to GL_r(C)$ and using a result of Borel-Hirzebruch [3]. This result asserts that every characteristic class of $P_1$ may be expressed in terms of the Chern classes of the unitary bundle associated to $P_1$ by some unitary representation of $G_1$.

COROLLARY. *Let $X$, $G$, $P$ be as in Theorem* 1, *and suppose $P$ has a connection. Then all the characteristic classes of $P$ are zero.*

## §5. Weil's theorem

Let $X$ be an algebraic curve, $P$ a principal bundle over $X$. Then, by Serre's duality theorem, $H^1(X, \mathbf{L}(\mathbf{P}) \otimes \mathbf{\Omega}^1)$ is dual to $H^0(X, \mathbf{L}(\mathbf{P})^*)$. In particular suppose that $G = GL_r(C)$, and let $E$ be the associated vector bundle. Then $L(P)^* = \text{End}\, E$, and so $H^1(X, \mathbf{L}(\mathbf{P}) \otimes \mathbf{\Omega}^1)$ is dual to $\Gamma$ End $E$, the vector space of all endomorphisms of $E$.

Let $E = E_1 \oplus E_2 \oplus \cdots \oplus E_q$ be a Remak decomposition of $E$, that is each $E_i$ is indecomposable. Then the $E_i$ are unique up to isomorphism [2], and it is easy to show that $a(P) = a(P_1) \oplus \cdots \oplus a(P_q)$ where $P_i$ is the principal bundle corresponding to $P_i$, and we regard End $\mathbf{E}_1 \oplus \cdots \oplus$ End $\mathbf{E}_q$ as a subsheaf of End $\mathbf{E}$. We may therefore suppose $E$ indecomposable. Then every $\phi \in \Gamma$ End $E$ is of the form $\phi = \lambda I + \psi$, where $I$ is the identity, $\lambda \in C$ and $\psi$ is nilpotent [2]. Finally we find

(i) $$\langle a(P), I\rangle = 2\pi i \deg E,$$

(ii) $$\langle a(P), \psi\rangle = 0.$$

Hence we obtain Weil's theorem:

THEOREM 2. *Let* $E = E_1 \oplus \cdots \oplus E_q$ *be a Remak decomposition of the vector bundle* $E$ *over the algebraic curve* $X$. *Let* $P$ *be the principal bundle corresponding to* $E$. *Then* $P$ *has a connection if and only if* $\deg(E_i) = 0$ *for* $i = 1, \cdots, q$.

This form of the theorem differs slightly from that in [12]. However it is easy to show that a principal $G$-bundle $P$ arises from a representation of the fundamental group of $X : \pi_1(X) \to G$ if and only if $P$ has an integrable connection. Moreover every connection over a curve is necessarily integrable. Thus Theorem 2 can be rephrased as follows: *$P$ arises from a representation of the fundamental group if and only if* $\deg(E_i) = 0$ *for* $i = 1, \cdots, q$.

## §6. Higher dimensional varieties

In this section we shall consider the case when the base space $X$ is an algebraic variety of dimension greater than or equal to two. We denote by $X_n$ the intersection of $X$ with a general hypersurface of degree $n$, and by $P_n$ or $E_n$ the restriction to $X_n$ of the principal bundle $P$ or the vector bundle $E$. We then have a standard exact sequence of sheaves:

$$O \to \mathbf{E}(-n) \to \mathbf{E} \to \mathbf{E}_n \to 0,$$

where $\mathbf{E}(-n)$ is the sub-sheaf of sections of $E$ vanishing on $X_n$, and $\mathbf{E}_n$ is the sheaf of germs of holomorphic sections of $E_n$. A basic theorem (cf. [9]) asserts that $H^q(X, \mathbf{E}(-n)) = 0$ if $q < \dim X$ and $n$ is sufficiently large (depending on $E$). From the exact cohomology sequence we then obtain, for sufficiently large $n$,

$$H^q(X, \mathbf{E}) \cong H^q(X_n, \mathbf{E}_n), \qquad q, q+1 \neq \dim X.$$

Thus, if a problem can be expressed in terms of cohomology groups of a vector bundle, an induction argument for reducing the dimension of $X$ is at our disposal. We make two applications of this method.

(1) By considering the vector bundle End $E$ we find that, if $\dim X \geqq 2$ and if $n$ is sufficiently large

$$\Gamma \text{ End } E \cong \Gamma \text{ End } E_n.$$

On the other hand the structure of the algebra $\Gamma$ End $E$ completely determines whether or not $E$ decomposes (cf. [2]). Thus we deduce the following: *if* dim $X \geqq 2$, *and provided $n$ is sufficiently large, $E$ decomposes if and only if $E_n$ decomposes.*

(2) By considering $L(P) \otimes T^*$ we find that *if* dim $X \geqq 3$, *and provided $n$ is sufficiently large, $P$ has a connection if and only if $P_n$ has a connection.*

Clearly the result in (1) becomes false if dim $X = 1$, since $X_n$ is then a set of points and so $E_n$ always decomposes. We shall now give an example to show that (2) becomes false if dim $X = 2$. Our example also shows that Weil's theorem does not generalize to higher dimensions.

Let $Y$ be a rational curve, $Z$ an elliptic curve, and put $X = Y \times Z$. We shall identify $Y$, $Z$ with the curves $Y \times z_0$, $y_0 \times Z$ on $X$, where $y_0$, $z_0$ are given points. We shall construct an extension

$$A : O \to [Z] \to E \to [-Z] \to 0,$$

where as usual $[Z]$ denotes the line-bundle on $X$ corresponding to the divisor $Z$. If we restrict to $Z$, we get the extension

$$O \to 1 \to E_Z \to 1 \to 0,$$

1 denoting the trivial line-bundle on $Z$. This restriction of extensions corresponds to the restriction homomorphism $\rho : H^1(X, [\mathbf{2Z}]) \to H^1(Z, \mathbf{O}_Z)$. Now, since $Z$ is elliptic, $H^1(Z, \mathbf{O}_Z)$ is of dimension one. By examining the exact cohomology sequence containing $\rho$ we can show that $\rho$ is an epimorphism. We choose an element of $H^1(X, [\mathbf{2Z}])$ not in the kernel of $\rho$ and we consider the corresponding extension $A$. By construction $A_Z$ is a non-trivial extension, and it is known that in this case $E_Z$ is indecomposable (cf. [1]).

Restricting $A$ to $Y$ we get

$$A_Y : 0 \to [D] \to E_Y \to [-D] \to 0.$$

where $D$ is a point divisor on $Y$. Since $Y$ is rational $H^1(Y, [\mathbf{2D}]) = 0$, and so $A_Y$ is trivial. Thus $E_Y \cong [D] \oplus [-D]$. Hence by Weil's Theorem $E_Y$ (or rather the corresponding principal bundle) does not have a connection, and so $E$ does not have a connection.

Finally, from the definition of $E$, we see that all the Chern classes are zero. The properties of $E$ thus show that Weil's Theorem does not have a generalization to higher dimension. Moreover, if $X_n$ has the same meaning as before, $E_n$ will be indecomposable for sufficiently large $n$ (since $E$ is indecomposable, and using (1) above). Also the Chern class of $E_n$, i.e., its degree, is zero. Hence by Weil's Theorem $E_n$ has a connection. But we have already shown that $E$ does not have a connection. This shows that (2) is false if dim $X = 2$.

## §7. Line-bundles

We conclude by mentioning an interesting special case of the exact sequence $\mathscr{A}(P)$.

Let $X$ be an algebraic variety of dimension $r$ embedded non-singularly in a projective space $\sum$ of dimension $n$. Let $E$ be the line-bundle on $X$ corresponding to the negative of a hyperplane section, let $T$ be the tangent bundle of $X$, and let $P$ be the principal bundle associated to $E$. Let $G(r+1, n+1)$ denote the Grassmannian of $r$-dimensional subspaces of $\sum$. Then the embedding of $X$ in $\sum$ defines a mapping of $X$ into $G(r+1, n+1)$ by assigning to each point $x$ the tangent projective $r$-space to $X$ at $x$. Let $W$ be the $(r+1)$-dimensional vector bundle induced by this mapping. Then it can be shown that the exact sequence $\mathscr{A}(P)$ reduces to the following:

$$O \to 1 \to E^* \otimes W \to T \to 0.$$

This sequence is due to Nakano [7] and Serre (unpublished) and is of interest because it enables us to express the Chern classes of $T$ in terms of those of $W$ and $E$ (cf. [7]). The formulas obtained this way are precisely those originally used by Todd to define Canonical Systems.

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# REMARQUES SUR CERTAINES ALGEBRES DE LIE

PAR RAYMOND RAFFIN

**1.** Les algèbres de Lie ont pris, depuis peu, en topologie algébrique, une importance croissante (voir par exemple CHEVALLEY and EILENBERG [2], KOSZUL [3], H. CARTAN and EILENBERG [4]).

Je présente ici quelques remarques d'un caractère élémentaire et purement algébrique sur les algèbres de Lie.

**2.** Les algèbres $A$ dont il sera question seront des algèbres (éventuellement) non associatives sur un anneau $B$ (commutatif, avec élément unité) qui sera généralement sur un corps $F$.

**3.** Dans la détermination des algèbres de Lie $A$ on rencontre le cas particulier suivant

$$xy \in (x, y) \quad \begin{cases} \text{quels que soient} \\ x \in A, y \in A, \end{cases} \tag{3.1}$$

où $(x, y)$ désigne le $B$-module engendré par $x$ et $y$.

*Une algèbre $A$ satisfaisant à* (3.1) *est à puissances associatives* et *si elle est commutative elle est une algèbre de* JORDAN. (comparer avec Théorème 5.1.)

**4.** La condition (3.1) peut s'écrire

$$xy = \lambda(x, y)x + \rho(x, y)y. \tag{4.1}$$

$A$ étant donnée, les fonctions $\lambda(x, y)$ et $\rho(x, y)$ ne sont généralement pas bien déterminées. Mais nous avons le

LEMME 4.1 *Soit une algèbre non associative d'ordre supérieur à $n + 1 - k$ sur un corps $F$ et telle qu'on ait*

$$a_1 . a_2 . \cdots . a_n \in (a_k, \cdots, a_n) \quad \begin{cases} 0 < k < n, \\ \text{quels que soient} \\ a_1, \cdots, a_n \text{ de } A, \end{cases}$$

*où le premier membre figure un produit non associatif quelconque, déterminé, où les parenthèses sont mises toujours de la même façon quels que soient $a_1, \cdots, a_n$ et où le second membre désigne le* F*-module engendré par $a_k, \cdots, a_n$; le premier membre peut donc s'écrire*

$$\textstyle\sum_{i=k}^{n} \rho_i(a_1, \cdots, a_n)a_i,$$

*où les $\rho_i$ sont des fonctions scalaires.* (c'est à dire à valeurs dans F.) *Alors parmi les ensembles ordonnés des fonctions $\rho$:*

$$\{\rho_k(a_1, \cdots, a_n), \cdots, \rho_n(a_1, \cdots, a_n)\}$$

*il en existe un tel que $\rho_i$ ne dépende pas de $a_i (i = k, \cdots, n)$, en outre chaque fonction $\rho_i$ de cet ensemble-là est linéaire et homogène par rapport à chacun de ses arguments $a_j (j \neq i)$.*

Pour pouvoir appliquer ce lemme *nous supposerons dans tout ce qui suit que $B$ est un corps $F$.*

Il en résulte en particulier que, pourvu que l'ordre de l'algèbre $A$ sur $F$ soit supérieur à 2, on peut remplacer la condition (4.1) par la condition

$$xy = \lambda(y)x + \rho(x)y \quad \begin{cases} \lambda \text{ et } \rho \text{ linéaires} \\ \text{et homogènes,} \end{cases} \tag{4.2}$$

où $\{\lambda(y), \rho(x)\}$ désigne celui des couples ordonnés $\{\lambda(x, y), \mu(x, y)\}$ qui est signalé par le lemme. Si l'ordre de l'algèbre $A$ est égal à 2 ce résultat est encore vrai à condition que $F$ soit différent du corps (0, 1).

**5.** Il résulte du §4 le

THÉORÈME 5.1. *Si dans une algèbre $A$ sur un corps $F$ et telle que $xy \in (x, y)$ quels que soient $x$ et $y$ de $A$, à tout $u \neq 0$ de $A$ on peut faire correspondre $v$ de $A$, indépendant de $u$ et anticommutant avec $u$, l'algèbre $A$ est une algèbre de Lie.*

On a alors

$$xy = \lambda(y)x - \lambda(x)y. \tag{5.1}$$

Ce résultat s'applique en particulier si on sait à priori que l'algèbre $A$ est anticommutative (cf. fin § 3).

**6.** Appelons "*algèbre*, sue $F$, *du produit vectoriel*" l'algèbre anticommutative, sur $F$, dont une portion de la table de multiplication est

Table (6.I)

| | $e_2$ | $e_3$ |
|---|---|---|
| $e_1$ | $e_2$ | $-e_2$ |
| $e_2$ | 0 | $e_1$ |

Appelons "*algèbre*, sur $F$, *du produit vectoriel généralisé*" l'algèbre anticommutative, sur $F$, définie par

Table (6.II)

| | $e_2$ | $e_3$ |
|---|---|---|
| $e_1$ | $e_3$ | $-\beta_2 e_2$ |
| $e_2$ | 0 | $\alpha_1 e_1$ |

où $\alpha_1$ et $\beta_2$ sont des éléments non nuls de $F$.

THÉORÈME 6.1 (I) Dans les hypothèses

(6.1) *$A$ est une algèbre non associative sur un corps $F$.*

(6.2) $x(yz) \in (y, z)$ } *quels que soient* $x, y, z$

(6.3) $x^2 = 0$ } *de $A$.*

(6.4) $\dim A > 2$ (dim $A$ *est l'ordre de $A$ sur $F$*).

*$A$ est une algèbre de Lie.*

(II) *Si, avec les hypothèses* ((6.1), (6.2), (6.3)), *on remplace* (6.4) *par*

$$(6.4') \qquad \begin{cases} (6.4'_1) & \dim A > 1 \\ (6.4'_2) & \begin{cases} a \text{ et } b \text{ indépendants sur } F \text{ entraîne} \\ a \text{ et } ba \text{ indépendants sur } F. \end{cases} \end{cases}$$

*A est une algèbre Lie d'ordre au moins égal à* 3 *et contenant une sous-algèbre, sur F, isomorphe à celle du produit vectoriel généralisé.*

(III) *Si en plus de* ((6.1), (6.2), (6.3), (6.4')) *on a*

$$(6.5) \qquad \lambda(a, a) \neq 0 \text{ pour tout } a \neq 0$$

*l'algèbre A est d'ordre* 3 (elle est donc isomorphe à l'algèbre du produit vectoriel généralisé).

(IV) *Si en plus de* ((6.1), (6.2), (6.3), (6.4'), (6.5)) *on a*

$$(6.6) \qquad \lambda(a, a) \textit{ est, pour tout } a, \textit{ un carré parfait de } F.$$

*l'algèbre A est isomorphe à l'algèbre, sur F, du produit vectoriel.*

**7.** Considérons l'algèbre anticommutative $\mathfrak{A}$, sur $F$ définie par

Table (7.I)

| $\mathfrak{A}$ | $e_2$ | $e_3$ |
|---|---|---|
| $e_1$ | $\gamma_1e_1 + \gamma_2e_2 + \gamma_3e_3$ | $-\beta_1e_1 - \beta_2e_2 - \beta_3e_3$ |
| $e_2$ | $0$ | $\alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3$ |

associons-lui la matrice

$$(7.1) \qquad \Gamma = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

Appelons algèbre *adjointe* $\mathfrak{G}$ de l'algèbre $\mathfrak{A}$, l'algèbre "scalaire" définie par

Table (7.II)

| $\mathfrak{G}$ | $e_1$ | $e_2$ | $e_3$ |
|---|---|---|---|
| $e_1$ | $g_{11}$ | $g_{12}$ | $g_{13}$ |
| $e_2$ | $g_{21}$ | $g_{22}$ | $g_{23}$ |
| $e_3$ | $g_{31}$ | $g_{32}$ | $g_{33}$ |

où les $g_{ij}$ sont des éléments de $F$ tels que la matrice

$$(7.2) \qquad G = (g_{ij})$$

qui sera dite associée à $\mathfrak{G}$ soit *l'adjointe* de $\Gamma$ (exemple: $g_{12}=$ cofacteur de $\alpha_2$ dans $\Gamma$). Cette *définition* est consistante car elle *est invariante* par *changement de base*.

On remarque les propositions suivantes:

*Pour qu'une algebre $\mathfrak{A}$ soit une algèbre de Lie il faut et il suffit que son algèbre adjointe soit commutative.*

*Pour qu'une algèbre $\mathfrak{A}$ soit isomorphe à l'algèbre, sur $F$, du produit vectoriel généralisé il faut et il suffit que sa matrice adjointe $G$ soit régulière et symétrique.*

Comme applications immédiates des remarques de cette section on a par exemple:

1. Dans le cas des algèbres de Lie d'ordre 3 les éventualités (3.1) et (6.3) se complètemt.

2. La réduction des algèbres de Lie d'ordre 3 se ramène à celle, par congruence, de la matrice adjointe $G$.

3. *Pour qu'une algèbre $\mathfrak{A}$, sur le corps des nombres réels, soit isomorphe à l'algèbre du produit vectoriel il faut et il suffit que la matrice adjointe $G$ soit définie positive.*

(C'est à dire que $G$ soit la matrice d'une forme quadratique definie positive.)

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# GEOMETRY OF SUBMANIFOLDS IN A COMPLEX PROJECTIVE SPACE

SHIING-SHEN CHERN

## Introduction

In the development of algebraic topology in the last ten years the notion of curvature has played a considerable rôle. In fact, some of the basic notions of modern topology, such as transgression and characteristic classes, were first discovered in their simplest forms in differential geometry. Among relations between curvature and characteristic classes one of the most inclusive results is the Weil homomorphism. It can be briefly described as follows: Let $\psi : E \rightarrow M$ be a differentiable fibre bundle with a compact differentiable manifold $M$ as base space and with a compact connected Lie group as structural group. Let a connection be given in the bundle from which the curvature is defined. Then the real characteristic classes of this bundle, which are elements of the cohomology ring $H^*(M, R)$, contain as representatives (in the sense of de Rham's theorem) exterior differential forms which can be constructed explicitly from the curvature [5].[1]

As an example we consider the tangent bundle of a compact Riemannian manifold with its Levi-Civita connection. Let $\Omega_{ij} = -\Omega_{ji}$, $1 \leqq i, j \leqq n$ ($=$ dim of $M$), be the curvature forms. Then the Pontrjagin class $p_k$ contains as representative the exterior differential form [8]

$$(1) \qquad \Psi_k = \frac{1}{(2\pi)^{2k}(2k)!} \sum \delta(i_1, \cdots, i_{2k}; j_1, \cdots, j_{2k})\, \Omega_{i_1 j_1} \wedge \cdots \wedge \Omega_{i_{2k} j_{2k}}.$$

In this formula $\delta(i_1, \cdots, i_{2k}; j_1, \cdots, j_{2k})$ is zero, except when $j_1, \cdots, j_{2k}$ is a permutation of $i_1, \cdots, i_{2k}$, and is then equal to $+1$ or $-1$, according as the permutation is even or odd, while the summation is extended over all indices $i_1, \cdots, i_{2k}, j_1, \cdots, j_{2k}$ from 1 to $n$.

Using this identification, we can interpret the Thom-Hirzebruch index theorem [10] as expressing the index $\tau(M)$ of a compact oriented Riemannian manifold $M$, a topological invariant, in terms of an integral over $M$. In fact, for $n = 4$ or 8, the theorem gives respectively the formulas

$$(2) \qquad \begin{aligned} \tau(M) &= \frac{1}{4\pi^2} \int_M \Omega_{12}^2 + \Omega_{13}^2 + \cdots + \Omega_{34}^2, \qquad n = 4, \\ \tau(M) &= \frac{1}{720\pi^4} \int_M 7\{(\Omega_{12}\Omega_{34} + \Omega_{13}\Omega_{42} + \Omega_{14}\Omega_{23})^2 + \cdots\} - (\Omega_{12}^2 + \cdots + \Omega_{78}^2), \\ & \qquad n = 8. \end{aligned}$$

[1] Reference is to Bibliography at the end of this paper.

In the last formula the sum in the braces is over all combinations, four at a time, of the 8 indices.

Similarly, the recent work of Hirzebruch on the Riemann-Roch Theorem for algebraic varieties can also be interpreted as a formula expressing the arithmetic genus of a complex algebraic variety as the integral of a certain curvature of the Kähler metric of the variety.

I propose to discuss in this paper certain aspects of the geometry of an analytic submanifold $M_m$ of (complex) dimension $m$ in a complex projective space of dimension $N$. The other interesting case of the study of submanifolds is that of differentiable submanifolds in an Euclidean space, which is the object of classical differential geometry. Although there is a certain analogy, we will see that the problems are quite different. When the occasion justifies, we will point out the analogies and the differences.

## 1. Local geometry of submanifolds

Let $M_m$ (or simply $M$) be an analytic submanifold of dimension $m$ in a complex projective space $P_N$ of dimension $N$. The latter has the Fubini-Study elliptic Hermitian metric, so that $M_m$ has an induced Kähler metric. It has curvature forms from its Kähler metric and has relative curvatures as a submanifold of $P_N$. To describe the situation analytically, we take the unitary vector space $V_{N+1}$, i.e., the complex vector space of dimension $N+1$ with a positive definite Hermitian scalar product $(Z, W) = \overline{(W, Z)}$, $Z, W \in V_{N+1}$, where the bar denotes the complex conjugate of a number. $V_{N+1}$ is acted on by the group $Z \to \rho Z$, $\rho \neq 0$, under which the zero vector is invariant. $P_N$ can be identified with the orbit space of $V_{N+1}-0$ under this group. This description allows us to define an analytic submanifold $M$ of $P_N$ by a vector-valued holomorphic function $Z(\zeta^1, \cdots, \zeta^m)$, where $\zeta^1, \cdots, \zeta^m$ are local complex coordinates on $M_m$ and $Z$ is defined up to a non-zero factor. Unless otherwise stated, the submanifold is allowed to have self-intersections. We do suppose, however, that $M_m$ consists entirely of regular points, that is, that the vectors $Z, \partial Z/\partial \zeta^1, \cdots, \partial Z/\partial \zeta^m$ are everywhere linearly independent. They span then a projective space of $m$ dimensions, the tangent space to $M$ at $Z$. The normal space to $M$ at $Z$ is spanned by the points $Y$ "orthogonal to $Z, \partial Z/\partial \zeta^1, \cdots, \partial Z/\partial \zeta^m$":

$$(3) \qquad (Z, Y) = \left(\frac{\partial Z}{\partial \zeta^i}, Y\right) = 0, \qquad 1 \leqq i \leqq m,$$

and is a projective space of dimension $N - m - 1$. (We use the convention that $(V, W)$ is linear in the first argument, so that $(\lambda V, W) = \lambda(V, W) = (V, \bar{\lambda} W)$ for any complex number $\lambda$.)

An important rôle in the theory of relative curvature of $M$ is played by the "second fundamental form":

$$(4) \qquad \Phi_Y = (d^2 Z, Y) = -(dZ, dY).$$

This is an ordinary quadratic differential form of type (2, 0) which is defined, for any point $Y$ of the normal space at $Z$, up to a non-zero factor. Let $Z_0 = Z$,

$Z_1, \cdots, Z_N$ be an orthonormal frame of $V_{N+1}$. Consider all orthonormal frames with the property that $Z$ defines a point of $M$ and $Z, Z_1, \cdots, Z_m$ span the tangent space at $Z$. Then we have

$$dZ_A = \omega_{A0}Z_0 + \omega_{A1}Z_1 + \cdots + \omega_{AN}Z_N, \qquad 0 \leqq A \leqq N, \tag{5}$$

where

$$\omega_{AB} + \bar{\omega}_{BA} = 0, \qquad \omega_{0\alpha} = 0, \qquad 0 \leqq B \leqq N, \qquad m+1 \leqq \alpha \leqq N. \tag{6}$$

We also find

$$\Phi_\alpha \equiv \Phi_{Z\alpha} = (d^2Z, Z_\alpha) = \textstyle\sum_{1 \leqq i \leqq m} \omega_{0i}\omega_{i\alpha}. \tag{7}$$

In this notation the induced Kähler metric on $M$ is given by

$$ds^2 = \textstyle\sum_i \omega_{0i}\bar{\omega}_{0i} \tag{8}$$

and its curvature form is

$$\Theta_{jk} = \omega_{0k} \wedge \bar{\omega}_{0j} + \delta_{jk} \textstyle\sum_l \omega_{0l} \wedge \bar{\omega}_{0l} + \frac{1}{4} \sum_\alpha \dfrac{\partial \Phi_\alpha}{\partial \omega_{0j}} \wedge \overline{\dfrac{\partial \Phi_\alpha}{\partial \omega_{0k}}}. \tag{9}$$

This shows that the second fundamental form determines the intrinsic curvature and contains more information.

If $m = 1$, i.e., if the submanifold is an analytic curve, the tangent space is only the first of a sequence of linear spaces $L \subset L' \subset L'' \subset \cdots$, the osculating spaces of successive orders. There may be points on an analytic curve where one of the osculating spaces becomes indeterminate; but it can be proved that such points are isolated. For a general manifold $M$ of dimension $m$ in $P_N$ we consider through a point all the curves on $M$. The linear space spanned by all the osculating spaces of a given order $p$ of these curves is called the osculating space of order $p$ of $M$ at the point. In general, the osculating space of order $p$ of a submanifold of dimension $m$ has the dimension

$$m + \binom{m+1}{2} + \cdots + \binom{m+p-1}{p}.$$

For $p = 1$ this becomes the tangent projective space. Thus the submanifold $M$ gives rise to a splitting of the tangent projective space of $P_N$ into a nested sequence of projective spaces.

## 2. Integral-geometric invariants

The consideration of the volume of a submanifold and of spaces associated to it gives rise to many geometrical theorems which can be described as belonging to integral geometry. In the case of an analytic submanifold in the projective space this study has a particularly significant aspect, because a theorem of Wirtinger [15] states that, for a compact (and therefore algebraic) submanifold, its volume is equal to $(2\pi)^m/m!$ times its order. In other words, the volume takes a double rôle and is useful also in the study of non-compact submanifolds.

Consider the simplest case of an analytic curve $M_1$. Suppose $Z \in M_1$ be a point at which the osculating spaces of all dimensions $\leq p$ are determined. Suppose the orthonormal frames $Z_0 = Z, Z_1, \cdots, Z_N$ be so chosen that $Z_0, \cdots, Z_q$ span the osculating space of dimension $q \leq p$ at $Z$. Then we have

$$dZ_i = \omega_{i0}Z_0 + \omega_{i1}Z_1 + \cdots + \omega_{i,i+1}Z_{i+1}, \qquad 0 \leq i \leq p. \tag{10}$$

The differential form $\omega_{i,i+1}$ is significant in the sense that it is a multiple of $\omega_{01} : \omega_{i,i+1} = \rho_i\omega_{01}$; moreover, the absolute value $|\rho_i|$ is an invariant of the curve $M_1$. We will call it the $i^{\text{th}}$ absolute curvature. From the above equations we derive

$$\begin{aligned} d(Z_0 \wedge \cdots \wedge Z_p) = (\omega_{00} + \cdots + \omega_{pp})(Z_0 \wedge \cdots \wedge Z_p) \\ + \omega_{p,p+1}(Z_0 \wedge \cdots \wedge Z_{p-1} \wedge Z_{p+1}). \end{aligned} \tag{11}$$

If $M_1$ is compact, the integral

$$\frac{i}{2\pi}\int_{M_1} \omega_{p,p+1} \wedge \bar{\omega}_{p,p+1} \tag{12}$$

is an integer, to be called the order of rank $p$ of $M_1$. It can be interpreted geometrically as a numerical multiple of the volume swept by the osculating spaces of order $p$ of $M_1$. There is another interpretation, due to Santalò [13], in terms of the Grassmann manifold of the linear spaces of dimension $p$ in $P_N$, which is as follows: All the osculating spaces of dimension $p$ of $M_1$ describe a curve in the Grassmann manifold, and the above integral is a numerical multiple of the arc length of this curve. The order of rank $p$ coincides with the invariant of the same name in H. Weyl's theory of meromorphic curves; it can be traced farther back in algebraic geometry. Exactly because of its expression as an integral, it can be defined for non-compact curves. It plays a vital rôle in the theory of Weyl and Ahlfors [1], [14].

The above considerations can be generalized to submanifolds of arbitrary dimension. However, the details are complicated and have not been completely carried out. If $Z$ describes $M_m$ and if $Z_0 = Z, Z_1, \cdots, Z_m$ are orthonormal points spanning the tangent projective space at $Z$, we have

$$\begin{aligned} d(Z_0 \wedge Z_1 \wedge \cdots \wedge Z_m) = (\omega_{00} + \omega_{11} + \cdots + \omega_{mm})(Z_0 \wedge Z_1 \wedge \cdots \wedge Z_m) \\ + \textstyle\sum_{1 \leq i \leq m} \sum_{m+1 \leq \alpha \leq N} \frac{\partial \Phi_\alpha}{\partial \omega_{0i}}(Z_0 \wedge \cdots \wedge Z_{i-1} \wedge Z_{i+1} \wedge \cdots \wedge Z_m). \end{aligned} \tag{13}$$

The integral

$$\frac{i^m}{(2\pi)^m}\int_M \left(\textstyle\sum_{\alpha,i} \frac{\partial \Phi_\alpha}{\partial \omega_{0i}} \wedge \overline{\frac{\partial \Phi_\alpha}{\partial \omega_{0i}}}\right)^m \tag{14}$$

is an integer for compact submanifolds and is equal to a numerical multiple of the volume swept by the tangent projective spaces of $M_m$.

I wish to remark that these integral-geometric methods are also fruitful in the study of submanifolds of the real Euclidean space, and that they lead to results of an entirely different nature [7]. In fact, let $M_R^m$ be a compact orientable submanifold of dimension $m$ differentiably imbedded in a real Euclidean space $E^N$ of dimension

$N$. To each point $x \in M_R^m$ and each unit normal vector $v$ to $M_R^m$ at $x$ let $G(x, v)$ be the Gauss-Kronecker curvature at $x$ of the orthogonal projection of $M_R^m$ into the linear space determined by $v$ and the tangent plane to $M_R^m$ at $x$. Let $d\sigma$ be the volume element of the unit hypersphere in the normal plane to $M_R^m$ at $x$. Its total volume is a constant given by

$$C_{N-m-1} = \frac{2\pi^{\frac{1}{2}(N-m)}}{\Gamma(\frac{1}{2}(N-m))}. \tag{15}$$

The integral

$$K(x) = \int G(x, v)\, d\sigma \tag{16}$$

over the unit hypersphere in the normal plane is zero if $m$ is odd. If $m$ is even, the integral $(C_m/2C_{N-1}) \int K(x)\, dV$ over $M_R^m$ is equal to the Euler-Poincaré characteristic of $M_R^m$.

Results of an entirely different kind can be obtained, if we consider instead of (16) the integral

$$K^*(x) = \int |G(x, v)|\, d\sigma \geqq 0. \tag{17}$$

We will call $K^*(x)$ the total curvature of $M_R^m$ at $x$. An essential feature of a submanifold in Euclidean space is the existence of a large number of differentiable functions, namely the coordinate functions. Because of this fact the integral of the total curvature over $M_R^m$ has an absolute lower bound as given by the inequality:

$$\int_{M_R^m} K^*(x)\, dV \geqq 2\, C_{N-1}. \tag{18}$$

For $m = 1$, i.e., for closed curves, this reduces to a classical result of Fenchel, which states that the integral of the absolute value of the curvature of a curve is $\geqq 2\pi$.

In the general case it is possible to draw some conclusions when the integral of total curvature is "small." Lashof and I have recently proved that [9]: (1) If the equality sign holds in (18), then $M_R^m$ belongs to a linear subspace of dimension $m + 1$ and is a convex hypersurface of the latter; (2) If

$$\int K^*(x)\, dV < 3C_{N-1}, \tag{19}$$

then $M_R^m$ is homeomorphic to an $m$-dimensional sphere. Moreover, these results are also true for immersed manifolds, when self-intersections are allowed. Intuitively speaking, if one considers only the absolute value of curvature, then both the submanifold itself and its position in the Euclidean space will be sharply restricted, when the total curvature is small.

### 3. Exterior differential forms on a submanifold

It is almost always significant to construct explicitly differential forms on a manifold. We have seen some examples in the Introduction. In the case of a submanifold the differential forms which suggest themselves most readily from the analytical viewpoint are Hermitian differential forms in the case of an analytic submanifold in a complex projective space and ordinary quadratic differential forms (for instance, the first and second fundamental forms) in the case of a differentiable submanifold in Euclidean space. However, experience has shown that exterior differential forms are usually of more geometric significance. Such exterior differential forms can be explicitly given, when there are auxiliary points or linear subspaces; they are therefore simultaneous invariants of the submanifold and the auxiliary elements under consideration. In the case of an analytic submanifold in $P_N$ we suppose furthermore that the forms are holomorphic or meromorphic. The following gives a few geometrical consequences which can be drawn from the consideration of such exterior differential forms.

Let us first write down some exterior differential forms we have in mind. We will be mostly concerned with a curve $M_1$, whose points have the coordinate vector $Z(\zeta)$ such that its components are holomorphic functions of the local coordinate $\zeta$ and that multiplication of the vector $Z$ by a non-zero factor does not change the point. If $A$, $B$, $C$ etc. denote fixed vectors or multivectors and dashes denote differentiations with respect to $\zeta$, then $(Z, A)/(Z, B)$, or, more generally,

$$(Z \wedge Z^1 \wedge \cdots \wedge Z^{(p)}, A)/(Z \wedge Z^1 \wedge \cdots \wedge Z^{(p)}, B), \tag{20}$$

is a meromorphic function in $M_1$. Similarly, the forms

$$(Z \wedge Z^1 \wedge \cdots \wedge Z^{(p-1)}, A)(Z \wedge Z^1 \wedge \cdots \wedge Z^{(p)} \wedge dZ^{(p)}, B)/(Z \wedge \cdots \wedge Z^{(p)}, C)^2 \tag{21}$$

$$(Z \wedge dZ, A)/(Z, B_1)^2 + \cdots + (Z, B_s)^2 \tag{22}$$

are meromorphic exterior differential forms. As an example of a meromorphic function involving higher derivatives we mention the following:

$$(Z, A)^{N+1} \{\det (Z, Z^1, \cdots, Z^{(N)})\}^{N-1}/\{\det (Z, Z^1, \cdots, Z^{(N-1)}, B)\}^{N+1}. \tag{23}$$

Each of these functions or forms has the property that it is invariant under an admissible change of the local coordinate on $M_1$ and under multiplication of $Z$ by a non-zero factor. *The study of the zeros and poles of the function* (20) *gives the* theorem that, for compact $M_1$, every hyperplane cuts the curve in the same number of points. Similarly, the study of the zeroes and poles of the form (21) gives for compact $M_1$ the Plücker formulas. These functions and forms also play a vital rôle in the theory of non-compact curves of $P_N$, as demonstrated by the work of H. Weyl, J. Weyl, and Ahlfors [1, 14].

The functions and forms in (20)–(23) have generalizations for $M_m$ in $P_N$. Since we will not make any applications of them in this paper, we restrict ourselves to a few typical cases. A meromorphic form in $M_m$ is given by

$$(Z \wedge \underbrace{dZ \wedge \cdots \wedge dZ}_{k \text{ times}}, A)/(Z, B_1)^{k+1} + \cdots + (Z, B_s)^{k+1}, \qquad k \leqq m \tag{24}$$

of which a particular case is

$$(25)\qquad (\underbrace{Z \wedge dZ \wedge \cdots \wedge dZ}_{k \text{ times}}, A)/(Z, B)^{k+1}.$$

For $m = 2$ and with $u, v$ denoting the local coordinates, a meromorphic differential form in $M_2$ involving higher derivatives is given by

$$(26)\qquad \frac{(Z, A)^3(Z \wedge Z_u \wedge Z_v \wedge Z_{uu} \wedge Z_{uv} \wedge Z_{vv}, B)}{(Z \wedge Z_u \wedge Z_v, C)^3}\, du \wedge dv.$$

For compact $M_m$ the consideration of these forms will give relations between invariants of manifolds generated by the osculating spaces of $M_m$ and the characteristic classes of $M_m$, relations which generalize the Plücker formulas.

We wish to state some results which arise from the study of the abelian sums of holomorphic and meromorphic differential forms of algebraic curves $M_1$. Let $d$ be the order of $M_1$, and let $u$ be a generic hyperplane. From a meromorphic form $\omega$ in $M_1$, we construct the "abelian sum"

$$(27)\qquad \omega(u) = \textstyle\sum_{1 \leqq i \leqq d} \omega_i,$$

where $\omega_i$ are the forms at the points common to $u$ and $M_1$. $\omega(u)$ is a rational form in the space of hyperplanes, and is holomorphic, if the original form $\omega$ is holomorphic. Since the space of hyperplanes is a rational variety and has no non-trivial holomorphic form of degree $\geqq 1$, it follows that $\omega(u) = 0$, if $\omega$ is holomorphic and of degree $\geqq 1$. Using this fact, it is possible to give a bound for the genus of an algebraic curve of order $d$, which lies in $P_N$, and not in $P_{N-1}$. The result is as follows [4]: Let $M_1$ be an algebraic curve of order $d$, which lies in $P_N$, but not in $P_{N-1}$. Let $s$ be the integer defined by the conditions: $s \equiv -d + 1$, mod $N - 1$, $0 \leqq s \leqq N - 2$. Then the genus of $M_1$ satisfies the inequality

$$(28)\qquad h_{01} \leqq \frac{1}{2(N-1)} \{(d-1)(d-N) + s(N-s-1)\}.$$

This result was obtained by Castlenuovo by the use of the Riemann-Roch Theorem [3]. The inequality (28) is the best possible in the sense that the bound at the right-hand side can be attained. The same method has been applied to find upper bounds for the geometrical genera of algebraic surfaces in $P_3$. They are 0, 1, 4, if the surfaces are of orders 3, 4, 5 respectively [2].

Instead of cutting the analytic curve $M_1$ by hyperplanes, we can also cut it by the hypersurfaces of a given order, which belong to an algebraic system. The abelian sum (27) can still be formed, except that the summation at the right-hand side is now taken over all the points of intersection of $M_1$ with the hypersurface. There are cases in which it can be proved that $\omega(u)$ is zero, even when $\omega$ is meromorphic. Results of this kind have been obtained by G. Humbert, together with numerous geometrical applications [11]. The basis of such geometrical applications lies in the fact that the differentials of various geometrical quantities are meromorphic differential forms. For instance, if $x$, $y$, $z$ are homogeneous coordinates in the plane, the element of area about the origin has the expression

$$d\alpha = \frac{1}{z^2}(x\,dy - y\,dx), \tag{29}$$

while the element of angle is given by

$$d\theta = \frac{1}{x^2 + y^2}(x\,dy - y\,dx). \tag{30}$$

Both differential forms are special cases of (22). We give in the following some simple theorems which can be derived by such considerations.

Let $M_1$ be a plane algebraic curve. Join a fixed point $O$ to the points where $M_1$ meets an algebraic curve $V$, which has no common asymptotic direction with $M_1$ (i.e. no point at infinity lies on both $M_1$ and $V$). The sum of areas described by the radius vectors is zero, if $V$ remains asymptotic to itself (i.e. if $V$ varies so that it passes through the same points at infinity and has there the same tangents).

Let $t$ be a fixed line in the plane. Let $R$, $S$ be two systems of $n$ lines each. If the sum of angles which the lines of $R$ make with $t$ is equal, up to a multiple of $\pi$, to the sum of angles which the lines of $S$ make with $t$, we say that $R$ and $S$ have the same inclination. This property is obviously independent of the choice of $t$. Then we have the theorem: *Let $M_1$ be a plane algebraic curve and $O$ a fixed point in the plane. Let $V$, $V'$ be two algebraic curves of the same order with the property that there is a curve of the pencil $V + \lambda V'$ which passes through all the points where $M_1$ meets the circle of radius zero about $O$. Then the lines joining $O$ to the points of intersection of $M_1$ and $V$ have the same inclination as the lines joining $O$ to the points of intersection of $M_1$ and $V'$.*

A much simpler theorem is the following: *Let $M$ and $M'$ be two plane curves of classes $m$ and $n$. The system of the $mn$ common tangents of $M$ and $M'$ has the same inclination as the system of $mn$ lines which join the $m$ real foci of $M$ to the $n$ real foci of $M'$.*

The theorems stated above have generalizations to higher dimensions.

A typical theorem for plane algebraic curves which does not seem to have a generalization to higher dimensions is an identity of Liebmann, together with its dual due to Bäckland. This is concerned with the meromorphic function (23) for the case $N = 2$. In this case the abelian sum as defined in (27) does not vanish identically. But it is possible to show that $\omega(u_0) = 0$, if $u_0$ is the line joining the points $A$ and $B$. In other words, we have

$$\sum_{1 \leq i \leq d} \frac{(Z_i, A)^3}{|Z_i, Z_i', B|^3} |Z_i, Z_i', Z_i''| = 0, \tag{31}$$

where the summation is extended over the points where the line $AB$ meets $M_1$, provided that it is nowhere tangent to $M_1$. Restricted to the real branch of the algebraic curve, this identity can be put in a geometrically more suggestive form, as follows: Let $\tilde{M}_1$ be a real algebraic curve of order $d$, and let $u$ be a line which meets $\tilde{M}_1$ in $d$ real and distinct points $p_i$, $1 \leqq i \leqq d$. Let $r_i$ be the curvature of $\tilde{M}_1$ at $p_i$ and $\tau_i$ the angle which $u$ makes with the tangent to $\tilde{M}_1$ at $p_i$. Then

$$\sum_{1 \leqq i \leqq d} \frac{r_i}{\sin^3 \tau_i} = 0. \tag{32}$$

This identity is due to Liebmann [12].

The identity (31) also gives an identity due to Bäcklund: Let $\tilde{M}_1$ be a real algebraic curve of class $m$, and $O$ a point not on the curve. Let $Op_i$, $1 \leqq i \leqq m$, be the tangents from $O$ to $\tilde{M}_1$, and $r_i$ the curvature of $\tilde{M}_1$ at $p_i$. Then we have

$$\sum_{1 \leqq i \leqq m} \frac{1}{r_i \overline{Op_i}^3} = 0. \tag{33}$$

Finally, it may be remarked that explicit exhibition of exterior differential forms on a submanifold in the real Euclidean space has also numerous geometrical consequences. It has recently been observed that many classical theorems in differential geometry in the large can be most quickly proved by picking a suitable differential form and using the fact that the integral of its exterior derivative over the manifold is zero. Among these theorems are [6]: the "Unverbiegbarkeit" of the sphere, Cohn-Vossen's rigidity theorem as proved by Herglotz, the Christoffel-Hurwitz and the Minkowski uniqueness theorems, etc. In all these problems the importance of the notion of exterior differential forms can hardly be exaggerated. Moreover, differential forms have their generalization in cochains, and it must be this fact which explains the counterparts of the above mentioned theorems for non-smooth convex surfaces.

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# ESPACES FIBRÉS ANALYTIQUES

PAR HENRI CARTAN

Je me propose de rendre compte de résultats récents de Hans Grauert[1] (à paraître aux Math. Annalen; voir une Note aux Comptes Rendus de l'Académie des Sciences de Paris, 30 janvier 1956). Ils concernent essentiellement les espaces fibrés analytiques principaux *dont la base est une variété de Stein.* Le cas où le groupe structural est *abélien* [12], ou, plus généralement, *résoluble* [9] était déjà connu; de plus, Frenkel a obtenu des résultats pour certains espaces de base qui ne sont pas des variétés de Stein.

## 1. Espaces analytiques holomorphiquement complets

La notion de variété analytique complexe est bien connue. Montrons cependant comment la catégorie des variétés analytiques complexes (et des applications analytiques) peut être définie dans le cadre général des conférences de S. Eilenberg (cf. ce Symposium). Prenons pour catégorie $\mathscr{M}$ de "modèles" celle dont les objets sont les ouverts des espaces numériques complexes $C^n$, et dont les applications sont les applications holomorphes d'un tel ouvert dans un autre. Si $T : \mathscr{M} \to \mathscr{T}$ désigne le foncteur évident dans la catégorie des espaces topologiques séparés (i.e. satisfaisant à l'axiome de Hausdorff), $T$ est *fidèle* et définit $\mathscr{M}$ comme *catégorie locale.* Alors la catégorie locale $\tilde{\mathscr{M}}$ (notation de Eilenberg) est celle des variétés analytiques complexes et des applications analytiques.

Nous aurons besoin d'une généralisation de la notion de variété analytique complexe (cf. [1], [3]). Prenons comme catégorie $\mathscr{M}$ de modèles celle-ci: un objet de $\mathscr{M}$ est un sous-ensemble $M \subset C^n$ tel qu'il existe un ouvert $U \subset C^n$, $U \supset M$, et des $f_i$ holomorphes dans $U$, de manière que

$$(x \in U, f_i(x) = 0 \text{ pour tout } i) \Leftrightarrow x \in M.$$

Une "application", dans la catégorie $\mathscr{M}$, est par définition une application continue $\varphi : M \to M'$ telle qu'on puisse choisir des ouverts $U \supset M$ et $U' \supset M'$ comme ci-dessus, et trouver une application holomorphe $\psi : U \to U'$ qui induise $\varphi$. La

[1] Je remercie vivement M. Grauert de m'avoir permis de prendre connaissance du manuscrit de son travail. Dans la manière d'exposer les résultats et la marche des démonstrations, je prends ici quelques libertés, tout en respectant les idées essentielles des démonstrations de Grauert. J'ai cru bon d'introduire la notion d'espace fibré $E$-principal (§2) qui généralise la notion classique d'espace fibré principal, et permet de donner leur pleine valeur aux Théorèmes, A et B de Grauert, ainsi qu'au Théorème 1 du §3. Il devient alors possible de déduire le Théorème A du Théorème 1 (cf. §3). J'ai aussi introduit un sous-espace analytique $Y$ qui permet de renforcer les Théorèmes 1 et 2 (voir les Théorèmes 1 bis et 2 bis). Enfin j'ai condensé en un seul énoncé ("Théorème principal", §4) des résultats techniques auxiliaires établis par Grauert.

catégorie $\mathcal{M}$ étant ainsi définie, le foncteur $T : \mathcal{M} \to \mathcal{T}$ est évident; alors $\mathcal{M}$ est une catégorie locale. On en déduit une catégorie locale $\tilde{\mathcal{M}}$: les objets de $\tilde{\mathcal{M}}$ s'appellent les *espaces analytiques*, les applications de $\tilde{\mathcal{M}}$ s'appellent les *applications analytiques* (ou *holomorphes*). Tout objet de $\tilde{\mathcal{M}}$ est localement isomorphe à un objet de $\mathcal{M}$. D'autre part, la catégorie des variétés analytiques complexes s'identifie évidemment à une *sous-catégorie pleine* de la catégorie des espaces analytiques.

On n'a pas supposé que les modèles $M$ soient analytiquement irréductibles en chacun de leurs points.

Soit $X$ un espace analytique; une fonction holomorphe (scalaire) est simplement une application analytique $X \to C$. Pour tout ouvert $U \subset X$ on a l'anneau $\mathcal{O}(U)$ des fonctions holomorphes dans $U$; ces anneaux (lorsque $U$ parcourt l'ensemble de tous les ouverts de $X$) définissent sur l'espace $X$ le faisceau $\mathcal{O}_X$ des germes de fonctions holomorphes. La donnée de l'espace topologique sous-jacent à $X$ et du faisceau $\mathcal{O}_X$ détermine complètement $X$ comme espace analytique.

Soit $X$ un espace analytique. On appelle *sous-espace analytique* un sous-ensemble *fermé* $Y \subset X$, tel que, au voisinage de chaque point $x_0 \in Y$, $Y$ puisse être défini par un nombre fini d'équations $f_i(x) = 0$, les $f_i$ étant holomorphes au voisinage de $x_0$ (dans l'espace $X$). Un tel $Y$ définit évidemment un objet de la catégorie $\tilde{\mathcal{M}}$ des espaces analytiques. Le faisceau $\mathcal{O}_Y$ est le faisceau induit sur $Y$ par le faisceau $\mathcal{O}_X$.

La catégorie des espaces analytiques est évidemment une *catégorie avec produits* (cf. 3$^e$ exposé de S. Eilenberg). On peut donc développer une théorie des *espaces fibrés analytiques* localement triviaux (la base et la fibre étant des espaces analytiques, dans le sens général qui vient d'être défini).

Si on restreignait la catégorie $\mathcal{M}$ des modèles à la sous-catégorie $\mathcal{M}'$ des modèles *normaux* ($M$ étant alors un sous-ensemble analytique normal dans $C^n$), on obtiendrait la sous-catégorie $\tilde{\mathcal{M}}'$ des *espaces analytiques normaux*, qui sont ceux considérés par Grauert; mais les résultats de Grauert s'étendent d'eux-mêmes au cas des espaces analytiques les plus généraux.

La notion bien connue de "variété de Stein" (les variétés de Stein constituent une sous-catégorie de la catégorie des variétés analytiques complexes) se généralise comme suit au cas des espaces analytiques:

DÉFINITION 1. On dit qu'un espace analytique $X$ est *holomorphiquement complet* s'il satisfait aux conditions suivantes:

(i) l'espace topologique $X$ est réunion dénombrable de compacts;

(ii) pour chaque point $x \in X$, il existe une application analytique $f : X \to C^k$ ($k$ désignant un entier convenable qui dépend de $x$), qui soit *non dégénérée* au point $x$ (i.e. telle que $x$ soit point isolé de l'ensemble $f^{-1}(f(x))$);

(iii) pour tout compact $K \subset X$, l'ensemble $\tilde{K}$ des $x \in X$ tels que $|f(x)| \leq \sup_{y \in K} |f(y)|$ pour toute $f$ holomorphe dans $X$, est *compact*.

Si, dans cette définition, on suppose en outre que $X$ est une vraie variété analytique complexe, on retrouve la définition d'une variété de Stein (à condition d'utiliser des résultats de Grauert: [10], Satz B). De plus, si un espace analytique $X$ est *connexe* et satisfait à (ii), $X$ satisfait à (i) (Grauert, [10], Satz A).

Il est évident que tout sous-espace analytique d'un espace holomorphiquement complet est holomorphiquement complet.

DÉFINITION 2. Soit $X$ un espace analytique. Un ouvert $U \subset X$ est dit *$X$-convexe* si, pour tout compact $K \subset U$, l'ensemble $\tilde{K}_X$ des $x \in U$ tels que

$$|f(x)| \leqq \sup_{y \in K} |f(y)| \text{ pour toute } f \text{ holomorphe dans } X,$$

est *compact*. (Remarque: la condition (iii) exprime que $X$ est $X$-convexe; d'autre part, tout ouvert $X$-convexe est un espace holomorphiquement complet).

DÉFINITION 3. Soit $X$ un espace analytique. Un compact $K \subset X$ est dit *spécial* (ou, plus exactement, $X$-spécial) s'il existe un système fini de fonctions $f_j$ holomorphes dans $X$ et de constantes réelles $a_j \leqq b_j$, $a'_j \leqq b'_j$, de manière que $K$ soit à la fois ouvert et fermé dans l'ensemble des points $x \in X$ satisfaisant aux inégalités

$$a_j \leqq \mathrm{Re}\,(f_j) \leqq b_j, \qquad a'_j \leqq \mathrm{Im}\,(f_j) \leqq b'_j$$

($\mathrm{Re}\,(f_j)$ et $\mathrm{Im}\,(f_j)$ désignent respectivement la partie réelle et la partie imaginaire de $f_j$).

On utilisera les propriétés connues que voici: soit $K$ un *compact spécial*; en adjoignant au besoin aux $f_j$ de nouvelles fonctions holomorphes dans $X$ (en nombre fini), on peut faire en sorte que l'application $f : X \to C^k$ définie par les $k$ fonctions $f_j$ induise un *isomorphisme* d'un voisinage de $K$ (comme espace analytique) sur un sous-espace analytique d'un voisinage du *cube* compact $\Gamma$ défini, dans l'espace numérique $C^k$, par les inégalités

$$a_j \leqq x_j \leqq b_j, \qquad a'_j \leqq y_j \leqq b'_j$$

(on note $x_j + iy_j$ les $k$ coordonnées complexes d'un point de $C^k$). Il résulte alors de théorèmes connus ([4], th. 2 et 3) que toute fonction $\varphi$ holomorphe au voisinage de $K$ peut s'écrire $\Phi(f_1, \cdots, f_k)$, $\Phi$ étant holomorphe au voisinage du cube $\Gamma$; comme $\Phi$ peut être uniformément approchée (au voisinage de $\Gamma$) par des polynômes par rapport aux coordonnées complexes de l'espace ambiant $C^k$, on voit que *toute fonction holomorphe au voisinage du compact spécial $K$ peut être uniformément approchée* (au voisinage de $K$) *par des fonctions holomorphes dans $X$.*

Soit maintenant $K$ un compact tel que $\tilde{K}_X$ soit compact. Alors tout voisinage de $\tilde{K}_X$ contient un compact spécial contenant $\tilde{K}_X$. Il en résulte que, si $U$ est un ouvert $X$-convexe, *$U$ est réunion d'une suite croissante de compacts spéciaux $K_n$*, tels en outre que chaque $K_n$ soit *intérieur* à $K_{n+1}$. On en déduit: *toute fonction holomorphe dans $U$* (ouvert $X$-convexe) *est limite* (uniformément sur tout compact de $U$) *de fonctions obtenues par restriction à $U$ de fonctions holomorphes dans $X$.*

Inversement, on voit facilement ceci: tout compact spécial $K$ possède un système fondamental de voisinages ouverts dont chacun est $X$-convexe.

## 2. Espaces fibrés $E$-principaux

Pour les définitions qui suivent, il est inutile de supposer que la base $X$ (qui, par hypothèse, est un espace analytique) soit holomorphiquement complète.

Soit donné un espace fibré analytique $E$, de base $X$, dont les fibres sont des *groupes de Lie* (complexes), tous isomorphes. Cela signifie qu'il existe un recouvrement ouvert $(U_i)$ de $X$ et un groupe de Lie $G$, tel que $E$ puisse être obtenu par la construction suivante: on forme la somme $F$ des espaces analytiques $U_i \times G$, et on fait le quotient de $F$ par une relation d'équivalence $R$ du type suivant: si $x \in U_{ij}$ $(= U_i \cap U_j)$, on identifie le point $(x, y) \in U_j \times G$ au point $(x, f_{ij}(x, y)) \in U_i \times G$, où les $f_{ij}$ sont des applications *analytiques* données

$$f_{ij} : U_{ij} \times G \rightarrow G$$

satisfaisant aux conditions suivantes: 1° $f_{ij}(x, f_{jk}(x, y)) = f_{ik}(x, y)$ pour $x \in U_{ijk}$, $y \in G$; 2° pour chaque $x \in U_{ij}$, l'application $y \rightarrow f_{ij}(x, y)$ est un *automorphisme* du groupe de Lie $G$.

$E$ étant donné, nous noterons $p : E \rightarrow X$ la projection du fibré $E$ sur sa base, et $G_x = p^{-1}(x)$ la fibre au-dessus du point $x \in X$; $G_x$ a une structure de groupe de Lie (complexe), mais il n'y a pas d'isomorphisme canonique de $G_x$ sur le groupe-modèle $G$.

Donnons-nous maintenant un recouvrement ouvert arbitraire de $X$, que nous noterons encore $(U_i)$, et, pour chaque couple $(i, j)$, une section holomorphe (resp. continue)

$$f_{ij} : U_{ij} \rightarrow E,$$

de manière que $f_{ij}f_{jk} = f_{ik}$ dans $U_{ijk}$ (la multiplication étant entendue au sens de la loi de groupe qui existe dans chaque fibre). Autrement dit, $(f_{ij})$ est un 1-*cocyle* du recouvrement $(U_i)$, à valeurs dans les sections du faisceau $\mathscr{E}^a$ (resp. du faisceau $\mathscr{E}^c$) des germes de sections holomorphes (resp. continues) du fibré $E$; $\mathscr{E}^a$ et $\mathscr{E}^c$ sont des *faisceaux de groupes*. Alors les $f_{ij}$ permettent de construire un nouvel espace fibré analytique (resp. topologique) $P$, comme suit: on prend la somme des $p^{-1}(U_i)$, et on en fait le quotient par la relation d'équivalence définie comme suit: si $x \in p^{-1}(U_{ij}) \subset E$, on identifie le point $x \in p^{-1}(U_j)$ au point

$$f_{ij}(p(x)) \cdot x \in p^{-1}(U_i),$$

la multiplication étant celle qui existe dans chaque fibre de $E$. Dans le fibré $P$, les fibres n'ont plus une structure de groupe; toutefois, si $u$ et $v \in P$ appartiennent à une même fibre, on peut définir $u^{-1}v$ qui est un point de $E$. Un tel espace $P$ sera dit *E-principal*; l'espace $E$ *opère à droite* dans $P$, dans le sens suivant: chaque fibre de $E$ est un groupe qui opère à droite dans la fibre correspondante de $P$.

Dans le cas particulier où $E$ est un produit $X \times G$, on retrouve la notion classique d'espace fibré principal de groupe $G$.

On a une notion évidente d'*isomorphisme* pour deux espaces fibrés $E$-principaux de même base $X$ (et relatifs au même fibré $E$): c'est un homéomorphisme qui induit l'application identique de la base $X$ et est compatible avec les opérations (à droite) de $E$; s'il s'agit d'espaces $E$-principaux *analytiques*, on astreint en outre cet homéomorphisme à être analytique. L'isomorphisme définit alors entre espaces $E$-principaux analytiques (resp. topologiques) une relation d'équivalence, et l'on peut parler de l'ensemble des *classes* d'espaces $E$-principaux analytiques (resp. topologiques). D'après un raisonnement classique, ces classes sont en correspondance biunivoque avec l'ensemble de cohomologie $H^1(X, \mathscr{E}^a)$, resp. $H^1(X, \mathscr{E}^c)$; si un espace $P$ est défini par un cocyle $(f_{ij})$ d'un recouvrement ouvert $\mathscr{U} = (U_i)$ de $X$, la classe de $P$ est l'image de la classe de cohomologie de $(f_{ij})$ par l'application naturelle

$$H^1(\mathscr{U}, \mathscr{E}) \to H^1(X, \mathscr{E}),$$

$\mathscr{E}$ désignant $\mathscr{E}^a$, resp. $\mathscr{E}^c$. De plus, l'inclusion de faisceaux $\mathscr{E}^a \to \mathscr{E}^c$ définit une application

$$H^1(X, \mathscr{E}^a) \to H^1(X, \mathscr{E}^c) \tag{1}$$

qui, à chaque classe d'espaces $E$-principaux analytiques, associe une classe d'espaces $E$-principaux topologiques.

Le résultat fondamental de Grauert est le suivant: *si $X$ est holomorphiquement complet, l'application* (1) *est une bijection.*

Afin d'expliciter ce qu'il convient de démontrer pour établir ce résultat, faisons un bref rappel concernant la cohomologie à coefficients dans un faisceau de groupes. Soit $\mathscr{F}$ un tel faisceau sur un espace topologique $X$; si $\mathscr{U} = (U_i)$ est un recouvrement ouvert de $X$, deux cocyles $f_{ij} : U_{ij} \to \mathscr{F}$ et $g_{ij} : U_{ij} \to \mathscr{F}$ sont dits *homologues* s'il existe une cochaîne $c_i : U_i \to \mathscr{F}$ telle que

$$g_{ij} = (c_i)^{-1} f_{ij} c_j \qquad \text{dans } U_{ij}. \tag{2}$$

L'ensemble $H^1(\mathscr{U}, \mathscr{F})$ est, par définition, l'ensemble des classes de cocycles homologues. Si $\mathscr{V}$ est un recouvrement plus fin que $\mathscr{U}$, on définit sans ambigüité (cf. [11], p. 41) une application $H^1(\mathscr{U}, \mathscr{F}) \to H^1(\mathscr{V}, \mathscr{F})$; alors $H^1(X, \mathscr{F})$ est défini comme la limite directe des $H^1(\mathscr{U}, \mathscr{F})$. Il est essentiel de remarquer que *l'application $H^1(\mathscr{U}, \mathscr{F}) \to H^1(\mathscr{V}, \mathscr{F})$ est injective*; on le prouve comme suit: soit $\mathscr{V} = (V_\alpha)$, et soit $\alpha \to \lambda(\alpha)$ une application de l'ensemble d'indices de $\mathscr{V}$ dans l'ensemble d'indices de $\mathscr{U}$, telle que $U_{\lambda(\alpha)} \supset V_\alpha$. Etant donnés deux cocycles $(f_{ij})$, $(g_{ij})$ de $\mathscr{U}$, on leur associe les cocyles

$$\varphi_{\alpha\beta} = f_{\lambda(\alpha),\lambda(\beta)}, \qquad \psi_{\alpha\beta} = g_{\lambda(\alpha),\lambda(\beta)}$$

de $\mathscr{V}$. Supposons qu'il existe une cochaîne $(\gamma_\alpha)$ de $\mathscr{V}$, telle que

$$\psi_{\alpha\beta} = (\gamma_\alpha)^{-1} \varphi_{\alpha\beta} \gamma_\beta \qquad \text{dans } V_{\alpha\beta};$$

on définit alors une cochaîne $(c_i)$ de $\mathscr{U}$ satisfaisant à (2), comme suit: pour $x \in U_i$, choisissons un $\alpha$ tel que $x \in V_\alpha$, et considérons

$$f_{i,\lambda(\alpha)}(x)\, \gamma_\alpha(x) g_{\lambda(\alpha),i}(x);$$

ceci ne dépend pas du choix de $\alpha$, et définit une section $x \to c_i(x)$ de $\mathscr{F}$ au-dessus de $U_i$, qui satisfait à (2).

Compte tenu de la remarque précédente, on voit que pour prouver que (1) est bijective, on doit démontrer les deux théorèmes suivants (qui expriment que (1) est injective, resp. surjective):

Théorème A. *Soit* $(U_i)$ *un recouvrement ouvert de* $X$ *holomorphiquement complet. Soient deux cocycles holomorphes*

$$f_{ij} : U_{ij} \to E, \quad g_{ij} : U_{ij} \to E.$$

*S'il existe des sections continues* $c_i : U_i \to E$ *satisfaisant à* (2), *il existe aussi des sections holomorphes satisfaisant aux mêmes relations.*

Théorème B. *Soit un recouvrement de* $X$ *holomorphiquement complet par des ouverts* $U_i$ *holomorphiquement complets. Soit un cocycle continu* $f_{ij} : U_{ij} \to E$. *Alors il existe des sections continues* $c_i : U_i \to E$ *telles que le cocycle.*

$$g_{ij} = (c_i)^{-1} f_{ij} c_j$$

*soit holomorphe.*

## 3. Homotopie entre sections d'un espace $E$-principal

Laissons d'abord de côté le Théorème B. On va transformer l'énoncé du Théorème A, en interprétant les cochaînes $(c_i)$ qui satisfont à (2) comme les *sections* d'un espace fibré auxiliaire. Le relation (2) s'écrit en effet

$$c_i = f_{ij} c_j (g_{ij})^{-1} = f_{ij} c_j g_{ji};$$

définissons un automorphisme $\theta_{ij}$ de l'espace des sections $U_{ij} \to E$ en associant à chaque section $c$ la section $f_{ij} c g_{ji}$ (cet automorphisme ne respecte pas la structure de groupe des fibres). Il est clair que l'on a $\theta_{ij} \circ \theta_{jk} = \theta_{ik}$ dans l'espace des sections $U_{ijk} \to E$; donc, en recollant les $p^{-1}(U_i) \subset E$ au moyen des automorphismes $\theta_{ij}$, un obtient un nouvel espace fibré $Q$, de base $X$; toute cochaîne $(c_i)$ satisfaisant à (2) définit une *section* de $Q$, et réciproquement. Plus précisément, les sections *holomorphes* de $Q$ correspondent aux cochaînes $(c_i)$ holomorphes qui satisfont à (2); de même pour les cochaînes *continues*.

De plus, soient $(c_i)$ et $(c_i')$ deux telles cochaînes; on a

$$(c_i)^{-1} c_i' = g_{ij}((c_j)^{-1} c_j')(g_{ij})^{-1};$$

cela signifie que la cochaîne $(c_i^{-1} c_i')$ définit une *section* de l'espace fibré $E_g$ déduit de $E$ en recollant les $p^{-1}(U_i)$ par les transformations

$$x \to g_{ij}(p(x)) \cdot x \cdot (g_{ij}(p(x)))^{-1}$$

qui respectent la structure de groupe des fibres de $E$; ainsi $E_g$ est un fibré analytique dont les fibres sont des groupes de Lie (isomorphes, mais non canoniquement, aux fibres de $E$), et le quotient $(c_i^{-1} c_i')$ de deux sections holomorphes (resp. continues) de $Q$ définit une section holomorphe (resp. continue) de $E_g$. Comme ce raisonnement vaut non seulement pour $X$, mais pour tout ouvert $U$ de l'espace de base $X$, on voit que (même lorsque $Q$ n'a pas de section globale au-dessus de $X$), *$Q$ est un espace $E_g$-principal.*

Il est maintenant clair que le Théorème A résultera du théorème plus précis:

Théorème 1. *Soit* $P$ *un espace* $E$*-principal analytique, dont la base* $X$ *est holomorphiquement complète. Alors toute section continue de* $P$ *est homotope à une section holomorphe de* $P$.

(Remarque. On munit l'ensemble des sections continues de $P$ de la topologie de la "convergence compacte" ("compact-open topology"); une homotopie dans l'ensemble des sections continues de $P$ n'est pas autre chose qu'un chemin dans cet espace topologique).

Le Théorème 1 implique évidemment que, s'il existe une section continue, il existe aussi une section holomorphe; appliquons ce résultat en remplaçant $E$ par $E_g$, et $P$ par $Q$: on obtient le Théorème A.

On établira aussi un théorème d'approximation:

Théorème 2. *Soit $P$ un espace $E$-principal analytique, dont la base $X$ est holomorphiquement complète; soit $U$ un ouvert de $X$, $X$-convexe, et soit $s$ une section holomorphe $U \to P$. Si $s$ peut être arbitrairement approchée* (dans l'espace des sections *continues* au-dessus de $U$) *par la restriction à $U$ de sections continues $X \to P$, alors $s$ peut être arbitrairement approchée par la restriction à $U$ de sections holomorphes $X \to P$.*

On notera que, même dans le cas trivial où $P = E$ et où $E$ est un produit $X \times G$, les Théorèmes 1 et 2 ne sont nullement évidents. Le Théorème 1 dit alors que toute application *continue* de $X$ dans un groupe de Lie complexe $G$ est homotope à une application *holomorphe* $X \to G$. Et le Théorème 2 dit que la possibilité d'approcher arbitrairement une application holomorphe $U \to G$ par les restrictions à $U$ d'applications holomorphes $X \to G$ est un problème dont l'obstruction est purement topologique.

Nous allons renforcer les deux théorèmes précédents: $E$ et $P$ ayant la même signification que dans les Théorèmes 1 et 2, nous introduisons un *sous-espace analytique* $Y$ de la base $X$ ($X$ est toujours supposé holomorphiquement complet):

Théorème 1 bis. *Soit $f: X \to P$ une section continue du fibré $P$, telle que la restriction $g: Y \to P$ de $f$ à $Y$ soit holomorphe. Alors, dans l'espace de toutes les sections continues de $P$ qui prolongent $g$, $f$ est homotope à une section holomorphe $X \to P$.*

Corollaire du Théorème 1 bis. Si une section *holomorphe* $Y \to P$ peut être prolongée en une section *continue* $X \to P$, elle peut aussi être prolongée en une section *holomorphe* $X \to P$.

Théorème 2 bis. *Soit $U$ un ouvert $X$-convexe de $X$. Soit $f: U \to P$ une section holomorphe, et soit $g: Y \to P$ une section holomorphe telle que $f$ et $g$ coïncident sur $Y \cap U$. Si $f$ peut être arbitrairement approchée par des sections continues $X \to P$ qui prolongent $g$, alors $f$ peut être arbitrairement approchée par des sections holomorphes $X \to P$ qui prolongent $g$.*

Le corollaire du Théorème 1 bis va entraîner un autre résultat:

Théorème 3. *L'espace $X$ étant toujours supposé holomorphiquement complet, soient $f, f': X \to P$ deux sections holomorphes. Supposons que $f$ et $f'$ soient homotopes dans l'espace de toutes les sections continues $X \to P$. Alors il existe une application holomorphe $h: X \times I \to P$* (où $I$ désigne le segment $[0, 1]$ de la droite réelle, considéré comme plongé dans le plan complexe $C$), *telle que*

$$\begin{cases} h(x, 0) = f(x), h(x, 1) = f'(x) \text{ pour } x \in X, \\ p(h(x, t)) = x \text{ pour } x \in X, t \in I. \end{cases}$$

*en notant $p$ la projection $P \to X$.*

(En particulier, $f$ et $f'$ sont homotopes dans l'espace des sections *holomorphes* de $P$; mais le théorème dit davantage, puisque l'homotopie définie par $h(x, t)$ dépend *analytiquement* du paramètre de déformation $t \in I$).

Démonstration du Théorème 3. Nous admettons le Théorème 1 bis, qui sera démontré plus loin, ainsi que le Théorème 2 bis. Soit $U$ un disque ouvert dans le plan $C$ de la variable complexe $t$; supposons que $U$ contienne les points $t = 0$ et $t = 1$, et identifions $I$ à un fermé de $U$. D'après l'hypothèse de l'énoncé, il existe une section continue $\varphi : X \times I \to P$ telle que

$$\varphi(x, 0) = f(x), \varphi(x, 1) = f'(x), p(\varphi(x, t)) = x.$$

Comme $I$ est rétracte de $U$, on peut prolonger $\varphi$ en une application continue $X \times U \to P$ jouissant des mêmes propriétés; on la notera encore $\varphi$. L'application $(x, t) \to (\varphi(x, t), t)$ est alors une section $\psi$ de $P \times U$, considéré comme fibré $(E \times U)$-principal, dont la base $X \times U$ est holomorphiquement complète. Considérons le sous-espace analytique

$$Y' = X \times \{0, 1\}$$

de $X' = X \times U$; la restriction de $\psi$ à $Y'$ est une section *holomorphe*. D'après le corollaire au Théorème 1 bis, il existe une section holomorphe $X \times U \to P \times U$ qui coïncide avec $\psi$ sur $Y'$; cette section a la forme

$$(x, t) \to (h(x, t), t),$$

où $h$ est une application holomorphe $X \times U \to P$. La restriction de $h$ à $X \times I$ démontre le théorème.

Remarque. Nous laissons au lecteur le soin d'énoncer et de démontrer un Théorème 3 bis, dans lequel intervient un sous-espace analytique $Y$ de $X$.

### 4. Le théorème principal

Les Théorèmes 1 bis et 2 bis du paragraphe précédent seront déduits d'un théorème assez technique, qu'on va exposer maintenant. $X$ désigne toujours un espace holomorphiquement complet, et $Y$ un sous-espace analytique (fermé) de $X$; $E$ désigne toujours un fibré analytique de base $X$ dont les fibres sont des groupes de Lie complexes. Pour chaque ouvert $U \subset X$, soit $\mathscr{G}^c(U)$ le groupe de toutes les sections continues $U \to E$ qui sont *neutres* sur $Y \cap U$ (noter que chaque fibre de $E$ possède un élément neutre); $\mathscr{G}^c(U)$ est un *groupe topologique*, pour la topologie de la convergence compacte; il possède un sous-groupe fermé $\mathscr{G}^a(U)$, formé des sections *holomorphes* $U \to E$ qui sont neutres sur $Y \cap U$.

Soit maintenant $C$ un espace auxiliaire, qu'on supposera *compact*: ce sera l'espace d'un paramètre $t$. Une application continue $\varphi$: $C \to \mathscr{G}^c(U)$ n'est pas autre chose qu'une application continue $(x, t) \to s(x, t)$ de $U \times C$ dans $E$, telle que:

$$p(s(x, t)) = x, \qquad s(x, t) \text{ neutre pour } x \in Y \cap U.$$

Soient de plus donnés deux *sous-espaces fermés* $N \subset H$ de $C$; on appellera $(N, H, C)$-application dans $\mathscr{G}^c(U)$ une application continue $\varphi : C \to \mathscr{G}^c(U)$ telle que:

1°. pour chaque $t \in N$, $\varphi(t)$ est la section neutre;

2°. pour chaque $t \in H$, $\varphi(t) \in \mathscr{G}^a(U)$, c'est-à-dire est une section *holomorphe*.

Notons pour un instant $\mathscr{F}(U)$ le groupe (topologique) de toutes les $(N, H, C)$-applications dans $\mathscr{G}^c(U)$. Lorsque $U$ parcourt l'ensemble des ouverts de $X$, les groupes $\mathscr{F}(U)$ constituent un *préfaisceau*, donc définissent un *faisceau* que nous noterons $\mathscr{F}$. Il est clair que $\mathscr{F}(U)$ n'est pas autre chose que le groupe $H^0(U, \mathscr{F})$ des sections du faisceau $\mathscr{F}$ au-dessus de $U$.

On notera que le faisceau $\mathscr{F}$ dépend de la donnée du fibré $E$ (de base $X$), du sous-espace analytique $Y \subset X$, et des espaces $N, H, C$.

Théorème principal. *Supposons que $X$ soit holomorphiquement complet, et que $N$ soit rétracte de déformation de $C$. Alors, $\mathscr{F}$ désignant le faisceau défini ci-dessus:*

(i) *le groupe topologique $H^0(X, \mathscr{F})$ est connexe par arcs*;

(ii) *si $U$ est un ouvert $X$-convexe de $X$, l'image de l'application $H^0(X, \mathscr{F}) \to H^0(U, \mathscr{F})$ est dense dans $H^0(U, \mathscr{F})$*;

(iii) $H^1(X, \mathscr{F}) = 0$.

La démonstration de ce théorème sera indiquée plus loin (§ 6). Pour le moment, on va montrer comment les Théorèmes 1 bis et 2 bis peuvent s'en déduire, en prenant

$$C = [0, 1], \quad H = \{0, 1\}, \quad N = \{0\}.$$

Démonstration du Théorème 1 bis. Soit $f : X \to P$ une section continue, dont la restriction $g$ à $Y$ soit holomorphe. Chaque point $x \in X$ appartient évidemment à un ouvert $U$ tel que la restriction $f|U$ soit homotope à une section holomorphe $U \to P$ dans l'espace de toutes les sections continues $U \to P$ qui prolongent la restriction $g|(U \cap Y)$; on le vérifie en observant que la restriction de $P$ au-dessus de $U$ est isomorphe à $U \times G$. Ainsi, nous avons un recouvrement ouvert $(U_i)$ de $X$, et pour chaque $U_i$ une homotopie

$$(x, t) \to f_i(x, t), \quad x \in U_i, \quad 0 \leqq t \leqq 1$$

avec $f_i(x, 0) = f(x)$, $f_i(x, 1)$ holomorphe, $f_i(x, t) = g(x)$ pour $x \in Y$. Dans $U_{ij}$, la section $f_{ij}(x, t) = f_i(x, t)^{-1} f_j(x, t)$ du fibré $E$ est neutre pour $x \in Y \cap U_{ij}$, neutre pour $t = 0$, holomorphe pour $t = 1$; donc $(f_{ij}(x, t))$ est un cocyle du faisceau $\mathscr{F}$. En vertu de l'assertion (iii) du Théorème principal, il existe une cochaîne $c_i(x, t)$ du faisceau $\mathscr{F}$, telle que

$$f_{ij}(x, t) = c_i(x, t)^{-1} c_j(x, t) \text{ pour } x \in U_{ij}.$$

Définissons, pour $x \in U_i$, $h_i(x, t) = f_i(x, t) c_i(x, t)^{-1}$ (rappelons que $E$ opère à droite dans $P$); alors $h_i(x, t) = h_j(x, t)$ pour $x \in U_{ij}$, et par suite les $h_i$ définissent une section $h(x, t)$ de $P$ au-dessus de $X$, dépendant du paramètre $t \in [0, 1]$. On a $h(x, 0) = f(x)$ parce que $f_i(x, 0) = f(x)$ et $c_i(x, 0) = e$; pour $x \in Y$, on a $h(x, t) = g(x)$ parce que $f_i(x, t) = g(x)$ et $c_i(x, t) = e$; enfin, $h(x, 1)$ est holomorphe, parce que $f_i(x, 1)$ et $c_i(x, 1)$ sont holomorphes. Alors $h(x, t)$ fournit l'homotopie désirée et prouve le théorème 1 bis.

Démonstration du Théorème 2 bis. Soit $K$ un compact donné, contenu dans $U$.

On cherche une section holomorphe $X \to P$, qui prolonge $g : Y \to P$, et soit arbitrairement voisine, au-dessus de $K$, de la section holomorphe donnée $f : U \to P$ (cela signifie que la section cherchée doit prendre sur $K$ des valeurs qui se trouvent dans un voisinage arbitraire de l'image $f(K)$). Il existe un ouvert $U' \subset X$ possédant les propriétés suivantes: $K \subset U'$, $U'$ est $X$-convexe, $\overline{U'}$ est compact et contenu dans $U$. Par hypothèse, il existe une section continue $\varphi : X \to P$ prolongeant $g$, et arbitrairement voisine de $f$ au-dessus de $\overline{U'}$; alors les restrictions $f'$ et $\varphi'$ de $f$ et $\varphi$ à $U'$ sont *homotopes* dans l'espace des sections continues, égales à $g$ au-dessus de $Y \cap U'$ (on le voit en utilisant les "paramètres canoniques" dans chaque fibre de $E$, qui est un groupe de Lie; noter que $f'^{-1}\varphi'$ est une section de $E$, voisine de la section neutre). D'autre part, d'après le Théorème 1 bis, $\varphi$ est homotope (dans l'espace de toutes les sections continues $X \to P$ qui prolongent $g$) à une section holomorphe $h : X \to P$ qui prolonge $g$. Soit $h'$ la restriction de $h$ à $U'$; alors $f'$ et $h'$ sont homotopes (dans l'espace des sections continues $U' \to P$ qui prolongent la restriction de $g$ à $U' \cap Y$). Ainsi $f'^{-1}h'$ est une section continue $U' \to E$, neutre sur $U' \cap Y$, et homotope à la section neutre (dans l'espace des sections qui sont neutres sur $U' \cap Y$). D'après l'assertion (ii) du théorème principal, il existe une section holomorphe $k : X \to E$, neutre au-dessus de $Y$, et dont la restriction $k'$ à $U'$ est arbitrairement voisine de $f'^{-1}h'$. Alors la restriction de $hk^{-1} : X \to P$ à $U'$ est arbitrairement voisine de $f'$, et en particulier $hk^{-1}$ est arbitrairement voisine de $f$ au-dessus du compact $K$. Comme $hk^{-1}$ est une section holomorphe de $P$, égale à $g$ au-dessus de $Y$, le Théorème 2 bis est établi.

## 5. Démonstration du Théorème B

Nous venons de montrer comment le Théorème principal du §4 (qui sera établi plus loin, §6) entraîne les Théorèmes 1 bis et 2 bis, et a fortiori le Théorème A du §2. On va maintenant démontrer le Théorème B du §2.

Compte tenu de la définition et des propriétés de la cohomologie $H^1(X, \mathscr{E}^a)$, resp. de $H^1(X, \mathscr{E}^c)$, il suffit de prouver le Théorème B pour des recouvrements ouverts $(U_i)$ tels qu'il y en ait d'arbitrairement fins. Dans ce qui suit, nous pourrons donc supposer que chaque ouvert $U_i$ est *holomorphiquement complet* et *relativement compact* (en effet, tout point de $X$ possède un système fondamental de voisinages ouverts qui jouissent de ces deux propriétés). De plus, ou supposera que le recouvrement $(U_i)$ est *localement fini*.

Soit alors $f_{ij} : U_{ij} \to E$ un cocyle *continu*. Convenons de dire qu'un ouvert $V \subset X$ est *bon* s'il existe des sections continues $c_i : U_i \cap V \to E$ telles que $c_i^{-1}f_{ij}c_j$ soit *holomorphe* dans $U_{ij} \cap V$. Nous voulons prouver que $X$ est bon; et nous savons déjà que tout ouvert assez petit de $X$ est bon.

Première partie de la démonstration. On va montrer que *si $V$ est l'intérieur d'un compact spécial $K$, $V$ est bon.* D'après le §1, $K$ possède un voisinage qui se réalise comme sous-espace analytique dans un voisinage d'un cube compact $\Gamma$. En recouvrant chaque côté de $\Gamma$ par un nombre fini d'intervalles assez petits, et en faisant le produit de ces recouvrements, on obtient un recouvrement ouvert fini de $\Gamma$ par des cubes $\Gamma_{\alpha_1, \ldots, \alpha_k}$ ($k$ désigne la dimension réelle du cube $\Gamma$), qui induit sur $V$ un recouvrement par des ouverts $V_{\alpha_1,\dots,\alpha_k} = V \cap \Gamma_{\alpha_1,\dots,\alpha_k}$ dont chacun est bon. Notons, pour chaque entier $m$ tel que $0 \leqq m \leqq k$, $V_{\alpha_{m+1},\dots,\alpha_k}$ la réunion $\bigcup_{\alpha_1,\dots,\alpha_m} V_{\alpha_1,\dots,\alpha_k}$. On va prouver, par récurrence sur $m$, que $V_{\alpha_{m+1},\dots,\alpha_k}$ est bon; c'est vrai pour $m = 0$. La récurrence utilise le lemme suivant:

LEMME. *Soit un cube compact* $\Gamma$, *dont un côté* $I$ *est réunion de deux intervalles ouverts* $I'$ *et* $I''$; *soient* $\Gamma'$ *et* $\Gamma''$ *les images réciproques de* $I'$ *et* $I''$ *dans* $\Gamma$; *on a donc* $\Gamma = \Gamma' \cup \Gamma''$, *et l'intersection* $\Gamma' \cap \Gamma''$ *est un cube. Supposons l'ouvert* $V$ *réalisé comme sous-ensemble analytique dans l'intérieur de* $\Gamma$, *et soit* $V' = V \cap \Gamma'$, $V'' = V \cap \Gamma''$. *Alors si* $V'$ *et* $V''$ *sont bons,* $V$ *est bon.*

Prouvons ce Lemme. Par hypothèse, on a des sections continues $c'_i : V' \cap U_i \to E$ et $c''_i : V'' \cap U_i \to E$ telles que $c'^{-1}_i f_{ij} c'_j = f'_{ij}$ soit holomorphe dans $U_{ij} \cap V'$, et $c''^{-1}_i f_{ij} c''_i = f''_{ij}$ soit holomorphe dans $U_{ij} \cap V''$. Dans $U_{ij} \cap V' \cap V''$ on a

$$f''_{ij} = h_i^{-1} f'_{ij} h_j, \qquad \text{avec } h_i = c'^{-1}_i c''_i.$$

Or $V' \cap V''$, comme sous-espace analytique d'un cube ouvert, est holomorphiquement complet; on peut donc lui appliquer le Théorème 1 (§3): il existe par suite des sections $h_i(x, t)$ de $E$ au-dessus de $U_i \cap V' \cap V''$, dépendant continûment d'un paramètre $t \in [0, 1]$, et telles que

$$\begin{cases} f''_{ij} = (h_i(t))^{-1} f'_{ij} h_j(t) \text{ pour tout } t, \qquad h_i(x, 0) = h_i(x), \\ h_i(x, 1) = k_i(x) \text{ holomorphe dans } U_i \cap V' \cap V''. \end{cases}$$

Considérons l'espace $U_i \cap V$, recouvert par deux ouverts $U_i \cap V'$ et $U_i \cap V''$; puisque $h_i(x)$ et $k_i(x)$ sont deux cocycles *homotopes* de ce recouvrement, ils définissent deux espaces fibrés topologiquement isomorphes (d'après un théorème classique sur les espaces fibrés principaux dépendant "continûment" d'un paramètre $t$). Or l'espace défini par $h_i$ est trivial, puisque $h_i = c'^{-1}_i c''_i$; donc l'espace défini par $k_i$ est trivial. Il est même *analytiquement trivial*, en vertu du Théorème A, puisque $U_i \cap V$ est holomorphiquement complet; il s'ensuit qu'il existe des sections holomorphes $h'_i : U_i \cap V' \to E$ et $h''_i : U_i \cap V'' \to E$, telles que

$$k_i = h'^{-1}_i h''_i \text{ dans } U_i \cap V' \cap V''.$$

On a donc $h'_i f'_{ij} h'^{-1}_j = h''_i f''_{ij} h''^{-1}_j$ dans $U_{ij} \cap V' \cap V''$. On définit alors un cocyle holomorphe $g_{ij} : U_{ij} \cap V \to E$, en posant $g_{ij} = h'_i f'_{ij} h'^{-1}_j$ dans $U_{ij} \cap V'$, $= h''_i f''_{ij} h''^{-1}_j$ dans $U_{ij} \cap V''$.

Il n'est pas encore certain que les cocyles $f_{ij}$ et $g_{ij}$ (cocyles du recouvrement $(U_i \cap V)$ de l'espace $V$) soient homologues; on sait seulement que, sur chacun des deux sous-espaces $V'$ et $V''$, ils induisent des cocyles homologues. Or, d'après le §3, on a un fibré analytique $E_g$ (dont les fibres sont des groupes de Lie) défini par le cocyle $(g_{ij})$; et $(f_{ij})$ définit un fibré (topologique) $Q$, qui est $E_g$-principal. La recherche d'un cocyle holomorphe (sur l'espace $V$) qui soit homologue à $(f_{ij})$ revient à la recherche d'un fibré analytique $E_g$-principal qui soit (topologiquement) isomorphe à $Q$. Or $Q$ induit sur $V'$ (resp. sur $V''$) un fibré $E_g$-principal topologiquement trivial; on peut donc définir $Q$, comme espace $E_g$-principal, par un cocyle continu $\varphi : V' \cap V'' \to E_g$ du recouvrement de $V$ formé des deux ouverts $V'$ et $V''$. D'après le Théorème 1, la section $\varphi$ est homotope à une section *holomorphe* $\psi$ : $V' \cap V'' \to E_g$, puisque $V' \cap V''$ est holomorphiquement complet; le cocyle $\psi$ définit alors un espace analytique $E_g$-principal, topologiquement isomorphe à $Q$, et ceci achève la démonstration du lemme: on a prouvé que $V$ est *bon*.

DEUXIÈME PARTIE DE LA DÉMONSTRATION. On sait que $X$ est réunion d'une suite croissante de compacts spéciaux $K_n$, tels que chaque $K_n$ soit contenu dans l'intérieur $V_{n+1}$ de $K_{n+1}$. D'après la première partie de la démonstration, chaque $V_n$ est *bon*; on veut maintenant prouver que $X$ est *bon*.

Puisque les $U_i$ sont relativement compacts, on peut supposer que la condition suivante est vérifiée:

$$(U_i \cap V_n \neq \emptyset) \Rightarrow (U_i \subset V_{n+1})$$

(car s'il n'en était pas ainsi, il suffirait d'extraire de la suite des $V_n$ une suite partielle). Pour chaque $n$, on a une cochaîne continue $c_i^n : U_i \cap V_n \to E$ telle que

$$(c_i^n)^{-1} f_{ij} c_j^n = g_{ij}^n \text{ soit } \textit{holomorphe} \text{ dans } U_{ij} \cap V_n,$$

puisque $V_n$ est bon. On a donc

$$g_{ij}^n = (d_i^n)^{-1} g_{ij}^{n+1} d_j^n \text{ dans } U_{ij} \cap V_n,$$

où $d_i^n = (c_i^{n+1})^{-1} c_i^n : U_i \cap V_n \to E$ est une section continue. Pour chaque $n$, la collection des $d_i^n$ ($i$ variable) définit une section continue d'un fibré (analytique) auxiliaire de base $V_n$ (§3); d'après le Théorème 1, cette section est homotope à une section holomorphe. Cela signifie qu'il existe des sections continues $x \to d_i^n(x, t)$ de $V_i \cap V_n$, dépendant continûment d'un paramètre $t \in [0, 1]$, telles que:

(3) $g_{ij}^n = (d_i^n(t))^{-1} g_{ij}^{n+1} d_j^n(t)$ dans $U_{ij} \cap V_n$, quel que soit $t$,

(4) $d_i^n(0) = (c_i^{n+1})^{-1} c_i^n$,

(5) $d_i^n(1)$ est une section *holomorphe* $U_i \cap V_n \to E$.

Sans changer les $c_i^n$, on va maintenant remplacer les $c_i^{n+1}$ par d'autres sections $c_i'^{n+1}$, de manière que:

($\alpha$) $g_i'^{n+1} = (c_i'^{n+1})^{-1} f_{ij} c_j'^{n+1}$ soit *holomorphe* dans $U_{ij} \cap V_{n+1}$;

($\beta$) $c_i'^{n+1} = c_i^n$ dans $U_i \cap V_{n-2}$.

Il suffit de poser $c_i'^{n+1} = c_i^{n+1}$ si $U_i \cap V_{n-1} = \emptyset$; et, si $U_i \cap V_{n-1} \neq \emptyset$, (ce qui entraîne $U_i \subset V_n$), de poser

$$c_i'^{n+1}(x) = c_i^{n+1}(x) \cdot d_i^n(x, \lambda(x)),$$

où $\lambda(x)$ est une fonction continue, définie dans $V_n$, à valeurs dans $[0, 1]$, telle que $\lambda(x) = 0$ pour $x \in V_{n-2}$, $\lambda(x) = 1$ pour $x \notin V_{n-1}$. La vérification de ($\alpha$) et ($\beta$) est immédiate.

On voit maintenant qu'on peut choisir la suite des $c_i^n$ ($n = 1, 2, \cdots$) de manière que $c_i^{n+1} = c_i^n$ dans $U_i \cap V_{n-2}$; il en résulte que cette suite, pour chaque $i$, converge vers une section continue $c_i : U_i \to E$, et que, dans $U_{ij}$, la section $(c_i)^{-1}f_{ij}c_j$ est holomorphe. Ainsi le théorème (B) est complètement démontré.

## 6. Démonstration du Théorème principal

Il reste à démontrer le Théorème principal du §4. On utilisera pour cela deux propositions auxiliaires, qui seront établies aux paragraphes 7 et 8.

PROPOSITION 1. Soit $X$ un espace holomorphiquement complet, et soit $K$ un compact *$X$-spécial.* Définissons le groupe topoloqique $H^0(K, \mathscr{F})$ comme la *limite directe* des groupes topologiques $H^0(U, \mathscr{F})$ relatifs aux ouverts $U$ contenant $K$ (on munit $H^0(U, \mathscr{F})$ de la topologie de la convergence compacte, comme au §4). Alors l'image de l'application $H^0(X, \mathscr{F}) \to H^0(K, \mathscr{F})$ est *dense dans tout un voisinage de l'élément neutre de* $H^0(K, \mathscr{F})$.

Avant d'énoncer la Proposition 2, introduisons une convention terminologique: soit $K$ un compact spécial, réalisé comme sous-ensemble analytique d'un cube compact $\Gamma$:

$$a_j \leqq x_j \leqq b_j, \quad a'_j \leqq y_j \leqq b'_j$$

(cf. §1). Soit $c$ un nombre tel que $a_1 \leqq c \leqq b_1$; soit $\Gamma'$ l'ensemble des points de $\Gamma$ tels que $x_1 \leqq c$, et soit $\Gamma''$ l'ensemble des points de $\Gamma$ tels que $x_1 \geqq c$. Alors $\Gamma'$, $\Gamma''$ et $\Gamma' \cap \Gamma''$ sont des cubes compacts, et $\Gamma = \Gamma' \cup \Gamma''$. Définissons les compacts

$$K' = K \cap \Gamma', \quad K'' = K \cap \Gamma'';$$

$K'$, $K''$ et $K' \cap K''$ sont des compacts spéciaux, et $K = K' \cup K''$. Un tel système $(K, K', K'')$ sera appelé une *configuration spéciale.*

PROPOSITION 2. Soit $(K, K', K'')$ une configuration spéciale. Alors tout élément $f \in H^0(K' \cap K'', \mathscr{F})$, *suffisamment voisin de l'élément neutre,* peut se mettre sous la forme

$$f = f' \cdot f''^{-1},$$

où $f' \in H^0(K', \mathscr{F}), f'' \in H^0(K'', \mathscr{F})$.

Nous admettons pour le moment les Propositions 1 et 2, et nous voulons en déduire le Théorème principal. Avant de prouver les assertions (i), (ii) et (iii) de ce théorème, on va d'abord montrer:

(1) si $K$ est un compact spécial, le groupe topologique $H^0(K, \mathscr{F})$ est connexe par arcs;

(2) si $K$ est un compact spécial, l'image de l'application $H^0(X, \mathscr{F}) \to H^0(K, \mathscr{F})$ est partout dense dans $H^0(K, \mathscr{F})$;

(3) si $(K, K', K'')$ est une configuration spéciale, tout élément $f \in H^0(K' \cap K'', \mathscr{F})$ peut s'écrire sous la forme $f' \cdot f''^{-1}$, avec $f' \in H^0(K', \mathscr{F}), f'' \in H^0(K'', \mathscr{F})$.

En fait, on va introduire, pour chaque entier $k \geqq 0$, les assertions $(1)_k$ et $(2)_k$, qui se rapportent au cas où le compact $K$ se réalise dans un cube compact *de dimension réelle* $k$; on introduit aussi l'assertion $(3)_k$, qui se rapporte au cas où le compact $K' \cap K''$ est réalisé dans un cube de dimension réelle $k$.

Pour prouver (1), (2), et (3), il suffit d'établir $(1)_k$, $(2)_k$, et $(3)_k$ pour tous les entiers

$k \geqq 0$. La démonstration va procéder comme suit: on prouvera d'abord $(1)_0$; puis on montrera que

$$(1)_k \Rightarrow (2)_k \Rightarrow (3)_k \Rightarrow (1)_{k+1}.$$

*Montrons d'abord que* $(1)_0$ *est vraie*: dans ce cas, $K$ se réduit à un point $x_0 \in X$, et il s'agit de montrer que toute section de $\mathscr{F}$ au voisinage de $x_0$ peut, dans un voisinage ouvert $U$ assez petit de $x_0$, être déformée dans la section neutre. Considérons d'abord une section de $\mathscr{F}$ au-dessus *du point* $x_0$ (et non dans tout un voisinage de $x_0$); c'est une application continue $t \rightarrow f(x_0, t)$ de $C$ dans la fibre de $E$ au-dessus de $x_0$, qui est neutre pour $t \in N$ (la condition d'holomorphie en $x$ pour $t \in H$ n'intervient pas, puisque $x$ reste fixe, au point $x_0$). Comme, par hypothèse, *$N$ est un rétracte de déformation de $C$*, cette application est homotope à l'application neutre; on a donc une application continue $(t, u) \rightarrow g(t, u)$ de $C \times I$ dans la fibre de $E$ au-dessus de $x_0$, telle que

$$\begin{cases} g(t, u) = e \text{ pour } t \in N, \\ g(t, 0) = f(x_0, t) \text{ pour } t \in C, \\ g(t, 1) = e \text{ pour } t \in C, \end{cases}$$

avec la condition supplémentaire $g(t, u) = e$ au cas où le point $x_0$ appartiendrait au sous-espace $Y$ de $X$.

Il existe un voisinage ouvert $U$ de $x_0$ tel que, au-dessus de $U$, le fibré $E$ soit trivial, c'est-à-dire isomorphe au produit $U \times G$, chaque fibre étant identifiée au groupe de Lie $G$; alors les sections de $E$ au-dessus de $U$ s'identifient aux applications $U \rightarrow G$, et en particulier la fonction $g(t, u)$ peut être considérée comme prenant ses valeurs dans $G$. De même la section donnée $f(x, t)$ du faisceau $\mathscr{F}$, au voisinage de $x_0$, peut être considérée comme prenant ses valeurs dans $G$; par suite $(f(x_0, t))^{-1}f(x, t)$ est défini; en outre, si $U$ a été choisi assez petit, $(f(x_0, t))^{-1}f(x, t)$ est voisin de l'élément neutre, et se trouve donc dans la région du groupe $G$ où l'on peut identifier $G$ et son algèbre de Lie $A(G)$ au moyen de l'application exponentielle. On peut alors définir le produit de $(f(x_0, t))^{-1}f(x, t)$ par un scalaire pas trop grand. Posons

$$G(x, t, u) = g(t, u) \cdot ((1 - u)(f(x_0, t)^{-1}f(x, t)), \text{ pour } 0 \leqq u \leqq 1.$$

On a $G(x, t, 0) = f(x, t)$, $G(x, t, 1) = e$, ce qui démontre l'assertion $(1)_0$.

*Montrons que* $(1)_k$ *entraîne* $(2)_k$: en effet, soit $K$ un compact $k$-spécial (i.e. réalisé comme sous-ensemble analytique d'un cube compact de dimension réelle $k$). L'assertion $(1)_k$ entraîne que tout élément $f \in H^0(K, \mathscr{F})$ est produit d'un nombre fini d'éléments de $H^0(K, \mathscr{F})$ arbitrairement voisins de l'élément neutre. A chacun d'eux on applique la Proposition 1; on obtient ainsi $(2)_k$.

*Montrons que* $(2)_k$ *entraîne* $(3)_k$; soit $(K, K', K'')$ une configuration spéciale, telle que $K' \cap K''$ soit $k$-spécial. Soit donné $f \in H^0(K' \cap K'', \mathscr{F})$; d'après $(2)_k$, on peut écrire $f = g \cdot f_1$, où $f_1 \in H^0(K' \cap K'', \mathscr{F})$ est arbitrairement voisin de l'élément neutre et $g \in H^0(K', \mathscr{F})$. D'après la Proposition 2, on a

$$f_1 = f' \cdot f''^{-1}, \text{ avec } f' \in H^0(K', \mathscr{F}), f'' \in H^0(K'', \mathscr{F}).$$

On en déduit $f = (g \cdot f') \cdot f''^{-1}$, ce qui prouve $(3)_k$.

*Montrons enfin que* $(1)_k$ *et* $(3)_k$ *entraînent* $(1)_{k+1}$: soit $K$ un compact $(k+1)$-spécial, et soit $f \in H^0(K, \mathscr{F})$ une section de $\mathscr{F}$ au-dessus d'un voisinage de $K$. Chaque point $\lambda$ du premier côté du $(k+1)$-cube compact dans lequel $K$ est réalisé a pour image réciproque, dans $K$, un compact $k$-spécial $K_\lambda$, auquel on peut appliquer l'assertion $(1)_k$. Donc $K_\lambda$ possède un voisinage $V_\lambda$ (dans $K$) tel que la restriction de $f$ à $V_\lambda$ soit homotope à la section neutre; on peut recouvrir $K$ avec un nombre fini de tels $V_{\lambda_i}$, et on peut supposer que les $V_{\lambda_i}$ sont des $(k+1)$-cubes compacts tels que l'intersection $V_{\lambda_i} \cap V_{\lambda_{i+1}}$ soit un cube de dimension $k$. Ecrivons désormais $K_i$ au lieu de $V_{\lambda_i}$, et soit $f_i(u) \in H^0(K_i, \mathscr{F})$ une section au-dessus d'un voisinage de $K_i$, dépendant continûment d'un paramètre $u \in [0, 1]$, telle que $f_i(0)$ soit la section induite par la section donnée $f$, et que $f_i(1)$ soit la section neutre.

Ces homotopies $f_i(u)$ $(i = 1, 2, \cdots)$ ne concordent pas dans les intersections $K_i \cap K_{i+1}$, qui sont des compacts $k$-spéciaux. On va maintenant les modifier successivement, de manière à les faire concorder, ce qui établira $(1)_{k+1}$. On est ramené à un *problème élémentaire*, du type suivant: soit $(K, K_1, K_2)$ une configuration spéciale, telle que $K_1 \cap K_2$ soit $k$-spécial; soit donné $f \in H^0(K, \mathscr{F})$, et soient données des homotopies $f_i(u) \in H^0(K_i, \mathscr{F})$ $(i = 1, 2)$ telles que $f_i(0)$ soit la restriction de $f$, et $f_i(1) = e$. On cherche $g(u) \in H^0(K, \mathscr{F})$ telle que $g(0) = f$ et $g(1) = e$.

Or $(f_1(u))^{-1}f_2(u) \in H^0(K_1 \cap K_2, \mathscr{F})$ est une homotopie de $e$ à $e$ dans le groupe $H^0(K_1 \cap K_2, \mathscr{F})$. C'est un élément de $H^0(K_1 \cap K_2, \mathscr{F}')$, où $\mathscr{F}'$ est un nouveau faisceau, relatif aux espaces compacts $N' \subset H' \subset C'$ définis par

$$C' = C \times I, \quad N' = (N \times I) \cup (C \times \{0\}) \cup (C \times \{1\}), \quad H' = (H \times I) \cup N'.$$

Ce faisceau $\mathscr{F}'$ satisfait aux hypothèses du Théorème principal, car $N'$ *est rétracte de déformation de* $C'$ (vérification immédiate). On peut donc appliquer à $\mathscr{F}'$ l'assertion $(3)_k$; elle montre que

$$(f_1(u))^{-1}f_2(u) = f'(u)\,(f''(u))^{-1},$$

où $f'(u) \in H^0(K_1, \mathscr{F})$ et $f''(u) \in H^0(K_2, \mathscr{F})$ dépendent du paramètre $u \in [0, 1]$, et sont neutres pour $u = 0$ et $u = 1$. Il suffit alors de poser $g(u) = f_1(u)f'(u)$ au voisinage de $K_1$, et $= f_2(u)f''(u)$ au voisinage de $K_2$, pour obtenir la déformation cherchée $g(u) \in H^0(K, \mathscr{F})$; ceci résout le "problème élémentaire," et par suite l'assertion $(1)_{k+1}$ est démontrée.

Ainsi les assertions (1), (2), et (3) sont maintenant établies. Il reste à en déduire les assertions (i), (ii), et (iii) du Théorème principal (§4). Tout d'abord, (ii) résulte immédiatement de (2).

DÉMONSTRATION DE (i). On soit que $X$ est réunion d'une suite croissante de compacts spéciaux $K_n$, tels que $K_n$ soit contenu dans l'intérieur $V_{n+1}$ de $K_{n+1}$. Soit $f \in H^0(X, \mathscr{F})$; d'après (1), l'image $f_n$ de $f$ dans $H^0(V_n, \mathscr{F})$ est homotope à l'élément neutre dans $H^0(V_n, \mathscr{F})$; soit $f_n(u)$ une telle homotopie, telle que $f_n(0) = f_n$, $f_n(1) = e$. Alors $(f_n(u))^{-1}f_{n+1}(u)$, au-dessus de $V_n$, est en fait un élément de $H^0(V_n, \mathscr{F}')$, où $\mathscr{F}'$ désigne le faisceau défini plus haut, et relatif aux espaces $N' \subset H' \subset C'$. En appliquant l'assertion (2) à ce faisceau, on voit que $(f_n(u))^{-1}f_{n+1}(u)$ peut être uniformément approché, au-dessus du compact $K_{n-1} \subset V_n$, par des éléments de de $H^0(V_{n+1}, \mathscr{F}')$. Ainsi, sans changer $f_n(u)$, on peut modifier $f_{n+1}(u)$ de manière que $(f_n(u))^{-1}f_{n+1}(u)$ soit arbitrairement voisin de la section neutre au-dessus de $K_{n-1}$. Nous pouvons donc faire en sorte que la suite des $f_n(u)$ converge dans $X$, uniformément sur tout compact; soit alors $f(u)$ la limite: $f(u)$ fournit l'homotopie désirée entre la section donnée $f \in H^0(X, \mathscr{F})$ et la section neutre.

DÉMONSTRATION DE (iii). On veut montrer que $H^1(X, \mathscr{F}) = 0$. Or (3) entraîne facilement que $H^1(K, \mathscr{F}) = 0$ pour tout compact spécial $K$. Il reste maintenant à "passer à la limite". Soient $(K_n)$ et $(V_n)$ des suites comme ci-dessus; étant donné un recouvrement ouvert $(U_i)$ de $X$, et un cocyle $(f_{ij})$ à valeurs dans $\mathscr{F}$, on a, pour chaque $n$, des sections

$$c_i^n \in H^0(U_i \cap V_n, \mathscr{F})$$

telles que $f_{ij} = (c_i^n)^{-1}c_j^n$ dans $U_{ij} \cap V_n$. On a donc $c_i^{n+1}(c_i^n)^{-1} = c_j^{n+1}(c_j^n)^{-1}$ dans $U_{ij} \cap V_n$, et par suite les $c_i^{n+1}(c_i^n)^{-1}$ définissent une section $\varphi^n \in H^0(V_n, \mathscr{F})$. En utilisant à nouveau l'assertion (2) comme ci-dessus, on peut assurer la convergence de la suite $c_i^n, c_i^{n+1}, \cdots$ uniformément sur tout compact de $U_i$. La limite $c_i \in H^0(U_i, \mathscr{F})$ est telle que

$$f_{ij} = (c_i)^{-1}\, c_j \text{ dans } U_{ij},$$

et par suite on a démontré que $H^1(X, \mathscr{F}) = 0$.

La démonstration du Théorème principal est ainsi achevée.

## 7. Démonstration de la Proposition 1

La démonstration des Propositions 1 et 2 repose, entre autres choses, sur le principe suivant: considérons le foncteur covariant qui, à chaque groupe de Lie complexe $G$, associe son algèbre de Lie $A(G)$ (espace vectoriel sur le corps complexe) et à chaque homomorphisme de groupes de Lie $G \rightarrow G'$, associe l'homorphisme correspondant d'algèbres de Lie $A(G) \rightarrow A(G')$. La définition du fibré $E$, dont les fibres sont des groupes de Lie complexes, conduit à un fibré associé $A(E)$, dont les fibres sont les algèbres de Lie des fibres de $E$; $A(E)$ est une *fibré analytique à fibres vectorielles* (complexes).

Pour un groupe de Lie $G$, on a l'application exponentielle $A(G) \rightarrow G$, qui induit un *isomorphisme* d'un voisinage de 0 dans $A(E)$ sur un voisinage de l'élément neutre de $G$ ("isomorphisme" au sens des variétés analytiques complexes). Comme $A(G) \rightarrow G$ est une application naturelle de foncteurs, il s'ensuit qu'elle définit une application analytique $A(E) \rightarrow E$ des espaces fibrés, notée exp; et que si on se restreint à un compact $K$ de l'espace de base $X$, l'application exp: $A(E) \rightarrow E$ est un *isomorphisme* d'un voisinage de la section nulle de $A(E)$ sur un voisinage de la section neutre de $E$ ("isomorphisme" au sens des espaces fibrés analytiques). Désormais toute section $K \rightarrow E$, assez voisine de la section neutre, pourra donc être identifiée à une section $K \rightarrow A(E)$.

LEMME 1. *Soit $V$ un fibré analytique vectoriel (complexe) ayant pour base $X$ un espace holomorphiquement complet, et soit $U$ un ouvert de $X$, relativement compact et holomorphiquement complet. Il existe un système fini $(g_\alpha)$ de sections de $V$ au-dessus de*

*$X$, jouissant de la propriété suivante: si $Y$ est un sous-espace analytique de $X$, toute $(N, H, C)$-section*[2] *$f(x, t)$ de $V$ au-dessus de $U$, nulle pour $x \in U \cap Y$, peut s'écrire*

$$f(x, t) = \sum_\alpha f_\alpha(x, t) g_\alpha(x), \qquad (x \in U, t \in C)$$

*où les $f_\alpha$ sont des $(N, H, C)$-fonctions scalaires,*[2] *nulles pour $x \in U \cap Y$.*

Démonstration: puisque $U$ est relativement compact, et que le faisceau des germes de sections holomorphes de $V$ est cohérent, il existe[3] un nombre fini $n$ de sections holomorphes $g_\alpha$ au-dessus de $X$, telles que, en chaque point $x \in U$, les $g_\alpha$ engendrent le module des germes de sections holomorphes de $V$ au point $x$ (comme module sur l'anneau des germes de fonctions holomorphes scalaires au point $x$). Soit alors $\mathscr{V}^a$ (resp. $\mathscr{V}^c$) le faisceau des germes de sections holomorphes (resp. continues) de $V$, nulles sur $Y$; soient $\mathcal{O}^a$ (resp. $\mathcal{O}^c$) le faisceau des germes de fonctions holomorphes scalaires (resp. continues scalaires) nulles sur $Y$. A chaque système $(f_\alpha)$ de $n$ éléments de $\mathcal{O}^a_x$ (resp. de $\mathcal{O}^c_x$) associons $\sum_\alpha f_\alpha g_\alpha \in \mathscr{V}^a_x$ (resp. $\in \mathscr{V}^c_x$); ceci définit un homomorphisme de faisceaux

$$\varphi^a\colon (\mathcal{O}^a)^n \to \mathscr{V}^a, \quad \text{resp.} \quad \varphi^c\colon (\mathcal{O}^c)^n \to \mathscr{V}^c;$$

$\varphi^a$ et $\varphi^c$ sont des *épimorphismes*: on le voit en utilisant une trivialisation locale du fibré vectoriel $V$. Soit $N^a$ (resp. $N^c$) le noyau de $\varphi^a$ (resp. $\varphi^c$); $N^a$ est un faisceau analytique cohérent (resp. $N^c$ est un faisceau fin), donc $H^1(U, N^a) = 0$, $H^1(U, N^c) = 0$. Il en résulte que $\varphi^a$ et $\varphi^c$ définissent des *épimorphismes*

$$\Phi^a\colon (H^0(U, \mathcal{O}^a))^n \to H^0(U, \mathscr{V}^a),$$

$$\Phi^c\colon (H^0(U, \mathcal{O}^c))^n \to H^0(U, \mathscr{V}^c).$$

Or $(H^0(U, \mathcal{O}^a))^n$, $H^0(U, \mathscr{V}^a)$, $(H^0(U, \mathcal{O}^c))^n$, $H^0(U, \mathscr{V}^c)$ sont des espaces de Fréchet; $\Phi^a$ et $\Phi^c$ sont des applications linéaires continues. On va leur appliquer un lemme sur les espaces de Fréchet (cf. Appendice).

Prenons d'abord $f_\alpha(x, t) = 0$ pour $t \in N$; on a ainsi une application continue (nulle) $N \to (H^0(U, \mathcal{O}^a))^n$. D'après le lemme de l'Appendice, on peut la prolonger en une application continue

$$H \to (H^0(U, \mathcal{O}^a))^n$$

qui "relève" l'application donnée $f\colon H \to H^0(U, \mathcal{O}^a)$. En composant ce relèvement $H \to (H^0(U, \mathcal{O}^a))^n$ et l'application naturelle $(H^0(U, \mathcal{O}^a))^n \to (H^0(U, \mathcal{O}^c))^n$, on obtient une application continue $H \to (H^0(U, \mathcal{O}^c))^n$; en utilisant à nouveau le

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[2] Une fonction $f_\alpha(x, t)$ définie pour $x \in U$, $t \in C$, continue sur $U \times C$, et àvaleurs scalaires, n'est pas autre chose qu'une application continue $t \to (x \to f_\alpha(x, t))$ de $C$ dans l'espace des fonctions continues sur $U$ à valeurs complexes, muni de la topologie de la convergence compacte. On dit que c'est une $(N, H, C)$-fonction si, pour tout $t \in H$, l'application $x \to f_\alpha(x, t)$ est holomorphe, et si $f_\alpha(x, t) = 0$ pour $t \in N$. D'autre part, une section $f$ de $V$ au-dessus de $U$, dépendant de $t \in C$, s'appelle une $(N, H, C)$-section si $f(x, t)$ est continue sur $U \times C$, et si, pour tout $t \in H$, la section $x \to f(x, t)$ est holomorphe, et si en outre, pour $t \in N$, on a $f(x, t) = 0$ pour tout $x \in U$.

(3) En vertu du Théorème A de [4].

où $\lambda(x, t, u)$ désigne, pour chaque $x$, l'automorphisme ad $(f(x, t, u))$ de la fibre vectorielle de $A(E)$ au-dessus du point $x$. Alors, si on intègre les équations (7)', en convenant que $f'$ et $f''$ sont neutres pour $u = 0$, on trouve des sections $f'(x, t, u)$ et $f''(x, t, u)$ de $E$, satisfaisant aux conditions suivantes:

$$\begin{cases} f'(x, t, u) \text{ est holomorphe en } x \text{ au voisinage de } K'; \\ f''(x, t, u) \text{ est holomorphe en } x \text{ au voisinage de } K''; \\ f' \text{ et } f'' \text{ sont neutres pour } t \in N \text{ et pour } x \in Y; \\ \text{enfin, pour } x \text{ voisin de } K' \cap K'', \text{ on a} \end{cases}$$

$$f'(x, t, u) = f(x, t, u) \cdot f''(x, t, u). \tag{11}$$

Revenons alors à notre "problème fondamental": $f(x, t)$ est une section donnée de $E$, holomorphe en $x$ au voisinage de $K' \cap K''$, pour chaque $t \in H$; de plus, $f(x, t)$ est neutre pour $t \in N$, et pour $x \in Y$. Enfin, on suppose $f(x, t)$ *voisin de la section neutre*; il existe donc une section $a(x, t)$ du fibré vectoriel $A(E)$, telle que $\exp(a(x, t)) = f(x, t)$; $a(x, t)$ est holomorphe en $x$ au voisinage de $K' \cap K''$, est nulle pour $t \in N$ et pour $x \in Y$. Soit $u$ un paramètre réel $(0 \leqq u \leqq 1)$; considérons la section $u \cdot a(x, t)$ (produit de $a(x, t)$ par le scalaire $u$, dans la structure vectorielle des fibres de $A(E)$), et soit

$$f(x, t, u) = \exp(u \cdot a(x, t)).$$

On a $\partial f/\partial u = a(x, t) \cdot f(x, t, u)$. Posons

$\lambda(x, t, u) = \mathrm{ad}(f(x, t, u))$. Supposons qu'on ait trouvé des sections $a'(x, t, u)$ (resp. $a''(x, t, u)$) de $A(E)$, holomorphes en $x$ au voisinage de $K'$ (resp. de $K''$), nulles pour $t \in N$ et pour $x \in Y$, telles que

$$a'(x, t, u) = a(x, t) + \lambda(x, t, u) \cdot a''(x, t, u) \text{ pour } x \text{ voisin de } K' \cap K''. \tag{12}$$

Alors, par intégration, on obtiendra des sections $f'(x, t, u)$ (resp. $f''(x, t, u)$) du fibré $E$, holomorphes en $x$ au voisinage de $K'$ (resp. de $K''$), neutres pour $t \in N$ et pour $x \in Y$, telles que (11) ait lieu pour $x$ voisin de $K' \cap K''$. En particulier, $f'(x, t, 1)$ et $f''(x, t, 1)$ fourniront une solution du "problème fondamental."

Ainsi, tout revient maintenant à trouver $a'(x, t, u)$ et $a''(x, t, u)$ satisfaisant à (12). Soit $U$ un ouvert $X$-convexe et relativement compact, contenant $K' \cap K''$, dans lequel la section donnée $f(x, t)$ soit holomorphe (pour chaque $t \in H$), ainsi par conséquent que $a(x, t)$ et $\lambda(x, t, u)$. Reprenons les sections $g_\alpha$ du fibré $A(E)$ au-dessus de $U$, comme dans le Lemme 1 (§7). On a

$$\lambda(x, t, u) \cdot g_\alpha(x) = \textstyle\sum_\beta \varphi_{\alpha\beta}(x, t, u) g_\beta(x),$$

où les coefficients $\varphi_{\alpha\beta}(x, t, u)$ sont holomorphes en $x$ pour $x \in U$, et continus en $t \in H$ et $u \in [0, 1]$. L'existence d'une telle relation résulte du Lemme 1. De plus, si $f(x, t)$ est assez voisine de la section neutre, $\lambda(x, t, u)$ est un automorphisme (de la fibre de $A(E)$) voisin de l'automorphisme identique; il en résulte que la matrice

$(\varphi_{\alpha\beta}(x, t, u))$ peut être choisie voisine de la matrice-unité.[4] On peut donc appliquer à cette matrice le Lemme 2; il en résulte que l'on a

$$\sum_\alpha \varphi_{\beta\alpha}\varphi'_{\alpha\gamma} = \varphi''_{\beta\gamma},$$

où la matrice $(\varphi'_{\alpha\beta}(x, t, u))$ (resp. $(\varphi''_{\alpha\beta}(x, t, u))$) est inversible, dépend continûment de $t \in H$ et $u \in [0, 1]$, et est holomorphe en $x$ au voisinage de $K'$ (resp. au voisinage de $K''$). Or, d'après le Lemme 1, on a

$$a(x, t) = \sum_\alpha a_\alpha(x, t)g_\alpha(x),$$

les $a_\alpha(x, t)$ étant holomorphes en $x$ au voisinage de $K' \cap K''$, et nulles pour $t \in N$ et pour $x \in Y$. Pour résoudre (12), il suffit de trouver des fonctions $a'_\alpha(x, t, u)$ (resp. $a''_\alpha(x, t, u)$), holomorphes en $x$ au voisinage de $K'$ (resp. de $K''$), nulles pour $t \in N$ et pour $x \in Y$, et telles que l'on ait

$$a'_\alpha(x, t, u) = a_\alpha(x, t) + \sum_\beta \varphi_{\beta\alpha}(x, t, u)\, a''_\beta(x, t, u)$$

pour $x$ dans un voisinage de $K' \cap K''$. Pour cela, il suffit que

$$\sum_\alpha \varphi'_{\alpha\gamma}a'_\alpha = \sum_\alpha \varphi'_{\alpha\gamma}a_\alpha + \sum_\alpha \varphi''_{\alpha\gamma}a''_\alpha. \tag{13}$$

Au lieu des $a'_\alpha$ et des $a''_\alpha$, prenons comme inconnues les

$$b'_\gamma = \sum_\alpha \varphi'_{\alpha\gamma}a'_\alpha \quad \text{et} \quad b''_\gamma = \sum_\alpha \varphi''_{\alpha\gamma}a''_\alpha,$$

qui doivent être nulles pour $x \in Y$; si on pose

$$\sum_\alpha \varphi'_{\alpha\gamma}(x, t, u)a_\alpha(x, t) = b_\gamma(x, t, u)$$

(fonction connue, holomorphe en $x$ au voisinage de $K' \cap K''$, nulle pour $x \in Y$), l'équation (13) devient

$$b'_\gamma(x, t, u) - b''_\gamma(x, t, u) = b_\gamma(x, t, u),$$

et elle se résout en $b'_\gamma$ et $b''_\gamma$ grâce à l'intégrale classique de Cauchy.[5]

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[4] Cela résulte du théorème de Banach ([2], théorème 1), comme suit: les matrices du type $(\varphi_{\beta\alpha}(x, t, u))$ forment un espace de Fréchet $F$; soit $F'$ l'espace de Fréchet formé des transformations analytiques du fibré $A(E)$ (au-dessus de l'ouvert $U$) qui sont linéaires dans chaque fibre et dépendent continûment des paramètres $t$ et $u$; on définit une application linéaire continue $F \to F'$ en associant à chaque matrice $(\varphi_{\alpha\beta}(x, t, u))$ la transformation linéaire $\lambda(x, t, u)$ définie par la formule du texte, et le Lemme 1 entraîne précisément que l'application $F \to F'$ applique $F$ *sur* $F'$. Alors le théorème de Banach entraîne que c'est une application *ouverte*; donc toute transformation $\lambda(x, t, u)$ assez voisine de l'identité est associée à au moins une matrice $(\varphi_{\alpha\beta}(x, t, u))$ voisine de la matrice-unité.

[5] Pour cela, on observe que $b_\gamma(x, t, u)$, comme fonction holomorphe de $x$ au voisinage de $K' \cap K''$, est induite par une fonction $B_\gamma(x, t, u)$, holomorphe en $x$ au voisinage du cube compact $\Gamma' \cap \Gamma''$ dans lequel le compact spécial $K' \cap K''$ est réalisé. Grâce à l'intégrale de Cauchy, on trouve $B'_\gamma(x, t, u)$ et $B''_\gamma(x, t, u)$, holomorphes en $x$ au voisinage de $\Gamma'$ et de $\Gamma''$ respectivement (et dépendant continûment de $t$ et $u$) telles que l'on ait

$$B'_\gamma(x, t, u) - B''_\gamma(x, t, u) = B_\gamma(x, t, u)$$

pour $x$ dans un voisinage de $\Gamma' \cap \Gamma''$. Il suffit alors de prendre pour $b'_\gamma$ et $b''_\gamma$ les fonctions induites par $B'_\gamma$ et $B''_\gamma$ sur $K'$ et $K''$ respectivement.

Finalement, le "problème fondamental" est résolu, et on a trouvé $f'(x, t)$ et $f''(x, t)$ pour $t \in H$. De plus, si la section donnée $f(x, t)$ est assez voisine de la section neutre, la solution précédente montre que $f'(x, t)$ et $f''(x, t)$ sont voisines de la section neutre. Pour établir la proposition 2 du §6, il reste à prolonger $f'(x, t)$ et $f''(x, t)$ aux valeurs de $t \in C$ (et non plus seulement pour $t \in H$). Or, pour $t \in H$, on peut considérer $f'(x, t)$ et $f''(x, t)$ comme des sections du fibré $A(E)$ à fibres vectorielles, nulles pour $x \in Y$. On peut donc les prolonger par continuité en des sections $F'(x, t)$ et $F''(x, t)$, définies pour $t \in C$, et nulles pour $x \in Y$. Considérons à nouveau $F'(x, t)$ et $F''(x, t)$ comme sections de $E$; le produit

$$F'(F'')^{-1}f^{-1} = \Phi(x, t)$$

est une section de $E$ (pour $x$ dans un voisinage de $K' \cap K''$), neutre pour $x \in Y$ et pour $t \in H$; de plus, on peut la supposer, pour $t \in C$, assez voisine de la section neutre pour que $\Phi(x, t)$ puisse être identifiée à une section de $A(E)$. Il en résulte qu'on peut trouver, pour $t \in C$, une section $\Psi(x, t)$ de $E$, définie pour $x$ dans un voisinage de $K'$, neutre pour $x \in Y$ et pour $t \in H$, de manière que $\Psi(x, t)$ coïncide avec $\Phi(x, t)$ quand $x$ est dans un voisinage de $K' \cap K''$. Posons alors

$$f'(x, t) = (\Psi(x, t))^{-1}F'(x, t) \text{ pour } x \text{ dans un voisinage de } K',$$

$$f''(x, t) = F''(x, t) \text{ pour } x \text{ dans un voisinage de } K''.$$

On a, pour $x$ dans un voisinage de $K' \cap K''$,

$$f(x, t) = f'(x, t) \cdot (f''(x, t))^{-1} \text{ quel que soit } t \in C;$$

par suite $f'(x, t)$ et $f''(x, t)$ définissent des éléments de $H^0(K', \mathscr{F})$ et $H^0(K'', \mathscr{F})$ dont l'existence démontre enfin la Proposition 2.

## 9. Applications

Le fait que la classification *analytique* des espaces fibrés $E$-principaux (pour un $E$ donné dont la base $X$ est holomorphiquement complète) coïncide avec la classification *topologique* (cf. Théorèmes A et B, §2) entraîne des conséquences dont nous mentionnons rapidement quelques-unes.

Soit $F$ un sous-espace fibré analytique de $E$, ayant même base $X$, et dont les fibres sont des sous-groupes de Lie complexes des fibres de $E$. Tout fibré $F$-principal $Q$ de base $X$ définit un fibré $E$-principal $P$ de même base $X$ ("extension du groupe structural"): si $Q$ est défini par un cocycle à valeurs dans les sections de $F$, $P$ est défini par le même cocycle, considéré comme prenant ses valeurs dans les sections de $E$.

Si un fibré $E$-principal $P$ peut être ainsi déduit d'un fibré $F$-principal, on dit qu'on peut, pour $P$, *restreindre le fibré structural* $E$ au sous-fibré $F$. Il faut naturellement distinguer entre la restriction au sens *analytique* et au sens *topologique*. Mais si la base $X$ est holomorphiquement complète, et si $P$ est *analytique*, la possibilité de restreindre le fibré structural au sens topologique entraîne la possibilité de le restreindre au sens analytique; en effet, le diagramme suivant est commutatif:

$$\begin{array}{ccc} H^1(X, \mathscr{F}^a) & \to & H^1(X, \mathscr{E}^a) \\ \downarrow & & \downarrow \\ H^1(X, \mathscr{F}^c) & \to & H^1(X, \mathscr{E}^c), \end{array}$$

et les flèches verticales désignent des bijections (en vertu des Théorèmes A et B).

Or il est classique que la possibilité, pour $P$, de restreindre le fibré structural $E$ au sous-fibré $F$, équivaut à l'existence d'une section $s : X \to P/F$ de l'espace $P/F$ (quotient de $P$ par la relation d'équivalence qu'y définissent les opérations de $F$ à droite), considéré comme fibré de base $X$. *Conséquence*: si le fibré $P/F$ de base $X$ (holomorphiquement complète) possède une section *continue*, il possède aussi une section *holomorphe*.

Signalons sans démonstration d'autres applications des Théorèmes A et B: si une variété de Stein est *parallélisable* (au sens topologique), elle l'est au sens analytique: il existe alors un champ *holomorphe* de vecteurs tangents non nuls. On a un résultat analogue pour les champs de $r$ vectors tangents linéairement indépendants (sur le corps $C$).

A ce sujet, signalons qu'on sait peu de choses sur les classes caractéristiques d'une variété de Stein (classes de Chern, de Pontrjagin, de Stiefel-Whitney), et que leur étude mériterait d'être entreprise.

Il y aurait lieu aussi de voir dans quelle mesure l'étude qui vient d'être faite pourrait conduire à des résultats concernant la classification des espaces fibrés *analytiques-réels*.

## 10. Appendice: lemme sur les espaces de Fréchet

LEMME 3. *Soient $F$ et $F'$ deux espaces de Fréchet* (i.e., deux espaces vectoriels topologiques localement convexes, métrisables et complets), *et soit $\varphi : F \to F'$ une application linéaire continue de $F$ sur $F'$. Soient d'autre part $A$ un espace compact et $B$ un sous-espace fermé de $A$. Supposons données une application continue $f' : A \to F'$ et une application continue $g : B \to F$ telles que la restriction de $f'$ à $B$ soit égale à l'application composée $\varphi \circ g$. Alors il existe une application continue $f : A \to F$ qui prolonge $g$ et satisfait à $\varphi \circ f = f'$.*

DÉMONSTRATION. Soit $(V_n)$ une suite fondamentale de voisinages fermés convexes de $O$ dans $F$, telle que $2V_{n+1} \subset V_n$. Puisque $\varphi$ est une application ouverte (en vertu du théorème de Banach: cf. [2], théorème 1), il existe un voisinage $V'_n$ (fermé convexe) de $O$ dans $F'$, tel que $V'_n \subset \varphi(V_n)$. On peut de plus supposer que les $V'_n$ forment un système fondamental de voisinages de $O$ dans $F'$.

Tout recouvrement ouvert (fini) de $B$ est induit par un recouvrement ouvert (fini) de $A$. Pour chaque $n$, il existe donc un recouvrement ouvert fini $\mathscr{U}_n$ de $A$ tel que: 1° l'image par $f'$ de tout ouvert de $\mathscr{U}_n$ soit petite d'ordre $V'_n$; 2° l'image par $g$ de tout ouvert du recouvrement de $B$ induit par $\mathscr{U}_n$ soit petite d'ordre $V_n$. Pour chaque $n$, il existe une partition de l'unité $(h_{n,i})_{i\in J_n}$ sur l'espace compact $A$, telle que le support de chaque fonction $h_{n,i}$ soit petit d'ordre $\mathscr{U}_n$: on peut de plus supposer que chaque $h_{n,i}$ est somme d'un certain nombre parmi les fonctions $h_{n+1,j}$

relatives à la partition d'ordre $n+1$. On a alors une application $u_n$ de l'ensemble d'indices $J_{n+1}$ sur l'ensemble d'indices $J_n$, telle que $h_{n,i}$ soit la somme des $h_{n+1,j}$ telles que $u_n(j)=i$.

Pour chaque $n$, et chaque indice $i \in J_n$, choisissons un point $a_{n,i} \in A$ assujetti aux conditions suivantes: $h_{n,i}(a_{n,i}) \neq 0$, et $a_{n,i} \in B$ s'il existe un $b \in B$ tel que $h_{n,i}(b) \neq 0$. Soit $x_{n,i} = f'(a_{n,i}) \in F'$; la fonction

$$f'_n(a) = \sum_{i \in J_n} h_{n,i}(a) x_{n,i}$$

est une application continue $A \to F'$, et on a, pour chaque $a \in A$,

$$f'(a) - f'_n(a) \in V'_n. \tag{14}$$

Par récurrence sur $n$, on choisit des $y_{n,i} \in F$ tels que $\varphi(y_{n,i}) = x_{n,i}$, de manière que: 1° si $a_{n,i} \in B$, alors $y_{n,i} = g(a_{n,i})$; 2° on ait

$$y_{n+1,j} - y_{n,u_n(j)} \in V_n. \tag{15}$$

C'est possible: on doit s'assurer que si le choix de $y_{n+1,j}$ est imposé par la condition 1°, alors la condition 2° est satisfaite; or $a_{n,u_n(j)} \in B$, et (15) résulte du fait que l'image par $f$ du recouvrement induit par $\mathcal{U}_n$ sur $B$ est petite d'ordre $V_n$. Par ailleurs, si $a_{n+1,j} \notin B$, on peut choisir $y_{n+1,j}$ de manière à satisfaire à (15), car $x_{n+1,j} - x_{n,u_n(j)} \in V'_n$, et $\varphi(V_n) \supset V'_n$ par hypothèse.

Posons $f_n(a) = \sum_{i \in J_n} h_{n,i}(a) y_{n,i}$. Alors $f_n$ est une application continue de $A$ dans $F$, et $\varphi \circ f_n = f'_n$. De plus,

$$f_n(a) - g(a) \in V_n \quad \text{pour } a \in B. \tag{16}$$

Quand $n$ tend vers l'infini, $f'_n$ converge uniformément vers $f'$ d'après (14). La suite $f_n$ converge uniformément, car

$$f_{n+1}(a) - f_n(a) = \sum_{j \in J_{n+1}} h_{n+1,j}(a)\,(y_{n+1,j} - y_{n,u_n(j)}) \in V_n$$

pour tout $a \in A$; on a donc $f_{n+k}(a) - f_n(a) \in V_{n-1}$ pour tout $k > 0$, ce qui prouve la convergence uniforme de la suite des $f_n$. La limite $f$ de cette suite est une application continue de $A$ dans $F$, qui prolonge $g$ d'après (16); et la relation $\varphi \circ f_n = f'_n$ donne, à la limite, $\varphi \circ f = f'$, ce qui achève la démonstration.

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*Rajouté à la correction des épreuves* (Octobre 1957): depuis que ce travail a été rédigé, H. Grauert a remanié son travail original, qui est en cours de publication sous forme de trois articles aux Mathematische Annalen:

*Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen*, 133 (1957), pp. 139–159.

*Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen*, 133 (1957), pp. 450–472;

*Analytische Faserungen über holomorph-vollständigen Räumen*, à paraître prochainement.

D'autre part, les résultats de J. Frenkel sur les espaces fibrés analytiques ont été publiés: *Cohomologie non abélienne et espaces fibrés*, Bull. Soc. Math. France, 85 (1957), pp. 135–220.

# ON SIMPLY CONNECTED 4-MANIFOLDS

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This paper will apply various known results to the problem of classifying simply connected 4-manifolds as to homotopy type. A complete classification is given for those manifolds with second Betti number $\leqq 7$, and a fairly good picture is obtained for the general case. But the problem is by no means solved.

The first section states results from the theory of quadratic forms. The author is indebted to O. T. O'Meara for assistance in the preparation of this section.

In §2 the quadratic form of a $4k$-manifold is defined, and elementary properties are verified.

In §3 it is shown (as a corollary to a theorem of J. H. C. Whitehead) that the homotopy type of a simply connected 4-manifold is determined by its quadratic form. The main problem is thus to decide whether or not a given quadratic form actually corresponds to some simply connected 4-manifold.

A subsequent paper will study the more general problem of classifying $2n$-manifolds which are $(n - 1)$-connected as to homotopy type.

## §1. Quadratic forms

Theorems 1 and 2 of this section will summarize known results concerning the classification of quadratic forms with determinant $\pm 1$ over the ring of integers.

It will be convenient to define[2] a *quadratic form of rank r* over an integral domain $D$ as a pair $(A, \phi)$ consisting of a free $D$-module $A$ of rank $r$, and a non-singular, symmetric, bilinear pairing $\phi : A \times A \to D$. Two forms $(A, \phi)$ and $(A', \phi')$ are *equivalent* if there is an isomorphism of $A$ onto $A'$ which carries $\phi'$ onto $\phi$. If $D$ is contained in a larger integral domain $D'$, note that every quadratic form over $D$ gives rise to a quadratic form over $D'$.

Given a basis $(a_1, \cdots, a_r)$ for $A$, the form is completely described by the symmetric matrix $\|\phi(a_i, a_j)\|$. This section will only be concerned with quadratic forms over the integers such that this matrix has determinant $\pm 1$.

We will say that a form is of *type* I (properly primitive) if some diagonal entry of its matrix is odd. If every diagonal entry is even, then the form is of *type* II (improperly primitive). Thus $(A, \phi)$ has type I if and only if $\phi(a, a)$ takes on odd values.

The index $\tau$ of a form is defined[3] as the number of positive diagonal entries minus the number of negative ones, after the matrix has been diagonalized over the real numbers.

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[1] The author holds an Alfred P. Sloan fellowship.

[2] This is compatible with the usual definition providing that $2 \neq 0$ in $D$.

[3] Topologists have called $\tau$ the "index," although "signature" is the classical term.

As examples, consider the quadratic forms corresponding to the following three matrices

$$T=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad U=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad V=\begin{pmatrix}2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 2 & 1 & 0 & -1 & 0 & 0 & 0\\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 1 & 2 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\end{pmatrix}.$$

The first has type I and index 0; the second has type II and index 0; while the third has type II and index 8.

Two forms have the same *genus* if they are equivalent over the $p$-adic integers for every prime $p$, and if they are equivalent over the real numbers. (Compare [8] p. 106–107.)

THEOREM 1. *The rank $r$, the index $\tau$, and the type* (I *or* II) *form a complete system of invariants for the genus of a quadratic form over the integers with determinant* $\pm 1$. *A given rank, index and type actually occur if and only if the following three conditions are satisfied:*

(1) *$r$ and $\tau$ are integers with*

$$-r \leqq \tau \leqq r, \qquad \tau \equiv r \pmod 2;$$

(2) *for forms of type* II, $\tau \equiv 0 \pmod 8$;

(3) *for forms of type* I, $r > 0$.

The following is an immediate consequence.

COROLLARY. *Every such form has the same genus as a form with matrix*

$$\operatorname{diag}(1, \cdots, 1, -1, \cdots, -1) \quad or \quad \pm\operatorname{diag}(U, \cdots, U, V, \cdots, V).$$

A form is called *definite* if $\tau = \pm r$. Otherwise it is *indefinite*.

THEOREM 2. *Two indefinite forms with determinant* $\pm 1$ *are equivalent if and only if they have the same genus. This is also true for definite forms, providing the rank is* $\leqq 8$.

(This theorem would definitely be false for definite forms of rank $\geqq 9$. For example the two positive definite $9 \times 9$ matrices diag $(1, \cdots, 1)$ and diag $(V, 1)$ represent forms $(A, \phi)$ and $(A', \phi')$ of the same genus. These forms are not equivalent since the equation $\phi(a, a) = 1$ has eighteen solutions, while $\phi'(a', a') = 1$ has only two solutions.)

PROOF OF THEOREM 1. Let $f_1, f_2$ be two forms with determinant $\pm 1$ having the same rank, index and type. Then $f_1$ is equivalent to $f_2$ over the $p$-adic integers, $p$ odd, by Corollary 36b of [8]. They are equivalent over the real numbers since they have the same index. Therefore (see [8] p. 39) we have $c_\infty(f_1) = c_\infty(f_2)$, which implies that $c_2(f_1) = c_2(f_2)$. Now by Theorems 15 and 36 of [8], $f_1$ is equivalent to $f_2$ over the 2-adic integers. Therefore $f_1$ and $f_2$ have the same genus.

NECESSITY OF CONDITIONS (1), (2), (3). Conditions (1) and (3) are trivial. If $f_1$ has type II then Theorem 33a of [8] implies that $f_1$ is equivalent to diag $(U, \cdots, U)$ over the 2-adic integers (making use of the fact that $\pm 3$ are not 2-adic squares).

Therefore the Gauss sum (see [2]) of $f_1$ modulo 8 is positive. Now the criterion ($\varepsilon$) of [2] implies that $\tau \equiv 0 \pmod 8$; which proves condition (2).

The sufficiency of conditions (1), (2), (3), is an easy exercise, using the forms mentioned in the Corollary. This completes the proof of Theorem 1.

PROOF OF THEOREM 2. For definite forms of rank $\leqq 8$, this result is due to Hermite [6] and Mordell [12]. (For further discussion see O'Connor and Pall [14].) For indefinite forms of type I, the result is due to Meyer [10].

For indefinite forms of type II the proof will be based on a theorem of Eichler ([4], [5]). We may assume that $r \geqq 4$, since the cases $r = 0, 2$ are easily taken care of.

Eichler considers a fixed quadratic form $(V, \psi)$ over the rational numbers $Q$. A *lattice* $L$ in $V$ is a finitely generated subgroup of maximal rank. The *norm* $n(L)$ is the fractional ideal generated by all $\frac{1}{2}\psi(x, x)$ with $x \in L$. A lattice is *maximal* if no properly containing lattice has the same norm. Two lattices are *similar* if one is carried onto the other by a similarity transformation of $V$. An appropriate concept of *genus* is defined.

Eichler's theorem ([4] Satz 3) can be stated as follows. *Two maximal lattices having the same genus, in an indefinite vector space $(V, \psi)$ of rank $\geqq 4$, are similar.*

To apply this theorem note that any free abelian group $A$ can be considered as a lattice in the vector space $V = A \otimes Q$. A quadratic form $(A, \phi)$ gives rise to a form $(V, \psi)$. If $(A, \phi)$ is of type II then it can be verified that the lattice $A$ is always maximal.

This implies that if $(A, \phi)$ and $(A', \phi')$ are indefinite quadratic forms of type II and rank $\geqq 4$ which have the same genus, then they are "similar," in the sense that $(A, \phi)$ is equivalent to $(A', c\phi')$ for some constant $c$. Clearly $c$ must be $\pm 1$. If $\tau \neq 0$ then $c$ must be $+1$, which completes the proof in this case.

For $\tau = 0$ the above argument shows that $(A, \phi)$ is equivalent to the form with matrix $\pm \operatorname{diag}(U, \cdots, U)$. Since $U$ is equivalent to $-U$, this completes the proof.

## §2. The quadratic form of a 4$k$-manifold

All manifolds considered are to be closed and connected. A manifold $M^n$ is *oriented* if it is orientable, and if one generator $\nu \in H_n(M^n)$ is distinguished; integer coefficients being understood. We will say that oriented manifolds $M_1^n$, $M_2^n$ have the same *oriented homotopy type* if there is a homotopy equivalence $f : M_1^n \to M_1^n$ with $f_*(\nu_1) = \nu_2$.

To every oriented $4k$-manifold is associated its quadratic form $(B^{2k}(M^{4k}), \phi)$; where $B^j(X)$ denotes the "co Betti group" $H^j(X)/(\text{torsion subgroup})$; and where[4]

$$\phi(x, y) = \langle x \cup y, \nu \rangle.$$

Clearly manifolds with the same oriented homotopy type have equivalent quadratic forms.

LEMMA 1. *The quadratic form of an oriented $4k$-manifold has determinant* $\pm 1$.

PROOF. (Compare Seifert and Threlfall [17] p. 252.) Consider the co Betti groups $B^h$, $B^{n-h}$ of an oriented manifold $M^n$. The bilinear pairing $\phi : B^h \times B^{n-h} \to Z$

[4] Here $\langle \alpha, \beta \rangle$ denotes the Kronecker index of the cohomology class $\alpha$ and the homology class $\beta$.

defined by $\phi(x, y) = \langle x \cup y, \nu\rangle$ has a determinant which is well defined up to sign. Choose a basis $y_1, \cdots, y_r$ for $B^{n-h}$. By the Poincaré duality theorem (as stated in [20] p. 119–120) the cap product with $\nu$ defines an isomorphism of $H^{n-h}(M^n)$ onto $H_h(M^n)$. Therefore the elements $y_i \cap \nu (i = 1, \cdots, r)$ form a basis for the Betti group $B_h$. Choose a dual basis $\{x_i\}$ for $B^h$. Then the identity $(x_i \cup y_j) \cap \nu = x_i \cap (y_j \cap \nu)$ implies that $\phi(x_i, y_j) = \langle x_i, y_j \cap \nu\rangle = \delta_{ij}$. Therefore the determinant equals $\pm 1$.

By a sum of two oriented $n$-manifolds $M_1^n, M_2^n$ will be meant an oriented $n$-manifold $M_1^n + M_2^n$ obtained as follows. (Compare Seifert [16].) Choose smooth[5] closed $n$-cells $\bar{e}_i^n \subset M_i^n$, $i = 1, 2$, with boundaries $\dot{e}_i^n$ and interiors $e_i^n$. Choose a homeomorphism $f_2\ \dot{e}_1^n \to \dot{e}_2^n$ of degree $-1$. Now let $M_1 + M_2$ be the manifold obtained from $M_1^n - e_1^n$ and $M_2^n - e_2^n$ by matching the boundaries $\dot{e}_1^n$ and $\dot{e}_2^n$ under the homeomorphism $f$. This manifold has an orientation compatible with that of $M_1^n$ and $M_2^n$.

The *sum* of two quadratic forms $(A, \phi)$ and $(A', \phi')$ will mean the form $(A \oplus A', \psi)$ where $\psi((x, x'), (y, y')) = \phi(x, y) + \phi'(x', y')$.

LEMMA 2. *The quadratic form of a sum of two oriented manifolds is naturally isomorphic to the sum of their quadratic forms.*

The proof is not difficult. (Compare [11] p. 400.)

The *index* or *type* of a manifold $M^{4k}$ will mean the index or type of its quadratic form.

LEMMA 3. *A differentiable manifold $M^{4k}$ which is $(2k - 1)$-connected has type* II *if and only if its Stiefel-Whitney class $W_{2k}$ is zero.*

PROOF. Clearly $M^{4k}$ has type II if and only if the homomorphism

$$\mathrm{Sq}^{2k} : H^{2k}(M^{4k}, Z_2) \to H^{4k}(M^{4k}, Z_2)$$

is zero. The conclusion now follows from Wu's formulas for the Stiefel-Whitney classes. (See [22].)

REMARK. In dimension 4 the following alternative interpretation holds. *A differentiable, simply connected manifold $M^4$ has type* II *if and only if, for every point $p$, the open manifold $M^4 - p$ is parallelizable.* The proof is not difficult.

## §3. Simply connected 4-manifolds

In order to simplify the proofs in this section we assume that all manifolds considered are triangulable.

THEOREM 3. *Two oriented, simply connected 4-manifolds have the same oriented homotopy type if and only if their quadratic forms are equivalent.*

PROOF. Whitehead has shown ([21] Theorem 2) that the homotopy type of a finite, simply connected 4-dimensional polyhedron $X$ is determined by the cohomology rings $H^*(X, Z_k)$, $k = 0, 1, 2, \cdots$; together with certain coefficient homomorphisms, Bochstein homomorphisms, and Pontrjagin squares. If $H^*(X,Z)$ has no torsion, then all of this structure is clearly determined by $H^*(X, Z)$.

For simply connected 4-manifold, the Poincaré duality theorem implies that

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[5] By "smooth" we mean that for some neighborhood $U$ of $\bar{e}_i$ the pair $(U, \bar{e}_i)$ is homeomorphic to the pair consisting of euclidean $n$-space $R^n$ and the unit ball in $R^n$.

there is no torsion. Furthermore the integral cohomology ring is completely described by the quadratic form. Therefore the quadratic form determines the homotopy type. Using Theorem 3 of [21] we see that the quadratic form actually determines the oriented homotopy type.

In view of Theorems 1, 2 and Lemma 1 this implies:

COROLLARY 1. *The oriented homotopy type of a simply connected 4-manifold is determined by its second Betti number r, its index* $\tau$ *and its type,* I *or* II; *except possibly in the case of a manifold with definite quadratic form of rank* $r \geqq 9$.

(Note that the Betti number, index, and type are subject to the restrictions given by conditions (1), (2), (3) of Theorem 1.)

REMARK. The well-known classification of 2-manifolds is somewhat analogous to the description given by Corollary 1. Define the *type* of a surface to be II or I according as it is orientable or not; and let $r$ denote the 1-dimensional mod 2 Betti number. (That is $r$ is the dimension of $H_1(M^2, Z_2)$ over $Z_2$.) These two invariants characterize the surface, and are subject to the following relations: (1) $r$ is a non-negative integer; (2) for surfaces of type II, $r \equiv 0 \pmod 2$; and (3) for surfaces of type I, $r > 0$.

The most familiar examples of simply connected 4-manifolds are the product $S^2 \times S^2$ and the complex projective plane $P_2(C)$. Let $\bar{P}_2(C)$ denote the complex projective plane with reversed orientation. The matrices of the corresponding quadratic forms are $U$, (1), and (—1) respectively. (Here $U$ and $V$ will denote the same matrices as in §1.) The following is a consequence of Corollary 1 and Lemma 2.

COROLLARY 2. *A simply connected 4-manifold of type* I *has the homotopy type of a sum of copies of* $P_2(C)$ *and* $\bar{P}_2(C)$; *except possibly when its quadratic form is definite of rank* $\geqq 9$.

(For example, the sum $P_2(C) + (S^2 \times S^2)$ must have the same homotopy type as $P_2(C) + P_2(C) + \bar{P}_2(C)$. The author does not know whether these two manifolds are homeomorphic. However, it is interesting to recall that the surface $P_2(R) + (S^1 \times S^1)$ is homeomorphic to $P_2(R) + P_2(R) + P_2(R)$.)

COROLLARY 3. *A simply connected 4-manifold of type* II *with index zero has the homotopy type of a sum of copies of* $S^2 \times S^2$.

Corollaries 2 and 3 take care of all possible homotopy types with Betti number $r \leqq 7$. The discussion could be completed very neatly if we could give an example of a simply connected 4-manifold with quadratic form V. However the following theorem asserts that such a manifold would be rather pathological.

THEOREM OF ROHLIN [15]. *If a differentiable simply connected 4-manifold has type* II, *then its index must satisfy*

$$\tau \equiv 0 \pmod{16}.$$

(Instead of $\tau \equiv 0 \pmod 8$ as in Theorem 1.) This restriction of Rohlin applies also to combinatorial[6] manifolds, since Cairns has proved ([3]) that every combinatorial 4-manifold possesses a differentiable structure.

[6] By a *combinatorial manifold* we mean a manifold in the sense of Newman [13] and Alexander [1].

The following example, suggested to the author by Hirzebruch, shows that the case $\tau = 16$ can actually occur. Let $M^4$ be any nonsingular algebraic surface of degree 4 in $P_3(C)$. *Then* $M^4$ *is simply connected, has second Betti number* 22, *index* —16, *and has type* II.

(Exactly the same description would hold for the Kummer surfaces, studied by Spanier in [18]. The relationship between these two examples might be interesting to study.)

PROOF. According to a remark of Lefschetz ([9] p. 57), $M^4$ is simply connected. Let $\alpha$ denote the canonical generator of $H^2(P_3(C))$, and let $i : M^4 \to P_3(C)$ denote the inclusion map. Since $M^4$ has degree 4 we have

$$\langle i^*(\alpha^2), \nu\rangle = 4.$$

Furthermore[7] the Chern class of the normal bundle of $M^4$ is $1 + 4i^*(\alpha)$. Recall that the Chern class of $P_3(C)$ is $(1 + \alpha)^4$. The Chern class $1 + c_1 + c_2$ of $M_4$ can now be computed by the product theorem. Solving the equation

$$(1 + c_1 + c_2)(1 + 4i^*(\alpha)) = i^*(1 + \alpha)^4,$$

we obtain

$$c_1 = 0, \qquad c_2 = 6i^*(\alpha^2).$$

Since the Stiefel-Whitney class $W_2$ of $M^4$ is equal to $c_1$ reduced modulo 2, Lemma 3 implies that $M^4$ has type II. The Euler characteristic of $M^4$ is equal to $\langle c_2, \nu\rangle = 24$, which implies that the middle Betti number $r$ equals 22. Finally the formulas

$$\tau = \tfrac{1}{3}\langle p_1, \nu\rangle, \qquad p_1 = c_1^2 - 2c_2$$

imply that $\tau = -16$; which completes the proof.

We conclude by asking several questions. The basic question of which quadratic forms are actually represented by simply connected 4-manifolds appears very difficult. However the following easier version would still be interesting. *Which genera of quadratic forms are represented by differentiable, simply connected 4-manifolds?* The first unanswered case of this is the following. *Does there exist a differentiable, simply connected 4-manifold of type* II *with index* 16 *and Betti number* 16? A positive answer to this question would imply that every genus compatible with the Rohlin theorem actually occurs.

Another interesting problem would be the following. *Which genera are represented by simply connected non-singular algebraic surfaces?*

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[7] The basic reference for the following discussion is Hirzebruch [7] pp. 66–73, 85. See also Steenrod [19] p. 212.

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# AUTOMORPHE FORMEN UND DER SATZ VON RIEMANN-ROCH

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Auf dem Symposium über "Algebraic Topology and its applications", das in Mexico City im August 1956 stattfand, habe ich eine Reihe von Vorträgen gehalten, in denen die Resultate einer gemeinsamen Arbeit mit A. Borel [4] dargestellt wurden. Ferner habe ich über Anwendungen dieser Resultate und des Satzes von Riemann-Roch auf beschränkte homogene symmetrische Gebiete berichtet, die in der vorliegenden Arbeit veröffentlicht werden sollen. Es handelt sich dabei um folgendes.

Die beschränkten homogenen symmetrischen Gebiete wurden von E. Cartan [6] klassifiziert. Einem solchen Gebiet $X$ des $\mathbf{C}^n$ ist bekanntlich eine $n$-dimensionale homogene algebraische Mannigfaltigkeit $X'$ in natürlicher Weise zugeordnet (vgl. [1]). $X$ kann als offene Teilmenge derart in $X'$ eingebettet werden, daß jeder Automorphismus von $X$ zu einem solchen von $X'$ erweitert werden kann. Wenn zum Beispiel $X$ die Hyperkugel ist, dann ist $X'$ der komplexe projektive Raum. Es sei $\Delta$ eine diskontinuierliche Gruppe von Automorphismen von $X$, für die $X/\Delta$ kompakt ist und die abgesehen von der Identität nur fixpunktfreie Transformationen enthält. Dann ist $X/\Delta$ eine algebraische Mannigfaltigkeit [11], und es wird gezeigt, daß jede Chernsche Zahl von $X/\Delta$ gleich dem arithmetischen Geschlecht von $X/\Delta$ (vgl. [8], S. 120/121) multipliziert mit der entsprechenden Chernschen Zahl von $X'$ ist (Satz 4 dieser Arbeit). Die Anregung zu diesem Proportionalitätssatz erhielt ich aus einer Arbeit von Igusa, wo man diesen Satz für einen Spezialfall "zwischen den Zeilen" finden kann ([9], Theorem 8). Für den Proportionalitätssatz wird ein einfacher Beweis mit Hilfe des Zusammenhangs zwischen Chernschen Zahlen und Krümmungstensor angegeben. Auf die Möglichkeit eines solchen Beweises (unter entscheidender Verwendung der Formel (9)) hat mich A. Borel aufmerksam gemacht. Mein ursprünglicher Beweis war komplizierter und verlief ohne jegliche Verwendung von Krümmungstensoren. Der Proportionalitätssatz liefert merkwürdige Beziehungen zwischen den Invarianten der komplexen Mannigfaltigkeit $X/\Delta$ (Korollar zu Satz 4) und schließlich eine Formel für die Anzahl der automorphen Formen vom Gewicht $r$, die einen Zusammenhang mit der Darstellungstheorie kompakter Gruppen ergibt (Satz 5). Auf die schwierige Frage, in welchen Fällen Gruppen $\Delta$ mit den verlangten Eigenschaften existieren, wird nicht eingegangen.

In den Bezeichnungen schließen wir uns dem Ergebnisheft [8] an.

## § 1. Proportionale komplexe Mannigfaltigkeiten

Für eine komplexe Mannigfaltigkeit $X$ sind Chernsche Klassen $c_i(X) \in H^{2i}(X, \mathbf{Z})$ definiert. Wenn $X$ kompakt ist und die komplexe Dimension $n$ hat, dann ist für jede Partition $\pi = (\lambda_1, \lambda_2, \cdots, \lambda_j)$ von $n$ die Chernsche Zahl $c_\pi(X) = c_{\lambda_1} c_{\lambda_2} \cdots c_{\lambda_j}(X)$

durch folgende Gleichung gegeben

$$c_\pi(X) \cdot h_X = c_{\lambda_1}(X) c_{\lambda_2}(X) \cdots c_{\lambda_j}(X), \tag{1}$$

wo $h_X$ das durch die natürliche Orientierung von $X$ ausgezeichnete Element der unendlich zyklischen Gruppe $H^{2n}(X, \mathbf{Z})$ ist.

Zwei kompakte komplexe Mannigfaltigkeiten $X$, $X'$ der Dimension $n$ sollen proportional heißen, wenn eine rationale Zahl $a \neq 0$ existiert, derart, daß für jede Partition $\pi$ von $n$ gilt

$$c_\pi(X) = a \cdot c_\pi(X'). \tag{2}$$

Wir schreiben abkürzend

$$X \approx aX', \tag{2*}$$

wenn $X$ und $X'$ proportional mit dem Faktor $a$ sind.

Da die gewöhnliche Euler-Poincarésche Charakteristik $E(X)$ gleich der Chernschen Zahl für $\pi = (n)$ ist, ergibt sich aus (2) insbesondere die Gleichung

$$E(X) = aE(X'). \tag{3}$$

Aus (2) ergibt sich auch, daß die Proportionalität mit dem Faktor $a$ entsprechend für die Pontrjaginschen Zahlen gilt. Es folgt dann weiter, daß für die durch $X$ und $X'$ bestimmten Elemente $[X]$ und $[X']$ der Thomschen "Cobordisme"—Algebra $\Omega \otimes \mathbf{Q}$ gilt

$$[X] = a \cdot [X'] \tag{4}$$

Zur Thomschen Algebra siehe [16] und [8]. Aus (4) erhält man für den Index $\tau$ die Gleichung

$$\tau(X) = a \cdot \tau(X') \tag{5}$$

Zur Definition des Index siehe [8], S. 83. Wenn $n$ nicht durch 2 teilbar ist, d.h. wenn die reelle Dimension von $X$ und $X'$ nicht durch 4 teilbar ist, dann ist per definitionem $\tau(X) = \tau(X') = 0$. Ferner ist dann nach Thom $[X] = [X'] = 0$. Die Gleichungen (4) und (5) sind also für ungerades $n$ trivial.

Es sei $X$ weiterhin eine kompakte komplexe Mannigfaltigkeit. Dann sind die Zahlen $\chi^p(X)$ definiert. $\chi^p(X)$ ist die Euler-Poincarésche Charakteristik von $X$ bezüglich der Cohomologie mit Koeffizienten in der Garbe der Keime von holomorphen $p$-Formen. $\chi^0(X) = \chi(X)$ ist das arithmetische Geschlecht. Es sei ferner $K$ das kanonische Geradenbündel von $X$ und $K^r$ seine $r$-te Potenz ($r$ ganz) im Sinne des Tensorprodukts. Die Zahl $\chi(X, K^r)$, welche wir abkürzend mit $\chi(X, r)$ bezeichnen wollen, ist die Euler-Poincarésche Charakteristik von $X$ bezüglich der Cohomologie mit Koeffizienten in der Garbe der Keime von holomorphen Schnitten des Geradenbündels $K^r$. Zu allen diesen Begriffen siehe [8] und die dort angegebene Literatur.

Unter einer algebraischen Mannigfaltigkeit verstehen wir in dieser Arbeit wie in [8] eine kompakte komplexe Mannigfaltigkeit, die singularitätenfrei in einen komplexen projektiven Raum geeigneter Dimension eingebettet werden kann. Nach dem Satz von Riemann-Roch [8] sind die Zahlen $\chi^p(X)$ und $\chi(X, r)$ Linearkombinationen der Chernschen Zahlen von $X$ mit rationalen (von $X$ unabhängigen) Koeffizienten. Dabei ist zu beachten, daß die charakteristische Klasse von $K$ gleich $-c_1(X)$ ist. Aus den vorstehenden Bemerkungen folgt der

SATZ 1. *Es seien $X$ und $X'$ proportionale algebraische Mannigfaltigkeiten mit dem Proportionalitätsfaktor $a$* (siehe (2), (2*)). *Dann gilt*

$$\chi^p(X) = a \cdot \chi^p(X')$$
$$\chi(X, r) = a \cdot \chi(X', r)$$
$$E(X) = a \cdot E(X')$$
$$\tau(X) = a \cdot \tau(X').$$

*In der Thomschen Algebra $\Omega \otimes \mathbf{Q}$ ist*

$$[X] = a \cdot [X'].$$

## § 2. Homogene hermitesche Mannigfaltigkeiten

Eine komplexe Mannigfaltigkeit, die mit einer hermiteschen Metrik versehen ist, wird kurz hermitesche Mannigfaltigkeit genannt. $X$ sei nun eine hermitesche Mannigfaltigkeit der komplexen Dimension $n$. Zu der hermiteschen Metrik gehört ein Krümmungstensor $R$, welcher für jeden Punkt $p$ von $X$ jedem Paar $x$, $y$ von Tangentialvektoren in $p$ einen komplexen Endomorphismus des Tangentialraumes von $X$ in $p$ zuordnet. Es ist $R(x, y) = -R(y, x)$. In Bezug auf ein lokales Koordinatensystem kann $R$ durch eine $n \times n$ - reihige Matrix $(\Omega_{rs})$ gegeben werden, deren Elemente $\Omega_{rs}$ äußere Differentialformen vom Grade 2 sind. Der Krümmungstensor $R$ kann auch aufgefaßt werden als eine äußere Differentialform vom Grade 2 mit Koeffizienten in dem komplexen Vektorraumbündel $W$ über $X$, das zu dem tangentiellen $\mathbf{U}(n)$-Prinzipalbündel über $X$ vermöge der adjungierten Darstellung von $\mathbf{U}(n)$ assoziiert ist. $W$ ist das Tensorprodukt $\mathfrak{T} \otimes \mathfrak{T}^*$, wo $\mathfrak{T}$ das tangentielle komplexe Vektorraumbündel von $X$ ist. Die Determinante

$$\left|\delta_{rs} - \frac{1}{2\pi i}\,\Omega_{rs}\right| \tag{6}$$

ist dann eine (vom Koordinatensystem unabhängige) reelle Differentialform (vom gemischten Grade) über $X$. Sie ist geschlossen und repräsentiert im Sinne von de Rham die (totale) Chernsche Klasse von $X$. Vgl [7]. Für Vorzeichenfragen siehe [4], Appendix.

$$1 + c_1 + c_2 + \cdots + c_n \cong \left|\delta_{rs} - \frac{1}{2\pi i}\,\Omega_{rs}\right| \tag{7}$$

Die $c_j$ sind hier als reelle Cohomologieklassen aufzufassen.

Bei der Berechnung der Determinante und in allen anderen Fällen ist ein Produkt von Differentialformen immer im äußeren Sinne zu nehmen. Wenn $X$ kompakt ist, dann ist die Chernsche Zahl $c_\pi(X)$ (vgl. § 1) gleich dem Integral über $X$ eines wohlbestimmten homogenen Polynoms $P_\pi$ vom Grade $n$ in den $\Omega_{rs}$. Dieses Polynom $P_\pi$ ist eine reelle Differentialform vom Grade $2n$, welche auch für nicht-kompaktes $X$ definiert ist.

Nun sei $X$ eine homogene hermitesche Mannigfaltigkeit, d.h. $X$ besitze eine transitive Gruppe von Automorphismen, welche die Metrik und damit auch den Krümmungstensor invariant lassen. Ein Automorphismus ist in dieser Arbeit immer eine komplex-analytische eineindeutige Abbildung einer komplexen Mannigfaltigkeit auf sich. Das Volumelement $v$ von $X$ ist eine bezügliche der Gruppe invariante Differentialform vom Grade $2n$. Die Differentialform $P_\pi$ ist ebenfalls invariant und $P_\pi/v$ ist deshalb eine reelle Zahl $r_\pi$. Damit kann auch für nicht-kompaktes homogenes hermitesches $X$ (bis auf einen von 0 verschiedenen Proportionalitätsfaktor) von den Chernschen Zahlen von $X$ gesprochen werden, und es ist klar, wann $X$ proportional zu einer kompakten komplexen Mannigfaltigkeit $X'$ gleicher Dimension genannt wird. Für die Berechnung der Chernschen Zahlen des cartesischen Produktes zweier kompakter komplexer Mannigfaltigkeiten aus denen der Faktoren gibt es eine wohlbekannte Formel (vgl. z.B. [8], Lemma 10.2.1). Dieselbe Formel gilt für das cartesische Produkt zweier homogener hermitescher Mannigfaltigkeiten.

Wir müssen jetzt über einige Dinge aus der Theorie der homogenen hermiteschen symmetrischen Mannigfaltigkeiten referieren. Es werde dazu auf [6], [5], [1], [2], [3] und auf die in [1] und [3] zitierte Literatur verwiesen. Bezüglich der beschränkten homogenen symmetrischen Gebiete werde auch auf die Untersuchungen von Harish-Chandra aufmerksam gemacht. (Vgl. [3], 16).

Eine komplexe Mannigfaltigkeit $X$ heißt symmetrisch, wenn es zu jedem Punkt $p$ von $X$ einen Automorphismus $\sigma_p$ von $X$ gibt, der involutiv ist (d.h. $\sigma_p^2 = Id$) und der $p$ als isolierten Fixpunkt hat. Eine hermitesche Mannigfaltigkeit heißt symmetrisch, wenn die Involution $\sigma_p$ außerdem noch die Metrik invariant läßt.

Es sei $G$ eine (zusammenhängende) einfache Liesche Gruppe, deren Zentrum nur aus dem Einselement bestehe, und es sei $H$ eine abgeschlossene Untergruppe von $G$. Der Quotientenraum $G/H$ ist in folgenden Fällen in natürlicher Weise mit einer homogenen hermiteschen (sogar kählerschen) symmetrischen Struktur versehen.

(1) *$G$ ist nicht-kompakt. $H$ ist eine maximale zusammenhängende kompakte Untergruppe von $G$. Das Zentrum von $H$ ist eindimensional.*

(2) *$G$ ist kompakt. $H$ ist eine maximale zusammenhängende echte kompakte Untergruppe von $G$. Der Rang von $H$ ist maximal. $H$ ist Zentralisator eines eindimensionalen Torus von $G$, der gleich der Zusammenhangskomponente des Einselements des Zentrums von $H$ ist.*

Jedes irreduzible beschränkte homogene symmetrische Gebiet $X$ in einem $\mathbf{C}^n$ ist einem Raum $G/H$ der ersten Art isomoph, d.h. es gibt eine eineindeutige komplex-analytische Abbildung von $X$ auf $G/H$, die die Bergmannsche Metrik von $X$ in die homogene hermitesche Metrik von $G/H$ überführt. Umgekehrt ist jeder Raum der ersten Art einem irreduziblen beschränkten homogenen symmetrischen Gebiet in diesem Sinne isomorph.

Die folgende Tabelle liefert eine vollständige Aufzählung der Räume zweiter Art. (Vgl. [1], S. 179–180).

$$\text{I. } \mathbf{U}(p+q)/\mathbf{U}(p)\times\mathbf{U}(q)$$

$$\text{II. } \mathbf{SO}(2p)/\mathbf{U}(p)$$

$$\text{III. } \mathbf{Sp}(p)/\mathbf{U}(p)$$

$$\text{IV. } \mathbf{SO}(p+2)/\mathbf{SO}(p)\times\mathbf{SO}(2),\ (p\neq 2)$$

$$\text{V. } \mathbf{E}_6/\mathbf{Spin}\,(10)\times\mathbf{T}^1$$

$$\text{VI. } \mathbf{E}_7/\mathbf{E}_6\times\mathbf{T}^1.$$

Diese Räume sind algebraische Mannigfaltigkeiten. (Um die Darstellung als $G/H$ mit einfachem $G$ "ohne Zentrum" zu erhalten, muß man Zähler und Nenner noch durch das Zentrum des Zählers dividieren.) Über die Räume zweiter Art siehe auch Borel—Siebenthal (Comm. Math. Helv., 23 (1949), 200–221).

Wie wir später sehen werden, kann jeder Raum $X$ der ersten Art auf natürliche Weise als offene Teilmenge in einem Taum $X'$ der zweiten Art eingebettet werden (vgl. [1]), und man erhält auf diese Weise eine eineindeutige Korrespondenz zwischen den Räumen erster und zweiter Art, so daß die vorstehende Tabelle auch eine Klassifikation der irreduziblen beschränkten homogenen symmetrischen Gebiete liefert.

Es sei $G/H$ ein Raum der ersten oder zweiten Art. Es sei $\mathfrak{g}$ bzw. $\mathfrak{h}$ die Liesche Algebra von $G$ bzw. $H$. Dann existiert ein Element $\sigma\in H$ mit $\sigma^2=1$, derart, daß $\mathfrak{h}$ der zum Eigenwert $+1$ gehörige Eigenraumdes Endomorphismus $\mathrm{Ad}(\sigma)$ von $\mathfrak{g}$ ist. Es sei $\mathfrak{m}$ der zum Eigenwert $-1$ gehörige Eigenraum von $\mathrm{Ad}(\sigma)$. Dann gilt

$$\mathfrak{g}=\mathfrak{h}+\mathfrak{m},\quad [\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h},\quad [\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m},\quad [\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h} \tag{8}$$

Es sei $B(x, y)=\mathrm{tr}(\mathrm{ad}(x)\circ\mathrm{ad}(y))$ die auf $\mathfrak{g}$ definierte Killingsche Bilinearform. ($x, y\in\mathfrak{g}$; "tr" bedeutet Spur = trace). Die linearen Teilräume $\mathfrak{h}$ und $\mathfrak{m}$ von $\mathfrak{g}$ sind dann bezüglich $B$ orthogonale Komplemente voneinander. $\mathfrak{m}$ ist mit dem Tangentialraum von $G/H$ in $e_0=(H)$ zu identifizieren. $B$ ist bei Beschränkung auf $\mathfrak{m}$ definit und zwar positiv für Räume der ersten und negativ für Räume der zweiten Art. $\mathrm{Ad}(H)$ bildet $\mathfrak{m}$ in sich ab. $\mathrm{Ad}(\sigma)$ ist die Symmetrie in $e_0$. Die Form $B$ ist invariant unter jedem Element von $\mathrm{Ad}(H)$. Deshalb liefert $B$ bzw. $-B$ eine homogene Riemannsche symmetrische Metrik für $G/H$. Der zugehörige Riemannsche Krümmungstensor $R$ ordnet dem Paar $x, y\in\mathfrak{m}$ den durch

$$R(x, y)z=-[[x, y], z],\quad z\in\mathfrak{m}, \tag{9}$$

gegebenen Endomorphismus von $\mathfrak{m}$ zu. (Man beachte (8).) Die Formel (9) stammt von E. Cartan. Für einen Beweis siehe [12].

Als Tangentialraum der komplexen Mannigfaltigkeit $G/H$ im Punkte $e_0$ trägt $\mathfrak{m}$ eine komplexe Struktur, d.h. es ist ein Endomorphismus $J$ von $\mathfrak{m}$ mit $JJ=-Id$

gegeben. Bekanntlich ist $J = \mathrm{ad}(h)|\mathfrak{m}$ für ein geeignetes $h \in \mathfrak{h}$. Ferner durchläuft $\exp(th)$ die Zusammenhangskomponente des Einselementes des Zentrums von $H$, und es ist $\exp(\pi h) = \sigma$. Nun ist

$$\mathrm{tr}(\mathrm{ad}[h, x] \circ \mathrm{ad}(y) + \mathrm{ad}(x) \circ \mathrm{ad}\,[h, y]) = 0. \tag{10}$$

Deshalb definiert $B$ bzw. $-B$ die homogene hermitesche symmetrische Struktur von $G/H$, die sogar kählersch ist. Zu dieser kählerschen Metrik gehört ein Krümmungstensor, der jedem $x, y \in \mathfrak{m}$ den Endomorphismus $R(x, y)$ von $\mathfrak{m}$ zuordnet, der durch den Riemannschen Krümmungstensor gegeben wird (siehe [7], S. 114). $R(x, y)$ respektiert die komplexe Struktur. Also ist durch (9) auch der Krümmungstensor $(\Omega_{rs})$, der zur kählerschen Metrik von $G/H$ gehört, gegeben.

Der kählerschen Metrik von $G/H$ ist eine geschlossene reelle Form $\omega$ vom Grade 2 zugeordnet (vgl. z.B. [8], 15.6). Für Räume erster Art ist die Form $\gamma = (2\pi i)^{-1} \sum_{s=1}^{n} \Omega_{ss}$ ein positives Vielfaches von $\omega$, für Räume zweiter Art ein negatives Vielfaches von $\omega$. Das kanonische Geradenbündel $K$ hat $-c_1$ (repräsentiert durch $\gamma$) als charakteristische Cohomologieklasse. Also ist $K$ für Räume erster Art positiv im Sinne von Kodaira (vgl. z.B. [8], S. 137), während für Räume zweiter Art $K^{-1}$ in diesem Sinne positiv ist. Nach einem fundamentalen Satz von Kodaira [11] kann man daraus wieder die Tatsache erhalten, daß die Räume zweiter Art algebraische Mannigfaltigkeiten sind.

Es sei $X = G/H$ ein Raum erster Art. Wir wollen nun darüber berichten, wie $X$ ein Raum $X'$ der zweiten Art zugeordnet wird. Es sei wieder $\mathfrak{g}$ die Liesche Algebra von $G$ und $\mathfrak{h}$ die von $H$. Dann haben wir die Zerlegung (8): $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, die eine Cartan Zerlegung von $\mathfrak{g}$ ist. (Vgl. Mostow, Bull. Amer. Math. Soc. 55 (1949) 969–980 und Memoirs of the AMS, 14, 1955.) Da $G$ kein Zentrum hat, ist die komplexe Erweiterung $C(G)$ wohldefiniert. $C(G)$ hat die Liesche Algebra $\mathfrak{c} = \mathfrak{h} + i\mathfrak{h} + \mathfrak{m} + i\mathfrak{m}$. Es sei $G'$ die zur Lieschen Unteralgebra $\mathfrak{g}' = \mathfrak{h} + i\mathfrak{m}$ von $\mathfrak{c}$ gehörige zusammenhängende Untergruppe von $C(G)$. Dann ist $G'$ kompakt, einfach und ohne Zentrum, enthält $H$, und $X' = G'/H$ ist der gesuchte Raum zweiter Art. In analoger Weise kann man umgekehrt von einem Raum zweiter Art zu einem Raum erster Art gelangen.

Das Element $\sigma \in H$, das die Symmetrie von $X$ liefert, liefert auch die Symmetrie von $X'$. Wenn wir $\mathfrak{m}' = i\mathfrak{m}$ setzen, dann ist $\mathfrak{g}' = \mathfrak{h} + \mathfrak{m}'$ die zu $X' = G'/H$ gehörige Zerlegung (8). Nun ist $\mathfrak{m}'$ mit dem Tangentialraum von $X'$ im Punkte $e_0' = (H)$ zu identifizieren. Das Element $h \in \mathfrak{h}$, das die komplexe Struktur von $\mathfrak{m}$ liefert, liefert auch die von $\mathfrak{m}'$. Ordnet man $x \in \mathfrak{m}$ das Element $ix \in \mathfrak{m}'$ zu, dann erhält man einen Isomorphismus der komplexen Vektorräume $\mathfrak{m}$ und $\mathfrak{m}'$. Wendet man (9) sowohl auf $X$ als auch auf $X'$ an, dann sieht man, dass bei diesem Isomorphismus der zur kählerschen Metrik von $X$ gehörige Krümmungstensor in das Negative des zur kählerschen Metrik von $X'$ gehörigen Krümmungstensors übergeht. Obwohl $X$ nicht kompakt ist, sind für $X$ als homogene hermitesche Mannigfaltigkeit Chernsche Zahlen (bis auf Proportionalität) definiert. Es folgt aus (7), dass $X$ und $X'$ proportional sind.

Satz 2. *Jedem Raum $X$ der ersten Art ist ein Raum $X'$ der zweiten Art in* *natürlicher Weise zugeordnet, und man erhält so eine eineindeutige Korrespondenz zwischen Räumen erster und zweiter Art. $X$ und $X'$ sind proportional!*

Die komplexe Liesche Gruppe $C(G)$ operiert komplex-analytisch auf $X' = G'/H$. Es gibt nämlich eine abgeschlossene komplexe Untergruppe $L$ von $C(G)$ mit $G' \cap L = G \cap L = H$. Man erhält einen natürlichen Homöomorphismus von $X'$ auf $C(G)/L$, welcher mit der Operationen von $G'$ und den komplexen Strukturen verträglich ist (vgl. [2]). Dann hat man auch eine natürliche Einbettung von $X = G/H$ in $X' = C(G)/L$, welche mit den Operationen von $G$ und den komplexen Strukturen verträglich ist. $X$ wird homöomorph auf eine offene Teilmenge von $X'$ abgebildet. Jeder zu $G$ gehörige Automorphismus von $X$ lässt sich also zu einem solchen von $X'$ erweitern ($G \subset C(G)$).

## § 3. Beschränkte homogene Gebiete

Es sei $X$ ein beschränktes Gebiet im $\mathbf{C}^n$. Die Gruppe $\mathfrak{A}(X)$ der Automorphismen von $X$ ist, versehen mit der Topologie der kompakten Konvergenz, eine Liesche Gruppe. Mit $\mathfrak{A}_0(X)$ werde die größte zusammenhängende Untergruppe von $\mathfrak{A}(X)$ bezeichnet. Eine Untergruppe von $\mathfrak{A}(X)$ ist genau dann diskret, wenn sie eine diskontinuierliche Gruppe von Automorphismen von $X$ ist. (Vgl. [1] und die dort angegebene Literatur).

Es sei nun $\Delta$ eine diskrete Untergruppe von $\mathfrak{A}(X)$ mit den folgenden Eigenschaften.

(11)
1. *Der Quotientenraum $X/\Delta$ ist kompakt.*
2. *Kein von der Identität verschiedenes Element von $\Delta$ hat einen Fixpunkt in $X$.*

Die kompakte komplexe Mannigfaltigkeit $Y = X/\Delta$ ist dann bekanntlich eine algebraische Mannigfaltigkeit, deren kanonisches Geradenbündel $K_Y$ positiv im Sinne von Kodaira ist ([11], S. 41). Zur Terminologie vgl. auch [8], S. 137. Wenn $X$ einfach-zusammenhängend ist, dann universelle Überlagerung von $X/\Delta$ zu $X$ komplex-analytisch homöomorph, und die Gruppe der Decktransformationen ist zu $\Delta$ isomorph.

Eine holomorphe Funktion $f$ auf $X$ heißt automorphe Form bezüglich $\Delta$ vom Gewicht $r$, wenn für alle $z \in X$ und $\gamma \in \Delta$ gilt:

$$f(\gamma z) = j_\gamma^{-r}(z) \cdot f(z), \tag{12}$$

wo $j_\gamma(z)$ die Funktionaldeterminaten der Abbildung $\gamma$ an der Stelle $z$ ist. Die automorphen Formen vom Gewicht $r$ bilden einen komplexen Vektorraum, welcher isomorph ist dem Vektorraume der holomorphen Schnitte des Geradenbündels $K_Y^r$ über $Y = X/\Delta$. Es bezeichne $\Pi_r(X, \Delta)$ die Dimension dieses Vektorraumes, d.h. $\Pi_r(X, \Delta)$ ist die "Anzahl" der komplex-linear-unabhängigen automorphen Formen von $X$ bezüglich $\Delta$ vom Gewicht $r$. Da $K_Y$ positiv ist (im Sinne von Kodaira), verschwinden nach einem Satz von Kodaira ([10]; vgl. auch [8], Sätze 18.2.1 und 18.2.2) die Cohomologiegruppen von $Y$ mit Koeffizienten in der Garbe der Keime von holomorphen Schnitten von $K_Y^r$ in den von 0 verschiedenen Dimensionen, falls $r \geqq 2$, und in den von $n$ verschiedenen Dimensionen, falls $r \leqq -1$. Also ist

$$\text{(13)} \qquad \begin{aligned} &\Pi_r(X, \Delta) = 0 \qquad \text{für } r \leqq -1 \\ &\Pi_0(X, \Delta) = 1 \\ &\Pi_1(X, \Delta) = g_n \\ &\Pi_r(X, \Delta) = \chi(X/\Delta, r) \qquad \text{für } r \geqq 2. \end{aligned}$$

Hier bezeichnet $g_j$ die "Anzahl" der holomorphen äußeren Differentialformen (Formen erster Gattung) vom Grade $j$ über $Y = X/\Delta$. In anderen Worten: $g_j$ ist die "Anzahl" der holomorphen äußeren Differentialformen vom Grade $j$ über $X$, welche automorph sind bezüglich $\Delta$.

Wir wollen das beschränkte Gebiet $X$ jetzt als homogen voraussetzen, d.h. $\mathfrak{A}(X)$ soll transitiv sein. Jedes Element von $\mathfrak{A}(X)$ läßt die Bergmannsche Metrik von $X$ invariant, also ist $X$ homogen hermitesch (sogar kählersch). Die Chernschen Zahlen von $X$ sind also bis auf Proportionalität wohl definiert. Da die hermitesche Metrik auf $X/\Delta$ durch die von $X$ induziert wird ($\Delta$ habe wieder die Eigenschaften 1 und 2, siehe (11)), ist $X$ proportional zu $X/\Delta$. Jeder Chernschen Zahl entspricht eine invariante äußere Differentialform $P_\pi$ vom Grade $2n$ auf $X$. Wenn $v$ das Volumenelement der Bergmannschen Metrik bezeichnet, dann ist $P_\pi/v$ eine Konstante. Wir erhalten so

SATZ 3. *Es sei $X$ ein beschränktes homogenes Gebiet des $\mathbf{C}^n$, und $\Delta_1$, $\Delta_2$ seien diskontinuierliche Gruppen von Automorphismen von $X$, die beide die Eigenschaften* 1 *und* 2 *haben* (11). *Dann sind $Y_1 = X/\Delta_1$ und $Y_2 = X/\Delta_2$ proportionale algebraische Mannigfaltigkeiten und zwar gilt*

$$\text{(14)} \qquad Y_1 \approx \frac{V_1}{V_2} Y_2,$$

*wo $V_1$* bzw. *$V_2$ das Volumen von $Y_1$* bzw. *$Y_2$ bezüglich der Bergmannschen Metrik von $X$ ist. Für $Y_1$, $Y_2$ gelten alle Relationen von Satz* 1. *Wegen* (13) *gilt insbesondere für die Anzahlen der automorphen Formen vom Gewicht $r$*

$$\text{(15)} \qquad V_2 \cdot \Pi_r(X, \Delta_1) = V_1 \cdot \Pi_r(X, \Delta_2) \qquad \text{für } r \geqq 2.$$

Nun werde zusätzlich vorausgesetzt, daß das beschränkte homogene Gebiet $X$ auch noch symmetrisch ist. Dann ist $X$ cartesisches Produkt $X_1 \times X_2 \times \cdots \times X_s$ von irreduziblen beschränkten homogenen symmetrischen Gebieten. (Siehe §2). Jeder Faktor $X_k$ ist ein Raum erster Art: $X_k = G_k/H_k$, wo $G_k$ nicht-kompakt, einfach und ohne Zentrum ist. Jeder Raum $X_k$ ist proportional zu einem Raum $X'_k = G'_k/H_k$ zweiter Art, wo $G'_k$ kompakt, einfach und ohne Zentrum ist. Wir ordnen $X$ die homogene algebraische Mannigfaltigkeit $X' = X'_1 \times X'_2 \times \cdots \times X'_s$ zu. $X$ kann als offene Teilmenge in $X'$ eingebettet werden. Bekanntlich ist

$$\mathfrak{A}_0(X) = G_1 \times G_2 \times \cdots \times G_s,$$

und somit kann jeder (zu $\mathfrak{A}_0(X)$ gehörige) Automorphismus von $X$ zu einem Automorphismus von $X'$ erweitert werden (vgl. den Schluss von §2). Der Satz 2 liefert

$$X \text{ proportional zu } X'. \tag{16}$$

Es sei $\Delta$ wieder eine diskontinuierliche Gruppe von Automorphismen von $X$ mit den Eigenschaften 1 und 2, siehe (11). Aus (16) und der Proportionalität von $X$ und $X/\Delta$ folgt, daß die algebraischen Mannigfaltigkeiten $X/\Delta$ und $X'$ proportional sind:

$$X/\Delta \approx a \cdot X'.$$

Zur Bestimmung der Proportionalitätskonstanten $a$ erinnern wir daran, daß die Zahlen $h^{p,q}$ von $X'$ verschwinden, wenn $p \neq q$. (Vgl. [4].) Daraus folgt, daß das arithmetische Geschlecht $\chi(X')$ gleich 1 ist (vgl. [8], Satz 15.7.1), was auch aus der Tatsache gefolgert werden kann, daß $X'$ rational ist (Goto), und was sich außerdem auch noch daraus ergibt, daß das kanonische Geradenbündel von $X'$ negativ im Sinne von Kodaira ist. Nach Satz 1 ist

$$\chi(X/\Delta) = a \cdot \chi(X') = a.$$

Der Proportionalitätsfaktor $a$ kann auch durch die Chernsche Zahl $c_1^n$ bestimmt werden:

$$a = c_1^n(X/\Delta) : c_1^n(X').$$

Da die erste Chernsche Klasse von $X/\Delta$ negativ, diejenige von $X'$ dagegen positiv im Sinne von Kodaira ist, folgt, daß $a$ für gerades $n$ positiv, für ungerades $n$ negativ ist ($n = \dim_{\mathbf{C}} X$). Wir haben damit bewiesen.

Satz 4. *Jedem beschränkten homogenen symmetrischen Gebiet $X$ des $\mathbf{C}^n$ ist eine $n$-dimensionale algebraische Mannigfaltigkeit $X'$ zugeordnet. Wenn $\Delta$ eine diskontinuierliche Gruppe von Automorphismen von $X$ ist, für die $X/\Delta$ kompakt ist und die abgesehen von der Identität nur fixpunktfreie Transformationen enthält, dann ist $X/\Delta$ eine algebraische Mannigfaltigkeit, und es gilt die Proportionalität*

$$X/\Delta \approx \chi(X/\Delta) \cdot X',$$

*wo $\chi(X/\Delta)$ das arithmetische Geschlecht von $X/\Delta$ ist, welches für gerades $n$ positiv und für ungerades $n$ negativ ist.*

Um den Satz 4 konkret anwenden zu können, werde kurz über die Berechnung einiger Invarianten der algebraischen Mannigfaltigkeit $X'$ referiert. Für die erforderlichen Informationen über kompakte Liesche Gruppen siehe etwa den Bericht von Samelson (Bull. Amer. Math. Soc. 58 (1952), 2–37).

Wir haben $X' = G'/H$, wo $G'$ das direkte Produkt der kompakten einfachen Gruppen $G'_1, \cdots, G'_s$ und $H$ das direkte Produkt der Gruppen $H_1, \cdots, H_s$ ist ($H_j \subset G'_j$).

Die Euler-Poincarésche Charakteristik $E(X')$ ist der Quotient der Ordnung der

Weylschen Gruppe von $G'$ durch die Ordnung der Weylschen Gruppe von H. Für Typ I–VI (siehe §2) ergeben sich für $E(X')$ die folgenden Werte

$$\begin{array}{ll} & \text{I. } (p+q)!/p!q! \\ & \text{II. } 2^{p-1} \\ & \text{III. } 2^p \\ (17) & \text{IV. } p+2 \text{ für gerades } p; \text{ und } p+1 \text{ für ungerades } p \\ & \text{V. } 27 \\ & \text{VI. } 56. \end{array}$$

Die Bettischen Zahlen $b_r(X')$ können mit Hilfe der Formel von Hirsch berechnet werden (vgl. A. Borel, Ann. of Math. 57 (1953), 115–207). Sie verschwinden für ungerades $r$. Die zweite Bettische Zahl von $X'$ ist gleich $s$, der Anzahl der irreduziblen Faktoren von $X'$. Aus dem Verschwinden der Zahlen $h^{p,q}(X')$ für $p \neq q$ folgert man

$$(18) \qquad b_{2j}(X') = h^{j,j}(X') = (-1)^j \chi^j(X').$$

Nach der Formel von Hodge (vgl. [8], Satz 15.8.2) ergibt sich daraus für den Index $\tau(X')$

$$(19) \qquad \tau(X') = \textstyle\sum_{j=0}^{n} (-1)^j \, b_{2j}, \qquad n = \dim_{\mathbf{C}} X'.$$

Wegen (19) ermöglicht die Formel von Hirsch auch eine Berechnung von $\tau(X')$. Für irreduzibles $X'$ ergibt sich $\tau(X') = 0$ außer in folgenden Fällen

$$\begin{array}{lll} & \text{I. } X' = \mathbf{U}(p+2r)/\mathbf{U}(p) \times \mathbf{U}(2r), & \tau(X') = \left(\left[\frac{p}{2}\right] + r\right)! \Big/ \left[\frac{p}{2}\right]! r! \\ (20) & \text{IV. } X' = \mathbf{SO}(4k+2)/\mathbf{SO}(4k) \times \mathbf{SO}(2), & \tau(X') = 2 \\ & \text{V. } X' = \mathbf{E}_6/\mathbf{Spin}(10) \times \mathbf{T}^1, & \tau(X') = 3. \end{array}$$

Die vorstehend zusammengetragenen Informationen geben dem folgenden Korollar einiges Interesse.

Korollar zu Satz 4. *Es sei $X$ ein beschränktes homogenes symmetrisches Gebiet des $\mathbf{C}^n$. Es sollen $\Delta$ und $X'$ dieselbe Bedeutung wie in Satz 4 haben. Ferner sei $b_r$ die $r$-te Bettische Zahl. Dann gilt*

$$\chi^j(X/\Delta) = (-1)^j \chi(X/\Delta) \cdot b_{2j}(X') \neq 0,$$
$$\chi^1(X/\Delta) = -s \cdot \chi(X/\Delta) \neq 0,$$

*wo $s$ die Anzahl der irreduziblen Faktoren von $X$ ist. Ferner gilt*

$$E(X/\Delta) = \chi(X/\Delta) \cdot E(X')$$
$$\tau(X/\Delta) = \chi(X/\Delta) \cdot \tau(X')$$

*$E(X/\Delta)$ ist positiv für gerades $n$ und negativ für ungerades $n$. Wenn wenigstens einer der irreduziblen Faktoren von $X$ eine zugehörige algebraische Mannigfaltigkeit hat, die nicht zu den in* (20) *aufgezählten Typen gehört, dann ist $\tau(X/\Delta) = 0$. Anderenfalls ist $\tau(X/\Delta) \geqq 1$.*

## § 4. Einiges über kompakte Liesche Gruppen (vgl. [4]).

Es sei $G$ eine kompakte Liesche Gruppe und $\rho$ eine Darstellung von G in einem komplexen Vektorraum $V$. Es sei $T$ ein maximaler Torus von $G$. Dann ist $V$ eine direkte Summe

$$V = E_0 + E_1 + \cdots + E_s,$$

wo $E_0$ der komplexe Vektorraum aller unter $T$ festen Vektoren ist und wo die $E_j$ $(1 \leqq j \leqq s)$ eindimensionale komplexe Unterräume von $V$ sind, die als ganzes durch jedes Element von $T$ in sich überführt werden. $T$ operiert auf $E_j$, und man erhält so Homomorphismen

$$\rho_j : T \to \mathbf{U}(1)$$

Es sei $\mathfrak{h}$ die Liesche Algebra von $T$ ($\mathfrak{h}$ ist ein reeller Vektorraum). Die Liesche Algebra von $\mathbf{U}(1)$ sei mit $\mathbf{R}$ identifiziert und zwar so, daß die Abbildung exp: $\mathbf{R} \to \mathbf{U}(1)$ jedem $t \in \mathbf{R}$ das Element $e^{it} \in \mathbf{U}(1)$ zuordnet. Dann bestimmt $\rho_j$ eine lineare Abbildung $2\pi a_j$ von $\mathfrak{h}$ in $\mathbf{R}$. Mit $\mathfrak{h}^*$ werde der duale Vektorraum von $\mathfrak{h}$ bezeichnet. Das Element $0 \in \mathfrak{h}^*$ (mit der Vielfachheit $\dim_{\mathbf{C}} E_0$) und die Elemente $a_j \in \mathfrak{h}^*$ $(1 \leqq j \leqq s)$ heißen die Gewichte der Darstellung $\rho$.

Von jetzt an werde $G$ als halbeinfach vorausgesetzt. Die Gewichte der Isotropiedarstellung von $T$ im komplex erweiterten reellen Tangentialraum von $G/T$ im Punkte $e_0 = (T)$ sind die Wurzeln von $G$. Die Wurzeln von $G$ sind $2m$ Elemente $\pm a_1, \pm a_2, \cdots, \pm a_m \in \mathfrak{h}^*$, wo $2m$ die reelle Dimension von $G/T$ ist. Der Vektorraum $\mathfrak{h}^*$ ist in natürlicher Weise mit der Cohomologiegruppe $H^1(T, \mathbf{R})$ zu identifizieren. Man hat ferner einen Isomorphismus-auf

$$\psi : \mathfrak{h}^* = H^1(T, \mathbf{R}) \to H^2(G/T, \mathbf{R}),$$

wo $\psi$ die negative Transgression im Faserbündel $(G, G/T, T)$ ist. Es seien jetzt $a_1, \cdots, a_m \in \mathfrak{h}^* = H^1(T, \mathbf{R})$ die positiven Wurzeln von $G$ bezüglich einer fest-gewählten lexikographischen Anordnung von $\mathfrak{h}^*$. Dann besitzt $G/T$ genau eine homogene komplexe Struktur, derart, daß die Gewichte der Isotropiedarstellung von $T$ im komplexen Tangentialraum von $G/T$ im Punkte $e_0 = (T)$ gleich $a_1, \cdots, a_m$ sind. Mit dieser komplexen Struktur ist $G/T$ eine einfach-zusammenhängende algebraische Mannigfaltigkeit. Die (totale) Chernsche Klasse von $G/T$ (als Cohomologieklasse mit reellen Koeffizienten) ist durch folgende Formel gegeben

$$c(G/T) = (1 + \psi a_1)(1 + \psi a_2) \cdots (1 + \psi a_m). \tag{21}$$

Die Cohomologiegruppe $H^2(G/T, \mathbf{Z})$ ist eine Untergruppe $U$ von $H^2(G/T, \mathbf{R})$. Ordnet man jedem komplex-analytischen Geradenbündel $F$ über $G/T$ seine charakteristische Klasse (erste Chernsche Klasse) $f = c_1(F) \in U$ zu, dann erhält man einen Isomorphismus $\gamma$ der Gruppe der Isomorphieklassen von komplex-analytischen Geradenbündeln über $G/T$ auf $U$. Mit $(,)$ werde das durch die negative Killing-Form auf $\mathfrak{h}$ und damit auch auf $\mathfrak{h}^*$ induzierte skalare Produkt bezeichnet. $W$ (Weylsche Kammer) sei die Menge aller $x \in H^2(G/T, R)$ mit $(\psi^{-1}x, a_j) \geq 0$ für $1 \leq j \leq m$, und $\mathring{W}$ sei das Innere von $W$, d.h. die Menge aller $x \in W$ mit $(\psi^{-1} x, a_j) > 0$ für $1 \leq j \leq m$. Die Menge der (Äquivalenzklassen von) irreduziblen unitären Darstellungen von $\mathfrak{g}$ = Lie Algebra von $G$ (oder in anderen Worten der universellen Überlagerung von $G$) entspricht eineindeutig der Menge $U \cap W$. Dem Element $f \in U \cap W$ entspricht die Darstellung mit Hauptgewicht $\psi^{-1} f$. Wenn $f \in U \cap W$, dann ist $\gamma^{-1} f$ ein Geradenbündel mit den folgenden Eigenschaften

$$H^j(G/T, \gamma^{-1}f) = 0 \quad \text{für } j > 0 \tag{22}$$

$$\chi(G/T, \gamma^{-1}f) = \dim_{\mathbf{C}} H^0(G/T, \gamma^{-1}f) \tag{22*}$$

(23) $\dim_{\mathbf{C}} H^0(G/T, \gamma^{-1}f)$ = Grad derjenigen irreduziblen Darstellung von $\mathfrak{g}$, welche das Hauptgewicht $\psi^{-1}f$ hat.

(22) ergibt sich daraus, daß $f + c_1(G/T) = f + \psi(a_1 + \cdots + a_m) \in \mathring{W}$ und daß die Elemente von $\mathring{W}$ positiv im Sinne von Kodaira sind (vgl. [2], [10] und [8], Satz 18.2.2). Nach der Definition von $\chi$ folgt dann (22*). Die Formel (23) kann durch direkte Konstruktion der $f$ zugeordneten Darstellung als Darstellung im Raume $H^0(G/T, \gamma^{-1}f)$ bewiesen werden (A. Borel—A. Weil, siehe einen Vortrag von J.-P. Serre im Seminar von N. Bourbaki, Mai 1954; vgl. auch die Untersuchungen von J. Tits, *Sur certaines classes d'espaces homogènes de groupes de Lie*, Acad. Roy. Belg. Cl. Sci. Mém. Coll. 29 (1955), no. 3).

Die Formel (23) kann (unter Benutzung von (22*)) auch durch Berechnung von $\chi(G/T, \gamma^{-1}f)$ nach dem Satz von Riemann-Roch bewiesen werden [4], wobei man die Formel von H. Weyl für den Grad der Darstellung mit Hauptgewicht $\psi^{-1}f$ zu verwenden hat [17].

(22*) und (23) ergeben für $f = 0$, dass das arithmetische Geschlecht $\chi(G/T)$ gleich 1 ist.

Es sei $K$ eine Untergruppe von $G$ mit maximalem Rang ($G \supset K \supset T$). Wir setzen voraus, dass $G/K$ eine homogene komplexe Struktur zuläßt. Dann ist $G/K$ mit dieser komplexen Struktur eine einfach-zusammenhängende algebraische Mannigfaltigkeit, und $K$ ist der Zentralisator eines in $T$ enthaltenen Torus. (Vgl. [2], Goto, Amer. J. Math. 76 (1954), 811–818 und Wang, ibid. 1–32.) Mit $\mathfrak{h}$ werde weiterhin die Liesche Algebra von $T$ bezeichnet. $b_1, \cdots, b_n \in \mathfrak{h}^*$ seien die Gewichte der Isotropiedarstellung von $K$ im komplexen Tangentialraum von $G/K$ im Punkte $e_0 = (K)$. Hier ist $n = \dim_{\mathbf{C}} G/K$. Eine lexikographische Anordnung von $\mathfrak{h}^*$ heisse verträglich mit der komplexen Struktur von $G/K$, wenn die $b_1, \cdots, b_n$ positive Wurzeln von $G$ bezüglich dieser Anordnung sind. Eine derartige verträgliche Anordnung existiert immer [4]. Zu ihr gehört eine homogene komplexe Struktur von $G/T$ und eine von $K/T$. Mit diesen komplexen Strukturen erhält man das komplex-analytische Faserbündel $(G/T, G/K, K/T, p)$. Wir bezeichnen die positiven Wurzeln der verträglichen lexikographischen Anordnung wieder mit $a_1, \cdots, a_m$. Die $b_1, \cdots, b_n$ sind dann genau diejenigen der $a_j$, welche nicht Wurzeln von $K$ sind. Die $b_1, \cdots, b_n$ heißen deshalb komplementäre Wurzeln von $G$ relativ $K$ bezüglich einer mit der komplexen Struktur von $G/K$ verträglichen lexikographischen Anordnung. Wie in [4] gezeigt wird, hat man die folgende Formel für die (totale) Chernsche Klasse von $G/K$ (mit reellen Koeffizienten).

$$p^*c(G/K) = (1 + \psi b_1)(1 + \psi b_2) \cdots (1 + \psi b_n) \tag{24}$$

Wir setzen $b = b_1 + b_2 + \cdots + b_n$. Also ist $\psi b = p^*c_1(G/K)$, und wir erinnern daran, daß $-c_1(G/K)$ die erste Chernsche Klasse des kanonischen Geradenbündels von $G/K$ ist. Wir wollen nun die Zahl $\chi(G/K, -r)$ für $r \geqq 0$ berechnen (siehe §1). Zunächst folgt aus multiplikativen Eigenschaften des arithmetischen (Toddschen) Geschlechtes ([4], vgl. auch [8], Satz 21.2.1) und aus $\chi(K/T) = 1$, dass

$$\chi(G/K, -r) = \chi(G/K, -r) \cdot \chi(K/T) = \chi(G/T, \gamma^{-1}(\psi(rb))).$$

Man rechnet nach [4], daß $b \in W$. Aus (22*) und (23) folgt dann

(25) $\chi(G/K, -r) =$ *Grad der irreduziblen Darstellung von G (bzw. einer Überlagerung von G) mit Hauptgewicht* $rb = r(b_1 + b_2 + \cdots + b_n)$.

## § 5. Automorphe Formen

Es sei $X$ ein beschränktes homogenes symmetrisches Gebiet des $\mathbf{C}^n$. Es sei $\Delta$ eine diskontinuierliche Gruppe von Automorphismen von $X$, die wieder die Eigenschaften 1 und 2 von §3, (11), haben soll, d.h. $X/\Delta$ soll kompakt und $\Delta$ fixpunktfrei sein. Wir wollen jetzt eine Formel für die Anzahl $\Pi_r(X, \Delta)$ der automorphen Formen vom Gewicht $r$ angeben. Wegen (13) beschränken wir uns auf den Fall $r \geqq 2$. Aus Satz 1 und Satz 4 folgt dann

$$\Pi_r(X, \Delta) = \chi(X/\Delta) \cdot \chi(X', r), \qquad (r \geqq 2). \tag{26}$$

Hier ist $X'$ wieder die $X$ zugeordnete algebraische Mannigfaltigkeit. (Vgl. §3). Es ist wohlbekannt, daß $\Pi_r(X, \Delta)$ ein Polynom in $r$ vom Grade $n$ ist. Wir müssen das Polynom $\chi(X', r)$ berechnen, das nur von $X$ und nicht mehr von $\Delta$ abhängt. Für algebraische Mannigfaltigkeiten $Y_1$ und $Y_2$ gilt:

$$\chi(Y_1 \times Y_2, r) = \chi(Y_1, r) \cdot \chi(Y_2, r), \tag{27}$$

wie man etwa aus [8], Satz 12.1.1, entnehmen kann. Da für beschränkte homogene symmetrische Gebiete $X_1$ und $X_2$ die algebraische Mannigfaltigkeit $(X_1 \times X_2)'$ gleich dem cartesischen Produkt von $X_1'$ und $X_2'$ ist, könnten wir uns also wegen (27) bei der Berechnung des Polynoms $\chi(X', r)$ auf den Fall beschränken, wo $X$ irreduzibel ist. Wir werden das vorläufig aber nicht tun.

Nach dem Serreschen Dualitätssatz [13] ([8], S. 120, (14)) ist

$$\chi(X', r) = (-1)^n \chi(X', 1 - r). \tag{28}$$

Die algebraische Mannigfaltigkeit $X' = G'/K$ ist ein Raum, wie er am Schluß von §4 betrachtet wurde. (In §3 wurde $H$ an Stelle von $K$ geschrieben). Für $r \geqq 2$ ist $\chi(X', 1 - r)$ nach (25) gleich dem Grad einer gewissen irreduziblen Darstellung. Wir erhalten so aus (26) und (28) den

SATZ 5. *Es sei $X$ ein beschränktes homogenes symmetrisches Gebiet im $\mathbf{C}^n$. Es sei $X'$ die $X$ zugeordnete homogene algebraische Mannigfaltigkeit* ($X' = G'/K$ mit $G'$ kompakt, halbeinfach und ohne Zentrum). *Es seien $b_k (1 \leqq k \leqq n)$ die komplementären Wurzeln von $G'$ relativ $K$ bezüglich einer mit der komplexen Struktur von $G'/K$ verträglichen lexikographischen Anordnung* (§4), d.h. *die $b_k$ sind die Gewichte der Isotropiedarstellung von $K$ im komplexen Tangentialraum von $G'/K$ im Punkte $e_0 = (K)$. Es sei $b = \sum_{k=1}^n b_k$. Gegeben sei nun eine diskontinuierliche Gruppe $\Delta$ von Automorphismen von $X$, für die $X/\Delta$ kompakt sei und die abgesehen von der Identität nur fixpunktfreie Transformationen enthalte. Dann ist die Anzahl $\Pi_r(X, \Delta)$ der automorphen Formen von Gewicht $r$ für $r \geqq 2$ durch folgende Formel gegeben*

(29) $$\Pi_r(X, \Delta) = (-1)^n \chi(X/\Delta) \cdot \operatorname{grad}(G', (r-1)b), \quad (r \geqq 2).$$

*Hier bezeichnet* grad $(G', (r-1)b)$ *den* Grad *derjenigen irreduziblen Darstellung von $G'$ (oder einer Überlagerung von $G'$), welche das Hauptgewicht $(r-1)b$ hat ("Hauptgewicht" ist im Sinne einer mit der komplexen Struktur von $G'/K$ verträglichen lexikographischen Anordnung gemeint). $\chi(X/\Delta)$ ist das arithmetische Geschlecht der algebraischen Mannigfaltigkeit $X/\Delta$. Es ist also*

$$\chi(X/\Delta) = 1 - g_1 + g_2 - \cdots + (-1)^n g_n,$$

*wo $g_j$ die Anzahl der komplex-linear-unabhängigen äusseren holomorphen Differentialformen vom Grade $j$ ist, welche in $X$ definiert und dort automorph sind bezüglich $\Delta$.*

Es seien jetzt $a_1, \cdots, a_m$ wieder die positiven Wurzeln von $G'$ bezüglich einer mit der komplexen Struktur von $G'/K$ verträglichen lexikographischen Anordnung, und es sei $a$ ihre Summe. $\mathfrak{h}$ sei wieder die Liesche Algebra des maximalen Torus $T$ ($T \subset K \subset G'$) und ( , ) das durch die negative Killingform auf $\mathfrak{h}^*$ induzierte skalare Produkt. Nach der Formel von H. Weyl ([17], Kap. IV, Satz 5) hat man

(30) $$\operatorname{grad}(G', (r-1)b) = \prod_{j=1}^m \left( \left( \frac{a}{2} + (r-1)b, a_j \right) \Big/ \left( \frac{a}{2}, a_j \right) \right)$$

Nun ist $(b, a_j) = 0$. wenn $a_j$ eine Wurzel von $K$ ist (vgl. [4], Chap. IV). Deshalb vereinfacht sich (30) wie folgt

(31) $$\operatorname{grad}(G', (r-1)b) = \prod_{k=1}^n \left( \left( \frac{a}{2} + (r-1)b, b_k \right) \Big/ \left( \frac{a}{2}, b_k \right) \right)$$

(31) ist ein Polynom in $r$ vom Grade $n$. Nach Satz 5 ist es für $r \geqq 2$ gleich $(-1)^n \Pi_r(X, \Delta)/\chi(X/\Delta)$. Nach den Bemerkungen, die im Zusammenhang mit Formel (27) gemacht wurden, genügt es, dieses Polynom für irreduzibles $X$ zu berechnen.

Es sei also $X$ ein irreduzibles beschränktes homogenes symmetrisches Gebiet. $X'$ ist dann eine algebraische Mannigfaltigkeit $G'/K$ von einem der Typen I–VI, die in §2 angegeben wurden. Wir betrachten die von Borel-Siebenthal (Comm. Math.

Helv. 23 (1949), 200–221) studierte explizite Einbettung von $K$ in $G'$ (siehe in der Tabelle loc. cit. S. 219 unter $G_{l-1} \times T$). Wie in [4], Chap. IV, gezeigt wird, gibt es auf $G'/K$ genau zwei homogene komplexe Strukturen, diese sind vermöge eines Automorphismus von $G'$ äquivalent, und man kann bei der Berechnung nach Formel (31) annehmen, dass die $a_j$ die positiven Wurzeln bezüglich einer beliebigen lexikographischen Anordnung sind, für die die Summe zweier komplementärer Wurzeln $b_i$, $b_j$ niemals eine Wurzel von $G'$ ist. Wir erhalten so (etwa unter Verwendung der in [14], S. 218–129, angegebenen lexikographischen Anordnungen und Wurzeln und unter Ausnutzung expliziter Formeln, die sich in [17], [18] finden) für $r \geqq 2$:

$$\text{I. } X' = \mathbf{U}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q)$$

$$\prod\nolimits_r(X, \Delta) = (-1)^{pq} \chi(X/\Delta) \cdot \prod\nolimits_{i,j} \frac{r(p+q)-i-j}{p+q-i-j}, \qquad \begin{cases} 0 \leqq i \leqq p-1 \\ 1 \leqq j \leqq q \end{cases}$$

Für $p = 1$ und $q = n$ ist $X$ die Hyperkugel im $\mathbf{C}^n$, und $X'$ ist der komplexe projektive Raum $\mathbf{P}_n(\mathbf{C})$. In diesem Fall ist

$$\prod\nolimits_r(X, \Delta) = (-1)^n \chi(X/\Delta) \cdot \binom{r(n+1)-1}{n}.$$

$$\text{II. } X' = \mathbf{SO}(2p)/\mathbf{U}(p)$$

$$\prod\nolimits_r(X, \Delta) = (-1)^{\frac{1}{2}p(p-1)} \chi(X/\Delta) \prod\nolimits_{0 \leqq i < j \leqq p-1} \frac{2(r-1)(p-1)+i+j}{i+j}.$$

$$\text{III. } X' = \mathbf{Sp}(p)/\mathbf{U}(p)$$

$$\prod\nolimits_r(X, \Delta) = (-1)^{\frac{1}{2}p(p+1)} \chi(X/\Delta) \prod\nolimits_{1 \leqq i \leqq j \leqq p} \frac{2(r-1)(p+1)+i+j}{i+j}$$

In diesem Fall ist $X$ äquivalent zur Siegelschen oberen Halbebene im $\mathbf{C}^{\frac{1}{2}p(p+1)}$ (siehe [15]).

$$\text{IV. } X' = \mathbf{SO}(p+2)/\mathbf{SO}(p) \times \mathbf{SO}(2)$$

$$\prod\nolimits_r(X, \Delta) = (-1)^p \chi(X/\Delta) \left(\binom{rp-1}{p} + \binom{rp}{p}\right)$$

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# SOME HIGHER ORDER COHOMOLOGY OPERATIONS

By W. S. Massey [1]

## 1. Introduction

Before discussing higher order cohomology operations in general, it is perhaps worth while to briefly review first and second order cohomology operations. First order cohomology (or universally defined cohomology operations) are usually defined as follows: Let $G$ and $G'$ be abelian groups, and let $m$ and $n$ be non-negative integers. A *first order cohomology operation associated with* $(G, G'; m, n)$ is a function $T$ which assigns to each space $X$ (of some suitable category of spaces) a mapping of cohomology groups:

$$T_X : H^m(X, G) \to H^n(X, G').$$

The only requirement imposed on the functions $T_X$ is that they commute with the homomorphisms induced on cohomology groups by continuous mappings of spaces; no algebraic properties are assumed. It is readily proved that such cohomology operations are in 1–1 correspondence with the elements of the Eilenberg-MacLane cohomology group $H^n(G, m; G')$; see Serre, [4]. A great deal is known about first order cohomology operations. We shall only be incidentally concerned with them in this lecture.

Next, we shall give two examples of second order cohomology operations, without trying to define precisely what is meant by a second order cohomology operation. First, there is the operation introduced by J. Adem in [1] to give the homotopy classification of maps of an $(n + 2)$-dimensional complex $K$ into an $n$-sphere. The domain of this operation is the kernel of the Steenrod square, $\mathrm{Sq}^2 : H^n(X, Z) \to H^{n+2}(X, Z_2)$, and the range is the co-kernel of the homomorphism $\mathrm{Sq}^2 : H^{n+1}(X, Z_2) \to H^{n+3}(X, Z_2)$. This operation is natural with respect to homomorphisms induced by continuous maps.

Our second example of a second order cohomology operation is the "triple product" (see [5]). Let $X$ be a topological space, and let $u \in H^p(X)$, $v \in H^q(X)$, $w \in H^r(X)$ be cohomology classes of $X$ with coefficients in an associative ring. If the cup products $w \cdot v$ and $v \cdot w$ are zero, then the triple product $\langle u, v, w \rangle$ is defined and is an element of the factor group.

$$H^{p+q+r-1}(X)/[u \cdot H^{q+r-1}(X) + H^{p+q-1}(X) \cdot v].$$

Since this operation will be important in what follows, we will give the details of

[1] This lecture is a report on research done while the author was partially supported by grant from the National Science Foundation of U.S.A.

the definition. Choose representative cocycles $u'$, $v'$, and $w'$ for $u$, $v$, and $w$ respectively; the conditions $u \cdot v = 0$ and $v \cdot w = 0$ imply the existence of cochains $a$ and $b$ such that

$$u' \cdot v' = \delta(a), \qquad v' \cdot w' = \delta(b).$$

Now form the cochain

$$z' = a \cdot w' - \bar{u}' \cdot b$$

(where, as usual $\bar{u}' = (-1)^p u'$; we will use this notation throughout this paper) an easy computation (using the associative law) shows that $\delta(z') = 0$, i.e. $z'$ is a cocycle. Let $z$ denote the cohomology class of $z'$. By definition, $z$ is a representative of $\langle u, v, w \rangle$. Altering the choices made in the definition, changes the cohomology class $z$, but it will always be an element of the same coset of the sub-group $[u \cdot H^{q+r-1}(X) + H^{p+q-1}(X) \cdot w]$; moreover, by making suitable choices in the course of the construction, we can obtain any cohomology class of this coset. This operation is also natural with respect to homomorphisms induced by continuous maps.[2]

These two examples illustrate the idea of a second order cohomology operation of $m$ variables; it is a collection of functions $T_X$ defined for each topological space $X$; if $u_1, u_2, \cdots, u_m$ are cohomology classes of $X$ (with prescribed degrees and coefficient groups) which satisfy certain conditions expressed by the vanishing of first order operations, then $T_X(u_1, \cdots, u_m)$ is defined, and it is a subset of some cohomology group of $X$. It may or may not be a coset of a subgroup. The only condition imposed on the functions $T_X$ is that they be natural with respect to the homomorphisms of cohomology groups induced by continuous maps.

Although the time is probably not ripe for giving a precise, general definition of an $n^{\text{th}}$ order cohomology operation, it is fairly clear what kind of an object such an operation will be. It is defined on a cohomology class or a set of cohomology classes which satisfy certain conditions expressed by the vanishing of cohomology operations of orders $1, 2, \cdots, n-1$; and it is required to be natural. There is a vague resemblance to the successive differential operators of a spectral sequence. We should also expect that many different sequences of operations of orders from 1 to $n$ will exist, and that these different sequences of operations will be in 1–1 correspondence with certain objects having auxiliary mathematical structure (e.g. Postnikov systems[3] or, in the examples that follow, ordered abstract simplicial complexes).

It seems quite likely that higher order cohomology operations will be of great importance in many problems of algebraic topology in the future. For example, the successive obstructions to the extension of a continuous map can probably be expressed in terms of higher order operations.

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[2] The author wishes to acknowledge that the idea for the definition of the triple product originated in a conversation he had with A. Shapiro at a Topology Conference in Chicago in 1950.

[3] For the use of Postnikov systems in defining higher order cohomology operations, see a paper by F. P. Peterson to appear in *Trans. Amer. Math. Soc.*, Vol. 86, Sept., 1957.

The aim of this lecture is to give some examples of higher order cohomology operations. These examples are defined in a purely algebraic way from the cochain ring of a given space; the definition is of a constructive nature. It is possible to study their algebraic properties, and in certain cases, to make computations. It is hoped that these examples will serve as models for further work in this field.

## 2. Provisional definition of some higher order operations

It is possible to generalize the "triple product" defined above to a "quadruple product."[4] Let $C^* = \sum_{i \geqq 0} C^i$ denote the cochain ring of the space $X$ with coefficients in some associative ring, and let $H^* = \sum_{i \geqq 0} H^i$ denote the cohomology ring of $X$. Suppose that $u_1$, $u_2$, $u_3$, and $u_4$ are cohomology classes on $X$ which satisfy the following conditions:

$$u_1 u_2 = u_2 u_3 = u_3 u_4 = 0, \tag{1}$$

$$\langle u_1, u_2, u_3 \rangle = 0, \qquad \langle u_2, u_3, u_4 \rangle = 0. \tag{2}$$

We may now try to construct a new cohomology class as follows: For each cohomology class $u_i$, choose a representative cocycle $a_i$, $1 \leqq i \leqq 4$. On account of condition (1), we may choose cochains $b_{12}$, $b_{23}$, and $b_{34}$ such that

$$\delta b_{21} = a_1 a_2 , \qquad \delta b_{23} = a_2 a_3 , \qquad \delta b_{34} = a_{34} . \tag{3}$$

Making use of condition (2), we may almost (but not quite) assume that $b_{12}$, $b_{32}$, and $b_{34}$ are chosen so that there exist cochains $c_{123}$ and $c_{234}$ such that

$$\delta c_{123} = b_{12} a_3 - \bar{a}_1 b_{23} , \qquad \delta c_{234} = b_{23} a_4 - \bar{a}_2 b_{34} . \tag{4}$$

Now form the following cochain:

$$z' = c_{123} a_4 - \bar{b}_{12} b_{34} + a_1 c_{234} .$$

An easy computation, using (3) and (4), shows that $\delta(z') = 0$, i.e. $z'$ is a cocycle. Let $z$ denote its cohomology class. By definition, the quadruple product, $\langle u_1, u_2, u_3, u_4 \rangle$, is the set of all possible cohomology classes one can obtain by this construction. If $u_i$ is of degree $p_i$, $1 \leqq i \leqq 4$, then $\langle u_1, u_2, u_3, u_4 \rangle$ is a subset of $H^n$, where $n = p_1 + p_2 + p_3 + p_4 - 2$. In general, it is not a coset of any subgroup.

The above definitions are not quite correct. In general, it is not possible to choose cochains $c_{123}$, and $c_{234}$, so as to satisfy *both* of the conditions listed in (4); by proper choice of the $b_{ij}$'s one can satisfy either condition separately, but not

[4] The quadruple product was developed independently by G. Hirsch for the purpose of computing the cohomology ring of a fibre space. The author has benefited from correspondence with Hirsch and from the privilege of reading several of his unpublished manuscripts. Hirsch's work on this subject will be published in the Proceedings of the Colloquium on Algebraic Topology held at Louvain, Belgium in June, 1956. [*Added in proof*: See the paper entitled "Certaines opérations homologiques et la cohomologie des espaces fibrés," Colloque de Topologie Algébrique Tenu à Louvain les 11, 12, et 13 juin 1956, pp. 167–190.]

both simultaneously. To get around this difficulty, it is necessary to strengthen condition (2) slightly in the following manner: For any four cohomology classes $u_1, \cdots, u_4$ which satisfy conditions (1) above, we define a subset of the direct product $H^{p_1+p_2+p_3-1} \times H^{p_2+p_3+p_4-1}$, denoted by $(\langle u_1, u_2, u_3\rangle, \langle u_2, u_3, u_4\rangle)$, as follows: Choose representative cocycles $a_1, \cdots, a_4$ and cochains $b_{12}$, $b_{23}$, and $b_{34}$ so as to satisfy condition (3). Then

$$y' = b_{12}a_3 - \bar{a}_1 b_{23},$$
$$z' = b_{23}a_4 - \bar{a}_2 b_{34},$$

are cocycles; let $y$ and $z$ denote their cohomology classes. By definition, $(\langle u_1, u_2, u_3\rangle, \langle u_2, u_3, u_4\rangle)$ is the set of all ordered pairs $(y, z)$ one can obtain by this construction. One can show by an easy calculation that $(\langle u_1, u_2, u_3\rangle, \langle u_2, u_3, u_4\rangle)$ is a coset of a certain subgroup of $H^{p_1+p_2+p_3-1} \times H^{p_2+p_3+p_4-1}$.

Using this definition, condition (2) above should be replaced by the following condition:

$$(2') \qquad (0, 0) \in (\langle u_1, u_2, u_3\rangle, \langle u_3, u_3, u_4\rangle).$$

Then one can carry out the construction given above for $\langle u_1, u_2, u_3, u_4\rangle$. The quadruple product as thus defined would be considered as a third order cohomology operation. It is obviously natural with respect to homomorphisms induced by continuous maps, and is a homotopy type invariant of the space $X$.

These definitions can be readily generalized to define an $n$-tuple product, $\langle u_1, u_2, \cdots, u_n\rangle$ which is an $(n-1)^{\text{th}}$ order cohomology operation of $n$ variables. We will now write down the necessary formulas for constructing the quintuple product, $\langle u_1, \cdots, u_5\rangle$, leaving the formulas for the general case to the reader.

Let $u_1, u_2, \cdots, u_5$ be cohomology classes which satisfy the following conditions:

$$(5) \qquad u_1u_2 = u_2u_3 = u_3u_4 = u_4u_5 = 0,$$

$$(6) \qquad (0, 0, 0) \in (\langle u_1, u_2, u_3\rangle, \langle u_2, u_3, u_4\rangle, \langle u_3, u_4, u_5\rangle),$$

$$(7) \qquad (0, 0) \in (\langle u_1, u_2, u_3, u_4\rangle, \langle u_2, u_3, u_4, u_5\rangle).$$

Here the operations used to express conditions (6) and (7) are modifications of the triple product and quadruple product respectively, similar to the modification of the triple product used above to express condition (2′). Under these conditions, the quintuple product $\langle u_1, \cdots, u_5\rangle$ is defined, and it is a subset of the cohomology group $H^n$, where $n = p_1 + \cdots + p_5 - 3$. Representatives of $\langle u_1, \cdots, u_5\rangle$ are constructed according to the following scheme: Let $a_i$ be a representative cocycle for $u_i$, $1 \leqq i \leqq 5$. On account of conditions (5), (6), and (7), there exist cochains $b_{12}, \cdots, b_{45}, c_{123}, \cdots, c_{345}, d_{1234}, d_{2345}$ such that the following conditions hold:

$$(8) \qquad \delta b_{12} = a_1a_2, \cdots, \delta b_{45} = a_4a_5,$$

$$(9) \qquad \delta c_{123} = b_{12}a_3 - \bar{a}_1 b_{23}, \cdots, \delta c_{345} = b_{34}a_5 - \bar{a}_3 b_{45},$$

$$(10) \qquad \delta d_{1234} = c_{123}a_4 - \bar{b}_{12}b_{34} + a_1c_{234},$$
$$\delta d_{2345} = c_{234}a_5 - \bar{b}_{23}b_{45} + a_2c_{345}.$$

Now form the cochain

$$z' = d_{1234}a_5 - \bar{c}_{123}b_{45} + b_{12}c_{345} - \bar{a}_1 d_{2345}.$$

A computation shows that $z'$ is a cocycle; its cohomology class $z$ is a representative of $\langle u_1, \cdots, u_5\rangle$.

It is evident that this procedure leads to formulas of increasing complexity which are not easy to handle. Moreover, the operations thus defined do not have nice algebraic properties. It does not make sense to ask whether or not they are multilinear. Even $\langle u_1, u_2, u_3, u_4\rangle$ need not be a coset of any subgroup.

In order to avoid these difficulties, the author has developed a different approach to these higher order cohomology operations. However, this new approach is much more abstract. The preceding paragraphs may be considered as motivation for this new method.

## 3. Abstract sheaves on a complex

In order to describe this new method we first need to discuss a slight generalization of homology with local coefficients. Although this generalization is undoubtedly well known, to our knowledge it has never been published in the explicit form in which we need it.

Let $K$ be an abstract cell complex as defined, for example, by Lefschetz, [3], Chapter III.

DEFINITION. An *abstract sheaf*[5] $\mathscr{G}$ on $K$ consists of a pair of functions $(G, I)$ such that $G$ assigns to each cell $\sigma$ of $K$ an abelian group $G(\sigma)$, and $I$ assigns to each pair of cells $\sigma$, $\tau$ of $K$ such that $\sigma < \tau$ a homomorphism.

$$I(\sigma, \tau) : G(\tau) \rightarrow G(\sigma).$$

These homomorphisms are required to satisfy the following two conditions:

(a) For any cell $\sigma \in K$, $I(\sigma, \sigma)$ is the identity map $G(\sigma) \rightarrow G(\sigma)$.

(b) For any three cells $\sigma$, $\tau$, and $\rho$ of $K$ such that $\sigma < \tau < \rho$, the following transitivity condition shall hold: $I(\sigma, \tau) \circ I(\tau, \rho) = I(\sigma, \rho)$.

The following example of an abstract sheaf is very important in what follows. Let $K$ be an ordered abstract simplicial complex, and let $R$ be an associative ring. For any $p$-simplex $\sigma$ of $K$, define

$$G(\sigma) = \otimes^{p+2}(R)$$

the tensor product (over the ring of integers) of $(p + 2)$-factors, each equal to $R$. If $\sigma^p$ is a $p$-simplex, and $\sigma_i^{p-1}$ denotes the face obtained by omitting the $i^{\text{th}}$ vertex (when the vertices are arranged in the given order), define $I(\sigma_i^{p-1}, \sigma^p)$ by the formula

$$I(\sigma_i^{p-1}, \sigma^p)(x_0 \otimes x_1 \otimes \cdots \otimes x_{p+1}) = x_0 \otimes \cdots \otimes x_{i-1} \otimes (x_i \cdot x_{i+1}) \otimes x_{i+2} \otimes \cdots \otimes x_{p+1}$$

for any elements $x_0, \cdots, x_{p+1} \in R$ and $0 \leq i \leq p$.

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[5] J.-P. Serre has suggested that it would be more appropriate to call these objects "abstract co-sheafs."

This formula may be extended by linearity to all of $\otimes^{p+2}(R)$, and then by using condition (b) above, the definition of $I(\sigma, \tau)$ is uniquely determined for any pair of cells which are incident.

This definition can be easily remembered by using the following mnemonic device: In the expression $x_0 \otimes \cdots \otimes x_{p+1}$, the symbol "$\otimes$" occurs $(p + 1)$ times. Imagine the occurrences of this symbol numbered from 0 to $p$, reading from left to right. Then corresponding to the omission of the $i^{\text{th}}$ vertex of $\sigma^p$, one omits the $i^{\text{th}}$ occurrence of the symbol "$\otimes$".

The abstract sheaf $\mathscr{G} = (G, I)$ thus obtained on $K$ we will call the *abstract sheaf on $K$ canonically associated with $R$.*

If $K$ is an arbitrary abstract cell complex, and $\mathscr{G} = (G, I)$ is any abstract sheaf on $K$, then we may define homology of $K$ with coefficients in $\mathscr{G}$ by a rather obvious procedure. First, one defines the $q$-dimensional chain group, $C_p(K, \mathscr{G})$ to be the direct sum of the groups $G(\sigma^p)$ for all $p$-cells $\sigma^p$ of $K$. Then one defines a boundary operator $\partial : C_p(K, \mathscr{G}) \to C_{p-1}(K, \mathscr{G})$ by the following formula:

$$\partial(a) = \sum [\tau^{p-1} : \sigma^p] \cdot I(\tau^{p-1}, \sigma^p)(a)$$

for any $a \in G(\sigma^p)$, where the summation is over all $(p - 1)$-cells $\tau^{p-1}$ of $K$, and $[\tau^{p-1} : \sigma^p]$ denotes the incidence number. As usual, one can prove that $\partial \circ \partial = 0$, and hence the homology groups $H_p(K, \mathscr{G})$ can be defined.

## 4. Definition of the new cohomology operations

Let $X$ be a topological space for which it is desired to define the new cohomology operations, and let $\Lambda$ denote an associative ring which is to be used as a coefficient ring for cohomology. Denote by

$$\Gamma^* = \sum_{p \geqq 0} \Gamma^p$$

a suitable *associative*, graded ring of cochains for $X$ with coefficients in $\Lambda$. For example, we could take the singular or Alexander-Spanier cochains of $X$. If $X$ were a simplicial polyhedron, we could use ordinary simplicial cochains. Or, if $X$ were a differentiable manifold and $\Lambda$ were the ring of real numbers, we could take $\Gamma^*$ to be the algebra of exterior differential forms on $X$. We will denote the coboundary operator by

$$\delta : \Gamma^p \to \Gamma^{p+1}$$

as usual. Later on we will wish to assume that $\Gamma^*$ does not have a unit; this may be achieved by choosing a basepoint and taking cochains of $X$ modulo the basepoint.

First of all, one must choose an ordered abstract simplicial complex $K$; for different choices of $K$, different sequences of higher order cohomology operations will be obtained. If convenient, one may choose $K$ to be augmented, i.e. let $K$ contain a unique simplex of dimension $-1$, the empty simplex.

Let $\mathscr{G} = (G, I)$ denote the abstract sheaf on $K$ which is canonically associated with the ring $\Gamma^*$, and let $C_p(K, \mathscr{G})$ be the group of chains of $K$ of degree $p$ with coefficients in the sheaf $\mathscr{G}$. To define $\mathscr{G}$ and $C_p(K, \mathscr{G})$, one only makes use of the ring structure of $\Gamma^*$; the fact that $\Gamma^*$ is graded and has a coboundary operator $\delta$ leads to additional structure on $C_p(K, \mathscr{G})$, as follows: In the first place, the graded structure on $\Gamma^*$ leads to a graded structure on $G(\sigma^p) = \otimes^{p+2}(\Gamma^*)$ according to the usual rules for the graded structure on a tensor product. Then this graded structure on each group $G(\sigma^p)$ gives rise to a graded structure on the direct sum

$$C_p(K, \mathscr{G}) = \sum G(\sigma^p)$$

in a trivial way. We will use the following notation to indicate the graded structure on $C_p(K, \mathscr{G})$:

$$C_p(K, \mathscr{G}) = \sum_{q\geq 0} C_p^q(K, \mathscr{G}).$$

An element of $C_p^q(K, \mathscr{G})$ will be said to be a *co-degree* $q$ and of *degree* $p$. Note that the boundary operator $\partial$ preserves the co-degree, i.e.

$$\partial[C_p^q(K, \mathscr{G})] \subset C_{p-1}^q(K, \mathscr{G}).$$

In the second place, the existence of the coboundary operator $\delta$ on $\Gamma^*$ enables one to define a coboundary operator (which we will also denote by $\delta$) on the group $G(\sigma^p) = \otimes^{p+2}(\Gamma^*)$ by the usual convention for defining the coboundary operator on a tensor product. Then this coboundary operator can be extended in the obvious way to a coboundary operator on the direct sum $C_p(K, \mathscr{G}) = \sum G(\sigma^p)$. This extended coboundary operator we will also denote by $\delta: C_p(K, \mathscr{G}) \to C_p(K, \mathscr{G})$. The following two facts about this extended coboundary operator are of importance:

(a) $\delta$ preserves degrees and increases co-degrees by one unit, i.e.

$$\delta[C_p^q(K, \mathscr{G})] \subset C_q^{q+1}(K, \mathscr{G}).$$

(b) $\delta$ and $\partial$ commute, i.e. $\delta \circ \partial = \partial \circ \delta$.

The verification of these two facts is routine.

If we now let $C(K, \mathscr{G})$ denote the direct sum, $\sum_p C_p(K, \mathscr{G})$, it is clear that the ordered triple $\{C(K, \mathscr{G}); \partial, \delta\}$ is a bi-complex (see, for example, Cartan and Eilenberg, [2], Chap. XV). As is usual when one has a bi-complex, one can define two different filtrations on $C(K, \mathscr{G})$, and these filtrations lead to two different spectral sequences. We will only be interested in the increasing filtration on $C(K, \mathscr{G})$ which is defined by means of the degree:

$$F_p[C(K, \mathscr{G})] = \sum_{i\leq p} C_i(K, \mathscr{G}).$$

The resulting spectral sequence of bigraded groups with differential operators will be denoted by $\{_iE(X, K, \Lambda), d_i\}$, $i = 1, 2, 3, \cdots$.

The successive differential operators, $d_1$, $d_2$, $d_3 \cdots$, are the new cohomology invariants which it was our purpose to define.

### 5. Identification of the first term of the spectral sequence

For the sake of simplicity, assume that $\Lambda$ is a field. Then we may identify $H^*(\otimes^n\Gamma^*)$ with $\otimes^n[H^*(\Gamma^*)]$ for any integer $n > 0$; this is a special case of the well-known fact that with a field for coefficients, the operations of taking tensor products and of taking derived groups commute with each other. Under this assumption, we assert that

$$_1E(X, K, \Lambda) = \sum_p C_p(K, \mathscr{G}'),$$

where $\mathscr{G}' = (G', I)$ is the abstract sheaf on $K$ canonically associated with the ring $H^*(\Gamma^*) = H^*(X, \Lambda)$; furthermore, the differential operator $d_1$ may be identified with the boundary operator $\partial : C_p(K, \mathscr{G}') \to C_{p-1}(K, \mathscr{G}')$. These statements follow directly from well-known facts about the term $({}_1E, d_1)$ of the spectral sequence of a differential filtered group, in case the filtration is defined by one of the degrees in a bi-complex (see Cartan and Eilenberg, [2], Chap. XV).

## 6. Some examples

As a first example, consider the case where $K$ consists of an ordered, augmented 1-simplex and all its faces. Then $_1E(X, K, \Lambda)$ has terms of degrees $+1$, 0, and $-1$ only, as follows:

degree $+1$: $H^*(X) \otimes H^*(X) \otimes H^*(X)$

degree 0: $[H^*(X) \otimes H^*(X)] + [H^*(X) \otimes H^*(X)]$ (direct sum)

degree $-1$: $H^*(X)$.

The differential operator $d_1$ decreases the degree by 1 unit. It may be described on generating elements of these groups as follows:

$$d_1(x_0 \otimes x_1 \otimes x_2) = [(x_0 \cdot x_1) \otimes x_2, \quad -x_0 \otimes (x_1 \cdot x_2)].$$

$$d_1(y_0 \otimes y_1, z_0 \otimes z_1) = y_0 \cdot y_1 + z_0 \cdot z_1.$$

It may be shown that if $x_0, x_1, x_2 \in H^*(X)$ are cohomology classes such that

$$d_1(x_0 \otimes x_1 \otimes x_2) = 0,$$

i.e. $x_0x_1 = 0$ and $x_1x_2 = 0$, then

$$d_2(x_0 \otimes x_1 \otimes x_2) \equiv \langle x_0, x_1, x_2 \rangle$$

modulo a certain subgroup of $H^*(X)$. It is at this point that one wants to assume that the ring $H^*(X)$ does *not* have a unit (as mentioned above). If $H^*(X)$ has a unit, then $d_2$ is identically zero.

This example may be generalized by taking $K$ to be the complex consisting of an ordered, augmented $n$-simplex and all its faces. Then there are terms of all degrees from $-1$ to $n$ inclusive in $_1E(X, K, \Lambda)$; the term of degree $n$ is $\otimes^{n+2}H^*(X, \Lambda)$, and the term of degree $-1$ is $H^*(X, \Lambda)$. Suppose that $x_0 \otimes x_1 \otimes \cdots \otimes x_{n+1}$ is a decomposable element of $\otimes^{n+2}H^*(X, \Lambda)$ such that the $(n + 2)$-tupe product $\langle x_0, x_1, \cdots, x_{n+1} \rangle$ is defined. Then

$$d_i(x_0 \otimes x_1 \otimes \cdots \otimes x_{n+1}) = 0$$

for $i = 1, 2, \cdots, n$, and $d_{n+1}(x_0 \otimes \cdots \otimes x_{n+1})$ is also defined, and $\langle x_0, \cdots, x_{n+1} \rangle$ is a subset of $d_{n+1}(x_0 \otimes \cdots \otimes x_{n+1})$, where the latter is considered as a coset of some sub-group of $H^*(X)$.

This makes clear the sense in which the spectral sequence $\{{}_iE(X, K, \Lambda), d_i\}$ generalizes the multiple products described in § 2. Naturally, in all these examples one must assume that $H^*(X)$ does not have a unit in order to avoid trivialities.

## 7. General properties of the spectral sequence

In this section we will briefly list some properties of the spectral sequence ${}_iE(X, K, \Lambda)$, $i = 1, 2, 3, \cdots$. Most of them are rather obvious.

(1) For fixed $K$ and $\Lambda$, the operation of assigning to each space $X$ the spectral sequence $\{{}_iE(X, K, \Lambda), d_i\}$ has all the usual naturality properties, e.g. a continuous map of $Y$ into $X$ induces a homomorphism of the spectral sequence ${}_iE(X, K, \Lambda)$ into the spectral sequence ${}_iE(Y, K, \Lambda)$, etc. From this fact, it follows readily that this spectral sequence is an invariant of the homotopy type of a space.

A similar argument shows that the definition of the spectral sequence ${}_iE(X, K, \Lambda)$ does not depend on the choice of the cochain ring $\Gamma^*$; for example, if the space $X$ is a simplicial polyhedron, we may use either the usual simplicial cochain ring of $X$, or the ring of singular cochains of $X$, and the end result will be the same. If $X$ is a differentiable manifold and $\Lambda$ is the field of real numbers, then we may use either the Alexander-Spanier cochains of $X$ or the algebra of exterior differential forms on $X$, and the result will again be the same.

(2) For different choices of $K$, one obtains different spectral sequences for any given space $X$. It is not clear whether or not all these different spectral sequences are equally important. It is even conceivable that a certain special collection of complexes $K$ may be "universal" in the sense that knowledge of the spectral sequences $\{{}_iE(X, K, \Lambda)\}$ for $K$ in the special collection determines $\{{}_iE(X, K, \Lambda)\}$ for any complex $K$.

(3) If $K$ is $n$-dimensional, then the differential operator $d_i = 0$ for $i > n + 1$, and

$${}_{n+2}E(X, K, \Lambda) = {}_{\infty}E(X, K, \Lambda).$$

## 8. Examples of spaces $X$ such that the spectral sequence $\{{}_iE(X, K, \Lambda)\}$ is non-trivial

The following example shows that given any integer $n$ (no matter how large) it is possible to choose a space $X$ and a complex $K$ such that $d_n \neq 0$ in the spectral sequence $\{{}_iE(X, K, \Lambda), d_i\}$.

Let $k$ be an integer greater than 1 and let

$$Y = S_1 \vee S_2 \vee \cdots \vee S_k$$

be the union of $k$ spheres with a single point in common. Let $p_i > 0$ be the dimension of the sphere $S_i$, $1 \leqq i \leqq k$, and let $\alpha_j$ be the element of the homotopy group $\pi_{p_j}(Y)$ represented by the inclusion map $S_j \to Y$ for $1 \leqq j \leqq k$. Form a space $X$ by attaching a cell $e$ to $Y$ by a map representing the multiple Whitehead product $[\alpha_1, [\alpha_2, [\cdots [\alpha_{k-1}, \alpha_k] \cdots]$; the cell $e$ is assumed to be of dimension $2 - k + \sum_i p_i$.

Let $K$ be a complex consisting of an ordered, augmented simplex of dimension $k - 2$ and all its faces. Choose $\Lambda$ to be any commutative ring with a unit, e.g. the ring of integers. Let $\Gamma^*$ denote the ring of singular cochains of $X$ modulo a point with coefficients in $\Lambda$.

Under these conditions one may prove (by an induction on $k$) that in the spectral sequence $\{_iE(X, K, \Lambda), d_i\}$, the differential operators $d_i = 0$ for $1 \leqq i \leqq k - 2$, while $d_{k-1} \neq 0$. The proof is too long and involved to be given here.

Note particularly the case where all the spheres $S_i$ are of dimension 1; then $X$ is a 2-dimensional cell complex. Hence one may construct examples of spaces which are highly non-trivial with respect to these operations, yet are only 2-dimensional.

Using a procedure due to R. Thom, one may imbed the space $X$ in a compact orientable manifold $M$ such that $X$ is a retract of $M$. It follows from naturality that in the spectral sequence $\{_iE(M, K, \Lambda), d_i\}$ we have

$$d_{k-1} \neq 0.$$

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## THE GENERALIZED PONTRJAGIN COHOMOLOGY OPERATIONS

BY EMERY THOMAS

### 1. The Pontrjagin $p^{\text{th}}$ powers

The cohomology operations described in this note are generalizations of the Pontrjagin square cohomology operation [5, 8]. They are defined using the method developed by N. E. Steenrod [6]. Besides describing the properties of the new operations, an example is given of information obtained using these operations which is not given by present cohomology invariants.

The Pontrjagin square cohomology operation is the function

$$\mathfrak{P}_2 : H^{2n}(K; Z_{2m}) \to H^{4n}(K; Z_{4m}),$$

where $H^r(K; G)$ denotes the $r^{\text{th}}$ cohomology group of a complex $K$ with coefficients in a group $G$, and $Z_s$ denotes the integers mod $s$. The Pontrjagin square has the following properties: for $u \in H^{2n}(K; Z_{2m})$,

$$\mathfrak{P}_2 f^*(u) = f^* \mathfrak{P}_2(u), \tag{1.1}$$

$$\eta_{2*}\mathfrak{P}_2(u) = u^2, \qquad \text{(2-fold cup-product)} \tag{1.2}$$

where $f^*$ is induced by a map $f : L \to K$, and $\eta_{2*}$ is induced by the natural homomorphism $\eta_2 : Z_{4m} \to Z_{2m}$.

In its simplest form the generalization of the Pontrjagin square is a function $\mathfrak{P}_p$, defined for each prime number $p$, such that

$$\mathfrak{P}_p : H^{2n}(K; Z_{pm}) \to H^{2pn}(K; Z_{p^2m}).$$

The functions $\mathfrak{P}_p$ have the following properties [7]: for $u \in H^{2n}(K; Z_{pm})$ and $v \in H^{2q}(K; Z_{pm})$,

$$\mathfrak{P}_p f^*(u) = f^* \mathfrak{P}_p(u), \tag{1.3}$$

$$\eta_{p*}\mathfrak{P}_p(u) = u^p, \qquad (p\text{-fold cup-product}) \tag{1.4}$$

$$\mathfrak{P}_p(u \smile v) = \mathfrak{P}_p(u) \smile \mathfrak{P}_p(v), \qquad (p \text{ odd}) \tag{1.5}$$

where $\eta_{p*}$ is induced by the natural homomorphism $Z_{p^2m} \to Z_{pm}$. Property (1.5) does not hold for $p = 2$; in this case there are two additional terms, each of order two [4]. Property (1.3) expresses the topological invariance of the functions $\mathfrak{P}_p$. That is, the functions are *cohomology operations*, which we term the *Pontrjagin* $p^{\text{th}}$ *powers*. It is clear from property (1.4) that these operations are non-trivial, since the $p$-fold cup-product is non-trivial. A simple example to illustrate this is the following.

Denote by $M_\infty$ the infinite complex projective space. Then, $H^2(M_\infty; Z)$ is cyclic infinite ($Z$ = integers). If $u$ is a generator for this group, the cohomology ring $H^*(M_\infty; Z)$ is a polynomial ring in the generator $u$. Set

$$u_r = \text{image } u \text{ in } H^2(M_\infty; Z_{p^r}) \ (r \geqq 1).$$

Then,

$$\mathfrak{P}_p(u_r) \quad \text{generates} \quad H^{2p}(M_\infty; Z_{p^{r+1}}), \tag{1.6}$$

a cyclic group of order $p^{r+1}$.

Our real interest, of course, lies in the question of whether these operations give any new information unobtainable by previous cohomology invariants. To show that they do, I construct an example of two complexes $K$ and $K'$ which known cohomology invariants fail to distinguish as to homotopy type, but which the operation $\mathfrak{P}_p$ does distinguish. The example is a mild generalization of an analogous result given by J. H. C. Whitehead [8] for the Pontrjagin square. For simplicity only the case $p = 3$ is given here.

Let $M$ be the complex projective plane. We can regard $M$ as a CW-complex,

$$M = S^2 \cup e^4,$$

where the 4-cell $e^4$ is attached to the 2-sphere $S^2$ by the Hopf map $S^3 \to S^2$. Now $S^5$ is a fibre bundle over $M$ with fibre $S^1$. Hence, $\pi_r(M) \approx \pi_r(S^5)$ $(r \geqq 1)$. Let $b$ denote the projection $S^5 \to M$. Set $\beta$ = homotopy class of $b$. Then, $\beta$ generates $\pi_5(M) \approx Z$.

Using the complex $M$ and the class $\beta$ we construct new complexes $M(r)$ $(r = 0, 1, \cdots)$ as follows: define

$$M(r) = M \underset{f_r}{\cup} e^6,$$

where the cell $e^6$ is attached to $M$ by a map $f_r$ of $S^5$ to $M$ such that $f_r \in r\beta$. If $r = 1$ and we choose $f_1 = b$, then $M(1)$ is simply the 3-dimensional complex projective space.

The complexes $K$ and $K'$ are now defined by choosing two specific values of $r$: namely, $r = 3$ and $r = 0$. Notice that the 2-skeleton of $M(r)$ is simply a 2-sphere, for all $r$. To this 2-sphere we attach a three cell, $e^3$, by a map of degree 3. Then,

$$K = M(3) \cup e^3, \quad K' = M(O) \cup e^3.$$

Any cohomology invariant which is to distinguish $K$ and $K'$ must have its value in the six-dimensional cohomology groups, since the five-skeletons of $K$ and $K'$ are identical. Consider the cohomology invariants going from dimension 2 to dimension 6. These are:

(i) the cup-product cube, mapping $H^2(X; Z_3)$ to $H^6(X; Z_3)$,

(ii) the operation $\mathfrak{P}_3$, mapping $H^2(X; Z_3)$ to $H^6(X; Z_9)$,

where $X = K$ and $K'$. Now,

$$H^2(K; Z) = 0 = H^2(K'; Z),$$
$$H^2(K; Z_3) \approx Z_3 \approx H^2(K'; Z_3),$$
$$H^6(K; Z) \approx Z \approx H^6(K'; Z).$$

Let $u$ and $u'$ generate respectively the groups $H^2(X; Z_3)$ and $v$, $v'$ generate respectively the groups $H^6(X; Z)$ $(X = K, K')$. Then,

$$u^3 = 3v \pmod 3 \equiv 0, \quad u'^3 = 0v' \pmod 3 \equiv 0;$$

$$\mathfrak{P}_3(u) = 3v \pmod 9 \not\equiv 0, \quad \mathfrak{P}_3(u') = 0v' \pmod 9 \equiv 0.$$

Hence, even though the cup-product rings of $K$ and $K'$ are isomorphic, the operation $\mathfrak{P}_3$ is different and so the complexes are not of the same homotopy type. Similarly, one can show that the cohomology invariants from dimensions 3 and 4 to dimension 6 are all zero.[1]

## 2. Extension of the Operations $\mathfrak{P}_p$

In a later paper J. H. C. Whitehead extended the definition of the Pontrjagin square to take coefficients in an arbitrary abelian group [9]. Independently, Eilenberg and MacLane also obtained such a generalization [2]. By using an algebraic device defined by Eilenberg and MacLane [3], the operations $\mathfrak{P}_p$ can also be extended.

Let $\Pi$ be an abelian group. Define a commutative, graded ring $\Gamma(\Pi)$ as follows (see [3; §18]): $\Gamma(\Pi)$ has as generators the elements $\gamma_t(x)$ for each non-negative integer $t$ and element $x \in \Pi$. These generators have the following relations:

(2.1) $$\gamma_r(x)\,\gamma_s(x) = (r, s)\,\gamma_{r+s}(x),$$

(2.2) $$\gamma_t(x + y) = \textstyle\sum_{r+s=t} \gamma_r(x)\,\gamma_s(y),$$

(2.3) $$\gamma_0(x) = 1,$$

where $(r, s)$ denotes the binomial coefficient $(r + s)!/(r!\,s!)$. Assign degree $2t$ to the generator $\gamma_t(x)$, and let

$$\Gamma_t(\Pi) = \text{subgroup of } \Gamma(\Pi) \text{ of elements of degree } 2t.$$

Using the ring $\Gamma(\Pi)$ as a ring of coefficients and restricting the group $\Pi$ to be finitely generated, we generalize the operations $\mathfrak{P}_p$ as follows: a sequence of operations $\mathfrak{P} = [\mathfrak{P}_t]_{t=0}^{\infty}$ is defined such that

$$\mathfrak{P}_t : H^{2n}(K; \Pi) \to H^{2tn}(K; \Gamma_t(\Pi)).$$

Define a cup-product, written $\smile'$, relative to the pairing given by multiplication in the ring $\Gamma(\Pi)$. Using this cup-product the above operations satisfy the following relations: for $u, v \in H^{2n}(K; \Pi)$,

(2.4) $$\mathfrak{P}_r(u) \smile' \mathfrak{P}_s(u) = (r, s)\mathfrak{P}_{r+s}(u),$$

(2.5) $$\mathfrak{P}_t(u + v) = \textstyle\sum_{r+s=t} \mathfrak{P}_r(u) \smile' \mathfrak{P}_s(v),$$

(2.6) $$\mathfrak{P}_0(u) = 1; \quad \mathfrak{P}_1(u) = u,$$

(2.7) $$\mathfrak{P}_t f^*(u) = f^*\,\mathfrak{P}_t(u).$$

where 1 is the unit of the cohomology ring of $K$ with integer coefficients.

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[1] The example is easily extended to give a similar result for any prime $p$. We now take $M$ to be the complex projective space of $(p - 1)$ complex dimensions. $M(r)$ is defined in the same way; and we choose the values of $r$ to be $r = p!$ and $r = 0$. The coefficient $p!$ is needed to kill off the images of cyclic reduced powers using primes less than $p$.

However, comparing properties (2.1)–(2.3) with (2.4)–(2.6) we see that the relations satisfied by $\mathfrak{P}$ are precisely those satisfied by the generators of $\Gamma(\Pi)$. Hence, we can define a ring homomorphism

$$\mathfrak{P}^* : \Gamma(H^{2n}(K; \Pi)) \rightarrow H^*(K; \Gamma(\Pi)), \tag{2.8}$$

by

$$\mathfrak{P}^* \gamma_t(u) = i_* \mathfrak{P}_t(u),$$

where $H^*(K; \Gamma(\Pi)) = \sum H^r(K; \Gamma(\Pi))$, and $i_*$ is induced by the inclusion homomorphism of $\Gamma_t(\Pi)$ into $\Gamma(\Pi)$.

If the group $\Pi$ is $Z_p$, then the group $\Gamma_p(Z_p)$ is isomorphic with $Z_{p^2}$. In this case the functions $\mathfrak{P}_p$ in the sequence $\mathfrak{P}$ coincides with the Pontrjagin $p^{\text{th}}$ power defined in §1. Also, $\mathfrak{P}_2$ coincides with the operation defined by Whitehead and by Eilenberg and MacLane [9], [2].

The ring $\Gamma(\Pi)$ is an example of a ring with *divided powers*, a concept introduced recently by H. Cartan (see [1; chapter 7]). The relations (2.4)–(2.6) are some of the properties possessed by divided power functions. Eilenberg and Cartan have suggested the possibility of using the functions $\mathfrak{P}$ to define divided powers in the cohomology ring of a space with coefficients in a ring with divided powers. It seems likely that this can be done: if so, it would do much to show the underlying nature of the functions $\mathfrak{P}_t$.

Applications of the operations $\mathfrak{P}_t$ should lie along two lines: first, using these operations to label specific elements of the cohomology groups of the $K(\Pi, n)$ spaces; and secondly, generalizing applications of the Pontrjagin square. Such applications include the calculation of certain obstructions, and an expression for the first Pontrjagin characteristic class mod 4. Thus, applications of the operations should initially be sought in these directions.

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# FUNCTIONAL HIGHER ORDER COHOMOLOGY OPERATIONS

BY FRANKLIN P. PETERSON[1,2]

## 1. Introduction

One problem we would like to investigate is the following: if $f: X \to Y$ and $g: Y \to Z$ are non-trivial (in the sense of homotopy), when is the composition $gf: X \to Z$ non-trivial? To be more specific, suppose $g$ is shown to be non-trivial by a functional primary cohomology operation (in the sense of Steenrod [13]) taking $z \in H^*(Z)$ into a non-zero element $y \in H^*(Y)$, and suppose $f$ is similarly shown to be non-trivial by a functional primary cohomology operation taking $y$ into a non-zero element $x \in H^*(X)$. We may ask, when is the operation $z$ into $x$ a homotopy invariant of $gf$? If the answer is "yes," then $gf$ is non-trivial. In this paper, we derive sufficient conditions for an answer of yes.

We first sketch a method of deriving an exact couple containing higher order cohomology operations as higher derivations from a given Postnikov system. We then give two equivalent algebraic constructions for defining functional operations corresponding to such exact couples. The first method generalizes the method introduced by Steenrod for defining functional primary cohomology operations and the second is introduced in order to prove the composition theorem referred to above. For applications and complete details, the reader is referred to [10].

## 2. The exact couple coming from a Postnikov system

We shall use the following notation. Let $\pi(X; Y)$ denote the set of homotopy classes of maps $\phi: (X, x_0) \to (Y, y_0)$, where $x_0 \in X$, $y_0 \in Y$, and $Y$ is an arcwise connected space. Let $S^rX$ denote the $r^{\text{th}}$ reduced suspension of $X$ [12; p. 656]; let ${}^rX$ denote the $r^{\text{th}}$ space of loops on $X$.

A sequence

$$\cdots \to Y_r \xrightarrow{\phi_r} Y_{r-1} \xrightarrow{\phi_{r-1}} Y_{r-2} \to \cdots$$

is called an *exact sequence of spaces* if the induced sequence

$$\cdots \to \pi(X; Y_r) \xrightarrow{\phi_{r\#}} \pi(X; Y_{r-1}) \to \cdots$$

is exact (kernels are well-defined as $\pi(X; Y)$ has a distinguished element represented by the constant map at $y_0$).

Let $p: E \to B$ be a fibre space with fibre $F$. There is a map $i: {}^1B \to F$ such that the following lemma holds.

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[1] The author was a postdoctoral National Science Foundation fellow during the preparation of this paper.

[2] This paper summarizes the results of [10].

LEMMA 2.1. $\cdots \to {}^rF \xrightarrow{{}^rj} {}^rE \xrightarrow{{}^rp} {}^rB \xrightarrow{{}^{r-1}i} {}^{r-1}F \xrightarrow{{}^{r-1}j} {}^{r-1}E \to \cdots \to E \to B$ *is an exact sequence of spaces.*

Let $\mathfrak{P} = (K(\pi, n), \theta; X, \theta'; \cdots; X^{(m-1)}, \theta^{(m)}; \cdots)$ be a Postnikov system ([4] and [11]). This means that $\theta \in H^q(K(\pi, n); G), \cdots, \theta^{(m)} \in H^z(X^{(m-1)}; G^{(m)})$, where $z = q^{(m)}$ and $K(\pi, n)$ denotes an Eilenberg-MacLane space of type $\pi$ and $n$, that $\rho^{(m)} : X^{(m)} \to X^{(m-1)}$ is a fibre map with fibre $K(G^{(m)}, q^{(m)} - 1)$ and $k$-invariant $\theta^{(m)}$, and that $q^{(m-1)} < q^{(m)}$. Applying Lemma 2.1 to each of the fibre spaces $\rho^{(m)} : X^{(m)} \to X^{(m-1)}$ we obtain an exact couple of spaces as in Figure 1.

$$\begin{array}{ccccc}
\vdots & & & & \vdots \\
\downarrow & & & & \downarrow \\
\cdots \to {}^1X' & \xrightarrow{{}^1\theta''} & K(G'', q''-1) & \xrightarrow{\omega''} & X'' \to \cdots \\
\downarrow {}^1\rho' & & & & \downarrow \rho'' \\
\cdots \to {}^1X' & \xrightarrow{{}^1\theta'} & K(G', q'-1) & \xrightarrow{\omega'} & X' \xrightarrow{\theta''} K(G'', q'') \\
\downarrow {}^1\rho & & & & \downarrow \rho' \\
\cdots \to K(\pi, n-1) & \xrightarrow{{}^1\theta} & K(G, q-1) & \xrightarrow{\omega} & X \xrightarrow{\theta'} K(G', q') \\
\downarrow & & & & \downarrow \rho \\
\cdots \to \text{pt.} & \to & K(\pi, n) & \to & K(\pi, n) \xrightarrow{\theta} K(G, q)
\end{array}$$

Fig. 1.

Applying the functor $X \to \pi(K; X)$ to each space in this exact couple of spaces, we obtain an exact couple $(A, C)$, where $f$ is induced by $\rho^{(m)}$, $g$ by $\theta^{(m)}$, and $h$ by $\omega^{(m)}$. In order to insure that the $A$ terms are groups, we must make the assumption that the Postnikov system $\mathfrak{P}$ represents a space of loops; we assume this throughout the rest of the paper.

Notice that $d = gh$: $H^n(K; \pi) \to H^q(K; G)$ is a homomorphism defined for all spaces $K$ which commutes with induced homomorphisms; i.e., $d$ is a primary cohomology operation. We define $d_m : C_m \to C_m$, where $(A_m, C_m)$ denotes the $m^{\text{th}}$ derived exact couple, as the $(m + 1)$-ary cohomology operation coming from the Postnikov system $\mathfrak{P}$. This definition is motivated by known examples of secondary operations [1; p. 723], by the fact that it generalizes the notion of a primary cohomology operation, by the fact that operations coming from the cohomotopy exact couple [8] are higher order cohomology operations in this sense, and by the heuristic feeling that an $m$-ary cohomology operation should be defined on the kernel of an $(m - 1)$-ary cohomology operation with values in the cokernel of an $(m - 1)$-ary cohomology operation [6; p. 343].

## 3. Functional cohomology operations: generalization of the Steenrod method

Let us first recall the method of Steenrod for defining functional primary cohomology operations [13]. In doing so, we point out an equivalent method of

Then $f'''k(a) = kf''(a) = kh''(u) = h'''k(u) = 0$. Hence we may define $d_{1,k}(\{u\}) = (h''')^{-1}k(a) \in \bar{C}'''$. It is easily checked that $d_{1,k}$ is well-defined.

In general, if $d_{m-1,k} = 0, \cdots, d_{1,k} = 0$, we define for $m > 1$

$$d_{m,k} : C''_m \to \bar{C}'''$$

as follows: Let $a \in A''$ be a lifting of $u$ $m$-stages. Then $f'''k(a) = kf''(a) = h'''d_{m-1,k}(\{u\}) = 0$. Hence we may define $d_{m,k}(\{u\}) = (h''')^{-1}k(a) \in \bar{C}'''$.

We apply our algebraic formalism to the same situation as in §3. Let $(A'', C'')$ be the exact couple for $K$, $(A''', C''')$ be the exact couple for $L$, and let $k = a^*$. $d_{m,k}$ is our second definition of the functional $m$-ary cohomology operation corresponding to the given Postnikov system $\mathfrak{P}$.

The equivalence $\Phi$ of §3 generalizes to an equivalence between $\Delta_m$ and $d_{m,k}$; it is not one to one in the algebraic formalism, but is in the application due to a special assumption.

Our first definition may also be applied to the higher order cohomology operations constructed by Massey [7] in order to define the corresponding functional operations. Unfortunately, the equivalence theorem doesn't hold in that case.

We now state a theorem which shows that functional $m$-ary cohomology operations are sufficient to decide whether or not a given map $a : L \to K$ is homotopic to a constant when $L$ is a finite dimensional CW-complex, and $K$ is a space of loops (this is a generalization of the usual notion of "stable range"). Let $B$ be the set of higher order cohomology operations coming from the exact couple corresponding to the Postnikov system for $K$.

THEOREM 4.1. *$a$ is homotopic to a constant if and only if all functional cohomology operations coming from $a$ and operations in $B$ are zero.*

## 5. The composition theorem

We now state our composition theorem; the proof is quite easy using the second definition. This theorem shows that under certain conditions, the composition of a functional $m$-ary cohomology operation and a functional $n$-ary cohomology operation is a functional $(m + n)$-ary cohomology operation.

Let $a : L \to K$ and $b : M \to L$. Then we have functional cohomology operations defined by $a^*$, denoted by $d_{m,a}$, and defined by $b^*$, denoted by $d_{m,b}$ (both coming from the exact couple corresponding to a given Postnikov system $\mathfrak{P}$).

THEOREM 5.1. *When both sides are defined, $d_{m+m,ab} = d_{n,b}\, d_{m,a}$.*

If both $a$ and $b$ are non-trivial, then is the composition $ab$ non-trivial? In order to apply Theorem 5.1 to this problem in a particular example, one must construct a Postnikov system which will contain $d_{n,b}$ and $d_{m,a}$ at the appropriate levels. Examples of the computations necessary to apply Theorem 5.1 appear in [10]; Adem [2] has a different approach which may be viewed as the case $m = n = 1$ of Theorem 5.1.

## 6. Duality

Hilton [5] has indicated that there exists a certain duality in homotopy theory. Motivated by this idea, we are led to the dual of a Postnikov system, where the definition; our algebraic constructions generalize these two methods to functional higher order cohomology operations.

Let $a : L \to K$. Let $M = K \cup_a CL$, where $CL$ denotes the cone on $L$, attached to $K$ by $a$. Let $b : K \to M$ be the inclusion, and $c : M \to SL$ be defined by collapsing $K$ to a point. Then the sequence

$$L \xrightarrow{a} K \xrightarrow{b} M \xrightarrow{c} SL \xrightarrow{Sa} SK \to \cdots \to S^rK \xrightarrow{S^rb} S^rM \to \cdots$$

is a *coexact sequence of spaces* in the sense that the induced sequence

$$\cdots \to \pi(S^rM;\, Y) \xrightarrow{S^rb^\#} \pi(S^rK;\, Y) \to \cdots \to \pi(K;\, Y) \xrightarrow{a^\#} \pi(L;\, Y)$$

is exact. Let $\theta : K(\pi, n) \to K(G, q)$ be a given additive cohomology operation. Then we have a commutative diagram with exact rows and columns (where $X$ is the space constructed in §2).

$$\begin{array}{ccccccccc}
 & & & & & & H^{n-1}(K;\pi) & \xrightarrow{a^*} & H^{n-1}(L;\pi) \\
 & & & & & & \downarrow {}^1\theta & & \downarrow {}^1\theta \\
 & & & & & & H^{q-1}(K;G) & \xrightarrow{a^*} & H^{q-1}(L;G) \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & \pi(K;X) & \xrightarrow{a^\#} & \pi(L;X) \\
 & & & & & & \downarrow & & \downarrow \\
H^{n-1}(K;\pi) & \xrightarrow{a^*} & H^{n-1}(L;\pi) & \xrightarrow{c^*} & H^n(M;\pi) & \xrightarrow{b^*} & H^n(K;\pi) & \xrightarrow{a^*} & H^n(L;\pi) \\
\downarrow {}^1\theta & & \downarrow {}^1\theta & & \downarrow \theta & & \downarrow \theta & & \downarrow \theta \\
H^{q-1}(K;G) & \xrightarrow{a^*} & H^{q-1}(L;G) & \xrightarrow{c^*} & H^q(M;G) & \xrightarrow{b^*} & H^q(K;G) & \xrightarrow{a^*} & H^q(L;G)
\end{array}$$

Fig. 2.

Let $u \in H^n(K;\pi)$ be such that $a^*(u) = 0$ and $\theta(u) = 0$. Define

$$\theta_a(u) \in H^{q-1}\,(L;\, G)/(\operatorname{Im} a^* + \operatorname{Im} {}^1\theta)$$

by

$$\theta_a(u) = (c^*)^{-1}\theta(b^*)^{-1}(u).$$

This is essentially the definition of Steenrod. By going up and to the right instead of down and to the left, we may define $\bar{\theta}_a(u) \in H^{q-1}(L;\, G)/(\operatorname{Im} a^* + \operatorname{Im} {}^1\theta)$ by

$$\bar{\theta}_a(u) = \omega^{-1}a^\#\rho^{-1}(u).$$

These two definitions are equivalent in the sense that there is an automorphism $\Phi$ of a subgroup of $H^{q-1}(L;\, G)/(\operatorname{Im} a^* + \operatorname{Im} {}^1\theta)$ such that $\Phi\theta_a(u) = \bar{\theta}_a(u)$; $\Phi$ is given by

$$\Phi = \omega^{-1}a^\#\rho^{-1}b^*\theta^{-1}c^*.$$

In §4 we will generalize this second definition to functional higher order cohomology operations.

We now give the algebraic formalism for our generalization of the Steenrod method. Let $(A', C')$, $(A, C)$, and $(A'', C'')$ be exact couples, and let $i : (A', C') \to (A, C)$ and $j : (A, C) \to (A'', C'')$ be couple maps. Assume

$$0 \to (A', C') \xrightarrow{i} (A, C) \xrightarrow{j} (A'', C'') \to 0$$

is exact in the sense that

$$0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$$

is exact. By the usual algebraic construction [3; p. 40], we get an exact triangle,

$$\begin{array}{ccc} C'_1 & \xrightarrow{i_1} & C_1 \\ \Delta_1 \nwarrow & & \swarrow j_1 \\ & C''_1 & \end{array}$$

where $\Delta_1(\{u\}) = (i)^{-1}d(j)^{-1}(u)$ for $u \in C''$ representing $\{u\} \in C''_1$. If $\Delta_1 = 0$, then

$$0 \to (A'_1, C'_1) \xrightarrow{i_1} (A_1, C_1) \xrightarrow{j_1} (A''_1, C''_1) \to 0$$

is exact in the above sense and we may repeat the process. In general, if $\Delta_1 = 0$, $\cdots$, $\Delta_{m-1} = 0$, then we may define

$$\Delta_m : C''_m \to C'_m \text{ such that}$$

$$\begin{array}{ccc} C'_m & \xrightarrow{i_m} & C_m \\ \Delta_m \nwarrow & & \swarrow j_m \\ & C''_m & \end{array}$$

is an exact triangle.

In our application, $(A', C')$ is the exact couple for $L$ coming from the Postnikov system $\mathfrak{P}$, $(A, C)$ is the exact couple for $M$, $(A'', C'')$ the exact couple for $K$, $i = c^*$, and $j = b^*$. We define $\Delta_m$ to be the functional $m$-ary cohomology operation corresponding to the given Postnikov system $\mathfrak{P}$. It should be noted that in our application, all exact couples are bigraded, and that $\Delta_m^{r,s}$ might be defined without assuming all $\Delta_{m-1}^{p,q}$ are zero; this can be done but is very complicated.

## 4. Functional cohomology operations: the second method

We first give the necessary algebraic formalism. Let $(A'', C'')$ be an exact couple. $u \in C''$ can be *lifted m-stages* if there exists an element $a \in A''$ such that $h''(u) = (f'')^m(a)$. $u$ can be lifted $m$-stages if and only if $d(u) = 0$, $d_1(\{u\}) = 0, \cdots$, $d_{m-1}(\{u\}) = 0$.

Let $k : (A'', C'') \to (A''', C''')$ be a couple map. We assume $k : C'' \to C'''$ is zero. For convenience, set $\bar{C}''' = C'''/\text{Im}\, g'''$. We now define a sequence of operations, $d_{m,k}$.

Define

$$d_{1,k} : C''_1 \to \bar{C}'''$$

as follows: Let $u \in C''$ represent $\{u\} \in C''_1$. Let $a \in A''$ be a lifting of $u$ one stage.

building blocks are spaces having a single non-vanishing cohomology group, and where the analogue of the $k$-invariants are elements of the generalized homotopy groups of the partially constructed space (generalized homotopy groups as in [9; p. 279]). Using this, we can construct an exact couple whose derivations are higher homotopy operations; all the algebraic constructions of this paper carry over and we have functional higher order homotopy operations and a corresponding composition theorem.

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## COHOMOLOGY OPERATIONS[1]

By N. E. Steenrod

### 1. Introduction

Speaking roughly, cohomology operations are operations which, when applied to cohomology classes of a space, produce other cohomology classes. Examples are plentiful. The simplest is *addition*: if $u$, $v$ are elements of the $q$-dimensional cohomology group $H^q(X; G)$ of the space $X$ with coefficient group $G$, then $u + v \in H^q(X; G)$. Another is the *cup-product*: if $u \in H^p(X;G)$ and $v \in H^q(X;G')$, then $u \smile v \in H^{p+q}(X; G \otimes G')$. Again, if $\eta : G \to G'$ is a homomorphism, then $\eta$ induces a homomorphism $\eta_* : H^q(X; G) \to H^q(X; G')$, and this operation is referred to as a *coefficient homomorphism*. Next is the Bockstein-Whitney *coboundary operation*. It is associated with an exact coefficient sequence $0 \to G' \to G \to G'' \to 0$, and is a homomorphism $\delta^* : H^q(X, G'') \to H^{q+1}(X; G')$. These four will be referred to as the *elementary* operations.

Less elementary are the cyclic reduced powers [12]

$$\mathrm{Sq}^i : H^q(X; Z_2) \to H^{q+i}(X; Z_2),$$
$$\mathscr{P}^i_p : H^q(X; Z_p) \to H^{q+2i(p-1)}(X; Z_p).$$

Here $p$ is an odd prime, and $Z_p$ is a cyclic group of order $p$. Related to these operations are the Pontrjagin square

$$\mathfrak{P}_2 : H^q(X; Z_{2^k}) \to H^{2q}(X; Z_{2^{k+1}});$$

and its generalization to odd primes $p$ found by Thomas [15]

$$\mathfrak{P}_p : H^q(X; Z_{p^k}) \to H^{pq}(X; Z_{p^{k+1}}).$$

It is clear that many more operations can be built by composing the ones above in various ways, e.g.,

$$\mathfrak{P}_p(\mathscr{P}^i_p u \smile \mathscr{P}^j_p(v + w))$$

is a cohomology operation of 3 variables. It is the main contention of this article that it is highly probable that the operations listed above generate all others by such compositions. Precise results supporting this view will be given.

This is a semi-expository article in which we will review most of the known facts with some mild improvements, and present a few new ones. The main novelty lies in the treatment of cohomology operations of several variables, and in the reduction theorem of §5. In addition to the papers referred to above which present specific cohomology operations, there are three published accounts treating the subject

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from a general view-point. These are by Eilenberg and MacLane [10], Serre [11], and Cartan [6].

It is important to note that we are restricting our attention to cohomology operations which are defined over the entire cohomology group and not to just a part. These are sometimes called *primary* operations as opposed to *secondary* operations which are defined only when some primary operation is zero (see the articles of Adem and of Massey).

At the end of this article a brief discussion is given of *homology* operations the import of which is that, although they exist, they are not as interesting as cohomology operations.

## 2. The general concept of cohomology operations

If a dimension $q$ and a coefficient group $G$ are specified, then the $q^{\text{th}}$ cohomology group with coefficient group $G$ is a contravariant functor $H^q(\quad; G)$ defined on the category of spaces and continuous mappings with values in the category of groups and homomorphisms. Precisely, if $X$ is a space, then $H^q(X; G)$ is a group, and if $f: X \rightarrow Y$ is a mapping, then $f^*: H^q(Y; G) \rightarrow H^q(X; G)$ is a homomorphism called the *induced* homomorphism. The latter satisfies the two axioms required in the definition of functor: first, if $f: X \rightarrow X$ is the identity, then $f^*$ is the identity; and second, if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, then $(gf)^* = f^*g^*$.

Briefly, a cohomology operation of one variable is a natural transformation of one such functor into another. To be precise, a cohomology operation $T$ is associated with a pair of dimensions $(q, r)$ and a pair of coefficient groups $(G, G')$, and $T$ is a natural transformation of $H^q(\quad; G)$ into $H^r(\quad; G')$. That is, for each space $X$, $T_X$ is a function

$$T_X: H^q(X; G) \rightarrow H^r(X; G')$$

such that, for each map $f: X \rightarrow Y$, the commutativity relation

$$T_X f^* = f^* T_Y$$

holds in the diagram

$$\begin{array}{ccc} H^q(Y; G) & \xrightarrow{T_Y} & H^r(Y; G') \\ \downarrow f^* & & \downarrow f^* \\ H^q(X; G) & \xrightarrow{T_X} & H^r(X; G'). \end{array}$$

For fixed $(q, G; r, G')$, the set of all such operations is denoted by $O(q, G; r, G')$. If $S$ and $T$ are in $O(q, G; r, G')$, we define their sum by

$$(S + T)_X u = S_X u + T_X u, \qquad u \in H^q(X; G).$$

Then $O(q, G; r, G')$ is an abelian group.

It must be emphasized that $T_X$ is not required to be a homomorphism. For example the Pontrjagin square is not a homomorphic operation. However it is a *quadratic* operation. The Pontrjagin cube is *cubic*, etc. It is shown in §6 that each cohomology operation of one variable satisfies some algebraic identity, i.e., it is a $k^{\text{th}}$ power operation for some integer $k$. But it is important that the definition of cohomology operation imposes no restriction on the algebraic nature of the function $T_X$. It is a consequence of this generality that $H^q(\ \ ; G)$ and $H^r(\ \ ; G')$ must be regarded as functors whose range category consists of abelian groups and arbitrary functions from one such to another. This is necessary if $T$ is to be a natural transformation in the strict sense.

The operations listed in §1 satisfy the above definition except for addition and cup-product which are cohomology operations of 2 variables. To define the general notion of a cohomology operation $T$ of $k$ variables, we must suppose given $k + 1$ dimensions $q_1, \cdots, q_k$, $r$ and corresponding coefficient groups $G_1, \cdots, G_k$, $G'$. Then, for each space $Y$ and elements $u_i \in H^{q_i}(Y; G_i)$, $i = 1, \cdots, k$, $T$ assigns an element

$$T(u_1, \cdots, u_k) \in H^r(Y; G')$$

such that, for each $f: X \to Y$, we have

$$f^*T(u_1, \cdots, u_k) = T(f^*u_1, \cdots, f^*u_k).$$

This implies that $T$ is a natural transformation of the *cartesian* product of the functors $H^{q_i}(\ \ ; G_i)$ into $H^r(\ \ ; G')$.

In the case of the cup-product, the operation is bilinear, and therefore the mapping of $H^p(X; G) \times H^q(X; G') \to H^{p+q}(X; G \otimes G')$ extends to the tensor product $H^p \otimes H^q \to H^{p+q}$. It is a somewhat paradoxical fact that the simplest operation of 2 variables, namely addition, is not bilinear.

The *composition* of cohomology operations is defined in the obvious way. For example, if $T$ is the operation of $k$ variables described above, and if $T_1$ is an operation on $l$ variables whose *range* functor is $H^{q_1}(\ \ ;G_1)$, then

$$T(T_1(v_1, \cdots, v_l), u_2, \cdots, u_k)$$

is an operation of $k + l - 1$ variables.

If $T$ is an operation on $k$ variables, as described above, and if $q_1 = q_2$ and $G_1 = G_2$, then an operation $T'$ on $k - 1$ variables is obtained by setting

$$T'(u_1, u_2, \cdots, u_{k-1}) = T(u_1, u_1, u_2, \cdots, u_{k-1}).$$

We refer to $T'$ as a *restriction* of $T$. It is understood that one may restrict on any pair of variables having the same domain functor.

A collection of cohomology operations is said to *generate* all operations if each cohomology operation can be built out of the operations of the collection by a finite number of compositions and restrictions.

The main problem under disucssion can now be formulated: *Find a simple collection of cohomology operations which generates all others.*

Once a system of generators has been found, there arises an important subsidiary problem: *Find a basis for the relations that the generators satisfy.* A *relation* of course is an identity of two operations built from the generators in formally distinct ways.

The operations mentioned in §1 satisfy many important identities. For example, $u \smile v$ is bilinear and associative, $\mathscr{P}^i$ is linear, and

$$\mathscr{P}^k(u \smile v) = \textstyle\sum_{i+j=k} \mathscr{P}^i u \smile \mathscr{P}^j v.$$

Additional identities can be found in [12], and in the papers of Adem [1, 2, 3,] and Cartan [5].

## 3. The complexes $K(\pi, n)$ of Eilenberg and MacLane

If $\pi$ is an abelian group and $n > 0$ is an integer, then a space $Y$ is said to be of type $K(\pi, n)$ if it is arcwise connected and all its homotopy groups are zero except $\pi_n(Y)$ which is isomorphic to $\pi$. For example, a circle is of type $K(Z, 1)$ where $Z =$ integers (it is covered by a line which is contractible space). Also the infinite dimensional real projective space is of type $K(Z_2, 1)$ (it is covered twice by the infinite dimensional sphere $S^\infty$, whose homotopy groups are zero). Another example is the complex projective space of infinite dimension. It is of type $K(Z, 2)$ since it is the base space of a fibration of $S^\infty$ by circles, i.e., by fibres of type $K(Z, 1)$. There are spaces, in fact CW-complexes, of type $K(\pi, n)$ for any prescribed $(\pi, n)$.

If $Y$ is of type $K(\pi, n)$, then the Hurewicz theorem asserts that the natural map $\phi : \pi_n(Y) \to H_n(Y)$ is an isomorphism. So $\phi^{-1}$ is defined and is an element of $\mathrm{Hom}(H_n(Y), \pi_n(Y))$. Since $H_{n-1}(Y) = 0$, it follows that the natural map

$$H^n(Y; \pi_n(Y)) \to \mathrm{Hom}(H_n(Y), \pi_n(Y))$$

is an isomorphism. The element $u_0 \in H^n(Y; \pi_n(Y))$ corresponding to $\phi^{-1}$ is called the *fundamental* class of $Y$ (see [13; p. 187]). Then for any space $X$ and any map $f : X \to Y$, the element $f^* u_0 \in H^n(X; \pi_n(Y))$ is defined and depends only on the homotopy class of $f$. The importance of the $K(\pi, n)$ spaces is expressed by the following known result:

3.1. *If $Y$ is of type $K(\pi, n)$, and $X$ is a complex, then the assignment to each $f : X \to Y$ of $f^* u_0$ defines a 1–1 correspondence between the homotopy classes of maps $X \to Y$ and the elements of $H^n(X; \pi_n(Y))$.*

A proof of this proposition in the geometric case can be found in [8; p. 243, Th. II] and for the purely algebraic case of semi-simplicial complexes see [10; pp. 520–521]. In essence the argument is the one used by Hopf in classifying the mappings of an $n$-complex into an $n$-sphere. A corollary is that, within the realm of CW-complexes, any two spaces of the type $K(\pi, n)$ have the same homotopy type, and therefore their homology and cohomology groups depend only on $(\pi, n)$. Because of this $H^r(Y; G')$ may be written $H^r(\pi, n; G')$.

The importance of $K(\pi, n)$ spaces to the study of cohomology operations is seen as follows. Let $T : H^q(\ \ ;G) \to H^r(\ \ ;G')$ be a cohomology operation of one variable as indicated. Let $Y$ be a $K(G, q)$ space. Then $T$ may be applied to the fundamental class $u_0 \in H^q(Y; G)$ to give an element $Tu_0 \in H^r(Y; G')$. Now, if $(q, G)$ and $(r, G')$ are fixed, we may regard $Tu_0$ as a function of $T$, and then we have a mapping of all cohomology operations relative to $(q, G; r, G')$ into $H^r(Y; G') = H^r(G, n; G')$. Then we have.

3.2. *The assignment* $T \to Tu_0$ *defines an isomorphism between all cohomology operations* $O(q, G; r, G')$ *and the elements of* $H^r(G, q; G')$.

This result is due to Serre [11; p. 220] and independently to Eilenberg-MacLane [9]. The proof runs as follows: Suppose $T$, $T'$ are two operations such that $Tu_0 = T'u_0$. Let $X$ be a complex, and $u \in H^q(X; G)$. Then there is a map $f : X \to Y$ such that $u = f^*u_0$. Therefore

$$Tu = Tf^*u_0 = f^*Tu_0 = f^*T'u_0 = T'f^*u_0 = T'u.$$

Thus $T$ and $T'$ coincide on complexes. Now let $w \in H^r(G, n; G')$. We must construct a $T$ such that $Tu_0 = w$. Let $X$ be a complex and $u \in H^q(X; G)$. Select $f : X \to Y$ so that $f^*u_0 = u$, and define $Tu = f^*w$. If $g : X' \to X$ and $u$ and $f$ are as before, then $fg : X' \to Y$ is such that $(fg)^*u_0 = g^*u$. Hence

$$Tg^*u = (fg)^*w = g^*f^*w = g^*Tu;$$

and therefore $T$ is *natural*. That $Tu_0 = w$ follows by choosing $u = u_0$, and $f =$ identity. This proves the proposition within the category of complexes. To obtain the extension of the result to the singular theory, one applies the result for complexes to the geometric realization of the singular complex of a space.

The preceding result for operations of 1 variable generalizes to $k$ variables quite readily. Consider cohomology operations $O$ of $k$ variables relative to $(q_1, \cdots, q_k, r; G_1, \cdots, G_k, G')$. Let $Y_i$ be a space of type $K(G_i, q_i)$, and set $Y = Y_1 \times \cdots \times Y_k$. Let $\phi_i : Y \to Y_i$ be the projection; and let $v_i \in H^{q_i}(Y_i; G_i)$ denote the fundamental class. The generalized result is as follows:

3.3. *The assignment to each* $T \in O$ *of the element* $T(\phi_1^*v_1, \cdots, \phi_k^*v_k) \in H^r(Y; G')$ *defines an isomorphism between* $O$ *and* $H^r(Y; G')$.

To prove this, let $X$ be a complex, and $u_i \in H^{q_i}(X; G_i)$. Choose a map $f_i : X \to Y_i$ such that $f_i^*v_i = u_i$; and let $f : X \to Y$ be the map whose components are $(f_1, \cdots, f_k)$, i.e., $\phi_i f = f_i$. Suppose $T$, $T' \in O$ are such that

$$T(\phi_1^*v_1, \cdots, \phi_k^*v_k) = T'(\phi_1^*v_1, \cdots, \phi_k^*v_k).$$

Apply $f^*$ to both sides, use the naturality of $T$ and $T'$, and the relation $f^*\phi_i^* = f_i^*$ to obtain $T(u_1, \cdots, u_k) = T'(v_1, \cdots, v_k)$; and therefore $T = T'$. On the other hand, if $w \in H^r(Y; G')$ is given, we can define a corresponding $T$ by $T(u_1, \cdots, u_k) = f^*w$.

## 4. Characterizations of operations in special cases

The preceding results will now be applied in certain cases to obtain complete information as to the possible cohomology operations. We shall restrict attention in this section to operations of one variable. As in §3, $Y$ will denote a $K(G, q)$ space.

4.1. *The operations* $O(0, G; 0, G')$ *are in 1–1 correspondence with the functions from* $G$ *to* $G'$.

The space $Y$ of type $K(G, 0)$ can be taken to be $G$ itself considered as a discrete space. Then $H^0(Y; G')$ is a direct sum of copies of $G'$, one copy for each point of $G$. This implies 4.1.

4.2. *If* $r > 0$, *then* $O(0, G: r, G') = 0$, *i.e., it consists of the single operation which is identically zero.*

With $Y$ as above, we have $H^r(Y, G') = 0$ for all $r > 0$.

4.3. *If* $q > 0$, *then* $O(q, G; O, G') \approx G'$, *i.e., each such operation is constant.*

In this case $Y$ is connected, and we have a natural isomorphism $H^0(Y; G') \approx G'$.

4.4. *If* $r > 0$ *and* $T \in O(q, G; r, G')$, *then* $T(0) = 0$.

Referring to the proof of 3.2, we have only to choose the map $f: X \to Y$ to be a map into a single point, then $f^*$ maps $H^r(Y)$ into zero.

4.5. *If* $q > r > 0$, *then* $O(q, G; r, G') = 0$.

By the Hurewicz theorem, $\pi_i(Y) = 0$ for $i < q$ implies $H_i(Y; G') = 0$ for $i < q$.

4.6. *If* $q = r > 0$, *then* $O(q, G; q, G')$ *is isomorphic to the group of all homomorphisms* $G \to G'$, *i.e. each such operation is induced by a coefficient homomorphism.*

By the Hurewicz theorem, $G = \pi_q(Y) \approx H_q(Y)$, Hence the group $\mathrm{Hom}(H_q(Y), G')$ may be identified with $\mathrm{Hom}(G, G')$. By the universal coefficient theorem, the natural map

$$H^q(Y;\ G') \to \mathrm{Hom}(H_q(Y), G') = \mathrm{Hom}(G, G')$$

is an isomorphism since $H_{q-1}(Y) = 0$. Thus any $w \in H^q(Y;\ G')$ may be identified with a homomorphism $\eta : G \to G'$. The fundamental class $u_0$ is the identity map $G \to G$; hence composing with $\eta$ gives $\eta_* u_0 = w$.

4.7. *If* $r > q = 1$ *and* $G = Z$, *then* $O(1, Z; r, G') = 0$.

This follows from the fact that a circle is a $K(Z, 1)$ space.

4.8. *Let* $q = 2$ *and* $G = Z$. *If r is odd, then* $O(2, Z; r, G') = 0$. *If* $r = 2m$ *is even and positive, then each* $T \in O(2, Z; r, G')$ *is an* $m^{\text{th}}$ *power followed by a coefficient homomorphism.*

In this case, we take $Y$ to be the infinite dimensional complex projective space. Its cohomology ring over $Z$ is the polynomial ring generated by the fundamental class $u_0$ of dimension 2. Then $H^r(Y;\ G') = 0$ when $r$ is odd, and $H^{2m}(Y;\ G') \approx \mathrm{Hom}(H_{2m}(Y), G')$. Since the $m$-fold power $u_0^m$ generates $H^{2m}(Y)$, as an element of $\mathrm{Hom}(H_{2m}(Y), Z)$ it gives an isomorphism $H_{2m}(Y) \approx Z$; hence any element of $\mathrm{Hom}(H_{2m}(Y), G')$ can be factored into this isomorphism followed by a homomorphism $Z \to G'$.

4.9. *Let* $q = 1$, $r > 0$, *and* $G = Z_2$. *Then each* $T \in O(1, Z_2; r, G')$ *is generated by cup-products, Bockstein-Whitney coboundaries and coefficient homomorphisms.*

The real projective space of infinite dimension is of type $K(Z_2, 1)$. It must be shown that the fundamental class $u_0$ generates all others using the three types of operations. We will assume two facts as known. First, there is a cellular decomposition of $Y$ with exactly one cell $e_i$ in each dimension $i$, with the boundary relations $\partial e_{2i} = 2e_{2i-1}$, $\partial e_{2i-1} = 0$. Secondly, the cohomology ring modulo 2 of $Y$ is the polynomial ring generated by $u_0$. Now any $w \in H^r(Y;\ G')$ is characterized uniquely by its Kronecker index $w \cdot e_r \in G'$.

If $r$ is odd, we must have $2w \cdot e_r = 0$ since $w$ is a cocycle (recall that $\delta w \cdot e_{r+1} = w \cdot \partial e_{r+1} = w \cdot 2e_r$). Since $u_0^r \cdot e_r = 1$ modulo 2, it follows that $\eta_* u_0^r = w$ where $\eta : Z_2 \to G'$ sends 1 into $w \cdot e_r$.

Now let $r = 2m$ be even. Let $\delta^*$ be the Bockstein-Whitney coboundary associated with the coefficient sequence $0 \to Z \to Z \to Z_2 \to 0$. Then $\partial e_{2m} = 2e_{2m-1}$, and $u_0^{2m-1} \cdot e_{2m-1} = 1$ modulo 2 imply $(\delta^* u_0^{2m-1}) \cdot e_{2m} = 1$. And, finally, we have $\eta_* \delta^* u_0^{2m-1} = w$ where $\eta : Z \to G'$ sends 1 into $w \cdot e_{2m}$.

The conclusion of 4.9 also holds for $q = 1$ and $G = Z_k$ for any integer $k > 0$. The argument is essentially the same using the infinite dimensional lens space in place of the projective space.

The preceding results, although striking in their generality, must be regarded as only the first and most rudimentary facts. We have used only the simplest properties of the $K(\pi, n)$ spaces. In recent years Eilenberg, MacLane, Cartan and Serre have obtained far-reaching results concerning the cohomology of these spaces. We will conclude this section by stating some of their results (without proofs), and interpreting these in terms of cohomology operations.

4.10. *Let $q > 0$ and $r = q + 1$. Then each element of $O(q, G; q + 1, G')$ is a Bockstein-Whitney coboundary operator.*

This was proved by Eilenberg-MacLane [10; pp. 528–529]. They establish the isomorphism

$$H^{q+1}(G, q; G') \approx \text{Extabel}\,(G, G')$$

so that each $w \in H^{q+1}$ corresponds to a unique abelian group extension $0 \to G' \to G'' \to G \to 0$. And then they verify, for the corresponding $\delta^*$, that $\delta^* u_0 = w$.

4.11. *If $q > 0$, $r > 0$, $G$ is finitely generated, and $G' = Z_2$, then each element of $O(q, G; r, Z_2)$ is generated by addition, cup-product, Bockstein-Whitney coboundaries, and the squaring operations* $\mathrm{Sq}^i$.

This result is due to Serre [11]. It is a corollary of explicit computations of the cohomology ring $H^*(G, q; Z_2)$ for $G = Z$, $Z_2$ and $Z_{2^h}$. For example $H^*(Z_2, q; Z_2)$ is the polynomial ring whose generators are all *admissible* iterated squares of the fundamental class

$$\mathrm{Sq}^{i_1}\, \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_r}\, u_0$$

where admissible means that

$$i_1 \geqq 2i_2,\ i_2 \geqq 2i_3, \cdots, i_{r-1} \geqq 2i_r$$

and

$$i_1 - \textstyle\sum_{k=2}^{r} i_k < q.$$

4.12. *If $q > 0$, $r > 0$, $G$ is finitely generated, and $G' = Z_p$ where $p$ is an odd prime, then each element of $O(q, G; r, Z_p)$ is generated by addition, cup-product, Bockstein-Whitney coboundaries, and the cyclic reduced $p^{\text{th}}$ powers $\mathscr{P}^i$.*

This result is due to Cartan [4]. As in the case $p = 2$, it is based on explicit computations of $H^*(G, q; Z_p)$ for $G = Z$, $Z_p$ and $Z_{p^h}$.

## 5. The reduction to operations of one variable with cyclic coefficients

We shall restrict consideration to cohomology operations (of any number of variables) whose coefficient groups are finitely generated. Let $C$ denote the family of cyclic groups each of whose orders is either infinite or a power of some prime.

5.1. *Each cohomology operation, whose coefficient groups are finitely generated, can be expressed as a composition of the following ones: addition, coefficient homomorphisms, cup-products with coefficient groups in $C$, and cohomology operations of one variable with coefficient groups in $C$.*

This gives a two-fold reduction. First, we can reduce from $k$ variables to 1 variable using only two explicit operations of 2 variables namely: addition and cup-product. Secondly, the only operations needed which involve general coefficients are addition and coefficient homomorphisms. The problem of finding a basis for cohomology operations is thereby reduced to finding a basis for operations of one variable with coefficients in $C$.

As a first step in proving 5.1, let us show that the operations of addition, coefficient homorphisms, and those operations, whose *terminal* coefficient group is in $C$, generate all operations. Let $T$ be an operation whose terminal coefficient group $G'$ is finitely generated. Then $G'$ is a direct sum $\sum_j G'_j$ where $G'_j \in C$. There are homorphisms $f_j : G' \to G'_j$ and $g_j : G'_j \to G'$ such that $f_j g_j$ is the identity of $G'_j$, and $\sum_j g_j f_j$ is the identity of $G'$. Then, for any $u_1, \cdots, u_k$, we have

$$\begin{aligned} T(u_1, \cdots, u_k) &= (\textstyle\sum_j g_j f_j)_* T(u_1, \cdots, u_k) \\ &= \textstyle\sum_j g_{j*}(f_{j*} T(u_1, \cdots, u_k)). \end{aligned}$$

Now $f_{j*}T$ has $G'_j \in C$ as terminal coefficient group, and $T$ is expressed in terms of the $f_{j*}T$ by means of coefficient homomorphisms and addition.

Therefore it suffices to prove 5.1 for a $T$ whose terminal coefficient group $G'$ is in $C$. Let $G_1, \cdots, G_k$ be its initial coefficient groups. Since each $G_i$ is finitely generated, $G_i$ may be written as a direct sum $\sum_\alpha G_{i,\alpha}$ of groups in $C$. Let $Y_{i,\alpha}$ be a complex of type $K(G_{i,\alpha}, q_i)$. It may be chosen to be finitely generated in each dimension [6, Exposé 11]. Then $Y_i = \prod_\alpha Y_{i,\alpha}$ is of type $K(G_i, q_i)$; and $Y = \prod_i Y_i$ is the complex constructed in 3.3. We need now a lemma.

5.2. If $K$, $L$ are free chain complexes which are finitely generated in each dimension, then each cohomology class of $K \times L$ with a coefficient group in $C$ is expressible in terms of cohomology classes of $K$ and $L$ having coefficient groups in $C$ by the use of the four operations of addition, tensor product, coefficient homomorphism and Bockstein-Whitney coboundary in each case restricted to coefficient groups in $C$.

The proof is a review of the classical argument due to Künneth. Because $K$, $L$ are finitely generated in each dimension, we may reduce each to a normal form

$$K = \sum K_i, \qquad L = \sum L_j$$

where each $K_i$, $L_j$ is an *elementary* subcomplex, i.e., either all chain groups of $K_i$ are zero save in one dimension and this group is infinite cyclic, or all chain groups of $K_i$ are zero save in two successive dimensions and these are infinite cyclic. Then $K \otimes L = \sum K_i \otimes L_j$ which shows that it suffices to prove the lemma when $K$, $L$ are elementary. We shall discuss only the least simple case where $K$ is generated by chains $a$ and $b$ with $\partial a = \alpha b$, and $L$ is generated by chains $c$ and $d$

with $\partial c = \gamma d$. Choose integers $m$, $n$ such that $m\alpha + n\gamma = \lambda$ is the G.C.D. of $\alpha$ and $\gamma$. Let $\alpha' = \alpha/\lambda$ and $\gamma' = \gamma/\lambda$. Then $K \otimes L$ is reduced to the normal form

$$\partial(a \otimes c) = \lambda f \quad \text{where} \quad f = \alpha' b \otimes c + (-1)^k \gamma' a \otimes d,$$

$$\partial g = \lambda b \otimes d \quad \text{where} \quad g = (-1)^{k+1} nb \otimes c + ma \otimes d,$$

and $k = \dim a$. Let $a'$, $b'$, $c'$, $d'$ be the cochains having the value 1 on $a$, $b$, $c$, $d$, respectively. Setting

$$f' = mb' \otimes c' + (-1)^k na' \otimes d',$$

$$g' = (-1)^{k+1} \gamma' b' \otimes c' + \alpha' a' \otimes d',$$

then $\operatorname{Hom}(K \otimes L, Z)$ is reduced to the normal form

$$\delta f' = \lambda a' \otimes c', \qquad \delta(b' \otimes d') = \lambda g'.$$

Consider first the cohomology of $K \otimes L$ with coefficients in $Z$. The only cocycles are the multiples of $a' \otimes c'$ and $g'$. Since the first is a tensor product of cocycles, we need consider only the second. Since $b'$, $d'$ are cocycles mod $\lambda$, so also is $b' \otimes d'$. Now $\delta(b' \otimes d') = \lambda g'$ implies that the class of $g'$ is the Bockstein-Whitney coboundary $\delta^*$ of the class mod $\lambda$ of $b' \otimes d'$ where $\delta^*$ is defined relative to the sequence $0 \to Z \to Z \to Z_\lambda \to 0$. In case $\lambda$ is a prime power, the argument in this case is complete since $Z$, $Z_\lambda$ are in $C$. Suppose not, and that $\lambda = p^k \mu$ where $\mu$ is prime to $p$. Again $b'$, $d'$, $b' \otimes d'$ are cocycles mod $p^k$, and $\mu g'$ is a B-W coboundary relative to $0 \to Z \to Z \to Z_{p^k} \to 0$. We can do this for each prime power factor of $\lambda$, and obtain cocycles $\mu_1 g', \cdots, \mu_s g'$ where only coefficient groups in $C$ are used. Since the $\mu$'s are relatively prime, a linear combination of these gives $g'$.

Finally consider, the cohomology with coefficients $Z_r$ where $r = p^h$ is a prime power. Having obtained $a' \otimes c'$ and $g'$ as integral cocycles by the allowed operations, we obtain them as cocycles mod $r$ by applying the coefficient homomorphism $Z \to Z_r$. Suppose $t$ is an integer and that

$$tf = m(tb') \otimes c' + (-1)^k na' \otimes (td')$$

is a cocycle mod $r$. Then $t\lambda \equiv 0 \bmod r$. This implies $t\alpha \equiv 0 \equiv t\gamma \bmod r$. Hence $tb'$ and $td'$ are cocycles mod $r$. It follows that $tf$ is a sum of tensor products of cocycles mod $r$. Again, if $t(b' \otimes d')$ is a cocycle mod $r$, then $t\lambda \equiv 0 \bmod r$. Let $\lambda = p^k \mu$ where $\mu$ is prime to $p$. Then a homomorphism $\eta : Z_{p^k} \to Z_{p^h}$ is defined by $\eta(1) = t$ since $tp^k \equiv 0 \bmod p^h$. Therefore the cocycle $t(b' \otimes d')$ is obtained in the required manner as the image under $\eta$ of the tensor product of the cocycles $b'$, $d'$ mod $p^k$.

Returning now to the proof of 5.1, an induction based on 5.2 shows that the cohomology $H^*(Y; G')$ is generated by the cohomologies $H^*(Y_{i,\alpha})$ with coefficient groups in $C$ using the operations $+$, $\otimes$, coefficient homomorphisms, and $\delta^*$ with coefficient groups in $C$. The tensor product is turned into a cup product by using the following well-known relation: If $\phi_1$, $\phi_2$ are the projections of $K_1 \otimes K_2$ into its factors and $v_1$, $v_2$ are cohomology classes of $K_1$, $K_2$, respectively, then

$$v_1 \otimes v_2 = (\phi_1^* u_1) \smile (\phi_2^* u_2).$$

Therefore the preceding statement holds with $\otimes$ replaced by $\smile$ if the cohomology groups $H^*(Y_{i,\alpha})$ are replaced by their images induced by the projection $\phi_{i,\alpha} : Y \to Y_{i,\alpha}$.

Let $v_{i,\alpha}$ be the fundamental class of $Y_{i,\alpha}$, and let $z$ be any class of $Y_{i,\alpha}$ with coefficient group in $C$. If $T_z$ is the corresponding cohomology operation of one variable (see 3.2), it follows from 3.2 that

$$T_z\phi^*_{i,\alpha}(v_{i,\alpha}) = \phi^*_{i,\alpha}(z).$$

Therefore, the classes $\phi^*_{i,\alpha}(v_{i,\alpha})$ generate $H^*(Y;\ G')$ using only the operations listed in 5.1.

Let $\eta_{i,\alpha} : G_i \to G_{i,\alpha}$ be the projection associated with the decomposition $G_i = \sum G_{i,\alpha}$. If $v_i$ is the fundamental class of $Y_i$, and $\phi_i : Y \to Y_i$ is the projection, it is easily seen that

$$\phi^*_{i,\alpha}(v_{i,\alpha}) = \eta_{i,\alpha *}\, \phi^*_i(v_i).$$

It follows that the elements $\phi^*_i(v_i)$ generate $H^*(Y;\ G')$ using only the operations listed in 5.1. If we recall from 3.3 that $T(\phi^*_1 v_1, \cdots, \phi^*_k v_k)$ is the element of $H^*(Y; G')$ corresponding to $T$, we see that the proof of 5.1 is complete.

## 6. Linearity and other algebraic properties of cohomology operations

Most of the explicit operations discussed in §1 are linear (i.e., homomorphic). The existence of some which are not calls for an investigation of the algebraic properties which operations satisfy.

Let $T \in O(q, G; r, G')$ where $q > 0$ and $r > 0$. Define a corresponding cohomology operation $T_2$ of 2 variables, relative to $(q, G, q, G; r, G')$ by

$$T_2(u, v) = T(u + v) - T(u) - T(v). \tag{6.1}$$

It is called the *deviation* of $T$ from linearity. It is symmetric: $T_2(u, v) = T_2(v, u)$.

Let $Y$ be a space of type $K(G, q)$. The results of §3 assert that $T$ is uniquely determined by the element $T(u_0) \in H^r(Y;\ G')$ where $u_0$ is the fundamental class of $Y$, and $T_2$ is uniquely determined by

$$T_2(\phi^*_1 u_0, \phi^*_2 u_0) \in H^r(Y \times Y; G'). \tag{6.2}$$

We shall present a direct relation between these two classes of $Y$ and $Y \times Y$.

We digress for a moment to recall some facts about the cohomology of a product space $X_1 \times X_2$. Let $\phi_1, \phi_2$ be its projections into its factors. Let $x_1 \in X_1$ and $x_2 \in X_2$ be base points. Set $X_1 \vee X_2 = (X_1 \times x_2) \cup (x_1 \times X_2)$. Define $g_i : X_i \to X_1 \times X_2$ for $i = 1, 2$ by $g_1(x) = (x, x_2)$ and $g_2(x) = (x_1, x)$. Let $\phi_0 : X_1 \times X_2 \to x_0$ be the mapping into the single point $x_0 = (x_1, x_2)$. Finally, let

$$X_1 \vee X_2 \xrightarrow{f} X_1 \times X_2 \xrightarrow{g} (X_1 \times X_2, X_1 \vee X_2)$$

be inclusion maps. Then the cohomology $H^*(X_1 \times X_2)$ with any coefficients decomposes into a direct sum

$$H^*(X_1 \times X_2) = A_0 + A_1 + A_2 + A_{12} \tag{6.3}$$

where

$$A_0 = \text{image } \phi_0^* \approx H^*(x_0) = H^0(x_0).$$

$$A_i = \text{image } \phi_i^* \approx H^*(X_i, x_i), \quad i = 1, 2,$$

$$A_{12} = (\text{kernel } g_1^*) \cap (\text{kernel } g_2^*)$$

$$= \text{image } g^* \approx H^*(X_1 \times X_2, X_1 \vee X_2).$$

It is well known that

$$H^*(X_1 \vee X_2) \approx H^*(x_0) + H^*(X_1, x_1) + H^*(X_2, x_2),$$

because it is a union of two spaces with a single point in common. The restriction mapping $f^*$ is readily seen to be an epimorphism: $f^*\phi_0^*$ picks out $H^*(x_0)$, and $f^*\phi_i^*$ picks out $H^*(X_i, x_i)$. Hence the exactness of the cohomology sequence implies that $g^*$ is a monomorphism. These facts imply the splitting 6.3 as stated. We shall apply this result with $X_1 = X_2 = Y$.

The space $Y$ is an $H$-space. Explicitly, if $y_0 \in Y$ is a base point, then there exists a mapping $h : Y \times Y \to Y$ such that

$$h(y, y_0) = y = h(y_0, y) \quad \text{for all} \quad y \in Y. \tag{6.4}$$

Its existence is derived from properties already stated as follows. By 3.1, there is a map $h : Y \times Y \to Y$ such that

$$h^*u_0 = \phi_1^*u_0 + \phi_2^*u_0. \tag{6.5}$$

Since $\phi_1 g_1 =$ identity, and $\phi_2 g_1 =$ constant, we have $g_1^* h^* u_0 = u_0$. Therefore 3.1 asserts that $h | Y \times y_0$ is deformable into one satisfying $h(y, y_0) = y$. The same is true of $h | y_0 \times Y$. A suitable extension of these homotopies to $Y \times Y$ gives the required $h$.

The relationship between the cohomology classes corresponding to $T$ and $T_2$ is given by

$$T_2(\phi_1^*u_0, \phi_2^*u_0) = h^*T(u_0) - \phi_1^*T(u_0) - \phi_2^*T(u_0). \tag{6.6}$$

For, by definition,

$$T_2(\phi_1^*u_0, \phi_2^*u_0) = T(\phi_1^*u_0 + \phi_2^*u_0) - T\phi_1^*u_0 - T\phi_2^*u_0;$$

and we obtain (6.6) by using (6.5), and the obvious commutativities $Th^* = h^*T$, $T\phi_i^* = \phi_i^*T$.

It follows now from (6.6) and the direct sum decomposition (6.3), that $T_2(\phi_1^*u_0, \phi_2^*u_0)$ is just the component of $h^*T(u_0)$ in the summand $A_{12}$. Now, by definition, an element $w \in H^*(Y)$ is called *primitive* if $h^*w = \phi_1^*w + \phi_2^*w$, i.e. the component of $h^*w$ in $A_{12}$ is zero. Thus we have

6.7. *The operation $T$ is linear if and only if its corresponding cohomology class $T(u_0) \in H^r(G, q; G')$ is a primitive element.*

Now the Künneth formulas for the cohomology of a product with integer coefficients give

$$H^r(Y \times Y, Y \vee Y) \approx \sum_{s=1}^{r-1} H^s(Y, y_0) \otimes H^{r-s}(Y, y_0)$$
$$+ \sum_{s=1}^{r-2} \operatorname{Tor}(H^s(Y, y_0), H^{r-s-1}(Y, y_0)).$$

If we recall that $Y$ is $(q-1)$-connected, it follows that $H^r(Y \times Y, Y \vee Y) = 0$ for $0 \leqq r < 2q$. This implies that $A_{12} = 0$ in the same range; and hence

6.8. *If $T \in O(q, G; r, G')$ where $q > 0$ and $0 < r < 2q$, then $T$ is a linear operation, i.e., $T_2 \approx 0$.*

In case $T$ is non-linear, we may ask if its $T_2$ is bilinear. We define $T_3$ to be the operation of 3 variables which measures the deviation of $T_2$ from linearity in either variable:

$$\begin{aligned} T_3(u, v, w) &= T_2(u + v, w) - T_2(u, w) - T_2(v, w) \\ &= T(u + v + w) - T(u + v) - T(u + w) - T(v + w) \\ &\qquad + T(u) + T(v) + T(w). \\ &= T_2(u, v + w) - T_2(u, v) - T_2(u, w). \end{aligned} \tag{6.9}$$

We shall say that $T$ is *quadratic* if its $T_3 = 0$. Note that a linear operation is automatically quadratic.

Now 3.3 asserts that $T_3$ is uniquely determined by

$$T_3(\phi_1^* u_0, \phi_2^* u_0, \phi_3^* u_0) \in H^r(Y \times Y \times Y; G'). \tag{6.10}$$

If we apply (6.3) twice, we obtain a direct sum decomposition for $H^*(Y^3)$, namely,

$$H^*(Y^3) = A_0 + A_1 + A_2 + A_3 + A_{12} + A_{13} + A_{23} + A_{123}$$

where

$$A_0 \approx H^*(y_0), \quad A_i \approx H^*(Y, y_0), \quad A_{ij} \approx H^*(Y^2, Y \vee Y)$$

$$A_{123} \approx H^*(Y^3, (Y^2 \times y_0) \cup (Y \times y_0 \times Y) \cup (y_0 \times Y^2)).$$

The subgroup $A_{123}$ can also be characterized as the intersection of the kernels of the homomorphisms $H^*(Y^3) \to H^*(Y^2)$ induced by the three cross-sectional imbeddings of $y_0 \times Y^2$, $Y \times y_0 \times Y$ and $Y^2 \times y_0$ in $Y^3$. Denote these by $g_1$, $g_2$, $g_3$. Since $\phi_1 g_1(Y^2) = y_0$, we have $g_1^* \phi_1^* u_0 = 0$. Therefore $g_1^*$ applied to (6.10) gives

$$T_3(0, g_1^* \phi_2^* u_0, g_1^* \phi_3^* u_0).$$

If, in (6.9), we replace $u$ by 0, the expression for $T_3$ reduces to $T(0)$; and by 4.4 this is zero. Similar arguments show that $g_2^*$ and $g_3^*$ also map the element (6.10) into zero. Therefore it lies in the subgroup $A_{123}$. The Künneth formulas for the triple product with integer coefficients show that $A_{123}$ is the direct sum of triple products (tensor, torsion and mixed) in which all three factors have positive dimension.

Since $Y$ is $(q-1)$-connected, we have $A_{123}=0$ in dimensions $<3q$ for integer coefficients, and hence for all coefficients. This proves

6.11. *If $q>0$ and $0<r<3q$, then each $T \in O(q, G; r, G')$ is quadratic.*

If we compare the results 4.5, 6.8, and 6.11, we have

$$\begin{array}{ll} 0<r<q & \text{implies } T \text{ is zero,} \\ q \leqq r < 2q & \text{implies } T \text{ is linear,} \\ 2q \leqq r < 3q & \text{implies } T \text{ is quadratic.} \end{array}$$

The general case is readily described as follows. The operation $T_k$ of $k$ variables associated with $T$ is defined inductively by $T_1 = T$ and $T_{k+1}$ is the deviation of $T_k$ from linearity in its first variable. Then $T_k$ is symmetric in its variables, and is given by

$$T_k(u_1, \cdots, u_k) = \sum (-1)^{k-s} T(u_{i_1} + \cdots + u_{i_s})$$

where the sum is taken over all distinct sets $i_1, \cdots, i_s$ satisfying $1 \leqq i_1 < i_2 < \cdots < i_s \leqq k$. We say that $T$ is a $k^{\text{th}}$ *power operation* if $T_{k+1}$ is identically zero. This is equivalent to requiring $T_k$ to be linear in each variable, $T_{k-1}$ to be quadratic in each variable, etc. The general result becomes

6.12. *If $q>0$ and $0<r<(k+1)q$, then each $T \in O(q, G; r, G')$ is a $k^{\text{th}}$ power operation.*

The proof is omitted since it is an obvious generalization of the ones given already in the cases $k=1$ and 2.

The above result can be found in the paper [10] of Eilenberg and MacLane in a somewhat different form (see Th. 8.1, p. 527).

## 7. Relative groups, coboundary and suspension

The definition of a cohomology operation $T$ of one variable, given in §2, demanded only that it be defined on the absolute cohomology group $H^q(X\colon G)$. If $A \subset X$, we have the relative group $H^q(X, A; G)$, and we can regard $H^q(\quad; G)$ as a functor defined on the category of *pairs* of topological spaces and mappings of pairs. We may now define a cohomology operation as a natural transformation of the enlarged functors $H^q(\quad; G) \to H^r(\quad; G')$. A priori, this definition would be expected to give a smaller set of operations than before. But this is not the case, each operation in the former sense is the restriction of a unique operation in the latter sense.

To see this, it suffices to consider the extension of Theorem 3.2 to the relative case. As a first step, one generalizes 3.1 by replacing $X$ by a pair $(X, A)$, choosing a base point $y_0 \in Y$, and considering homotopy classes of maps $f\colon (X, A) \to (Y, y_0)$ relative to $A$. Then the conclusion of 3.1 holds with $H^n(X; \pi_n(Y))$ replaced by $H^n(X, A; \pi_n(Y))$. The proof is no more difficult than in the special case $A = \emptyset$. In positive dimensions $H^p(Y, y_0) \approx H^p(Y)$; so the fundamental class $u_0$ of $Y$ may be regarded as in $H^q(Y, y_0; G)$, and then

$$Tu_0 \in H^r(Y, y_0; G') \approx H^r(G, q; G').$$

With these modifications, the statement and proof of 3.2 remain valid when $q$ and $r$ are positive.

In the cohomology sequence of $(X, A)$, every third homomorphism is a coboundary

$$\delta : H^p(A) \to H^{p+1}(X, A) \tag{7.1}$$

and the others are induced by inclusion mappings. The latter commute with cohomology operations. It makes no sense to ask if a particular $T$ commutes with $\delta$ because the dimensions involved do not agree. However, there is an important relation which we will proceed to develop.

Let $T \in O(q, G; r, G')$ where $q > 0$ and $r > 0$. Let $CX$ denote the *cone* on $X$, i.e., the join of $X$ with a point. Since $CX$ is contractible, its cohomology is zero, and therefore $\delta$ is isomorphic:

$$\delta : H^p(X) \approx H^{p+1}(CX, X) \qquad p \geqq 0.$$

When $p = 0$, $H^0(X)$ must be interpreted as the reduced cohomology group. Then, for each $u \in H^{q-1}(X; G)$ we define $T'u$ by

$$T'u = \delta^{-1}T\delta u \in H^{r-1}(X; G'). \tag{7.2}$$

Since the cone construction is functorial we obtain a cohomology operation $T' \in O(q-1, G; r-1, G')$. It is called the *suspension* of $T$.

7.3. *If $T'$ is the suspension of $T$, $u \in H^{q-1}(A; G)$, and $\delta$ is as in* 7.1, *then*

$$\delta T'u = T\delta u.$$

For 7.3 to be true, we must impose a mild condition on $(X, A)$, namely, that the identity map of $A$ can be extended to a map $f : (X, A) \to (CA, A)$. If $A$ is triangulable, so is $CA$; and since $CA$ is contractible, the homotopy extension property, applied to the constant map of $X$ into the vertex of $CA$, gives the required $f$. If $\delta_1$ is the coboundary operator for $(CA, A)$, then (7.2) becomes

$$T'u = \delta_1^{-1}T\delta_1 u, \quad \text{or} \quad \delta_1 T'u = T\delta_1 u.$$

If we apply $f^*$ to both sides, use $f^*\delta_1 = \delta(f/A)^* = \delta$, and $f^*T = Tf^*$, we obtain the formula 7.3.

An infinite sequence of operations $\{T_k\}$ $(k = 1, 2, \cdots)$ such that $T_k \in O(k, G; k + i, G')$ and $T_k$ is the suspension of $T_{k+1}$ for each $k$, is customarily treated as a single operation called a *stable* operation of degree $i$. For example $\mathrm{Sq}^i$ is stable of degree $i$, and $\mathscr{P}_p^i$ is stable of degree $2i(p-1)$. A coefficient homomorphism is stable of degree 0; and a Bockstein-Whitney coboundary is stable of degree 1. However the Pontrjagin $p^{\text{th}}$ powers are unstable: they cannot be represented as suspensions. This is a consequence of their non-linearity and the following result.

7.4. *The suspension of a cohomology operation is always a linear operation. In particular each stable operation is linear.*

Since $\delta$ is linear, inspection of (7.2) shows that it suffices to prove that

$$T : H^q(CX, X; G) \to H^r(CX, X; G')$$

is linear (even though $T$ may not be linear for more general spaces). Let $(SX, x_0)$ be the space obtained from $(CX, X)$ by collapsing $X$ to a point $x_0$; and let $f : (CX, X) \to (SX, x_0)$ be the collapsing map. Then $SX$ is the suspension of $X$. For all coefficient groups,

$$f^* : H^n(SX, x_0) \approx H^n(CX, X).$$

Therefore it suffices to prove that

$$T : H^q(SX, x_0; G) \to H^r(SX, x_0; G')$$

is linear. Again, the inclusion map of $SX$ in $(SX, x_0)$ induces isomorphisms in all dimensions $>0$; hence it suffices to prove that

$$T : H^q(SX; G) \to H^r(SX; G')$$

is linear. Now $SX$ has topological category 2, i.e., it can be decomposed into two sets each contractible in $SX$. So it suffices to prove that any $T$ is linear when restricted to the cohomology of a space of category 2. This is a special case of the following proposition:

7.5. *If $X$ is a space of category $k + 1$, then each operation $T$ of one variable, when restricted to the cohomology of $X$, is a $k^{\text{th}}$ power operation.*

Referring to §6, it suffices to prove that the associated $T_{k+1}$, when restricted to the cohomology of $X$, is zero. Let $Y$ be a space of type $K(G, q)$. In $Y^{k+1}$, let $W$ be the union of the $k$-fold subproducts obtained by requiring at least one coordinate to be the base point $y_0 \in Y$. As shown in §6, $T_{k+1}$ corresponds to an element $z \in H^r(Y^{k+1}; G')$ which belongs to the kernel of the homomorphism

$$g^* : H^r(Y^{k+1}; G') \to H^r(W; G')$$

induced by the inclusion. If $u_1, \cdots, u_{k+1} \in H^q(X; G)$, then 3.3 gives

$$T_{k+1}(u_1, \cdots, u_{k+1}) = f^*z$$

for some map $f : X \to Y^{k+1}$. We can suppose the components $(f_1, \cdots, f_{k+1})$ of $f$ have the property $f_j(x_0) = y_0$ where $x_0$ is a selected base point in $X$. Since $X$ has category $k + 1$, it is the union of closed sets $X_j$ ($j = 1, \cdots, k + 1$) such that, for each $j$, there is a homotopy $F_j : X \times I \to X$ satisfying

$$F_j(x, 0) = x \quad \text{for} \quad x \in X, \qquad F_j(x, 1) = x_0 \quad \text{for} \quad x \in X_j.$$

Define a homotopy of $f$ by

$$f_j(x, t) = f_j(F_j(x, t)) \qquad j = 1, \cdots, k + 1.$$

Then $f(x, 0) = f(x)$. When $t = 1$ and $x \in X$, we have $x \in X_j$ for some $j$; and therefore $f(x, 1)$ lies in $W$. Let $f'(x) = f(x, 1)$. Then $f'$ can be factored $f' = gh$ where $h : X \to W$. Thus

$$f^*z = f'^*z = h^*g^*z = 0.$$

Remarks. The term "suspension" is used since it corresponds, under the isomorphism 3.2, to the suspension homomorphism

$$\sigma : H^r(G, q; G') \to H^{r-1}(G, q - 1; G')$$

studied in the theory of the Eilenberg-MacLane complexes. The linearity of a suspension (see 7.4) corresponds to the property that an image element of $\sigma$ is always primitive (see 6.7). It is known that not every primitive element is a suspension (see John C. Moore, *On the homology of* $K(\pi, n)$, Proc. Nat. Acad. Sci. U.S.A., 43 (1957), 409–411).

The proposition 7.5 has a generalization to arbitrary operations which we will describe briefly. The decomposition of the cohomology of $X_1 \times X_2$ given in (6.3) generalizes to products of $k$ factors

$$H^*(X_1 \times \cdots \times X_k) = A_0 + \textstyle\sum_{i=1}^{k} A_i + \sum_{i<j} A_{ij} + \cdots + A_{12\cdots k}.$$

If this is applied to a product of Eilenberg-MacLane complexes, it follows that any operation $T$ of $k$ variables can be decomposed into a sum of operations $T = \sum T_j + T'$ where $T'$ is an operation on $k$ variables, each $T_j$ is an operation on $<k$ variables, and $T'$, $T_j$ are *normal* operations in the sense that they are zero when any one of their respective variables is zero. The argument of 7.5 shows that, *in a space of category k, any normal operation of k variables is identically zero*. This generalizes the known fact that, in a space of category $k$, the cup product of $k$ cohomology classes of positive dimension is always zero.

## 8. A likely basis for cohomology operations

In view of the reduction theorem of §5 a set of cohomology operations which generate all operations of one variable with coefficient groups in $C$ must necessarily be a basis for all operations. A detailed study of the $K(\pi, n)$-spaces with $\pi \in C$ should lead to a solution. This is certainly feasible; and already it has been pushed quite far by Cartan [6]. Generalizations of the results 4.11 and 4.12 are needed in which the terminal coefficient group $Z_p$ is replaced by $Z$ and also by $Z_{p^k}$.

It is to be emphasized that the explicit cohomology operations $\mathrm{Sq}^i$, $\mathfrak{P}_2$, $\mathscr{P}^i_p$, $\mathfrak{P}_p$ were discovered and studied using methods apparently unrelated to the $K(\pi, n)$-spaces. Recently I have developed a single method which can be used to define all of these operations [14]. In fact, the method led to the discovery of $\mathfrak{P}_p$ by Thomas. Briefly the method involves the study of the $k^{\text{th}}$ power $X^k$ of a complex (the product of $k$ copies of $X$), the action of the symmetric group of permutations of the factors, chain transformations approximating the diagonal map $X \to X^k$, and measurements of the impossibility of finding symmetric approximations.

This new method provides many more cohomology operations of one variable than the ones mentioned above. However it gives no essentially new ones. All such are generated by addition, cup-products, coefficient homomorphisms, Bockstein-Whitney coboundaries, $\mathrm{Sq}^i$, $\mathfrak{P}_2$ and the $\mathscr{P}^i_p$, $\mathfrak{P}_p$ for odd primes $p$. This has been proved by Thomas and myself by means of an elaborate analysis of the method [14]. This result in itself gives hope that the named operations form a basis.

The hope is converted into an expectation by a remarkable result proved recently by Dold and Thom [7]. If $X$ is any complex, let $Y_k$ denote its $k$-fold symmetric product, i.e. $Y_k$ is the decomposition space of $X^k$ under the action of the symmetric group. The selection of a base point in $X$ determines imbeddings $Y_k \subset Y_{k+1}$ for each $k$. Then $Y = \bigcup_{k=1}^{\infty} Y_k$ is called the *infinite symmetric product of* $X$. The Dold-Thom result asserts:

$$H_i(X) \approx \pi_i(Y) \qquad \text{for} \qquad i > 0.$$

This result applies to give explicit constructions of $K(\pi, n)$-spaces. It is easy to construct a space $X$ whose only non-zero homology group $H_n(X) = \pi$; then its corresponding $Y$ is a $K(\pi, n)$-space. This shows exactly how the symmetric groups enter into the structure of the $K(\pi, n)$-spaces, and brings their construction into close conjunction with the method I have used to construct cohomology operations. Not much more should be needed to effect a coalescing of the methods.

If, as expected, the operations listed above do form a basis, remarks are in order concerning the second problem stated in §2, that is, finding a basis for the relations they satisfy. If the $\mathfrak{P}_p$ are omitted from the list for all primes $p$, then we do have a complete set of relations on the others. I am referring now to the relations found by Adem [1, 2, 3] and by Cartan [5] involving iterations of the $\mathrm{Sq}^i$ and $\mathscr{P}_p^i$. The completeness of the relations is proved by the Theorems 4.11 and 4.12.

Thomas has found a system of relations involving the $\mathfrak{P}_p$. It is not likely that this is a complete system since there are no known relations as yet concerning compositions of $\mathscr{P}_p^i$ and $\mathfrak{P}_p$. The test of adequacy is to give generalizations of 4.12 wherein $Z_p$ is replaced by $Z_{p^k}$ and by $Z$.

## 9. Homology operations

By analogy with §2, a homology operation of one variable is a *natural* transformation of one homology functor into another: $T : H_q(\ \ ; G) \to H_r(\ \ ; G')$. If $H_q(\ \ ; G)$ is replaced by a cartesian product of homology functors, we obtain the notion of a homology operation of many variables. Examples of these are also plentiful: addition, coefficient homomorphisms, and the Bockstein-Whitney boundary operator of a coefficient sequence.

Less trivial examples are obtained by dualizing the $\mathrm{Sq}^i$ and $\mathscr{P}_p^i$. Here we take advantage of the fact that $Z_2$ and $Z_p$ are fields. Because of this we have the well-known natural equivalence of functors

$$H_q(\ \ ; Z_p) \approx \mathrm{Hom}(H^q(\ \ ; Z_p), Z_p)$$

given by the Kronecker index of a cycle and cocycle mod $p$. Since

$$\mathscr{P}_p^i : H^q(X; Z_p) \to H^{q+2i(p-1)}(X; Z_p)$$

is a homomorphism, it induces a homomorphism of the dual groups

$$Q_p^i : H_{q+2i(p-1)}(X; Z_p) \to H_q(X, Z_p).$$

The naturality of $\mathscr{P}$ under mappings $X \to Y$ implies that of $Q$. Notice that this method cannot be applied to $\mathfrak{P}_p$ since it is not homomorphic.

We proceed now to study the structure of a homology operation $T$ of $k$ variables $(z_1, \cdots, z_k)$. So as not to encumber the discussion with an awkward special case, we will suppose that the dimensions $q_1, \cdots, q_k$ of $z_1, \cdots, z_k$ and the dimension $r$ of

$T(z_1, \cdots, z_k)$ are positive. If all the variables are replaced by zero excepting $z_j$, we obtain a homology operation of one variable which we denote by $T_j$. Then we have the reduction:

$$(9.1) \qquad T(z_1, \cdots, z_k) = \textstyle\sum_{j=1}^{k} T_j(z_j).$$

To prove this, let $X_1, \cdots, X_k$ be copies of a space $X$, and let $f_j : X_j \to X$ be a homeomorphism. Let $X'$ be the disjoint union of $X_1, \cdots, X_k$, and let $f : X' \to X$ be defined by $f \mid X_j = f_j$. Let $g_j$ be the inclusion map $X_j \subset X$. Choose a base point $x_j \in X_j$, and let $h_j : X' \to X_j$ be the retraction defined by $h_j(x') = x_j$ if $x' \in X_i$ with $i \neq j$. Choose $z'_j \in H_{q_j}(X_j; G_j)$ such that

$$f_{j*} z'_j = z_j \in H_{q_j}(X; G_j), \qquad j = 1, \cdots, k.$$

Let $z''_j = g_{j*} z'_j$; then $f_* z''_j = z_j$. Clearly

$$h_{j*} z''_j = z'_j, \quad \text{and} \quad h_{j*} z''_i = 0 \quad \text{if} \quad i \neq j.$$

Now the homology of $X'$ splits into a direct sum of the homologies of the components $X_j$; and therefore

$$\begin{aligned} T(z''_1, \cdots, z''_k) &= \textstyle\sum_{j=1}^{k} g_{j*} h_{j*} T(z''_1, \cdots . z''_k) \\ &= \textstyle\sum g_{j*} T(h_{j*} z''_1, \cdots, h_{j*} z''_k) \\ &= \textstyle\sum g_{j*} T(0, \cdots, 0, z'_j, 0, \cdots, 0) \\ &= \textstyle\sum g_{j*} T_j(z'_j) = \sum T_j(z''_j). \end{aligned}$$

This proves (9.1) in $X'$. Applying $f_*$ gives the required relation in $X$.

The above proof can be paraphrased by saying that the variables of $T$ can be separated in such a way that cross effects must be zero. It is clear that (9.1) reduces the problem of finding a base for homology operations to the case of operations of one variable. It has an additional consequence.

9.2. *Each homology operation of one variable is linear*: $T(y + z) = T(y) + T(z)$.

Let $T_2(x, y)$ be its deviation from additivity. By (9.1), $T_2(x, y) = T_2(x, 0) + T_2(0, y)$. If we put $y = 0$ in the definition of $T_2(x, y)$ we obtain $T_2(x, 0) = T(0)$. Similarly $T_2(0, y) = T(0)$. It remains to prove that $T(0) = 0$ for any space $X$ and any $T$. It is certainly true if $X$ is a single point; because only positive dimensions are considered. If we map a single point space into $X$, and apply the naturality of $T$, we obtain the result in general.

Let us restrict ourselves henceforth to finitely generated coefficient groups.

9.3. *Each homology operation of one variable is generated by the operations of addition, coefficient homomorphisms, and operations of one variable with coefficient groups in $C$.*

Exactly as in the proof of 5.1, we can reduce to the case where the terminal coefficient group of $T$ is in $C$. Let $G$ be the initial coefficient group, and let $G = \sum G_j$ be a direct sum splitting where $G_j \in C$. Then we have homomorphisms $f_j : G_j \to G$

and $g_j : G \to G_j$ such that $g_j f_j = 1$, and $\sum f_j g_j = 1$. If $z \in H_q(X; G)$, we have by (9.1) that

$$T(z) = T(\sum f_{j*} g_{j*} z) = \sum T(f_{j*} g_{j*} z).$$

If $y \in H_q(X; G_j)$, and we define $T_j(y)$ by $T_j(y) = T(f_{j*} y)$, then $T_j$ is a homology operation with coefficient groups in $C$. Since $T(z) = \sum T_j(g_{j*} z)$, the proof is complete.

9.4. *A homology operation of one variable which increases dimension is identically zero.*

If $T : H_q \to H_r$ where $q < r$, then $T(z) = 0$ is obvious whenever $H_r = 0$. If $X^q$ is the $q$-skeleton of $X$, we always have $H_r(X^q) = 0$. But any $z \in H_q(X)$ is an image of a $z' \in H_q(X^q)$. Then $T(z') = 0$ and the naturality of $T$ imply $T(z) = 0$.

9.5. *Each homology operation of one variable is the dual of a linear cohomology operation.*

Suppose $T : H_q( \;\; ; G) \to H_r( \;\; ; G')$ is a homology operation. Let ch $G$ be the character group of $G$ treated as a discrete group. For any complex $X$ there is a natural isomorphism.

$$H^q(X; \text{ch } G) \approx \text{ch } H_q(X: G), \tag{9.6}$$

and since $T$ is a homomorphism (see (9.1)), it has a *dual* homomorphism of the character groups

$$T' : H^r(X; \text{ch } G') \to H^q(X; \text{ch } G).$$

The naturality of $T$ and the isomorphism (9.6) imply the naturality of $T'$. So $T'$ is a cohomology operation.

REMARKS. It is to be noted that ch $G$ is a compact group, and so also $H^q(X; \text{ch } G)$. Furthermore $T'$ is continuous. Our discussion of cohomology operations has been limited to those with discrete coefficient groups which are finitely generated. Therefore an analysis of linear cohomology operations of that type would not provide an analysis of homology operations. In case $G$, $G'$ are finite groups, the same is true of ch $G$ and ch $G'$; and then $T'$ is of the type considered earlier. To obtain a satisfactory reduction to cohomology operations, something more needs to be done in the cases where $G$, $G'$ are cyclic and one or both are infinite cyclic.

It should be noted that, in the subcategory of $H$-spaces and $H$-mappings, there are homology operations which do not reduce to sums of linear operations of one variable. The Pontrjagin multiplication provides a non-trivial ring structure in the homology of an $H$-space.

## 10. Other operations

The definitions of cohomology and homology operations, used in this article, do not cover all of the standard operations of importance in algebraic topology. As an example, the *cap* product which mixes homology and cohomology is such an operation. Explicitly, if $G_1$, $G_2$ are coefficient groups, and $G = \text{Hom}(G_1, G_2)$, then the cap-product, from a functorial point of view, is a natural transformation

$$H_r(X; G) \to \text{Hom}(H^q(X; G_1), H_{r-q}(X; G_2)).$$

The cohomology cross-product is another example. It is a natural transformation

$$H^p(X\,;G) \otimes H^q(Y\,;G') \to H^{p+q}(X \times Y\,;G \otimes G').$$

The *homology* cross-product is similar. The cup-product in the relative case for triads $(X\,;A,B)$ is a natural transformation

$$H^p(X,A\,;G) \otimes H^q(X,B\,;G') \to H^{p+q}(X,A \cup B\,;G \otimes G').$$

Another example involves the torsion product and the natural transformation appearing in the Künneth theorem for a product space. Again, if $SX$ denotes the suspension of $X$, then $S$ is a functor, and we have a natural equivalence

$$H_q(X) \approx H_{q+1}(SX), \qquad q > 0.$$

In each case the functors appearing on the left and right are constructed by composing functors of three types. The first type consists of geometric functors from spaces to spaces, e.g., the product of two spaces, the suspension, the loop space, the passage from a triad $(X\,;A,B)$ to the pair $(X; A \cup B)$. The second type consists of the homology and cohomology functors from spaces to groups with various coefficients. The third type consists of functors from groups to groups, e.g., tensor, hom, tor, ext. For the lack of a better name, let us refer momentarily to such a composition as a *homology* functor.

If $H$ and $H'$ are two homology functors having the same domain and range, then a *homology* operation would be a natural transformation $T : H \to H'$ (in the sense of arbitrary functions from one group to another). For fixed $H,H'$, the problem exists of determining all such $T$'s. The purpose of this section is to call attention to the class of these problems. Since the very special case treated in this paper is of considerable importance to the progress of algebraic topology, it is reasonable to expect that other cases will be fruitful.

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# OPERACIONES COHOMOLÓGICAS DE SEGUNDO ORDEN ASOCIADAS CON CUADRADOS DE STEENROD

JOSÉ ADEM[1]

## 1. Introducción

Actualmente se cuenta con un estudio casi completo sobre operaciones cohomológicas de primer orden (ver [11]). En contraste, se conoce muy poco sobre operaciones de orden superior.[2]

Un ejemplo de operación cohomológica de segundo orden es el triple producto de Massey ([6]). Se construcción se basa en la asociatividad del producto (cup-product) del anillo de cohomología.[3]

Otro ejemplo de operación cohomológica de segundo orden es la operación $\Phi_2^3$ (usando la notación de este trabajo) introducida, por Shimada (para dimensión 2) y por el autor (para dimensión arbitraria) para calcular la "tercera obstrucción" al transformar un complejo en una $n$-esfera ([1], [8]). La construcción de $\Phi_2^3$ se basa en la relación $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^3\mathrm{Sq}^1 \bmod 2$.

Posteriormente, Shimada y Uehara ([9]), modificando la construcción de $\Phi_2^3$, definieron una nueva operación de segundo orden ($\Theta_2^3$ en nuestra notación), con el fin de calcular la "tercera obstrucción" al transformar un complejo en un espacio $(n-1)$-conexo. Su construcción se basa en la relación $2\mathrm{Sq}^2\mathrm{Sq}^2 = 0 \bmod 4$.

En este trabajo, generalizando $\Phi_2^3$ y $\Theta_2^3$, se construyen dos familias de operaciones cohomológicas de segundo orden:

$$\Phi_{2a}^i \quad \text{con} \quad a = 1, 2, \cdots; \qquad i = 3a, 3a+1, \cdots,$$

$$\Theta_{2a}^{2i+1} \quad \text{con} \quad a = 1, 2, \cdots; \qquad i = a + [(a+2)/2] - 1, \cdots.$$

La construcción de $\Phi_{2a}^i$ se basa en el desarrollo mod 2 de $\mathrm{Sq}^{2a}\mathrm{Sq}^b$ con $a < b$. La construcción de $\Theta_{2a}^{2i+1}$ utiliza el desarrollo mod 4 de $2\mathrm{Sq}^{2a}\mathrm{Sq}^{2b}$ con $a < 2b$.

Se demuestra la invariancia topológica de las nuevas operaciones y se establecen algunas de sus propiedades. En las dimensiones inferiores las operaciones no son necesariamente aditivas; se determina la desviación. Se obtienen relaciones con cuadrados funcionales, lo que permite calcularlas en un gran número de casos. Por último, en la cohomología relativa se obtienen las relaciones con el operador cofrontera invariante.

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[2] Un estudio general sobre operaciones de orden superior, utilizando sistemas de Postnikov, ha sido iniciado por Peterson ([7]).

[3] Recientemente, el triple producto ha sido generalizado, en forma independiente, por Hirsch ([4]) y Massey ([5]), obteniendo ejemplos de operaciones de diferentes órdenes. Estas operaciones tienen aplicaciones importantes en los espacios fibrados.

## 2. Los grupos que se utilizan

Sea $S_4$ el grupo simétrico de grado cuatro, representado como grupo de transformaciones del conjunto de permutaciones del renglón (1, 2, 3, 4). Obviamente, todo elemento de $S_4$ queda determinado mediante su operación en un renglón particular. Sean $x$, $x_1$, $x_2$ los elementos de $S_4$ tales que

$$x(1, 2, 3, 4) = (3, 4, 1, 2),$$

$$x_1(1, 2, 3, 4) = (2, 1, 3, 4),$$

$$x_2(1, 2, 3, 4) = (1, 2, 4, 3).$$

Sean $\pi$, $\pi_1$, $\pi_2$ los subgrupos cíclicos de orden dos de $S_4$ generados, respectivamente, por $x$, $x_1$, $x_2$.

Como subgrupos de $S_4$, el producto

$$\rho = \pi_1\pi_2 = \pi_2\pi_1,$$

luego $\rho$ es un grupo. Igualmente, el producto

$$G = \rho\pi = \pi\rho,$$

es un grupo. El orden de $G$ es $2^3$, por lo tanto, $G$ es un 2-grupo de Sylow de $S_4$.

Consideremos los elementos $y$, $z$ de $S_4$ definidos por

$$y(1, 2, 3, 4) = (2, 1, 4, 3),$$

$$z(1, 2, 3, 4) = (1, 3, 2, 4).$$

Claramente, $y = x_1x_2$ y resulta

$$xy = yx.$$

El elemento $z$ es de orden dos, luego $z = z^{-1}$. Además, se tiene que

$$x = zyz, \qquad y = zxz.$$

Sea

$$\sigma : S_4 \to S_4$$

el automorfismo interior inducido por $z$, ésto es, $\sigma(\alpha) = z\alpha z^{-1} = z\alpha z$, para $\alpha \in S_4$. Luego, resulta que

$$\sigma(x) = y, \qquad \sigma(y) = x.$$

Definimos $G_z$ como el subconjunto de $G$ invariante bajo $\sigma$. Esto es,

$$G_z = G \cap (zGz).$$

Obviamente, $G_z$ resulta ser el grupo generado por $x$, $y$. Luego, si $\pi'$ representa el subgrupo cíclico generado por $y$, se tiene que

$$G_z = \pi\pi'.$$

La restricción de $\sigma$ a $G_z$, denotada por $\tau = \sigma|G_z$, es el isomorfismo

$$\tau : G_z \to G_z$$

que intercambia $x$ con $y$.

Finalmente, si

$$\xi : G_z \to G,$$
$$\theta : G \to S_4,$$

representan las inclusiones respectivas, se tiene el siguiente diagrama conmutativo de grupos y homomorfismos:

$$\begin{array}{ccccc} G_z & \xrightarrow{\xi} & G & \xrightarrow{\theta} & S_4 \\ \downarrow \tau & & & & \downarrow \sigma \\ G_z & \xrightarrow{\xi} & G & \xrightarrow{\theta} & S_4 \end{array} . \qquad (2.1)$$

## 3. Complejos para los grupos

Si $\pi$ es un grupo cíclico de orden dos y generador $x$, representamos con $V$ el complejo canónico $\pi$-libre y acíclico definido en [13]. Esto es, para cada dimensión $i$ ($i \geqq 0$) $V$ tiene como $\pi$-base un solo elemento $e_i$ y el operador frontera está definido por

$$\partial e_{2i+1} = \Delta e_{2i}, \qquad \partial e_{2i+2} = \Sigma e_{2i+1},$$

donde,

$$\Delta = x - 1, \qquad \Sigma = x + 1,$$

son elementos de $Z(\pi)$, el anillo de grupo de $\pi$ sobre los enteros.

El producto tensorial

$$V \otimes V^2 = V \otimes (V \otimes V)$$

se transforma en un $G$-complejo, acíclico y $G$-libre al definir operaciones de $G$ en $V \otimes V^2$ como sigue. En los elementos $e_i \otimes e_j \otimes e_k$, los generadores $x$, $x_1$, $x_2$ de $G$ operan del modo siguiente:

$$x(e_i \otimes e_j \otimes e_k) = (-1)^{jk}(xe_i) \otimes e_k \otimes e_j,$$
$$x_1(e_i \otimes e_j \otimes e_k) = e_i \otimes (xe_j) \otimes e_k,$$
$$x_2(e_i \otimes e_j \otimes e_k) = e_i \otimes e_j \otimes (xe_k).$$

La forma en que operan los generadores es consistente con las relaciones de $G$ y, por lo tanto, las operaciones se extienden a un elemento cualquiera de $G$ y de $V \otimes V^2$.

El complejo para $G_z$ puede ser $V \otimes V$ (cf. [2; p. 220]). Sin embargo, con el fin de obtener una expresión más simple para la transformación de cadena $\xi_{\#}$: $V \otimes V \to V \otimes V^2$ inducida por $\xi$: $G_z \to G$, necesitamos modificar $V \otimes V$ mediante cierta "aumentación". Para evitar posibles confusiones usaremos símbolos diferentes para denotar las celdas de este complejo aumentado.

Así, sea $U$ un complejo tal que para cada dimensión $i$ ($i \geqq -1$) tiene como $\pi$-base un solo elemento $d_i$, y el operador frontera definido análogamente como en $V$. En $U$ se tiene $\partial d_0 = \Sigma\, d_{-1}$, $\partial d_{-1} = 0$. El producto $U \otimes U$ tiene elementos distintos de cero en dimensiones $\geqq -2$.

Por construcción $\pi$ opera libremente en $U$ y $U \otimes U$ resulta $G_z$-libre al definir las operaciones para los generadores de $G_z$, como sigue:

$$x(d_i \otimes d_j) = (xd_i) \otimes d_j,$$

$$y(d_i \otimes d_j) = d_i \otimes (xd_j).$$

Sea $W$ un $S_4$-complejo, acíclico y $S_4$-libre. De (2.1) se obtiene el siguiente diagrama

$$\begin{array}{ccccc} U \otimes U & \xrightarrow{\xi_\#} & V \otimes V^2 & \xrightarrow{\theta_\#} & W \\ \downarrow \tau_\# & & & & \downarrow \sigma_\# \\ U \otimes U & \xrightarrow{\xi_\#} & V \otimes V^2 & \xrightarrow{\theta_\#} & W \end{array} \tag{3.1}$$

donde, $\xi_\#$, $\theta_\#$, $\sigma_\#$, $\tau_\#$ son transformaciones de cadena equivariantes, respectivamente, con relación a los homomorfismos $\xi$, $\theta$, $\sigma$, $\tau$.

Como $U$ contiene elementos de dimensión $-1$, la transformación $\xi_\#$ es especial. En efecto, $\xi_\#(d_0d_0)$ se considera con índice de Kronecker igual a uno, mientras que $\xi_\#(d_{-1}d_1)$ y $\xi_\#(d_1d_{-1})$ se construyen con índice de Kronecker igual a cero. En general, ésto último es necesario para poder definir $\xi_\#(d_0d_1)$ y $\xi_\#(d_1d_0)$. Una expresión concreta para $\xi_\#$ se construirá en la sección siguiente.

Consideremos el homomorfismo

$$\nu : G_z \to S_4,$$

donde $\nu = \sigma\theta\xi = \theta\xi\tau$ (cf. (2.1)). Claramente,

$$\nu(x) = y, \qquad \nu(y) = x.$$

Las composiciones $\sigma_\#\theta_\#\xi_\#$ y $\theta_\#\xi_\#\tau_\#$ definen dos $\nu$-transformaciones de cadena de $U \otimes U$ en $W$. Ahora, como $U \otimes U$ es $G_z$-libre y $W$ es acíclico, se puede construir una homotopía $\nu$-equivariante

$$\Omega : U \otimes U \to W,$$

tal que,

$$\partial\Omega + \Omega\partial = \sigma_\#\theta_\#\xi_\# - \theta_\#\xi_\#\tau_\#. \tag{3.2}$$

Obviamente, si $w \in W$,

$$\sigma_\#(w) = zw.$$

Además,

$$\tau_\#(d_i \otimes d_j) = (-1)^{ij} d_j \otimes d_i.$$

Luego, para los elementos de la $G_z$-base resulta,

$$\partial\Omega(d_i \otimes d_j) + \Omega\partial(d_i \otimes d_j) = z\theta_\#\xi_\#(d_i \otimes d_j) - (-1)^{ij}\theta_\#\xi_\#(d_j \otimes d_i). \tag{3.3}$$

## 4. La construcción de $\xi_\#$

Por brevedad en la escritura eliminaremos, en general, el símbolo $\otimes$ al escribir elementos de un producto tensorial. Usaremos un exponente para indicar el producto tensorial de un elemento consigo mismo cierto número de veces. Así, en lugar de $e_k \otimes e_j \otimes e_j$ escribiremot $e_k e_j^2$.

Primero, construiremos una transformación auxiliar,

$$\eta : U \otimes U \to V \otimes V^2,$$

en términos de la cual definiremos $\xi_\#$. Definamos $\eta$ en los elementos de la $G_z$-base de $U \otimes U$ como sigue:

$$(4.1) \quad \eta(d_k d_{4j}) = (-1)^{k+1}(xe_{k+1})(\partial e_{2j})e_{2j} + e_k e_{2j}^2,$$

$$(4.2) \quad \eta(d_k d_{4j+1}) = (-1)^k(xe_{k+1})e_{2j}\partial e_{2j+1} + e_k(\partial e_{2j+1})e_{2j+1} + (-1)^k(xe_{k-1})e_{2j+1}^2,$$

$$(4.3) \quad \eta(d_k d_{4j+2}) = (-1)^k(xe_{k+1})e_{2j+1}\partial e_{2j+1} - e_k e_{2j+1} x e_{2j+1},$$

$$(4.4) \quad \eta(d_k d_{4j+3}) = (-1)^{k+1}(xe_{k+1})(\partial e_{2j+2})e_{2j+1}.$$

En forma equivariante la transformación $\eta$ queda definida en un elemento arbitrario de $U \otimes U$. Un cálculo directo permite verificar las relaciones siguientes:

$$(4.5) \quad \partial\eta(d_k d_{4j}) = \eta\partial(d_k d_{4j}),$$

$$(4.6) \quad \partial\eta(d_k d_{4j+1}) = \eta\partial(d_k d_{4j+1}),$$

$$(4.7) \quad \partial\eta(d_k d_{4j+2}) = \eta\partial(d_k d_{4j+2}) + (-1)^k\eta(d_{k-2}\partial d_{4j+4}),$$

$$(4.8) \quad \partial\eta(d_k d_{4j+3}) = \eta\partial(d_k d_{4j+3}).$$

La transformación $\xi_\#$ se define en la $G_z$-base de $U \otimes U$ como sigue:

$$(4.9) \quad \begin{aligned} \xi_\#(d_i d_{2j}) &= (-1)^j \textstyle\sum_{k=0}^{j} \binom{k}{j-k} \eta(d_{i+2j-4k} d_{4k}) \\ &\quad + (-1)^{j+1} \textstyle\sum_{k=0}^{j-1} \binom{k}{j-k-1} \eta(d_{i+2j-4k-2} d_{4k+2}), \end{aligned}$$

$$(4.10) \quad \begin{aligned} \xi_\#(d_i d_{2j+1}) &= (-1)^j \textstyle\sum_{k=0}^{j} \binom{k}{j-k} \eta(d_{i+2j-4k} d_{4k+1}) \\ &\quad + (-1)^{j+1} \textstyle\sum_{k=0}^{j-1} \binom{k}{j-k-1} \eta(d_{i+2j-4k-2} d_{4k+3}). \end{aligned}$$

Con las convenciones usuales, los símbolos $\binom{m}{n}$ representan coeficientes binomiales (cf. [2; p. 233]). Usando las relaciones (4.5–8), se comprueba fácilmente que $\partial\xi_\# = \xi_\#\partial$. Se tiene que

$$\xi_\#(d_0 d_0) = e_0 e_0^2,$$

además $\quad \xi_\#(d_1 d_{-1}) = 0, \quad$ y $\quad \xi_\#(d_{-1} d_1) = -(xe_0)e_0\partial e_1$

es una 0-cadena con índice de Kronecker igual a cero.

## 5. Simplificaciones en la notacion

De aquí en adelante, salvo cuando se indique lo contrario, todas las expresiones se considerarán módulo 2. En general, omitiremos el símbolo # en las transformaciones de cadena, así, escribiremos $\xi$ y $\theta$ en lugar de $\xi_\#$ y $\theta_\#$.

Por comodidad introduciremos la notación siguiente:

(5.1) $$x_t^k = \eta(d_{t-k}d_k),$$

(5.2) $$y_t^k = \theta x_t^k$$

(5.3) $$\bar{y}_t^k = z y_t^k,$$

(5.4) $$\omega_t^k = \Omega(d_{t-k}d_k),$$

(5.5) $$Y_t^k = z\theta\xi(d_{t-k}d_k) + \theta\xi(d_k d_{t-k}).$$

Sean

$$\Sigma_1 = x + 1, \qquad \Delta_1 = x - 1,$$

$$\Sigma_2 = y + 1, \qquad \Delta_2 = y - 1,$$

donde $x$, $y$ son los elementos de $S_4$ definidos en §2. Módulo 2, obviamente,

$$\Sigma_1 = \Delta_1, \qquad \Sigma_2 = \Delta_2.$$

Ahora, puesto que $\Omega$ es $\nu$-equivariante, módulo 2 se tiene que

$$\Omega\partial(d_{t-k}d_k) = \Sigma_2\Omega(d_{t-k-1}d_k) + \Sigma_1\Omega(d_{t-k}d_{k-1}).$$

Por lo tanto, con la nueva notación, (3.3) módulo 2 resulta:

(5.6) $$Y_t^k + \partial\omega^k = \Sigma_2\omega_{t-1}^k + \Sigma_1\omega_{t-1}^{k-1}.$$

Usando (4.9) y (4.10) podemos escribir $Y_t^k$ explícitamente. Primero, definimos

(5.7) $$a_s^r = \binom{s}{r-s}.$$

Claramente, $a_s^r = 0$ si $s > r$. Luego, considerando un signo de suma común se obtienen las siguientes expresiones (mod 2):

(5.8) $$Y_{2t}^{2k} = \textstyle\sum_{j\geqq 0}[a_j^k\bar{y}_{2t}^{4j} + a_j^{k-1}\bar{y}_{2t}^{4j+2} + a_j^{t-k}y_{2t}^{4j} + a_j^{t-k-1}y_{2t}^{4j+2}],$$

(5.9) $$Y_{2t+1}^{2k} = \textstyle\sum_{j\geqq 0}[a_j^k\bar{y}_{2t+1}^{4j} + a_j^{k-1}\bar{y}_{2t+1}^{4j+2} + a_j^{t-k}y_{2t+1}^{4j+1} + a_j^{t-k-1}y_{2t+1}^{4j+3}],$$

(5.10) $$Y_{2t}^{2k+1} = \textstyle\sum_{j\geqq 0}[a_j^k\bar{y}_{2t}^{4j+1} + a_j^{k-1}\bar{y}_{2t}^{4j+3} + a_j^{t-k-1}y_{2t}^{4j+1} + a_j^{t-k-2}y_{2t}^{4j+3}],$$

(5.11) $$Y_{2t+1}^{2k+1} = \textstyle\sum_{j\geqq 0}[a_j^k\bar{y}_{2t+1}^{4j+1} + a_j^{k-1}\bar{y}_{2t+1}^{4j+3} + a_j^{t-k}y_{2t+1}^{4j} + a_j^{t-k-1}y_{2t+1}^{4j+2}].$$

Como se verá más adelante, estas expresiones combinadas con (5.6) producirán las relaciones entre los cuadrados iterados de Steenrod. Las expresiones (5.9) y (5.11) son esencialmente iguales, ya que $Y_{2t+1}^{2k+1} = zY_{2t+1}^{2t-2k}$.

## 6. Ciertas relaciones en $W \otimes_{S_4} K^{*4}$

Sea $K$ un complejo regular de celdas y sea $K^*$ su complejo de cocadenas enteras. El grupo $S_4$ opera en $K^{*4} = K^* \otimes K^* \otimes K^* \otimes K^*$ permutando los factores de los productos, de modo que $K^{*4}$ resulta un $S_4$-complejo.

Sea $W$ el complejo $S_4$-libre y acíclico de §3. Primero, formemos el complejo de cocadenas $W \otimes K^{*4}$, introducido por Steenrod en [12; p. 197], con el operador cofrontera definido por

$$\delta(w \otimes a) = \partial w \otimes a + (-1)^i w \otimes \delta a, \tag{6.1}$$

donde $w \in W$, $a \in K^{*4}$, $i = \dim w$. Luego, formamos $W \otimes_{S_4} K^{*4}$ al considerar las relaciones,

$$\alpha(w \otimes a) = \alpha w \otimes \alpha a = w \otimes a, \tag{6.2}$$

para toda $\alpha \in S_4$.

Ahora, si $u \in K^*$ y $u^4 = u \otimes u \otimes u \otimes u$, es fácil comprobar, usando (6.2), que se tienen en $W \otimes_{S_4} K^{*4}$ las siguientes relaciones (mod 2):

$$\bar{y}_t^j \otimes u^4 = y_t^j \otimes u^4, \tag{6.3}$$

$$y_t^{4j} \otimes u^4 = \theta(e_{t-4j}e_{2j}^2) \otimes u^4, \tag{6.4}$$

$$y_t^{4j+1} \otimes u^4 = y_t^{4j+2} \otimes u^4 = \theta(e_{t-4j-2}e_{2j+1}^2) \otimes u^4, \tag{6.5}$$

$$y_t^{4j+3} \otimes u^4 = 0. \tag{6.6}$$

Igualmente, si $V \otimes V^2$ es el $G$-complejo definido en §3, se obtienen en $(V \otimes V^2) \otimes_G K^{*4}$ relaciones análogas. Así, $x_t^{4j} \otimes u^4 = e_{t-4j}e_{2j}^2 \otimes u^4$, $x_t^{4j+1} \otimes u^4 = x_t^{4j+2} \otimes u^4 = e_{t-4j-2}e_{2j+1}^2 \otimes u^4$, $x_t^{4j+3} \otimes u^4 = 0$.

En $W \otimes_{S_4} K^{*4}$ consideremos $Y_t^k \otimes u^4$ (mod 2). Usando (6.3–6) junto con (5.8), (5.10) y (5.11), se demuestra fácilmente que

$$\begin{aligned} Y_{2t+1}^{2k+1} \otimes u^4 = \textstyle\sum_{j \geq 0} \Bigg[ \binom{j}{k-j} y_{2t+1}^{4j+2} + \binom{j}{2j+k-t} y_{2t+1}^{4j} \\ + \binom{j}{2j+k+1-t} y_{2t+1}^{4j+2} \Bigg] \otimes u^4, \end{aligned} \tag{6.7}$$

$$\begin{aligned} (Y_{2t}^{2k} + Y_{2t}^{2k-1}) \otimes u^4 = \textstyle\sum_{j \geq 0} \Bigg[ \binom{j}{k-j} y_{2t}^{4j} + \binom{j}{2j+k-t} y_{2t}^{4j} \\ + \binom{j+1}{2j+k-t} y_{2t}^{4j+2} \Bigg] \otimes u^4. \end{aligned} \tag{6.8}$$

Ahora, si suponemos que $u$ es un cociclo mod 2, de (5.6) y (6.2) se sigue que

$$\delta(\omega_{2t+1}^{2k+1} \otimes u^4) = Y_{2t+1}^{2k+1} \otimes u^4, \tag{6.9}$$

$$\delta(\omega_{2t}^{2k} + \omega_{2t}^{2k-1}) \otimes u^4 = (Y_{2k}^{2t} + Y_{2t}^{2k-1}) \otimes u^4. \tag{6.10}$$

Las relaciones (6.9) y (6.10) implican las relaciones entre los cuadrados iterados de Steenrod. Las escribiremos mediante productos de matrices y vectores (cf. [2; p. 228]), con el fin de presentarlas del mismo modo que la fórmula (23.8) de [2].

## 7. Las relaciones expresadas mediante matrices

Con *t fijo*, consideremos la matriz cuadrada $a_j^k$ (ver (5.7)), donde $0 \leqq k, j \leqq [t/2]$, y las matrices rectangulares

$$a*_i^k(m, n) = \binom{i + n}{2i + 2n + k + m - t}, \tag{7.1}$$

donde $m$, $n$ son parametros fijos tales que $t - 2n - m \geqq 0$, $0 \leqq k \leqq [t/2]$, $0 \leqq i \leqq [(t - 2n - m)/2]$.

Igual que en [2; p. 228], las relaciones (6.7) y (6.8) se pueden escribir mediante productos de matrices y vectores del modo siguiente:

$$\|Y_{2t+1}^{2k+1} \otimes u^4\| = \|a_j^k\| \, \|y_{2t+1}^{4j+2} \otimes u^4\| + \|a*_j^k(0, 0)\| \, \|y_{2t+1}^{4j} \otimes u^4\| + \|a*_i^k(1, 0)\| \, \|y_{2t+1}^{4i+2} \otimes u^4\|, \tag{7.2}$$

$$\|(Y_{2t}^{2k} + Y_{2t}^{2k-1}) \otimes u^4\| = \|a_j^k\| \, \|y_{2t}^{4j} \otimes u^4\| + \|a*_j^k(0, 0)\| \, \|y_{2t}^{4j} \otimes u^4\| + \|a*_i^k(-1, 1)\| \, \|y_{2t}^{4i+2} \otimes u^4\|, \tag{7.3}$$

donde $0 \leqq k, j \leqq [t/2]$, $0 \leqq i \leqq [(t - 1)/2]$.

Observación. Los rangos de las sumas de los distintos términos de (6.7) y (6.8), se pueden arreglar de modo que resulten los determinados por (7.2) y (7.3). Esto es, si los términos tienen como factor $a_j^k$, entonces $0 \leqq j \leqq [t/2]$, puesto que $a_j^k = 0$ para $j > [t/2]$. Si tienen como factor $a*_i^k(m, n)$, como por ejemplo $a*_i^k(1, 0) y_{2t+1}^{4i+2} \otimes u^4$, entonces $0 \leqq i \leqq [(t - 1)/2]$, puesto que $y_{2t+1}^{4i+2} \otimes u^4 = 0$ para $i > [(t - 1/)2]$ (ver 6.5).

El determinante $|a_j^k| = 1$, luego existe el inverso

$$\|d_j^k\| = \|a_j^k\|^{-1}.$$

Del mismo modo que se establece 22.18 de [2], usando 25.3 de [2] con $p = 2$, se demuestra que

$$\|d_j^k\| \, \|a*_i^k(m, n)\| = \|b_i^k(m, n)\|, \tag{7.4}$$

donde,

$$b_i^k(m, n) = \binom{i + n - 1 - k}{2i + 2n + m + k - t}$$

es una matriz rectangular con

$$0 \leqq k \leqq [t/2], \qquad 0 \leqq i \leqq [(t - 2n - m)/2].$$

Ahora, suponemos que $u$ es un cociclo mod 2. Al multiplicar (7.2) y (7.3) por $\|d_j^k\|$, teniendo en cuenta (6.9) y (6.10) se obtienen

$$\|d_j^k\| \, \|\delta\omega_{2t+1}^{2k+1} \otimes u^4\| = \|y_{2t+1}^{4j+2} \otimes u^4\| + \|b_j^k(0, 0)\| \, \|y_{2t+1}^{4j} \otimes u^4\| + \|b_i^k(1, 0)\| \, \|y_{2t+1}^{4i+2} \otimes u^4\|, \tag{7.5}$$

$$\|d_j^k\| \, \|\delta(\omega_{2t}^{2k} + \omega_{2t}^{2k-1}) \otimes u^4\| = \|y_{2t}^{4j} \otimes u^4\| + \|b_j^k(0, 0)\| \, \|y_{2t}^{4j} \otimes\| \, u^4 + \|b_i^k(-1, 1)\| \, \|y_{2t}^{4i+2} \otimes u^4\|. \tag{7.6}$$

## 8. Cuadrados iterados de Steenrod

Abrimos aquí un paréntesis con el fin de indicar de que modo se interpreta $y_i^j \otimes u^4$ como iteración de cuadrados.

Consideramos una *aproximación diagonal* $\pi$-equivariante,

$$\varphi' : V \otimes K \to K^2, \tag{8.1}$$

y su *dual*,

$$\varphi : V \otimes_\pi K^{*2} \to K^*, \tag{8.2}$$

ambas definidas como en [12]. Formemos la composición

$$V \otimes V^2 \otimes K \xrightarrow{\alpha} V^2 \otimes V \otimes K \xrightarrow{1 \otimes \varphi'} V^2 \otimes K^2 \xrightarrow{\beta} (V \otimes K)^2 \xrightarrow{\varphi' \otimes \varphi'} K^4,$$

donde $\alpha$, $\beta$ son, respectivamente, los isomorfismos naturales que intercambian los factores en la forma indicada. Luego, si definimos

$$\psi_0' = (\varphi' \otimes \varphi')\beta(1 \otimes \varphi')\alpha, \tag{8.3}$$

es fácil verificar que

$$\psi_0' : V \otimes V^2 \otimes K \to K^4,$$

es una aproximación diagonal $G$-equivariante. Además, si $\psi_0$ es su dual, se tiene que

$$\psi_0 = \varphi(1 \otimes \varphi^2)\lambda, \tag{8.4}$$

donde

$$\lambda : (V \otimes V^2) \otimes_G K^{*4} \to V \otimes_\pi (V \otimes_\pi K^{*2})^2,$$

es el isomorfismo que se obtiene intercambiando los factores de $V^2$ y de $(K^{*2})^2$ (cf. [12; p. 215]).

Sea $u$ un $q$-cociclo mod 2 de $K$ y $\bar{u} \in H^q(K; Z_2)$ la $q$-clase de cohomología que determina. Para los cuadrados de Steenrod se usan las notaciones: $\mathrm{Sq}_i$ y $\mathrm{Sq}^j$. La relación entre ambas es

$$\mathrm{Sq}_i \bar{u} = \mathrm{Sq}^{q-i} \bar{u}. \tag{8.5}$$

Como es bien sabido, $\varphi(e_i \otimes u^2)$ es un cociclo (mod 2) que representa la clase $\mathrm{Sq}_i \bar{u}$ (cf. [12; p. 200]). Por comodidad, con una $\varphi$ determinada, usaremos la notación

$$\mathrm{Sq}_i u = \varphi(e_i \otimes u^2). \tag{8.6}$$

También (mod 2),

$$\psi_0(e_j e_i^2 \otimes u^4) = \varphi(e_j[\varphi(e_i \otimes u^2)]^2), \tag{8.7}$$

resulta ser un representante de

$$\mathrm{Sq}_j \mathrm{Sq}_i \bar{u} = \mathrm{Sq}^{2q-i-j} \mathrm{Sq}^{q-i} \bar{u}. \tag{8.8}$$

Ahora, consideremos una aproximación diagonal $S_4$-equivariante,

$$\psi' : W \otimes K \to K^4. \tag{8.9}$$

La composición

$$V \otimes V^2 \otimes K \xrightarrow{\theta \otimes 1} W \otimes K \xrightarrow{\psi'} K^4,$$

define

(8.10) $$\psi_1' = \psi'(\theta \otimes 1),$$

que, también como $\psi_0'$, resulta ser una aproximación diagonal $G$-equivariante. Luego, usando el portador diagonal podemos construir $\gamma'$, una homotopía $G$-equivariante, tal que

(8.11) $$\partial\gamma' + \gamma'\partial = \psi_0' - \psi_1'.$$

Al considerar las transformaciones duales, se tiene (mod 2)

(8.12) $$\delta\gamma + \gamma\delta = \varphi(1 \otimes \varphi^2)\lambda - \psi(\theta \otimes 1),$$

donde $\psi$ representa el dual de $\psi'$.

Claramente, $\delta(x_t^j \otimes u^4) = 0$ en $(V \otimes V^2) \otimes_G K^{*4}$ (ver §6). Además, de (5.2), $\psi(y_t^j \otimes u^4) = \psi(\theta x_t^j \otimes u^4)$. Por lo tanto, con la notación (8.6), usando (8.12) y (6.4–5), se obtienen (mod 2),

(8.13) $$\psi(y_t^{4j} \otimes u^4) = \mathrm{Sq}_{t-4j}\mathrm{Sq}_{2j}u + \delta\gamma(x_t^{4j} \otimes u^4),$$

(8.14) $$\psi(y_t^{4j+2} \otimes u^4) = \mathrm{Sq}_{t-4j-2}\mathrm{Sq}_{2j+1}u + \delta\gamma(x_t^{4j+2} \otimes u^4).$$

Por último, obviamente, de (6.5–6) se sigue que

$$\psi(y_t^{4i+1} \otimes u^4) = \psi(y_t^{4i+2} \otimes u^4) \quad \text{y} \quad \psi(y_t^{4i+3} \otimes u^4) = 0.$$

## 9. Las relaciones entre los cuadrados iterados

Aplicando $\psi$ a cada uno de los vectores de (7.5–6), se obtienen

(9.1) $$\delta \textstyle\sum_{j=0}^{[t/2]} d_j^s \psi(\omega_{2t+1}^{2j+1} \otimes u^4) = \psi(y_{2t+1}^{4s+2} \otimes u^4)$$
$$+ \textstyle\sum_{j=0}^{[t/2]} b_j^s(0, 0)\psi(y_{2t+1}^{4j} \otimes u^4) + \sum_{i=0}^{[(t-1)/2]} b_i^s(1, 0)\psi(y_{2t+1}^{4i+2} \otimes u^4)$$

(9.2) $$\delta \textstyle\sum_{j=0}^{[t/2]} d_j^s \psi[(\omega_{2t}^{2j} + \omega_{2t}^{2j-1}) \otimes u^4] = \psi(y_{2t}^{4s} \otimes u^4)$$
$$+ \textstyle\sum_{j=0}^{[t/2]} b_j^s(0, 0)\psi(y_{2t}^{4j} \otimes u^4) + \sum_{i=0}^{[(t-1)/2]} b_i^s(-1, 1)\psi(y_{2t}^{4i+2} \otimes u^4).$$

Definamos,

(9.3) $$E_{2t+1}^{2s+1}(u^4) = \textstyle\sum_{j=0}^{[t/2]} d_j^s\, \psi(\omega_{2t+1}^{2j+1} \otimes u^4),$$

(9.4) $$E_{2t}^{2s}(u^4) = \textstyle\sum_{j=0}^{[t/2]} d_j^s\, \psi[(\omega_{2t}^{2j} + \omega_{2t}^{2j-1}) \otimes u^4],$$

(9.5) $$X_{2t+1}^{2s+1} = x_{2t+1}^{4s+2} + \textstyle\sum_{j=0}^{[t/2]} b_j^s(0, 0)x_{2t+1}^{4j} + \sum_{i=0}^{[(t-1)/2]} b_i^s(1, 0)x_{2t+1}^{4i+2},$$

(9.6) $$X_{2t}^{2s} = x_{2t}^{4s} + \textstyle\sum_{j=0}^{[t/2]} b_j^s(0, 0)x_{2t}^{4j} + \sum_{i=0}^{[(t-1)/2]} b_i^s(-1, 1)x_{2t}^{4i+2}.$$

Con estas definiciones, y teniendo presente (8.13–14), de (9.1–2) se siguen

(9.7) $$\delta[E_{2t+1}^{2s+1}(u^4) + \gamma(X_{2t+1}^{2s+1} \otimes u^4)] = \mathrm{Sq}_{2t-4s-1}\mathrm{Sq}_{2s+1}u$$
$$+ \textstyle\sum_{j=0}^{[t/2]} b_j^s(0, 0)\mathrm{Sq}_{2t-4j+1}\mathrm{Sq}_{2j}u + \sum_{j=0}^{[(t-1)/2]} b_j^s(1, 0)\mathrm{Sq}_{2t-4j-1}\mathrm{Sq}_{2j+1}u$$

(9.8) $$\delta[E_{2t}^{2s}(u^4) + \gamma(X_{2t}^{2s} \otimes u^4)] = \mathrm{Sq}_{2t-4s}\mathrm{Sq}_{2s}u$$
$$+ \textstyle\sum_{j=0}^{[t/2]} b_j^s(0, 0)\mathrm{Sq}_{2t-4j}\mathrm{Sq}_{2j}u + \sum_{j=0}^{[(t-1)/2]} b_j^s(-1, 1)\mathrm{Sq}_{2t-4j-2}\mathrm{Sq}_{2j+1}u.$$

De aquí, para las operaciones en clases de cohomología, se derivan las siguientes relaciones. Si suponemos que $3s + 1 < t$, de (9.7) se obtiene,

$$\text{(9.9)} \qquad \mathrm{Sq}_{2t-4s-1}\mathrm{Sq}_{2s+1} = \textstyle\sum_{j=t-s}^{t} \binom{j - 2s - 2}{2j + 2s - 2t} \mathrm{Sq}_{2t-2j+1}\mathrm{Sq}_j.$$

Igualmente, si suponemos que $3s < t$, de (9.8) resulta,

$$\text{(9.10)} \qquad \mathrm{Sq}_{2t-4s}\mathrm{Sq}_{2s} = \textstyle\sum_{j=t-s}^{t} \binom{j - 2s - 1}{2j + 2s - 2t} \mathrm{Sq}_{2t-2j}\mathrm{Sq}_j.$$

Para justificar ésto únicamente necesitamos verificar los coeficientes binomiales. Si $b_j^s(0, 0) \neq 0$, de

$$b_j^s(0, 0) = \binom{j - s - 1}{2j + s - t},$$

resulta $2j + s - t \geqq 0$. Supongamos que $3s + 1 < t$, ó bien que $3s < t$. En ambos casos, de lo anterior se obtiene que $2j - 2s > 0$, y por lo tanto $j - s - 1 \geqq 0$. Ahora, usando el criterio de Lucas ([2; p. 233]), para $m \geqq 0$, fácilmente se demuestra que,

$$\binom{m}{n} \equiv \binom{2m}{2n} \equiv \binom{2m + 1}{2n} \qquad (\mathrm{mod}\ 2).$$

Así, se obtiene ($3s + 1 < t$, ó bien $3s < t$),

$$\text{(9.11)} \qquad b_j^s(0, 0) \equiv \binom{2j - 2s - 2}{4j + 2s - 2t} \equiv \binom{2j - 2s - 1}{4j + 2s - 2t} \qquad (\mathrm{mod}\ 2).$$

Análogamente, si $3s + 1 < t$, se sigue que

$$\text{(9.12)} \qquad b_j^s(1, 0) \equiv \binom{2j - 2s - 1}{4j + 2 + 2s - 2t} \qquad (\mathrm{mod}\ 2),$$

y si $3s < t$, resulta

$$\text{(9.13)} \qquad b_j^s(-1, 1) \equiv \binom{2j - 2s}{4j + 2 + 2s - 2t} \qquad (\mathrm{mod}\ 2).$$

Por lo tanto, las fórmulas (9.9–10) quedan demostradas.

Observación. Las relaciones (9.9–10) son casos particulares de las relaciones implicadas por la fórmula (22.20) de [2] (con $p = 2$). Al transformarlas, de acuerdo con (8.8), en relaciones entre operaciones con índices superiores, es fácil verificar que se obtiene (cf. [2; (23.8)]),

$$\text{(9.14)} \qquad \mathrm{Sq}^{2a}\mathrm{Sq}^b = \textstyle\sum_{i=0}^{a} \binom{b - i - 1}{2a - 2i} \mathrm{Sq}^{2a+b-i}\mathrm{Sq}^i,$$

donde $a < b$.

Como es bien sabido, (9.14) junto con la relación

$$\mathrm{Sq}^1\mathrm{Sq}^a = (a + 1)\mathrm{Sq}^{a+1},$$

caracterizan un conjunto completo de relaciones. Esto, justifica el haber elegido, por simplicidad, nuestras relaciones particulares.

## 10. Algunas construcciones auxiliares

Con $u$ una $q$-cocadena entera de $K$, en $V \otimes K^{*2}$ definimos,

(10.1) $$p_i(u) = \varphi(e_{i-1} \otimes u^2) + (-1)^i \varphi(e_i \otimes u\delta u).$$

Obviamente, si $u$ es cociclo mod 2, se tiene

$$p_i(u) = \varphi(e_{i-1} \otimes u^2) = \mathrm{Sq}_{i-1}u \qquad (\mathrm{mod}\ 2).$$

Usando (6.1), se comprueba fácilmente que

(10.2) $$\delta p_i(u) = p_{i+1}(\delta u) + [(-1)^q - (-1)^i]p_{i-1}(u).$$

Así, se obtiene

(10.3) $$\delta p_i(u) = \mathrm{Sq}_i \delta u \qquad (\mathrm{mod}\ 2).$$

Si suponemos que $q + i$ es *impar* y que $u$ es cociclo mod 2, de (10.2) resulta

(10.4) $$\delta p_i(u) = 2\mathrm{Sq}_{i-2}u \qquad (\mathrm{mod}\ 4).$$

Por lo tanto,

(10.5) $$\delta^* \mathrm{Sq}_{i-1}u + \delta \tfrac{1}{2}\varphi(e_i \otimes u\delta u) = \mathrm{Sq}_{i-2}u \qquad (\mathrm{mod}\ 2),$$

donde $\delta^* = \frac{1}{2}\delta$, y pasando a clases de cohomología, resulta

(10.6) $$\delta^* \mathrm{Sq}_{i-1}\bar{u} = \mathrm{Sq}_{i-2}\bar{u}.$$

Aquí, $\delta^*$ es el operador de Bockstein-Whitney de la sucesión exacta de coeficientes $0 \to Z_2 \to Z_4 \to Z_2 \to 0$. Por consiguiente, $\mathrm{Sq}_{i-1}\bar{u} = 0$ implica que $\mathrm{Sq}_{i-2}\bar{u} = 0$. Para referencia futura haremos explícita esta implicación por medio de cocadenas. Por hipótesis, existen cocadena $a$, $b$, tales que

(10.7) $$\mathrm{Sq}_{i-1}u = \delta a + 2b \qquad (\mathrm{mod}\ 4),$$

luego, $\delta^* \mathrm{Sq}_{i-1} = \delta b$, y de (10.5) se obtiene,

(10.8) $$\delta[b + \tfrac{1}{2}\varphi(e_i \otimes u\delta u)] = \mathrm{Sq}_{i-2}u \qquad (\mathrm{mod}\ 2).$$

Con una elección diferente en (10.7), por ejemplo $a'$, $b'$, puesto que $\delta a + 2b = \delta a' + 2b'$ (mod 4), se sigue que $b' = b + \delta^*(a - a')$. Luego, en (10.8) la cocadena $b$ puede variar únicamente por $\delta^* w$, donde $w$ es un cociclo mod 2.

Ahora, para toda $\varphi$, se tiene

(10.9) $$\mathrm{Sq}_q u = \varphi(e_q \otimes u^2) \equiv u \qquad (\mathrm{mod}\ 2).$$

Entonces, 10.6 con $i = q + 1$ resulta,

$$\delta^* \bar{u} = \mathrm{Sq}_{q-1}\bar{u}.$$

Por lo tanto, $\mathrm{Sq}_{q-1}\bar{u} = 0$ si y sólo si $\bar{u}$, clase mod 2, es también una clase mod 4.

Finalmente, si $u$ es un $q$-cociclo mod 4, definimos

(10.10) $$\tilde{u} = \tfrac{1}{2}[\varphi(e_q \otimes u^2) - u].$$

Es fácil demostrar que

$$\text{(10.11)} \qquad \mathrm{Sq}_{q-1}u = \delta\tilde{u} \qquad \text{(mod 2)}.$$

## 11. Las construcciones principales

Introduciremos las siguientes definiciones:

$$\text{(11.1)} \qquad A^{2s+1}_{2t+1}(u^4) = E^{2s+1}_{2t+1}(u^4) + \gamma(X^{2s+1}_{2t+1} \otimes u^4),$$

$$\text{(11.2)} \qquad A^{2s}_{2t}(u^4) = E^{2s}_{2t}(u^4) + \gamma(X^{2s}_{2t} \otimes u^4),$$

$$\text{(11.3)} \qquad g^s_j = \binom{j - 2s - 2}{2j + 2s - 2t},$$

$$\text{(11.4)} \qquad h^s_j = \binom{j - 2s - 1}{2j + 2s - 2t}.$$

Usando esta notación, con las restricciones de (9.9–10), podemos escribir las relaciones (9.7–8) mediante una sola suma. El rango de la suma, $0 \leqq j \leqq t$, se puede sustituir por $0 \leqq j \leqq q$, donde $q = \dim u$. En efecto, si $q \leqq t$ entonces, $\mathrm{Sq}_j u = 0$ para $q < j$; y si $t < q$ entonces, $\mathrm{Sq}_{2t-2j+1} = \mathrm{Sq}_{2t-2j} = 0$ para $t < j$. Escribiendo aparte los términos $j = q$, $q-1$ (cf. (10.9), (10.11)), se obtienen las siguientes expresiones (mod 2): si $3s + 1 < t$,

$$\text{(11.5)} \qquad \delta A^{2s+1}_{2t+1}(u^4) = \mathrm{Sq}_{2t-4s-1}\mathrm{Sq}_{2s+1}u + g^s_q \mathrm{Sq}_{2t-2q+1}u + g^s_{q-1}\mathrm{Sq}_{2t-2q+3}\mathrm{Sq}_{q-1}u + \textstyle\sum_{j=0}^{q-2} g^s_j \mathrm{Sq}_{2t-2j+1}\mathrm{Sq}_j u,$$

si $3s < t$,

$$\text{(11.6)} \qquad \delta A^{2s}_{2t}(u^4) = \mathrm{Sq}_{2t-4s}\mathrm{Sq}_{2s}u + h^s_q \mathrm{Sq}_{2t-2q}u + h^s_{q-1}\mathrm{Sq}_{2t-2q+2}\mathrm{Sq}_{q-1}u + \textstyle\sum_{j=0}^{q-2} h^s_j \mathrm{Sq}_{2t-2j}\mathrm{Sq}_j u.$$

Ahora, multiplicando por 2 las relaciones (11.5–6), y usando (10.4), se obtienen las siguientes relaciones (mod 4): si $q$ es impar y $3s + 1 < t$,

$$\text{(11.7)} \qquad \delta[2A^{2s+1}_{2t+1}(u^4) - \textstyle\sum_{j=0}^{(q-1)/2} g^s_{2j} p_{2t-4j+3}\mathrm{Sq}_{2j}u] = 2\mathrm{Sq}_{2t-4s-1}\mathrm{Sq}_{2s+1}u + 2g^s_q \mathrm{Sq}_{2t-2q+1}u + 2\textstyle\sum_{j=0}^{(q-3)/2} g^s_{2j+1}\mathrm{Sq}_{2t-4j-1}\mathrm{Sq}_{2j+1}u,$$

si $q$ es par y $3s < t$,

$$\text{(11.8)} \qquad \delta[2A^{2s}_{2t}(u^4) - \textstyle\sum_{j=0}^{(q-2)/2} h^s_{2j+1} p_{2t-4j}\mathrm{Sq}_{2j+1}u] = 2\mathrm{Sq}_{2t-4s}\mathrm{Sq}_{2s}u + 2h^s_q \mathrm{Sq}_{2t-2q}u + 2\textstyle\sum_{j=0}^{(q-2)/2} h^s_{2j}\mathrm{Sq}_{2t-4j}\mathrm{Sq}_{2j}u.$$

A continuación, empleando (11.5–8) y ciertas hipótesis adicionales, construiremos un conjunto de cociclos que usaremos como representantes de las operaciones cohomológicas de segundo orden, que se definen en las secciones siguientes. Se considerarán cuatro tipos (esencialmente dos) de operaciones cohomológicas, cada uno definido mediante una de las cuatro hipótesis siguientes:

Hipótesis A (se usará con (11.5))

(A1) $u$ es un $q$-cociclo mod 4 (mod 2 si $g^s_{q-1} = 0$).

(A2) $$\mathrm{Sq}_{2s+1} u = \delta a_{2s+1} \qquad \text{(mod 2)}.$$

(A3) Si $g^s_q = 1$ y $q$ es impar, entonces $q > t$; si $g^s_q = 1$ y $q$ es par, entonces

$$\mathrm{Sq}_{2t-2q+2} u = \delta a_{2t-2q+2} + 2b_{2t-2q+2}. \qquad \text{(mod 4)}.$$

(A4) $g^s_j \mathrm{Sq}_j u = \delta g^s_j a_j$ (mod 2), para $j = 0, \cdots, q-2$.

Hipótesis A′ (se usará con (11.6))

(A′1) $u$ es un $q$-cociclo mod 4 (mod 2 si $h^s_{q-1} = 0$).

(A′2) $$\mathrm{Sq}_{2s} u = \delta a_{2s} \qquad \text{(mod 2)}.$$

(A′3) Si $h^s_q = 1$ y $q$ es par, entonces $q > t$; si $h^s_q = 1$ y $q$ es impar, entonces

$$\mathrm{Sq}_{2t-2q+1} u = \delta a_{2t-2q+1} + 2b_{2t-2q+1} \qquad \text{(mod 4)}.$$

(A′4) $h^s_j \mathrm{Sq}_j u = \delta h^s_j a_j$ (mod 2), para $j = 0, \cdots, q-2$.

Hipótesis B (se usará en (11.7))

(B1) $u$ es un $q$-cociclo mod 2 ($q$ impar).

(B2) $$\mathrm{Sq}_{2s+1} u = \delta a_{2s+1} \qquad \text{(mod 2)}.$$

(B3) Si $g^s_q = 1$, entonces $q > t$.

(B4) $g^s_{2j+1} \mathrm{Sq}_{2j+1} u = \delta g^s_{2j+1} a_{2j+1}$ (mod 2), para $j = 0, \cdots, (q-3)/2$.

Hipótesis B′ (se usará en (11.8))

(B′1) $u$ es un $q$-cociclo mod 2 ($q$ par).

(B′2) $$\mathrm{Sq}_{2s} u = \delta a_{2s} \qquad \text{(mod 2)}.$$

(B′3) Si $h^s_q = 1$, entonces $q > t$.

(B′4) $h^s_{2j} \mathrm{Sq}_{2j} u = \delta h^s_{2j} a_{2j}$ (mod 2), para $j = 0, \cdots, (q-2)/2$.

Por comodidad, si $q$ es par y $g^s_q = 1$, definimos ($t$ es fijo) (cf. (10.8), (A3)),

(11.9) $$\bar{b}_{2t-2q+2} = b_{2t-2q+2} + \tfrac{1}{2}\varphi(e_{2t-2q+3} \otimes u\delta u),$$

y si $q$ es impar o si $g^s_q = 0$, definimos $\bar{b}_{2t-2q+2} = 0$. Análogamente, si $q$ es impar y $h^s_q = 1$, definimos

(11.10) $$\bar{b}_{2t-2q+1} = b_{2t-2q+1} + \tfrac{1}{2}\varphi(e_{2t-2q+2} \otimes u\delta u),$$

y si $q$ es par o si $h^s_q = 0$, definimos $\bar{b}_{2t-2q+1} = 0$.

Ahora, usando la hipótesis A construimos las cocadenas (mod 2),

(11.11) $$B^{2s+1}_{2t+1}(u^4) = p_{2t-4s-1}(a_{2s+1}) + g^s_q \bar{b}_{2t-2q+2} + g^s_{q-1} p_{2t-2q+3}(\tilde{u}) + \textstyle\sum_{j=0}^{q-2} g^s_j p_{2t-2j+1}(a_j).$$

Con la hipótesis A′, construimos las cocadenas (mod 2),

(11.12) $$B^{2s}_{2t}(u^4) = p_{2t-4s}(a_{2s}) + h^s_q\tilde{b}_{2t-2q+1} + h^s_{q-1}p_{2t-2q+2}(\tilde{u}) + \textstyle\sum_{j=0}^{q-2} h^s_j p_{2t-2j}(a_j).$$

Con la hipótesis B, construimos las cocadenas (mod 4),

(11.13) $$C^{2s+1}_{2t+1}(u^4) = 2p_{2t-4s-1}(a_{2s+1}) + 2\textstyle\sum_{j=0}^{(q-3)/2} g^s_{2j+1}p_{2t-4j-1}(a_{2j+1}) + \textstyle\sum_{j=0}^{(q-1)/2} g^s_{2j}p_{2t-4j+3}\mathrm{Sq}_{2j}u.$$

Finalmente, con la hipótesis B′, construimos las cocadenas (mod 4),

(11.14) $$C^{2s}_{2t}(u^4) = 2p_{2t-4s}(a_{2s}) + 2\textstyle\sum_{j=0}^{(q-2)/2} h^s_{2j}p_{2t-4j}(a_{2j}) + \textstyle\sum_{j=0}^{(q-2)/2} h^s_{2j+1}p_{2t-4j}\mathrm{Sq}_{2j+1}u.$$

Con $3s + 1 < t$, y toda $q$ si $q$ es par o si $g^s_q = 0$, y solo para $q > t$ si $q$ es impar y $g^s_q = 1$, definimos

(11.15) $$\Phi_{2t+1,2s+1}(u) = A^{2s+1}_{2t+1}(u^4) + B^{2s+1}_{2t+1}(u^4) \qquad \text{(mod 2)}.$$

Con $3s < t$, y toda $q$ si $q$ es impar o si $h^s_q = 0$, y solo para $q > t$ si $q$ es par y $h^s_q = 1$, definimos,

(11.16) $$\Phi_{2t,2s}(u) = A^{2s}_{2t}(u^4) + B^{2s}_{2t}(u^4) \qquad \text{(mod 2)}.$$

Con $3s + 1 < t$, y toda $q$ (impar) si $g^s_q = 0$, y solo para (impar) $q > t$ si $g^s_q = 1$, definimos,

(11.17) $$\Theta_{2t+1,2s+1}(u) = 2A^{2s+1}_{2t+1}(u^4) - C^{2s+1}_{2t+1}(u^4) \qquad \text{(mod 4)}.$$

Con $3s < t$, y toda $q$ (par) si $h^s_q = 0$, y solo para (par) $q > t$ si $g^s_q = 1$, definimos

(11.18) $$\Theta_{2t,2s}(u) = 2A^{2s}_{2t}(u^4) - C^{2s}_{2t}(u^4) \qquad \text{(mod 4)}.$$

Es inmediato verificar que las cocadenas $\Phi_{t,s}(u)$ son cociclos módulo 2 (ver (10.3), (10.8), (10.11)). Análogamente, las cocadenas $\Theta_{t,s}(u)$ resultan ser cociclos módulo 4.

Obviamente, en la construcción de estos cociclos se emplean varios elementos cuya determinación no es única. Concretamente, estos elementos son $\Omega$, $\theta$, $W$, $\varphi$, $\psi$, $\gamma$ y las cocadenas $a_j$, $b_i$ mencionadas en las hipótesis. Estudiaremos las variaciones de los cociclos $\Phi_{t,s}(u)$, $\Theta_{t,s}(u)$, al considerar otros elementos.

## 12. Estudio de las desviaciones

Con una $\varphi$ fija (ver (8.6)), las cocadenas $a_j$ usadas en las hipótesis A, A′, B, B′, exceptuando las de A3, A′3, pueden variar libremente mediante cualquier cociclo mod 2. Por ejemplo, en A2

$$\mathrm{Sq}_{2s+1}u = \delta a_{2s+1} = \delta(a_{2s+1} + c),$$

para toda $c$ que sea un $(2q - 2s - 2)$-cociclo mod 2. De acuerdo con lo mencionado en §10, para una elección diferente en las cocadenas de A3, A′3, la $b_i$ varía por un elemento de la forma $\delta^* c$, donde $c$ es un cociclo mod 2. Por lo tanto, teniendo en cuenta las construcciones (11.11–14), se obtiene lo siguiente:

La clase $\{\Phi_{2t+1,2s+1}(u)\}$ queda determinada módulo

(12.1) $$\mathrm{Sq}_{2t-4s-2}H^{2q-2s-2}(K; Z_2) + g_q^s \mathrm{Sq}_{4q-2t-4}H^{4q-2t-3}(K; Z_2) + \textstyle\sum_{j=0}^{q-2} g_j^s \mathrm{Sq}_{2t-2j}H^{2q-j-1}(K; Z_2).$$

La clase $\{\Phi_{2t,2s}(u)\}$ queda determinada módulo

(12.2) $$\mathrm{Sq}_{2t-4s-1}H^{2q-2s-1}(K; Z_2) + h_q^s \mathrm{Sq}_{4q-2t-3}H^{4q-2t-2}(K; Z_2) + \textstyle\sum_{j=0}^{q-2} h_j^s \mathrm{Sq}_{2t-2j-1}H^{2q-j-1}(K; Z_2).$$

La clase $\{\Theta_{2t+1,2s+1}(u)\}$ queda determinada módulo

(12.3) $$2\mathrm{Sq}_{2t-4s-2}H^{2q-2s-2}(K; Z_2) + 2\textstyle\sum_{j=0}^{(q-3)/2} g_{2j+1}^s \mathrm{Sq}_{2t-4j-2}H^{2q-2j-2}(K; Z_2)$$

La clase $\{\Theta_{2t,2s}(u)\}$ queda determinada módulo

(12.4) $$2\mathrm{Sq}_{2t-4s-1}H^{2q-2s-1}(K; Z_2) + 2\textstyle\sum_{j=0}^{(q-2)/2} h_{2j}^s \mathrm{Sq}_{2t-4j-1}H^{q-2j-1}(K; Z_2).$$

Ahora analizaremos las variaciones con respecto a las transformaciones $\theta_\#$, $\Omega$, $\psi'$ y $\gamma'$, respectivamente, de (3.1), (3.2), (8.9) y (8.11), así como también con respecto al complejo $W$ de (3.1).

LEMA 12.5. *Para una elección diferente en las transformaciones* $\Omega$, $\gamma'$, $\theta_\#$, $\psi'$ *y en el complejo* $W$, *los cociclos* $\Phi_{t,s}(u)$, $\Theta_{t,s}(u)$ *se alteran únicamente por una cofrontera.*

DEMOSTRACIÓN. Si $\Omega_1$ es otra homotopía en (3.2), existe $M$, una homotopía $\nu$-equivariante, tal que

$$\partial M - M\partial = \Omega_1 - \Omega.$$

Luego, si $u$ es un cociclo mod 2, en $W \otimes_{S_4} K^{*4}$ se tiene (mod 2),

$$\Omega_1(d_i \otimes d_j) \otimes u^4 = \Omega(d_i \otimes d_j) \otimes u^4 + \delta[M(d_i \otimes d_j) \otimes u^4].$$

Obviamente, ésto implica el Lema 12.5 para $\Omega$.

Análogamente, sea $\gamma_1'$ en (8.11), otra homotopía. Existe $N'$, una homotopía $G$-equivariante y con valores en el portador diagonal, tal que

$$\partial N' - N'\partial = \gamma_1' - \gamma'.$$

Considerando las transformaciones duales valuadas en $x_t^s \otimes u^4$, resulta (mod 2),

$$\gamma_1(x_t^s \otimes u^4) = \gamma(x_t^s \otimes u^4) + \delta N(x_t^s \otimes u^4),$$

lo que demuestra el Lema 12.5 para $\gamma'$.

Ahora, sea $\theta_{1\#}$ en (3.1), otra transformación $\theta$-equivariante. Existe $\rho$, una homotopía $\theta$-equivariante, tal que

$$\partial\rho + \rho\partial = \theta_{1\#} - \theta_\#.$$

Es fácil verificar que las transformaciones

$$\Omega_1 = \Omega + \sigma_\# \rho \xi_\# - \rho \xi_\# \tau_\#,$$
$$\gamma_1' = \gamma' - \psi'(\rho \otimes 1),$$

pueden usarse con $\theta_{1\#}$ como las homotopías de (3.2) y (8.11), respectivamente.

Denotemos con $A_t^s(u^4)$ y con $\bar{A}_t^s(u^4)$ las expresiones (11.1–2), construídas, respectivamente, con $\theta_{\#}$, $\Omega$, $\gamma'$ y con $\theta_{1\#}$, $\Omega_1$, $\gamma_1'$. Resulta $A_t^s(u^4) = \bar{A}_t^s(u^4)$, lo que demuestra el lema para $\theta_{\#}$.

Sea $\psi_*'$ en (8.9), otra transformación. Como en los casos anteriores, existe $P$, una homotopía $S_4$-equivariante y con valores en el portador diagonal tal que

$$\partial P + P\partial = \psi' - \psi_*'.$$

Si definimos (ver (8.11)),

$$\gamma_*' = \gamma' + P(\theta \otimes 1),$$

resulta,

$$\partial\gamma_*' + \gamma_*'\partial = \psi_0' - \psi_{*1}',$$

donde $\psi_{*1}' = \psi_*'(\theta \otimes 1)$. Usando estas expresiones el lema se demuestra fácilmente para $\psi'$.

Por último, sea $W^*$ otro complejo $S_4$-libre y acíclico. Como es bien sabido, existen transformaciones $S_4$-equivariantes, $f$: $W \to W^*$, $g$: $W^* \to W$, tales que $fg$, $gf$ resultan, en forma equivariante, homotópicas a la identidad. Si

$$\partial\zeta + \zeta\partial = gf - 1,$$

es una de las homotopías, podemos considerar las transformaciones para $W^*$ en función de las de $W$, como sigue:

$$\begin{aligned}
\theta_{\#}^* &= f\theta_{\#},\\
\Omega^* &= f\Omega,\\
\psi^{*\prime} &= \psi'(g \otimes 1),\\
\psi_1^{*\prime} &= \psi^{*\prime}(\theta_{\#}^* \otimes 1) = \psi'(gf \otimes 1),\\
\gamma^{*\prime} &= \gamma' + \psi'(\zeta\theta_{\#} \otimes 1).
\end{aligned}$$

Sean $A_t^s(u^4)$ y $A^{*s}_t(u^4)$ las expresiones (11.1–2), construídas con $W$ y $W^*$, respectivamente. Se comprueba que $A_t^s(u) - A^{*s}_t(u) \sim 0$ (mod 2). Esto implica el Lema 12.5 para $W$ y termina la demostración del lema.

Lo anterior se ha establecido con respecto a una $\varphi'$ fija. Las desviaciones de $\Phi_{t,s}(u)$, $\Theta_{t,s}(u)$ al variar la $\varphi'$ se estudiarán en §15, y se obtendrán como corolario de una situación más general, al considerar el comportamiento de los cociclos bajo transformaciones de un complejo en otro.

## 13. Las operaciones en una cofrontera

Con $u = \delta a$, definimos en $K^{*4}$ las cocadenas $\alpha_0 = au^3$, $\alpha_1 = a^2u^2$, $\alpha_2 = (au)^2$, $\alpha_3 = a^3u + ua^3$. Sean $\Sigma_1$, $\Sigma_2$ como en §5. Se tienen las siguientes igualdades (mod 2):

$$\begin{aligned}
&\delta\alpha_0 = u^4, \qquad \delta\alpha_1 = \Sigma_2\alpha_0, \qquad \delta\alpha_2 = \Sigma_1\alpha_0,\\
&\delta\alpha_3 = \Sigma_1\alpha_1 + \Sigma_2\alpha_2, \qquad \delta a^4 = \Sigma_1\alpha_3 = \Sigma_2\alpha_3.
\end{aligned} \tag{13.1}$$

En $W \otimes_{S_4} K^{*4}$ definimos,

(13.2) $Q_t^k(a) = \omega_t^k \otimes \alpha_0 + \omega_{t-1}^k \otimes \alpha_1 + \omega_{t-1}^{k-1} \otimes \alpha_2 + \omega_{t-2}^{k-1} \otimes \alpha_3 + (\omega_{t-3}^{k-1} + \omega_{t-3}^{k-2}) \otimes a^4,$

(13.3) $R_t^k(a) = Y_t^k \otimes \alpha_0 + Y_{t-1}^k \otimes \alpha_1 + Y_{t-1}^{k-1} \otimes \alpha_2 + Y_{t-2}^{k-1} \otimes \alpha_3 + (Y_{t-3}^{k-1} + Y_{t-3}^{k-2}) \otimes a^4.$

Se demuestra fácilmente, usando (5.6) con (13.1), que

$$\omega_t^k \otimes (\delta a)^4 = \delta Q_t^k(a) + R_t^k(a) \qquad (\text{mod } 2).$$

Para construir las operaciones $\Phi_{t,s}$, $\Theta_{t,s}$ en $\delta a$, necesitamos los vectores (ver (9.3–4)),

(13.4) $$\|\omega_{2t+1}^{2k+1} \otimes (\delta a)^4\| = \|\delta Q_{2t+1}^{2k+1}(a)\| + \|R_{2t+1}^{2k+1}(a)\|,$$

(13.5) $$\|(\omega_{2t}^{2k} + \omega_{2t}^{2k-1}) \otimes (\delta a)^4\| = \|\delta(Q_{2t}^{2k}(a) + Q_{2t}^{2k-1}(a))\| + \|R_{2t}^{2k}(a) + R_{2t}^{2k-1}(a)\|,$$

donde $t$ se considera fijo y $0 \leqq k \leqq [t/2]$. Con el fin de escribir estas relaciones en función de los términos que figuran en la construcción, introducimos las definiciones siguientes:

(13.6)
$$z_t^{4j} = x_t^{4j} \otimes \alpha_0 + x_{t-1}^{4j} \otimes \alpha_2 + x_{t-2}^{4j-1} \otimes \alpha_3 + x_{t-3}^{4j-2} \otimes a^4,$$
$$z_t^{4j+1} = x_t^{4j+1} \otimes \alpha_0 + x_{t-1}^{4j} \otimes \alpha_1 + x_{t-1}^{4j+1} \otimes \alpha_2 + x_{t-2}^{4j} \otimes \alpha_3 + x_{t-3}^{4j} \otimes a^4,$$
$$z_t^{4j+2} = x_t^{4j+2} \otimes \alpha_0 + x_{t-1}^{4j+1} \otimes \alpha_1 + x_{t-1}^{4j+2} \otimes \alpha_2 + (x_{t-2}^{4j+1} + x_{t-2}^{4j+3}) \otimes \alpha_3 + x_{t-3}^{4j} \otimes a^4,$$
$$z_t^{4j+3} = x_t^{4j+3} \otimes \alpha_0 + x_{t-1}^{4j+2} \otimes \alpha_1 + x_{t-1}^{4j+3} \otimes \alpha_2 + x_{t-2}^{4j+2} \otimes \alpha_3.$$

La imagen de estas cocadenas en $W \otimes_{S_4} K^{*4}$, se denotará con

$$\tilde{z}_t^j = (\theta \otimes 1) z_t^j.$$

Para $0 \leqq j \leqq [t/2]$, definimos las cocadenas $w_t^j$, como sigue. Si $t$ es par, $w_t^j = 0$, y si $t$ es impar,

(13.7) $$w_t^j = \binom{t+1}{2j+2}(\theta \otimes 1)(e_0 e_{t+1}\, \partial e_{t+1} \otimes \alpha_2 + e_0\, \partial e_{t+1} e_t \otimes \alpha_3).$$

Con cada entero $n \geqq 0$, consideremos la matriz rectangular $\|a_i^k(n)\|$, definida por

$$a_i^k(n) = \binom{i}{k-i-n} \qquad (\text{mod } 2),$$

con $0 \leqq k \leqq [t/2]$, $0 \leqq i \leqq [t/2] - n$. Igual que en (7.4), se puede demostrar que

$$\|d_j^k\|\,\|a_i^k(n)\| = \|c_i^k(n)\|,$$

donde,

(13.8) $$c_i^k(n) = \binom{i-k-1}{k-i-n}$$

En particular, si $n = 0$, se tiene que $\|a_j^k\| = \|a_j^k(0)\|$, y por consiguiente, $c_i^k(0)$ es la delta de Kronecker (mod 2).

Ahora, sustituyendo en (13.3) las expresiones (5.8–11) para las $Y_t^k$, agrupando los términos, y usando (13.6), se obtienen,

$$\left\|R_{2t+1}^{2k+1}(a)\right\| = \left\|a_j^k\right\| \left\|\tilde{z}_{2t+1}^{4j+1}\right\| + \left\|a_j^k(1)\right\| \left\|\tilde{z}_{2t+1}^{4j+3}\right\| + \left\|a*_j^k(0, 0)\right\| \left\|\tilde{z}_{2t+1}^{4j}\right\| + \left\|a*_j^k(1, 0)\right\| \left\|\tilde{z}_{2t+1}^{4j+2}\right\| + \left\|\delta w_t^j\right\|, \tag{13.9}$$

$$\left\|R_{2t}^{2k}(a) + R_{2t}^{2k-1}(a)\right\| = \left\|a_j^k\right\| \left\|\tilde{z}_{2t}^{4j}\right\| + \left\|a_j^k(1)\right\| \left\|\tilde{z}_{2t}^{4j+1} + \tilde{z}_{2t}^{4j+2}\right\| + \left\|a_j^k(2)\right\| \left\|\tilde{z}_{2t}^{4j+3}\right\| + \left\|a*_j^k(0, 0)\right\| \left\|\tilde{z}_{2t}^{4j} + \tilde{z}_{2t}^{4j+1}\right\| + \left\|a*_j^k(1, 0)\right\| \left\|\tilde{z}_{2t}^{4j+2} + \tilde{z}_{2t}^{4j+3}\right\|. \tag{13.10}$$

Sea $\psi_0 = \varphi(1 \otimes \varphi^2)\lambda$, la transformación de (8.4). Definimos las cocadenas $v_t^j$ de $K$, como sigue:

$$\begin{aligned} v_t^{4j} &= \psi_0[e_{t-4j-2}e_{2j}^2 \otimes a^4 + e_{t-4j-2}e_{2j}e_{2j+1} \otimes a^3u + e_{t-4j-3}e_{2j+1}^2 \oplus aua^2], \\ v_t^{4j+1} &= \psi_0[e_{t-4j-2}e_{2j+1}e_{2j} \otimes aua^2 + e_{t-4j-1}e_{2j}^2 \otimes a^3u + e_{t-4j-2}e_{2j+1}^2 \otimes (au)^2 \\ &\quad + e_{t-4j-1}e_{2j}e_{2j+1} \otimes a^2u^2], \\ v_t^{4j+2} &= \psi_0[e_{t-4j}e_{2j-1}^2 \otimes a^4 + e_{t-4j}e_{2j}e_{2j-1} \otimes aua^2 \\ &\quad + e_{t-4j+1}e_{2j-1}e_{2j} \otimes a^2u^2], \\ v_t^{4j+3} &= \psi_0[e_{t-4j-4}e_{2j+1}^2 \otimes a^4 + e_{t-4j-3}e_{2j+1}^2 \otimes aua^2 \\ &\quad + e_{t-4j-2}e_{2j+1}^2 \otimes u^2a^2]. \end{aligned} \tag{13.11}$$

Se comprueba (mod 2) que,

$$\begin{aligned} \psi_0(z_t^{4j}) &= \delta v_t^{4j} + p_{t-4j}p_{2j}(a), \\ \psi_0(z_t^{4j+1}) &= \delta v_t^{4j+1} + p_{t-4j-2}p_{2j+1}(a), \\ \psi_0(z_t^{4j+2}) &= \delta v_t^{4j+2} + p_{t-4j-2}p_{2j+1}(a), \\ \psi_0(z_t^{4j+3}) &= \delta v_t^{4j+3}. \end{aligned} \tag{13.12}$$

Además, se tiene que,

$$\begin{aligned} \delta z_t^{4j} &= e_{t-4j}e_{2j}^2 \otimes (\delta a)^4, \\ \delta z_t^{4j+1} &= \delta z_t^{4j+2} = e_{t-4j-2}e_{2j+1}^2 \otimes (\delta a)^4, \\ \delta z_t^{4j+3} &= 0. \end{aligned} \tag{13.13}$$

Definimos,

$$P_{2t+1}^{2s+1}(a) = p_{2t-4s-1}p_{2s+1}(a) + \textstyle\sum_{j=0}^{[t/2]} b_j^s(0, 0)p_{2t-4j+1}p_{2j}(a) + \sum_{j=0}^{[(t-1)/2]} b_j^s(1, 0)p_{2t-4j-1}p_{2j+1}(a), \tag{13.14}$$

$$P_{2t}^{2s}(a) = p_{2t-4s}p_{2s}(a) + \textstyle\sum_{j=0}^{[t/2]} b_j^s(0, 0)p_{2t-4j}p_{2j}(a) + \sum_{j=0}^{[(t-1)/2]} b_j^s(-1, 1)p_{2t-4j-2}p_{2j+1}(a), \tag{13.15}$$

$$V_{2t+1}^{2s+1}(a) = v_{2t+1}^{4s+1} + \gamma(z_{2t+1}^{4s+1}) + \textstyle\sum_{j=0}^{[t/2]} b_j^s(0, 0)(v_{2t+1}^{4j} + \gamma(z_{2t+1}^{4j})) + \sum_{j=0}^{[(t-1)/2]} b_j^s(1, 0)(v_{2t+1}^{4j+2} + \gamma(z_{2t+1}^{4j+2})), \tag{13.16}$$

$$V_{2t}^{2s}(a) = v_{2t}^{4s} + \gamma(z_{2t}^{4s}) + \textstyle\sum_{j=0}^{[t/2]} b_j^s(0, 0)(v_{2t}^{4j} + v_{2t}^{4j+1} + \gamma(z_{2t}^{4j}) + \gamma(z_{2t}^{4j+1})) + \sum_{j=0}^{[(t-1)/2]} b_j^s(1, 0)(v_{2t}^{4j+2} + v_{2t}^{4j+3} + \gamma(z_{2t}^{4j+2}) + \gamma(z_{2t}^{4j+3})). \tag{13.17}$$

En estas expresiones, $\gamma$ es la transformación de (8.12). Ahora, sustituyendo (13.9–10) en (13.4–5), multiplicando por $\|d_j^k\|$, aplicando $\psi$, y usando (8.12) junto con (13.12–13), se obtienen las relaciones siguientes (ver (11.1–2)):

$$\begin{aligned}A_{2t+1}^{2s+1}((\delta a)^4) + P_{2t+1}^{2s+1}(a) &= \delta \textstyle\sum_{j=0}^{[t/2]} d_j^s \psi(Q_{2t+1}^{2j+1}(a) + w_t^j) \\ &\quad + \delta V_{2t+1}^{2s+1}(a) + \delta \textstyle\sum_{j=0}^{[t/2]-1} c_j^s(1)(v_{2t+1}^{4j+3} + \gamma(z_{2t+1}^{4j+3})),\end{aligned} \tag{13.18}$$

$$\begin{aligned}&A_{2t}^{2s}((\delta a)^4) + P_{2t}^{2s}(a) = \delta \textstyle\sum_{j=0}^{[t/2]} d_j^s \psi(Q_{2t}^{2j}(a) + Q_{2t}^{2j-1}(a)) \\ &+ \delta V_{2t}^{2s}(a) + \delta \textstyle\sum_{j=0}^{[t/2]-1} c_j^s(1)(v_{2t}^{4j+1} + v_{2t}^{4j+2} + \gamma(z_{2t}^{4j+1}) + \gamma(z_{2t}^{4j+2})) \\ &+ \delta \textstyle\sum_{j=0}^{[t/2]-2} c_j^s(2)(v_{2t}^{4j+3} + \gamma(z_{2t}^{4j+3})).\end{aligned} \tag{13.19}$$

LEMA 13.20. *Si $u = \delta a$, se pueden satisfacer las hipótesis usadas en la construcción de $\Phi_{t,s}$, $\Theta_{t,s}$, de modo que resulten,*

$$\Phi_{t,s}(\delta a) \sim 0, \qquad \Theta_{t,s}(\delta a) \sim 0.$$

DEMOSTRACIÓN. Si $q + k$ es impar, donde $q = \dim u$, de (10.2) se sigue que (ver (10.7)),

$$\varphi(e_{k+1} \otimes u^2) = \delta p_{k+1}(a) + 2p_k(a) \qquad (\text{mod } 4),$$

y como $\delta u = 0$, podemos tomar $\tilde{b}_{k+1} = p_k(a)$ (ver (11.9–10)).

También, de (10.2) se obtiene que

$$\tilde{u} = p_{q-1}(a) + \delta\tilde{a} \qquad (\text{mod } 2),$$

con $\tilde{u}$, $\breve{a}$ definidos por (10.10).

Representemos con $P_t^s(a)$ las expresiones (13.14–15), y con $B_t^s(u^4)$ las expresiones (11.11–12), estas últimas construídas con $\tilde{u}$ como en (10.10), con $\tilde{b}_{k+1} = p_k(a)$, y con $a_i = p_i(a)$.

En vista de (13.17–18), para demostrar que $\Phi_{t,s}(\delta a) \sim 0$, es suficiente verificar que $B_t^s(u^4) \sim P_t^s(a)$. Ahora, para los dos términos donde $B_t^s(u^4)$ y $P_t^s(a)$ pueden diferir, se tiene

$$\tilde{b}_{k+1} = p_k(a) = p_k p_q(a),$$
$$p_{k+2}(\tilde{u}) \sim p_{k+2} p_{q-1}(a).$$

Esto demuestra la primera parte del lema.

Análogamente, representamos con $C_t^s(u^4)$ las expresiones (11.13–14), construídas con $a_i = p_i(a)$.

Usando 10.2, se obtienen las relaciones (mod 4),

$$p_{2k+3}\mathrm{Sq}_{2j}(\delta a) \sim 2p_{2k+1}p_{2j}(a), \qquad \text{si } q \text{ es impar},$$
$$p_{2k+2}\mathrm{Sq}_{2j+1}(\delta a) \sim 2p_{2k}p_{2j+1}(a), \qquad \text{si } q \text{ es par}.$$

De ésto, se sigue que $C_t^s(u^4) \sim 2P_t^s(a)$ (mod 4). Luego, al multiplicar por 2 las relaciones (13.17–18) resulta $\Theta_{t,s}(\delta a) \sim 0$.

## 14. Las operaciones en una suma

Sean $u$, $v$ dos cocadenas de $K^*$. En $K^{*4}$ definimos

$$\beta_1 = u^2v^2, \qquad \beta_2 = (uv)^2, \qquad \beta_3 = uv^2u, \qquad \beta_4 = u^3v + v^3u.$$

Resulta (mod 2),

$$(u+v)^4 = u^4 + v^4 + \Sigma_1 \beta_1 + \Sigma_2 (\beta_2 + \beta_3) + \Sigma_1 \Sigma_2 \beta_4.$$

Con ésto y (5.6), calcularemos $\omega_t^k \otimes (u+v)^4$ mod 2 en $W \otimes_{S_4} K^{*4}$.

Ya que $\Sigma_2 \beta_1 = 0$, $\Sigma_1 \beta_2 = 0$, $\Sigma_2\Sigma_2 \beta_4 = 0$, se tienen,

$$\omega_t^k \otimes \Sigma_1\beta_1 = (\partial\omega_{t+1}^{k+1}) \otimes \beta_1 + Y_{t+1}^{k+1} \otimes \beta_1,$$

$$\omega_t^k \otimes \Sigma_2\beta_2 = (\partial\omega_{t+1}^{k}) \otimes \beta_2 + Y_{t+1}^{k} \otimes \beta_2,$$

$$\omega_t^k \otimes \Sigma_1\Sigma_2\beta_4 = (\partial\Sigma_2\omega_{t+1}^{k+1}) \otimes \beta_4 + (\Sigma_2 Y_{t+1}^{k+1}) \otimes \beta_4.$$

Análogamente, de $\Sigma_2\beta_3 = \Sigma_1\beta_3$, y aplicando varias veces (5.6), se obtiene

$$\omega_t^k \otimes \Sigma_2\beta_3 = \textstyle\sum_{i=-1}^{k} (\partial\omega_{t+1}^{i}) \otimes \beta_3 + \sum_{i=-1}^{k} Y_{t+1}^{i} \otimes \beta_3.$$

Por lo tanto, si $u$, $v$ son cociclos mod 2,

$$\begin{aligned}(14.1)\quad \omega_t^k \otimes (u+v)^4 = \omega_t^k \otimes u^4 + \omega_t^k \otimes v^4 + D_t^k(u+v) \\ + \delta[\omega_{t+1}^{k+1} \otimes \beta_1 + \omega_{t+1}^{k} \otimes \beta_2 \\ + \textstyle\sum_{i=-1}^{k} \omega_{t+1}^{i} \otimes \beta_3 + (\Sigma_2\omega_{t+1}^{k+1}) \otimes \beta_4],\end{aligned}$$

donde,

$$(14.2)\quad D_t^k(u+v) = Y_{t+1}^{k+1} \otimes \beta_1 + Y_{t+1}^{k} \otimes \beta_2 + \textstyle\sum_{i=-1}^{k} Y_{t+1}^{i} \otimes \beta_3 + (\Sigma_2 Y_{t+1}^{k+1}) \otimes \beta_4.$$

Por comodidad, en esta sección *calcularemos módulo cofronteras*, escribiendo $\sim$ en lugar de igualdad. Así, para nuestras operaciones consideramos

$$(14.3)\qquad \omega_{2t+1}^{2k+1} \otimes [(u+v)^4 + u^4 + v^4] \sim D_{2t+1}^{2k+1}(u+v),$$

$$(14.4)\qquad (\omega_{2t}^{2k} + \omega_{2t}^{2k-1}) \otimes [(u+v)^4 + u^4 + v^4] \sim D_{2t}^{2k}(u+v) + D_{2t}^{2k-1}(u+v).$$

Definamos,

$$(14.5)\qquad \begin{aligned} G_t^i &= e_{t-2i}e_i^2 \otimes \beta_1 + (e_{t-2i-2}e_{i+1}^2 + e_{t-2i-1}e_{i+1}\,\partial e_{i+1}) \otimes \beta_2 \\ &\qquad + e_{t-2i}e_i\,\partial e_{i+1} \otimes \beta_4, \\ \bar{G}_k^i &= (\theta \otimes 1)G_k^i. \end{aligned}$$

Fácilmente se verifica que,

$$(14.6)\qquad \partial G_t^i = e_{t-2i-1}e_i^2 \otimes [(u+v)^4 + u^4 + v^4].$$

Para $t$ fija, y $0 \leq j \leq [t/2]$, definimos los vectores $F_{2t+2}^j$, $H_{2t+2}^j$, como sigue:

$$(14.7)\qquad F_{2t+2}^j = (\theta \otimes 1)e_{2t+2}e_0^2 \otimes (uv)^2,$$

$$(14.8)\qquad H_{2t+2}^j = (\theta \otimes 1)\binom{t+1}{2j+2} e_0 e_{t+1}^2 \otimes u^2v^2.$$

Ahora, sustituyendo en (14.2) las expresiones (5.8–11), y efectuando cálculos en $W \otimes_{S_4} K^{*4}$, se obtienen

$$\text{(14.9)} \quad \left\|D_{2t+1}^{2k+1}(u+v)\right\| \sim \left\|a_j^k\right\| \left\|\tilde{G}_{2t+2}^{2j+1}\right\| + \left\|a^{*k}_j(0,0)\right\| \left\|\tilde{G}_{2t+2}^{2j}\right\| + \left\|a^{*k}_j(1,0)\right\| \left\|\tilde{G}_{2t+2}^{2j+1}\right\| + \left\|F_{2t+2}^{j}\right\| + \left\|H_{2t+2}^{j}\right\|,$$

$$\text{(14.10)} \quad \left\|(D_{2t}^{2k} + D_{2t}^{2k-1})(u+v)\right\| \sim \left\|a_j^k\right\| \left\|\tilde{G}_{2t+1}^{2j}\right\| + \left\|a^{*k}_j(0,0)\right\| \left\|\tilde{G}_{2t+1}^{2j}\right\| + \left\|a^{*k}_j(-1,1)\right\| \left\|\tilde{G}_{2t+1}^{2j+1}\right\|.$$

Al multiplicar (14.9) por $\left\|d_j^s\right\|$, necesitamos calcular $\left\|d_j^s\right\| \left(\left\|F_{2t+2}^{j}\right\| + \left\|H_{2t+2}^{j}\right\|\right)$. Con este fín establecemos los resultados siguientes.

Sea $c$ una constante entera, positiva o negativa. Para $0 \leqq j \leqq [t/2]$, definimos los vectores $\left\|r_c^j\right\|$, $\left\|s_c^j\right\|$, con

$$r_c^j = \binom{c-j-1}{j}, \qquad s_c^j = \binom{c}{j} \qquad \text{(mod 2)}.$$

Usando el Teorema 25.3 de [2], se obtiene que $\left\|a_j^k\right\| \left\|r_c^j\right\| = \left\|s_c^j\right\|$, luego, multiplicando por $\left\|d_j^s\right\|$ resulta,

$$\text{(14.11)} \qquad \left\|d_j^s\right\| \left\|s_c^j\right\| = \left\|r_c^s\right\| \qquad \text{(mod 2)},$$

Por lo tanto, con $c = -1$, de (14.11) se sigue que,

$$\text{(14.12)} \qquad \textstyle\sum_{j=0}^{[t/2]} d_j^s = \binom{2s+1}{s},$$

Si $r = [(t+1)/2]$, claramente,

$$\binom{t+1}{2j+2} = \binom{r}{j+1} = \textstyle\sum_{i=0}^{r-1} \binom{i}{j} \qquad \text{(mod 2)},$$

luego, usando (14.11),

$$\textstyle\sum_{j=0}^{[t/2]} d_j^s \binom{t+1}{2j+2} = \sum_{i=0}^{r-1} \sum_{j=0}^{[t/2]} d_j^s \binom{i}{j} = \sum_{i=0}^{r-1} \binom{i-s-1}{s}$$
$$= \binom{2s+1}{s} + \binom{r-s-1}{s+1}.$$

En consecuencia, si $r - s - 1 \geqq 0$,

$$\text{(14.13)} \qquad \textstyle\sum_{j=0}^{[t/2]} d_j^s \binom{t+1}{2j+2} = \binom{2s+1}{s} + \binom{t-2s-1}{2s+2} \qquad \text{(mod 2)}.$$

Ahora, interpretaremos los términos $\psi_0(G_t^j)$, en función de los elementos que intervienen en la construcción de $\Phi_{t,s}$, $\Theta_{t,s}$.

Para una $\varphi$ fija, supongamos que

$$\delta a_i = \mathrm{Sq}_i u = \varphi(e_i \otimes u^2), \qquad \delta a_i' = \mathrm{Sq}_i v = \varphi(e_i \otimes v^2).$$

Entonces, $\delta a_i'' = \mathrm{Sq}_i(u+v) = \varphi(e_i \otimes (u+v)^2)$, donde

$$a_i'' = a_i + a_i' + \varphi(e_{i+1} \otimes uv)$$

Además, se comprueba fácilmente que

$$p_{t-2i}(a_i'') + p_{t-2i}(a_i') + p_{t-2i}(a_i) \sim \psi_0(G^i_{t+1}) \qquad (\text{mod } 2), \tag{14.14}$$

donde $\psi_0$ es la transformación (8.4).

Con respecto a las cocadenas definidas por (10.10), si $u$, $v$ son $q$-cociclos, se tiene

$$\widetilde{u+v} = \tilde{u} + \tilde{v} + \varphi(e_q \otimes uv),$$

y resulta

$$p_{t-2q+2}(\widetilde{u+v}) + p_{t-2q+2}(\tilde{u}) + p_{t-2q+2}(\tilde{v}) \sim \psi_0(G^{q-1}_{t+1}) \quad (\text{mod } 2).$$

En forma análoga, si $\tilde{b}_{t-2q+1}$, $\tilde{b}'_{t-2q+1}$, son las cocadenas no triviales de (11.9–10), construídas con $u$, $v$, respectivamente, entonces,

$$\tilde{b}''_{t-2q+1} = \tilde{b}'_{t-2q+1} + \tilde{b}_{2t-2q+1} + \varphi(e_{t-2q+1} \otimes uv) + \delta\tfrac{1}{2}\varphi(e_{t-2q+3} \otimes v\,\delta u), \tag{14.15}$$

es una cocadena tal para $u + v$, y resulta

$$\tilde{b}''_{t-2q+1} + \tilde{b}'_{t-2q+1} + \tilde{b}_{t-2q+1} \sim \psi_0(G^q_{t+1}) \qquad (\text{mod } 2). \tag{14.16}$$

Ahora, si $q$ es impar y $g^s_q = 1$, las cocadenas (11.9) son cero. Por otra parte, ya que $q > t$ (ver A3), $\psi_0(G^q_{2t+2})$ resulta cero solo para $q \neq t + 1$. Si $q = t + 1$, $\psi_0(G^q_{2t+2}) = u \smile v$, y necesitaremos tener presente este término. Con este fin, para $q > t$, escribimos

$$\psi_0(G^q_{2t+2}) = (\mathrm{Sq}_{2q-t-1}u) \smile (\mathrm{Sq}_{2q-t-1}v). \tag{14.17}$$

Si $q$ es par y $h^s_q = 1$, las cocadenas (11.10) son triviales, se tiene $q > t$ (ver A′3), y resulta $\psi_0(G^q_{2t+1}) = 0$.

Por último, para los términos especiales en $\Theta_{t,s}$ (ver (11.13–14)), se comprueba lo siguiente. Sea $q = \dim u = \dim v$. Si $q$ es impar, mod 4 se tiene que,

$$p_{t+2-4j}\mathrm{Sq}_{2j}(u + v) - p_{t+2-4j}\mathrm{Sq}_{2j}u - p_{t+2-4j}\mathrm{Sq}_{2j}v \sim 2\psi_0(G^{2j}_{t+1}), \tag{14.18}$$

y si $q$ es par, mod 4 se tiene que,

$$p_{t-4j}\mathrm{Sq}_{2j+1}(u + v) - p_{t-4j}\mathrm{Sq}_{2j+1}u - p_{t-4j}\mathrm{Sq}_{2j+1}v \sim 2\psi_0(G^{2j+1}_{t+1}). \tag{14.19}$$

Sean $u$, $v$, dos $q$-cociclos de $K$, satisfaciendo una misma hipótesis, de las hipótesis A, A′, B, B′. Claramente, la suma, $u + v$, satisfacerá también la hipótesis común de $u$, $v$.

Definimos (ver (14.12–13)),

$$m_s = \binom{2s+1}{s}, \qquad n^t_s = \binom{2s+1}{s} + \binom{t-2s-1}{2s+2}. \tag{14.20}$$

TEOREMA 14.21. *Las cocadenas usadas para construir $\Phi_{t,s}$, $\Theta_{t,s}$, se pueden elegir, según el caso, de modo que resulte lo siguiente:*

(1) *Si $u$, $v$ satisfacen la hipótesis* A, *entonces* mod 2,

$$\Phi_{2t+1,2s+1}(u+v) \sim \Phi_{2t+1,2s+1}(u) + \Phi_{2t+1,2s+1}(v) + m_s\mathrm{Sq}_{2t+2}(u \smile v) \\ + n_s^t(\mathrm{Sq}_{t+1}u)\smile(\mathrm{Sq}_{t+1}v) + q\cdot g_q^s(\mathrm{Sq}_{2q-t-1}u)\smile(\mathrm{Sq}_{2q-t-1}v)$$

(2) *Si $u$, $v$ satisfacen la hipótesis* A$'$, *entonces* mod 2

$$\Phi_{2t,2s}(u+v) \sim \Phi_{2t,2s}(u) + \Phi_{2t,2s}(v)$$

(3) *Si $u$, $v$ satisfacen la hipótesis* B, *entonces* mod 4,

$$\Theta_{2t+1,2s+1}(u+v) \sim \Theta_{2t+1,2s+1}(u) + \Theta_{2t+1,2s+1}(v) + 2m_s\mathrm{Sq}_{2t+2}(u \smile v) \\ + 2n_s^t(\mathrm{Sq}_{t+1}u)\smile(\mathrm{Sq}_{t+1}v) + 2g_q^s(\mathrm{Sq}_{2q-t-1}u)\smile(\mathrm{Sq}_{2q-t-1}v).$$

(4) *Si $u$, $v$ satisfacen la hipótesis* B$'$, *entonces* mod 4,

$$\Theta_{2t,2s}(u+v) \sim \Theta_{2t,2s}(u) + \Theta_{2t,2s}(v).$$

DEMOSTRACIÓN. Primero, sustituímos (14.9–10) en las expresiones vectoriales formadas con (14.3–4), y multiplicamos por $\|d_j^s\|$, teniendo en cuenta (14.12–13) para los términos (14.7–8). Luego, aplicamos $\psi$, y usamos (8.12) en los términos $\psi(\tilde{G}_t^j)$, $\psi(F_t^j)$, $\psi(H_t^j)$, teniendo en cuenta (14.6) al calcular $\gamma\delta(G_t^j)$. Por último, interpretando $\psi_0(G_t^j)$ por medio de (14.14), . . . , (14.19), según el caso, se obtiene la demostración del teorema.

Ahora, tomando $v = \delta a$, y combinando (13.19) con Teorema 14.21, se obtiene el siguiente

COROLARIO 14.21

$$\Phi_{t,s}(u+\delta a) \sim \Phi_{t,s}(u) \qquad (\text{mod } 2),$$

$$\Theta_{t,s}(u+\delta a) \sim \Theta_{t,s}(u) \qquad (\text{mod } 4).$$

## 15. Las operaciones bajo transformaciones de cadena

Sean $K$, $L$ dos complejos regulares y $f_\# : K \to L$ una transformación de cadena (propia). Consideremos el diagrama

$$\begin{array}{ccc} W \otimes K & \xrightarrow{\psi'} & K^4 \\ \downarrow 1 \otimes f_\# & & \downarrow f_\#^4 \\ W \otimes L & \xrightarrow{\bar{\psi}'} & L^4 \end{array}$$

donde, $\psi'$, $\bar{\psi}'$ son aproximaciones diagonales $S_4$-equivariantes (ver (8.9)). Existe $\Gamma'$, una homotopía equivariante y con valores en el portador diagonal bajo $f_\#^4$, tal que

$$f_\#^4\psi' - \bar{\psi}'(1 \otimes f_\#) = \partial\Gamma' + \Gamma'\partial.$$

Pasando al dual, se tiene (mod 2),

$$\psi(1 \otimes f^{\#4}) + f^{\#}\bar{\psi} = \delta\Gamma + \Gamma\delta, \tag{15.1}$$

donde, $\psi$, $\bar{\psi}$, $f^{\#}$, $\Gamma$ son, respectivamente, los duales de $\psi'$, $\bar{\psi}'$, $f_{\#}$, $\Gamma'$.

Si $u$ es un cociclo mod 2 de $L$, entonces

$$\delta(\omega_t^k \otimes u^4) = Y_t^k \otimes u^4 \qquad \text{(mod 2)},$$

y de (15.1) se sigue que,

$$\psi(\omega_t^k \otimes (f^{\#}u)^4) \sim f^{\#}\bar{\psi}(\omega_t^k \otimes u^4) + \Gamma(Y_t^k \otimes u^4).$$

Por lo tanto (ver (9.3–4)),

$$E_{2t+1}^{2s+1}((f^{\#}u)^4) \sim f^{\#}E_{2t+1}^{2s+1}(u^4) + \textstyle\sum_{j=0}^{[t/2]} d_j^s\Gamma(Y_{2t+1}^{2j+1} \otimes u^4), \tag{15.2}$$

$$E_{2t}^{2s}((f^{\#}u)^4) \sim f^{\#}E_{2t}^{2s}(u^4) + \textstyle\sum_{j=0}^{[t/2]} d_j^s\Gamma[(Y_{2t}^{2j} + Y_{2t}^{2j-1}) \otimes u^4]. \tag{15.3}$$

Necesitamos calcular explícitamente los términos $\Gamma(Y_t^j \otimes u^4)$.

Con las aproximaciones diagonales $\pi$-equivariantes,

$$\varphi' : V \otimes K \to K^2, \qquad \bar{\varphi}' : V \otimes L \to L^2,$$

construímos las transformaciones $G$-equivariantes (ver (8.3)),

$$\psi_0' = (\varphi' \otimes \varphi')\beta(1 \otimes \varphi')\alpha,$$

$$\bar{\psi}_0' = (\bar{\varphi}' \otimes \bar{\varphi}')\beta(1 \otimes \bar{\varphi}')\alpha.$$

Como en (8.11), sean $\gamma'$, $\bar{\gamma}'$, dos homotopías $G$-equivariantes tales que,

$$\psi_0' - \psi_1' = \partial\gamma' + \gamma'\partial,$$

$$\bar{\psi}_0' - \bar{\psi}_1' = \partial\bar{\gamma}' + \bar{\gamma}'\partial,$$

donde, $$\psi_1' = \psi'(\theta \otimes 1), \qquad \bar{\psi}_1' = \bar{\psi}'(\theta \otimes 1).$$

Sea $\Pi$ una homotopía $G$-equivariante y con valores en el portador diagonal bajo $f_{\#}^4$, tal que

$$f_{\#}^4\psi_0' - \bar{\psi}_0'(1 \otimes f_{\#}) = \partial\Pi + \Pi\partial. \tag{15.4}$$

Si definimos,

$$P = \Pi - \Gamma'(\theta \otimes 1) - f_{\#}^4\gamma' + \bar{\gamma}'(1 \otimes f_{\#}), \tag{15.5}$$

se verifica fácilmente que $\partial P + P\partial = 0$. Luego, existe $Q$, una homotopía $G$-equivariante, tal que

$$P = \partial Q - Q\partial. \tag{15.6}$$

Por otra parte, $\Pi$ se puede definir explícitamente. Sea $\mu'$ una homotopía $\pi$-equivariante y con valores en el portador diagonal bajo $f_{\#}^2$, tal que

$$f_{\#}^2\varphi' - \bar{\varphi}'(1 \otimes f_{\#}) = \partial\mu' + \mu'\partial. \tag{15.7}$$

Ahora, usando la convención para productos tensoriales, introducida por Eilenberg y MacLane en [3; p. 517], definimos,

$$M = (\bar{\varphi}' \otimes \bar{\varphi}')\beta(1 \otimes \mu')\alpha + ((\bar{\varphi}'(1 \otimes f_{\#})) \otimes \mu')\beta(1 \otimes \varphi')\alpha$$
$$+ (\mu' \otimes (\bar{\varphi}'(1 \otimes f_{\#})))\beta(1 \otimes \varphi')\alpha.$$

Claramente, $M : V \otimes V^2 \otimes K \to L^4$ es equivariante. Luego, en un elemento $e_k e_i e_j \otimes \sigma$ de la $G$-base de $V \otimes V^2 \otimes K$, definimos la transformación $N$, como,

$$N(e_k e_i e_j \otimes \sigma) = \mu' \otimes (\partial\mu' + \mu'\partial)\beta(1 \otimes \varphi')\alpha(e_k e_i e_j \otimes \sigma)$$
$$+ (-1)^{i+j+1}(\mu' \otimes \mu')\beta(1 \otimes \varphi')\alpha(x e_{k-1} e_i e_j \otimes \sigma),$$

y extendemos $N$ por equivariancia a todo elemento de $V \otimes V^2 \otimes K$. Se verifica que

$$\Pi = M + N, \tag{15.8}$$

puede considerarse como la homotopía de (15.4).

Combinamos (15.5) con (15.6), usando la expresión (15.8) para $\Pi$. Al considerar las transformaciones duales, valuadas en $e_i e_j^2 \otimes u^4$, donde $u$ es un cociclo mod 2 de $L$, se obtiene (mod 2),

$$\Gamma((\theta e_i e_j^2) \otimes u^4) \sim f^{\#}\bar{\gamma}(e_i e_j^2 \otimes u^4) + \gamma(e_i e_j^2 \otimes (f^{\#}u)^4)$$
$$+ \{\mu(1 \otimes \bar{\varphi}^2) + \varphi[1 \otimes (f^{\#}\bar{\varphi} \otimes \mu + \mu \otimes f^{\#}\bar{\varphi})]$$
$$+ \varphi(1 \otimes \mu \otimes \delta\mu)\} e_i \otimes (e_j \otimes u^2)^2 + \varphi(e_{i-1}[\mu(e_j \otimes u^2)]^2), \tag{15.9}$$

donde, $\varphi$, $\bar{\varphi}$, $\gamma$, $\bar{\gamma}$, $\mu$, son, respectivamente, los duales de $\varphi'$, $\bar{\varphi}'$, $\gamma'$, $\bar{\gamma}'$, $\mu'$.

En la construcción de $\Phi_{t,s}$, $\Theta_{t,s}$ en $f^{\#}u$, podemos determinar la elección de las cocadenas en $K$ en función de las elegidas en $L$. Así, sea

$$\varphi(1 \otimes f^{\#2}) + f^{\#}\bar{\varphi} = \delta\mu + \mu\delta, \tag{15.10}$$

el dual de (15.7), mod 2. Si $u$ es un cociclo mod 2 de $L$, tal que

$$\overline{\mathrm{Sq}}_j u = \bar{\varphi}(e_j \otimes u^2) = \delta\bar{a}_j,$$

usando 15.10, se comprueba con $a_j = f^{\#}\bar{a}_j + \mu(e_j \otimes u^2)$, que

$$\mathrm{Sq}_j f^{\#}u = \varphi(e_j \otimes (f^{\#}u)^2) = \delta a_j.$$

Además, con (15.9–10), podemos verificar que (mod 2),

$$\Gamma((\theta e_i e_j^2) \otimes u^4) \sim f^{\#}\bar{\gamma}(e_i e_j^2 \otimes u^4) + \gamma(e_i e_j^2 \otimes (f^{\#}u)^2) + f^{\#}\bar{p}_i(\bar{a}_j) + p_i(a_j), \tag{15.11}$$

donde, $\bar{p}_i$, $p_i$ están definidos, respectivamente, con $\bar{\varphi}$, $\varphi$.

Si $u$ es un $q$-cociclo mod 4, se tiene

$$\widetilde{(f^{\#}u)} = f^{\#}\tilde{u} + \mu(e_{q-1} \otimes u^2),$$

y resulta (mod 2),

$$\Gamma((\theta e_i e_{q-1}^2) \otimes u^4) \sim f^{\#}\bar{\gamma}(e_i e_{q-1}^2 \otimes u^4) + \gamma(e_i e_{q-1}^2 \otimes (f^{\#}u)^4)$$
$$+ f^{\#}\bar{p}_i(\tilde{u}) + p_i(\widetilde{(f^{\#}u)}). \tag{15.12}$$

Ahora, consideramos las cocadenas de (11.9–10), que, por claridad, las denotaremos con $\tilde{b}_{i+1}(u)$. Con las hipótesis necesarias en el $q$-cociclo $u$, se demuestra para $i + q$ impar, que

$$\tilde{b}_{i+1}(f^{\#}u) \sim f^{\#}\tilde{b}_{i+1}(u) + \mu(e_i \otimes u^2) \qquad \text{(mod 2)}.$$

Luego, se obtiene (mod 2),

$$\text{(15.13)} \quad \Gamma((\theta e_i e_q^2) \otimes u^4) \sim f^{\#}\bar{\gamma}(e_i e_q^2 \otimes u^4) + \gamma(e_i e_q^2 \otimes (f^{\#}u)^4) + f^{\#}\tilde{b}_{i+1}(u) + \tilde{b}_{i+1}(f^{\#}u).$$

Por último, para los elementos especiales de $\Theta_{t,s}$, se demuestra que si $u$ es un $q$-cociclo mod 2, y si $q$ es impar, entonces (mod 4),

$$\text{(15.14)} \quad 2\Gamma((\theta e_{2i+1} e_{2j}^2) \otimes u^4) \sim 2f^{\#}\bar{\gamma}(e_{2i+1}e_{2j}^2 \otimes u^4) + 2\gamma(e_{2i+1}e_{2j}^2 \otimes (f^{\#}u)^4)$$
$$+ f^{\#}\bar{p}_{2i+3}\bar{\text{S}}\text{q}_{2j}u + p_{2i+3}\text{Sq}_{2j}f^{\#}u,$$

y si $q$ es par, también (mod 4),

$$\text{(15.15)} \quad 2\Gamma((\theta e_{2i}e_{2j+1}^2) \otimes u^4) \sim 2f^{\#}\bar{\gamma}(e_{2i}e_{2j+1}^2 \otimes u^4) + 2\gamma(e_{2i}e_{2j+1}^2 \otimes (f^{\#}u)^4)$$
$$+ f^{\#}\bar{p}_{2i+2}\bar{\text{S}}\text{q}_{2j+1}u + p_{2i+2}\text{Sq}_{2j+1}f^{\#}u.$$

Combinando (15.2–3) con (15.11), . . . , (15.15) se obtiene la demostración del siguiente:

Teorema 15.16. *Sea $f_{\#} : K \to L$ una transformación de cadena (propia) y $f^{\#} : L^* \to K^*$ su dual. Si $u$ es un cociclo de $L$ tal que se pueda definir una de las operaciones $\Phi_{t,s}(u)$, $\Theta_{t,s}(u)$, entonces, según el caso, se puede construir $\Phi_{t,s}(f^{\#}u)$, $\Theta_{t,s}(f^{\#}u)$ de modo que resulte,*

$$f^{\#}\Phi_{t,s}(u) \sim \Phi_{t,s}(f^{\#}u) \qquad \text{(mod 2)},$$
$$f^{\#}\Theta_{t,s}(u) \sim \Theta_{t,s}(f^{\#}u) \qquad \text{(mod 4)}.$$

Claramente, si en el Teorema 15.16 tomamos $K = L$, y $f_{\#} = 1$, se obtiene el siguiente:

Corolario 15.17. *Para otra elección en la transformación $\varphi' : V \otimes K \to K^2$, las construcciones se pueden determinar de modo que $\Phi_{t,s}(u)$, $\Theta_{t,s}(u)$ se alteren únicamente por una cofrontera.*

Esto complementa el estudio de las desviaciones desarrollado en §12.

## 16. Las operaciones en clases de cohomología

Empezaremos por introducir una notación más conveniente para establecer los resultados.

En la construcción de las operaciones $\Phi_{t,s}$ hemos usado las relaciones (9.9–10), que son equivalentes, usando la notación superior, con ($a < b$)

$$\text{(16.1)} \qquad \text{Sq}^{2a}\text{Sq}^{b} = \textstyle\sum_{j=0}^{a} \lambda_j^{2a,b}\text{Sq}^{2a+b-j}\text{Sq}^{j} \qquad \text{(mod 2)}$$

donde,

$$\lambda_j^{2a,b} = \binom{b-j-1}{2a-2j}.$$

Consideremos las operaciones (con $k = 1, 2$),

(16.2) $$\mathrm{Sq}^b : H^q(K; Z_{2k}) \to H^{q+b}(K; Z_2),$$

(16.3) $$\lambda_j^{2a,b}\mathrm{Sq}^j : H^q(K; Z_{2k}) \to H^{q+j}(K; Z_2), \quad \text{con} \quad j = 2, \cdots, a,$$

(16.4) $$\lambda_0^{2a,b}\mathrm{Sq}^{2a+b-1} : H^q(K; Z_{2k}) \to H^{q+2a+b-1}(K; Z_2),$$

donde, $$k = 1, \text{ si } \lambda_1^{2a,b} = 0; \ k = 2, \text{ si } \lambda_1^{2a,b} = 1.$$

Para $a < b$, definimos

(16.5) $$N^q(2a, b, K, Z_{2k}),$$

subgrupo de $H^q(K; Z_{2k})$, como sigue. Si $b$ es impar, (16.5) se define para toda dimensión $q$, como la intersección de los núcleos de las operaciones (16.2), (16.3) y (16.4). Si $b$ es par y $\lambda_0^{2a,b} = 0$, para toda $q$, (16.5) es la intersección de los núcleos de las operaciones (16.2) y (16.3). Por último, si $b$ es par y $\lambda_0^{2a,b} = 1$, entonces, se considera únicamente $q < 2a + b$, y como en el caso anterior, (16.5) es la intersección de los núcleos de (16.2) y (16.3). Denotaremos (16.5) con $N^q(K; Z_{2k})$, ó más brevemente con $N^q(K)$.

Se verifica fácilmente que si $\{u\} \in N^q(K)$ y $q - b$ es impar, entonces $u$ satisface la hipótesis A, con $b = q - 2s - 1$, $a = q + s - t$; y si $q - b$ es par, entonces $u$ satisface la hipótesis A′, con $b = q - 2s$, $a = q + s - t$.

Con relación a los módulos (12.1–2), definimos

(16.6) $$M^{q+2a+b-1}(2a, b, K, Z_2),$$

como el subgrupo de $H^{q+2a+b-1}(K; Z_2)$, generado por los siguientes subgrupos:

$$\mathrm{Sq}^{2a}H^{q+b-1}(K; Z_2),$$

$$\lambda_0^{2a,b}\mathrm{Sq}^1H^{q+2a+b-2}(K; Z_2),$$

$$\lambda_j^{2a,b}\mathrm{Sq}^{2a+b-j}H^{q+j-1}(K; Z_2), \quad \text{con} \quad j = 2, \cdots, a.$$

Brevemente, denotaremos (16.6) con $M^{q+i}(K; Z_2)$, donde $i = 2a + b - 1$. Se comprueba que (16.6) coincide, con (12.1) si $q - b$ es impar, y con (12.2) si $q - b$ es par.

Ahora, para las operaciones $\Theta_{t,s}$, observando que $2\mathrm{Sq}^{2j+1} = 0 \bmod 4$ (ver (10.4)), de (16.1) se obtiene ($a < 2b$),

(16.7) $$2\mathrm{Sq}^{2a}\mathrm{Sq}^{2b} = 2 \textstyle\sum_{j=0}^{[a/2]} \lambda_j^{a,b}\mathrm{Sq}^{2a+2b-2j}\mathrm{Sq}^{2j} \qquad (\mathrm{mod}\ 4)$$

donde (cf. 23.8 de [2]),

$$\lambda_j^{a,b} = \binom{b-j-1}{a-2j}.$$

La fórmula (16.7) es equivalente con las fórmulas que se derivan de (11.7–8), y representa las relaciones que se utilizan para definir $\Theta_{t,s}$.

Consideremos las operaciones,

(16.8) $\quad \mathrm{Sq}^{2b} : H^q(K; Z_2) \to H^{q+2b}(K; Z_2),$

(16.9) $\quad \lambda_j^{a,b}\mathrm{Sq}^{2j} : H^q(K; Z_2) \to H^{q+2j}(K; Z_2), \quad \text{con} \quad j = 1, \cdots, [a/2].$

Para $a < 2b$, definimos

(16.10) $$N_1^q(2a, 2b, K, Z_2),$$

como la intersección de los núcleos de (16.8) y (16.9). Se considera toda $q$ si $\lambda_0^{a,b} = 0$, y únicamente $q < 2a + 2b$ si $\lambda_0^{a,b} = 1$. Claramente, (16.10) es un subgrupo de $H^q(K; Z_2)$. Como en el caso anterior, denotaremos (16.10) con $N_1^q(K; Z_2)$, o simplemente con $N_1^q(K)$.

Para $\{u\} \in N_1^q(K)$ se verifica que, si $q$ es impar, $u$ satisface la hipótesis B, con $2b = q - 2s - 1$, $a = q + s - t$, y si $q$ es par, entonces $u$ satisface la hipótesis B', con $2b = q - 2s$, $a = q + s - t$.

Análogamente a (16.6), con $i = 2a + 2b - 1$, definimos

(16.11) $$M_1^{q+i}(2a, 2b, K, Z_4),$$

como el subgrupo de $H^{q+i}(K; Z_4)$, generado por los siguientes subgrupos:

$$2\mathrm{Sq}^{2a}H^{q+2b-1}(K; Z_2)$$

$$2\lambda_j^{a,b}\mathrm{Sq}^{2a+2b-2j}H^{q+2j-1}(K; Z_2), \quad \text{con} \quad j = 1, \cdots, [a/2].$$

Denotaremos (16.11) simplemente con $M_1^{q+i}(K; Z_4)$. Se verifica que (16.11) coincide, con (12.3) si $q$ es impar, y con (12.4) si $q$ es par.

Para $\Phi_{t,s}$, $\Theta_{t,s}$ introduciremos la notación $\Phi_a^i$, $\Theta_a^i$, usando índices superiores, como sigue. Si las operaciones $\Phi_{t,s}(u)$, $\Theta_{t,s}(v)$ están definidas, entonces

(16.12) $$\Phi_{2q+s-t}^{3q-t-1}(u) = \Phi_{t,s}(u),$$

(16.13) $$\Theta_{2q+s-t}^{3q-t-1}(v) = \Theta_{t,s}(v),$$

donde $q = \dim u = \dim v$.

Con estas definiciones, para $\{u\} \in N^q(2a, b, K, Z_{2k})$, las operaciones (11.15–16) resultan ser $\Phi_{2a}^i(u)$, con $i = 2a + b - 1$. Igualmente, para $\{v\} \in N_1^q(2a, 2b, K, Z_2)$, las operaciones (11.17–18) resultan ser $\Theta_{2a}^i(v)$, con $i = 2a + 2b - 1$.

Definimos las operaciones $\Phi_{2a}^i$, $\Theta_{2a}^i$, en clases de cohomología, como las clases laterales

$$\Phi_{2a}^i(\{u\}) = \{\Phi_{2a}^i(u)\} + M^{q+i}(K; Z_2),$$

$$\Theta_{2a}^i(\{u\}) = \{\Theta_{2a}^i(v)\} + M_1^{q+i}(K; Z_4).$$

Las operaciones definidas de esta manera, resultan ser independientes de las elecciones arbitrarias que intervienen en su construcción (ver §12, (14.21), Corolario 15.17).

Luego, para $a > 0$, con $i = 2a + b - 1$, donde $a < b$, se tiene

(16.14) $\quad \Phi_{2a}^i : N^q(K; Z_{2k}) \to H^{q+i}(K; Z_2)/M^{q+i}(K; Z_2).$

Igualmente, para $a > 0$, con $i = 2a + 2b - 1$, donde $a < 2b$, se tiene

$$\Theta^i_{2a} : N^q_1(K; Z_2) \to H^{q+i}(K; Z_4)/M^{q+i}_1(K; Z_4). \tag{16.15}$$

Una transformación de cadena (propia), $f_\# : K \to L$, determina las transformaciones,

$$f^* : N^q(L) \to N^q(K), f^* : H^p(L)/M^p(L) \to H^p(K)/M^p(K),$$

$$f^* : N^q_1(L) \to N^q_1(K), f^* : H^p(L)/M^p_1(L). \to H^p(K)/M^p_1(K).$$

Si $u \in N^q(L)$, $v \in N^q_1(L)$, por el Teorema 15.16, se tiene que

$$f^*\Phi^i_{2a}(u) = \Phi^i_{2a}(f^*u), \qquad f^*\Theta^i_{2a}(v) = \Theta^i_{2a}(f^*v). \tag{16.16}$$

Por lo tanto, $\Phi^i_{2a}$, $\Theta^i_{2a}$ resultan naturales en el sentido de que conmutan con los homomorfismos inducidos por transformaciones contínuas $f : |K| \to |L|$. Esto implica la invariancia topológica de las operaciones.

Ahora, si expresamos los resultados del Teorema 14.21 por medio de la nueva notación y consideramos clases, obtenemos el siguiente:

TEOREMA 16.17. *Sean u, v dos elementos de* $N^q(K)$. *Si* $q + i$ *es impar,*

$$\Phi^i_{2a}(u + v) = \Phi^i_{2a}(u) + \Phi^i_{2a}(v).$$

*Si* $q + i$ *es par,*

$$\begin{aligned} \Phi^i_{2a}(u + v) &= \Phi^i_{2a}(u) + \Phi^i_{2a}(v) + m_s\lambda^*\mathrm{Sq}^{i-q}(u \smile v) \\ &\quad + n_s^{s+q-a}\, \lambda^*(\mathrm{Sq}^{(i-q)/2}u) \smile (\mathrm{Sq}^{(i-q)/2}v) \\ &\quad + (q \cdot \lambda_0^{2a,b})\lambda^*(\mathrm{Sq}^{(q-i)/2}u) \smile (\mathrm{Sq}^{(q-i)/2}v) \end{aligned} \tag{16.18}$$

*donde,* $\lambda^* : H^p(K) \to H^p(K)/M^p(K)$ *es la factorización natural,* $s = 1/2(q - i) + a - 1$, *y los coeficientes son los definidos en* (14.20).

*Análogamente, sean u, v son dos elementos de* $N^q_1(K)$. *Si q es par,*

$$\Theta^{2i+1}_{2a}(u + v) = \Theta^{2i+1}_{2a}(u) + \Theta^{2i+1}_{2a}(v).$$

*Si q es impar,*

$$\begin{aligned} \Theta^{2i+1}_{2a}(u + v) &= \Theta^{2i+1}_{2a}(u) + \Theta^{2i+1}_{2a}(v) + 2m_s\lambda^*\mathrm{Sq}^{2i+1-q}(u \smile v) \\ &\quad + 2n_s^{s+q-a}\lambda^*(\mathrm{Sq}^{i-1/2(q-1)}u) \smile (\mathrm{Sq}^{i-1/2(q-1)}v) \\ &\quad + 2\lambda_0^{a,b}\lambda^*(\mathrm{Sq}^{1/2(q-1)-i}u) \smile (\mathrm{Sq}^{1/2(q-1)-i}v), \end{aligned} \tag{16.19}$$

*donde,* $\lambda^* : H^p(K) \to H^p(K)/M^p_1(K)$, $s = 1/2(q - 1) + a - i - 1$.

Si $q$ es impar y $\lambda_0^{2a,b} = 1$, entonces $q \leqq i$ en $N^q(K)$. Luego, si el último término de (16.18) es distinto de cero, debe ser $u \smile v$. Se tiene una observación análoga para (16.19).

Claramente, si las operaciones están definidas para $q > i$ $(q > 2i + 1)$, resultan ser homomorfismos.

## 17. Relaciones con operaciones funcionales

Sea $f_{\#}: K \to L$ una transformación de cadena (propia), y $f^{\#}: L^* \to K^*$ su dual. Sea $u$ un $q$-cociclo de $L$ tal que $\Phi_{2t+1,2s+1}(u)$ esté definido. De acuerdo con el Teorema 15.16, se pueden construir las operaciones de modo que resulte

$$\begin{aligned} f^{\#}\Phi_{2t+1,2s+1}(u) &\sim \Phi_{2t+1,2s+1}(f^{\#}u) \\ &= A_{2t+1}^{2s+1}((f^{\#}u)^4) + B_{2t+1}^{2s+1}((f^{\#}u)^4). \end{aligned}$$

Ahora, si suponemos que

$$f^{\#}u = \delta a \qquad (\text{mod } 2k),$$

usando (13.18), se obtiene que

$$(17.1) \qquad f^{\#}\Phi_{2t+1,2s+1}(u) \sim P_{2t+1}^{2s+1}(a) + B_{2t+1}^{2s+1}((f^{\#}u)^4) \qquad (\text{mod } 2).$$

Análogamente, si suponemos que $u$ satisface las hipótesis necesarias para definir las operaciones respectivas, se tienen

$$(17.2) \qquad f^{\#}\Phi_{2t,2s}(u) \sim P_{2t}^{2s}(a) + B_{2t}^{2s}((f^{\#}u)^4) \qquad (\text{mod } 2),$$

$$(17.3) \qquad f^{\#}\Theta_{2t+1,2s+1}(u) \sim 2P_{2t+1}^{2s+1}(a) - C_{2t+1}^{2s+1}((f^{\#}u)^4) \qquad (\text{mod } 4),$$

$$(17.4) \qquad f^{\#}\Theta_{2t,2s}(u) \sim 2P_{2t}^{2s}(a) - C_{2t}^{2s}((f^{\#}u)^4) \qquad (\text{mod } 4).$$

Con el fin de interpretar los términos que se encuentran a la derecha de estas cohomologías, haremos las observaciones siguientes. Primero, si $\delta v = \delta w$ mod 2, entonces

$$p_i(v + w) \sim p_i(v) + p_i(w) \qquad (\text{mod } 2).$$

Ahora, como en §15, sean

$$\begin{aligned} a_j &= f^{\#}\tilde{a}_j + \mu(e_j \otimes u^2), \\ \widetilde{(f^{\#}u)} &= f^{\#}\tilde{u} + \mu(e_{q-1} \otimes u^2), \\ \tilde{b}_{i+1}(f^{\#}u) &\sim f^{\#}\tilde{b}_{i+1}(u) + \mu(e_i \otimes u^2). \end{aligned}$$

Entonces, $p_j(a) + a_j$ es un representante de la operación funcional $\mathrm{Sq}_j^f\{u\}$ (ver [10; p. 982]). Luego,

$$p_i(p_j(a) + a_j) \sim p_i p_j(a) + p_i(a_j)$$

es un representante de

$$\mathrm{Sq}_{i-1}(\mathrm{Sq}_j^f\{u\}) \in H^*(K; Z_2)/f^*\mathrm{Sq}_{i-1}H^*(L; Z_2) + \mathrm{Sq}_{i-1}\mathrm{Sq}_{j-1}H^*(K; Z_{2k}).$$

En la misma forma,

$$p_i(p_{q-1}(a) + \widetilde{(f^{\#}u)}) \sim p_i p_{q-1}(a) + p_i(\widetilde{(f^{\#}u)})$$

resulta un representante de

$$\mathrm{Sq}_{i-1}(\mathrm{Sq}_{q-1}^f\{u\}) \in H^*(K; Z_2)/\mathrm{Sq}_{i-1}\mathrm{Sq}_{q-1}H^*(K; Z_{2k}).$$

La elección canónica $\tilde{u}$ reduce el módulo. Por otra parte, si $\lambda_1^{2a,b} = 1$, para las operaciones $\Phi_{t,s}$ se considera $k = 2$, y en este caso el módulo se reduce a cero. Además se tiene $\mathrm{Sq}_{q-1}^f\{u\} = 0$ mod 2.

La cocadena $\tilde{b}_{i+1}(u)$ puede variar únicamente por un elemento de la forma $\frac{1}{2}\delta c$ donde $c$ es un cociclo mod 2 (ver §10). Por lo tanto,

$$\tilde{b}_{i+1}(f^{\#}u) + p_i p_q(a)$$

es un representante de

$$\mathrm{Sq}_i^f\{u\} \in H^*(K;\, Z_2)/f^*\mathrm{Sq}^1 H^*(L;\, Z_2) + \mathrm{Sq}_{i-1}H^*(K;\, Z_{2k}).$$

Por último, si $q$ es impar, se tiene que

$$2p_{2i+1}p_{2j}(a) - p_{2i+3}\mathrm{Sq}_{2j}(f^{\#}u) \sim 0 \qquad \text{(mod 4)},$$

y si $q$ es par, entonces

$$2p_{2i}p_{2j+1}(a) - p_{2i+2}\mathrm{Sq}_{2j+1}(f^{\#}u) \sim 0 \qquad \text{(mod 4)}$$

Esto es inmediato a partir de (10.2).

Por lo tanto, los términos a la derecha en (17.1–4) se identifican con una suma de cociclos, representantes de diferentes operaciones funcionales. Luego, para obtener un resultado en clases necesitamos considerar un módulo suficientemente grande para que resulte común a las diversas operaciones. Con este fin, y usando la notación de §16, introducimos las siguientes definiciones.

Con $i = 2a + b - 1$ definimos $Q^{q+i}(2a, b, K, Z_2)$, ó brevemente $Q^{q+i}(K;\, Z_2)$, como el subgrupo de $H^{q+i}(K;\, Z_2)$ generado por los siguientes subgrupos:

$$\mathrm{Sq}^{2a}\mathrm{Sq}^{b}H^{q-1}(K;\, Z_{2k}),$$

$$\lambda_j^{2a,b}\mathrm{Sq}^{2a+b-j}\mathrm{Sq}^{j}H^{q-1}(K;\, Z_{2k}), \quad \text{con } j = 0, \cdots, a.$$

Análogamente, con $i = 2a + 2b - 1$ definimos $Q_1^{q+i}(2a, 2b, K, Z_4)$, ó brevemente $Q_1^{q+i}(K;\, Z_4)$, como el subgrupo de $H^{q+i}(K;\, Z_4)$ generado por los siguientes subgrupos:

$$2\mathrm{Sq}^{2a}\mathrm{Sq}^{2b}H^{q-1}(K;\, Z_2),$$

$$2\lambda_j^{a,b}\mathrm{Sq}^{2a+2b-2j}\mathrm{Sq}^{2j}H^{q-1}(K{:}Z_2), \text{ con } j = 1, \cdots, [a/2].$$

Denotemos con $f^{**}$ las siguientes transformaciones inducidas por $f^* : H^p(L) \to H^p(K)$.

$$f^{**} : H^p(L)/M^p(L) \to H^p(K)/Q^p(K) + f^*M^p(L),$$

$$f^{**} : H^p(L)/M_1^p(L) \to H^p(K)/Q_1^p(K) + f^*M_1^p(L).$$

Se ha demostrado el teorema siguiente:

Teorema 17.5. *Sea $f^* : H^p(L) \to H^p(K)$ el homomorfismo inducido en cohomología por una transformación continua $f : |K| \to |L|$. Supongamos que para $u \in H^q(L)$ se tiene $f^*u = 0$. Si $u \in N^q(L)$,*

$$f^{**}\Phi_{2a}^i(u) = \mathrm{Sq}^{2a}(\mathrm{Sq}_f^b u) + \textstyle\sum_{j=0}^{a}\lambda_j^{2a,b}\mathrm{Sq}^{i-j+1}(\mathrm{Sq}_f^j u).$$

*Si $u \in N_1^q(L)$,*

$$f^{**}\Theta_{2a}^i(u) = 2\mathrm{Sq}^{2a}(\mathrm{Sq}_f^{2b}u) + 2\textstyle\sum_{j=1}^{[a/2]}\lambda_j^{a,b}\mathrm{Sq}^{i+1-2j}(\mathrm{Sq}_f^{2j}u).$$

## 18. Las operaciones en la cohomología relativa

Sea $K$ un complejo y $L \subset K$ un subcomplejo. Las definiciones y propiedades usadas en la construcción de $\Phi^i_{2a}$, $\Theta^i_{2a}$, se pueden extender a la cohomología relativa $H^q(K, L)$.

Consideraciones en el portador demuestran, que con (8.9), se tiene la transformación relativa

$$\psi : W \otimes_{S_4} (K, L)^{*4} \to (K, L)^*.$$

Resultados análogos se tienen para las transformaciones $\varphi'$, $\gamma'$ de (8.1), (8.11). Entonces, (9.7–8) son también relaciones válidas en $(K, L)$.

Se verifica fácilmente que las construcciones efectuadas en §11, así como también los resultados de §12, . . . , §15, se extienden a $(K, L)$.

Por lo tanto, con las definiciones (16.5–6), (16.10–11), para $(K, L)$, se obtienen las operaciones

$$\Phi^i_{2a} : N^q(K, L; Z_{2k}) \to H^{q+i}(K, L; Z_2)/M^{q+i}(K, L; Z_2),$$

$$\Theta^i_{2a} : N^q_1(K, L; Z_2) \to H^{q+i}(K, L; Z_4)/M^{q+i}_1(K, L; Z_4).$$

Estas operaciones satisfacen las propiedades (16.16) y el Teorema 16.17 en el caso relativo. En lo que sigue, estudiaremos las relaciones de $\Phi^i_{2a}$, $\Theta^i_{2a}$ con el operador cofrontera

$$\delta : H^q(L) \to H^{q+1}(K, L).$$

Para ésto, utilizaremos los resultados de §13, pero primero vamos a indicar algunas propiedades de las matrices que usamos.

Las matrices $\|a^k_j\|$, $\|d^k_j\|$ son triangulares. Como $|a^k_j| = 1$, el elemento $d^k_j$ es simplemente el cofactor de $a^j_k$ en el determinante $|a^k_j|$. Con la relación,

$$\textstyle\sum_{i=0}^{[t/2]} a^k_i d^i_j = \delta^k_j,$$

los elementos $d^k_j$ se pueden caracterizar inductivamente, como sigue:

$$d^k_j = 0 \quad \text{si } k < j; \qquad d^k_k = 1; \qquad d^k_j = \textstyle\sum_{i=0}^{k-1} a^k_i d^i_j \quad \text{si } k > j.$$

Un argumento inductivo demuestra que

$$d^k_j = d^{k+1}_{j+1} + d^{k+1}_{j+2}.$$

Por último, observamos que la matriz que se obtiene de $\|a^k_j\|$ suprimiendo la última columna y el último renglón, tiene como inverso la matriz que resulta al suprimir en $\|d^k_j\|$ la última columna y el último renglón.

Representamos con $\delta$ el operador cofrontera de $K$. Calcularemos módulo cofronteras de $(K, L)$, escribiendo $\sim$ en lugar del signo de igualdad.

Sean $u$, $v$ dos cociclos de $L$ satisfaciendo las hipótesis necesarias para poder definir $\Phi_{2t-2,2s}(u)$, $\Theta_{2t-2,2s}(v)$. Usando, los resultados anteriores sobre las matrices, las relaciones (5.8–11), y las hipótesis para definir las operaciones, de (13.18) se obtienen,

$$(18.1)\quad A^{2s+1}_{2t+1}((\delta u)^4) + P^{2s+1}_{2t+1}(u) \sim \delta A^{2s}_{2t-2}(u^4) + \delta \sum_{j=0}^{[(t-2)/2]} \binom{j-s-1}{2j+s+3-t} \mathrm{Sq}_{2t-4j-3}\mathrm{Sq}_{2j+1}u \pmod 2.$$

$$(18.2)\quad 2A^{2s+1}_{2t+1}((\delta v)^4) + 2P^{2s+1}_{2t+1}(v) \sim 2\delta A^{2s}_{2t-2}(v^4) \pmod 4.$$

Análogamente, si $u$, $v$ son dos cociclos de $L$ satisfaciendo las hipótesis necesarias para definir $\Phi_{2t-3,2s-1}(u)$, $\Theta_{2t-3,2s-1}(v)$, de (13.19) se obtienen ($s > 0$),

$$(18.3)\quad \begin{aligned} A^{2s}_{2t}((\delta u)^4) + P^{2s}_{2t}(u) &\sim \delta A^{2s-1}_{2t-3}(u^4) + \delta \mathrm{Sq}_{2t-4s-4}\mathrm{Sq}_{2s+1}u \\ &+ \delta \sum_{j=0}^{[(t-2)/2]} \binom{j-s-1}{2j-1+s-t} \mathrm{Sq}_{2t-4j-4}\mathrm{Sq}_{2j+1}u \\ &+ \delta \binom{t-1-2s}{2s} \mathrm{Sq}_0\mathrm{Sq}_{t-1}u + \delta d_1^s \psi(\omega^{-1}_{2t-3} \otimes u^4) \pmod 2, \end{aligned}$$

$$(18.4)\quad 2A^{2s}_{2t}((\delta v)^4) + 2P^{2s}_{2t}(v) \sim 2\delta A^{2s-1}_{2t-3}(v^4) + 2\delta d_1^s \psi(\omega^{-1}_{2t-3} \otimes v^4) \pmod 4.$$

El término $\omega^{-1}_{2t-3} = \Omega(d_{2t-2} \otimes d_{-1})$ se calcula explícitamente a partir de la relación (ver (3.3)),

$$\partial\Omega(d_j \otimes d_{-1}) + \Omega\, \partial(d_j \otimes d_{-1}) = (-1)^{j+1}\theta_{\#}\xi_{\#}(d_{-1} \otimes d_j).$$

Así, en $(K, L)$ se obtiene que

$$(18.5)\quad \delta\psi(\omega^{-1}_{2t-3} \otimes u^4) \sim \delta \sum_{j=0}^{[(t-2)/2]} (\mathrm{Sq}_{2j}u) \smile (\mathrm{Sq}_{2t-2j-2}u) + \delta \sum_{j=0}^{[(t-2)/2]} (\mathrm{Sq}_{2j+1}u) \smile (\mathrm{Sq}_{2t-2j-3}u),$$

donde $u$ es un cociclo mod 2 de $L$.

Denotamos con $\delta_L$ el operador cofrontera de $L$, y con $p_j^L$ el operador (10.1), construído con $\delta_L$. Para interpretar los términos $P_t^s(u)$ en (18.1–4), establecemos los resultados siguientes.

Sea $u$ un $q$-cociclo de $L$ tal que

$$\mathrm{Sq}_j u = \varphi(e_j \otimes u^2) = \delta_L a_j \pmod 2.$$

La cocadena $\bar{a}_j = \delta a_j + p_{j+1}(u)$ está en $(K, L)$, y se tiene

$$\mathrm{Sq}_{j+1}(\delta u) = \delta \bar{a}_j.$$

En $(K, L)$ se verifica que

$$p_i(\bar{a}_j) + \delta p^L_{i-1}(a_j) \sim p_i p_{j+1}(u) \pmod 2.$$

Supongamos que $u$ es un cociclo mod 4. Si definimos $\tilde{u}$, $\widetilde{(\delta u)}$ como en (10.10), la cocadena $\widetilde{(\delta u)}$ pertenece a $(K, L)$. Calculando módulo cofronteras de $(K, L)$, se obtiene que

$$p_i\widetilde{(\delta u)} + \delta p^L_{i-1}(\tilde{u}) \sim p_i p_q(u) \pmod 2.$$

Ahora consideraremos las cocadenas $\tilde{b}_i$ definidas en (11.9–10). Sea $u$ un q-cociclo mod 2 de $L$. Si representamos $u$ como cocadena mod 4, se tiene que $\delta_L u = 2w$, $\delta u = 2w + v$, donde $w$ es una cocadena de $L$, y $v$ es una cocadena de $(K, L)$. Luego $v$ representa a $\delta u$ como cocadena mod 4 de $(K, L)$.

Con $q + i$ par, supongamos que se tiene en $L$,

$$\varphi(e_i \otimes u^2) = \delta_L a_i + 2b_i \qquad \text{(mod 4)}.$$

Se comprueba que la cocadena análoga a $b_i$, en $(K, L)$ es

$$b_{i+1}(v) = \delta b_i + \delta_L b_i + \varphi(e_i \otimes uv) + \varphi(e_{i+1} \otimes wv) + \varphi(e_{i+1} \otimes u\ \delta w).$$

Resulta,

$$\tilde{b}_{i+1}(v) + \delta \tilde{b}_i \sim p_i p_{q+1}(u).$$

Para las operaciones $\Theta_{t,s}$ se establecen, en $(K, L)$, los siguientes resultados mod 4. Sean $v$, $w$ como en caso anterior. Si $q$ es par,

$$p_{2i+3}\mathrm{Sq}_{2j+2}v - \delta p^L_{2i+2}\mathrm{Sq}_{2j+1}u \sim 2p_{2i+1}p_{2j+2}(u),$$

y si $q$ es impar,

$$p_{2i+2}\mathrm{Sq}_{2j+1}v - \delta p^L_{2i+1}\mathrm{Sq}_{2j}u \sim 2p_{2i}p_{2j+1}(u).$$

En cada caso, estos resultados permiten demostrar que $P^s_t$ es igual a la diferencia formada con los términos de (11.11–14).

El operador cofrontera invariante $\delta$, transforma $N^q(L)$, $M^q(L)$, respectivamente, en $N^{q+1}(K, L)$, $M^{q+1}(K, L)$ (suponiendo que estos últimos están definidos), y por lo tanto, induce en los grupos factores, la transformación

$$\delta : H^q(L)/N^q(L) \to H^{q+1}(K, L)/M^{q+1}(K, L).$$

Resultados análogos se tienen para $N^q_1(L)$, $M^q_1(L)$.

Sea $\lambda^*$ la transformación del Teorema 16.17, en el caso relativo. Partiendo de (18.1–4), y usando lo que le sigue, se demuestra el teorema siguiente:

TEOREMA 18.6. *Suponemos, por simplicidad, que el anillo de cohomología de L es trivial (ver* (18.5)). *Sea* $u \in N^q(L)$. *Si* $q + i$ *es impar, entonces*

$$\delta\Phi^i_{2a}(u) = \Phi^i_{2a}(\delta u) + \begin{cases} \lambda^* \sum_{j=0}^{[(a+1)/2]} \binom{b-2j-3}{2a-4j+2} \mathrm{Sq}^{i-2j}\mathrm{Sq}^{2j}\delta u, & \text{si } q \text{ es impar,} \\ \lambda^* \sum_{j=0}^{[a/2]} \binom{b-2j-4}{2a-4j} \mathrm{Sq}^{i-2j-1}\mathrm{Sq}^{2j+1}\delta u, & \text{si } q \text{ es par.} \end{cases}$$

*Si* $q + i$ *es par, entonces*

$$\delta\Phi^i_{2a}(u) = \Phi^i_{2a}(\delta u) + \lambda^*\mathrm{Sq}^{2a+1}\mathrm{Sq}^{b-2}\delta u + \begin{cases} \lambda^* \sum_{j=0}^{[(a-3)/2]} \binom{b-2j-4}{2a-4j-6} \mathrm{Sq}^{i-2j}\mathrm{Sq}^{2j}\delta u, & \text{si } q \text{ es impar,} \\ \lambda^* \sum_{j=0}^{[(a-8)/2]} \binom{b-2j-5}{2a-4j-8} \mathrm{Sq}^{i-2j-1}\mathrm{Sq}^{2j+1}\delta u, & \text{si } q \text{ es par.} \end{cases}$$

*Por último, si* $v \in N_1^q(L)$, *resulta*

$$\delta\Theta_{2a}^{i}(v) = \Theta_{2a}^{i}(\delta v).$$

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# ON THE HOMOTOPY GROUPS OF SPHERES

I. M. JAMES

Let $S^q$ denote a $q$-dimensional sphere, where $q \geqq 1$. The homotopy groups of $S^q$ are the abelian groups $\pi_m(S^q)$, $m = 1, 2, \cdots$, whose elements are the homotopy classes of maps of $S^m$ into $S^q$. If $m < q$ there is only one homotopy class, likewise if $m > q$ and $q = 1$. The problem is to determine the groups when $m \geqq q > 1$. Present methods of computation only work when $m - q$ is small, and throw little light on the general case. There are some results, however, in which no such restriction is involved, and these appear to be of two kinds. The first kind are relations between the homotopy groups of various spheres, ranging from the classical isomorphism

$$\pi_m(S^2) \approx \pi_m(S^3) \qquad (m \geqq 3),$$

to the exact sequences of [4] and [9]. The second kind are theorems which give an indication, admittedly slight, of the general form taken by the homotopy groups of $S^q$. The following is an account of those results which come under the latter heading. The objective, of course, is a procedure for computing the homotopy groups of spheres which is as effective as, for example, that which we now have for the homology groups of Eilenberg-MacLane spaces. This problem has already been an important stimulus to research in algebraic topology. When it is solved the subject should open up very considerably.

A map of $S^q$ into itself determines an integer, the degree, which is unchanged when the map is subjected to a homotopy. By composing representatives with such a map we obtain, for each degree, an endomorphism of $\pi_m(S^q)$, which is not necessarily the same as multiplication by the degree. No doubt a computation of the homotopy groups of $S^q$ should include the determination of these endomorphisms. Another element of structure to be considered is the Whitehead product, which constitutes a bilinear, anti-commutative pairing of $\pi_m(S^q)$ with $\pi_n(S^q)$ to $\pi_{m+n-1}(S^q)$. But I will leave these and similar matters out of the present discussion.

The correspondence between maps of $S^q$ into itself and their degrees induces an isomorphism of $\pi_q(S^q)$ with the group of integers. A map of $S^{2q-1}$ into $S^q$ also determines an integer, the Hopf invariant, and from this correspondence we obtain a homomorphism of $\pi_{2q-1}(S^q)$ into the group of integers which is non-trivial if, and only if, $q$ is even. In certain cases, therefore, $\pi_m(S^q)$ is of infinite order. A decisive advance was made by Serre in [5] when he used spectral sequence methods to prove

THEOREM 1. *Let $m \neq q$, and let $m \neq 2q - 1$ if $q$ is even. Then $\pi_m(S^q)$ is finite. Let $q$ be even. Then $\pi_{2q-1}(S^q)$ is the direct sum of a finite group and an infinite cyclic group.*

The problem, therefore, is to determine these finite groups, and we may conveniently split them up into their $p$-primary components. That all the prime numbers $p$ are involved is shown by

THEOREM 2. *Let $q \geqq 2$, and let $p$ be a prime number. Then there are an infinite number of values of $m$ such that $\pi_m(S^q)$ contains elements of order $p$.*

The above theorem is proved by the method of killing homotopy groups, and makes use of information concerning the homology, modulo $p$, of the Eilenberg-MacLane spaces. The case $p = 2$ is due to Serre [7], and the general case is proved in a similar way with the help of results obtained by Cartan in [1]. The method does not reveal, however, whether the $p$-torsion of $\pi_m(S^q)$ can be bounded independently of $m$. The case $q = 1$, of course, is exceptional because the higher homotopy groups of $S^1$ are all trivial.

Our next theorem arises from the study of the endomorphisms induced by mapping $S^q$ onto itself. The proof is contained in [4] for the case $p = 2$, and in [9] for the case when $p$ is odd.

THEOREM 3. *Let $q$ be odd, and let $p$ be a prime number. Then the homotopy groups of $S^q$ contain no elements of order $p^q$.*

For example, let $q = 3$ and let $m > 3$. The theorem states that $\pi_m(S^3)$ is the direct sum of a number of cyclic groups of order $p^2$ and a number of cyclic groups of order $p$, where $p$ runs through the primes. In [5], however, Serre has shown that the primes greater than $m/2$ are not involved. Given $p$, values of $m$ can certainly be found such that $\pi_m(S^3)$ contains an element of order $p$, whereas elements of order $p^2$ have only been found in case $p = 2$. It would be interesting to decide whether elements of order $p^2$ are ever present when $p$ is odd and $q = 3$. It is interesting to note that $\pi_{28}(S^{13})$ contains an element of order 32, according to Toda [8], which suggests that the theorem is not too wide of the mark.

There are standard relations between the homotopy groups of even-dimensional spheres and those of odd-dimensional spheres (see [4] and [6]). By applying these in connection with Theorem 3 we obtain

THEOREM 4. *Let $q$ be even, and let $p$ be an odd prime number. Then the homotopy groups of $S^q$ contain no elements of order $p^{2q-1}$ nor any of order $4^q$. Suppose that $S^{2q-1}$ can be mapped into $S^q$ with Hopf invariant unity (as is the case when $n = 2, 4$ or $8$). Then the homotopy groups of $S^q$ contain no elements of order $2^{2q-1}$.*

It is useful, in the applications of obstruction theory, to have a version of these last two theorems formulated in terms of induced endomorphisms. Hence we apply formulae from [3] and [6] to those theorems and obtain

THEOREM 5. *Let $n \geqq 2$, $q \geqq 1$. Consider the endomorphisms of the homotopy groups of $S^q$ which are induced by mapping $S^q$ onto itself with degree $n^k$, where $k = q - 1$ if $n$ or $q$ is odd, and $k = q$ otherwise. The images of these endomorphisms contain no elements of order $n$.*

We have so far been considering the homotopy groups of one sphere at a time. The usual approach, introduced by Freudenthal in [2], is to proceed stem by stem. The $r$-stem, we recall, is the sequence of suspension homomorphisms

$$E\colon \pi_{q+r}(S^q) \to \pi_{q+r+1}(S^{q+1}) \qquad (q = 1, 2, \cdots).$$

We denote $\pi_{q+r}(S^q)$ by $G_r$ if $q > r + 1$, since $E$ is then an isomorphism, and we refer to $G_r$ as the stable group of the $r$-stem. By Theorem 1, $G_r$ is finite if $r > 0$, and $G_0$ is infinite cyclic. In some respects it appears that the stable groups may be easier to determine than the others. For example, the operation of composition induces a pairing of $G_r$ with $G_s$ to $G_{r+s}$, which turns the set of stable groups into a graded anti-commutative ring with a unit element.

I have recently proved the following result (to be published in the PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY).

THEOREM 6. *Let $p$ be a prime number. Then there are an infinite number of values of $k$ such that $G_{4k-1}$ contains elements of order $p$.*

The above theorem originates in a study of the quaternionic Steifel manifolds. Despite its apparent similarity to Theorem 2, the proof is entirely different, and neither of these results appear to be a corollary of the other. I have not been able to obtain any analogue of Theorem 3 in the case of the stable groups. On the contrary, it appears likely that values of $r$ exist, for each prime $p$ and integer $m$, such that $G_r$ contains elements of order $p^m$. Some information about the order of the elements can be obtained by combining Theorem 3 with various suspension theorems from [6] and [9]. The following, although capable of being improved, is included as an example of what can be deduced by such methods.

THEOREM 7. *Let $p$ be a prime number, and let $k$ be an integer. If $r < k(p-1)$ then $G_r$ contains no elements of order $p^k$.*

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# THE HUREWICZ THEOREM

BY DANIEL M. KAN

## 1. Introduction

Let $L$ be a simplicial complex, $\pi_n(L)$ its $n^{\text{th}}$ homotopy group (relative to some base point) and $H_n(L)$ its $n^{\text{th}}$ homology group. For each integer $n > 0$ let $h^* : \pi_n(L) \to H_n(L)$ be the Hurewicz homomorphism. Then the *Hurewicz theorem* states (see [2]):

(a) $h_* : \pi_1(L) \to H_1(L)$ *is onto and has the commutator subgroup* $[\pi_1(L), \pi_1(L)]$ *as kernel.*

(b) *if* $\pi_i(L) = 0$ *for* $1 \leqq i \leqq n$, *then* $h_* : \pi_{n+1}(L) \to H_{n+1}(L)$ *is an isomorphism and* $h_* : \pi_{n+2}(L) \to H_{n+2}(L)$ *is onto.*

The usual definitions of the homotopy groups of $L$ only involve its underlying topological space and disregard the simplicial structure of $L$; consequently the corresponding proofs of the Hurewicz theorem are also of a topological nature. In [3] a definition of the homotopy groups of $L$ and of the Hurewicz homomorphisms was given in terms of simplicial structure of $L$ only. The object of this paper is, starting from these definitions to give a completely combinatorial proof of the Hurewicz theorem. In fact it will be shown that the Hurewicz theorem may be considered as a special case of a purely group theoretical theorem.

We shall only consider the case of a c.s.s. complex which has only one 0-simplex. This is no real restriction as every simplicial complex may be converted into a c.s.s. complex by a (partial ordering of its vertices and as every connected c.s.s. complex is of the same homotopy type as one which has only one 0-simplex.

The paper is divided into two parts. In Part I the necessary definitions are given and the Hurewicz theorem is formulated and reduced to a purely group theoretical theorem. The proof of this theorem is given in Part II.

## PART I

## 2. C.s.s. complexes and c.s.s. groups

A *c.s.s. complex* $K$ (see [1]) is a collection of elements (called *simplices*) to each of which is attached a *dimension* $n \leqq 0$, such that for every $n$-simplex $\sigma \in K$ and every integer $i$ with $0 \leqq i \leqq n$ there are defined in $K$ an $(n-1)$-simplex $\sigma\varepsilon^i$ (called *face*) and an $(n+1)$-simplex $\sigma\eta^i$ (called *degenerate*). The operators $\varepsilon^i$ and $\eta^i$ are required to satisfy the following identities

$$\begin{aligned} \varepsilon^i\varepsilon^{j-1} &= \varepsilon^j\varepsilon^i & i &< j \\ \eta^{j-1}\eta^i &= \eta^i\eta^j & i &< j \\ \eta^j\varepsilon^i &= \varepsilon^i\eta^{j-1} & i &< j \\ \eta^j\varepsilon^i &= \text{identity} & i &= j, j+1 \\ \eta^j\varepsilon^i &= \varepsilon^{i-1}\eta^j & i &> j+1. \end{aligned}$$

The set of the $n$-simplices of $K$ is denoted by $K_n$. The face and degeneracy operators $\varepsilon^i$ and $\eta^j$ thus may be considered as functions $\varepsilon^i : K_n \to K_{n-1}$ and $\eta^i : K_n \to K_{n+1}$.

A c.s.s. map $f : K \to L$ is a dimension preserving function which commutes with all face and degeneracy operators, i.e., for every simplex $\sigma \in K_n$ and integer $i$ with $0 \leqq i \leqq n$

$$(f\sigma)\varepsilon^i = f(\sigma\varepsilon^i)$$

$$(f\sigma)\eta^i = f(\sigma\eta^i).$$

A *c.s.s. group* $G$ is a c.s.s. complex such that for every integer $n \geqq 0$

(a) $G_n$ is a group.

(b) all face and degeneracy operations $\varepsilon^i : G_n \to G_{n-1}$ and $\eta^i : G_n \to G_{n+1}$ are homomorphisms.

Let $G$ and $H$ be c.s.s. groups. A *c.s.s. homomorphism* $f : G \to H$ is a c.s.s. map such that for every integer $n \geqq 0$ the restriction $f_n : G_n \to H_n$ is a homomorphism.

A c.s.s. group $G$ is called *free* if $G_n$ is a free (non abelian) group for all $n$.

Let $G$ be a c.s.s. group. Define[1] for each integer $n \geqq 0$ a subgroup $\tilde{G}_n \subset G_n$ by

$$\tilde{G}_n = \cap_{i=1}^{n} \text{kernel } \varepsilon^i.$$

Then $\sigma \in \tilde{G}_{n+1}$ implies $\sigma\varepsilon^0 \in \tilde{G}_n$. Hence we may define a homomorphism $\tilde{\partial}_{n+1} : \tilde{G}_{n+1} \to \tilde{G}_n$ by

$$\tilde{\partial}_{n+1}\, \sigma = \sigma\varepsilon^0 \qquad \sigma \in \tilde{G}_{n+1}$$

For each integer $m < 0$ let $\tilde{G}_m = 1$ and let $\tilde{\partial}_{m+1} : \tilde{G}_{m+1} \to \tilde{G}_m$ be the trivial map. Then it can be shown that image $\tilde{\partial}_{n+1}$ is a normal subgroup of kernel $\tilde{\partial}_n$ for all $n$, i.e., $\tilde{G} = \{\tilde{G}_n, \tilde{\partial}_n\}$ is a (not necessarily abelian) chain complex. Its homology groups are

$$H_n(\tilde{G}) = \text{kernel } \tilde{\partial}_n / \text{image } \tilde{\partial}_{n+1}.$$

Let $\sigma \in$ kernel $\tilde{\partial}_n$. Then the element of $H_n(\tilde{G})$ containing $\sigma$ will be denoted by $\{\sigma\}$.

### 3. The homotopy groups

Let $K$ be a c.s.s. complex which has only one 0-simplex. Then we define a c.s.s. group $G$ as follows. $G_n$ is the (not necessarily abelian) group which has a generator $\bar{\sigma}$ for every $\sigma \in K_{n+1}$ and a relation $\overline{\tau\eta^0} = 1$ for every $\tau \in K_n$. As clearly the groups $G_n$ are free, it suffices to define the face and degeneracy homomorphisms $\varepsilon^i : G_n \to G_{n-1}$ and $\eta^i : G_n \to G_{n+1}$ on the generators of $G_n$. This is done by the following formulas:

$$\bar{\sigma}\varepsilon^0 = \overline{(\sigma\varepsilon^0)}^{-1}\, \overline{\sigma\varepsilon^1}$$

$$\bar{\sigma}\varepsilon^i = \overline{\sigma\varepsilon^{i+1}} \qquad 0 < i \leqq n$$

$$\bar{\sigma}\eta^i = \overline{\sigma\eta^{i+1}} \qquad 0 \leqq i \leqq n.$$

[1] This construction is due to J. C. Moore.

For every integer $n > 0$ we now define $\pi_n(K)$, *the* $n^{\text{th}}$ *homotopy group of* $K$, by

$$\pi_n(K) = H_{n-1}(\tilde{G}).$$

## 4. The homology groups

We define a c.s.s. group $A$ as follows. For each integer $n \geq 0$ let

$$A_n = G_n/[G_n, G_n]$$

where $[G_n, G_n]$ denotes the commutator subgroup of $G_n$, and let the face and degeneracy homomorphisms $\varepsilon^i : A_n \to A_{n-1}$ and $\eta^i : A_n \to A_{n+1}$ be those induced by the corresponding homomorphisms of $G$. Thus $A$ is "*G made abelian*" and we write

$$A \to G/[G, G].$$

Foı each integer $n > 0$ we define $H_n(K)$, *the* $n^{\text{th}}$ *homology group of* $K$, by

$$H_n(K) = H_{n-1}(\tilde{A}).$$

## 5. The Hurewicz homomorphisms

Let $k : G \to A$ denote the projection, i.e., $k$ maps an $n$-simplex of $G$ on the coset of $[G_n, G_n]$ containing it. Clearly $k$ is a c.s.s. homomorphism. It induces a chain map $\tilde{k} : \tilde{G} \to \tilde{A}$ (i.e., $\tilde{\partial}_n \tilde{k}\sigma = \tilde{k}\tilde{\partial}_n\sigma$ for every $\sigma \in \tilde{G}_n$) and hence induces homomorphisms

$$\tilde{k}_* : H_{n-1}(\tilde{G}) \to H_{n-1}(\tilde{A})$$

for each integer $n > 0$.

For each integer $n > 0$ we now define the *Hurewicz homomorphism* $h_* : \pi_n(K) \to H_n(K)$ by

$$h_* = \tilde{k}_*.$$

## 6. The Hurewicz theorem and its reduction to a group theoretical theorem

We first formulate both halves of the *Hurewicz theorem* in Theorem 1a and 1b below.

THEOREM 1a. *Let $K$ be a c.s.s. complex which has only one 0-simplex. Then the homomorphism $h_* : \pi_1(K) \to H_1(K)$ is onto and has $[\pi_1(K), \pi_1(K)]$ as kernel.*

THEOREM 1b. *Let $K$ be a c.s.s. complex which has only one 0-simplex and let $\pi_i(K) = 0$ for $0 < i \leq n$. Then $h_* : \pi_{n+1}(K) \to H_{n+1}(K)$ is an isomorphism and $h_* : \pi_{n+2}(K) \to H_{n+2}(K)$ is onto.*

It follows immediately from the definition of the Hurewicz homomorphism (see §5) that Theorem 1a and 1b are a special case of the following group theoretical theorems.

THEOREM 2a. *Let $F$ be a free c.s.s. group, let $B = F/[F, F]$ and let $\tilde{l} : \tilde{F} \to \tilde{B}$ be the chain map induced by the projection $l : F \to B$. Then $\tilde{l}_* : H_0(\tilde{F}) \to H_0(\tilde{B})$ is onto and has $[H_0(\tilde{F}), H_0(\tilde{F})]$ as kernel.*

THEOREM 2b. *Let $F$ be a free c.s.s. group, let $B = F/[F, F]$ and let $\tilde{l} : \tilde{F} \to \tilde{B}$ be* *the chain map induced by the projection $l: F \to B$. Let $H_i(\tilde{F}) = 0$ for $0 \leqq i < n$ Then $\tilde{l}_* : H_n(\tilde{F}) \to H_n(\tilde{B})$ is an isomorphism into and $\tilde{l}_* : H_{n+1}(\tilde{F}) \to H_{n+1}(\tilde{B})$ is onto.*

## PART II

### 7. Proof of Theorem 2a

The following lemmas will be needed for the proof of Theorem 2a.

LEMMA 1.[2] *Let $F$ be a c.s.s. group and let $\alpha_1, \cdots, \alpha_n \in F_{n-1}$ be such that $\alpha_i \varepsilon^{j-1} = \alpha_j \varepsilon_i$ for $0 < i < j \leqq n$. Then there exists an $\alpha \in F_n$ such that $\alpha\varepsilon^i = \alpha_i$ for $i = 1, \cdots, n$.*

PROOF. Let $\beta_n = \alpha_n \eta^{n-1}$. Then $\beta_n \varepsilon^n = \alpha_n$. Now suppose that $\beta_{k+1} \in F_n$ already has been defined such that $\beta_{k+1}\varepsilon^i = \alpha_i$ for $i \geqq k+1$. Define

$$\beta_k = (\alpha_k \eta^{k-1})(\beta_{k+1}{}^{-1}\, \varepsilon^k \eta^{k-1})\beta_{k+1}.$$

Then

$$\begin{aligned} \beta_k \varepsilon^k &= (\alpha_k \eta^{k-1}\varepsilon^k)(\beta_{k+1}{}^{-1}\varepsilon^k \eta^{k-1}\varepsilon^k)(\beta_{k+1}\varepsilon^k) = \alpha_k \\ \beta_k \varepsilon^i &= (\alpha_k \eta^{k-1}\varepsilon^i)(\beta_{k+1}{}^{-1}\varepsilon^k \eta^{k-1}\varepsilon^i)(\beta_{k+1}\varepsilon^i) \\ &= (\alpha_k \varepsilon^{i-1}\eta^{k-1})(\beta_{k+1}{}^{-1}\varepsilon^i \varepsilon^k \eta^{k-1})\alpha_i \\ &= (\alpha_i \varepsilon^k \eta^{k-1})(\alpha_i^{-1}\varepsilon^k \eta^{k-1})\, \alpha_i = \alpha_i \quad i \geqq k+1, \end{aligned}$$

i.e., $\beta_k \varepsilon^i = \alpha_i$ for $i \geqq k$. By induction on $k$ we finally obtain $\alpha = \beta_1 \in F_n$ such that $\alpha\varepsilon^i = \beta_1 \varepsilon^i = \alpha_i$ for $i = 1, \cdots, n$.

REMARK. In the above proof the element $\alpha \in F_n$ was obtained from the elements $\alpha_1, \cdots, \alpha_n \in F_{n-1}$ by application of the following operations only: $\varepsilon^i$, $\eta^i$, multiplication and taking inverses. We shall denote this element $\alpha \in F_n$ obtained from $\alpha_1, \cdots, \alpha_n$ in this specific way, by $e(\alpha_1, \cdots, \alpha_n)$. Clearly if $l: F \to B$ is a c.s.s. homomorphism, then $le(\alpha_1, \cdots, \alpha_n) = e(l\alpha_1, \cdots, l\alpha_n)$. Also if $\alpha_i = 1_{n-1}$, the unit element of $F_{n-1}$, for all $i$, then $e(\alpha_1, \cdots, \alpha_n) = 1_n$, the unit element of $F_n$.

LEMMA 2. *Let $F$ be a c.s.s. group, let $B = F/[F, F]$ and let $l: F \to B$ be the projection. Let $\psi \in \tilde{B}_n$. Then there exists a $\phi \in \tilde{F}_n$ such that $l\phi = \psi$.*

PROOF. Clearly $l$ is a c.s.s. homomorphism onto. Hence there exists an $\alpha \in F_n$ such that $l\alpha = \psi$. Let $\beta = e(\alpha\varepsilon^1, \cdots, \alpha\varepsilon^n)$. Because $l(\alpha\varepsilon^i) = (l\alpha)\varepsilon^i = \psi\varepsilon^i = 1_{n-1}$ for $i \neq 0$ it follows that $l\beta = le(\alpha\varepsilon^1, \cdots, \alpha\varepsilon^n) = e(l(\alpha\varepsilon^1), \cdots, l(\alpha\varepsilon^n)) = 1_n$. Let $\phi = \alpha\beta^{-1}$, then clearly $l\phi = l\alpha = \psi$ and $\phi\varepsilon^i = (\alpha\varepsilon^i)(\beta^{-1}\varepsilon^i) = 1_{n-1}$ for $i \neq 0$, q.e.d.

PROOF OF THEOREM 2a. The first part of Theorem 2a follows immediately from the fact that $l: F \to B$ is a c.s.s. homomorphism onto.

Let $\sigma \in F^0$ be such that $\{l\sigma\} = 0$, i.e., there exists a $\psi \in \tilde{B}_1$ such that $l\sigma = \psi\varepsilon^0$. Let $\phi \in \tilde{F}_1$ be such that $l\phi = \psi$ and let $\tau = (\phi^{-1}\varepsilon^0)\sigma$. Then $\{\sigma\} = \{\tau\}$. Furthermore $l\tau = (l\phi^{-1}\varepsilon^0)(l\sigma) = 1_0$. Hence $\tau\varepsilon[F_0, F_0]$ and $\{\sigma\} = \{\tau\} \in [H_0(\tilde{F}), H_0(\tilde{F})]$. As $H_0(\tilde{B})$ is abelian (because $B_0$ is abelian) it follows that the kernel of $\tilde{l}_* : H_0(\tilde{F}) \to H_0(\tilde{B})$ is exactly $[H_0(\tilde{F}), H_0(\tilde{F})]$. This completes the proof.

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[2] This lemma is due to J. C. Moore.

## 8. Proof of Theorem 2b

The following lemmas will be needed.

LEMMA 3. *Let $F$ be a c.s.s. group and let $\alpha \in F_n$ and $\phi \in \tilde{F}_1$ be such that $\alpha\varepsilon^n \cdots \varepsilon^1 = \phi\varepsilon^0$. Then there exist elements $\beta_0, \cdots, \beta_n \in F_{n+1}$ such that*

$$\beta_0\varepsilon^0 = \alpha$$
$$\beta_i\varepsilon^i = \beta_{i-1}\varepsilon^i \quad 0 < i \leqq n$$
$$\beta_i\varepsilon^{n+1} \cdots \varepsilon^{i+1} = 1_i \quad 0 \leqq i \leqq n$$

PROOF. Let

$$\beta_0 = (\alpha\eta^0)(\alpha^{-1}\varepsilon^n \cdots \varepsilon^1\eta^0 \cdots \eta^n)(\phi\eta^1 \cdots \eta^n).$$

Then

$$\beta_0\varepsilon^0 = \alpha(\alpha^{-1}\varepsilon^n \cdots \varepsilon^1\eta^0 \cdots \eta^{n-1})(\phi\varepsilon^0\eta^0 \cdots \eta^{n-1}) =$$
$$\beta_0\varepsilon^{n+1} \cdots \varepsilon^1 = (\alpha\varepsilon^n \cdots \varepsilon^1)(\alpha^{-1}\varepsilon^n \cdots \varepsilon^1)(\phi\varepsilon^1) = 1_0.$$

Now suppose $\beta_{k-1}$ has already been defined in such a manner that $\beta_{k-1}\varepsilon^{k-1} = \beta_{k-2}\varepsilon^{k-1}$ and $\beta_{k-1}\varepsilon^{n+1} \cdots \varepsilon^k = 1_{k-1}$.
Let

$$\beta_k = (\beta_{k-1}\varepsilon_k\eta^k)(\beta_{k-1}{}^{-1}\varepsilon^{n+1} \cdots \varepsilon^{k+2}\varepsilon^k\eta^k \cdots \eta^n)(\beta_{k-1}\varepsilon^{n+1} \cdots \varepsilon^{k+2}\varepsilon^k\eta^{k-1}\eta^{k+1} \cdots \eta^n).$$

Then a straightforward computation yields

$$\beta_k\varepsilon^k = \beta_{k-1}\varepsilon^k$$
$$\beta_k\varepsilon^{n+1} \cdots \varepsilon^{k+1} = l_k.$$

The lemma now follows by induction on $k$.

LEMMA 4. *Let $F$ be a c.s.s. group, let $\gamma \in$ kernel $\tilde{\partial}_n$ and let $\alpha \in F_n$ and $\phi \in \tilde{F}_1$ be such that $\alpha\varepsilon^n \cdots \varepsilon^1 = \varphi\varepsilon^0$. Then there exists a $\lambda \in \tilde{F}_{n+1} \cap [F_{n+1}, F_{n+1}]$ such that $\lambda\varepsilon^0 = \gamma\alpha\gamma^{-1}\alpha^{-1}$.*

PROOF. For each integer $i$ with $0 \leqq i \leqq n$ let

$$\lambda_i = (\gamma\eta^i)\beta_i(\gamma^{-1}\eta^i)\beta_i^{-1} \in [F_{n+1}, F_{n+1}$$

where $\beta_i$ is as in Lemma 3. Then

$$\lambda_0\varepsilon^0 = \gamma\alpha\gamma^{-1}\alpha^{-1}$$
$$\lambda_i\varepsilon^i = \gamma(\beta_i\varepsilon^i)\gamma^{-1}(\beta_i^{-1}\varepsilon^i)$$
$$\lambda_i\varepsilon^{i+1} = \gamma(\beta_i\varepsilon^{i+1})\gamma^{-1}(\beta_i^{-1}\varepsilon^{i+1}) = \gamma(\beta_{i+1}\varepsilon^{i+1})\,\gamma^{-1}(\beta_{i+1}^{-1}\varepsilon^{i+1}),\ i \neq n$$
$$\lambda_i\varepsilon^j = l_n \quad j \neq i \quad i+1$$
$$\lambda_n\varepsilon^{n+1} = l_n.$$

Let

$$\lambda = \textstyle\prod_{i=0}^n(\lambda_i)^{\varepsilon_1} \quad \text{where} \quad \varepsilon_i = (-1)^i.$$

Then it is readily verified that

$$\lambda \in \tilde{F}_{n+1} \cap [F_{n+1}, F_{n+1}]$$
$$\lambda\varepsilon^0 = \gamma\alpha\gamma^{-1}\alpha^{-1}, \qquad \text{q.e.d.}$$

LEMMA 5. *Let $F$ be a c.s.s. group such that $H_0(\tilde{F}) = 0$. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in F_n$ be such that $\alpha_1\varepsilon^i = \beta_1\varepsilon^i$ and $\alpha_2\varepsilon^i = \beta_2\varepsilon^i$ for all $i$. Then there exists a $\nu \in \tilde{F}_{n+1} \cap [\tilde{F}_{n+1}, F_{n+1}]$ such that*

$$\nu\varepsilon^0 = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}\beta_2\beta_1\beta_2^{-1}\beta_1^{-1}.$$

PROOF. Because $H_0(\tilde{F}) = 0$ it follows (using Lemma 4) that there exist elements $\lambda, \mu \in \tilde{F}_{n+1} \cap [F_{n+1}, F_{n+1}]$ such that

$$\lambda\varepsilon^0 = (\beta_2^{-1}\alpha_2)\alpha_1^{-1}(\alpha_2^{-1}\beta_2)\alpha_1$$
$$\mu\varepsilon^0 = (\beta_1^{-1}\alpha_1)\beta_2(\alpha_1^{-1}\beta_1)\beta_2^{-1}.$$

Let

$$\nu = (\alpha_1\eta^0)(\beta_2\eta^0)\lambda(\beta_2^{-1}\eta^0)(\alpha_1^{-1}\eta^0)(\beta_1\eta^0)\mu(\beta_1^{-1}\eta^0).$$

Then a direct computation yields

$$\nu\varepsilon^0 = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}\beta_2\beta_1\beta_2^{-1}\beta_1^{-1}$$
$$\nu\varepsilon^i = 1_n \qquad i \neq 0, \qquad \text{q.e.d.}$$

LEMMA 6. *Let $F$ be a free c.s.s. group and let $H_i(\tilde{F}) = 0$ for $0 \leq i < n$. Then there exist homomorphisms $D_i : F_i \to F_{i+1}$ $(0 \leq i < n)$ such that for every $\alpha \in F_i$*

$$(D_i\alpha)\varepsilon^0 = \alpha$$
$$(D_i\alpha)\varepsilon^j = D_{i-1}(\alpha\varepsilon^{j-1}) \qquad j \neq 0.$$

PROOF. Let $K$ be an integer such that $0 \leq k < n$ and suppose that for $i < k$ homomorphisms $D_i : F_i \to F_{i+1}$ have been defined satisfying the above conditions. As $F_k$ is a free group it is sufficient to define $D_k$ on a set of generators $\sum$ of $F_k$. This is done as follows. Let $\alpha \in \sum$ be a generator, and let

$$\delta = e(D_{k-1}(\alpha\varepsilon^0), \cdots, D_{k-1}(\alpha\varepsilon^k)).$$

Then for $0 \leq i \leq k$

$$(\alpha(\delta^{-1}\varepsilon^0))\varepsilon^i = (\alpha\varepsilon^i)(\delta^{-1}\varepsilon^0\varepsilon^i) = (\alpha\varepsilon_i)(\delta^{-1}\varepsilon^{i+1}\varepsilon^0)$$
$$= (\alpha\varepsilon^i)((D_{k-1}(\alpha^{-1}\varepsilon^i))\varepsilon^0) = (\alpha\varepsilon^i)(\alpha^{-1}\varepsilon^i) = l_{k-1}.$$

As $H_k(\tilde{F}) = 0$ there exists a $\phi \in \tilde{F}_{k+1}$ such that $\phi\varepsilon^0 = \alpha(\delta^{-1}\varepsilon^0)$. Now define

$$D_k\alpha = \phi\delta.$$

In order to prove that the homomorphism $D_k : F_k \to F_{k+1}$ defined in this manner has the desired properties it clearly suffices to show that this is the case for each generator $\alpha \in \sum$. Indeed for each $\alpha \in \sum$ we have

$$(D_k\alpha)\varepsilon^0 = (\phi\delta)\varepsilon^0 = \alpha(\delta^{-1}\varepsilon^0)(\delta\varepsilon^0) = \alpha$$
$$(D_k\alpha)\varepsilon^j = (\phi\delta)\varepsilon^j = \delta\varepsilon^j = D_{k-1}(\alpha\varepsilon^{j-1}) \quad j \neq 0.$$

The lemma now follows by induction on $k$.

LEMMA 7. *Let $F$ be a free c.s.s. group and let $H_i(\tilde{F}) = 0$ for $0 \leqq i < n$. Let $\rho \in$ kernel $\tilde{\partial}_n \cap [F_n, F_n]$. Then there exists a $\chi \in \tilde{F}_{n+1} \cap [F_{n+1}, F_{n+1}]$ such that $\chi\varepsilon^0 = \rho$.*

PROOF. As $\rho \in [F_n, F_n]$ there exists an integer $q$ and elements $\alpha_1, \cdots, \alpha_{2q} \in F_n$ such that

$$\rho = \textstyle\prod_{s=1}^{q} [\alpha_{2s-1}, \alpha_{2s}]$$

where [ , ] denotes the commutator. For $0 \leqq t \leqq 2q$ let

$$\delta_t = e(D_{n-1}(\alpha_t\varepsilon^0), \cdots, D_{n-1}(\alpha_t\varepsilon^n))$$

and let $\beta_t = \delta_t\varepsilon^0$. Then by Lemma 5 there exists for each integer $s$ with $0 \leqq s \leqq q$ a $\nu_s \in F_{n+1} \cap [F_{n+1}, F_{n+1}]$ such that

$$\nu_s\varepsilon^0 = [\alpha_{2s-1}, \alpha_{2s}][\beta_{2s}, \beta_{2s-1}].$$

Let

$$\chi = \textstyle\prod_{s=1}^{q} (\nu_s[\delta_{2s-1}, \delta_{2s}]).$$

Then a direct computation yields that $\chi\varepsilon^0 = \rho$ and $\chi\varepsilon^i = 1_n$ for $i \neq 0$, q e.d.

PROOF OF THEOREM 2b. Let $\sigma \in$ kernel $\tilde{\partial} \cap F_n$ be such that $\{l\sigma\} = 0$, i.e., there exists a $\psi \in \tilde{B}_{n+1}$ such that $l\sigma = \psi\varepsilon^0$. Let $\phi \in \tilde{F}_{n+1}$ be such that $l\phi = \psi$ and let $\nu = (\phi^{-1}\varepsilon^0)\sigma$. Then $\{\sigma\} = \{\tau\}$ and $l\tau = (l\phi^{-1}\varepsilon^0)(l\sigma) = 1_n$, i.e., $\tau \in$ kernel $\tilde{\partial}_n \cap [F_n, F_n]$. Hence by Lemma 7 $\{\tau\} = 0$. This proves the first part of Theorem 2b.

Let $\xi \in$ kernel $\tilde{\partial}_{n+1} \cap B_{n+1}$. Then there exists a $\rho \in \tilde{F}_{n+1}$ such that $l\rho = \xi$. As $\rho\varepsilon^0\varepsilon^i = \rho\varepsilon^{i+1}\varepsilon^0 = 1_{n-1}$ for all $i$ and $l(\rho\varepsilon^0) = (l\rho)\varepsilon^0 = \xi\varepsilon^0 = 1_n$, it follows that $\rho\varepsilon^0 \in$ kernel $\tilde{\partial}_n \cap [F_n, F_n]$. By Lemma 7 there exists a $\chi \in \tilde{F}_{n+1} \cap [F_{n+1}, F_{n+1}]$ such that $\chi\varepsilon^0 = \rho\varepsilon^0$. Hence $(\rho\chi^{-1})\varepsilon^i = 1_n$ for all $i$ and $l(\rho\chi^{-1}) = l\rho = \xi$, i.e., $\bar{l}_* \{\rho\chi^{-1}\} = \{\xi\}$. This completes the proof.

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# SEMI-SIMPLICIAL COMPLEXES AND POSTNIKOV SYSTEMS

By John C. Moore[1]

Classically, in algebraic topology, one studied simplicial complexes. However, many of the spaces which arise naturally in modern algebraic topology are not simplicial complexes. For example, the loop space of a simplicial complex is not a simplicial complex. This illustrates the fact that simplicial complexes are not adequate to deal with homotopy from a modern point of view. In 1950, the concept of semi-simplicial complex was introduced by Eilenberg and Zilber [1]. At present it seems certain that the category of semi-simplicial complexes and semi-simplicial maps is the most convenient category to work in when studying homotopy problems. Sometimes it is convenient to work with an arbitrary semi-simplicial complex, and sometimes with one satisfying the extension condition of Kan [2].

In this paper part of the theory of semi-simplicial complexes will be outlined, including in particular an outline of the development of homotopy theory for those complexes which satisfy the extension condition. After this is done, the results will be used to describe and discuss Postnikov systems [3].

Much of the material in this paper was presented in a course of lectures at Princeton during 1955–1956, or in the Cartan seminar of 1954–1955 [4].

## §1. Semi-simplicial complexes and homotopy

Definitions 1.1. Let $Z^+$ denote the set of non-negative integers. Now a *semi-simplicial complex* consists of the following:

(1) A set $X = \bigcup_{q \in Z^+} X_q$, where the $X_q$ are disjoint sets (an element of $X_q$ is called a $q$-simplex of $X$);

(2) functions $\partial_i : X_{q+1} \to X_q$, $i = 0, \cdots, q+1$, called *face operators*, and

(3) functions $s_i : X_q \to X_{q+1}$, $i = 0, \cdots, q$ called *degeneracy operators*.

The face and degeneracy operators are assumed to satisfy the relations

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i && i < j, \\ s_i s_j &= s_{j+1} s_i && i \leqq j, \\ \partial_j s_j &= \partial_{j+1} s_j = \text{identity}, && \\ \partial_i s_j &= s_{j-1} \partial_i && i < j, \text{ and} \\ \partial_i s_j &= s_j \partial_{i-1} && i > j+1. \end{aligned}$$

We will denote a semi-simplicial complex by its set $X$ of simplexes. A simplex $x \in X_{n+1}$ is called *degenerate* if $x = s_j y$ for some $y \in X_n$ and some degeneracy operator $s_j$; otherwise $x$ is called *non-degenerate*.

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EXAMPLE 1. Recall that a simplicial complex $K$ is a set whose elements are finite subsets of a given set $\bar{K}$, subject to the condition that if $x \in K$ and $y$ is a non-empty subset of $x$, then $y \in K$. Sets with 1 element are called vertices, and sets with $(n+1)$ elements are called $n$-simplexes of $K$.

Linearly order the elements of $\bar{K}$, i.e., the vertices of $K$. Now define a semi-simplicial complex $X(K)$ by letting the $n$-simplexes of $X(K)$ be $(n+1)$-tuples $(a_0, \cdots, a_n)$ of elements of $\bar{K}$ such that $a_0 \leqq \cdots \leqq a_n$, and such that the set $\{a_0, \cdots, a_n\}$ is an $r$ simplex of $K$ for some $r \leqq n$. Define

$$\partial_i(a_0, \cdots, a_n) = (a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n), \text{ and}$$

$$s_i(a_0, \cdots, a_n) = (a_0, \cdots, a_i, a_i, a_{i+1}, \cdots, a_n).$$

EXAMPLE 2. Let $\Delta_n$ denote the standard $n$-simplex, in other words a point of $\Delta_n$ is an $(n+1)$-tuple $(t_0, \cdots, t_{n+1})$ of real numbers such that $0 \leqq t_i \leqq 1$, $i = 0, \cdots, n$, and $\sum t_i = 1$. Let $A$ be a topological space. A singular $n$-simplex of $A$ is a map $U : \Delta_n \to A$. Denote by $S(A)_n$ the set of singular $n$-simplexes of $A$, and set $S(A) = \bigcup_{n \in Z+} S(A)_n$. Define

$$\partial_i : S(A)_n \to S(A)_{n-1}$$

by $\partial_i U(t_0, \cdots, t_{n-1}) = U(t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1})$, and define

$$s_i : (SA)_n \to S(A)_{n+1}$$

by $s_i U(t_0, \cdots, t_{n+1}) = U(t_0, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+i}, \cdots, t_{n+1})$. One verifies easily that $S(A)$ is a semi-simplicial complex; it is known as the total singular complex of the space $A$.

DEFINITION 1.2. A semi-simplicial complex $X$ is said to satisfy the *extension condition* if given $x_0, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1} \in X_n$ such that $\partial_i x_j = \partial_{j-1} x_i$ for $i < j$, $i, j \neq k$, there exists $x \in X_{n+1}$ such that $\partial_i x = x_i$ for $i \neq k$. Such a complex will be called a *Kan complex*.

PROPOSITION 1.3. *If $A$ is a topological space, then the total singular complex $S(A)$ satisfies the extension condition.*

The proposition follows from the fact that the union of $(n+1)$ faces $\Delta_{n+1}$ is a retract of $\Delta_{n+1}$.

Although it has long been realized that the total singular complex satisfies the extension condition, it was only recently that it was pointed out by D. M. Kan that the extension condition is sufficient for the definition of homotopy groups. In fact, in the category of Kan complexes and semi-simplicial maps, one can treat all problems involving only questions of homotopy type. The original work of Kan in this direction was done on semi-cubical complexes, but it was clear from the outset that one could work equally well with semi-simplicial complexes. At present almost everyone is agreed that for various technical reasons the category of semi-simplicial complexes is more convenient than the category of semi-cubical complexes.

DEFINITION 1.4. Let $\Delta_n$ denote the semi simplicial complex whose $q$-simplexes are $(q+1)$-tuples $(a_0, \cdots, a_q)$ of integers such that $0 \leqq a_0 \leqq \cdots \leqq a_q \leqq n$. Suppose further face and degeneracy operations are defined as in Example 1, by

$$\partial_i(a_0, \cdots, a_n) = (a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n), \text{ and}$$
$$s_i(a_0, \cdots, a_n) = (a_0, \cdots, a_{i-1}, a_i, a_i, a_{i+1}, \cdots, a_n).$$

This is just exactly the standard simplicial complex for the $n$-simplex. Further let

$$\varepsilon_i : \Delta_{n-1} \to \Delta_n, \text{ and}$$
$$\eta_i : \Delta_n \to \Delta_{n-1}$$

be the simplicial maps such that $\varepsilon_i(j) = j$ for $j < i$, $\varepsilon_i(j) = j+1, j \geqq i$; $\eta_i(j) = j$ for $j \leqq i$, and $\eta_i(j) = j-1, j > i$.

Denote by $\dot{\Delta}_n$ the subcomplex of $\Delta_n$ such that a $q$-simplex is a $(q+1)$-tuple $(a_0, \cdots, a_q)$ such that the set $\{a_0, \cdots, a_q\}$ has at most $n$ elements. In other words $\dot{\Delta}_n$ is the boundary of $\Delta_n$, or the $(n-1)$ skeleton of $\Delta_n$. All simplexes of $\dot{\Delta}_n$ of dimension greater than $(n-1)$ are degenerate.

DEFINITION 1.5. If $X$, $Y$ are semi simplicial complexes the Cartesian product $X \times Y$ of $X$ and $Y$ is the semi simplicial complex such that

(1) $(X \times Y)_n = \{(a, b) | a \in X_n, b \in Y_n\}$,

(2) $\partial_i : (X \times Y)_n \to (X \times Y)_{n-1}$ is defined by $\partial_i(a, b) = (\partial_i a, \partial_i b)$, and

(3) $s_i : (X \times Y)_n \to (X \times Y)_{n+1}$ is defined by $s_i(a, b) = (s_i a, s_i b)$.

PROPOSITION 1.6. *If $A$ and $B$ are topological spaces, then* $S(A \times B) = S(A) \times S(B)$.

This proposition follows immediately from the fact that a singular simplex in the product space $A \times B$ is uniquely determined by its projections on the factors [5].

DEFINITION 1.7. If $X$ and $Y$ are semi-simplicial complexes and $f : X \to Y$ is a function, then $f$ is a semi-simplicial map (or simply map) if

(1) $f(X_n) \subset Y_n$ for $n \in Z^+$,

(2) $\partial_i f = f \partial_i$, and

(3) $s_i f = f s_i$.

DEFINITION 1.8. If $X$ and $Y$ are semi-simplicial complexes, then the complex of maps from $X$ to $Y$ is the semi-simplicial complex $Y^X$ such that

(1) $(Y^X)_n = \{f | f : X \times \Delta_n \to Y \text{ is a map}\}$,

(2) $\partial_i f = f \circ (1 \times \varepsilon_i)$ for $f$ an $n$-simplex, and

(3) $s_i f = f \circ (1 \times \eta_i)$ for $f$ an $n$-simplex.

If $A$, $X$, $B$, and $Y$ are semi-simplicial complexes with $A \subset X$, $B \subset Y$ then $(Y, B)^{(X,A)}$ is the subcomplex of $Y^X$ such that an $n$-simplex is a map $f : (X \times \Delta_n, A \times \Delta_n) \to (Y, B)$.

THEOREM 1.9. *If $(X, A)$ and $(Y, B)$ are pairs of semi-simplicial complexes such that $Y$ and $B$ are Kan complexes, then $(Y, B)^{(X,A)}$ is a Kan complex.*

The proof of this theorem is somewhat long and tedious, but not particularly difficult.

DEFINITION 1.10. Let $X$ be a semi-simplicial complex, a *point* in $X$ is a zero simplex of $X$, i.e., an element of $X_0$, and a *path* in $X$ is a 1-simplex, i.e., an element of $X_1$. If $x$ is a path in $X$, then $\partial_1 x$ is the *initial point* or origin of $x$, and $\partial_0 x$ is the final or *terminal* point of $x$.

Equivalently a point of $X$ is a map of $\Delta_0$ into $X$, and a path in $X$ is a map of $\Delta_1$ into $X$.

Two points $a$, $b$ of $X$ are in the *same path component* of $X$ if there is a path in $X$ with initial point $a$ and final point $b$, this will be denoted by $a \sim b$.

PROPOSITION 1.11. *If $X$ is a Kan complex, and $a$, $b$, $c$, are points of $X$, then*

(i) $a \sim a$,

(ii) *if* $a \sim b$, *then* $b \sim a$, *and*

(iii) *if* $a \sim b$, $b \sim c$, *then* $a \sim c$.

NOTATION. For any Kan complex $X$ let $\pi_0(X)$ denote the set of path components of $X$. Further if $x \in X_0$ let $[x] \in \pi_0(X)$ denote the equivalence class of $x$.

DEFINITION 1.12. If $(Y, B)$ is a pair of Kan complexes, and $(X, A)$ is a pair of semi-simplicial complexes, then, $f, g : (X, A) \to (Y, B)$ are *homotopic* if and only if both $f$ and $g$ are in the same path component of $(Y, B)^{(X,A)}$, i.e., $[f] = [g] \in \pi_0((Y, B)^{(X,A)})$. A homotopy between $f$ and $g$ is a path in $(Y, B)^{(X,A)}$ joining $f$ to $g$. In other words a homotopy is a map $F : (\Delta_1 \times X, \Delta_1 \times A) \to (Y, B)$ such that $\partial_0 F = g$, $\partial_1 F = f$.

Now using the preceding proposition we have that homotopy between maps of a semi-simplicial pair into a Kan pair is an equivalence relation.

HOMOTOPY EXTENSION THEOREM 1.13. *If $(X, A)$ is a semi-simplicial pair, $Y$ a Kan complex, $f : X \to Y$ and $F : \Delta_1 \times A \to Y$ maps such that $\partial_1 F = f|A$, then there exists $\bar{F} : \Delta_1 \times X \to Y$ such that $\bar{F}|\Delta_1 \times A = F$, and $\partial_1 \bar{F} = f$.*

DEFINITION 1.14. Let $X$ be a Kan complex, and $x$ a point of $X$. Also let $x$ denote the subcomplex of $X$ generated by $x$, i.e., there is a unique $q$-simplex $s_0^q x$ in this subcomplex for every positive integer $q$. Now define.

$$\pi_n(X, x) = \pi_0((X, x)^{(\Delta_n, \dot{\Delta}_n)})$$

for $n > 0$.

LEMMA 1.15. *If $X$ is a Kan complex, $x$ a point of $X$, $f, g : (\Delta_n, \dot{\Delta}_n) \to (X, x)$, and $i$, $j$, $k$ distinct integers, then there exists $F : \Delta_{n+1} \to X$ such that $\partial_i F = f$, $\partial_j F = g$, $\partial_q F \in x$ for $q \neq i, j, k$, and if $F'$ is another such map, then $[\partial_k F] = [\partial_k F'] \in \pi_n(X, x)$.*

DEFINITION 1.16. Let $X$ be a Kan complex, $x$ a point of $X$, $f, g : (\Delta_n, \dot{\Delta}_n) \to (X, x)$ maps, and $F : \Delta_{n+1} \to X$ a map such that $\partial_{n+1} F = f$, $\partial_{n-1} F = g$, and $\partial_i F \in x$ for $i < n - 1$. Define

$$[f] + [g] = [\partial_n F] \in \pi_n(X, x).$$

PROPOSITION 1.17. *Let $X$ be a Kan complex, $x$ a point of $X$, then*

(1) *$\pi_n(X, x)$ is a group for $n > 0$ and*

(2) *$\pi_n(X, x)$ is abelian for $n > 1$.*

The group $\pi_n(X, x)$ is the *n-dimensional homotopy group of $X$* at the base point $x$.

The homotopy groups of Kan complexes enjoy all the usual properties of homotopy groups of spaces. In fact the homotopy groups of a topological space are just the homotopy groups of its singular complex. A few of the elementary properties of homotopy groups are summarized in the next proposition. Exact sequences of homotopy groups will not be considered until the next section which will be devoted to fibre spaces.

PROPOSITION 1.18. *Let* $(X, x)$, $(Y, y)$, $(Z, z)$ *be Kan complexes with base point, then*

(1) *if* $f: (X, x) \to (Y, y)$ *is a map,* $f$ *induces a homomorphism* $f^{\#}: \pi_n(X, x) \to \pi_n(Y, y)$ *for* $n > 0$.

(2) *if* $f: (X, x) \to (Y, y)$ *and* $g: (Y, y) \to (Z, z)$ *are maps then* $(gf)^{\#} = g^{\#}f^{\#}: \pi_n(X, x) \to \pi_n(Z, z)$ *for* $n > 0$,

(3) *if* $f, g: (X, x) \to (Y, y)$ *are maps such that* $f, g$ *belong to the same component of* $(Y, y)^{(X,x)}$, *then* $f^{\#} = g^{\#}$.

(4) *if* $f: (X, x) \to (X, x)$ *is the identity map, then* $f^{\#}$ *is the identity homomorphism, and*

(5) *if* $f: (X, x) \to (y, y)$, *then* $f^{\#}$ *is the zero homomorphism.*

Now following Eilenberg and Zilber ([1]) we will outline the proof that any Kan complex has a minimal subcomplex which is equivalent to the original complex as far as homotopy is concerned.

DEFINITION 1.19. Let $X$ be a Kan complex. The complex $X$ is *minimal* if whenever $x, y\ X_q$ are such that $\partial_i x = \partial_i y$ for $i \neq k$, then $\partial_k x = \partial_k y$. Two maps $f, g: \Delta_q \to X$ are *compatible* if $f|\dot{\Delta}_q = g|\dot{\Delta}_q$, and $f, g$ are homotopic if there exists $F: \Delta_1 \times \Delta_q \to X$ such that $F|(0) \times \Delta_q = f$, $F|(1) \times \Delta_q = g$, and $F(\sigma \times \tau) = f(\tau)$ for $\tau \in \dot{\Delta}_q$.

LEMMA 1.20. *The Kan complex* $X$ *is minimal if and only if for each compatible homotopic pair of maps* $f, g: \Delta_q \to X$ *we have* $f = g$.

DEFINITION 1.21. Let $X$ be a semi-simplicial complex and $A$ a subcomplex of $X$. Then $A$ is a deformation retract of $X$ if there is a map $F: \Delta_1 \times X \to X$ such that $F(\sigma \times \tau) = \tau$ for $\tau \in A$, $F(s_0^q(0) \times \tau) = \tau$, and $F(s_0^q(1) \times \tau) \in A$ for any $\tau \in X$.

THEOREM 1.22. *If* $X$ *is a Kan complex, then there is a minimal subcomplex* $M$ *of* $X$ *which is a deformation retract of* $X$, *and if* $M'$ *is another such complex, then* $M'$ *is isomorphic to* $M$.

## §2. Fibre spaces

Now having developed a little of the theory of semi-simplicial complexes, we now turn to the study of fibre spaces. It is here that the study of Postnikov systems naturally arises.

DEFINITIONS 2.1 A *fibre space* (or semi-simplicial fibre space) is a triple $(E, p, B)$ where $E$ and $B$ are semi-simplicial complexes, and $p: E \to B$ is a semi-simplicial map such that if $x \in B_{q+1}$, $y_0, \cdots, y_{k-1}, y_{k+1}, \cdots, y_{q+1} \in E_q$ are such that $p(y_i) = \partial_i x$ and $\partial_i y_j = \partial_{j-1} y_i$ for $i < j$, $i, j \neq k$, then there exists $y \in E_{q+1}$ such that $\partial_i y = y_i$ for $i \neq k$, and $p(y) = x$. The map $p$ is called a fibre map.

A fibre map $p: E \to B$ is *minimal* if $y, y' \in E_{q+1}$ are such that $p(y) = p(y')$ and $\partial_i y = \partial_i y'$ for $i \neq k$, then $\partial_k y = \partial_k y'$. The fibre space $(E, p, B)$ is *minimal* if the fibre map $p$ is minimal and the complex $B$ is minimal.

Let $b \in B_0$, and let $F$ be the counter image in $E$ of the complex generated by $b$. The complex $F$ is called the fibre over $b$.

PROPOSITION 2.2. *Let $(E, p, B)$ be a fibre space.*

(1) *If $F$ is the fibre over a point of $B$, then $F$ is a Kan complex.*

(2) *The complex $E$ is a Kan complex if and only if $B$ is a Kan complex.*

DEFINITION 2.3. Let $(E, p, B)$ be a fibre space, where $B$ is a Kan complex, $b$ a point of $B$ and $a$ a point of $F$ the fibre over $b$.

For $q \geqq 2$ define

$$\partial^{\#} : \pi_q(B, b) \rightarrow \pi_{q-1}(F, a).$$

Recall that $\alpha \in \pi_q(B, b)$ is represented by $x \in B_q$ such that $\partial_i x = s_0^{q-1} b$ for all $i$. Since $p$ is a fibre map, there exists $y \in E_q$ such that $p(y) = x$ and $\partial_i y = s_0^{q-1} a$ for $i > 0$. Then $\partial_0 y \in F_{q-1}$ and represents an element of $\pi_{q-1}(F, a)$. Checking independence of representative, define $\partial^{\#}([x]) = [\partial_0 y)$.

THEOREM 2.4. *Let $(E, p, B)$ be a fibre space, and suppose $B$ is a Kan complex. Let $b \in B_0$, $F$ be the fibre over $b$, and $a \in F_0$, then*

(1) *$\partial^{\#} : \pi_q(B, b) \rightarrow \pi_{q-1}(F, a)$ is a homomorphism for $q \geqq 2$, and*

(2) *The sequence*

$$\cdots \rightarrow \pi_q(F, a) \xrightarrow{i^{\#}} \pi_q(E, b) \xrightarrow{p^{\#}} \pi_q(B, b) \xrightarrow{\partial^{\#}} \pi_{q-1}(F, a) \rightarrow \cdots$$

*is exact, where $i : F \rightarrow E$ is the inclusion map.*

DEFINITIONS 2.5. Let $X$ be a semi-simplicial complex. Define a new semi-simplicial complex $X^{(n)}$ as follows:

(1) A $q$-simplex of $X^{(n)}$ is an equivalence class of $q$-simplexes of $X$, where two $q$ simplexes $x, x'$ are equivalent if their faces of dimension less than or equal to $n$ agree, i.e., $x, x' : \Delta_q^n \rightarrow X$ and $x|\Delta_q^n = x'|\Delta_q^n$ where $\Delta_q^n$ is the $n$-skeleton of $\Delta_q$.

(2) The face and degeneracy operations in $X^{(n)}$ are induced by those in $X$.

Let $X^{(\infty)} = X$, and let $p_k^n : X^{(n)} \rightarrow X^{(k)}$ be the natural map for $n \geqq k$, where $\infty \geqq k$ for every $k$. When there is no danger of confusion, $p_k^n$ will be abbreviated by $p$.

THEOREM 2.6. *Let $X$ be a Kan complex, then*

(1) *$X^{(n)}$ is a Kan complex for every $n$,*

(2) *$(X^{(n)}, p, X^{(k)})$ is a fibre space for $n \geqq k$, and*

(3) *if $x$ is a point of $X$, then $\pi_q(X^{(n)}, x) = 0$ for $q > n$, and*

$$p^{\#} : \pi_q(X^{(n)}, x) \xrightarrow{\approx} \pi_q(X^{(k)}, x) \quad \text{for} \quad q \leqq k.$$

In the context of complexes satisfying the extension condition, the proof of the preceding theorem is very easy. This theorem contains many classical results. For example, consider the case $(X, p, X^{(k)})$. We then have that $p^{\#} : \pi_q(X, x) \rightarrow \pi_q(X^{(k)}, x)$ for $q \leqq k$, and $\pi_q(X^{(k)}, x) = 0$ for $q > k$. In other words the fibre spaces $(X, p, X^{(k)})$ are the precise analogue of the construction (II) of Cartan and Serre [8].

DEFINITION 2.7. Let $X$ be a Kan complex, and $x$ a point of $X$. Let $E_n(X, x)$ denote the fibre over the point $x$ of $p : X \rightarrow X^{(n-1)}$. The complex $E_n(X, x)$ is the $n^{\text{th}}$ *Eilenberg subcomplex of $X$* based at $x$, and is that subcomplex of $X$ consisting of simplexes whose faces of dimension less than $n$ are at the base point $x$, [9].

PROPOSITION 2.7′. *Let $X$ be a Kan complex with base point $x$. We then have*

(1) $\pi_q(E_n(X, x), x) = 0$ *for* $q < n$, *and*

(2) $i^{\#} : \pi_q(E_n(X, x)x) \approx \pi_q(X, x)$ *for* $q \geqq n$, *where* $i : E_n(X, x) \to X$ *is the inclusion map.*

DEFINITION 2.8. If $X$ is a Kan complex, let $\chi^n$ be the fibre space $(X^{(n+1)}, p, X^{(n)})$. The sequence of fibre spaces $\chi = (\chi^0, \chi^1, \cdots, \chi^n, \cdots)$ is by definition the *natural Postnikov system of* $X$, [3].

DEFINITION 2.9. Let $X$ be a connected Kan complex with base point $x$. Then $X$ is an *Eilenberg-MacLane complex of type* $(\pi, n)$ if and only if $\pi_q(X,x) = 0$ for $q \neq n$, and $\pi_n(X, x) = \pi$.

THEOREM 2.10. *If $X$ is a Kan complex with base point $x$, $\chi$ is the natural Postnikov system of $X$, and $F^{(n+1)}$ is the fibre of the map $p : X^{(n+1)} \to X^{(n)}$ which is the $n^{\text{th}}$ term of $\chi$, then $F^{(n+1)}$ is an Eilenberg-MacLane complex of type $(\pi_{n+1}(X, x), n + 1)$.*

With this theorem we see that any Kan complex may be constructed in some sense from Eilenberg-MacLane complexes. The process of so doing will be studied further later. However, before doing so we want to consider a generalization of the preceding which applies to a fibre map. It may well at this stage to point out that if $X$ is a semi-simplicial complex and $x$ the complex of a point, then the unique map $f : X \to x$ is a fibre map if and only if $X$ is a Kan complex.

DEFINITION 2.11. Let $p : E \to B$ be a fibre map, $e$ a point of $E$, $b = p(e)$ and $F$ the fibre over $b$. Suppose that $B$ and $F$ are connected and that $B$ is a Kan complex (recall that this means $E$ is connected and a Kan complex). Define a new semi-simplicial complex $E^{(n)}$ as follows:

(1) A $q$-simplex of $E^{(n)}$ is an equivalence class of $q$-simplexes of $E$ where two $q$-simplexes $x, x'$ are equivalent if

(i) $p(x) = p(x')$, and

(ii) $x|\Delta^n_q = x'|\Delta^n_q$.

(2) The face and degeneracy operations in $E^{(n)}$ are induced by those in $E$.

Let $E^{(\infty)} = E$, and let $p^n_k : E^{(n)} \to E^{(k)}$ be the natural map for $n \geqq k$, where $\infty \leqq k$.

THEOREM 2.12. *Let $(E, p, B)$ be a fibre space of connected Kan complexes, $e$ a point of $E$, $b = p(e)$, and $F$ the fibre over $b$. Then*

(1) *$E^{(n)}$ is a Kan complex for every $n$,*

(2) $E^{(0)} = B$ if $E_0 = \{ e \}$,

(3) *$(E^{(n)}, p, E^{(k)})$ is a fibre space for $n \geqq k$,*

(4) $\pi_q(E, e) \overset{\approx}{\to} \pi_q(E^{(n)}, e)$ *for* $q \leqq n$,

(5) $\pi_q(E^{(n)}, e) \overset{\approx}{\to} \pi_q(B, b)$ *for* $q > n + 1$, *and*

(6) *the fibre of $p : E^{(n)} \to B$ is $F^{(n)}$, the $n^{\text{th}}$ term in the Postnikov system for the fibre $F$.*

DEFINITION 2.13. Let $(E, p, B)$ be a fibre space of connected Kan complexes, and let $\varepsilon^n$ be the fibre space $(E^{(n+1)}, p, E^{(n)})$. The sequence of fibre spaces $\varepsilon = (\varepsilon^0, \varepsilon^1, \cdots, \varepsilon^n, \cdots,)$ is by definition the natural Postnikov system of $(E, p, B)$.

THEOREM 2.14. *If $(E, p, B)$ is a fibre space of connected Kan complexes, $e$ a point* *of $E$, $b = p(e)$, and $F$ the fibre over $b$, then the fibre over $b$ in $\varepsilon^n$, the $n^{\text{th}}$ term of the Postnikov system of the fibre space $(E, p, B)$ is an Eilenberg-MacLane complex of type $(\pi_{n+1}(F, e), n + 1)$.*

Consequently as a result of this theorem we see that a fibre space may be constructed by giving a base complex $B$, and then "adding" in the homotopy groups of the fibre one at a time. In fact one has given a fibre space $(E, p, B)$ an infinite sequence of fibre spaces, $E \to \cdots \to E^{(n+1)} \to E^{(n)} \to \cdots \to E^{(1)}\ B$, and the special case of this were $B$ is the complex of a point gives exactly the ordinary Postnikov system of $E$.

Just as any Kan complex has a minimal complex, which is a deformation retract, any fibre space has a minimal fibre space. The situation however is somewhat better than this as will be seen by the following theorems.

THEOREM 2.15. *Let $(E, p, B)$ be a fibre space of connected Kan complexes, and let $B'$ be a minimal subcomplex of $B$ which is equivalent to $B$. Define $E' = p^{-1}(B')$. Then*

(1) *$(E', p, B')$ is a fibre space, and*

(2) *there exists a commutative diagram*

$$\begin{array}{ccc} E \times \Delta_1 & \xrightarrow{F} & E \\ \Big\downarrow{\scriptstyle p \times 1} & & \Big\downarrow{\scriptstyle p} \\ B \times \Delta_1 & \xrightarrow{f} & B \end{array}$$

*such that* $F(\sigma, \tau) = \sigma$ *for* $\sigma \in E'$,

$F(\sigma, s_0^q\, 0) = \sigma$ *for* $\sigma \in E_q$, *and*

$F(\sigma, s_0^q\, 1) \in E'_q$ *for* $\sigma \in E_q$.

THEOREM 2.16. *Let $(E, p, B)$ be a fibre space of connected Kan complexes, and suppose $B$ is minimal. Then there exists $E' \subset E$ such that*

(1) *$(E', p, B)$ is a minimal fibre space, and*

(2) *there exists a commutative diagram*

$$\begin{array}{ccc} E \times \Delta_1 & \xrightarrow{F} & E \\ \Big\downarrow{\scriptstyle p \times 1} & & \Big\downarrow{\scriptstyle p} \\ B \times \Delta_1 & \xrightarrow{f} & B \end{array}$$

*such that* $F(\sigma, \tau) = \sigma$ *for* $\sigma \in E'$, $\tau \in \Delta_1$,

$F(\sigma, s_0^q 0) = \sigma$ *for* $\sigma \in E_q$

$F(\sigma, s_0^q 1) \in E'_q$ *for* $\sigma \in E_q$, *and*

$f(\sigma, \tau) = \sigma$ *for* $\sigma \in B$, $\tau \in \Delta_1$.

Henceforth when we speak of a minimal subcomplex of a complex $X$ we will usually mean one which is a deformation retract of $X$, and similarly in the case of fibre spaces minimal sub fibre spaces will usually mean one which is a deformation retract of the original.

THEOREM 2.17. *Let $(E, p, B)$ be a fibre space of connected Kan complexes, and let $(E', p, B')$ and $(E'', p, B'')$ be minimal fibre spaces which are deformation retracts of $(E, p, B)$, then $(E', p, B')$ and $(E'', p, B'')$ are isomorphic.*

PROPOSITION 2.18. *If $(E, p, B)$ is a minimal fibre space of connected complexes, and $\varepsilon^n = (E^{(n+1)}, p, E^{(n)}$ is the $n^{\text{th}}$ term in the natural Postnikov system $\varepsilon$, then $p : E^{(n+1)} \to E^{(n)}$ is a minimal fibre map.*

We have now reduced the problem of studying either complexes or fibre spaces with the extension condition to the problem of studying minimal ones. Further we have seen that all of these things are put together out of Eilenberg-MacLane complexes. Consequently we wish to make this process more explicit, to see how unique it is, and to see its relationship without homotopy type. These questions will be dealt with in inverse order.

DEFINITION 2.19. If $(E, p, B)$ and $(E', p, B')$ are fibre spaces then a map of the first into the second is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & E' \\ \downarrow p & & \downarrow p' \\ B & \xrightarrow{f} & B' \end{array}$$

of semi-simplicial complexes. Such a mapping $(\bar{f}, f)$ is homotopic to a mapping $(\tilde{g}, g)$ if there exists a commutative diagram of semi simplicial complexes

$$\begin{array}{ccc} E \times \Delta_1 & \xrightarrow{\tilde{F}} & E' \\ \downarrow p \times 1 & & \downarrow p' \\ B \times \Delta_1 & \xrightarrow{F} & B' \end{array}$$

such that

(1) $\tilde{F}(\sigma, s_0^q 0) = \bar{f}(\sigma)$, and

(2) $\tilde{F}(\sigma, s_0^q 1) = \tilde{g}(\sigma)$.

DEFINITIONS 2.20. Two semi simplicial complexes $X$ and $Y$ have the same *homotopy type* if and only if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $fg$ is homotopic to the identity map of $Y$ and $gf$ is homotopic to the identity map of $X$.

Two fibre spaces $(E, p, B)$ and $(E', p', B')$ have the same *homotopy type* if and only if there exist maps $(\bar{f}, f) : (E, p, B) \to (E', p', B')$ and $(\tilde{g}, g) : (E', p', B') \to (E, p, B)$ such that $(\bar{f}, f)(\tilde{g}, g)$ is homotopic to the identity map of $(E', p', B')$ and $(\tilde{g}, g)(\bar{f}, f)$ is homotopic to the identity map of $(E, p, B)$.

LEMMA 2.21. *If $X$ and $Y$ are connected minimal complexes such that $\pi_q(X) = \pi_q(Y) = 0$ for $q \neq n$, and $\varphi : \pi_n(X) \to \pi_n(Y)$ is a homomorphism, then there is a unique map $f : X \to Y$ such that $f^{\#} = \varphi : \pi_n(X) \to \pi_n(Y)$.*

COROLLARY. *If $X$ and $Y$ are connected minimal complexes such that $\pi_n(X) \cong \pi_n(Y)$ and $\pi_q(X) = \pi_q(Y) = 0$ for $q \neq n$, then $X$ is isomorphic with $Y$.*

Thus we see that two minimal complexes of type $(\pi, n)$ are isomorphic. Later we will see how to prove the existence of such complexes after the fashion of Eilenberg-MacLane.

PROPOSITION 2.22. *If $X$ and $Y$ are connected minimal complexes and $f : X \to Y$ is a map such that $f^{\#} : \pi_q(X) \to \pi_q(Y)$ for all $q$, then $f$ is an isomorphism.*

THEOREM 2.23. *Let $X$ and $Y$ be connected Kan complexes. The following conditions are equivalent:*

(1) *$X$ and $Y$ have the same homotopy type,*

(2) *there is a map $f : X \to Y$ such that*

$$f^{\#} : \pi_q(X) \overset{\approx}{\to} \pi_q(Y) \text{ for all } q, \text{ and}$$

(3) *$X$ and $Y$ have isomorphic minimal subcomplexes.*

The fact that conditions 1 and 2 in the preceding theorem are equivalent is in the framework of CW-complexes a well-known theorem of J. H. C. Whitehead [10]. We now give the analogue of this theorem for fibre spaces.

THEOREM 2.24. *Let $(E, p, B,)$ and $(E', p', B')$ be fibre spaces of connected Kan complexes. The following conditions are equivalent:*

(1) *$(E, p, B)$ and $(E', p', B')$ have the same homotopy type,*

(2) *there is a map $(\bar{f}, f) : (E, p, B) \to (E', p', B')$ such that at least two of the following conditions are verified.*

$$\text{(i) } \bar{f}^{\#} : \pi_q(E) \overset{\approx}{\to} \pi_q(E') \text{ for all } q$$

$$\text{(ii) } f^{\#} : \pi_q(B) \overset{\approx}{\to} \pi_q(B') \text{ for all } q$$

$$\text{(iii) } \bar{f}^{\#} : \pi_q(F) \overset{\approx}{\to} \pi_q(F') \text{ for all } q$$

*where $F$ and $F'$ are the fibres in the respective fibre spaces, and*

(3) *$(E, p, B)$ and $(E', p', B')$ have isomorphic minimal sub fibre spaces.*

With this theorem we complete our study of fibre spaces from an elementary point of view. Now we pass on to study them in more detail using cohomology and twisted Cartesian products. The notion of twisted Cartesian product is not an invariant one, but any principal fibre space, or any minimal fibre space may be given such a structure. Further any Postnikov system which is minimal may be constructed as a series of twisted Cartesian products.

## §3. Twisted Cartesian products and monoid complexes

DEFINITION 3.1. A *twisted Cartesian product* is a triple $(F, B, E)$ such that $F, B, E)$ are semi-simplicial complexes with $E_q = \{(a, b)\}|a \in F_q$ and $b \in B_q.\}$ Defining $p : E \to B$ by $p(a, b) = b$ and $i_b : F \to E$ by $i_b(a) = (a, s_0^q b)$ for $b$ a point of $B$ and $a \in F_q$ we assume further

(1) $p$ is a semi-simplicial map,

(2) $i_b$ is a semi-simplicial map for any point $b$ in $B$, and

(3) $\partial_i(a, b) = (\partial_i a, \partial_i b)$ for $i > 0$, and

$$s_i(a, b) = (s_i a, s_i b) \text{ for } i \geq 0.$$

$F$ is called the fibre of the twisted Cartesian product, $B$ the base, and $E$ the total space or total complex. Usually, but not always, the map $p$ will be a fibre map as defined earlier. Notice that $E$ is the Cartesian product of $F$ and $B$ if and only if $\partial_0(a, b) = (\partial_0 a, \partial_0 b)$ for $(a, b)$ a positive dimensional simplex of $E$.

PROPOSITION 3.2. *Let $(F, B, E)$ be a twisted Cartesian product, and define $\tau : E_{q+1} \to F_r$ by the equation $\partial_0(a, b) = (\tau(a, b), \partial_0 b)$, then $\tau$ satisfies the identities.*

(1) $\tau(\partial_1 a, \partial_1 b) = \tau(\tau(a, b), \partial_0 b)$,
(2) $\tau(\partial_{i+1} a, \partial_{i+1} b) = \partial_i \tau(a, b)$ *for* $i > 0$
(3) $\tau(s_0 a, s_0 b) = a$,
(4) $\tau(s_{i+1} a, s_{i+1} b) = s_i \tau(a, b)$, *and*
(5) $\tau(a, s_0^q b) = \partial_0 a$ *for $b$ a point of $B$.*

*Further if $F$ and $B$ are semi simplicial complexes and $\tau : (F \times B)_{q+1} \to F_q$ is a function satisfying identities 1 through 5 above, then one can define a unique twisted Cartesian product $(F, B, E)$ so that in $E$ $\partial_0(a, b) = (\tau(a, b), \partial_0 b)$.*

The function $\tau$ of the preceding proposition is known as a *twisting function, and* the proposition establishes a one to one correspondence between twisting functions $\tau : F \times B \to F$ and twisted Cartesian products $(F, B, E)$.

DEFINITIONS 3.3. A semi-simplical complex $\Gamma$ is a *monoid complex* if
(1) $\Gamma_q$ is a monoid with identity for each $q$, and
(2) $\partial_i : \Gamma_{q+1} \to \Gamma_q$ and $s_i : \Gamma_q \to \Gamma_{q+1}$
are homomorphisms of monoids with identity elements. We will denote by $e_q$ or $1_q$ the identity of $\Gamma_q$.

$\Gamma$ is a *group complex* if $\Gamma$ is a monoid complex and each $\Gamma_q$ is a group. When each $\Gamma_q$ is abelian, $\Gamma$ will be called an abelian monoid complex, or an abelian group complex as the case may be. When $\Gamma$ is a group complex and $x \in \Gamma_q$, the inverse of $x$ will be denoted by $\bar{x}$.

Notice that if $G$ is a topological space and there is given a map of $G \times G \to G$ which makes $G$ into a monoid with identity, then the total singular complex of $G$ is a monoid complex which is abelian if and only if $G$ is abelian. Further if $G$ is a topological group, then the total singular complex of $G$ is a group complex.

THEOREM 3.4. *If $\Gamma$ is a group complex, then $\Gamma$ is a Kan complex.*

A proof of this fact may be found in [4].

DEFINITION 3.5. A *monoid complex with homotopy* is a monoid complex which is a Kan complex. In this case $\pi_q(\Gamma, e_0)$ will be denoted by $\pi_q(\Gamma)$.

PROPOSITION 3.6. *If $\Gamma$ is a monoid complex with homotopy, then $\pi_1(\Gamma)$ is abelian, and if $x, y \in \Gamma_q$ are elements such that $\partial_i x = \partial_i y = e_{q-1}$ for $i = 0, \cdots, q$, then $[x], [y] \in \pi_q(\Gamma)$ and $[x][y] = [xy]$.*

The preceding proposition gives the analogue of the classical results that the group operations in the homotopy groups of a topological group come from the group operation in the group, and that the fundamental group of a topological group is abelian.

Now for group complexes we wish to define homotopy groups in an alternative fashion.

DEFINITION 3.7. If $\Gamma$ is a group complex, define

$$\tilde{\pi}_q(\Gamma) = \cap_{j=0}^{q-1} \text{kernel } \partial_i : \Gamma_q \to \Gamma_{q-1}, \text{ and}$$
$$\tilde{\pi}(\Gamma) = \sum_q \tilde{\pi}_q(\Gamma).$$

PROPOSITION 3.8. *If $\Gamma$ is a group complex, then image $\partial_{q+1} : \tilde{\pi}_{q+1}(\Gamma) \to \Gamma_q$ is a normal subgroup of $\Gamma_q$ contained in kernel $\partial_q : \tilde{\pi}_q(\Gamma) \to \tilde{\pi}_{q-1}(\Gamma)$.*

DEFINITION 3.9. For any group complex $\Gamma$, consider $\tilde{\pi}(\Gamma)$ as a chain complex (not necessarily abelian) with respect to the last face operator. Define

$$\pi'_q(\Gamma) = H_q(\tilde{\pi}(\Gamma)).$$

PROPOSITION 3.10. *If $\Gamma$ is a group complex, then*

*(1) $\pi'_q(\Gamma) = \pi_q(\Gamma)$ for all $q$, and*

*(2) $\Gamma$ is minimal if and only if $\partial_{q+1} : \tilde{\pi}_{q+1}(\Gamma) \to \tilde{\pi}_q(\Gamma)$ is zero for all $q$.*

DEFINITIONS 3.11. A twisted Cartesian product $(\Gamma, B, E)$ is principal if

(1) $\Gamma$ is a monoid complex, and

(2) the function $f : \Gamma \times E \to E$ defined by $f(a', (a, b)) = (a'a, b)$ is a semi-simplicial map.

PROPOSITION 3.12. *If $(\Gamma, B, E)$ is a principal twisted Cartesian product and $\tau$ its twisting function, then $\tau(a,b) = \partial_0 a\tau(e_q, b)$. Defining $\tau' : B_q \to \Gamma_{q-1}$ by $\tau', (b) = \tau(e_q, b)$ we have*

*(1) $\tau'(\partial_1 b) = \partial_0\tau'(b)\tau'(\partial_0)b)$,*

*(2) $\tau'\partial_{i+1} = \partial_i\tau'$ for $i > 0$,*

*(3) $\tau'(s_0)b = e_q$ for $b \in B_q$, and*

*(4) $\tau' s_{i+1} = s_i\tau'$.*

*Further if $\tau' : B_{q+1} \to \Gamma_q$ is a function satisfying the preceding identities, then there is a unique twisted Cartesian product $(\Gamma, B, E)$ such that $\partial_0(a, b) = (\partial_0 a\tau'(b), \partial_0 b)$.*

PROPOSITION 3.13. *Let $(F^1, B^1, B^2)$ and $(F^2, B^2, B^3)$ be twisted Cartesian products, such that $B^2$ and $B^3$ have a single vertex, with twisting functions $\tau_1$, and $\tau_2$. Denote by $i_1 : F^1 \to B^2$ the inclusion map. Then there are twisted Cartesian products $(F^2, F^1, F^{1,2})$ with twisting function $\tau^1$ and $(F^{1,2}, B^1, B^3)$ with twisting function $\tau^2$, where $\tau^1(a, b) = \tau_1(a, i_1(b))$ and $\tau^2(a, b, c) = (\tau_2(a, b, c), \tau_1(b, c))$.*

This proposition is the analogue of the well-known theorem that if $B^3 \to B^2$ is a fibre map and $B^2 \to B^1$ is a fibre map, then $B^3 \to B^1$ by composition is a fibre map.

PROPOSITION 3.14. *If $(\Gamma, B, E)$ is a principal twisted, Cartesian product where $\Gamma$ is a group complex, then $E \to B$ is a fibre map.*

DEFINITION 3.15. Let $(F, B, E)$ and $(F', B', E')$ be twisted Cartesian products. A map of $(F, B, E)$ into $(F', B', E')$ is a map $f : F \to F'$, a map $g : B \to B'$ and a map $h : E \to E'$ such that $h(a, b) = (f(a), g(b))$. Any such map $h$ is said to be compatible with the map $f : F \to F'$ of the fibres.

DEFINITION 3.16. A principal twisted Cartesian product $(\Gamma, B, E)$ is of type $(W)$ if $B_0$ has a single element and $\partial_0 : e_{q+1} \times B_{q+1} \to E_r$ is one to one.

THEOREM 3.17. *Let $(\Gamma, B, E)$ and $(\Gamma', B', E')$ be principal twisted Cartesian products, the second being of type $(W)$, and suppose $f : \Gamma \to \Gamma'$ be a map of monoid*

*Complexes, then there is a unique map* $\bar{f}: (\Gamma, B, E) \to (\Gamma', B', E')$ *compatible with* $f$.

COROLLARY 3.18. *If* $(\Gamma, B, E)$ and $(\Gamma, B', E')$ *are twisted Cartesian products of type* $(W)$ *there is a unique isomorphism between them compatible with the identity map* $i: \Gamma \to \Gamma$.

THEOREM 3.19. *If* $\Gamma$ *is a monoid complex, then there is a twisted Cartesian product* $(\Gamma, \bar{W}(\Gamma), W(\Gamma))$ *of type* $(W)$, *and* $W(\Gamma)$ *is acyclic.*

This theorem was originally proved by MacLane [11], and is an extension of work of Eilenberg and MacLane who gave an explicit description of $\bar{W}(\Gamma)$[6], without introducing $W(\Gamma)$. More details concerning the $W$-construction $(\Gamma, \bar{W}(\Gamma), W(\Gamma))$ may be found in [4]. Geometrically one thinks of $\Gamma$ as corresponding to a topological group, $\bar{W}(\Gamma)$ to its classifying space, and $W(\Gamma)$ as the contractible fibre bundle over $\bar{W}(\Gamma)$ with fibre $\Gamma$.

THEOREM 3.20. *Let* $\Gamma$ *be a connected monoid complex. Then*

(1) *if* $\Gamma$ *is minimal,* $\Gamma$ *is group complex and* $(\Gamma, \bar{W}(\Gamma), W(\Gamma))$ *is a minimal fibre space,*

(2) *if* $\Gamma$ *is a group complex, then* $\bar{W}(\Gamma)$ *and* $W(\Gamma)$ *are Kan complexes, and*

(3) *if* $\Gamma$ *is abelian there is a unique map of the twisted Cartesian product* $(\Gamma \times \Gamma, \bar{W}(\Gamma) \times \bar{W}(\Gamma), W(\Gamma) \times W(\Gamma))$ *into* $(\Gamma, \bar{W}(\Gamma), W(\Gamma))$ *compatible with the multiplication map* $\Gamma \times \Gamma \to \Gamma$, *and this map makes* $\bar{W}(\Gamma)$ *and* $W(\Gamma)$ *into abelian monoid complexes.*

DEFINITION 3.21. If $\Gamma$ is an abelian monoid complex, let $\bar{W}(\Gamma)^0 = \Gamma$, $\bar{W}(\Gamma)^1 = \bar{W}(\Gamma)$, and $\bar{W}(\Gamma)^{n+1} = \bar{W}(\bar{W}(\Gamma)^n)$.

DEFINITION 3.22. If $\pi$ is a group, let $K(\pi, 0)$ be the group complex such that $K(\pi, 0)_0 = \pi$ and $\partial_i: K(\pi, 0)_{q+1} \to K(\pi,0)_q$ $s_i$: $K(\pi, 0)_q \to K(\pi, 0)_{q+1}$ are isomorphisms.

THEOREM 3.23. *If* $\pi$ *is an abelian group, then* $\bar{W}(K(\pi, 0))^n = K(\pi, n)$.

Recall that $K(\pi, n)$ was the unique minimal complex with its $n$-dimensional homotopy group $\pi$ and all others zero. Then the preceding theorem ([6]) gives the existence of such complexes. There is also a well-known explicit description of $K(\pi, n)$ by letting $K(\pi, n)_q = Z^n(\Delta_q, \pi)$ (for details concerning this see [6]).

Now we want to reconstruct Postnikov systems, but before doing so it is necessary to introduce the notion of induced twisted Cartesian product.

DEFINITION 3.24. Let $(F, B, E)$ be a twisted Cartesian product and $f: X \to B$ a map. Define $(F, X, E_f)$ to be the twisted Cartesian product with twisting function $\tau' = \tau(i \times f)$ where $i \times f: F \times F \to F \times B$ and $\tau: F \times B \to F$ is the twisting function of the twisted Cartesian product $(F, B, E)$.

Henceforth we will assume familiarity with the homology and cohomology theory of semi simplicial complexes.

NOTATION. If $X$ is a semi simplicial complex $C(X)_N$ denotes the normalized chain complex of $X$. Further if $\pi$ is an abelian group we will denote by $C^q(X; \pi)$ the group of normalized $q$-cochains of $X$, i.e., an element of $C^q(X; \pi)$ is a function on $X_q$ with values in $\pi$ which vanishes on degenerate $q$-simplexes. Let $Z^q(X; \pi)$ be the sub-group of $C^q(X; \pi)$ consisting of cocycles, i.e. such that if $x \in X_{q+1}$ and $f \in Z^q(X; \pi)$, then $\sum(-1)^i f(\partial_i x) = 0$.

Now we are in a position to state the well-known theorem of Eilenberg-MacLane concerning mappings into $K(\pi, n)$.

THEOREM 3.25. *Let $X$ be a semi-simplicial complex. For any map $f\colon X \to W(K(\pi, n))$ let $\bar{f}$ be the n-cochain of $X$ which is $f|X_n$. Then we have*

(1) *the correspondence $f \to \bar{f}$ between maps of $X \to W(K(\pi, n))$ is one to one,*

(2) *$f\colon X \to K(\pi, n)$ if and only if $\bar{f} \in Z^n(X; \pi)$, and*

(3) *the correspondence induces a natural isomorphism between homotopy classes of maps of $X$ into $K(\pi, n)$ and $H^n(X; \pi)$.*

In other words $(W(K(\pi, n)^X)_0 = C^n(X; \pi)$ $(K(\pi, n)^X)_0 = Z^n(X; \pi)$, and $\pi_0(K(\pi, n)^X) = H^n(X; \pi)$. Notice that since $K(\pi, n)$, $W(K(\pi, n)$ and $K(\pi, n+1)$ are group complexes, the space of mapping of $X$ into one of these complexes is a group complex, and the above isomorphisms are isomorphisms of groups. Further the map $W(K(\pi, n)) \to K(\pi, n+1)$ just induces the map $\delta\colon C^n(X; \pi) \to C^{n+1}(X; \pi)$.

THEOREM 3.26. *Let $X$ be a connected minimal complex, $\pi_n = \pi_n(X)$, and $k^{n+2}$ a cocycle representing the obstruction to a cross section of the fibre map $p\colon X^{(n+1)} \to X^{(n)}$.* Then

(1) $X^{(1)} = \bar{W}(K(\pi_0, 0))$, *and*

(2) if $f^{n+2}\colon X^{(n)} \to K(\pi_{n+1}, n+2)$ *is the mapping corresponding to $k^{n+2}$, there is an isomorphism between $X^{(n+1)}$ and the total space of the twisted Cartesian product $(K(\pi_n, n), X^{(n)}, W_f n+2)$ induced by $f^{n+2}$ from the twisted Cartesian product $(K(\pi_n, n), K(\pi_n, n+1), W(K(\pi_n, n)))$, and this isomorphism makes the fibre spaces $(X^{(n+1)}, p, X^{(n)})$ into a twisted Cartesian product $(K(\pi_n, n), X^{(n)}, X^{(n+1)})$.*

Now it is clear how one can construct any minimal complex. Suppose there is given an infinite sequence $(\pi_1, \pi_2, \cdots, \pi_n, \cdots)$ of groups such that $\pi_i$ is abelian for $i > 1$, then let $X^{(1)} = \bar{W}(K(\pi_1, 0))$. Suppose $k^3$ is a 3-cocycle on $X^{(1)}$ with coefficients in $\pi_2$, we have $f^3 : X^{(1)} \to K(\pi_2, 3)$ and an induced twisted Cartesian product $(K(\pi_2, 2), X^{(1)}, X^{(2)})$. Now if $k^4$ is a 4-cocycle on $X^{(2)}$ with coefficients in $\pi_3$ we have $f^4\colon X^{(2)} \to K(\pi_3, 4)$ and a twisted Cartesian product $(K(\pi_3, 3), X^{(2)}, X^{(3)})$ etc.

As always, a theorem such as the preceding one is a special case of a more general theorem involving a fibre map instead of the special fibre map into a point. We, therefore proceed to the general case.

THEOREM 3.27. *Let $(E, p, B)$ be a minimal fibre space with connected base and fibre, let $F$ denote the fibre, and $\pi_n = \pi_n(F)$. Suppose further that $k^{n+2}$ is a cocycle representing the obstruction to a cross section of the fibre map $p\colon E^{(n+1)} \to E^{(n)}$. We then have*

(1) $E^{(0)} = B$, *and*

(2) *if $f^{n+2}\colon E^{(n)} \to K(\pi_{n+1}, n+2)$*

*is the mapping corresponding to $k^{n+2}$, there is an isomorphism between $E^{(n+1)}$ and the total space of the twisted Cartesian product $(K(\pi_{n+1}, n+1), E^{(n)}, W_f\, n+2)$ induced by $f^{n+2}$, and this isomorphism makes the fibre space $(E^{(n+1)}, p, E^{(n)}$ into a twisted Cartesian product.*

COROLLARY 3.28. *Any minimal fibre space $(E, p, B)$ with connected fibre and base may be given the structure of a twisted Cartesian product.*

This corollary says only that the structure of a twisted Cartesian product may be given to any minimal fibre space. The way of doing this is by no means unique. In fact suppose that each fibre space $(E^{(n+1)}, p, E^{(n)})$ in the preceding theorem has been given the structure of a twisted Cartesian product. We then have the twisted Cartesian product $(K(\pi_{n+1}, n + 1), E^{(n)}, E^{(n+1)})$. Suppose $\tau$ is its twisting function define $k^{(n+2}(X) = \tau(X)$ for $X$ an $n + 2$ simplex of $E^{(n)}$, then $k^{n+2}$ is a cocycle which is the obstruction to a cross section. Further define $\tilde{k}^{n+1}(a, x)$ to be $a$, for $(a, x)$ an $(n + 1)$ simplex of $E^{(n+1)}$. Then $\delta\tilde{k}^{n+1} = p^*(k^{n+2})$. Therefore we have $k^{n+2}$ chosen and an $(n + 1)$ chain $\tilde{k}^{n+1} \in C^{n+1} (E^{(n+1)}; \pi_{n+1})$ whose coboundary is the cochain $p^*(k^{n+2})$: It is not difficult to see that in order to make $(E^{(n+1)}, p, E^{(n)})$ into a twisted Cartesian product it suffices to choose $k^{n+2}$ and $\tilde{k}^{n+1}$ so that $k^{n+2} \in Z^{n+2}(E^{(n)}; \pi_{n+1})$ is the obstruction to a cross section and $\tilde{k}^{n+1} \in c^{n+1}(E^{(n+1}; \pi_{n+1})$ has the property that $\delta\tilde{k}^{n+1} = p^*k^{n+2}$. In other words so that with the obvious notation we have a commutative diagram

$$\begin{array}{ccc} E^{(n+1)} & \xrightarrow{\tilde{k}^{n+1}} & W(K(\pi_{n+1}, n + 1)) \\ \downarrow & & \downarrow \\ E^{(n)} & \xrightarrow{k^{n\;2}} & (K\pi_{n+1}, n + 2). \end{array}$$

There are many applications of the preceding theory, but we will not go into them here. Instead we will content ourselves with one not particularly surprising result which uses only a small part of the preceding theory. Namely, if one has an abelian group complex, then all of its $k$-invariants are zero.

THEOREM 3.29. *Let $\Gamma$ be a connected abelian group complex, and let $\pi_n = \pi_n(\Gamma)$. Then $\Gamma$ has the homotopy type of the infinite Cartesian product $X_{n=1}^{\infty} K(\pi_n, n)$.*

To prove this theorem it suffices to produce a mapping of $H_n(\Gamma) \to \pi_n(\Gamma)$ so that the composite of this map with the natural map of $\pi_n(\Gamma) \to H_n(\Gamma)$ is the identity. For then we can choose an $n$-cocycle $f^n \in Z^n(\Gamma, \pi_n)$ corresponding to this map, and this determines a map $f^n : \Gamma \to K(\pi_n, n)$ which maps the $n$-dimensional homotopy group isomorphically. The fact that we can choose such a map follows easily from the following proposition.

PROPOSITION 3.30. *Let $\Gamma$ be a connected abelian group complex, and define $\delta : \Gamma_n \to \Gamma_{n-1}$ by $\delta x = \sum(-1)^i \partial_i x$, then $\pi_n(\Gamma)$ is the kernel of $\partial : \Gamma_n \to \Gamma_{n-1}$ modulo the image of $\partial : \Gamma_{n+1} \to \Gamma_n$.*

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## DUALITY BETWEEN CW-LATTICES

By J. H. C. Whitehead

### 1. Introduction

This is a continuation of E. H. Spanier's address and it describes work on which he and I are collaborating. It is concerned with duality in relative homotopy theory and is based on a "functorial" duality in the category of join-homomorphisms between finite lattices. One of the new features is what we call "external" duality. This provides a duality between inclusion maps and "pinching" maps of the form $X \to X/A$, where $X/A$ denotes the space obtained from a space $X$ by pinching a sub-space $A$ to a point (all our spaces and sub-spaces will be CW-complexes and sub-complexes). Before discussing these topics I will make a few remarks concerning extension and compression.

### 2. Extension and compression

Extension and compression problems, in $S$-theory, are dual to each other in the way indicated by the diagram

(2.1)
$$\begin{array}{ccc} A & \xrightarrow{\theta} & Y \\ \iota\downarrow & \nearrow \phi & \\ X & & \end{array} \qquad\qquad \begin{array}{ccc} A^* & \xleftarrow{\theta^*} & Y^* \\ \iota^*\uparrow & \swarrow \phi^* & \\ X^* & & \end{array}$$

Here the arrows indicate $S$-maps between finite CW-complexes, $A$ is a sub-complex of $X$, the complexes $X^*$, $A^*$, $Y^*$ are weakly $n$-dual to $X$, $A$, $Y$ in such a way that $X^* \subset A^*$ and the dual of the inclusion $S$-map $\iota$ is the inclusion $S$-map $\iota^*$. The $S$-maps $\theta^*$, $\phi^*$ are dual to $\theta$, $\phi$. In the extension problem we try to factor a given $\theta$ into $\phi\iota$, for some $\phi$. In the compression problem we try to factor a given $\theta^*$ into $\iota^* \phi^*$, for some $\phi^*$. I emphasize the fact that this duality is "functorial" and not merely "schematic". Thus a given $\theta$ has an $n$-dual $\theta^*$, for a sufficiently large value of $n$, and the existence of $\phi$, such that $\theta = \phi\iota$, is equivalent, not merely analogous, to the existence of $\phi^*$ such that $\theta^* = \iota^* \phi^*$. Moreover the duality is symmetric between $\theta$, $\iota$, $\phi$ and $\theta^*$, $\iota^*$, $\phi^*$.

An extension problem is equivalent to what I will call a *zero restriction* problem. By this I mean the problem of deciding whether or no $\phi \mid A = 0$ (i.e. $\phi$ has a representative map $f : S^pX \to S^pY$ such that $f \mid S^pA$ is homotopic to a constant), for a given $\phi : X \to Y$. This is obviously equivalent to an extension problem for the pair $(TA \cup X, X)$, where $TA$ is a cone with $A$ as base and $TA \cap X = A$. Conversely, the problem of extending an $S$-map $A \to Y$ to an $S$-map $X \to Y$ is equivalent to a zero restriction problem for the pair $(Z, C)$, where $Z = (TX)/A$, $C = X/A$ ([4; p. 659] N.B. $SA \subset Z$ and $SA$ is a deformation retract of $Z$). Thus an extension problem and, by duality, a compression problem are equivalent to a zero restriction problem.

The zero restriction problem for $\phi : X \to Y$ and $\phi \mid A$ may also be formulated as the problem of factoring $\phi$ into the composite of the pinching $S$-map $X \to X/A$ and some $S$-map $X/A \to Y$.

## 3. External duality for pairs

Let $P$ be a CW-complex and let $E$, $E'$ be contractible CW-complexes containing $P$ such that $E \cap E' = P$. Then $SP$ is of the same homotopy type as $E \cup E'$ and hence as $E/P$. If $E$ is $S$-contractible,[1] then $E/P$ is of the same $S$-homotopy type as $SP$. Let $X, A$ be as above and let $A_0$ be a contractible sub-complex (e.g. a 0-cell) of $A$. Let $X^*$, $A^*$, $A_0^*$ be weak $(n-1)$-duals of $X, A, A_0$ such that $X^* \subset A^* \subset A_0^*$ and the inclusion $S$-maps $X^* \to A^* \to A_0^*$ are dual to the inclusion $S$-maps $X \leftarrow A \leftarrow A_0$. Then $A_0^*$ is $S$-contractible and $A_0^*/X^*$, $A_0^*/A^*$ are weakly $n$-dual to $X$, $A$. Moreover, if $X' = A_0^*/X^*$, $A' = A^*/X^*$ then $A_0^*/A^*$ may be identified with $X'/A'$. Thus $X$, $A$ are weakly $n$-dual to $X'$, $X'/A'$ and in such a way that the dual of $\iota : A \subset X$ is the pinching $S$-map $X' \to X'/A'$. This is a simple case of external duality. It is external duality between the lattice consisting of the three elements $X$, $A$, $A_0$ and the lattice consisting of $(A_0', A', X')$, where $A_0'$ is the point $X^*/X^*$. Under external duality extension and compression problems are dual to "lifting" and zero restriction problems, as indicated by the diagram

$$(3.1)\qquad \begin{array}{ccc} A & \xrightarrow{\theta} & Y \\ {\scriptstyle\iota}\downarrow & \overset{\phi}{\nearrow} & \uparrow{\scriptstyle\kappa} \\ X & \xrightarrow[\psi]{} & B \end{array} \qquad\qquad \begin{array}{ccc} X'/A' & \xleftarrow{\theta'} & Y' \\ {\scriptstyle\iota'}\uparrow & \overset{\phi'}{\swarrow} & \downarrow{\scriptstyle\kappa'} \\ X' & \xleftarrow[\psi']{} & Y'/B'. \end{array}$$

## 4. The category $CJ$

I recall that a *carrier* is an inclusion-preserving map $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{A}$, $\mathfrak{B}$ are sets of sub-sets of spaces $X$, $Y$. An $\mathfrak{f}$-map or $\mathfrak{f}$-homotopy $X \to Y$ is one in which $A \to \mathfrak{f}A$ for every $A \in \mathfrak{A}$ and $[\mathfrak{f}]$ will denote the set of $\mathfrak{f}$-homotopy classes $X \to Y$. A carrier $f' : \mathfrak{A}' \to \mathfrak{B}'$, where $\mathfrak{A}'$, $\mathfrak{B}'$ are also sets of sub-sets of $X$, $Y$, is said to be *equivalent* to $\mathfrak{f}$ if, and only if, every $\mathfrak{f}$-map is an $\mathfrak{f}'$-map and every $\mathfrak{f}'$-map is an $\mathfrak{f}$-map.

Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of sub-complexes of CW-complexes $X$, $Y$. Then any carrier $\mathfrak{A} \to \mathfrak{B}$ is equivalent to a join-homomorphism $\mathfrak{A}' \to \mathfrak{B}'$, where $\mathfrak{A}'$, $\mathfrak{B}'$ are complete lattices [2] of sub-complexes of $X$, $Y$ (when we refer to a complete lattice of sub-sets of a space it is to be understood that the join, $\bigcup\mathfrak{C}$, and meet, $\bigcap\mathfrak{C}$, of any set $\mathfrak{C}$, of elements in the lattice are the union and intersection of the elements in $\mathfrak{C}$. A join-homomorphism, $\mathfrak{f}' : \mathfrak{A}' \to \mathfrak{B}'$, is one such that $\mathfrak{f}' \cup \mathfrak{C} = \bigcup \mathfrak{f}' \mathfrak{C}$ for every $\mathfrak{C} \subset \mathfrak{A}'$. In particular $\mathfrak{f}' \cap \mathfrak{A}' = \bigcap \mathfrak{B}'$). Moreover it may also be assumed that $X \in \mathfrak{A}'$, $Y \in \mathfrak{B}'$ in which case $\mathfrak{A}'$, $\mathfrak{B}'$ will be called CW-lattices. We describe $\mathfrak{A}'$ as *strictly finite* if, and only if, $X$ is a finite complex.

Let $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ be a given join-homomorphism, where $\mathfrak{A}$, $\mathfrak{B}$ are CW-lattices. Then

[1] i.e. $E$ is of the same $S$-homotopy type as a single point. This is the case if, and only if, all the homology groups of $E$ (reduced in dimension 0) are zero.

$S^p\mathfrak{A}$ will denote the CW-lattice consisting of the sub-complexes $S^pA \subset S^pX$, for every $A \in \mathfrak{A}$, and $\mathfrak{f}^q_p : S^p\mathfrak{A} \to S^q\mathfrak{B}$ the join-homomorphism defined by $\mathfrak{f}^q_p S^pA = S^q\mathfrak{f}A$. We write $\mathfrak{f}^p_p = S^p\mathfrak{f}$. A map $S : [S^p\mathfrak{f}] \to [S^{p+1}\mathfrak{f}]$ is defined in the usual way and $\{\mathfrak{f}\}$ will denote the Abelian group, with track addition [1], which is the direct limit of the sequence

$$[\mathfrak{f}] \xrightarrow{S} [S\mathfrak{f}] \to \cdots \to [S^p\mathfrak{f}] \xrightarrow{S} [S^{p+1}\mathfrak{f}] \to \cdots.$$

An element of $\{\mathfrak{f}\}$ will be called an $\mathfrak{f}$-*S-map*.

Let $\mathfrak{g} : \mathfrak{A} \to \mathfrak{B}$ be a join-homomorphism such that $\mathfrak{f} \leqq \mathfrak{g}$ (i.e. $\mathfrak{f}A \subset \mathfrak{g}A$ for every $A \in \mathfrak{A}$) and let $T\mathfrak{A} \vee \mathfrak{A}$ consist of the sub-complexes $TA_1 \cup A_2; \subset TX$ for every pair of elements $A_1, A_2 \in \mathfrak{A}$ such that $A_1 \subset A_2$. Then it may be verified that $T\mathfrak{A} \vee \mathfrak{A}$ is a complete lattice. Define

$$(4.1) \qquad \{\mathfrak{g};\ \mathfrak{f}\}_1 : T\mathfrak{A} \vee \mathfrak{A} \to \mathfrak{B},\ T\mathfrak{f} \vee \mathfrak{g} : \mathfrak{A} \to T\mathfrak{B} \vee \mathfrak{B}$$

by $\{\mathfrak{g}:\ \mathfrak{f}\}_1\ (TA_1 \cup A_2) = \mathfrak{g}A_1 \cup \mathfrak{f}A_2$, $(T\mathfrak{f} \vee \mathfrak{g})A = T(\mathfrak{f}A) \cup \mathfrak{g}A$. Then $\{\mathfrak{g};\ \mathfrak{f}\}_1$ and $T\mathfrak{f} \vee \mathfrak{g}$ are join-homomorphisms. We define $\{\mathfrak{f}\}_m$, $\{\mathfrak{g};\ \mathfrak{f}\}_m$, for $m = 0, \pm 1, \pm 2, \cdots$ by

$$\begin{aligned} \{\mathfrak{f}\}_m &= \{\mathfrak{f}^0_m\} \qquad \textit{if} \quad m \geqq 0 \\ &= \{\mathfrak{f}^{-m}_0\} \qquad \textit{if} \quad m < 0 \end{aligned}$$

and $\{\mathfrak{g};\ \mathfrak{f}\}_m = (\{\mathfrak{g};\ \mathfrak{f}\}_1)_{m-1}$. Let $\dim X < \infty$. Then there is a diagram

$$(4.2) \qquad \begin{array}{ccccccc} & & & \overset{v}{\nearrow} & \{\mathfrak{g};\mathfrak{f}\}_m & \overset{w}{\searrow} & & \\ \cdots \to \{\mathfrak{f}\}_m & \xrightarrow{u} & \{\mathfrak{g}\}_m & & \downarrow h & & \{\mathfrak{f}\}_{m-1} \xrightarrow{u} \cdots, \\ & & & \underset{v'}{\searrow} & \{T\mathfrak{f} \vee \mathfrak{g}\}_m & \underset{w'}{\nearrow} & & \end{array}$$

in which $\cdots, u, v, w, \cdots$ and $\cdots, u, v', w', \cdots$ are exact sequences of homomorphisms and $h$ is an isomorphism such that

$$(4.3) \qquad hv = -v', \qquad w'h = w.$$

The homomorphisms $u$, $v'$ are injections (i.e. corresponding elements are represented by the same map), $w$ is defined by restriction to $S^pX \subset TS^pX$ and $v$, $w'$ by composition with the canonical maps $TS^pX \to S^{p+1}X$, $TS^qY \to S^{q+1}Y$.

We now define a category $CJ$, whose objects are all CW-lattices. A mapping in $CJ$ is an ordered triple $(\phi, \mathfrak{A}, \mathfrak{B})$, also written as $\phi : \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{A}$, $\mathfrak{B}$, are CW-lattices and $\phi$ is an $\mathfrak{f}$-$S$-map for some join-homomorphism $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$. If $f : S^pX \to S^pY$ is a representative of $\phi$. where $X = \cup\mathfrak{A}$, $Y = \cup\mathfrak{B}$, then we also denote $\phi$ by $\{f, \mathfrak{f}\}$. We now use $\{\mathfrak{f}\}$ to denote the group of $CJ$-maps of the form $\{f, \mathfrak{f}\}$ (N.B. these are indexed by $\mathfrak{A}, \mathfrak{B}$). An isomorphism $S : \{\mathfrak{f}\} \approx \{S\mathfrak{f}\}$ is defined by $S\{f, \mathfrak{f}\} = \{f, S\mathfrak{f}\}$, where $f : S^pX \to S^pY$ for some $p > 0$.

Our duality applies to the sub-category of $CJ$ consisting of the strictly finite CW-lattices and all mappings between them. It depends on a duality in the purely algebraic theory of lattices, which I proceed to describe.

## 5. The algebraic duality

Let $\mathfrak{f}, \mathfrak{g} : \mathfrak{A} \to \mathfrak{B}$ be maps of a complete lattice $\mathfrak{A}$ in a complete lattice $\mathfrak{B}$. We write $\mathfrak{f} \leqq \mathfrak{g}$, or $\mathfrak{g} \geqq \mathfrak{f}$, if, and only if, $\mathfrak{f}A \leqq \mathfrak{g}A$ for every $A \in \mathfrak{A}$. We denote the identical map of $\mathfrak{A}$ by $\mathfrak{i}_{\mathfrak{A}}$.

We describe a map $\mathfrak{f}^{\#} : \mathfrak{B} \to \mathfrak{A}$ as *dual* to an order-preserving map $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ if, and only if, it is order-preserving and

$$\mathfrak{f}^{\#}\mathfrak{f} \geqq \mathfrak{i}_{\mathfrak{A}}, \qquad \mathfrak{f}\mathfrak{f}^{\#} \leqq \mathfrak{i}_{\mathfrak{B}}. \tag{5.1}$$

In consequence of (5.1) we have $\mathfrak{f}\mathfrak{f}^{\#}B \leqq B$, for every $B \in \mathfrak{B}$, and, since $\mathfrak{f}^{\#}$ is order-preserving, $A \leqq \mathfrak{f}^{\#}\mathfrak{f}A \leqq \mathfrak{f}^{\#}B$ if $\mathfrak{f}A \leqq B$. Therefore, if $\mathfrak{A}_B$ denotes the set of all elements $A \in \mathfrak{A}$ such that $\mathfrak{f}A \leqq B$, then $\mathfrak{A}_B$ has a largest element, namely $\mathfrak{f}^{\#}B$. Thus $\mathfrak{f}^{\#}$, if it exists, is unique. It is easily proved that, if $\mathfrak{f}$ is a join-homomorphism then $\mathfrak{f}^{\#}$, defined by $\mathfrak{f}^{\#}B = \bigcup \mathfrak{A}_B$, is dual to $\mathfrak{f}$ and is a meet-homomorphism (i.e. $\mathfrak{f}^{\#} \cap \mathfrak{B}_0 = \bigcap \mathfrak{f}^{\#}\mathfrak{B}_0$ for every sub-set $\mathfrak{B}_0 \subset \mathfrak{B}$. In particular $\mathfrak{f}^{\#} \cup \mathfrak{B} = \bigcup \mathfrak{A}$). Similarly a given meet-homomorphism $\mathfrak{f}^{\#} : \mathfrak{B} \to \mathfrak{A}$ has a unique dual $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$, which is a join-homomorphism satisfying (5.1).

The dual of a join-homomorphism $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ may also be regarded as a join-homomorphism $\mathfrak{f}^{\#} : \mathfrak{B}^{\#} \to \mathfrak{A}^{\#}$, where $\mathfrak{A}^{\#}, \mathfrak{B}^{\#}$ are the lattices dual [2; p. 3] to $\mathfrak{A}, \mathfrak{B}$. Let $\boldsymbol{L}$ denote the category of join-homomorphisms between complete lattices and let $D : \boldsymbol{L} \to \boldsymbol{L}$ be defined by $D\mathfrak{A} = \mathfrak{A}^{\#}$, $D\mathfrak{f} = \mathfrak{f}^{\#}$. Then it is easily verified that $D$ is a contravariant functor such that $DD$ is the identity. Thus $D$ determines a "functorial" duality in the category $\boldsymbol{L}$.

Let $\mathfrak{A}^*, \mathfrak{B}^*$ be complete lattices which are the images of $\mathfrak{A}^{\#}, \mathfrak{B}^{\#}$ in isomorphisms $\mathfrak{a} : \mathfrak{A}^{\#} \approx \mathfrak{A}^*$, $\mathfrak{b} : \mathfrak{B}^{\#} \approx \mathfrak{B}^*$. We also describe $\mathfrak{a}, \mathfrak{b}$ as anti-isomorphisms of $\mathfrak{A}, \mathfrak{B}$, We denote the fundamental partial ordering relation in $\mathfrak{A}^*, \mathfrak{B}^*$ by $\leqq$ so that $A_1 \leqq A_2$ if, and only if, $\mathfrak{a}A_2 \leqq \mathfrak{a}A_1$, where $A_1, A_2 \in \mathfrak{A}$. Then we have

$$\begin{array}{ccccc} \mathfrak{A} & \xrightarrow{\mathfrak{f}} & \mathfrak{B} & \xrightarrow{\mathfrak{f}^{\#}} & \mathfrak{A} \\ \downarrow \mathfrak{a} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{a} \\ \mathfrak{A}^* & \xrightarrow{(\mathfrak{f}^*)^{\#}} & \mathfrak{B}^* & \xrightarrow{\mathfrak{f}^*} & \mathfrak{A}^* \end{array} \tag{5.2}$$

where $\mathfrak{f}, \mathfrak{f}^{\#}$ are as above and $\mathfrak{f}^* = \mathfrak{a}\mathfrak{f}^{\#}\mathfrak{b}^{-1}$, $(\mathfrak{f}^*)^{\#} = \mathfrak{b}\mathfrak{f}\mathfrak{a}^{-1}$. We describe $\mathfrak{f}^*$ as *dual to* $\mathfrak{f}$ *under* $\mathfrak{a}, \mathfrak{b}$. A simple calculation shows that $(\mathfrak{f}^*)^{\#}$ is dual to $\mathfrak{f}^*$, whence $\mathfrak{f}$ is dual to $\mathfrak{f}^*$ under $\mathfrak{b}^{-1}, \mathfrak{a}^{-1}$.

Let $(\mathfrak{A}, \mathfrak{B})_{\mathfrak{f}}$ denote the set of ordered pairs $(A, B)$, for all $A \in \mathfrak{A}$, $B \in \mathfrak{B}$ such that $\mathfrak{f}A \leqq B$. Define $(A, B) \leqq (A', B')$ if, and only if, $A \leqq A'$, $B \leqq B'$, where $(A', B') \in (\mathfrak{A}, \mathfrak{B})_{\mathfrak{f}}$. Then it may be verified that $(\mathfrak{A}, \mathfrak{B})_{\mathfrak{f}}$ is a complete lattice. Since $\mathfrak{f}A \leqq B$ if, and only if, $\mathfrak{f}^{\#}B \geqq A$, which is equivalent to

$$\mathfrak{f}^*\mathfrak{b}B = \mathfrak{a}\mathfrak{f}^{\#}B \leqq \mathfrak{a}A,$$

it follows that an anti-isomorphism $\mathfrak{c} : (\mathfrak{A}, \mathfrak{B})_{\mathfrak{f}} \to (\mathfrak{B}^*, \mathfrak{A}^*)_{\mathfrak{f}^*}$ is defined by

$$\mathfrak{c}(A, B) = (\mathfrak{b}B, \mathfrak{a}A). \tag{5.3}$$

It is clear that join-homomorphisms

$$\mathfrak{B} \xrightarrow{\mathfrak{i}} (\mathfrak{A}, \mathfrak{B})_{\mathfrak{f}} \xrightarrow{\mathfrak{k}} \mathfrak{A}$$

are defined by $\mathfrak{i}B = (\cap\mathfrak{A}, B)$, $\mathfrak{k}(A, B) = A$. Let

$$\mathfrak{B}^* \xleftarrow{\mathfrak{k}'} (\mathfrak{B}^*, \mathfrak{A}^*)_{\mathfrak{f}^*} \xleftarrow{\mathfrak{i}'} \mathfrak{A}^*$$

be similarly defined. Then a straightforward calculation shows that $\mathfrak{k}'$, $\mathfrak{i}'$ are dual to $\mathfrak{i}$, $\mathfrak{k}$ under $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{c}$, $\mathfrak{a}$.

Let $(\mathfrak{A}, \mathfrak{A}) = (\mathfrak{A}, \mathfrak{A})_{\mathfrak{i}}$ $(\mathfrak{i} = \mathfrak{i}_{\mathfrak{A}})$ and let $\mathfrak{f} \leqq \mathfrak{g} : \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{g}$ is a join-homomorphism. Define

$$(\mathfrak{g};\ \mathfrak{f}) : (\mathfrak{A}, \mathfrak{A}) \to \mathfrak{B}, \quad T\mathfrak{f} \vee \mathfrak{g} : \mathfrak{A} \to (\mathfrak{B}, \mathfrak{B})$$

by $(\mathfrak{g};\ \mathfrak{f})(A_1, A_2) = \mathfrak{g}A_1 \cup \mathfrak{f}A_2$, $(T\mathfrak{f} \vee \mathfrak{g})A = (\mathfrak{f}B, \mathfrak{g}B)$. Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{f}^*$ be as above and let $\mathfrak{g}^* : \mathfrak{B}^* \to \mathfrak{A}^*$ be the dual of $\mathfrak{g}$ under $\mathfrak{a}$, $\mathfrak{b}$. Since $\mathfrak{f} \leqq \mathfrak{g}$ it follows that $\mathfrak{f}^\# \geqq \mathfrak{g}^\#$, whence $\mathfrak{f}^* \leqq \mathfrak{g}^*$. A calculation shows that

$$(5.4) \qquad \begin{cases} (\mathfrak{g};\ \mathfrak{f})^* = T\mathfrak{f}^* \vee \mathfrak{g}^* : \mathfrak{B}^* \to (\mathfrak{A}^*, \mathfrak{A}^*) \\ (T\mathfrak{f} \vee \mathfrak{g})^* = (\mathfrak{g}^*\ ;\ \mathfrak{f}^*) : (\mathfrak{B}^*, \mathfrak{B}^*) \to \mathfrak{A}^* \end{cases}$$

under $\mathfrak{a}$, $\mathfrak{b}$ and the anti-isomorphisms $(A_1, A_2) \to (\mathfrak{a}A_2, \mathfrak{a}A_1)$, $(B_1, B_2) \to (\mathfrak{b}B_2, \mathfrak{b}B_1)$, where $(\mathfrak{g};\ \mathfrak{f})^*$, $(T\mathfrak{f} \vee \mathfrak{g})^*$ are the duals of $(\mathfrak{g};\ \mathfrak{f})$, $T\mathfrak{f} \vee \mathfrak{g}$.

If $\mathfrak{A}$ is a CW-lattice, then $(\mathfrak{A}, \mathfrak{A})$ may be identified with $T\mathfrak{A} \vee \mathfrak{A}$, in (4.1), so that $(A_1, A_2) = TA_1 \cup A_2$. Then $(\mathfrak{g}, \mathfrak{f}) = (\mathfrak{g};\ \mathfrak{f})_1$ and $T\mathfrak{f} \vee \mathfrak{g}$ means the same as in §4.

## 6. The geometric duality

The duality between CW-lattices is based on a relation of *n-duality* between polyhedral lattices in $S^n$. Two such lattices, $\mathfrak{A}$, $\mathfrak{A}^*$ are said to be *n-dual* to each other if, and only if, there is an anti-isomorphism $\mathfrak{a} : \mathfrak{A} \to \mathfrak{A}^*$ such that each $A \in \mathfrak{A}$ is $n$-dual to $\mathfrak{a}A$. We then say that $\mathfrak{A}^*$ is $n$-dual to $\mathfrak{A}$ under $\mathfrak{a}$. Every polyhedral lattice in $S^n$ has an $n$-dual and the symmetry of this relation follows from the corresponding symmetry of $n$-duality in the absolute case. We use $D_n\mathfrak{A}$ to denote any $n$-dual of $\mathfrak{A}$.

Let $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ be a join-homomorphism between polyhedral lattices in $S^n$, let $D_n\mathfrak{A}$, $D_n\mathfrak{B}$ be $n$-dual to $\mathfrak{A}$, $\mathfrak{B}$ under anti-isomorphisms $\mathfrak{a} : \mathfrak{A} \to D_n\mathfrak{A}$, $\mathfrak{b} : \mathfrak{B} \to D_n\mathfrak{B}$ and let $D_n\mathfrak{f} : D_n\mathfrak{B} \to D_n\mathfrak{A}$ be the dual of $\mathfrak{f}$ under $\mathfrak{a}$, $\mathfrak{b}$. Then there is a homomorphism

$$(6.1) \qquad D_n : \{\mathfrak{f}\} \to \{D_n\mathfrak{f}\}$$

with the following properties:

(6.2) *if* $A \subset \mathfrak{f}A$, $B^* \subset (D_n\mathfrak{f})B^*$, *for every* $A \in \mathfrak{A}$, $B^* \in D_n\mathfrak{B}$, *then*

$$D_n\{i, \mathfrak{f}\} = \{i^*, D_n\mathfrak{f}\},$$

*where* $i : \cup\mathfrak{A} \subset \cup\mathfrak{B}$, $i^* : \cup D_n\mathfrak{B} \subset \cup D_n A$;

(6.3) *if* $\mathfrak{f} \leqq \mathfrak{g} : \mathfrak{A} \to \mathfrak{B}$, *whence* $D_n\mathfrak{f} \leqq D_n\mathfrak{g}$, *and if*

$u : \{\mathfrak{f}\} \to \{\mathfrak{g}\}$, $u^* : \{D_n\mathfrak{f}\} \to \{D_n\mathfrak{g}\}$ *are the injections, then*

$$D_n(u\,\phi) = u^* D_n\phi \qquad (\phi \in \{\mathfrak{f}\});$$

(6.4) *if* $\mathfrak{A} \xrightarrow{\phi} \mathfrak{B} \xrightarrow{\psi} \mathfrak{C}$, $\quad D_n\mathfrak{A} \xleftarrow{D_n\phi} D_n\mathfrak{B} \xleftarrow{D_n\psi} D_nC$,

*where* $\phi$, $\psi$, $D_n\phi$, $D_n\psi$ *are CJ-maps, then*

$$D_n(\psi\,\phi) = (D_n\,\phi)(D_n\psi);$$

(6.5)
$$\begin{cases} SD_n = D_{n+1} : \{\mathfrak{f}\} \to \{SD_n\mathfrak{f}\} \\ D_n = D_{n+1}S : \{\mathfrak{f}\} \to \{D_n\mathfrak{f}\}; \end{cases}$$

(6.6) *if* $D_nD_n\mathfrak{A}$, $D_nD_n\mathfrak{B}$ *are taken to be* $\mathfrak{A}$, $\mathfrak{B}$ *then*

$$D_nD_n\theta = \theta \textit{ for every } \theta \in \{\mathfrak{f}\} \textit{ or } \{D_n\mathfrak{f}\}.$$

The homomorphism (6.1) is uniquely determined by the conditions (6.2), $\cdots$, (6.5) and it follows from (6.6) that

(6.7)
$$D_n : \{\mathfrak{f}\} \approx \{D_n\mathfrak{f}\}.$$

Let $\mathfrak{A}$ be a strictly finite CW-lattice. Then, for a sufficiently large value of $n$, there is a polyhedral lattice $\mathfrak{A}_0$, in $S^n$, which is related to $\mathfrak{A}$ by a *CJ*-equivalence $\xi : \mathfrak{A} \to \mathfrak{A}_0$. Let $\mathfrak{A}_0^*$ be $n$-dual to $\mathfrak{A}_0$ under an anti-isomorphism $\mathfrak{d} : \mathfrak{A}_0 \to A_0^*$ and let $\mathfrak{A}^*$ be a strictly finite CW-lattice which is related to $\mathfrak{A}_0^*$ by a *CJ*-equivalence $\xi^* : \mathfrak{A}_0^* \to \mathfrak{A}^*$. Let $\mathfrak{x} : \mathfrak{A} \approx \mathfrak{A}_0$, $\mathfrak{x}^* : \mathfrak{A}_0^* \approx A^*$ be the lattice isomorphisms associated with $\xi$, $\xi^*$ (i.e. $\xi \in \{\mathfrak{x}\}$, $\xi^* \in \{\mathfrak{x}^*\}$). Then we describe $\mathfrak{A}^*$ as *weakly n-dual* to $\mathfrak{A}$ under the anti-isomorphism $\mathfrak{x}^*\mathfrak{d}\mathfrak{x} : \mathfrak{A} \to \mathfrak{A}^*$. The propositions stated above are valid for weak duality with the appropriate reservations concerning the choice of $\xi$, $\xi^*$ etc. For example in (6.4) it is to be understood that the maps $\phi \to D_n\phi$ and $\psi \to D_n\psi$ both refer to the same *CJ*-equivalences between $\mathfrak{B}$, $D_n\mathfrak{B}$ and polyhedral lattices in $S^n$, analogous to $\xi$, $\xi^*$ above.

## 7. External duality

Strictly finite CW-lattices $\mathfrak{A}$, $\mathfrak{A}^*$ are said to be *externally n-dual* to each other if, and only if,

(7.1a) $\bigcap\mathfrak{A}$, $\bigcap\mathfrak{A}^*$ *are S-contractible,*

(7.1b) *there is an anti-isomorphism* $\mathfrak{a} : \mathfrak{A} \to \mathfrak{A}^*$ *such that the lattice* $T\mathfrak{A}^* \vee \mathfrak{A}^*$ *is weakly n-dual to* $TA \vee \mathfrak{A}$ *under the anti-isomorphism* $TA_1 \cup A_2 \to T(\mathfrak{a}A_2) \cup \mathfrak{a}A_1$. *In this case* $\mathfrak{A}^*$ *is said to be externally n-dual to* $\mathfrak{A}$ *under* $\mathfrak{a}$.

The propositions in §6, with the appropriate qualifications, are also true for external duality.

Let $\mathfrak{A}^*$ be externally $n$-dual to $\mathfrak{A}$ under $\mathfrak{a}$. Then, for every pair $A_1, A_2 \in \mathfrak{A}$ such that $A_1 \subset A_2$, the CW-complex $TA_1 \cup A_2$ is weakly $n$-dual to $T(\mathfrak{a}A_2) \cup \mathfrak{a}A_1$ whence $A_2/A_1$ is weakly $n$-dual to $\mathfrak{a}A_1/\mathfrak{a}A_2$. Moreover this duality is such that, if $A_1 \subset A_3 \subset A_4, A_2 \subset A_4$ $(A_3, A_4 \in \mathfrak{A})$, then the canonical $S$-map $A_2/A_1 \to A_4/A_3$ is dual to $\mathfrak{a}A_3/\mathfrak{a}A_4 \to \mathfrak{a}A_1/\mathfrak{a}A_2$. Since $\cap\mathfrak{A}$, $\cap\mathfrak{A}^*$ are $S$-contractible it follows that $A$ is weakly $n$-dual to $X^*/\mathfrak{a}A$ and $X$ to $X^*$, where $A \in \mathfrak{A}$, $X = \cup\mathfrak{A}$, $X^* = \cup\mathfrak{A}^*$.

Let $\mathfrak{A}$ be any strictly finite CW-lattice such that $\cap\mathfrak{A}$ is $S$-contractible and let $\mathfrak{A}_1^*$ be weakly $(n-1)$-dual to $\mathfrak{A}$ under an anti-isomorphism $\mathfrak{a}_1 : \mathfrak{A} \to \mathfrak{A}_1^*$. Then it is not difficult to prove that $T\mathfrak{A}_1^* \vee \mathfrak{A}_1^*$ is $n$-dual to $T\mathfrak{A} \vee \mathfrak{A}$ under the anti-isomorphism $TA_1 \cup A_2 \to T(\mathfrak{a}_1 A_2) \cup \mathfrak{a}_1 A_1$. Let $\mathfrak{A}^*$ be the CW-lattice obtained from $\mathfrak{A}_1^*$ by pinching $\cap\mathfrak{A}_1^*$ to a point, let $g : \cup\mathfrak{A}_1^* \to \cup\mathfrak{A}^*$ be the identification map and let $\mathfrak{g} : \mathfrak{A}_1^* \approx \mathfrak{A}^*$ be the isomorphism defined by $\mathfrak{g}A = gA$. Then $\mathfrak{A}^*$ is externally $n$-dual to $\mathfrak{A}$ under $\mathfrak{g}\mathfrak{a}_1 : \mathfrak{A} \to \mathfrak{A}^*$.

In contrast to external $n$-duality we describe the duality discussed in §6 as *internal duality*.

## 8. Dual attachments

Let $\mathfrak{A}$, $\mathfrak{B}$ be strictly finite CW-lattices, let $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ be a join-homomorphism and $f : X \to Y$ an $\mathfrak{f}$-map, where $X = \cup\mathfrak{A}$, $Y = \cup\mathfrak{B}$. Let $f$ be cellular and assume that $fx = x$ if $x \in X \cap Y$, as will be the case if $X \cap Y = \emptyset$ or $f : X \subset Y$. Let $Z$ be the CW-complex obtained from $TX \cup Y$, where $TX \cap Y = X \cap Y$, by identifying each point $x \in X$ with $fx \in Y$. Let $\mathfrak{C}_f$ be the lattice consisting of the subcomplexes $TA \vee B$, for every pair $A \in \mathfrak{A}$, $B \in \mathfrak{B}$ such that $\mathfrak{f}A \subset B$, where $TA \vee B$ denotes the image of $TA \cup B$ in the identification map $TX \cup Y \to Z$. Clearly $\mathfrak{C}_f$ may be identified with $(\mathfrak{A}, \mathfrak{B})_{\mathfrak{f}}$, in §5, so that $TA \vee B = (A, B)$.

Let $\mathfrak{A}^*$, $\mathfrak{B}^*$ be weakly $n$-dual to $\mathfrak{A}$, $\mathfrak{B}$ under anti-isomorphisms $\mathfrak{a} : \mathfrak{A} \to \mathfrak{A}^*$, $\mathfrak{b} : \mathfrak{B} \to \mathfrak{B}^*$ and let $\mathfrak{f}^* : \mathfrak{B}^* \to \mathfrak{A}^*$ be the dual of $\mathfrak{f}$ under $\mathfrak{a}$, $\mathfrak{b}$. Let $n > 2 \dim Z$. Then $\mathfrak{A}^*$, $\mathfrak{B}^*$ may be chosen so that the injection $[\mathfrak{f}^*] \to \{\mathfrak{f}^*\}$ is onto, whence $\{f, \mathfrak{f}\}^* = \{f^*, \mathfrak{f}^*\}$ for some (cellular) map $f^* : \cup\mathfrak{B}^* \to \cup\mathfrak{A}^*$. Then we have

$$\begin{array}{ccccc} \mathfrak{B} & \xrightarrow{\ \mathfrak{i}\ } & \mathfrak{C} & \xrightarrow{\ \mathfrak{s}\mathfrak{k}\ } & S\mathfrak{A} \\ \downarrow{\scriptstyle \mathfrak{s}\mathfrak{b}} & & \downarrow{\scriptstyle \mathfrak{c}} & & \downarrow{\scriptstyle \mathfrak{a}\mathfrak{s}^{-1}} \\ S\mathfrak{B}^* & \xleftarrow{\ \mathfrak{s}\mathfrak{k}'\ } & \mathfrak{C}^* & \xleftarrow{\ \mathfrak{i}'\ } & \mathfrak{A}^* \end{array} \qquad \begin{array}{l} (\mathfrak{C} = \mathfrak{C}_f) \\ \\ (\mathfrak{C}^* = \mathfrak{C}_{f^*}), \end{array} \tag{8.1}$$

where $\mathfrak{i}$, $\mathfrak{k}$, $\mathfrak{c}$ etc. are as in §5, $\mathfrak{s} : \mathfrak{L} \approx S\mathfrak{L}$ is defined by $\mathfrak{s}L = SL$, for any CW-lattice $\mathfrak{L}$, and

$$\mathfrak{c}(TA \vee B) = T(\mathfrak{b}B) \vee \mathfrak{a}A. \tag{8.2}$$

The join-homomorphisms $\mathfrak{i}'$, $\mathfrak{s}\mathfrak{k}'$ are dual to $\mathfrak{s}\mathfrak{k}$, $\mathfrak{i}$ under $\mathfrak{a}\mathfrak{s}^{-1}$, $c$, $\mathfrak{s}\mathfrak{b}$. Let $i : Y \subset Z$ and let $k : Z \to SX$ be the canonical map in which $Y$ and the vertex of $Z$ correspond respectively to the first and second poles of $SX$. Let $i' : \cup\mathfrak{A}^* \subset \cup\mathfrak{C}^*$ and $k' : \cup\mathfrak{C}^* \to S \cup \mathfrak{B}^*$ be similarly defined. Then $i$ is an $\mathfrak{i}$-map, $k$ an $\mathfrak{s}\mathfrak{k}$-map, $i'$ an $\mathfrak{i}'$-map and

$k'$ an $\mathfrak{s}\mathfrak{k}'$-map. It may be proved that $\mathfrak{C}^*$ is weakly $(n+1)$-dual to $\mathfrak{C}$ under $\mathfrak{c}$ in such a way that

(8.2) $$D_{n+1}\{i, \mathfrak{i}\} = -\{k', \mathfrak{s}\mathfrak{k}'\}, D_{n+1}\{k, \mathfrak{s}\mathfrak{k}\} = \{i', \mathfrak{i}'\}.$$

A similar result holds for external duality provided $n > 2 \dim Z + 1$.

## 9. The dual sequences

Let $\mathfrak{f} \leqq \mathfrak{g} : \mathfrak{A} \to \mathfrak{B}$ be join-homomorphisms, where $\mathfrak{A}$, $\mathfrak{B}$ are strictly finite CW-lattices. Let $\mathfrak{A}^*$, $\mathfrak{B}^*$ be either (weak) internal or external $n$-duals of $\mathfrak{A}$, $\mathfrak{B}$ and let $\mathfrak{f}^*, \mathfrak{g}^* : \mathfrak{B}^* \to \mathfrak{A}^*$ be the duals of $\mathfrak{f}$, $\mathfrak{g}$. Then $\mathfrak{f}^* \leqq \mathfrak{g}^*$ and we have

$$(\mathfrak{g};\ \mathfrak{f})_1 : T\mathfrak{A} \vee \mathfrak{A} \to \mathfrak{B}, \quad (T\mathfrak{f}^* \vee \mathfrak{g}^*)_1 : S\mathfrak{B} \to T\mathfrak{A}^* \vee \mathfrak{A}^*.$$

It follows from (8.2) and (5.4) that

(9.1) $$D_{n+1} : \{\mathfrak{g};\ \mathfrak{f}\}_1 \approx \{T\mathfrak{f}^* \vee \mathfrak{g}^*\}_1.$$

The homomorphisms $v$, $w'$ in (4.2) are defined by composition with the canonical $CJ$-maps $T\mathfrak{A} \vee \mathfrak{A} \to S\mathfrak{A}$, $T\mathfrak{B} \vee \mathfrak{B} \to S\mathfrak{B}$, suitably suspended. Hence it follows that there is a commutative diagram

(9.2) $$\begin{array}{ccccccc} \cdots \longrightarrow \{\mathfrak{f}\}_m & \xrightarrow{u} & \{\mathfrak{g}\}_m & \xrightarrow{v} & \{\mathfrak{g};\ \mathfrak{f}\}_m & \xrightarrow{w} \cdots \\ \downarrow D_{m,n} & & \downarrow D_{m,n} & & \downarrow F_{m,n} & \\ \cdots \longrightarrow \{\mathfrak{f}^*\}_m & \xrightarrow{u} & \{\mathfrak{g}^*\}_m & \xrightarrow{v'} & \{\mathfrak{g}^*;\ \mathfrak{f}^*\}_m & \xrightarrow{w'} \cdots \end{array}$$

where $D_{m,n} = D_{|m|+n}$ and $F_{m,n} = h^{-1} D_{|m|+n}$.

It will sometimes be convenient to express (9.2) in terms of $n$-duality. For this purpose we introduce the pair $(\mathfrak{A}, p)$, which we also denote by $S^p\mathfrak{A}$, for any negative integer $p$. Thus $S^p\mathfrak{A}$ is defined for every integral $p$ and we write $S^p(S^q\mathfrak{A}) = S^{p+q}\mathfrak{A}$, even if $p < 0$ or $q < 0$. If $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ is a join-homomorphism, then $\{\mathfrak{f}_p^q\}$ will denote the group consisting of classes of $\mathfrak{f}_{q+r}^{p+r}$-$S$-maps $S^{p+r} \cup \mathfrak{A} \to S^{q+r} \cup \mathfrak{B}$, for all $r \geqq \max(-p, -q)$, classified as in the definition of $S$-maps and indexed by $S^p\mathfrak{A}$, $S^q\mathfrak{B}$. Thus $\{\mathfrak{f}_q^p\}$ means the same as before if $p \geqq 0$, $q \geqq 0$ and if $p < 0$ or $q < 0$ then

$$S^r : \{\mathfrak{f}_r^p\} \approx \{\mathfrak{f}_{q+r}^{p+r}\} \qquad (r \geqq \max(-p, -q)),$$

where $S^r$ is a conventional suspension isomorphism. We write

$$\{\mathfrak{f}_m^0\} = \textstyle\sum_m(\mathfrak{f}), \qquad \{\mathfrak{f}_0^m\} = \sum^m(\mathfrak{f})$$
$$\textstyle\sum_{m-1}((g;\ f)_1) = \sum_m(\mathfrak{g};\ \mathfrak{f}),\ \sum^{m+1}((\mathfrak{g};\ \mathfrak{f})_1) = \sum_m(\mathfrak{g};\ \mathfrak{f}).$$

With these conventions we may apply the isomorphism $S^{-|m|}$ to (9.2) and thus obtain the commutative diagram

(9.3) $$\begin{array}{ccccccc} \cdots \longrightarrow \sum_m(\mathfrak{f}) & \xrightarrow{u} & \sum_m(\mathfrak{g}) & \xrightarrow{v} & \sum_m(\mathfrak{g};\ \mathfrak{f}) & \xrightarrow{w} \cdots \\ \downarrow D_n & & \downarrow D_n & & \downarrow F_n & \\ \cdots \longrightarrow \sum^{-m}(\mathfrak{f}^*) & \xrightarrow{u} & \sum^{-m}(\mathfrak{g}^*) & \xrightarrow{v'} & \sum^{-m}(\mathfrak{g}^*;\ \mathfrak{f}^*) & \xrightarrow{w} \cdots. \end{array}$$

Similar results hold for the sequences of triples $(\mathfrak{h}, \mathfrak{g}, \mathfrak{f})$, $(\mathfrak{h}^*, \mathfrak{g}^*, \mathfrak{f}^*)$, where $\mathfrak{f} \leqq \mathfrak{g} \leqq \mathfrak{h}$.

## 10. Combinatorial duals

If $X$ is a finite CW-complex and $x_0$ a 0-cell of $X$, then $\mathfrak{A}(X, x_0)$ will denote the lattice of all sub-complexes of $X$ which contain $x_0$. Thus, if $x_0$ is the only 0-cell in $X$, then $\mathfrak{A}(X, x_0)$ consists of all the non-vacuous sub-complexes of $X$. In this case we may write $\mathfrak{A}(X, x_0) = \mathfrak{A}(X)$. A finite CW-complex $X^*$ will be called a *combinatorial $n$-dual* of $X$ if, and only if, $\mathfrak{A}(X^*, x_0^*)$ is externally $n$-dual to $\mathfrak{A}(X, x_0)$, for some pair of 0-cells $x_0$, $x_0^*$ in $X$, $X^*$. It follows from an inductive argument, which depends on the (external) duality between $\mathfrak{C}$, $\mathfrak{C}^*$ in §8, that such an $X^*$ exists. If $\mathfrak{A}(X^*, x_0^*)$ is externally $n$-dual to $\mathfrak{A}(X, x_0)$ under an anti-isomorphism $\mathfrak{a} : \mathfrak{A}(X, x_0) \to \mathfrak{A}(X^*, x_0^*)$, then we say that $X^*$ or, more explicitly, $(X^*, x_0^*)$ is combinatorially $n$-dual to $(X, x_0)$ *under* $\mathfrak{a}$. In this case $(X, x_0)$ is combinatorially $n$-dual to $(X^*, x_0^*)$ under $\mathfrak{a}^{-1}$.

Let $(X^*, x_0^*)$ be combinatorially $n$-dual to $(X, x_0)$ under $\mathfrak{a}$. Let $X_p = X^p$ or $x_0$ according as $p \geq 0$ or $p < 0$ $(p = 0, \pm 1, \cdots)$ and let $X_p^*$ be similarly defined. We describe $A_1, A_2 \in \mathfrak{A}(X, x_0)$ as *adjacent* if, and only if, $A_1 \neq A_2$, $A_1 \subset A_2$ or $A_2 \subset A_1$, say $A_1 \subset A_2$, and $A_1 \subset A \subset A_2$ implies $A = A_1$ or $A_2$. If $A_1$, $A_2$ are adjacent so are $\mathfrak{a}A_1$, $\mathfrak{a}A_2$. This is the case, with $A_1 \subset A_2$, if, and only if, $A_2 = A_1 \cup e$, where $e$ is an open cell of $X$. If this is so, then $A_2/A_1$ is a $p$-sphere, where $p = \dim e$, and $\mathfrak{a}A_1/\mathfrak{a}A_2$ is an $(n - p - 1)$-sphere. Hence it follows without difficulty that

$$\mathfrak{a}X_p = X^*_{n-p-2} \qquad (p = 0, \pm 1, \cdots). \tag{10.1}$$

By a *closed cell* of $X$ we mean the smallest sub-complex which contains some point in $X$, and hence the open cell of $X$ which contains that point. If $\sigma$ is a closed cell of $X$, then $C(X, \sigma)$ will denote the largest sub-complex of $X$ which does not contain $\sigma$, We write $\sigma \in X - x_0$ to indicate that $\sigma$ is a closed cell of $X$ other than $x_0$. A sub-complex of the form $x_0 \cup \sigma^p$, where $\sigma^p \in X - x_0$ $(\dim \sigma^p = p)$, is characterized as a minimal element $A \in \mathfrak{A}(X, x_0)$ such that $A \subset X_p$, $A \not\subset X_{p-1}$. A sub-complex of the form $C(X^*, \tau^q)$, where $\tau^q \in X^* - x_0^*$, is a maximal $A^* \in \mathfrak{A}(X^*, x_0^*)$ such that $X^*_{q-1} \subset A^*$, $X_q^* \not\subset A^*$. Also $\sigma^p \not\subset C(X, \sigma^p)$ and $\tau_1^q = \tau^q$ if $\tau_1^q \not\subset C(X^*, \tau^q)$. Hence if follows that $\mathfrak{a}$ determines a 1–1 correspondence, $\theta$, from the set of closed cells of $X - x_0$ to the set of closed cells of $X^* - x_0^*$ such that

$$\begin{cases} \dim \sigma + \dim (\theta \sigma) = n - 1 \\ \mathfrak{a}(x_0 \cup \sigma) = C(X^*, \theta\sigma),\ \mathfrak{a}C(X, \sigma) = x_0^* \cup \theta\sigma. \end{cases} \tag{10.2}$$

It may be verified that $\theta$ is order reversing (i.e. $\sigma_1 \subset \sigma_2$ if, and only if, $\theta\sigma_2 \subset \theta\sigma_1$). If $A \in \mathfrak{A}(X, x_0)$, then $\mathfrak{a}A$ is the union of $x_0^*$ and the closed cells $\theta\sigma$, for every $\sigma \in X - x_0$ such that $\sigma \not\subset A$.

If $\dim X < n - 1$, then $X_{n-2} = X_{n-1}$ and it follows from (10.1) that $X_0^* = X_{-1}^*$. That is to say $x_0^*$ is the only 0-cell of $X^*$. In this case, if $X^{\#}$ is any other combinatorial $n$-dual of $X$, then the dual of the identical $CJ$-map of $\mathfrak{A}(X, x_0)$ is a $CJ$-equivalence $\mathfrak{A}(X^*) \to \mathfrak{A}(X^{\#})$.

## 11. Dual exact couples

Let $\mathfrak{A}$, $\mathfrak{B}$ be strictly finite CW-lattices and $\mathfrak{f}$, $\mathfrak{g} : \mathfrak{A} \to \mathfrak{B}$ join-homomorphisms. Then $\mathfrak{f} \cup \mathfrak{g} : \mathfrak{A} \to \mathfrak{B}$ is defined by $(\mathfrak{f} \cup \mathfrak{g})A = \mathfrak{f}A \cup \mathfrak{g}A$. It is a join-homomorphism and, if $\mathfrak{f}^*$, $\mathfrak{g}^* : \mathfrak{B}^* \to \mathfrak{A}^*$ are dual to $\mathfrak{f}$, $\mathfrak{g}$ under $\mathfrak{a} : \mathfrak{A} \to \mathfrak{A}^*$, $\mathfrak{b} : \mathfrak{B} \to \mathfrak{B}^*$, where $\mathfrak{A}^*$, $\mathfrak{B}^*$ are $n$-duals of $\mathfrak{A}$, $\mathfrak{B}$, then

$$(\mathfrak{f} \cup \mathfrak{g})^* = \mathfrak{f}^* \cup \mathfrak{g}^*. \tag{11.1}$$

The duality between $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{A}^*$, $\mathfrak{B}^*$ may be either internal or external. To fix ideas let us take it to be external, $\bigcap\mathfrak{A}$, $\bigcap\mathfrak{B}$ being $S$-contractible.

Let $K$, $L$ be given elements of $\mathfrak{A}$, $\mathfrak{B}$. Then we define

$$\mathfrak{f} \cup L, \mathfrak{f} \cap L, \mathfrak{f} \div K : \mathfrak{A} \to \mathfrak{B} \tag{11.2}$$

by $(\mathfrak{f} \cup L)A = \mathfrak{f}A \cup L$, $(\mathfrak{f} \cap L)A = \mathfrak{f}A \cap L$ and the condition that $(\mathfrak{f} \div K)A$ is the smallest $B \in \mathfrak{B}$ such that $A \subset \mathfrak{f}^{\#}B \cup K$. Then $\mathfrak{f} \cup L$, $\mathfrak{f} \cap L$ are join-homomorphisms and $\mathfrak{f}^{\#} \cup K$, defined in the same way as $\mathfrak{f} \cup L$, is a meet-homomorphism. Clearly $\mathfrak{f} \div K$ is the join-homomorphism dual to $\mathfrak{f}^{\#} \cup K$ and it follows that $\mathfrak{f}^{\#} \cup K = (\mathfrak{f} \div K)^{\#}$. Therefore $\mathfrak{f}^* \cap \mathfrak{a}K = (\mathfrak{f} \div K)^*$ and

$$(\mathfrak{f} \cap L)^* = \mathfrak{f}^* \div \mathfrak{b}L : \mathfrak{B}^* \to \mathfrak{A}^*. \tag{11.3}$$

Now let $\mathfrak{f} \leqq \mathfrak{g}$ and let $L_p \in \mathfrak{B}$ ($p = 0, \pm 1, \cdots$) be such that $L_p \subset L_{p+1}$. Define

$$\mathfrak{h}_p = \mathfrak{f} \cup (\mathfrak{g} \cap L_p) : \mathfrak{A} \to \mathfrak{B}. \tag{11.4}$$

Then $\mathfrak{f} \leqq \mathfrak{h}_p \leqq \mathfrak{h}_{p+1} \leqq \mathfrak{g}$. Let $\langle \mathfrak{h} \rangle$ denote the exact couple which consists of the groups

$$A_{p,q} = \sum_{p+q} (\mathfrak{h}_p), C_{p,q} = \sum_{p+q} (\mathfrak{h}_p, \mathfrak{h}_{p-1}) \tag{11.5}$$

and the exact sequences (as in (9.3)) of the pairs $(\mathfrak{h}_p, \mathfrak{h}_{p-1})$.

It follows from (11.1), (11.3) that

$$\mathfrak{h}_p^* = \mathfrak{f}^* \cup (\mathfrak{g}^* \div L_p^*) : \mathfrak{B}^* \to \mathfrak{A}^* \qquad (L_p^* = \mathfrak{b}L_p) \tag{11.6}$$

and, by the commutativity of (9.3), the exact couple $\langle \mathfrak{h}^* \rangle$ is isomorphic to $\langle \mathfrak{h} \rangle$ under external $n$-duality. Thus

$$\begin{cases} D_n : \sum_m(\mathfrak{h}_p) \approx \sum^{-m} (h_p^*) \\ D_n : \sum_m(\mathfrak{h}_p, \mathfrak{h}_{p-1}) \approx \sum^{-m}(\mathfrak{h}_p^*, \mathfrak{h}_{p-1}^*). \end{cases} \tag{11.7}$$

There is also an exact couple, $\langle \bar{h} \rangle$, which consists of the groups

$$\bar{A}_{p,q} = \sum_{p+q}(\mathfrak{g}, \mathfrak{h}_{p-1}), \bar{C}_{p,q} = \sum_{p+q}(\mathfrak{h}_p, \mathfrak{h}_{p-1}) \tag{11.8}$$

and the exact sequences of the triples $(\mathfrak{g}, \mathfrak{h}_p, \mathfrak{h}_{p-1})$. This is isomorphic under $D_n$ to the corresponding exact couple $\langle \bar{h}^* \rangle$.

Let $\mathfrak{B} = \mathfrak{A}(Y, y_0)$, $\mathfrak{B}^* = \mathfrak{A}(Y^*, y_0^*)$, where $Y$ is a finite CW-complex and $(Y^*, y_0^*)$ is a combinatorial dual of $(Y, y_0)$. As in §10 let $Y_p = Y^p$ or $y_0$ according as $p \geqq 0$ or $p < 0$. Let $\mathfrak{A} = \mathfrak{A}(S^0, x_0)$, $\mathfrak{A}^* = \mathfrak{A}(S^n, x_0^*)$, where $S^0$ consists of $x_0$ and some other point, and let $\mathfrak{f}$, $\mathfrak{g}$ be defined by

$$\mathfrak{f}S^0 = \mathfrak{f}x_0 = \mathfrak{g}x_0 = y_0, \qquad \mathfrak{g}S^0 = Y.$$

Let $L_p = Y_p$. Then $\langle \mathfrak{h} \rangle$ is the $S$-homotopy exact couple of $Y$ with

$$\begin{cases} A_{p,q} = A_{p,q}(Y) = \sum_{p+q}(Y_p) \\ C_{p,q} = C_{p,q}(Y) = \sum_{p+q}(Y_p, Y_{p-1}). \end{cases} \tag{11.9}$$

We have $L_p^* = Y^*_{n-p-1}$ and $\langle \mathfrak{h}^* \rangle$ is the $S$-cohomotopy exact couple of $Y^*$, as defined (for ordinary homotopy theory) by Franklin P. Peterson [3]. It consists of the groups

$$A^{r,q}(Y^*) = \sum\nolimits^{r-q}(Y^*, Y^*_{r-1}), C^{r,q}(Y^*) = \sum\nolimits^{r-q}(Y^*_r, Y^*_{r-1}) \tag{11.10}$$

and the exact $S$-cohomotopy sequences of the triples $(Y^*, Y^*_r, Y^*_{r-1})$. The duality is expressed by

$$\begin{cases} D_n : A_{p,q}(Y) \approx A^{n-p-1,q}(Y^*) \\ D_n : C_{p,q}(Y) \approx C^{n-p-1,q}(Y^*). \end{cases} \tag{11.11}$$

The couples $\langle \bar{h} \rangle$, $\langle \bar{h}^* \rangle$ consist of the groups

$$\bar{A}_{p,q}(Y) = \sum\nolimits_{p+q}(Y, Y_{p-1}), \bar{C}_{p,q} = \sum\nolimits_{p+q}(Y_p, Y_{p-1}) \tag{11.12}$$

$$\bar{A}^{r,q}(Y^*) = \sum\nolimits^{r-q}(Y^*_r), \bar{C}^{r,q} = \sum\nolimits^{r-q}(Y^*_r, Y^*_{r-1}) \tag{11.13}$$

and the corresponding sequences, with

$$D_n : \bar{A}_{p,q}(Y) \approx \bar{A}^{n-p-1,q}(Y^*). \tag{11.14}$$

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# DUALITY AND THE SUSPENSION CATEGORY

By E. H. Spanier

## 1. Introduction

This report is a summary of some of the joint work of J. H. C. Whitehead and the author on duality in homotopy theory [5; 6; 7; 8; 9; 12]. We shall present a statement of the results for the absolute theory and sketch some applications.

The underlying motivation is the existence of an analogy in homotopy theory which is like a duality principle. The analogy seems to be between the following: homology group—cohomology group, homotopy group—cohomotopy group, inclusion map—identification map, lowest non-trivial dimension for homology—highest non-trivial dimension for cohomology. That is, theorems concerning these concepts seem to occur in pairs, one being obtained from the other by interchanging the paired concepts above. As a well-known example of a pair of dual theorems we cite the Hurewicz isomorphism theorem (the lowest dimensional non-trivial homotopy group is isomorphic to the integral homology group of the same dimension, if the dimension in question is larger than 1) and the Hopf classification theorem (for a finite dimensional space the highest dimensional non-trivial cohomotopy group is isomorphic to the integral cohomology group of the same dimension).

Though not universally valid this dualization leads to correct theorems often enough to suggest that there should be a precise formulation of the duality. An examination of the sets of dual theorems shows that these theorems are stable under suspension. For example, the Hurewicz theorem for dimension 1 is different from the higher dimensional result, and we exclude this as it is not stable. We are limited to the stable case because the cohomotopy groups are defined only in that range, there being no natural way of imposing a group structure on the set of homotopy classes of maps $X \to S^n$ unless $\dim X < 2n - 1$. Hence, the first step in seeking a rigorous duality seems to be to consider only the stable situation. This is done systematically by means of the suspension category which we discuss next.

## 2. The suspension category

For given topological spaces $X$ and $Y$ let $[X, Y]$ denote the set of homotopy classes of (continuous) mappings $X \to Y$. The *homotopy category* is the category whose objects are topological spaces and whose maps are homotopy classes of continuous mappings from one space to another. If $f: X \to Y$ is a mapping we use $[f]$ to denote the homotopy class of $f$.

Let $SX$ denote the join of $X$ with an ordered pair of points; thus if $X \neq 0$, $SX$ can be regarded as the space obtained from $X \times I$ by identifying $X \times 0$ to a point and $X \times 1$ to a point. Given a map $f: X \to Y$ we define $Sf: SX \to SY$ by

$$(Sf)(x, t) = (f(x), t).$$

If $f, g : X \to Y$ are homotopic, then $Sf$, $Sg$ are homotopic so $S$ gives rise to a map $S : [X, Y] \to [SX, SY]$ by setting $S[f] = [Sf]$. We set $S^k X = S(S^{k-1}X)$ for $k > 1$ obtaining the sequence

$$[X, Y] \to \cdots \to [S^k X, S^k Y] \xrightarrow{S} [S^{k+1}X, S^{k+1}Y] \to \cdots .$$

We define $\{X, Y\}$ to be the direct limit of this sequence. The elements of $\{X, Y\}$ are called *S-maps* from $X$ to $Y$ and consist of equivalence classes of maps $S^k X \to S^k Y$ for $k \geqq 0$ where $f : S^k X \to S^k Y$ and $g : S^j X \to S^j Y$ are equivalent if and only if there exists an integer $p \geqq 0$ such that the two maps

$$S^{j+p} f, S^{k+p} g : S^{j+k+p} X \to S^{j+k+p} Y$$

are homotopic. We denote the element of $\{X, Y\}$ determined by a map $f : S^k X \to S^k Y$ by $\{f\}$ and shall use Greek letters to denote $S$-maps.

There is a natural pairing of $\{X, Y\}$ and $\{Y, Z\}$ to $\{X, Z\}$ by composition. This follows from the fact that given $\alpha \in \{X, Y\}$ and $\beta \in \{Y, Z\}$ there exist an integer $k \geqq 0$ and maps $f : S^k X \to S^k Y$ and $g : S^k Y \to S^k Z$ such that $\alpha = \{f\}$ and $\beta = \{g\}$. Then $gf : S^k X \to S^k Z$ represents an element $\{gf\} \in \{X, Z\}$, and it is easy to see that $\{gf\}$ depends only on $\alpha$ and $\beta$ and not on the choice of $k, f$, or $g$. We set $\beta\alpha = \{gf\}$. With this law of composition we define the *suspension category* or *S-category* to be the category whose objects are topological spaces and whose maps are $S$-maps between spaces [6; 8].

If $X$ is a non-trivial space, there are isomorphisms

$$S : H_p(X) \to H_{p+1}(SX), \quad S : H^p(X) \to H^{p+1}(SX)$$

which are defined by taking the join of the homology (cohomology) class in question with the basic 0-dimensional homology (cohomology) class of the ordered pair of points joined to form the suspension (all homology and cohomology groups are taken reduced [2, p. 18] and the coefficient group can be arbitrary unless it is specified). Furthermore, if $f : X \to Y$, we have commutativity in the diagram

$$\begin{array}{ccc} H_p(X) & \xrightarrow{f_*} & H_p(Y) \\ S \downarrow & & \downarrow S \\ H_{p+1}(SX) & \xrightarrow{(Sf)_*} & H_{p+1}(SY) \end{array}$$

and similar commutativity for the corresponding diagram of cohomology groups. Hence, it follows that we can unambiguously define $\alpha_* : H_p(X) \to H_p(Y)$ for $\alpha \in \{X, Y\}$ by choosing a map $f : S^k X \to S^k Y$ which represents $\alpha$ and then defining $\alpha_*$ so that the following diagram is commutative

$$\begin{array}{ccc} H_p(X) & \xrightarrow{\alpha_*} & H_p(Y) \\ S^k \downarrow & & \downarrow S^k \\ H_{p+k}(S^k X) & \xrightarrow{f_*} & H_{p+k}(S^k Y). \end{array}$$

Similarly we define $\alpha^* : H^p(Y) \to H^p(X)$ for $\alpha \in \{X, Y\}$. It follows from [11, Theorem 3] that an $S$-map $\alpha : X \to Y$, where $X$ and $Y$ are CW-complexes, is an $S$-equivalence if and only if $\alpha_*$ is an isomorphism onto for every $p$ or, equivalently, if and only if $\alpha^*$ is an isomorphism onto for every $p$.

The $S$-category has several desirable properties. Firstly, it is stable under suspension. That is, the map $S : \{X, Y\} \to \{SX, SY\}$ defined by $S\{f\} = \{Sf\}$ is a 1–1 correspondence, as is seen from the definition of $\{X, Y\}$. Secondly, for any two spaces $X$, $Y$ the set $\{X, Y\}$ can be given the structure of an abelian group by the track addition [1], and in terms of this group structure the pairing of $\{X, Y\}$ and $\{Y, Z\}$ to $\{X, Z\}$ by composition is bilinear. Also the pairing of $\{X, Y\}$ and $H_p(X)$ to $H_p(Y)$ defined by $\alpha_* z$ for $\alpha \in \{X, Y\}$, $z \in H_p(X)$ is bilinear with a similar property valid for cohomology.

Let $A$ denote a subcomplex of a finite CW-complex $X$ and let $X/A$ denote the complex obtained by identifying $A$ to a single point. Then we have the identification map $X \to X/A$, and we construct a map $X/A \to SA$ by regarding $SA$ as the union of two cones $T_+A$ and $T_-A$ over $A$ which intersect in $A$ and extending the identity map $A \to A$ to a map $X \to T_+A$. Regarding the latter map as into $SA$ we deform it by contracting $T_-A$ over itself to a point thus obtaining a map $X \to SA$ which sends $A$ to a single point so corresponding to a map $X/A \to SA$. Since $S(X/A) = SX/SA$, we can form the sequence

$$A \to X \to X/A \to SA \to SX \to S(X/A) \to \cdots$$

This sequence is exact in the sense that for any finite CW-complex $Y$ the following two sequences are exact [8]

$$\cdots \to \{X/A, Y\} \to \{X, Y\} \to \{A, Y\} \to \{X/A, SY\} \to \cdots$$
$$\cdots \to \{Y, A\} \to \{Y, X\} \to \{Y, X/A\} \to \{Y, SA\} \to \cdots.$$

The first is a generalization of the cohomotopy sequence of $(X, A)$ (which would result if $Y$ were a sphere), and the second is a generalization of the homotopy sequence of $(X, A)$.

Frequently it is convenient to consider spaces $X$ with a base point $x_0$ and to form the *reduced suspension* $S_0X$ (which is obtained from the suspension $SX$ by identifying $Sx_0$ to a single point to serve as the base point of $S_0X$) instead of $SX$. Since the identification map $SX \to S_0X$ is a homotopy equivalence, by taking the limit with respect to $S_0$ instead of $S$ we obtain an isomorphic copy of the $S$-category for spaces with a base point. This treatment has the advantage that when $X$ is a CW-complex $S_0X$ doesn't contain any extraneous cells.

Lastly, there is a natural map of the homotopy category into the $S$-category which is the identity map on the objects and which sends $[f]$ into $\{f\}$. This map has the property that it is an isomorphism of $[X, Y]$ onto $\{X, Y\}$ for finite CW-complexes in the stable range (i.e. $Y$ is $(n-1)$-connected and dim $X < 2n - 1$) [8]. Hence, the duality principle formulated below for the $S$-category gives rise to dual pairs of theorems in the homotopy category in the stable range.

## 3. $\mathscr{C}$-theory and the suspension category

In a certain sense the $S$-category for finite CW-complexes is dominated by the stable homotopy groups of spheres. We indicate one way in which this statement can be made precise.

Let $X$ be a finite CW-complex with a single 0-cell to be used as base point of $X$. If $X$ consists of more than a single point, let *dim* $X$ = maximum dimension of the cells of $X$ of positive dimension and *codim*$X$ = minimum dimension of the cells of $X$ of positive dimension. Then

$$\dim S_0X = \dim X + 1, \qquad \operatorname{codim} S_0X = \operatorname{codim} X + 1.$$

We define *height* $X = \dim X - \operatorname{codim} X$, and for two complexes $X$ and $Y$, we define *stem* $(X, Y) = \dim X - \operatorname{codim} Y$. Then we see that

$$\text{height } S_0X = \text{height } X, \qquad \text{stem}(S_0X, S_0Y) = \text{stem}(X, Y).$$

The concepts of stem and height are useful for making inductive arguments in the suspension category.

Let $\mathscr{C}$ denote a class of abelian groups closed under subgroups, quotient groups, extension groups, and isomorphisms as used by Serre [4]. We shall be working modulo $\mathscr{C}$. A sequence

$$G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} G_3$$

will be called *$\mathscr{C}$-exact* if $\phi_2\phi_1 = 0$ and if kernel $\phi_2$ modulo image $\phi_1G_1$ is in $\mathscr{C}$. This is not the most general concept of $\mathscr{C}$-exactness that could be defined, but it suffices for our applications. Note that an equivalent formulation is that the homology group of

$$G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} G_3$$

at $G_2$ is in $\mathscr{C}$. We shall need the following form of the 5-lemma for $\mathscr{C}$ theory.

LEMMA. *Consider the commutative diagram*

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 & \xrightarrow{a_3} & A_4 & \xrightarrow{a_4} & A_5 \\
\downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_2} & & \downarrow{\scriptstyle \lambda_3} & & \downarrow{\scriptstyle \lambda_4} & & \downarrow{\scriptstyle \lambda_5} \\
B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & B_3 & \xrightarrow{b_3} & B_4 & \xrightarrow{b_4} & B_5
\end{array}$$

*in which the two rows are $\mathscr{C}$-exact and $\lambda_1, \lambda_2, \lambda_4, \lambda_5$ are $\mathscr{C}$-isomorphisms. Then $\lambda_3$ is also a $\mathscr{C}$-isomorphism.*

PROOF. Let $K_i =$ kernel $\lambda_i$ and $Q_i = A_i/K_i$ so we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1 & \longrightarrow & K_2 & \longrightarrow & K_3 & \longrightarrow & K_4 & \longrightarrow & K_5 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 & \xrightarrow{a_3} & A_4 & \xrightarrow{a_4} & A_5 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q_1 & \longrightarrow & Q_2 & \longrightarrow & Q_3 & \longrightarrow & Q_4 & \longrightarrow & Q_5 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

in which the columns are exact, and the horizontal maps are induced by the $a$'s. By assumption $K_1$, $K_2$, $K_4$, $K_5$ are in $\mathscr{C}$, and to prove $\lambda_3$ is a $\mathscr{C}$-monomorphism we must show that $K_3$ is in $\mathscr{C}$. Now $(A_i, a_i)$ can be regarded as a chain complex with subcomplex $(K_i)$ and quotient complex $(Q_i)$. To prove $K_3 \in \mathscr{C}$ it suffices to show that $H(K_3) \in \mathscr{C}$ because $K_3$ is an extension of $Z(K_3)$ by a group in $\mathscr{C}$, and $Z(K_3)$ is an extension of a group in $\mathscr{C}$ by $H(K_3)$.

The maps $\lambda_i : A_i \to B_i$ induce monomorphisms $\mu_i : Q_i \to B_i$, and we denote the quotients $B_i/\mu_i Q_i$ by $C_i$. Then we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q_1 & \longrightarrow & Q_2 & \longrightarrow & Q_3 & \longrightarrow & Q_4 & \longrightarrow & Q_5 \\
\downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_3} & & \downarrow{\scriptstyle \mu_4} & & \downarrow{\scriptstyle \mu_5} \\
B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & B_3 & \xrightarrow{b_3} & B_4 & \xrightarrow{b_4} & B_5 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & C_4 & \longrightarrow & C_5 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

in which the columns are exact and the horizontal maps are induced by the $b$'s. To prove $\lambda_3$ is a $\mathscr{C}$-epimorphism we must show that $C_3$ is in $\mathscr{C}$, and since $C_1$, $C_2$, $C_4$, $C_5 \in \mathscr{C}$, this is equivalent to showing that $H(C_3) \in \mathscr{C}$.

To complete the argument we use the fact that there are exact sequences

$$\cdots \to H(K_i) \to H(A_i) \to H(Q_i) \to H(K_{i+1}) \to \cdots$$

and

$$\cdots \to H(Q_i) \to H(B_i) \to H(C_i) \to H(Q_{i+1}) \to \cdots .$$

Since $(A_i, a_i)$ is $\mathscr{C}$-exact, $H(A_4) \in \mathscr{C}$, and because $K_5 \in \mathscr{C}$ it follows that $H(K_5) \in \mathscr{C}$. Hence, from the first exact sequence $H(Q_4) \in \mathscr{C}$. Since $(B_i, b_i)$ is $\mathscr{C}$-exact, $H(B_3) \in \mathscr{C}$

so from the second exact sequence $H(C_3) \in \mathscr{C}$ so $\lambda_3$ is a $\mathscr{C}$-epimorphism. A similar argument shows that it is a $\mathscr{C}$-monomorphism and completes the proof.

For a given abelian coefficient group $G$ let $H$ denote the homology functor from the $S$-category of finite CW-complexes to the category of graded abelian groups and homomorphisms defined by

$$H(X) = H_*(X; G), \quad H(\alpha) = \alpha_* : H(X) \to H(Y).$$

We assume the coefficient group $G$ and the class $\mathscr{C}$ so related that for any $A \subset X$ and any $Y$ the two sequences

$$\cdots \to \operatorname{Hom}(H(X/A), H(Y)) \to \operatorname{Hom}(H(X), H(Y)) \to \operatorname{Hom}(H(A), H(Y)) \to \cdots$$

$$\cdots \to \operatorname{Hom}(H(Y), H(A)) \to \operatorname{Hom}(H(Y), H(X)) \to \operatorname{Hom}(H(Y), H(X/A)) \to \cdots$$

are $\mathscr{C}$-exact. For example, taking $\mathscr{C}$ to be the class of finitely generated groups we can take $G$ to be the trivial coefficient group, or, taking $\mathscr{C}$ to be the class of torsion groups, $G$ can be the integers or the rationals.

THEOREM I. *With the same relation between $G$ and $\mathscr{C}$ as above assume that the map*

$$H : \{X, Y\} \to \operatorname{Hom}(H(X), H(Y))$$

*is a $\mathscr{C}$-isomorphism when $X$ and $Y$ are spheres of stem $\leqq k$. Then $H$ is a $\mathscr{C}$-isomorphism for arbitrary finite complexes $X$ and $Y$ of stem $\leqq k$.*

PROOF. The proof will be carried out in two stages. First we prove it is true for arbitrary $X$ when $Y$ is a sphere and stem $(X, Y) \leqq k$, and then we prove it is valid for arbitrary $X$ and $Y$ with stem $(X, Y) \leqq k$.

For the first part we use induction on the height of $X$. If height $X = 0$, $X$ is a union of spheres of the same dimension with a point in common. Then both $\{X, Y\}$ and $\operatorname{Hom}(H(X), H(Y))$ are isomorphic to direct sums of groups corresponding to the spheres of which $X$ is the union, and the map $H : \{X, Y\} \to \operatorname{Hom}(H(X), H(Y))$ corresponds under these isomorphisms to the direct sum of the maps corresponding to the spheres. Since, for spheres of stem $\leqq k$ this map is assumed to be a $\mathscr{C}$-isomorphism, it is a $\mathscr{C}$-isomorphism for arbitrary $X$ of height 0 if $Y$ is a sphere and stem $(X, Y) \leqq k$.

Now we assume the result for $X$ of height $<h$ (where $h > 0$) and let $X$ have height $h$ and stem $(X, Y) \leqq k$. Let $\dim X = n + 1$ and consider the commutative diagram

$$\begin{array}{ccccccc}
\{S_0X^n, Y\} & \to & \{X/X^n, Y\} & \to & \{X, Y\} & \to & \{X^n, Y\} \\
\downarrow H & & \downarrow H & & \downarrow H & & \downarrow H \\
(H(S_0X^n), H(Y)) & \to & (H(X/X^n), H(Y)) & \to & (H(X), H(Y)) & \to & (H(X^n), H(Y))
\end{array}$$

$$\begin{array}{c}
\to \{X/X^n, S_0Y\} \\
\downarrow H \\
\to (H(X/X^n), H(S_0Y))
\end{array}$$

where in the bottom row we have abbreviated $\operatorname{Hom}(G, G')$ by omitting the prefix

Hom. The top row is exact while the bottom row is $\mathscr{C}$-exact by the assumption on $G$ and $\mathscr{C}$. The height of $X^n =$ height $S_0X^n < h$ and stem $(S_0X^n, Y) \leqq$ stem $(X, Y) \leqq k$, stem $(X^n, Y) \leqq k$. Also $X/X^n$ has height 0 and stem $(X/X^n, S_0Y) <$ stem $(X/X^n, Y) \leqq k$. Hence, by the inductive assumption the first two and last two vertical maps are $\mathscr{C}$-isomorphisms. By the lemma the middle one is also a $\mathscr{C}$-isomorphism.

The second part of the argument follows similarly by induction on the height of $Y$.

As applications of this theorem we point out that if $\mathscr{C}$ is the class of finitely generated groups and $G$ is the trivial coefficient group, we obtain the result that $\{X, Y\}$ is finitely generated for $X$, $Y$ finite complexes. Also taking $G$ to be the rationals and $\mathscr{C}$ to be the class of torsion groups we obtain a result of Thom that if two maps $\alpha, \beta : X \to Y$ induce the same homomorphisms on the rational homology groups there exists a positive integer $N$ such that $N\alpha = N\beta$ and given any homomorphism $\phi : H_*(X) \to H_*(Y)$ (rational homology) there exist a positive integer $N$ and a map $\alpha : X \to Y$ such that $\alpha_* = N\phi$.

It is clear that Theorem I can be strengthened in that we can define stem $(X, Y)$ in terms of connectedness and coconnectedness instead of using dimension and codimension. That is, we define *virtual dim* $X \leqq p$ if all the $S$-cohomotopy groups of $X$ of dimension $>p$ vanish and *virtual codim* $X \geqq q$ if all the $S$-homotopy groups of $X$ of dimension $<q$ vanish. Then we define *virtual stem* $(X, Y) =$ virtual dim $X -$ virtual codim $Y$, and Theorem I can be extended to virtual stem instead of stem. Similarly we can extend the result by defining dimension and codimension mod $\mathscr{C}$ by taking the largest dimension for cohomotopy not in $\mathscr{C}$ and smallest dimension for homotopy not in $\mathscr{C}$, respectively, but we shall not go into details of this generalization here.

## 4. Duality

The duality we construct will be in the $S$-category. It has its origin in the duality between a subset of a sphere and its complement. Homologically this duality is well-known under the name of the Alexander duality theorem [3] (which asserts that for a closed subset $A \subset S^n$ there is an isomorphism of $H^p(A)$ onto $H_{n-p-1}(S^n - A)$). This homological duality is basic for the general duality we construct below.

We consider $S^n$ with a fixed triangulation and rectilinear subdivisions relative to this triangulation. Let $X$ denote a subpolyhedron of $S^n$. An *n-dual* $D_nX$ of $X$ is a subpolyhedron of $S^n - X$ which is an $S$-deformation retract of $S^n - X$ (this means that the inclusion map $D_nX \subset S^n - X$ is an $S$-equivalence). Hence, an $n$-dual is a polyhedral model of $S^n - X$, and the inclusion map $D_nX \subset S^n - X$ induces isomorphisms of the homology and cohomology groups of these two spaces. Then the Alexander duality theorem gives isomorphisms (depending on an orientation of $S^n$)

$$\mathscr{D}_n : H^p(X) \approx H_{n-p-1}(D_nX) \quad \text{for every } p.$$

If $X$ is a subpolyhedron of $S^n$, it has an $n$-dual. To show this we subdivide $S^n$ so that $X$ is a *complete subcomplex* (every simplex having all its vertices in $X$ itself belongs to $X$). Then the *complementary complex* of $X$ (the set of all simplices having no vertex in $X$) is an $n$-dual of $X$. A similar argument shows that given a collection $\{X_i\}$ of subcomplexes of $S^n$ and given for each $i$ a compact subset $C_i \subset S^n - X_i$ there exist $n$-duals $\{D_nX_i\}$ such that for each $i$ the compact set $C_i$ is contained in $D_nX_i$ and whenever $X_i \subset X_j$ then $D_nX_j \subset D_nX_i$. This is important in proving the main results later.

It is easy to verify [7, §3] that if $D_nX$ is an $n$-dual of $X$, then $X$ is an $n$-dual of $D_nX$ and also $SD_nX$ is $(n+1)$-dual to $X$ and $D_nX$ is $(n+1)$-dual to $SX$. Thus we see that the $S$-category enters naturally when considering $n$-duals because a space and its suspension have the same complex as $n$-dual, $(n+1)$-dual, respectively.

The principal facts about the duality are summarized in the following theorem. Detailed proofs can be found in [7]

THEOREM II. *Given n-duals $D_nX$, $D_nY$ of $X$, $Y$, respectively, there exists a unique homomorphism*

$$D_n : \{X, Y\} \to \{D_nY, D_nX\}$$

*having the following properties*:

1. *If $i : X \subset Y$ and $i' : D_nY \subset D_nX$ are inclusion maps, then*

$$D_n\{i\} = \{i'\}.$$

2. *If $D_nZ$ is n-dual to $Z$ and if $\alpha \in \{X, Y\}$, $\beta \in \{Y, Z\}$, then*

$$D_n(\beta\, \alpha) = (D_n\alpha)(D_n\beta).$$

3. *Since $Y$, $X$ are n-dual to $D_nY$, $D_nX$, respectively, there is also a homomorphism $D_n : \{D_nY, D_nX\} \to \{X, Y\}$, and this is inverse to $D_n : \{X, Y\} \to \{D_nY, D_nX\}$.*

4. *If $\alpha \in \{X, Y\}$, we have commutativity in the diagram*

$$\begin{array}{ccc} H^p(Y) & \xrightarrow{\alpha^*} & H^p(X) \\ \Big\downarrow \mathscr{D}_n & & \Big\downarrow \mathscr{D}_n \\ H_{n-p-1}(D_nY) & \xrightarrow{(D_n\alpha)_*} & H_{n-p-1}(D_nX). \end{array}$$

5. *Taking $SD_nX$, $SD_nY$ as $(n+1)$-dual to $X$, $Y$, then*

$$SD_n = D_{n+1} : \{X, Y\} \to \{SD_nY, SD_nX\}.$$

6. *Taking $D_nX$, $D_nY$ as $(n+1)$-dual to $SX$, $SY$, then*

$$D_{n+1}S = D_n : \{X, Y\} \to \{D_nY, D_nX\}.$$

Note that $S^p \subset S^n$ is $n$-dual to a sphere $S_1^{n-p-1}$. Hence, if we define the $p^{\text{th}}$ stable homotopy group of $X$ by $\sum_p(X) = \{S^p, X\}$ and the $q^{\text{th}}$ stable cohomotopy group of $Y$ by $\sum^q(Y) = \{Y, S_1^q\}$, then we see that

$$D_n : \textstyle\sum_p(X) \approx \sum^{n-p-1}(D_n X), \qquad D_n : \sum^p(X) \approx \sum_{n-p-1}(D_n X).$$

Furthermore, let $s^* \in H^q(S_1^q)$ be a generator of $H^q(S_1^q)$ (with integral coefficients) determined by an orientation of $S_1^q$ and let $s \in H_{n-q-1}(S^{n-q-1})$ be the generator defined by $s = \mathscr{D}_n s^*$, which is used to orient $S^{n-q-1}$. We define two homomorphisms

$$\phi^* : \textstyle\sum^q(X) \to H^q(X), \qquad \phi : \sum_p(Y) \to H_p(Y)$$

by $\phi^*\alpha = \alpha^* s^*$ and $\phi\,\beta = \beta_* s$ for $\alpha \in \{X, S_1^q\}$, $\beta \in \{S^p, Y\}$. That the maps $\phi^*, \phi$ are homomorphisms is a consequence of remarks made in §2. Property 4 of Theorem II yields commutativity in the following diagram

$$\begin{array}{ccc} \sum^q(X) & \xrightarrow{\phi^*} & H^q(X) \\ D_n \downarrow & & \downarrow \mathscr{D}_n \\ \sum_{n-q-1}(D_n X) & \xrightarrow{\phi} & H_{n-q-1}(D_n X). \end{array}$$

Hence, it is a consequence of Theorem II that the Hurewicz isomorphism theorem is dual to the Hopf extension theorem.

Though there is not a unique $n$-dual of a space $X$, it follows from 1 and 2 of Theorem II that if $D_n X$ and $D'_n X$ are both $n$-duals of $X$ then corresponding to the identity map $i : X \subset X$ there is an $S$-map

$$D_n\{i\} : D_n X \to D'_n X$$

which is an $S$-equivalence (because $\{i\}$ is an $S$-equivalence). Hence, any two $n$-duals of $X$ have the same $S$-homotopy type, and if $D_n X$ is an $n$-dual of $X$ and $D'_m X$ is an $m$-dual of $X$, then $S^m D_n X$ and $S^n D'_m X$ have the same $S$-homotopy type.

## 5. Weak duality

In order to extend the duality map $D_n$ to spaces other than subpolyhedra of spheres we introduce the concept of weak duality. Given finite CW-complexes $X$, $X^*$ and $S$-equivalences $\xi : X \to X'$, $\xi^* : D_n X' \to X^*$ where $X'$, $D_n X'$ are $n$-dual subpolyhedra of $S^n$, we say that $\xi, \xi^*$ form a *weak n-duality* between $X, X^*$. If we have a similar weak duality $\eta, \eta^*$ between $Y, Y^*$ where $\eta : Y \to Y'$, $\eta^* : D_n Y' \to Y^*$, then we can define a map $D_n : \{X, Y\} \to \{Y^*, X^*\}$ by defining first $\alpha'$ and then $D_n\alpha$ so that each of the diagrams below is commutative

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \xi \downarrow & & \downarrow \eta \\ X' & \xrightarrow{\alpha'} & Y' \end{array} \qquad \begin{array}{ccc} X^* & \xleftarrow{D_n\alpha} & Y^* \\ \xi^* \uparrow & & \uparrow \eta^* \\ D_n X' & \xleftarrow{D_n\alpha'} & D_n Y'. \end{array}$$

Then properties similar to those of Theorem II are valid for this extension of the map $D_n$ provided that the equivalences $\xi$, $\xi^*$ are suitably chosen (as the map $D_n$ depends not only on the choice of $X^*$ but also on the choice of $\xi$, $\xi^*$).

Any finite CW-complex has a weak $n$-dual for sufficiently large $n$. This is because any finite CW-complex is of the same homotopy type as some simplicial complex [11] (which can be imbedded in $S^n$ for large enough $n$) and any $n$-dual of the imbedded simplicial complex is a weak $n$-dual of the original complex. Hence, weak $n$-duals exist. The next result gives some indication of how to obtain a weak dual for a complex which is constructed from other complexes for which weak duals are known.

Let $f: X \to Y$ be a cellular map of the finite CW-complex $X$ into the finite CW-complex $Y$. Let $TX$ denote a cone over $X$ which is taken disjoint from $Y$ and let $Z_f$ denote the CW-complex obtained from $TX \cup Y$ by identifying $x \in X$ with $fx \in Y$. We parametrize the points of $TX$ by pairs $(x, t)$, $x \in X$, $t \in I$ where the vertex of the cone is the set of points of the form $(x, 1)$ (all identified to a single point of $TX$). Then $Z_f$ consists of points $(x, t) \in TX$ and $y \in Y$ with the identifications $(x, 0) = fx$. Clearly there is an inclusion map $i : Y \subset Z_f$. We define a natural map $g : Z_f \to SX$ by $g(x, t) = (x, t)$ and $g(y) = (x, 0)$ in $SX$. With these preliminaries out of the way we can state the next result, which is proved in [7].

THEOREM III. *Let $f^* : Y^* \to X^*$ be a cellular map weakly $n$-dual to $f: X \to Y$, and let $Z = Z_f$ (as above) and $Z^* = Z_{f^*}$. There is a weak $(n + 1)$-duality between $Z$ and $Z^*$ such that in the two sets of mappings*

$$Y \xrightarrow{i} Z \xrightarrow{g} SX, \qquad SY^* \xleftarrow{g^*} Z^* \xleftarrow{i^*} X^*$$

*we have the relations*

$$D_{n+1}\{i\} = -\{g^*\}, \qquad D_{n+1}\{g\} = \{i^*\},$$

*it being understood that $Y$, $SY^*$ and $SX$, $X^*$ are weakly $(n + 1)$-dual by the given weak duality.*

This theorem shows how to construct a weak dual of the complex obtained by adjoining a cell to another complex for which a weak dual is already known. For example, $S^{p+1}$ is weakly $(2p + 2)$-dual to $S^p$, and relative to such a weak duality the essential map $S^{p+1} \to S^p$ (for $p \geq 3$) is self dual. Letting $M^{p+2}$ denote the space obtained by adjoining $TS^{p+1}$ to $S^p$ by this map (which gives a space of the same homotopy type as the $(p - 2)$-fold suspension of the complex plane) we see by Theorem II that $M^{p+2}$ is weakly $(2p + 3)$-dual to itself. Hence,

$$\textstyle\sum_k(M^{p+2}) \approx \sum^{2p+2-k}(M^{p+2}).$$

## 6. Examples and applications

We present some examples of duality and some mappings defined by the duality. Also we indicate an application of the theory to imbedding questions. First we show how the duality acts on joins of mappings.

THEOREM IV. *If $X$, $D_nX$ are $n$-dual in $S^n$ and $X'$, $D_mX'$ are $m$-dual in $S^m$, then the joins $X * X'$ and $D_nX * D_mX'$ are $(n + m + 1)$-dual in $S^{n+m+1} = S^n * S^m$. Moreover, if $Y$, $D_nY$ and $Y'$, $D_mY'$ are similar pairs of duals, then for $\alpha \in \{X, Y\}$ and $\beta \in \{X', Y'\}$ we find for the join $\alpha * \beta$ that*

$$D_{n+m+1}(\alpha * \beta) = D_n(\alpha) * D_m(\beta).$$

PROOF. We can find a triangulation $K$ of $S^n$ such that $X$ is a complete subcomplex whose complementary complex $X^*$ contains $D_nX$. Similarly we can find a triangulation $K'$ of $S^m$ in which $X'$ is a complete subcomplex whose complementary complex $X'^*$ contains $D_mX'$. Then $K * K'$ is a triangulation of $S^n * S^m$ in which $X * X'$ is a complete subcomplex with complementary complex $X * X'^*$. Since $D_nX \subset X^*$ and both are $n$-dual to $X$, $D_nX$ is an $S$-deformation retract of $X^*$. Similarly $D_mX'$ is an $S$-deformation retract of $X'^*$. Hence, $D_nX * D_mX'$ is an $S$-deformation retract of $X^* * X'^*$, and since $X^* * X'^*$ is an $(n + m + 1)$-dual of $X * X'$, so is $D_nX * D_mX'$ an $(n + m + 1)$-dual of $X * X'$.

To prove the second part about the dual of a join map, note that if $i : X \subset Y$, $i' : X' \subset Y'$ are inclusion maps dual to inclusions $j : D_nY \subset D_nX$, $j' : D_mY' \subset D_mX'$, then the inclusion map $i * i' : X * X' \subset Y * Y'$ is dual to $j * j' : D_nY * D_mY' \subset D_nX * D_mX'$. Since any map can be factored as a composite of inclusion maps and inverses of inclusion maps [7, Lemma (9.13)], the general result follows from the above special case.

Let $S^{2p+1} = S^p * S^p_1$. Then we take $S^p_1$ as $(2p + 1)$-dual to $S^p$ and obtain an involution $D_{2p+1} : \{S^p, S^p_1\} \to \{S^p, S^p_1\}$. Theorem IV is used in determining this involution.

First let $S^1 = S_a * S_b$ where $S_a$ is the pair of points $a$, $a'$ and $S_b$ is the pair of points $b$, $b'$. Taking $D_1S_a = S_b$ and $D_1S_b = S_a$ we want to determine the map $D_1 : \{S_a, S_b\} \to \{S_a, S_b\}$. Now $\{S_a, S_b\}$ is infinite cyclic generated by $\{f\}$ where $f : S_a \to S_b$ is defined by $f(a) = b$, $f(a') = b'$. Also the map $g : S_a \to S_b$ defined by $g(a) = b'$, $g(a') = b$ represents a generator of $\{S_a, S_b\}$, and $\{g\} = -\{f\}$. We carry through an explicit proof that $D_1\{f\} = -\{f\}$.

We use the notation $|x, y|$ to denote the closed segment from $x$ to $y$ in $S_a * S_b$ (if $x$ and $y$ belong to a 1-simplex of $S_a * S_b$) and $xy/2$ to denote the midpoint of this segment. Letting $C$ denote the subset $|a, b| \cup |a', b'|$ of $S_a * S_b$ we see that the map $f$ is the composite

$$S_a \subset C \to S_b$$

where the arrow represents a retraction. Then as 1-duals for $S_a$, $C$, $S_b$ we take

$$S^*_a = |ab'/2, b'| \cup |a'b/2, b|, \quad C^* = ab'/2 \cup a'b/2, \quad S^*_b = |ab'/2, a| \cup |a'b/2, a'|$$

which satisfy the conditions

$$C^* \subset S^*_a \cap S^*_b, \quad D_1S_a = S_b \subset S^*_a, \quad D_1S_b = S_a \subset S^*_b$$

so that $D_1\{f\}$ is represented by the composite

$$S_a \subset S^*_b \to C^* \subset S^*_a \to S_b$$

where each arrow represents a retraction. The composite is easily seen to be the map $g$, which completes the calculation.

Now let $S^p = S_{a_0} * \cdots * S_{a_p}$ be the join of $(p+1)$-pairs of points and let $S_1^p = S_{b_0} * \cdots * S_{b_p}$ be a similar join. Then $S^p$, $S_1^p$ are $(2p+1)$-duals in $S^{2p+1} = S^p * S_1^p$, and if $f_i : S_{a_i} \to S_{b_i}$ is as above, then $\{f_0 * \cdots * f_p\}$ is a generator of the infinite cyclic group $\{S^p, S_1^p\}$. Using Theorem IV and the calculation above we find

$$D_{2p+1}\{f_0 * \cdots * f_p\} = D_{2p+1}(\{f_0\} * \cdots * \{f_p\}) = (-1)^{p+1}\{f_0 * \cdots * f_p\}.$$

Therefore, we have proved that $D_{2p+1}$ on $\{S^p, S_1^p\}$ is the involution obtained by multiplying by $(-1)^{p+1}$. (Note that this corrects the sign in [5, page 200 line 9 from bottom] for $p$ odd.)

A similar involution can be defined for any space $X$ and $n$-dual $D_nX$ because $D_n : \{X, D_nX\} \to \{X, D_nX\}$. We do not know this involution in general; however, it follows from the above considerations that the diagram

$$\begin{array}{ccc} \{X, D_nX\} & \xrightarrow{S} & \{S_aX, S_bD_nX\} \\ {\scriptstyle D_n}\downarrow & & \downarrow{\scriptstyle -D_{n+2}} \\ \{X, D_nX\} & \xrightarrow{S} & \{S_aX, S_bD_nX\} \end{array}$$

is commutative.

The duality gives rise to an involution in the stable homotopy groups of spheres also. To see this let $S^n$ be an oriented $n$-sphere (with $n = p + q + k + 1$ where $p, q, k \geqq 0$) and let $S^{p+k}$, $S^p$ be oriented spheres contained in $S^n$ with $n$-duals $S_1^q$, $S_1^{q+k}$ oriented by the orientations of $S^{p+k}$, $S^p$, and $S^n$. Then we have a map

$$D_n : \{S^{p+k}, S^p\} \to \{S_1^{q+k}, S_1^q\},$$

and since all the spheres in question are oriented, this can be interpreted as a map of the stable group $\{S^{p+k}, S^p\}$ into itself (note that reorienting $S^n$ reorients both $S_1^{q+k}$ and $S_1^q$ so doesn't change the map). Letting $\Lambda^k$ denote the stable group $\{S^{p+k}, S^p)$, we have constructed an involution $D_n : \Lambda^k \to \Lambda^k$. Since $D_nS = D_{n+1}$, this involution doesn't depend on $n$ so we end up with an involution which we denote by

$$D : \Lambda^k \to \Lambda^k \text{ for every } k \geqq 0.$$

For $k = 0, 1, 2$ this map is the identity but we do not know it for general $k$.

We now indicate how the duality theory can be applied to imbedding questions. Suppose $X$ is a finite CW-complex which is $S$-equivalent to a subpolyhedron $X' \subset S^n$. Let $D_nX'$ be an $n$-dual of $X'$. If $X$ is $S$-equivalent to $X'' \subset S^{n-1}$, then any $(n-1)$-dual $D_{n-1}X''$ of $X''$ has the property that $SD_{n-1}X''$ is a weak $n$-dual of $X$. Since $D_nX'$ is also a weak $n$-dual of $X$, it follows that $SD_{n-1}X''$ is $S$-equivalent to $D_nX'$. Hence, if $X$ is $S$-equivalent to a subpolyhedron of $S^{n-1}$, then $D_nX'$ can be $S$-desuspended, and, more generally, any weak $m$-dual $(m \geqq n)$ can be $S$-desuspended $m - n - 1$ times. Explicit study of a weak $m$-dual of $X$ may show that it cannot be desuspended (in the $S$-category) more than $k$ times in which case we can conclude that $X$ is not $S$-equivalent to a subpolyhedron of $S^{m-k-1}$ (so, in particular, $X$ cannot be imbedded in $S^{m-k-1}$).

As an example let $X$ denote the complex projective plane. We have seen that $SX$, which we denoted previously by $M^5$, is weakly 9-dual to itself. With coefficients mod 2 the cohomology of $M^5$ is non-trivial in dimensions 3 and 5 and trivial in other dimensions and $Sq^2 : H^3(M^5) \to H^5(M^5)$ is non-trivial. It follows that $M^5$ cannot be $S$-desuspended twice because if it could there would have to exist a space $Y$ with $Sq^2 : H^1(Y) \to H^3(Y)$ (coefficients mod 2) non-trivial, and this is impossible. Hence, $SX$ cannot be imbedded in $S^7$ so $X$ cannot be imbedded in $S^6$.

This proof is essentially the same as that of Thom [10, p. 163]. The operations $\theta_p$ defined by Thom in homology mod 2 are dual by means of $D_n$, $\mathscr{D}_n$ to $Sq^p$ in the dual of the original space.

This can be generalized to dualize any stable operation. By a *stable cohomology operation* $\theta$ *of type* $(k, G, G')$ we shall mean that for any finite CW-complex $X$ and any $p \geqq 0$ we have a homomorphism

$$\theta : H^p(X; G) \to H^{p+k}(X; G')$$

such that if $f : X \to Y$ then commutativity holds in the diagram

$$\begin{array}{ccc} H^p(Y; G) & \xrightarrow{\theta} & H^{p+k}(Y; G') \\ f^* \downarrow & & \downarrow f^* \\ H^p(X; G) & \xrightarrow{\theta} & H^{p+k}(X; G'), \end{array}$$

and for any complex $X$ commutativity holds in the diagram

$$\begin{array}{ccc} H^p(X; G) & \xrightarrow{\theta} & H^{p+k}(X; G') \\ S \downarrow & & \downarrow S \\ H^{p+1}(SX; G) & \xrightarrow{\theta} & H^{p+k+1}(SX; G'). \end{array}$$

Similarly we define a *stable homology operation* $\phi$ *of type* $(k, G, G')$ as a homomorphism $\phi : H_p(X; G) \to H_{p-k}(X; G')$ which commutes with induced homomorphisms and suspension. Given a stable cohomology operation $\theta$ of type $(k, G, G')$ we get a stable homology operation $D(\theta)$ of type $(k, G, G')$ by choosing a weak $n$-dual $X^*$ for $X$ and letting $D(\theta)$ be defined in $X$ to correspond to $\theta$ in $X^*$ by means of $\mathscr{D}_n$. The resulting operation $D(\theta)$ is easily seen to be independent of the choice of $n$ and $X^*$, and the map $\theta \to D(\theta)$ thus defined is an isomorphism of the group of stable cohomology operations of type $(k, G, G')$ onto the group of stable homology operations of type $(k, G, G')$.

If we take $G = G' =$ a field $F$, then $H_p(X; F)$ and $H^p(X; F)$ are conjugate vector spaces. Hence, there is an isomorphism between homology operations of type $(k, F, F)$ and cohomology operations of type $(k, F, F)$ by taking the conjugate operation. Hence, combining these two maps we get an isomorphism

$\theta \to D'\theta$ of stable cohomology operations of type $(k, F, F)$ onto itself by letting $D'\theta$ be conjugate to the homology operation $D\theta$. Thom [10] has shown that for $F$ the field with two elements and $\theta = Sq^k$ that $D'Sq^k$ is completely determined by the formulas

$$\textstyle\sum_i (D'Sq^{r-i})Sq^i = 0 \quad \text{for} \quad r > 0.$$

It should prove worthwhile to determine $D'\theta$ for other operations $\theta$.

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# HOMOTOPY THEORY OF MODULES AND DUALITY

By P. J. Hilton

## 1. Introduction

In this article a homotopy theory is presented for modules (over a ring with unit element) which in many respects parallels the usual homotopy theory for spaces with base points. The usual constructions for homotopy theory find their analogues in the new setting; however, the basic result

$$\pi(SX;\, Y) \cong \pi(X; \Omega Y) \tag{1.1}$$

is missing from the theory. It is replaced by a duality in the exact category (in the sense of D. A. Buchsbaum) of modules. This duality leads to the introduction of dual definitions of homotopy for module-maps and the expressions on either side of (1.1) have analogues in the homotopy theory of modules relative to the dual definitions. In the second section of this article the homotopy theory of modules is developed and in the third section the corresponding duality is studied in the homotopy theory of topological spaces.

The work described in this article was carried out jointly by B. Eckmann and the author. Complete details are to be published later in a joint article.

## 2. Homotopy theory of modules

The definition of homotopy presented here is motivated by the observation that a map $f: X \to Y$ from the topological space $X$ to the topological space $Y$ is nullhomotopic if and only if it may be extended to every $X' \supseteq X$ such that $(X', X;\, Y)$ is an H. E. triple in the sense of Barratt, and that it is in fact only necessary to extend $f$ to the cone on $X$ to ensure its extendability to every such $X'$.

Now let $\Lambda$ be a ring with unit element. We will consider $\Lambda$-modules $A, B, \ldots$ (precisely, left $\Lambda$-modules) and maps $\phi : A \to B$, i.e., homomorphisms from $A$ to $B$ which are, of course, required to preserve the module structure.

Definition 2.1. *A map $\phi : A \to B$ is i-nullhomotopic if it may be extended to every module $A'$ containing $A$. Two maps $\phi_0, \phi_1 : A \to B$ are i-homotopic if $\phi_0 - \phi_1$ is i-nullhomotopic.*

*We write $\phi \simeq_i 0$, $\phi_0 \simeq_i \phi_1$.*

It is well-known that every module may be embedded in an injective module. We will write $\bar{A}$ for an injective module containing $A$ and may prove

Theorem 2.1. *Let $\bar{A}$ be an injective module containing $A$. Then $\phi : A \to B$ is i-nullhomotopic if and only if it may be extended to $\bar{A}$.*

For let $\eta : A' \to \bar{A}$ be an extension of the embedding $A \subseteq \bar{A}$. Then if $\bar{\phi} : \bar{A} \to B$ is an extension of $\phi$, $\bar{\phi}\eta : A' \to B$ is an extension of $\phi$ to $A'$.

Let Hom $(A, B)$ be the group of maps $\phi : A \to B$. Then the set of $i$-nullhomotopic maps obviously forms a subgroup. The quotient group

$$\bar{\pi}(A; B)$$

is called the $i$-homotopy group of maps from $A$ to $B$. It is easy to verify that if $\phi_0 \simeq_i \phi_1 : A \to B, \theta : D \to A, \psi : B \to C$, then

$$\psi\phi_0 \simeq_i \psi\phi_1, \qquad \phi_0\theta \simeq_i \phi_1\theta.$$

Thus $\bar{\pi}$ is a functor of two variables, contravariant in the first and covariant in the second. We may define $i$-homotopy type (or $i$-type, briefly) in the obvious way and $\bar{\pi}(A; B)$ is then an invariant of the $i$-types of $A$ and $B$. Among the "translations" from topology which lead to important theorems we mention

THEOREM 2.2. (Homotopy extension theorem). *Let $\phi : A \to B$, $\phi' = \phi$ cut down to $A' \subseteq A$, and $\phi' \simeq_i \psi' : A' \to B$. Then $\phi \simeq_i \psi : A \to B$ such that $\psi \mid A' = \psi'$.*

In fact, one can even extend the homotopy between $\phi'$ and $\psi'$ in the following sense. Such a homotopy is a map $\chi' : \bar{A}' \to B$ where $\bar{A}'$ is an injective module containing $A'$ and $\chi' \mid A' = \phi' - \psi'$. Assuming that $\bar{A}' \cap A = A'$ (this is reasonable and quite unrestrictive), we form $A^*$, the direct sum of $\bar{A}'$ and $A$ with the sub-module $A'$ amalgamated. Then we may write $\bar{A}$ for $\bar{A}^*$ since this is an injective module containing $A$ and we may extend $\chi'$ to $\chi : \bar{A} \to B$. Let $\psi = \phi - \chi \mid A$. Then $\phi \simeq_i \psi$ and $\psi \mid A' = \phi' - \chi \mid A' = \phi' - (\phi' - \psi') = \psi'$. Moreover, the homotopy $\chi$ between $\phi$ and $\psi$ "extends" the homotopy $\chi'$ between $\phi'$ and $\psi'$.

A basic idea in homotopy theory is that of suspension. In reality the suspension is a function from homotopy type to homotopy type and we find this situation reproduced for modules. Given a module $A$, let $\bar{A}$ be an injective module containing $A$ and let $S(A) = \bar{A}/A$. We prove

THEOREM 2.3. *The $i$-homotopy type of $S(A)$ depends only on that of $A$.*

We prove more. Let $\phi : A \to B$ be a map. Then $\phi$ may be extended to $\bar{\phi} : \bar{A} \to \bar{B}$ and hence induces $S\phi : S(A) \to S(B)$. Let $\phi = 0$. Then $S\phi$ may be factored through $\bar{B}$ and so $S\phi \simeq_i 0$. Now let $\phi$ be arbitrary and let $\bar{\phi}_1, \bar{\phi}_2$ be two extensions of $\phi$. Then $\bar{\phi}_1 - \bar{\phi}_2$ is an extension of 0 and if $S_1\phi, S_2\phi$ are the maps induced by $\bar{\phi}_1, \bar{\phi}_2$, we have $S_1\phi - S_2\phi = S(0)$, so that $S_1\phi \simeq_i S_2\phi$. Thus the homotopy class of $S\phi$ depends only on $\phi$ (and not on the choice of $\bar{\phi}$). Now suppose that $\phi \simeq \psi : A \to B$ with homotopy $\chi : \bar{A} \to B$, i.e., $\chi \mid A = \phi - \psi$. Choose $\bar{\phi} - \chi$ as an extension of $\psi$; with this choice $S\psi = S\phi$. Thus we have proved that the homotopy class of $S\phi$ depends only on that of $\phi$. We write $S\{\phi\}$ for the homotopy class containing $S\phi$, $\phi \in \{\phi\} \in \bar{\pi}(A; B)$. Let $\{\psi\} \in \bar{\pi}(B; C)$. Then it is clear that $\{\psi\}\{\phi\} = \{\psi\phi\} \in \bar{\pi}(A; C)$. The assertion of the theorem now follows by classical arguments. Indeed if $\{\phi\}$ is a class of homotopy equivalences $A \simeq_i B$, then $S\{\phi\}$ is a class of homotopy equivalences $S(A) \simeq_i S(B)$.

It should be noticed that the relation between $S(A)$ and $S(B)$ is in fact closer than homotopy equivalence. We remarked above that if $\phi = 0$ then $S\phi$ may be factored though $\bar{B}$. For arbitrary $A,B$ and $\phi : A \to B$ choose fixed modules $S(A), S(B)$ and induced map $S\phi : S(A) \to S(B)$. We prove

THEOREM 2.4. *$\phi \simeq_i 0 : A \to B$ if and only if $S\phi$ may be factored through $\bar{B}$.*

Let us write $\kappa_1$, $\kappa_2$ for the maps $\bar{A} \to S(A)$, $\bar{B} \to S(B)$ and let $\bar{\phi} : \bar{A} \to \bar{B}$ be the chosen extension of $\phi$ so that $S(\phi)\, \kappa_1 = \kappa_2 \bar{\phi}$.

Now let $\phi \simeq_i 0$ and let $\chi : \bar{A} \to B$ be an extension of $\phi$. Then $\bar{\phi} - \chi$ is zero on $A$ so that a map $\theta : S(A) \to \bar{B}$ is defined by $\theta\kappa_1 = \bar{\phi} - \chi$. Then $\kappa_2\theta\kappa_1 = \kappa_2\bar{\phi} = S(\phi)\kappa_1$, so that $\kappa_2\theta = S(\phi)$.

Conversely, suppose that $\theta : S(A) \to \bar{B}$ exists such that $\kappa_2\theta = S(\phi)$. Then $\bar{\phi} - \theta\kappa_1$ actually maps $\bar{A}$ into $B$ since $\kappa_2(\bar{\phi} - \theta\kappa_1) = 0$ and $(\bar{\phi} - \theta\kappa_1) \mid A = \phi$. Thus $\phi \simeq_i 0$.

We remark that a map $S(A) \to S(B)$ may well be $i$-nullhomotopic and yet not admit a factorization through $\bar{B}$.

Let $S^n(A)$ represent the iterated suspension of $A$. We may then define $\bar{\pi}_n(A; B)$ unambiguously as $\bar{\pi}(S^n(A); B)$. The groups $\bar{\pi}_n(A; B)$ may actually be regarded as homology groups of a suitable chain-group. For let

$$0 \to A \to C_0 \xrightarrow{\delta} \cdots \xrightarrow{\delta} C_n \xrightarrow{\delta} \cdots$$

be an injective resolution of $A$. Then $C_0/A$ is a suitable $S(A)$ and, generally, $C_n/\delta C_{n-1}$ has the $i$-homotopy type of $S^{n+1}(A)$. Let $D_n = \mathrm{Hom}\,(C_n, B)$ and let $\delta : D_n \to D_{n-1}$ be the boundary operator induced by $\delta$. Then

$$H_n(D) = \bar{\pi}_{n+1}(A; B), \; n > 0. \tag{2.1}$$

We may relativize the concept of homotopy group, but instead of defining relative groups of a pair $(Y, B)$ we prefer to consider the more general concept of the homotopy groups of a map $\phi : B \to Y$. Precisely, we will first define a group $\bar{\pi}(\theta; \phi)$ where $\theta : A \to X$, $\phi : B \to Y$. We understand by Hom $(\theta; \phi)$ the group of pairs of maps $\alpha$, $\beta$, where $\alpha : A \to B$, $\beta : X \to Y$, such that $\phi\alpha = \beta\theta$. We must now distinguish the nullhomotopic elements of Hom $(\theta, \phi)$. First we will say that $\theta' : A' \to X'$ extends (or contains) $\theta : A \to X$ if (i) $A \subseteq A'$, $X \subseteq X'$, (ii) $\theta'A' \cap X = \theta A$, (iii) $\theta'^{-1}(0) \subseteq A$. Restriction (iii) asserts that the kernel of $\theta$ is not enlarged in passing to $\theta'$, so that, in particular, $\theta'$ is an embedding if $\theta$ is. In the latter case, restriction (ii) asserts that the overlap between $A'$ and $X$ is minimal. The effect of (ii) and (iii) is to justify a generalization of Theorem 2.1. We say that a map $\bar{\theta} : \bar{A} \to \bar{X}$ is an injective extension of $\theta$ if it is an extension[1]; certainly injective extensions exist and we may prove

THEOREM 2.5. *A map $(\alpha, \beta)$ from $\theta$ to $\phi$ may be factored through any extension of $\theta$ if it may be factored through some injective extension.*

It is important to observe that, if we call $\bar{\theta} : \bar{A} \to \bar{X}$ an injective semi-extension when we drop condition (ii), then we may replace "injective extension" by "injective semi-extension" in Theorem 2.5.

It is now clear how we should define the subgroup of Hom $(\theta; \phi)$ consisting of $i$-nullhomotopic elements. The factor group is $\bar{\pi}(\theta; \phi)$. We may also define $\bar{\pi}_n(\theta; \phi)$, but we will be more concerned with a group $\bar{\pi}_n(A; \phi)$ which we now define.

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[1] A map from one injective module to another is an injective object in the category of maps of modules with our definition of inclusion of maps.

Let $\iota_n : S^{n-1}(A) \to \overline{S^{n-1}(A)}$ be the embedding map. Then it may be shown that $\bar{\pi}(\iota_n; \phi)$ depends only on the homotopy type of $A$ (indeed, of $S^{n-1}(A)$) and we write

$$\bar{\pi}_n(A; \phi) = \bar{\pi}(\iota_n; \phi)$$

Any map $S^n(A) \to Y$ may be regarded as a map $\overline{S^{n-1}(A)}$, $S^{n-1}(A) \to Y$, $0$. By this association a natural isomorphism

$$\bar{\pi}_n(A; Y) \cong \bar{\pi}_n(A; \omega) \tag{2.2}$$

is established where $\omega : 0 \to Y$ is the embedding. Using (2.2) we obtain an exact sequence

$$\cdots \to \bar{\pi}_n(A; B) \to \bar{\pi}_n(A; Y) \to \bar{\pi}_n(A; \phi) \to \bar{\pi}_{n-1}(A; B) \to \cdots, \tag{2.3}$$

ending with $\cdots \to \bar{\pi}(A; B) \to \bar{\pi}(A; Y)$. The first homomorphism is induced by $\phi$, the second by the obvious map $\omega \to \phi$, and the third by "restriction" to $S^{n-1}(A)$. The proof of exactness may be carried out directly or by appeal to (2.1) and the theory of Cartan-Eilenberg. The latter technique involves a lemma identifying the sequence with that obtained from a certain three-term sequence; there is also the question of ensuring that the last few places of the sequence preserve exactness.

We may pursue the analogy with ordinary homotopy theory further by defining a cohomotopy sequence; namely, if $(X, A)$ is a pair, there is a sequence

$$\bar{\pi}(X/A; Y) \to \bar{\pi}(X; Y) \to \bar{\pi}(A; Y) \to \bar{\pi}(X/A; SY) \to \bar{\pi}(X; SY) \to \cdots \tag{2.4}$$

The homomorphism $\bar{\pi}(A; Y) \to \bar{\pi}(X/A; SY)$ is defined by extending a map $A \to Y$ to a map $X \to \bar{Y}$ and then taking the induced map. The sequence is always exact at $\bar{\pi}(X; Y)$; it is exact at $\bar{\pi}(X/A; SY)$ if $S : \bar{\pi}(A; Y) \to \bar{\pi}(SA; SY)$ is onto; and it is exact at $\bar{\pi}(A; Y)$ if $S : \bar{\pi}(A; Y) \to \bar{\pi}(SA; SY)$ is $(1-1)$ and $S : \bar{\pi}(X; Y) \to \bar{\pi}(SX; SY)$ is onto. Thus we may apply $S$-theory, in the sense of Spanier-Whitehead, and obtain an exact sequence.

We may define a map $\phi : B \to Y$ to be a fibre-map if any map of any injective module into $Y$ may be lifted into $B$. Let $K = \phi^{-1}(0)$; if $\phi$ is a fibre-map it may be shown that there is an isomorphism

$$\lambda : \bar{\pi}_{n-1}(A; K) \cong \bar{\pi}_n(A; \phi); \tag{2.5}$$

the homomorphism $\lambda$ is defined by regarding a map $\theta : S^{n-1}(A) \to K$ as a map

$$\begin{array}{ccc} S^{n-1}(A) & \xrightarrow{\theta} & B \\ \downarrow & & \downarrow \\ \overline{S^{n-1}(A)} & \xrightarrow{0} & Y \end{array}$$

This isomorphism gives rise to the exact sequence

$$\cdots \to \bar{\pi}_n(A; B) \to \bar{\pi}_n(A; Y) \to \bar{\pi}_{n-1}(A; K) \to \bar{\pi}_{n-1}(A; B) \to \cdots \tag{2.6}$$

Notice that the above differs formally from the usual procedure in which $\phi$ is taken to be an embedding and $Y \to Y/B$ is a fibre-map.

If we consider the purely algebraic aspects of our definition we may state the following results.

THEOREM 2.6. *Let $A, B$ be modules. Then the cohomology groups* $\text{Ext}^n(A; B)$ *depend only on the i-type of* $S^{n-1}(B)$. *A map* $\phi : B_1 \to B_2$ *induces the zero homomorphism on* $\text{Ext}^n(A; B_1)$ *for all A if and only if* $S^{n-1}(\phi) \simeq_i 0$.

THEOREM 2.7. *The following statements are equivalent*: (i) $\bar{\pi}_n(A; B) = 0$, *all B*; (ii) $\text{Ext}^{n+1}(B; A) = 0$, *all B*; (iii) $S^n(A) \simeq_i 0$.

The latter theorem establishes, in a sense, the nontriviality of the homotopy groups.

We now refer to the duality which exists in the exact category of $\Lambda$-modules and maps. This enables us to give a dual definition of homotopy, namely

DEFINITION 2.1*. *A map* $\phi : B \to A$ *is p-nullhomotopic if it may be factored through every module A′ of which A is a quotient. Two maps* $\phi_0, \phi_1 : B \to A$ *are p-homotopic if* $\phi_0 - \phi_1$ *is p-nullhomotopic. We write* $\phi \simeq_p 0$, $\phi_0 \simeq_p \phi_1$.

THEOREM 2.1*. *Let* $\underline{A}$ *be a projective module of which A is a quotient. Then* $\phi : B \to A$ *is p-nullhomotopic if and only if it may be factored through* $\underline{A}$.

We now define the group $\underline{\pi}(B; A)$ in the obvious way. We call this the $p$-homotopy group of maps from $B$ to $A$. We may then continue to dualize the concepts and statements introduced for $i$-homotopy. In particular, we remark

THEOREM 2.2*. (Homotopy lifting theorem). *Let* $\phi : B \to A$ *and let* $\phi' : B \to A'$ *where* $\mu : A \to A'$ *is an epimorphism with* $\phi' = \mu\phi$. *Then if* $\phi' \simeq_p \psi' : B \to A'$, *there exists* $\psi : B \to A$ *with* $\phi \simeq_p \psi$ *and* $\psi' = \mu\psi$.

Given a module $A$, let $\mu : \underline{A} \to A$ be a map of a projective module $\underline{A}$ onto $A$ and let $\Omega(A)$ be the kernel of $\mu$.

THEOREM 2.3*. *The* p-*type of* $\Omega(A)$ *depends only on that of A*.

We define $\underline{\pi}_n(B; A)$ to be $\underline{\pi}(B; \Omega^n(A))$. Let

$$\cdots \to C_n \xrightarrow{\partial} C_{n-1} \to \cdots \to C_0 \to A \to 0$$

be a projective resolution of $A$. Then the kernel of $\partial : C_n \to C_{n-1}$ has the $p$-type of $\Omega^{n+1}(A)$. Let $E_n = \text{Hom}(B, C_n)$ and let $\partial' : E_n \to E_{n-1}$ be the boundary induced by that in $C_n$. Then

$$H_n(E) = \underline{\pi}_{n+1}(B; A), n > 0. \tag{2.1*}$$

We relativize as before by passing from the category of modules to the category of maps. Omitting the details, we may prescribe conditions under which a map from the map $\theta' : X' \to A$ to the map $\theta : X \to A$ is to be called a projection and hence define the group $\underline{\pi}(\phi; \theta)$, where $\phi : Y \to B$. In particular if $\kappa_n : \underline{\Omega^{n-1}(A)} \to \Omega^{n-1}(A)$ is the projection from the projective module $\underline{\Omega^{n-1}(A)}$ onto $\Omega^{n-1}(A)$, we define

$$\underline{\pi}_n(\phi; A) = \underline{\pi}(\phi; \kappa_n).$$

Any map $Y \to \Omega^n(A)$ may be regarded as a map from $\zeta$ to $\kappa_n$; where $\zeta$ is the map[2] $Y \to 0$. This leads to an identification

$$\underline{\pi}_n(Y; A) \cong \underline{\pi}_n(\zeta; A) \tag{2.2*}$$

and hence an exact sequence

$$\cdots \to \underline{\pi}_n(B; A) \to \underline{\pi}_n(Y; A) \to \underline{\pi}_n(\phi; A) \to \underline{\pi}_{n-1}(B; A) \to \cdots, \tag{2.3*}$$

ending with $\cdots \to \underline{\pi}(B; A) \to \underline{\pi}(Y; A)$.

There is also a $p$-cohomotopy sequence. Let $\mu$ be an epimorphism of $X$ onto $A$ with kernel $K$. There is then a sequence

$$\underline{\pi}(Y; K) \to \underline{\pi}(Y; X) \to \underline{\pi}(Y; A) \to \underline{\pi}(\Omega Y; K) \to \cdots \tag{2.4*}$$

The homomorphism $\underline{\pi}(Y; A) \to \underline{\pi}(\Omega Y; K)$ is defined by lifting a map $Y \to A$ to a map $Y \to X$ and taking the induced map $\Omega Y \to K$. The sequence is always exact at $\underline{\pi}(Y; X)$; it is exact at $\underline{\pi}(\Omega Y; K)$ if $\Omega : \underline{\pi}(Y; A) \to \underline{\pi}(\Omega Y; \Omega A)$ is $(1 - 1)$; and it is exact at $\underline{\pi}(Y; A)$ if $\Omega : \underline{\pi}(Y; A) \to \underline{\pi}(\Omega Y; \Omega A)$ is onto and $\Omega : \underline{\pi}(Y; X) \to \underline{\pi}(\Omega Y; \Omega X)$ is $(1 - 1)$. There is thus an $\Omega$-theory dual to the $S$-theory for $i$-homotopy.

We have seen that "homotopy lifting" is dual to "homotopy extension." To obtain a sequence dual to (2.6) we must impose a homotopy-extension property on the map $\phi : Y \to B$ of (2.3*). We say it is a $p$-fibre-map if any map of $Y$ into a projective module may be factored through $\phi$. The $p$-fibre is then $L = B/\phi Y$ and we have an isomorphism

$$\underline{\pi}_{n-1}(L; A) \cong \underline{\pi}_n(\phi; A), \tag{2.5*}$$

giving rise to the exact sequence

$$\cdots \to \underline{\pi}_n(B; A) \to \underline{\pi}_n(Y; A) \to \underline{\pi}_{n-1}(L; A) \to \underline{\pi}_{n-1}(B; A) \to \cdots \tag{2.6*}$$

We also quote

THEOREM 2.6*. *The cohomology groups* $\mathrm{Ext}_n\ (A; B)$ *depend only on the p-type of* $\Omega^{n-1}(A)$. *A map* $\phi : A_1 \to A_2$ *induces the zero homomorphism on* $\mathrm{Ext}^n\ (A_2; B)$ *for all B if and only if* $\Omega^{n-1}(\phi) \simeq_p 0$. *Moreover, the homology groups*[3] $\mathrm{Tor}_n\ (A; B)$ *depend only on the p-types of* $\Omega^{n-1}(A)$, $\Omega^{n-1}(B)$.

THEOREM 2.7*. *The following statements are equivalent*: (i) $\underline{\pi}_n(A; B) = 0$, *all* $A$; (ii) $\mathrm{Ext}^{n+1}\ (B; A) = 0$, *all* $A$; (iii) $\Omega^n(B) \simeq_p 0$.

There is a character theory relating $i$-homotopy and $p$-homotopy. Precisely, let $R$ be a divisible abelian group, possessing elements of order $n$ for every[4] $n \geq 1$, and for any abelian group $A$, let $A^* = \mathrm{Hom}\ (A, R)$. Then if $A$ is a left $\Lambda$-module, $A^*$ may, in an obvious way, be given the structure of a right $\Lambda$-module. We may then prove

THEOREM 2.8. *If* $A \simeq_p B$, *then* $A^* \simeq_i B^*$.

[2] For the purpose of the exact sequences we may regard $\zeta$ as the null map $Y \to B$, but in principle the image of $\zeta$ is a fixed trivial group.

[3] Here we take $A$ to be a right $\Lambda$-module, $B$ a left $\Lambda$-module.

[4] This condition is only necessary if we require $A \subseteq A^{**}$.

More precisely, if $\phi : A \to B$ is a map, then a map $\phi^* : B^* \to A^*$, dual to $\phi$, is defined and $\phi^*$ is an $i$-homotopy equivalence if $\phi$ is a $p$-homotopy equivalence.

THEOREM 2.9. $(\Omega A)^* = S(A^*)$.

It follows that $\phi \to \phi^*$ induces a homomorphism

$$\underline{\pi}_n(A; B) \to \bar{\pi}_n(B^*; A^*). \tag{2.7}$$

((*Added in Proof.*) Eckmann and the author have recently proved

THEOREM 2.8. *The following* 5 *statements are equivalent*: (1) $\phi : A \simeq_i A'$; (2) $\phi^* : \bar{\pi}(A'; B) \cong \bar{\pi}(A; B)$, *all* $B$; (3) $\phi_* : \bar{\pi}(B; A) \cong \bar{\pi}(B; A')$, *all* $B$; (4) $\phi_*$: $\text{Ext}\,(B, A) \cong \text{Ext}\,(B, A')$, *all* $B$; (5) $\phi$ *factorizes as* $A \xrightarrow{\iota} A + I \xrightarrow{\bar{\phi}} A' + J \xrightarrow{\kappa} A'$, *where* $I, J$ *are injective*, $\iota$ *embeds*, $\kappa$ *projects, and* $\bar{\phi}$ *is isomorphic. The equivalence* (1) $\leftrightarrow$ (5) *has been pointed out by A. Heller.* (There is also a dual Theorem 2.8*).)

## 3. Duality in topology

We saw at the start of the previous section that $i$-homotopy for modules is a natural analogue for homotopy in topology; equally well, $p$-homotopy for modules is analogous to homotopy in topology. For we saw that every module-map is a "fibre-map" with respect to $p$-homotopy (Theorem 2.2*) and, for topological spaces, a map $f : A \to B$ may be lifted into every fibre-space over $B$ if and only if it is nullhomotopic. Thus the dual theories of $i$-homotopy and $p$-homotopy may be said to coincide (with homotopy in the usual sense) for topological spaces, and the relation between statements in the $i$-theory and corresponding statements in the $p$-theory may be expressed in terms of Kan's notion of "adjoint functors". Precisely, (1.1) derives from the fact that $S$ and $\Omega$ are adjoint ($S$ is a left adjoint of $\Omega$), and so indeed are the functors $X \to X \times I$, $X \to X^I$.

Let us briefly examine the exact sequences of the previous section from this standpoint. We recall that we are always concerned with maps and homotopies with basepoints. Certainly we have an exact sequence

$$\cdots \to \pi_n(A; B) \to \pi_n(A; Y) \to \pi_n(A; f) \to \pi_{n-1}(A; B) \to \cdots, \tag{3.1}$$

where $f : B \to Y$ is a map and $\pi_n(A; B)$, for example, is the group (or set if $n = 0$) of homotopy classes of maps $S^n(A) \to B$. Indeed if $A$ is locally compact this is just the homotopy sequence in the usual sense of the map $f^A : B^A \to Y^A$ induced by $f$. If $f$ is a fibre-map (with respect to maps of suspensions of $A$), then $f^A$ is a fibre-map for maps of polyhedra and (3.1) becomes

$$\cdots \to \pi_n(A; B) \to \pi_n(A; Y) \to \pi_{n-1}(A; F) \to \pi_{n-1}(A; B) \to \cdots, \tag{3.2}$$

where $F$ is the fibre.

Dually, if $f : Y \to B$ is a map, we have an exact sequence

$$\cdots \to \pi_n(B; A) \to \pi_n(Y; A) \to \pi_n(f; A) \to \pi_{n-1}(B; A) \to \cdots; \tag{3.3}$$

if $B$ and $Y$ are locally compact this is the homotopy sequence of the map $A^f : A^B \to A^Y$ induced by $f$. If $(B, Y)$ is a pair with the homotopy extension property with respect to maps into $\Omega^n(A)$, $n = 0, 1, \ldots$, then $A^f$ is a fibre-map and (3.3) reduces to

$$(3.4) \qquad \cdots \to \pi_n(B; A) \to \pi_n(Y; A) \to \pi_{n-1}(B/Y; A) \to \pi_{n-1}(B; A) \to \cdots,$$

which is Barratt's track group sequence. Thus homotopy extension and homotopy lifting properties are adjoint and (3.3), (3.4) may be deduced from (3.1), (3.2) by using Kan's theory of adjoint functors. It is to be noted that (3.4) contains the cohomology sequence (obtained by replacing $A$ by a suitable Eilenberg-MacLane complex) and, essentially, the cohomotopy sequence (obtained by replacins $A$ by a suitable sphere and desuspending within the suspension range). The $i$-cohomotopy sequence also yields the usual cohomotopy sequence, while the $p$-cohomotopy sequence yields a sequence which is rarely exact owing to the fact that $\Omega: \pi(X; Y) \to \pi(\Omega X; \Omega Y)$ is rarely $(1-1)$ onto.

Let us reexamine the derivation of the cohomology sequence from (3.4). Confining attention to pairs $(B, Y)$ which are CW-complexes or C.S.S. complexes ($B$ being a subcomplex of $Y$) we have the sequence (3.4) in particular when $A$ is the Eilenberg-MacLane complex $K(G; m+n)$. Now $\pi_n(B; K(G; m+n)) = \pi(B; \Omega^n K(G; m+n)) = \pi(B; K(G; m))$. The group $\pi(B; K(G; m))$ is naturally isomorphic with $(H^m B; G)$ and the boundary homomorphism $\pi_n(Y; A) \to \pi_{n-1}(B/Y; A)$ is thereby converted into the homomorphism $\delta: H^m(Y; G) \to H^{m+1}(B/Y; G)$ of the cohomology sequence. In this way we obtain the cohomology sequence

$$(3.5) \qquad \cdots \to H^m(B; G) \to H^m(Y; G) \to H^{m+1}(B/Y; G) \to H^{m+1}(B; G) \to \cdots,$$

we remark also that $H^r(B/Y; G) = H^r(B, Y; G)$, the cohomology group on the left being the reduced group. Thus we might take the point of view that the cohomology group $H^r(B, Y; G)$ may be defined as $\pi(B/Y; K(G; r))$; that is, as the group of homotopy classes of maps $B/Y \to K(G; r)$, the group structure being induced by that of $K(G; r)$ (i.e., by the presentation of $K(G; r)$ as $\Omega K(G; r+1)$). Thus cohomology groups have been defined in terms of homotopy groups, using the $H$-structure of Eilenberg-MacLane complexes. Dually, homotopy groups may be defined in terms of cohomology groups.[5] Given an abelian group $G$, let $Y(G; r)$ be a space with a single non-vanishing cohomology group $G$ in dimension $r$. If $r > 2$, $Y(G; r)$ is a suspension of $Y(G; r-1)$ and thus $\pi(Y(G; r); M)$ has a group structure; the resulting group is precisely the homotopy group $\pi_r(M)$ if $G$ is the group of integers. We may relativize by considering, not a pair, but a fibre-map $f: Y \to B$ with fibre $F$. Then we define the $r^{\text{th}}$ homotopy group of $f: Y \to B$ with coefficients in $G$ as $\pi(Y(G; r); F)$. The exact sequence (3.2) with $A = Y(G; r)$ is thus dual to (3.4) with $A = K(G; r)$. It would be of some importance to place this duality on a rigorous basis, in which its relation to the duality between

[5] ((*Added in Proof.*) Eckmann has remarked that there are advantages, in certain contexts, in defining homotopy groups with coefficients not by means of $Y(G; r)$, as above, but by means of the space $K'(G; r)$, first considered by J. C. Moore, with a single non-vanishing *homology* group, namely $G$ in dimension $r$.)

$H$-spaces and $S$-spaces (spaces of the second category) would become apparent. We may demonstrate this heuristic duality with one further example.

Let $f: X \to Y$ be a map from the CW-complex $X$ to the CW-complex $Y$. If $\pi_r(Y) = 0$, $0 \leqq r < n$, then $f$ is homotopic to a map $f'$ sending $X^{n-1}$ to $y$, the base-point. Write $X_n = X/X^{n-1}$ and let $f_0: X_n \to Y$ be the map induced by $f'$. If $f_0, f_1: X^n \to Y$ are both approximations in the above sense to $f$ and if $p: X^{n-1} \to X^n$ is the projection, then $f_0 p \simeq f_1 p$.

The dual statement is as follows. Let $f: Y \to X$ be a map from the CW-complex $Y$ to the CW-complex $X$. If $H^r(Y) = 0$, $r > n$, then $f$ is homotopic to a map $f'$ sending $Y$ into $X^n$. Let $f^0: Y \to X^n$ be the map $f'$ regarded as a map into $X^n$. If $f^0, f^1: Y \to X^n$ are both approximations in the above sense to $f$ and if $i: X^n \to X^{n+1}$ is the injection, then $if^0 \simeq if^1$. The dual statement requires that $X$ be simply-connected (if $n = 2$, we must take $X^1$ simply-connected).

Finally we consider the two propositions

(3.6) $$S(X \times Y) \simeq S(X) \vee S(Y) \vee S(X \wedge Y),$$

(3.7) $$\Omega(X \vee Y) \simeq \Omega(X) \times \Omega(Y) \times \Omega E(X \times Y, X \vee Y).$$

Here $X \wedge Y$ is the factor space $X \times Y/X \vee Y$ and $E(X \times Y, X \vee Y)$ is the space of paths on $X \times Y$ starting in $X \vee Y$. These decompositions are in some sense dual to each other. Recently John Milnor has shown that, if $X$, $Y$ are suspensions, say $X = S(A)$, $Y = S(B)$, then $\Omega(X \vee Y)$ admits an infinite decomposition as a product of loop spaces constructed from $A$ and $B$ using the $\wedge$-product[6]. A similar expression for $S(X \times Y)$ should be available.

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[6] Milnor argues from the free group complex construction; it is possible to obtain a geometrical proof by setting up a homology isomorphism.

# APPLICATIONS OF MORSE THEORY TO SYMMETRIC SPACES

BY R. BOTT AND H. SAMELSON

1. The paper described here (a complete account will appear elsewhere) forms a a sequel to [3]. Let $K$ be a compact, connected Lie group, and let it act, by isometries, on the (complete) Riemannian manifold $M$. Let $\Omega$ be the space of paths ($D'$-curves, parametrized proportional to arclength) from some point $p$ of $M$ to some $K$-orbit $N$ in $M$. Our purpose is to study the Morse theory of $\Omega$, using the information contained in the action of $K$ on $M$. We consider transversal geodesics in $M$, i.e. geodesics which are orthogonal to one (and then automatically to every) orbit they meet. A variational vector field or Jacobi field along a geodesic will be called transversal if it is derived from a one parameter family of transversal geodesics. Let $S$ be the set of all transversal geodesic segments from $p$ to $N$. For any element $s$ of $S$ we let $C_s$ be the stabilizer of $s$, i.e. the group of those elements of $K$ that leave $s$ pointwise fixed; the stabilizer $C_x$ of any point $x$ of $M$ is defined analogously. A parameter value $t \in [0, 1)$ will be called exceptional for $s$, if, putting $s(t) = q$, the dimension of $C_q$ is greater than that of $C_s$. Suppose there is only a finite number of exceptional values for $s$, and that 0 is not exceptional; let values be $0 < t_1 < \cdots < t_k < 1$. Write $C_i$ for $C_{s(t_i)}$, put $W(s) = C_1 \times \cdots \times C_k$ and $U(s) = C_s \times \cdots \times C_s$ ($k$ factors). We let $U(s)$ operate on $W(s)$ by $w \cdot u = (c_1, \cdots, c_k)\cdot(u_1, \cdots, u_k) = (c_1u_1, u_1^{-1}c_2u_2, \cdots, u_{k-1}^{-1}c_ku_k)$; this defines $W(s)$ as $U(s)$-principal fibre bundle, whose base space (a manifold) we call $K(s)$. With $w = (c_1, \cdots, c_k)$ we associate the geodesic polygon $w \cdot s = s_0 \cup c_1 \cdot s_1 \cup c_1 \cdot c_2 \cdot s_2 \cup \cdots \cup c_1 \cdots c_k \cdot s_k$; here $s_i$ is the part of $s$ from $t_i$ to $t_{i+1}$. This association constitutes a map of $W(s)$ into $\Omega$, which is invariant under the action of $U(s)$ and so induces a map $f_s$ of $K(s)$ into $\Omega$. Let $\bar{k}(s)$ denote the $f_s$-image of the fundamental cycle mod 2 of $K(s)$: if $K(s)$ is orientable, let $k(s)$ denote the $f_s$-image of a fundamental integral cycle of $K(s)$.

We say that $K$ is variationally complete on $M$ if every transversal Jacobi field (along a tranversal geodesic), which is tangent to the $K$-orbits at two points, is derived from a one-parameter group of $K$. (If $K$ is the identity this means absence of conjugate points.)

THEOREM I. *With the notation as above, let $K$ be variationally complete on $M$; let $p$ lie on an orbit of highest dimension. Then the $\bar{k}(s)$, $s \in S$, form a basis for $H_*(\Omega; Z_2)$; if all the $K(s)$ are orientable, then the $k(s)$ form a basis for $H_*(\Omega; Z)$ [in particular there is then no torsion in $\Omega$].*

The proof is an application of Morse theory. Briefly speaking, the $\bar{k}(s)$ are absolute cycles completing the relative cycles which in the Morse theory are attached to the critical points of the length function on $\Omega$.

2. The foregoing will now be applied to symmetric spaces. Let $G$ be a compact connected Lie group, let $\sigma$ be an involution (automorphism of order two) of $G$, and let $K$ be the component of the identity of the fixed point set of $\sigma$. The quotient

space $M = G/K$ is then a symmetric space, and we shall call $(G, K)$ a symmetric pair. $G$, and therefore also $K$, acts in a well-known way on $M$. In addition $K$ acts, by the "adjoint action", on the tangent space $M_0$ to $M$ at the point $x_0$, represented by $K$.

THEOREM II. *If $(G, K)$ is a symmetric pair, the action of $K$ on $M = G/K$ and on $M_0$ is variationally complete (in the metric derived from a biinvariant metric on $G$).*

For the proof one has to analyze the action of $G$ in some detail. For this and for the subsequent discussion an important role is played by the maximal flat toruses in $G/K$ [or equivalently by the maximal toruses of $G$, which intersect $K$ as little as possible]; it is known that any two such toruses through $x_0$ are conjugate under $K$. We apply now Theorem I to $M$ under the action of $K$, with $N$ consisting of the point $x_0$. If $T$ is a maximal flat torus containing $p$ and $x_0$, then all the geodesics from $p$ to $x_0$ lie in $T$; the intersection of the $K$-orbits of non-maximal dimension with $T$ (the singular set) consists of a finite number of toruses whose dimension is one less than that of $T$. As is well-known, the situation is best described in the universal covering space $E^l$ of $T$, in which $p$ appears as a lattice of points, and the singular set as union of a finite number of families of parallel equidistant hyperplanes [the diagram of $(G, K)$]. If $\bar{p}$ is one of the points representing $p$, if $s$ is the straight segment from $\bar{p}$ to the origin $\bar{x}_0$ of $E^l$, then the exceptional points on $s$, in the sense of Theorem I, are the points (except $\bar{x}_0$) where $s$ meets the diagram of $(G, K)$ and one can easily determine the dimensions of the various stabilizers, and therefore the dimensions of the manifolds $K(s)$. In slightly different terms this can be expressed as follows: Let $H$ be a Cartan algebra of $G$, containing the Cartan algebra $H^-$ corresponding to $T$ ($H^-$ can be identified with the $E^l$ above; it is also characterized as the eigenspace to the eigenvalue $-1$ for the involution, induced by $\sigma$ on $H$). Let $\theta_i$ be a set of positive roots of $G$ (linear forms on $H$). Define a function $\lambda$ on $H$ by $\lambda(x) = \sum_i [|\theta_i;\ (x)|]$; here [ ] stands for the integral part.

THEOREM III. *The Poincaré series of the loopspace $\Omega$ of $M$,* mod 2, *is given by*

$$P(\Omega, t, Z_2) = \int_{H^-} t^{\lambda(x)} dv_x,$$

*the integral taken with a suitably normalized Euclidean measure in $H^-$.*

In the case where $G = K \times K$, and $\sigma$ is the interchange of the two factors, one knows that $M$ can be identified with $K$. It turns out that in this case all $K(s)$ are orientable and even dimensional; in particular $\Omega(K)$ has no torsion (cf. [1], [2]); and it is easy to read off the Betti numbers of $\Omega(K)$ from the diagram of $K$ [each cell, in the positive chamber, corresponds to a cycle, whose dimension is twice the number of singular planes separating it from the origin]. One derives easily the fact that the 3rd homotopy group of a simple Lie group is cyclic-infinite. Furthermore, the cohomology ring of $K(s)$ can be computed from the fact that $K(s)$ is an iterated 2-sphere bundle; the description is as follows: $K(s)$ corresponds, as noted, to a cell in a positive chamber; let $\omega_1, \cdots, \omega_k$, in this order and with possible repetition, be the (positive) roots belonging to the singular planes met by the segments $s$ from a point $\bar{p}$ in the cell to the origin; let $\alpha_{ij}$ be the Weyl integer belonging to the pair $(\omega_i, \omega_j)$.

THEOREM IV. *$H^*(K(s))$ is the polynomial ring in two-dimensional variables $x_1, \cdots, x_k$ modulo the ideal generated by the elements* $\rho_i = x_i^2 + \sum_1^{i-1} \alpha_{ij}x_ix_j$, $i = 1, \cdots, k$ (cf. [4]).

It is possible to use this information to compute the low-dimensional part of $H^*(\Omega(K))$, in particular to find out whether the powers of the generator of $H^2(\Omega(K))$ (assuming $K$ simple) are divisible by any integers (below the dimension of the next primitive exponent of $K$), and thereby to determine the next nonvanishing homotopy group of $K$ after the third. For $E_6$, $E_7$, $E_8$ these dimensions are 9, 11, 15, and the computation shows that the groups in question are cyclic-infinite. This in turn means that the homology of these groups up to the critical dimensions is that of the Eilenberg-MacLane space $K(Z, 3)$.

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# SINGULARITIES OF MAPPINGS OF EUCLIDEAN SPACES

BY HASSLER WHITNEY

## 1. Introduction

We shall describe here some results and methods pertaining to the following general problem (details will appear elsewhere). Suppose a mapping $f_0$ of an open set $R$ in $n$-space $E^n$ into $m$-space $E^m$ is given (we shall write $f$: $R^n \to E^m$). How can we alter $f_0$ slightly, obtaining a mapping $f$ with nicer and simpler properties? By the Weierstrass approximation theorem (generalized), we may require that $f$ be analytic in $R$; if $f_0$ was $r$-smooth (had continuous partial derivatives through the $r^{\text{th}}$ order), we may require the partial derivatives of $f$ through the $r^{\text{th}}$ order to approximate those of $f_0$ (we then call $f$ an *r-approximation*). Now take any *regular* point $p$ of $f$, that is, a point $p$ such that $f$ is of maximum rank $\nu = \inf(n, m)$ at $p$. (Equivalently, using coordinate systems in $E^n$ and in $E^m$, the Jacobian matrix of $f$ at $p$ is of rank $\nu$.) Then, by the implicit function theorem, we may choose coordinates so that $f$ has the form

$$(1.1) \qquad y^i = x^i \quad (i = 1, \cdots, \nu), \qquad y^i = 0 \quad (i > \nu, \text{ if } m > n).$$

Hence at any regular point $p$, $f$ has the structure of the particular mapping (1.1); we shall be satisfied with $f$ here.

Any non-regular point we call a *singular* point of $f$. In general (unless $m \geqq 2n$; see §4) we cannot avoid the presence of singular points. We would then like to reduce them as much as possible, making them lie in small and simple point sets, and requiring the structure of $f$ to be as simple as possible in the neighborhood of a singular point. We shall show that the set $S'$ of singular points may be made to form a smooth manifold plus boundary. There will be subsets of $S'$, also manifolds with boundaries, consisting of points in the neighborhood of which $f$ is more complicated. This gives a splitting of $S'$ into sets, in each of which $f$ satisfies specified conditions. Thus the singularities of $f$ are divided into various types. We give the geometric basis of this splitting in Part I. Examples of the types of singularities will be found in Part II.

The manner of defining the singularities is as follows. Let $L^r$ be the space of possible values of the differentials of a mapping $f: R^n \to E^m$ through the order $r$ at a point $p$ (see §2). Each $L^r$ contains a certain set of manifolds, say $L^r_{(1)}, L^r_{(2)}, \cdots$, of various dimensions. Given $f$, the values at $p$ of the differentials of $f$ give a mapping $f^r$: $R^n \to L^r$, for each $r$. (We may keep $r \leqq 2n$; see §11.) By a slight change in $f$, we may require $f^r$ to be "crosswise" to the $L^r_{(i)}$ (see §5); then $f^r(R)$ does not intersect any $L^r_{(j)}$ of small dimension, and it intersects other $L^r_{(i)}$ in as simple a manner as possible. Using a suitable $r$, we then say $f$ is *locally generic*. The sets in $R$ which map into the $L^r_{(i)}$ are the singular sets of $f$; at a point of one of these sets, we say $f$ has a *generic singularity*.

Suppose $p$ is a generic singularity; say $q = f^r(p) \in L^r_{(i)}$. Then $f^r$ is crosswise to $L^r_{(i)}$ at $q$. It follows that for any good $(r+1)$-approximation $f_1$ to $f$, $f_1^r$ will also be crosswise to $L^r_{(i)}$ at some point $q_1 = f_1^r(p_1)$; thus a generic singularity cannot be removed in this manner. (It may be removable under a $k$-approximation for smaller $k$; see for instance §22 of [13].)

We say the above $f_1$ has a singularity at $p_1$ of the same type that $f$ had at $p$. A basic question now is, to what extent is $f_1$ near $p_1$ like $f$ near $p$? If the division into singular sets is sufficiently complete, then we would like $f_1$ to be obtainable from $f$ by "changes of coordinates." Explicitly, there is then a mapping $F$ defined by

$$y^i = \phi^i(x^1, \cdots, x^n) \qquad (i = 1, \cdots, m),$$

such that both $f$ and $f_1$ may be put into this form by proper choices of coordinate systems. We then choose a particular mapping $F$, and call it a *normal form* for the type of singularity, and we say that $f$ is *stable* at $p$.

Our *principal conjecture* is that the division of singularities into types satisfies the above condition. The general program may be described as follows.

(a) Carry out the definition of types of singularities, as proposed below. This will be seen to run into questions about the relation of planes to certain algebraic cones. (The basic theoretical considerations are not difficult, but carrying out the description of the singularities in high dimensional cases seems very complicated.) It is then easy to show that arbitrarily near any $f_0$ there is a locally generic $f$.

(b) Show that the division into types is complete, in that any locally generic mapping is stable at each point. This is the most difficult part of the program. (The choice of a normal form for a given type of singularity is relatively easy.)

A further study should include the following:

(c) Find topological properties relating to the singularities, both locally and in the large (with $E^n$ and $E^m$ replaced by smooth manifolds). We shall not discuss this problem here. See [11] and [13]; also Thom, [4], [5] and [6].

The program has been carried through in certain cases, as follows: For $m = 1$, we have a real function $f$ in $E^n$; the singular points are the critical points of $f$. The theory of Marston Morse, in [1] and [2], covers this case (see §16). For $m \geq 2n$, there are no singularities; see §4. For $m = 2n - 1$, we can have singularities at isolated points; see [9] and §20. For $n = m = 2$, $S'$ consists of smooth curves, and there are isolated points of other type on the curves; see [13] and §17.

Suggestions have been made about the possible types of singularities in more general cases. F. Roger [3] described the types we call $S_1^{(k)}$. R. Thom found explicitly the types in low dimensional cases in [4] (the entry $S_2(S_1)$ appears first for $(n, m) = (5, 4)$ (see §25), not $(4, 3)$). But no proofs that the division into types is complete in these cases has been given.

Singularities which can be removed by small deformations (for instance branch points; see §7 of [13]) of course may nevertheless be of importance. We shall not consider these here.

## I. Geometric structure of singularities

### 2. The differentials of a mapping

Let $V(E^n) = V^n$ denote the space of vectors in $E^n$. The *differential* $df(p)$ of $f$ at $p$ is the linear transformation of $V^n$ into $V^m$ defined by

$$df(p) \cdot v = \lim_{t \to 0+} \frac{1}{t}[f(p + tv) - f(p)] \in V^m, \qquad v \in V^n. \tag{2.1}$$

The *second differential* $d^2f(p)$ is the bilinear transformation of $V^n \times V^n$ into $V^m$ defined by

$$d^2f(p) \cdot (v, w) = \lim_{s,t \to 0+} \frac{1}{st}[f(p + sv + tw) - f(p + sv) - f(p + tw) + f(p)]. \tag{2.2}$$

It is symmetric:

$$d^2f(p) \cdot (v, w) = d^2f(p) \cdot (w, v). \tag{2.3}$$

Higher differentials are defined similarly, or by induction.

If coordinate systems in the spaces are given, then naming the first $r$ differentials of $f$ is equivalent to naming the partial derivatives of orders up through $r$. Relation (2.3) corresponds to the symmetry of cross partial derivatives.

Let $L^1 = \mathfrak{L}^1$ denote the space of linear transformations of $V^n$ into $V^m$; it is a linear space, of dimension $nm$. More generally, let $\mathfrak{L}^r$ denote the space of multilinear symmetric transformations of the Cartesian product $V^n \times \cdots \times V^n$ ($r$ factors) into $V^m$. Thus an element of $\mathfrak{L}^r$ is a multilinear symmetric function $F(v_1, \cdots, v_r)$, whose values are vectors of $V^m$. Set

$$L^r = \mathfrak{L}^1 \oplus \cdots \oplus \mathfrak{L}^r. \tag{2.4}$$

Now with $f$ given, $d^kf(p)$ is an element of $\mathfrak{L}^k$, for each $k$. Let $f^r(p)$ denote the set of $d^kf(p)$, $k \leq r$. Thus

$$d^kf(p) \in \mathfrak{L}^k, \qquad f^r(p) = (df(p), d^2f(p), \cdots, d^rf(p)) \in L^r. \tag{2.5}$$

### 3. The structure of $L^1$

Each point $T \in L^1$ is a transformation of $V^n$ into $V^m$. The *rank* of $T$ is the dimension of the image space $T(V^n)$. Let $L^1_\rho$ denote the set of points of $L^1$ of rank $\rho$. Now

$$L^1 = L^1_0 \cup L^1_1 \cup \cdots \cup L^1_\nu, \quad \nu = \inf(n, m). \tag{3.1}$$

It is easy to see that each $L^1_\rho$ is a manifold, of dimension

$$\dim(L^1_\rho) = (n + m)\rho - \rho^2 = nm - (n - \rho)(m - \rho). \tag{3.2}$$

The *codimension* of a manifold in a fixed containing space is the difference of the two dimensions. Hence

$$\operatorname{codim}(L^1_\rho) = (n - \rho)(m - \rho). \tag{3.3}$$

Clearly any limit points of $L^1_\rho$ not in $L^1_\rho$ lie in the sets $L^1_{\rho'}$, $\rho' > \rho$. We express this by saying that the $L^1_\rho$ form a *manifold collection*.

## 4. The rank of $f$

Given $f_0 : R^n \to E^m$, there corresponds a mapping $f_0^1 = df_0 : R^n \to L^1$. Set

$$\rho(n, m) = \sup \{r + 1 : (n - r)(m - r) > n\}; \tag{4.1}$$

this is the largest number $r + 1$ such that the condition shown holds.

Set $\rho' = \rho(n, m) - 1$; then

$$\text{codim}\,(L^1_{\rho'}) = (n - \rho')(m - \rho') > n.$$

It follows [13, Theorem 11A] that a small deformation of $f_0$ will give a mapping $f$ such that no point $df(p)$ lies in $L^1_{\rho'}$.

We say *$f$ is of rank $\rho$ at $p$* if $df(p)$ is of rank $\rho$. The *rank* of $f$ is the smallest rank at any point. The result above is that *by a small deformation, we may make $f$ of rank $\geqq \rho(n, m)$.*

The *deficiency* of $f$ is (with $\nu = \inf (n, m)$)

$$\text{dfc}\,(f, p) = \nu - \text{rank}\,(f, p), \qquad \text{dfc}\,(f) = \nu - \text{rank}\,(f). \tag{4.2}$$

Set

$$\delta(n, m) = \nu - \rho(n, m). \tag{4.3}$$

Then *we may make $f$ of deficiency $\leqq \delta(n, m)$.*

Some special cases of this are the following:

(a) If $m \geqq 2n$, then $\rho(n, m) = n$, $\delta(n, m) = 0$; hence we can remove all singularities in this case. Take any smooth manifold $M$, and set $f_0(p) = q_0 \in E^{2n}$ $(p \in M)$. Applying the result above gives an "immersion" $f$ of $M$ in $E^{2n}$. (There may be self-intersections of $f(M)$.)

(b) Suppose $m \geqq \frac{3}{2}(n - 1)$. Using $r = n - 2$ in (4.1) shows that $\rho(n, m) \geqq n - 1$. Thus all singular points may be made of deficiency $\leqq 1$.

(c) For $n \geqq 2m - 3$, we find $\rho(n, m) \geqq m - 1$, and $\delta(n, m) \leqq 1$ again.

(d) If $n = m < (k + 1)^2$, we find $\delta(n, m) \leqq k$.

The results given are the best possible, in that there exist mappings such that any sufficiently nearby mapping has the rank $\rho(n, m)$ and deficiency $\delta(n, m)$.

Some values of $\delta(n, m)$ are given in the table. (This was found in large part by Wolfsohn, [14].)

| $n \backslash m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 5 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 6 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 0 |

Table of $\delta(n, m)$.

## 5. Crosswise mappings

Consider $F : R^n \to E^N$, and let $M$ be a smooth manifold in $E^N$. Suppose $F(p) = q \in M$. At $q$ we have the tangent vector space $P(M, q)$ of $M$, and the image $dF(p)(V^n)$ of $V^n$ under $dF$ at $p$. We say $F$ is *crosswise* to $M$ if, for all such $p$, these two vector spaces span $V^N$:

$$dF(p)(V^n) + P(M, F(p)) = V^N. \tag{5.1}$$

Note that if

$$n + \dim(M) < N, \quad \text{i.e.,} \quad n < \operatorname{codim}(M),$$

then $F(R) \cap M$ must be void.

LEMMA (Thom, [6]). *Let $M$ be a manifold collection in $L^r$. Then arbitrarily near any $f_0 : R^n \to E^m$ there is an $f$ for which $f^r$ is crosswise to each manifold of $M$.*

Suppose $f^r : R^n \to L^r$ is crosswise to the smooth manifold $M$, and $\operatorname{codim}(M) = s \leqq n$ (otherwise $f(R)$ does not intersect $M$). Say $f^r(p) = q \in M$. Let $\phi_1, \cdots, \phi_s$ be real functions defining $M$ near $q$; they vanish only in $M$, and their gradients are independent there. Set $\psi_i(p) = \phi_i(f^r(p))$ in $R$. Now the functions $\psi_1, \cdots, \psi_s$, near $p$, vanish only in $S = (f^r)^{-1}(M)$, and their gradients are independent at $p$, since $f^r$ is crosswise to $M$; hence $S$ is a manifold in $E^n$, of codimension $s$.

## 6. The singular sets $S_k$

Let $f : R^n \to E^m$ be locally generic (§1). Then (§5) $f^1 = df$ is crosswise to the $L^1_\rho$. Set

$$S_k = (f^1)^{-1}(L^1_{\nu-k}), \qquad \nu = \inf(n, m). \tag{6.1}$$

Then (§5) the $S_k$ are smooth manifolds in $E^n$; they form a manifold collection. Now $S_0$ is the set of regular points of $f$, and $S' = S_1 \cup S_2 \cup \cdots$ is the set of singular points; $f$ is of deficiency $k$ in $S_k$. By (3.3) and §5,

$$\operatorname{codim}(S_{\nu-\rho}) = (n-\rho)(m-\rho), \tag{6.2}$$

provided this number is $\leqq n$; otherwise, $S_{\nu-\rho}$ is void. This gives

$$\operatorname{codim}(S_k) = k(|n-m| + k). \tag{6.3}$$

As special cases (provided the numbers shown are between 0 and $n$),

$$n \geqq m: \begin{cases} \operatorname{codim}(S_1) = n - m + 1, & \dim(S_1) = m - 1, \\ \operatorname{codim}(S_2) = 2(n - m + 2), & \dim(S_2) = 2m - n - 4, \end{cases}$$

$$n \leqq m: \begin{cases} \operatorname{codim}(S_1) = m - n + 1, & \dim(S_1) = 2n - m - 1, \\ \operatorname{codim}(S_2) = 2(m - n + 2), & \dim(S_2) = 3n - 2n - 4. \end{cases}$$

## 7. Structure of the $S_k$

Take any point $p$ where $f$ is of rank $\rho$; then $p \in S_{\nu-\rho}$, $q = f^1(p) \in L^1_\rho$. It is easy to see that coordinate systems $(x^1, \cdots, x^n)$ and $(y^1, \cdots, y^m)$ may be chosen near $p$ and $f(p)$ respectively so that $f$ has the form

$$(7.1) \qquad y^i = x^i \quad (i \leqq \rho), \qquad y^j = \phi^j(x^1, \cdots, x^n) \qquad (j > \rho).$$

Let $y^i_k$ denote $\partial y^i/\partial x^k$. The Jacobian matrix of (7.1) is

$$J = \begin{Vmatrix} I & 0 \\ J'' & J' \end{Vmatrix}, \qquad J' = \begin{Vmatrix} y^{\rho+1}_{\rho+1} & \cdots & y^{\rho+1}_n \\ \cdots & \cdots & \cdots \\ y^m_{\rho+1} & \cdots & y^m_n \end{Vmatrix};$$

$I$ is the unit matrix of $\rho$ rows and columns, and 0 is a matrix of zeros. Moreover, the elements of $J''$ and of $J'$ vanish at $p$.

The $y^i_k$ occurring in $J'$ are a set of $h_\rho = (n-\rho)(m-\rho)$ real functions in $E^n$ near $p$. Now assume that $f$ is locally generic. Then $f^1$ is crosswise to $L^1_\rho$, and as a result, it is easy to see that the gradients of these functions are independent at $p$. We may choose functions $z^j$ $(h_\rho < j \leqq n)$ which vanish at $p$ and whose gradients are independent of the above gradients; now *these* $y^i_k$ *and* $z^j$ *may be chosen as new coordinates in* $E^n$ *near* $p$, with origin at $p$.

Note that

$$(7.2) \qquad \text{rank}(J) = \text{rank}(J') + \rho.$$

Hence the set $\bar{S}_{\nu-(\rho+k)}$ is given (near $p$) by setting all $(k+1)$-rowed determinants of $J'$ equal to zero. This is a set of algebraic equations of degree $k+1$ in $h_\rho$ of the new coordinates. For an example, see §23 below.

## 8. The null spaces $N(p)$

The *null space* $N(p)$ of $f$ at $p$ is the set of all vectors $v$ mapped into 0 by $df(p)$. Clearly

$$(8.1) \qquad \text{for } p \in S_k, \qquad \dim(N(p)) = \begin{cases} n-m+k & \text{if } n \geqq m, \\ k & \text{if } n \leqq m. \end{cases}$$

For $p \in S_k$, $N(p)$ can be related in various ways to $S_k$ near $p$, and more generally to the sets $S_{k-1}, \cdots, S_1$ near $p$. This will be used to split $S_k$ into several sets.

## 9. The differentials $df^r$

Given $f$, we have $f^r(p) \in L^r$, and since $dd^kf = d^{k+1}f$,

$$df^r = d(df, \cdots, d^rf) = (d^2f, \cdots, d^{r+1}f).$$

Thus knowing the value of $f^r$ at $p$, the possible values for the tangent vectors $df^r(p) \cdot v$ are severely restricted; only the last term $d^{r+1}f(p) \cdot v$ is arbitrary.

There are also restrictions due to the symmetry of higher differentials. Thus, $f^1(p) = df(p) \in L^1$, hence $df^1(p) \cdot v \in L^1$, $[df^1(p) \cdot v] \cdot u \in V^m$, and

$$[df^1(p) \cdot v] \cdot u = d^2f(p) \cdot (u, v) = d^2f(p) \cdot (v, u) = [df^1(p) \cdot u] \cdot v.$$

Suppose we know the point $f^{r+1}(p) \in L^{r+1}$. Then we also know the $d^k f(p)$ $(k \leqq r+1)$, hence $df^r(p)$, and thus the tangent vectors $df^{(r)}(p) \cdot v$ in $L^r$.

In particular, the statement that $f^r$ is crosswise to a given manifold collection $M \subset L^r$ is a statement about $df^r$, and thus is equivalent to the statement that $f^{r+1}(R)$ avoids a certain point set $M_1 \subset L^{r+1}$. One might prove Thom's lemma (§5) by showing that any such $M_1$ is a manifold collection of codimension $>n$. Such a proof seems difficult to carry out.

## 10. The mapping $f^2$ into $L^2$

In the expression (2.4) for $L^r$, if we drop out the last terms, we obtain any $L^k$, $k < r$. Hence to each point of $L^r$ corresponds a definite point of each $L^k$, $k < r$. Let $L^r_\rho$ be the subset of $L^r$ thus corresponding to $L^1_\rho$ in $L^1$. Clearly

$$\text{codim } (L^r_\rho) = \text{codim } (L^1_\rho), \tag{10.1}$$

and $f^r$ is transversal to the $L^r_\rho$ if $f^1$ is crosswise to the $L^1_\rho$.

Supposing that $f^1$ is crosswise to the $L^1_\rho$, we now consider, for each point $p$, the relation of the null space $N(p)$ to the sets $\bar{S}_k$ which contain $p$. To give the relation of a vector $v$ to $\bar{S}_k$ $(p \in \bar{S}_k)$ is to give the relation of $df^1(p) \cdot v$ to $L^1_{\nu-k}$; hence a given position of $N(p)$ relative to the $\bar{S}_k$ corresponds to a property of $df^{(1)}(p)$, i.e., a property in $L^2$ (see §9).

By §§6 and 8,

$$\dim (S_1) + \dim (N(p)) = n \qquad \text{if} \quad n \geqq m; \tag{10.2}$$

hence we may expect, at most of the points $p \in S_1$, that $N(p)$ and the tangent plane $P(S_1, p)$ have only the zero vector in common. More generally,

$$\dim (N(p)) = \text{codim } (S_k) + (k-1)(n-m+k) \qquad \text{if} \quad n \geqq m. \tag{10.3}$$

Then

$$h = \dim [N(p) \cap P(S_k, p)]$$

might be 0, or $>0$.

To say that $f^1(p) = q \in L^1_\rho$ and $f^1$ is crosswise to $L^1_\rho$ at $q$ is to say that $f^2(p) \in L^2_\rho$ but $f^2(p)$ avoids a certain subset of $L^2_\rho$ (of codimension $>n$). Let $L^{*2}$ be the part of $L^2$ in no such subset; set $L^{*2}_\rho = L^{*2} \cap L^2_\rho$.

The different values of $h$ above correspond to different subsets of $L^{*2}_{\nu-k}$, and give a splitting

$$L^{*2}_{\nu-k} = L^{*2}_{\nu-k,0} \cup L^{*2}_{\nu-k,1} \cup \cdots. \tag{10.4}$$

Set

$$S_{k,h} = (f^2)^{-1}(L^{*2}_{\nu-k,h}); \quad \text{then } S_k = S_{k,0} \cup S_{k,1} \cup \cdots. \tag{10.5}$$

Now $S_{k,h}$ is the subset of $S_k$ where, if we consider $f$ in $S_k$ alone, $f$ has deficiency $h$. The $S_{k,h}$ are called $S_h(S_k)$ in Thom, [4] (provided that $n \geqq m$).

We have similar splittings for $n < m$.

Now consider $p \in S_2$, assuming $n \geqq m$. We must consider the relation of $N(p)$ not only to $P(S_2, p)$, but also in relation to $\bar{S}_1$ near $p$. This is a more complex situation; see §23 below. The possible relationships of $N(p)$ to the structure of $\bar{S}_1$ near $p$ is reflected in a splitting of the $L^{*2}_{r-2,h}$ near $f(p)$, and this gives a corresponding splitting of the $S_{2,h}$. We shall not give names to the new sets here. In a similar manner, each $S_{k,h}$ is split into subsets, on considering the possible relationships of $N(p)$ to the $\bar{S}_l$ $(l \leqq k)$ near $p \in S_k$.

Note that $S_0$ (the set of regular points) is not split. Also, since (using coordinate systems) the possible relationships of the $N(p)$ to the $\bar{S}_k$ are expressible by algebraic equations, and hence the new sets in $L^{*2}$ are algebraic varieties and hence manifold collections, we see that the new sets in $E^n$ are manifold collections. There is an open subset of $S'$ where $N(p)$ is in the most general position; the rest of $S'$ forms a manifold collection of dimension less than that of $S_1$.

## 11. Further splitting of the $S_k$

In $L^3$, we have the subset $L^{*3}$ corresponding to mappings $f$ such that $f^2$ is crosswise to the sets in $L^{*2}$. We have also sets $L^{*3}_\rho \subset L^{*3}$, corresponding to the $L^1_\rho$ (see §10); they also correspond to the $L^{*2}_\rho$ in $L^2$. But these sets $L^{*2}_\rho$ have been split up; this gives a splitting of the $L^{*3}_\rho$. We shall split these further.

The $S_k$ in $E^n$ have been split into subsets, forming a manifold collection. At any point $p$, we have considered the relation of $N(p)$ to the $\bar{S}_k$; we now consider it also in relation to the new sets. The various possible relationships correspond to facts about $df^2$, and hence to subsets of $L^{*3}$ (see §9); this gives the desired further splitting in $L^{*3}$. Through $(f^3)^{-1}$, we find a further splitting of the $S_k$.

As in the last section, the first new manifolds we obtain are of dimension one less than that of the previous new ones at most, and hence of dimension two less than that of $S_1$ at most.

Remark. Though we have described the manner of splitting in $L^{*2}$ and in $L^{*3}$ through a discussion of the function $f$, the definitions of the sets in $L^2$ and $L^3$ are clearly intrinsic; they depend on the properties of the $\mathfrak{L}^r$ alone.

We next consider the relation of the $N(p)$ to the new sets, giving a splitting in $L^{*4}$ and hence a new splitting of the $S_k$, the largest dimension of a new manifold being three less than that of $S_1$ at most, etc. The process must stop after the new manifolds are of dimension at most zero. Hence the largest $f^r$ and $L^r$ we need use are those for which $r - 1 = \dim (S_1)$. In $L^{*r}$, we can make $f^r$ crosswise to all manifolds obtained, and this gives us the desired locally generic $f$; see also §13. The final requirements on $f$ employ $d^{r+1}f$. Thus (see §6) the requirements on $f$ to be locally generic involve derivatives through the order at most

$$\gamma(n, m) = n - |n - m| + 1 = \begin{cases} m + 1 & (n \geqq m), \\ 2n - m + 1 & (n \leqq m). \end{cases} \tag{11.1}$$

## 12. The singular sets $S_1^{(i)}$

Suppose first that $n \geqq m$. In a neighborhood $U \subset S_1$ of a point $p$ of $S_1$, we may choose a set $v_1(p'), \cdots, v_{m-1}(p')$ of independent vectors orienting $P(S_1, p')$; choose also $v_m(p'), \cdots, v_n(p')$ orienting $N(p')$ $(p' \in U)$; see (8.1). For most points $p' \in U$, the whole set $v_1(p'), \cdots, v_n(p')$ is independent, and an orientation of $E^n$ is determined. In some parts of $U$, one orientation of $E^n$ may be determined, and in other parts, the opposite orientation; these parts are divided by the set $S_{1,1} \cup S_{1,2} \cup \cdots$. Hence

$$\text{(12.1)} \qquad \dim (S_{1,1}) = \dim (S_1) - 1 = m - 2 \quad (n \geqq m).$$

We shall write $S_1^{(2)}$ for the manifold $S_{1,1}$.

For $p \in S_1^{(2)}$, $N(p) \cup P(S_1, p) = Q(p)$ is of dimension 1; it may point from $p$ into $P(S_1, p)$ in either side of $P(S_1^{(2)}, p)$; or at exceptional points, it may lie in $P(S_1^{(2)}, p)$. This exceptional set is a manifold in $S_1^{(2)}$, of dimension $m - 3$; it is the inverse image under $f^3$ of a certain set in $L^3$. At these points $p$, $N(p) \cap P(S_1^{(2)}, p) = Q'(p)$ is of dimension 1 instead of 0; we call this set $S_{1,1,1}$, or $S_1^{(3)}$. At a point of $S_1^{(3)}$, $Q'(p)$ may point into $P(S_1^{(2)}, p)$ in either side of $P(S_1^{(3)}, p)$, or be tangent to the latter; thus we find $S_1^{(4)}$, etc. We have

$$\text{(12.2)} \qquad \dim (S_1^{(i)}) = m - i \qquad (1 \leqq i \leqq m \leqq n).$$

These singular sets (inverse images of sets in $L^1$) were found by Roger [3], and appear in Thom [4] with the notation $S_1(S_1)$, $S_1(S_1(S_1))$, etc. Note that the last one corresponds to a set in $L^m$; this gives actual occurrences of $\gamma(n, m)$ in (11.1), for $n \geqq m$.

This gives the total splitting of $S_1$, if $n \leqq 4$.

For $m \geqq n$, codim $(S_1) = m - n + 1$ and dim $(N(p)) = 1 (p \in S_1)$; hence $N(p)$ will lie in $P(S_1, p)$ (for locally generic $f$) in a manifold $S_{1,1} = S_1^{(2)}$ of codimension $m - n + 1$ in $S_1$, and hence of dimension $3n - 2m - 2$. For $p \in S_1^{(2)}$, $N(p)$ lies in $P(S_1, p)$; it will lie in $P(S_1^{(2)}, p)$ at each point of a manifold $S_{1,1,1} = S_1^{(3)}$ of codimension $m - n + 1$ in $S_1^{(2)}$, etc. Thus

$$\text{(12.3)} \qquad \operatorname{codim} (S_1^{(i)}) = i(m - n + 1) \text{ if present } (1 \leqq i \leqq m \geqq n).$$

The definition of the $S_1^{(i)}$ shows that $f$ is locally one-one with nonvanishing Jacobian in each $S_1^{(i)} - S_1^{(i+1)}$; but the image has a cusp manifold in the next set $S_1^{(i+1)}$.

## 13. On the classification of singularities

As before, if we cut out a certain subset of $L^r$, $r = \gamma(n, m)$, of codimension $>n$, corresponding to mappings not crosswise to the sets we have found in $L^{*r-1}$, we have left $L^{*r}$; any mapping $f_0 : R^n \to E^m$ is arbitrarily near a mapping $f$ with $f^r(p) \in L^{*r}$ $(p \in R)$, and any such $f$ is locally generic. Morevoer, $f^r$ is automatically crosswise to the sets of the splitting of $L^{*r}$.

The various sets of the splitting in $L^{*r}$ define the different types of singularities for locally generic $f$.

Note that each set $A$ of the splitting in $L^{*r-1}(r = \gamma(n, m))$ corresponds to a set $B$ in $L^r$ which is not split further; but part of $B$ may be cut out when we reduce $L^r$ to $L^{*r}$, and what is left may be composed of several connected pieces, which may correspond to different types of singularities. This happens for instance in the case $m = 1, n > 1$; see §16.

### 14. Encasing of singularities

Given $f : R^n \to E^m$, the *s-dimensional encasing* of $f$ is a mapping $F : R^{n+s} \to E^{m+s}$ defined as follows. Write $E^{n+s} = E^n \times E^s$, $E^{m+s} = E^m \times E^s$, and

$$F(p, q) = (f(p), q) \qquad (p \in E^n,\ q \in E^s). \tag{14.1}$$

To each generic singular point of $f$ corresponds a generic singular point of $F$; the latter is the *s-dimensional encasing* of the former. Examples will be given below.

Clearly

$$\text{rank}\ (F, (p, q)) = \text{rank}\ (f, p) + s; \tag{14.2}$$

hence, for each $q \in E^s$,

$$S_k(F) \cap (E^n \times q) = S_k(f) \times q, \qquad \text{codim}\ (S_k(F)) = \text{codim}(S_k(f)). \tag{14.3}$$

Clearly $N_F(p, q)$ lies in the space $V^n$ of $E^n$ always; hence the relation of null spaces to the $S_k$ is the same for $F$ as for $f$, and we see step by step that the $S_k$ are split into subsets in the same way for each. Finally, at any $p$, lying in a certain singular set, $f^r$ (for some $r$) is crosswise to the corresponding subset in $L^r(n, m)$; hence $F^r$ is crosswise to the corresponding set in $L^r(n + s, m + s)$, showing that $F$ is locally generic.

### 15. Increasing $n$

Given $f : R^n \to E^m$, with $n \geq m$, define $F : R^{n+1} \to E^m$ near a point $p \in S_1$ as follows. Since dim $[df(p) \cdot V^n] = m - 1$, we may choose a vector $w_0$ in $V^m$ not in $df(p) \cdot V^n$. (There are essentially two choices.) Set

$$F(p, t) = f(p) + t^2 w_0. \tag{15.1}$$

With a coordinate system in $E^n$,

$$\frac{\partial F}{\partial x^i} = \frac{\partial f}{\partial x^i} \quad (i = 1, \cdots, n), \qquad \frac{\partial F}{\partial t} = 2tw_0.$$

For $p'$ near $p$ and $t \neq 0$, the last vector is independent of the former; hence rank $(F, (p', t)) = \text{rank}\ (f, p') + 1 = m = \nu$, and $(p', t)$ is not a singular point of $F$. For $t = 0$, $N_F(p', 0) = N_f(p') + V^1$; the relation of these spaces to the $S_1{}^{(i)}$ for $F$ is the same as that of the $N(p')$ to the $S_1{}^{(i)}$ for $f$. Thus $S_1{}^{(i)}$ for $f$ becomes part of $S_1{}^{(i)}$ for $F$ at $t = 0$.

The above discussion fails for the $S_k$, $k \geq 2$. For instance, $S_2$ may be present for mappings $E^4 \to E^4$, but not for mappings $E^5 \to E^4$ (see §4).

## II. Examples of singularities

### 16. Critical points of real functions

(See M. Morse, [1] and [2]). For $E^n \to E^1$, we let $E^1$ be the set of real numbers. Here, $L^1$ is simply the set of real linear functions (i.e., covectors) in $V^n$; it is of dimension $n$, and $L_0^1$ consists of 0 only. Since codim $(S_1) = \text{codim}\ (L_0^1) = n$, $S_1$ consists of isolated points.

With a Cartesian coordinate system in $E^n$ and hence in $L^1$, the components of $f^1(p) = df(p)$ are

$$f^1(p) = \left(\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n}\right). \tag{16.1}$$

Now (compare §9)

$$\partial f^1/\partial x^i = (\partial^2 f/\partial x^1 \partial x^i, \cdots, \partial^2 f/\partial x^n \partial x^i),$$

and if $f$ is locally generic, these vectors in $L^1$ are independent for $i = 1, \cdots, n$. That is, the matrix of second partial derivatives is of rank $n$.

With proper choice of coordinates, $f$ near $p$ may be written in the form (for some $k$)

$$y = (x^1)^2 + \cdots + (x^k)^2 - (x^{k+1})^2 - \cdots - (x^n)^2. \tag{16.2}$$

For $n = 1$, $y = \pm(x^1)^2$. (If we allow coordinate systems reversing orientation, we may write $y = (x^1)^2$.) If we now increase $n$ as in §15, choosing $w_0 \in E^1$ sometimes positive and sometimes negative, we obtain the above critical points.

Note that $L_0^{*2}$ is $L_0^2$ except for all points where the determinant $|\partial^2 f/\partial x^i \partial x^j| = 0$. The different parts of $L_0^{*2}$ correspond to critical points of different index, i.e., with different numbers of minus signs in (16.2).

## 17. The case $n = m = 2$

For $E^1 \to E^1$, a typical singularity is given by $y = x^2$. If we encase this (§14), we obtain

$$y^1 = x^1, \qquad y^2 = (x^2)^2; \tag{17.1}$$

this has a singular set $S_1$ on the line $x^2 = 0$.

Consider the mapping defined by

$$y^1 = x^1, \qquad y^2 = x^1x^2 - (x^2)^3. \tag{17.2}$$

The Jacobian is $J = \partial y^2/\partial x^2 = x^1 - 3(x^2)^2$, which vanishes on the curve $S_1$, defined by $x^1 = 3(x^2)^2$. With unit vectors $e_1$, $e_2$ along the axes, the image of any vector $v$ under $df$ is

$$df(x^1, x^2) \cdot (v^1e_1 + v^2e_2) = \left(v^1 \frac{\partial y^1}{\partial x^1} + v^2 \frac{\partial y^1}{\partial x^2}, v^1 \frac{\partial y^2}{\partial x^1} + v^2 \frac{\partial y^2}{\partial x^2}\right).$$

Along $S_1$, this equals $(v^1, v^1x^2)$; hence $N(p)$ consists of all vectors $ae_2$ for $p \in S_1$. Except for $p = (0, 0)$, $N(p)$ is not tangent to $S_1$; hence we are not in $S_1^{(2)}$ here. But we have tangency at $(0, 0)$. (The image under $f$ of $S_1$ is a curve with a cusp point at the origin.) Also, $N(p)$ is crossing $S_1$ here (the mapping $f^2$ is crosswise to the corresponding set), as is easily seen; hence the origin is in $S_1^{(2)}$.

For fuller details, see [13].

## 18. The case $m = 2$, $n \geqq 3$

If we increase $n$ as in §15, we find singular curves $S_1$, with isolated points $S_1^{(2)}$ on them, as in §17. We find typical mappings, first for a point $p$ of $S_1 - S_1^{(2)}$, with $n = 3$. By (17.1), the images of all vectors at $p$ lie in the $y^1$-direction; hence we may choose $w_0 = te_2$ in (15.1); $t$ is replaced by $x^3$. Inserting the sign $\pm$ in (17.1), this gives

$$y^1 = x^1, \qquad y^2 = \pm (x^2)^2 \pm (x^3)^2, \qquad \text{in } S_1 - S_1^{(2)}. \tag{18.1}$$

Similarly, using (17.2),

$$y^1 = x^1, \qquad y^2 = x^1x^2 - (x^2)^3 \pm (x^3)^2 \qquad \text{in } S_1^{(2)}. \tag{18.2}$$

Continuing this process gives, for typical singularities in the general case,

$$y^1 = x^1, \qquad y^2 = \pm (x^2)^2 \pm \cdots \pm (x^n)^2 \qquad \text{in } S_1 - S_1^{(2)}, \tag{18.3}$$

$$y^1 = x^1, \qquad y^2 = x^1x^2 - (x^2)^3 \pm (x^3)^2 \pm \cdots \pm (x^n)^2 \qquad \text{in } S_1^{(2)}. \tag{18.4}$$

## 19. The case $n \geqq m = 3$

Here, we have singularities of types $S_1$, $S_1^{(2)}$, $S_1^{(3)}$. Typical examples are, for $n = 3$,

$$y^1 = x^1, \qquad y^2 = x^2, \qquad y^3 = (x^3)^2 \qquad \text{in } S_1 - S_1^{(2)}, \tag{19.1}$$

$$y^1 = x^1, \qquad y^2 = x^2, \qquad y^3 = x^2x^3 - (x^3)^3 \qquad \text{in } S_1^{(2)} - S_1^{(3)}, \tag{19.2}$$

$$y^1 = x^1, \qquad y^2 = x^2, \qquad y^3 = x^1x^3 + x^2(x^3)^2 - (x^3)x^4 \qquad \text{in } S_1^{(3)}. \tag{19.3}$$

In the last example, the Jacobian is $\partial y^3/\partial x^3$, and $S_1$ is the surface

$$S_1 : \phi = x^1 + 2x^2x^3 - 4(x^3)^3 = 0.$$

Clearly $N(x^1, x^2, x^3)$ consists of all vectors $ae_3$, in the $x^3$-direction, at points of $S_1$. These lie in $S_1$ if they are orthogonal to the gradient of $\phi$. Thus $S_1^{(2)}$ is the curve given by:

$$S_1^{(2)} : \phi = 0, \qquad \psi = 2x^2 - 12(x^3)^2 = 0.$$

For points of $S_1^{(3)}$, $e_3$ must be orthogonal to the gradient of $\psi$ also, this gives $x^3 = 0$. Thus $S_1^{(3)}$ contains the origin alone.

For general $n$, typical singularities are as follows. As before, $y^1 = x^1$ and $y^2 = x^2$; in the above expressions for $y^3$, we add $\pm(x^4)^2 \pm \cdots \pm (x^n)^2$.

## 20. The case $m = 2n - 1$

Here, $S_1$ consists of isolated points. By [9], a typical singularity is given by

$$\begin{cases} y^i = x^i & (i = 1, \cdots, n-1), \\ y^j = x^n x^{j-n+1} & (j = n, n+1, \cdots, 2n-1). \end{cases} \tag{20.1}$$

For instance, for $n = 2$,

$$y^1 = x^1, \qquad y^2 = x^1x^2, \qquad y^3 = (x^2)^2. \tag{20.2}$$

## 21. The case $m = 2n - 2$

If we take the singularity above for $E^{n-1} \to E^{2n-3}$, and encase it as in §14, we find a typical singularity in $S_1$ for $E^n \to E^{2n-2}$. We have extra variables, say $x^0$ and $y^0$, and the extra equation $y^0 = x^0$; $S_1$ is the $x^0$-axis. For $n \geqq 3$, (12.3) shows that $S_1^{(2)}$ is void; for $n = 2$, we are back to the case $n = m = 2$.

## 22. Some other cases with $m > n$

The lowest case not considered is the case $(n, m) = (4, 5)$. Here,

$$\dim (S_1) = 2, \qquad \dim (S_1^{(2)}) = 0 \qquad (n = 4, m = 5). \tag{22.1}$$

In general, by (12.3), $S_1^{(2)}$ appears only if $2m \leqq 3n - 2$; $S_1^{(3)}$ appears only if $3m \leqq 4n - 3$ (the first case is $n = 6$, $m = 7$), etc. From the table in §4, we see that the first appearance of $S_2$ with $m > n$ is also for the case $n = 6$, $m = 7$.

## 23. The case $n = m = 4$

In this case, $S_1$ may contain sets $S_1^{(2)}, S_1^{(3)}$ and $S_1^{(4)}$, but no $S_{1,2}$. We may now also have isolated points belonging to $S_2$. We examine these more closely.

As in §7, we may choose coordinates in the two spaces so that, near the origin $p_0$, we have $y^1 = x^1$, $y^2 = x^2$. Now the matrix $J'$ in §7 is

$$J' = \begin{Vmatrix} y_3^3 & y_4^3 \\ y_3^4 & y_4^4 \end{Vmatrix}. \tag{23.1}$$

We may also (§7) use these $y_k^i$ as new coordinates near $p_0$:

$$X^1 = y_3^3, \qquad X^2 = y_4^3, \qquad X^3 = y_3^4, \qquad X^4 = y_4^4. \tag{23.2}$$

The parts of $\bar{S}_1$ and of $S^2$ near $p_0$ are

$$\bar{S}_1 \colon \begin{vmatrix} X^1 & X^2 \\ X^3 & X^4 \end{vmatrix} = X^1X^4 - X^2X^3 = 0, \tag{23.3}$$

$$S_2 \colon X^1 = X^2 = X^3 = X^4 = 0. \tag{23.4}$$

Thus, in these coordinates, $\bar{S}_1$ is a quadratic cone, with vertex at $p_0$, $p_0$ being the single point of $S_2$. Let $T$ be the tangent cone of $\bar{S}_1$ at $p_0$; it consists of all vectors $v$ such that $p_0 + v \in \bar{S}_1$ (and hence $p_0 + av \in \bar{S}_1$, all $a$).

We now ask what relation $N(p_0)$ (of dimension 2) can have to $\bar{S}_1$. If we look for vectors in both $N(p_0)$ and $T$, we find a quadratic relation; there may be no common vectors $\neq 0$, or there may be common vectors in one or in two distinct directions. There can also be more special relations (for instance, if $N(p_0)$ is the $(X^1, X^2)$-plane, then $N(p_0) \subset T$).

We give two examples:

$$y^1 = x^1, \qquad y^2 = y^2, \qquad y^3 = x^1x^3 + x^2x^4 + x^3x^4, \tag{23.5}$$

and

$$\begin{cases} \text{(a) } y^4 = x^1x^4 - x^2x^3 + \frac{1}{2}(x^3)^2 - \frac{1}{2}(x^4)^2, \\ \text{(b) } y^4 = x^1x^4 + x^2x^3 - \frac{1}{2}(x^3)^2 - \frac{1}{2}(x^4)^2. \end{cases} \tag{23.6}$$

We find at once that $\bar{S}_1$ is given in the two cases by

$$\text{(a)} \quad (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 = 0,$$
$$\text{(b)} \quad (x^1)^2 - (x^2)^2 + (x^3)^2 - (x^4)^2 = 0.$$

In both cases, $N(p_0)$ consists of the vectors in the $(x^3, x^4)$-plane. In (a), $N(p_0) \cap T$ has no vector $\neq 0$; in (b), it contains those vectors with $v^3 = \pm v^4$. We thus have two distinct types of singularities, which might be called $S_{2,a}$, $S_{2,b}$.

## 24. The case $n = m = 5$

Here, $S_2$ consists of curves. We shall illustrate a special singularity at an isolated point of a curve of $S_2$. Define $f$ by

$$(24.1) \quad \begin{aligned} &y^i = x^i \ (i = 1, 2, 3), \qquad y^4 = x^2x^4 + (x^4)^2 + x^4x^5, \\ &y^5 = x^3x^5 + x^4x^5 + \tfrac{1}{2}(x^4)^2 + \tfrac{1}{2}(x^5)^2 + \tfrac{1}{2}x^1(x^4)^2. \end{aligned}$$

Then $S_2$ is the $x^1$-axis (near the origin $p_0$), and for $p \in S_2$, $N(p)$ consists of the vectors in the $(x^4, x^5)$-plane. Note that $S_2$ is not tangent to $N(p)$; hence $f$ maps $S_2$ into a smooth curve.

The singular points are given by $y^4_4y^5_5 - y^4_5y^5_4 = 0$, i.e.,

$$(x^2 + 2x^4 + x^5)(x^3 + x^4 + x^5) - x^4(x^4 + x^5 + x^1x^4) = 0.$$

Let $S'(x^1)$ be that part of $S'$ with $x^1$ fixed; it is a quadratic cone. Let $T(x^1)$ be the corresponding cone of vectors, as in §23, and $N(x^1)$, the null space at $(x^1, 0, \cdots, 0)$. A simple computation shows that $N(x^1)T \cap (x^1)$ has no non-zero vectors if $x^1 < 0$, and has two independent directions if $x^1 > 0$; for $x^1 = 0$, it is the 1-dimensional space of all vectors $a(e_4 - e_5)$. The origin is a new type of singular point.

## 25. The case $n = 5$, $m = 4$

Here we have no $S_2$, but there can be points of $S_{1,2}$. Note that $\dim (S_1) = 3$, $\dim (N(p)) = 2 (p \in S_1)$. Consider the mapping $f$ defined by $y^i = x^i$ $(i = 1, 2, 3)$, and

$$(25.1) \quad y^4 = x^1x^4 + x^2x^5 + x^3x^4x^5 + \tfrac{1}{6}(x^4)^3 + \tfrac{1}{6}(x^5)^3.$$

Then $S_1$ is given by $y^4_4 = y^4_5 = 0$, that is, by

$$x^1 + x^3x^5 + \tfrac{1}{2}(x^4)^2 = x^2 + x^3x^4 + \tfrac{1}{2}(x^5)^2 = 0,$$

and $N(p)$ consists of all vectors in the $(x^4, x^5)$-plane $(p \in S_1)$. Note that at the origin $p_0$, the tangent 3-plane to $S_1$ is the $(x^3, x^4, x^5)$-plane, which contains $N(p_0)$.

This is a phenomenon which cannot be removed by arbitrarily small alterations of $f$. To see this, note that the components of the differentials of $y^4_4$ and $y^4_5$ are

$$\left\| \begin{matrix} dy^4_4 \\ dy^4_5 \end{matrix} \right\| = \left\| \begin{matrix} y^4_{14} & y^4_{24} & y^4_{34} & y^4_{44} & y^4_{45} \\ y^4_{15} & y^4_{25} & y^4_{35} & y^4_{45} & y^4_{55} \end{matrix} \right\|$$

To have the $(x^4, x^5)$-plane tangent to $S_1$, we must have

$$dy_4^4(p) \cdot e_i = dy_5^4(p) \cdot e_i = 0, \quad i = 4, 5.$$

These equations give

$$y_{44}^4 = y_{45}^4 = y_{55}^4 = 0,$$

occurring at the origin $p_0$ in the present example. Note that

$$y_{44}^4 = x^4, y_{45}^4 = x^3, y_{55}^4 = x^5,$$

and

$$J^* = \begin{Vmatrix} dy_{44}^4 \\ dy_{45}^4 \\ dy_{55}^4 \end{Vmatrix} = \begin{Vmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

Letting $p$ move in $S_1$, this shows that the mapping $(y_{44}^4, y_{45}^4, y_{55}^4)$ is crosswise to $(0, 0, 0)$ in 3-space. Now take any sufficiently good 3-approximation $f'$ to $f$. It is easy to see that new coordinates $\bar{x}^i$, $\bar{y}^i$ close to the old may be found so that $y^i = \bar{x}^i$ $(i = 1, 2, 3)$. We will have $S_1$ for $f'$ close to $S_1$ for $f$, and $J^*$ for $f'$ close to $J^*$ for $f$. Hence there will be a solution of $\bar{y}_{44}^4 = \bar{y}_{45}^4 = \bar{y}_{55}^4 = 0$, with crosswiseness, as required. (The crosswiseness can of course be expressed in $L^3$.) Note that the fact that the second differential is symmetric $(y_{54}^4 = y_{45}^4)$ plays an essential role here.

## III. General definitions

### 26. Local equivalence of mappings

Take $f_i : R_i^n \to E_i^m$ $(i = 1, 2)$, and take $p_i \in R_i$. We say $f_2$ at $p_2$ is *equivalent* to $f_1$ at $p_1$ if we can write

(26.1) $$f_2 = \psi f_1 \phi \text{ near } p_2,$$

where $\phi : U_2^n \to R_1^n$ and $\psi : U_1^m \to E_2^m$ are smooth homeomorphisms of rank $n$ and $m$ respectively. Let $\mathfrak{A}^s$ denote arithmetic $s$-space (of all ordered sets of $s$ real numbers). If we take $E_2^n = \mathfrak{A}^n$, $E_2^m = \mathfrak{A}^m$, we have introduced coordinates into $E_1^n$ and $E_1^m$ through $\phi$ and $\psi$, and the mapping $f_2$ becomes

(26.2) $$y^i = \phi^i(x^1, \cdots, x^n) \quad (i = 1, \cdots, m).$$

We noted in §1 that if $f : R^n \to E^m$ is regular at $p$, then $f$ is equivalent at $p$ to the mapping (1.1). If $p$ is a singular point of $f$, we would like to choose coordinates in some way to give a particularly simple representation (26.2) for $f$. Such a choice we call a *normal form* for the singular point. For instance, for $f : E^1 \to E^1$, if $f$ has a minimum at $p$ and $d^2f(p) \neq 0$, an equivalent mapping is given by $y = x^2$. Various normal forms were given in Part II.

### 27. Local stability

We say $f : R^n \to E^m$ is *stable at* $p$ if there is a neighborhood $U$ of $p$ with the following property. Take any neighborhood $U'$ of $p$, $U' \subset U$. Then for any sufficiently good $s$-approximation $f'$ to $f$ in $U$, for some $s$, there is a point $p' \in U'$ such that $f'$ at $p'$ is equivalent to $f$ at $p$.

Clearly the mapping (1.1) is stable at all points (use good 1-approximations). So is the mapping $y = x^2$ at $x = 0$, using 2-approximations; but if we used only 1-approximations, we could obtain a new mapping with $y$ constant in some small interval, and local equivalence would then fail.

One form of the principal conjecture is that any locally generic mapping (defined through the discussion of Part I) is stable at all points. We give a strengthened form in the next section.

## 28. Topologies in function space.

Let $\mathfrak{S}$ be the set of all pairs $(f, R)$, where $f : R^n \to E^m$ is $\infty$-smooth. We shall give rough descriptions of two topologies in $\mathfrak{S}$, the *normal topology* and the *stability topology*, turning $\mathfrak{S}$ into topological spaces $\mathfrak{S}_n$ and $\mathfrak{S}_s$ respectively.

In $\mathfrak{S}_n$, we say that $(f', R')$ is near $(f, R)$ if $R'$ is near $R$ (with a simple definition), and for some large $r$, $f'$ is a good $r$-approximation to $f$ in $R \cap R'$. In $\mathfrak{S}_s$, we say $(f', R')$ is near $(f, R)$ if we can make (26.1) hold, with the following requirements. There are open sets $R_1$, $R_1'$ near $R$, $R'$ respectively, $\phi$ maps $R_1'$ onto $R_1$, and $\phi$ and $\psi$ are good $r$-approximations to the identity in $R_1'$ and in the domain of $\psi$ respectively, with large $r$.

There is an identity mapping $I$ of $\mathfrak{S}_n$ into $\mathfrak{S}_s$, which is not continuous at most points $(f, R)$. Let $\mathfrak{S}(U_0)$ denote the set of all $(f, U_0)$. We say $f$ is *strongly stable* at $p$ if there is a neighborhood $U$ of $p$ such that the mapping $I$, considered in $\mathfrak{S}_n(U)$ alone, is continuous at $(f, U)$. Thus any $(f', U)$ which is sufficiently close to $(f, U)$ in the normal topology is also close in the stability topology.

If $I$ is continuous in $\mathfrak{S}_n(U)$ at $(f, U)$, it is also continuous in $\mathfrak{S}_n(U')$ at $(f, U')$, for any open ball $U' \subset U$. This may be seen by extending $f' - f$ from $U'$ to $U$; see [12].

Say $f$ is *locally strongly stable* in $R$ if it is strongly stable at each point of $R$. If this holds, then we can cover $R$ by open balls $U_1, U_2, \cdots$ such that $I$ is continuous at $(f, U_i)$ in $\mathfrak{S}_n(U_i)$ for each $i$. It follows that for any sufficiently good $r$-approximation $f'$ to $f$ in $R$, for large enough $r$, we can write $f' = \psi_i f \phi_i$ in each $U_i$. We cannot expect to have $f' = \psi f \phi$ in $R$, unless the self-intersections of $f(R)$ have been made generic. For example, if $f : E^1 \to E^1$ is given by $y = (x^2 - 1)^2$, then $df = 0$ at $x = -1$ and at $x = 1$, and $f(-1) = f(1)$; this will not happen for all nearby $f'$.

## 29. General problems

For each $R \subset E^n$, we have a subset $\mathfrak{S}(R)$ of $\mathfrak{S}$. Are the locally strongly stable mappings, or the locally stable mappings, in $\mathfrak{S}(R)$, dense in $\mathfrak{S}_n(R)$? Do they form an open set in both cases? If the complete pattern of singularities is as suggested in Part I, these questions are answered in the affirmative.

In §13 we discussed the description of singularities, in terms of the method of splitting of the $L^r$ used; each connected piece $L^r_{(i)}$ corresponds to a singularity type. The test of this is as follows. Let $q_1$ and $q_2$ be points of the same $L^r_{(i)}$. Let

$$f_1 : R_1^n \to E^m \quad \text{and} \quad f_2 : R_2^n \to E^m$$

be mappings such that, with $\gamma$ defined by (11.1),

$$f_1^{\gamma(n,m)}(p_1) = q_1, \qquad f_2^{\gamma(n,m)}(p_2) = q_2.$$

Then $\phi$ and $\psi$ should exist as in §26, such that $f_2 = \psi f_1 \phi$ near $p_2$. In other words a single normal form will do for all points of any $L^r_{(i)}$. This has been proved in the following cases: for $m = 1$, in [1] and [2]; for $m = 2n - 1$, in [9]; for the $s$-dimensional encasing (§14) of the above singularities in [14]; and for $n = m = 2$, in [13].

Say $f : R^n \rightarrow E^m$ is *stable* if the following is true. For any sufficiently good approximation $f'$ to $f$ (we would use a $\gamma(n, m)$-approximation, which gets better and better as we approach the boundary of $R$), we can find $\phi$, mapping $R^n$ onto $R^n$, and $\psi$, mapping a neighborhood of $f(R^n)$ into $E^m$, such that $f' = \psi f \phi$ in $R$. We could require that $\phi$ and $\psi$ be near the identity; we could also require that $\psi$ be defined in a larger set, perhaps the whole of $E^m$, We then wish to show that the stable mappings form a dense open set in $\mathfrak{S}(R)$, in a suitable topology. As noted in the last section, we must take care of intersection properties of $f(R)$ to obtain a *generic* mapping (i.e., a stable mapping, assuming the truth of the statement). The study of self-intersections seems less difficult than that of singularities.

Finally, recall that topological questions have been discussed only in certain cases; see §1. (For topological properties related to self-intersections in the case $m = 2n$, see [10].)

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# GENERALIZATIONS OF THE BORSUK-ULAM THEOREM

By B. A. Rattray

We prove two generalizations of the Borsuk-Ulam theorem.[1] Let $S^n$ be the $n$-sphere $\sum_{i=1}^{n+1} x_i^2 = 1$. The point $-x$ is said to be antipodal to $x$. Let $R^n$ be $S^n - \{0, \cdots, 0, 1\}$, which is homeomorphic to Euclidean $n$-space. Let $P^n$ be the projective $n$-space, obtained from $S^n$ by identifying antipodal points.

Theorem 1. *Any continuous map $f: S^n \to S^n$ of even degree carries at least one pair of antipodal points into the same point.*

Proof. Suppose that for every pair of antipodal points the images $f(x), f(-x)$ are different. Then $f(x)$ and $-f(-x)$ are not antipodal, so that the segment joining them does not contain the origin. Its midpoint $h(x) = \frac{1}{2}(f(x) - f(-x))$ projects from the origin into a point $g(x) = h(x)/\|h(x)\|$ in $S^n$. Obviously $g(x)$ is continuous and homotopic to $f(x)$. Also $g(-x) = -g(x)$ for each $x$ so that $g(x)$, and hence $f(x)$, has odd degree [1, p. 135].

As a corollary we obtain the classical Borsuk-Ulam theorem: *Any continuous map $f: S^n \to R^n$ carries some pair of antipodal points into the same point.* For any such map is a map $S^n \to S^n$ which is nullhomotopic and hence of degree 0.

Theorem 2. *Any continuous map $f: S^n \to S^n$ of odd degree carries some pair of antipodal points into antipodal points.*

Proof. Otherwise $f(x)$ is homotopic to $g(x) = h(x)/\|h(x)\|$ where $h(x) = \frac{1}{2}(f(x) + f(-x))$. The map $g$ carries every two antipodal points into the same point, and such a map is easily seen to be of even degree. Thus $f$ also has even degree.

Theorem 3. *Any continuous map $f: S^n \to P^n$ carries some pair of antipodal points into the same point.*

Proof. Let $\alpha: S^n \to P^n$ be the covering map which carries each point of $S^n$ into the antipodal pair containing it. Any map $f$ can be lifted into a map $g: S^n \to S^n$ such that $f = \alpha g$. Since $g$ has either even or odd degree, it carries some pair of antipodal points into the same point or two antipodal points. In either case $f$ carries these antipodal points into the same point of $P^n$.

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[1] ((*Added in Proof.*) I have recently learned that similar results were published by Guy Hirsch in his paper, *Un théorème sur les transformations des sphères*, *Académie Royale de Belgique*, Bulletin de la classe des sciences, 5e Serie, Tome XXXII, (1946) pp. 394–399.)

# ON THE GEOMETRY OF FUNCTION SPACES

BY JAMES EELLS, JR.

## Introduction

In the present work we consider function spaces $\mathfrak{F}$ of continuous maps of a compact topological space $S$ into a Riemannian manifold $M$, with a suitable uniform structure on $\mathfrak{F}$. Such function spaces have long played a central role in the geometry of $M$, largely because many properties (algebraic, metric, topological) of $M$ are simply reflected in $\mathfrak{F}$. It will be shown here that *for a large variety of function spaces the Riemannian structure on $M$ determines a differentiable structure on $\mathfrak{F}$*; i.e., $\mathfrak{F}$ is a differentiable manifold, not necessarily connected and of infinite dimension, in general. Certain function spaces (of rigid maps) have been interpreted before as manifolds of finite dimension; e.g., the Stiefel and Grassmann manifolds (Steenrod [11, §7]), the configuration spaces of dynamical systems.

If $\mathfrak{M}$ is a paracompact differentiable manifold modeled on a Hilbert space $E$ (i.e., $\mathfrak{M}$ is locally homeomorphic with $E$), then we can extend de Rham's Theorem, showing that the real singular cohomology algebra of $\mathfrak{M}$ can be described by classes of closed differential forms on $\mathfrak{M}$. In §4C we apply that result to a space of loops on an $n$-dimensional manifold.

This work has benefited greatly by discussions with several members of the Institute for Advanced Study during 1956; I hereby gratefully express my appreciation.

## 1. Smooth transformations of linear spaces

Let $E$ and $F$ denote complete locally convex linear spaces[1] over the real numbers $R$, and let $\mathfrak{L}(E, F)$ be the linear space of all continuous linear maps of $E$ into $F$. We will endow $\mathfrak{L}(E, F)$ with the topology—in fact, uniform structure—of uniform convergence on the bounded sets of $E$; this topology is called the *conjugate topology* on the conjugate space $E' = \mathfrak{L}(E, R)$. If $E$ is a Fréchet space (i.e., $E$ is metrizable), then $E'$ is complete in the conjugate topology.

(A) DEFINITIONS.[2] Suppose that $\phi$ is a transformation of an open set $U$ of $E$ into $F$. Given[3] a point $x \in U$ and a vector $v \in E$ we define the *directional derivative of $\phi$ at $x$ in the direction $v$ by*

$$(1) \qquad \nabla \phi (x, v) = \lim_{h \to 0+} \frac{\phi (x + hv) - \phi (x)}{h}$$

[1] For the fundamental properties of such spaces see Bourbaki [2].

[2] If $E$ and $F$ are finite dimensional, then these concepts coincide with the usual ones. For the class $C^k$ see Graves and Hildebrandt [6].

[3] The distinction between a *point* $x \in E$ and a *vector* $v \in E$ is made to emphasize the different roles played by $x$ and $v$.

if the limit (1) exists as $h$ approaches zero through positive real values. For each $x \in U$ we suppose that $\nabla \phi(x, v)$ exists and is linear and continuous in $v$. Then the *differential* $\phi_*(x)$ *of* $\phi$ *at* $x$ is the element of $\mathfrak{L}(E, F)$ defined by $\phi_*(x) \cdot v = \nabla \phi(x, v)$ for all $v \in E$. We say that $\phi$ is of *class* $C^1$ *on* $U$ if $\phi$ is continuous on $U$ and the map $\phi_* : U \to \mathfrak{L}(E, F)$ defined by $(\phi_*)x = \phi_*(x)$ is continuous. (If $E$ and $F$ are Banach spaces and $\phi_*$ is continuous on $U$, then $\phi$ is Fréchet differentiable and is therefore continuous.) We say that $\phi$ is of *class* $C^k$ *on* $U$ $(k > 1)$ if $\phi$ is $C^1$ and the map $\phi_* : U \to \mathfrak{L}(E, F)$ is $C^{k-1}$; $\phi$ is *smooth* if it is $C^k$ for all $k$.

(B) We are especially concerned (cf. de Rahm's Theorem) with linear spaces $E$ which satisfy

CONDITION (S). *Given any closed bounded subset* $B$ *of* $E$ *and any neighborhood* $U$ *of* $B$ *there is a smooth real function* $\phi$ *defined on* $E$ *such that* $\phi(x) = 1$ *for* $x \in B$ *and* $\phi(x) = 0$ *for* $x \in E - U$.

Any real Hilbert space satisfies Condition (S), for its inner product is a smooth function, and we can compose it with a suitable smooth real function of a real variable to construct $\phi$. More generally, a Banach space will satisfy Condition (S) if the square of its norm is smooth; however, differentiability properties of the norm imply strong restrictions on a Banach space; see Šmulian [10]. I do not know whether the Banach space $C$ of continuous functions on the unit interval (with uniform norm) satisfies Condition (S).

## 2. Manifolds modeled on linear spaces

(A) As in the finite dimensional case we make the

DEFINITIONS. Let $E$ denote a complete locally convex linear space. We say that a topological space $\mathfrak{M}$ *has a smooth manifold structure modeled on* $E$ if for every point $x \in \mathfrak{M}$ there is a neighborhood $U$ of $x$ and a homeomorphism $\theta$ mapping $U$ into $E$ such that if $x, x' \in \mathfrak{M}$ have overlapping neighborhoods $U, U'$ in which the homeomorphisms $\theta, \theta'$ are defined, then the map $\theta' \circ \theta^{-1}$ is a smooth transformation of the open set $\theta(U \cap U')$ of $E$ into $E$. The homeomorphisms $\theta$ are called *coordinate systems of* $\mathfrak{M}$ and their domains are called *coordinate patches*.

$\mathfrak{M}$ is locally compact if and only if $E$ is finite dimensional (Bourbaki [2, Ch. I, 2]), whence $\mathfrak{M}$ is a smooth manifold in the usual sense. If $\mathfrak{M}$ has a metric for which it is complete, then $\mathfrak{M}$ is of the second (Baire) category; in particular, $\mathfrak{M}$ is not expressible as a countable union of compact sets unless it is finite dimensional.

(B) We are able to give a standard definition[4] of the *tangent space* $T(\mathfrak{M}, x)$ *to* $\mathfrak{M}$ *at the point* $x$, defining a tangent vector as an equivalence class of smooth paths at $x$; $T(M, x)$ has a complete uniform structure induced from that of $E$. If $E$ is a Banach space, then $T(\mathfrak{M}, x)$ is normable.

(C) Tensor analysis can now be developed on $\mathfrak{M}$ via the theory of tensor bundles (Steenrod [11, §6]). It is convenient in the case of $r$-contravariant tensors to take

[4] Apparently the functional definition of tangent vector as given by Chevalley [4, p. 76] is not applicable in infinite dimensions.

as fibre the $r$-fold *projective tensor product* of $E$ (Grothendieck [7]); the fibre in the $r$-covariant case is then the conjugate space with conjugate topology.

## 3. The manifold $\mathfrak{C}(S, M)$

We give here examples of function spaces which are manifolds modeled on Banach spaces.

(A) Let $M$ denote a smooth Riemannian $n$-dimensional manifold, and $\langle,\rangle_m$ the associated inner product in the tangent space $T(M, m)$. We denote by $\rho$ the metric on $M$, giving the distance between two points $m, p \in M$ as the infimum of the lengths of the rectifiable paths joining $m$ and $p$. If $g: (I, O, 1) \to (M, m, p)$ is a (shortest) geodesic from $m$ to $p$, then $g$ determines the vector $u = d/dt[g(t)]_{t=0}$ in $T(M, m)$, and $\rho(m, p) = |\langle u, u\rangle_m|^{1/2} = |u|_m$. Set $u = \psi_m(p)$; then the inverse function (with $m$ fixed) $p = \phi_m(u)$ exists in a suitable neighborhood of $O \in T(M, m)$. It is a basic result in the theory of differential equations that $\psi_m(p)$ and $\phi_m(u)$ are smooth functions of their arguments.

(B) Let $S$ be a compact topological space, and let $\mathfrak{C} = \mathfrak{C}(S, M)$ be the totality of continuous maps of $S$ into $M$. The metric $\rho$ on $M$ induces a metric on $\mathfrak{C}$ by

$$|x, y| = \sup \{\rho(x(s), y(s)) : s \in S\}.$$

THEOREM. *The function space $\mathfrak{C}(S, M)$ is a smooth manifold locally homeomorphic to a real Banach space (of infinite dimension, in general).*

PROOF. Given any map $x \in \mathfrak{C}(S, M)$ the *tangent space to $\mathfrak{C}$ at $x$* is the totality $T(\mathfrak{C}, x)$ of maps (the "continuous variations of $x$") $u \in \mathfrak{C}(S, \mathscr{T}(M))$ such that $\pi \circ u(s) = x(s)$ for all $s \in S$, where $\mathscr{T}(M)$ is the tangent bundle over $\mathfrak{M}$ with projection $\pi$. Clearly $T(\mathfrak{C}, x)$ is a real linear space, complete with respect to the norm $|u|_x = \sup \{|u(s)|_{x(s)} : s \in S\}$.

Every point $x \in \mathfrak{C}$ has a neighborhood mapped isometrically onto a neighborhood of $O \in T(\mathfrak{C}, x)$. For since $x(S)$ is a compact subset of $M$, there is a number $\lambda_x > 0$ such that for every $s \in S$ the set $\{m \in M : \rho(m, x(s)) < \lambda_x\}$ is a normal coordinate patch at $x(s)$. Thus given $y \in \mathfrak{C}$ such that $|x, y| < \lambda_x$ there is for each $s \in S$ one and only one shortest geodesic (of length $<\lambda_x$) joining $x(s)$ and $y(s)$; then $u(s) = \psi_{x(s)}(y(s))$ is in $T(M, x(s))$, and $u \in T(\mathfrak{C}, x)$; furthermore, $|u|_x = |x, y|$. Conversely, given $u \in T(\mathfrak{C}, x)$ satisfying $|u|_x < \lambda_x$ we define $y(s) = \phi_{x(s)}(u(s))$; then $y \in \mathfrak{C}$, and $|x, y| < \lambda_x$.

We set $u = \psi_x(y)$ and will call $\psi_x$ a *coordinate system of $\mathfrak{C}$ at $x$*; the metric ball $U_x = \{y \in \mathfrak{C}: |x, y| < \lambda_x\}$ is called a *coordinate patch at $x$*. If $U_x$ and $U_{x'}$ are overlapping coordinate patches at the points $x, x' \in \mathfrak{C}$, then the coordinate transformation

$$\psi = \psi_{x'} \circ \psi_x^{-1} : \psi_x(U_x \cap U_{x'}) \to \psi_{x'}(U_x \cap U_{x'}) \tag{1}$$

is a smooth map of an open set of $T(\mathfrak{C}, x)$ into $T(\mathfrak{C}, x')$.

First we will show that for any point $u \in \psi_x(U_x \cap U_{x'})$ and vector $v \in T(\mathfrak{C}, x)$ the directional derivative $\nabla\psi(u; v)$ exists as an element of $T(\mathfrak{C}, x')$. Take a positive number $h_0$ such that if $h \in H = \{h \in R : 0 \leq h \leq h_0\}$, then $u + hv \in \psi_x(U_x \cap U_{x'})$.

We next define the function $F$ on $H \times S$ by

$$F(h, s) = \psi_{x'(s)} [\psi^{-1}_{x(s)}(u(s) + hv(s))],$$

and set $f(h, s) = \partial F(h, s)/\partial h$.

The difference quotient $[F(h, s) - F(o, s)]/h$ converges to $f(o, s)$ uniformly on $S$ as $h \to 0+$; for, given $\varepsilon > 0$ we can choose $h(\varepsilon) \in H$ so that for each positive $h < h(\varepsilon)$ we have $|f(h, s) - f(o, s)| < \varepsilon$ for all $s \in S$. Using the theorem of mean value applied to $F$, for each $(h, s) \in H \times S$ there is a number $k_{hs}$ satisfying $0 < k_{hs} < h$ and $[F(h, s) - F(o, s)]/h = f(k_{hs}, s)$. Thus for all $h < h(\varepsilon)$ and $s \in S$ we have

$$\left|\frac{F(h,s) - F(o, s)}{h} - f(o, s)\right| < \varepsilon,$$

whence $\nabla\psi\,(u, v)$ exists, and $\nabla\,\psi\,(u, v)s = f(o, s)$. For each $u \in \psi_x(U_x \cap U_{x'})$ the derivative $\nabla\psi(u, v)$ is a linear function of $v$, continuous because

$$(2) \qquad |f(o, s)|_{x'(s)} \leqq |\psi_* \, u(s)| \, |v(s)|_{x(s)},$$

where the first norm on the right is that of the Banach space $\mathfrak{L}(T(M, x(s)), T(M, x'(s)))$; the inequality

$$|\nabla\,\psi\,(u, v)|_{x'} \leqq |\psi_* \, u| \, |v|_x$$

follows at once from (2). Similarly, the differential $\psi_*$ is a continuous map of $\psi_x(U_x \cap U_{x'})$ into $\mathfrak{L}[T(\mathfrak{C}, x), T(\mathfrak{C}, x')]$.

Thus we see that the coordinate transformation (1) is $C^1$ in its domain of definition; that it is $C^k$ for all $k$ follows by a standard induction argument, completing the proof of the theorem.

We remark that the norms $|\ \ |_x$ determine a natural Finsler structure in $\mathfrak{C}(S, M)$.

(C) 1. REMARKS. If $M$ is a Lie group with analytic Riemannian metric, then $\mathfrak{C}(S, M)$ is an *analytic group* (in the large) in the sense of G. Birkhoff [1], using the notion of analytic transformation defined in Hille [8].

2. If $S_0$ is a closed subspace of $S$ and $M_0$ is a closed Riemannian submanifold of $M$, then the function space of continuous maps $(S, S_0) \to (M, M_0)$ is a closed submanifold of $\mathfrak{C}(S, M)$. The (Serre) fibre space of continuous paths on $M$ with fixed endpoint is a *differentiable fibre space* (not locally trivial, in general).

3. We can modify the above examples by considering smooth maps of a compact smooth manifold $S$ into $M$. For example, let $\mathfrak{D}(I, M)$ be the function space of smooth maps of the unit interval into $M$, with uniform structure given by the increasing sequence of metrics

$$|x, y|^k = \max \{|x, y|, |x^{(1)}, y^{(1)}|, \cdots, |x^{(k)}, y^{(k)}|\}$$

for any $x, y \in \mathfrak{D}(I, M)$, where $x^{(i)}(s) = d^i x(s)/ds^i$. It follows as in (B) that $\mathfrak{D}(I, M)$ is an infinite dimensional manifold modeled on a Fréchet space.

## 4. On the cohomology ring of certain function spaces

We give here an extension of the Theorem of de Rham.

(A) As a tensor field (see §2C) on a manifold $\mathfrak{M}$ a smooth differential $r$-form $\omega$

can be defined as a section of an appropriate tensor bundle; thus $\omega$ associates to a point $x \in \mathfrak{M}$ an alternating $r$-linear real function on $T(\mathfrak{M}, x)$ which is *simultaneously continuous* in its arguments. The exterior differential $d\omega$ is defined as in the finite dimensional case, and its usual properties are valid; the relation $dd\omega = 0$ requires use of the symmetry of the iterated directional derivative $\nabla\omega\ (x; v_1, v_2)$ in the model $E$ (Theorem of Graves [5]), and in particular uses the *completeness* of $T(\mathfrak{M}, x)$. The Formula of Stokes and the "Poincaré Lemma" carry over to infinite dimensions without change.

(B) THEOREM. *Let $\mathfrak{M}$ be a paracompact smooth manifold modeled on a linear space $E$ satisfying Condition* (S) *of* §1. *Then the real singular cohomology algebra $H^*(\mathfrak{M}, R)$ (with cup product) is algebra isomorphic with the derived cohomology of the exterior algebra of smooth differential forms on $\mathfrak{M}$.*

The proof follows a standard method in the theory of sheaves (see, for example, H. Cartan [3]); the Condition (S) is used to insure that the sheaf of germs of smooth forms on $\mathfrak{M}$ is fine.

REMARK. Using methods of H. Whitney [12, Part II] we can establish a version of de Rahm's Theorem for any smooth manifold modeled on a Banach space.

(C) As an application of this theorem, let us consider the following function space (used in the calculus of variations), resuming the notation of §3. Let $\mathfrak{P} = \mathfrak{P}(I, 0; M, m)$ be the totality of absolutely continuous maps $x : (I, 0) \rightarrow (M, m)$ such that the tangent vector field $x^{(1)}(s) = dx(s)/ds$ is square integrable on the tangent bundle $\mathscr{T}(M)$. The *tangent space* $T(\mathfrak{P}, x)$ *to* $\mathfrak{P}$ *at* $x$ *is* $\{u \in \mathfrak{P}(I, 0\colon \mathscr{T}(M), 0)\colon \pi \circ u(s) = x(s)$ for all $s \in I\}$; then $T(\mathfrak{P}, x)$ is a separable real Hilbert space with inner product

$$(1) \qquad (u, v)_x = \int_0^1 < u^{(1)}(s), v^{(1)}(s) > ds,$$

using the Riemannian structure of the tangent bundles of $M$.

$\mathfrak{P}$ is a *smooth infinite dimensional manifold;* in fact, $\mathfrak{P}$ has a natural Riemannian structure with metric tensor induced from (1). The space $\mathfrak{P}_0 = \{x \in \mathfrak{P}\colon\ x(1) = m\}$ of loops on $M$ based at $m$ is a closed submanifold of $\mathfrak{P}$. The theorem in (B) is applicable to the loop space $\mathfrak{P}_0$ and shows that the real cohomology algebra of $\mathfrak{P}_0$ is isomorphic to its differential form cohomology algebra.

REMARK. In our examples we have supposed—for simplicity in handling normal coordinates—that the range manifold $M$ is finite dimensional. That assumption is inessential, for it is known (Michal-Hyers [9]) that for any Riemannian manifold (such as $\mathfrak{P}$ or $\mathfrak{P}_0$) every point has a neighborhood in which a kind of normal coordinate can be introduced. In particular, we can apply our construction to the space of loops on $\mathfrak{P}_0$.

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# ON THE EXACT COHOMOLOGY SEQUENCE OF A SPACE WITH COEFFICIENTS IN A NONABELIAN SHEAF

By Paul Dedecker

The object of this communication is to give an account of the relation between the classification problem of principal fibre bundles and the cohomology of the base with coefficients in a suitable sheaf (generally nonabelian.) The first part of the material covered here has been published elsewhere so that details and proofs are omitted. In the second part (from §6 on) we discuss two dimensional cohomology in the line of cohomology *groupoids* described in the beginning. This improves a first approach [2] to the matter; for instance the two dimensional classes now become equivalence classes of cocycles. However, the definition of these is perhaps not yet the definitive one; they do not coincide with *all* the classical cocycles in the abelian case.

The bundles of groups and the "new" principal bundles described here are precisely those coming in the lectures of H. Cartan and S. Eilenberg.

## §1. Introduction

It has been recognized by several authors [1], [5], [7] that the various classes of principal fibre bundles over a base $B$ with a (topological) structure group $G$ are in one-one correspondance with the *set* $H^1(B, \mathscr{G})$ of the cohomology classes of $B$ with coefficients in the sheaf $\mathscr{G}$ of local jets (or germs) of continuous maps of $B$ in $G$. In the general case this set seems to have little algebraic structure but possesses a privileged (or neuter) element $e_0$ corresponding to the trivial bundle $B \times G$. If, however, $G$ is abelian it turns out that it coincides with the usual cohomology *group*.

The question therefore arises to describe a general property of $H^1(B, \mathscr{G})$ which reduces to the group property in the abelian case. The answer to this has been given in [3]. *There exists a groupoid $\mathscr{H}^1(B, \mathscr{G})$ in which $H^1(B, \mathscr{G})$ can be naturally imbedded with the following properties*: (a) *the neuter element* $(e_0)$ *is a unit of the groupoid and actually the right unit of all the elements of* $H^1(B, \mathscr{G})$; (b) *any element of* $\mathscr{H}^1(B, \mathscr{G})$ *the right unit of which is* $e_0$ *belongs to* $H^1(B, \mathscr{G})$; (c) *every unit of the groupoid is left unit to an element in* $H^1(B, \mathscr{G})$.

Note that by *groupoid* we understand essentially a Brandt groupoid, i.e., a set $\Phi$ with a not everywhere defined associative multiplication satisfying the left and right cancellation property and in which, moreover, every element has a left and right unit as well as a left inverse. These turn out to be unique and the left inverse is at the same time the unique right inverse. The groupoid $\mathscr{H}^1(B, \mathscr{G})$ satisfies as a consequence of (a) and (c) the *connectedness* condition: every pair of units is the pair of left and right units of some element. Moreover properties (a), (b), and (c) together imply that every element of the groupoid is of the form $E' \otimes E^{-1}$ with

$E, E' \in H^1(B, \mathscr{G})$. Putting $E' = e_0$, we get a meaning (at least abstract) to the operation: take the inverse of a principal bundle $E$ over $B$ with structure group $G$.

## §2. Geometric description of $\mathscr{H}^1(B, \mathscr{G})$

Let $E = E(B, G)$ be any principal bundle (in the usual sense) over $B$ with structure group $G$. Now $G$ acts onto itself from the left by means of interior automorphisms: this induces an associated fibre bundle[1] $\Gamma = \Gamma(B, G, G_\iota, E)$ the fibres $\Gamma_x (x \in B)$ of which actually have a group structure isomorphic (but not canonically) to $G$; we express this fact saying that $\Gamma$ is a *bundle of G-groups*, more precisely, the bundle of $G$-groups *associated* to $E$. It is well-known that $G$ acts from the right on each fibre $E_x$ in $E$. Now we remark that $\Gamma_x$ as a group acts in a canonical way on $E_x$ from the left.[2] We refer briefly to this property saying that $\Gamma$ acts on $E$ from the left; in a similar sense we may say that the trivial group bundle $B \times G$ acts on $E$ from the right.

Now let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of $B$ together with continuous sections $\gamma_{ij} : U_{ij} = U_i \cap U_j \to \Gamma$ satisfying the conditions

$$\gamma_{ij}(x) = \gamma_{ji}^{-1}(x), \qquad x \in U_{ij}$$

$$\gamma_{ij}(x) \cdot \gamma_{jk}(x) \cdot \gamma_{ki}(x) = \varepsilon_x,$$

$$x \in U_{ijk} = U_{ij} \cap U_k, \qquad \varepsilon_x \text{ identity of } \Gamma_x.$$

Let $\pi : \Gamma \to B$ be the projection of $\Gamma$ and $X_i$ the inverse image of $U_i$. In the sum $X = \sum_{i \in I} X_i$ consider the equivalence relation $\rho$

$$\text{(2.1)} \qquad \gamma_i \sim \gamma_j \Leftrightarrow \pi(\gamma_i) = \pi(\gamma_j) = x \in U_{ij}, \gamma_i = \gamma_{ij}(x) \cdot \gamma_j.$$

Then the quotient $E' = X/\rho$ is a topological space with a natural projection $p' : E' \to B$ on each fibre $E'_x = p'^{-1}(x)$ of which the fibre $\Gamma_x$ acts in a natural way from the right.

For any triple $(e'_x, e_x, \gamma_x) \in (E'_x \times E_x \times \Gamma_x)$ we may therefore consider the products $e'_x\gamma_x \in E'_x$ and $\gamma_x e_x \in E_x$. Then in $E' \times {}_B E = \sum_{x \in B} E'_x \times E_x$ we have the equivalence relation $(e'_x\gamma_x, e_x) \sim (e'_x, \gamma_x e_x)$ the quotient of which can be denoted by $E' \otimes E$. This is a topological space with a natural projection onto $B$ and on which $G$ acts in a natural way from the right. It possesses actually a structure of a (classical) principal bundle over $B$ with structure group $G$. It defines therefore an associated group bundle $\Gamma'$ acting on it from the left.

DEFINITION. The space $E'$ together with the projection $p'$ and the group bundle $\Gamma$ acting from the right is called a *principal bundle over B relative to* $\Gamma$, *or briefly a* $\Gamma$-*principal bundle.*

[1] We use the notation of Ehresmann in which $G_\iota$ stands for $G$ acting onto itself by interior automorphisms.

[2] See [4]. Taking the natural onto-map $E \times G \to \Gamma$, this is induced by the operation $[(e_x, g_x), e'_x] \to e_x g e_x^{-1} e'_x$ of $E_x \times G$ on $E_x$.

If instead of the equivalence (2.1) in $X$ we use the one for which $\gamma_i = \gamma_{ij}(x) \cdot \gamma_j \cdot \gamma_{ji}(x)$, the quotient is a group bundle acting on $E'$ from the left and which turns out to be isomorphic to $\Gamma'$; moreover, the way in which it acts on $E' \otimes E$ follows from the natural way it acts on $E' \times_B E$ and the map $E' \times_B E \to E' \otimes E$.

Let $\mathscr{E}$ be the sheaf of jets of local sections of $\Gamma$. Then the classes of $\Gamma$-principal bundles are in one-one correspondance with the 1-dimensional cohomology *set* $H^1(B, \mathscr{E})$, the neuter element of which corresponds to the *trivial* $\Gamma$-principal bundle isomorphic to $\Gamma$ (defined by $\gamma_{ij}(x) = \varepsilon_x$).

By definition, $\mathscr{H}^1(B, \mathscr{G})$ as a set is the union of all the $H^1(B, \mathscr{E})$ for all possible choices of $\Gamma$ defined as above by means of some $E \in H^1(B, \mathscr{G})$.

## §3. Products in $\mathscr{H}^1(B,\mathscr{G})$. Interpretation of the inverse

Every $E \in \mathscr{H}^1(B, \mathscr{G})$ will be identified to a corresponding principal bundle relative to a certain $\Gamma$; it therefore corresponds to two group bundles $\Gamma$, $\Gamma'$ acting one from the right and the other from the left. We write

$$E : \Gamma \to \Gamma'.$$

Now if we have another element $E'$ such that $E' : \Gamma' \to \Gamma''$ we define $E' \otimes E$ as the quotient of $\sum_{x \in B} E'_x \times E_x$ by the equivalence relation $(e'_x\gamma'_x, e_x) \sim (e'_x, \gamma'_x e_x)$ for $(e'_x, e_x, \gamma'_x)\ E'_x \times E_x \times \Gamma'_x$. One proves:

THEOREM 3.1 *The not everywhere defined product $E' \otimes E$ makes $\mathscr{H}^1(B, \mathscr{G})$ to be a groupoid, the units of which can be identified with the group bundles $\Gamma$.*

From the construction it is obvious that this groupoid satisfies the conditions (a), (b), and (c) of §1.

We now give a geometric interpretation of the inverse $E^{-1}$ of an element $E \in \mathscr{H}^1(B, \mathscr{G})$; in particular we will get a picture for $E \in H^1(B, \mathscr{G})$ of the inverse of a classical principal bundle.

In the above considerations, the left plays a preferred role since for instance in (2.1) the $\gamma_{ij}$ are used as left multipliers. For this reason the (relative) bundles above could be called *left-bundles* and we might define *right-bundles* in a symmetric way, using $\gamma_i = \gamma_j \cdot \gamma_{ji}(x)$ in (2.1). On such bundles, the structure group bundle acts from the left. This leads to the following:

PROPOSITION 3.2. *There exists a one-one correspondence between the left and right bundles relative to a given $\Gamma$.*

PROPOSITION 3.3. *If $E : \Gamma \to \Gamma'$ is a left $\Gamma$-bundle, then the operations of $\Gamma'$ define on $E$ a structure of right $\Gamma'$-bundle.*

To obtain the inverse $E^{-1}$ of $E$, we first consider on it the structure $E'$ of right $\Gamma'$-bundle deduced from Proposition 3.3. then we consider the left $\Gamma'$-bundle $E''$ corresponding to $E'$ by Proposition 3.2; this is precisely $E^{-1}$. Therefore the inverse of a classical bundle $E$ is only classical in case its associated group bundle is isomorphic to $B \times G$. All such classical bundles form a subgroup of the groupoid contained in $H^1(B, \mathscr{G})$. Of course when $G$ is abelian $\Gamma$ is always trivial and this subgroup coincides with the whole set $H^1(B, \mathscr{G})$ as well as with $\mathscr{H}^1(B, \mathscr{G})$.

## §4. Homomorphisms of structure groups

The preceding theory can be made slightly more general if we use (instead of $G\iota$) a topological group $A$ acting continuously on $G$ by means of continuous homomorphisms. Then any principal bundle (in the classical sense) over $B$ with structure group $A$ defines an associated bundle $\Gamma$ with fibres isomorphic to $G$ which we will call a generalized bundle of $G$-groups (with respect to $A$). We can define principal bundles relative to these new $\Gamma$'s and this gives rise to a larger groupoid $\mathscr{H}^1_A(B, \mathscr{G})$.

For instance if $G$ is a normal subgroup of $H$ we may take $A = H$ acting on $G$ by interior automorphisms. Then in a bundle of $H$-groups each fibre contains canonically a subgroup isomorphic to $G$ and these subgroups span a generalized bundle of $G$-groups with respect to $H$. However, two different bundles of $H$-groups may induce isomorphic bundles of $G$-groups[3] and this makes it impossible to define a canonical map

$$\mathscr{H}^1_H(B, \mathscr{G}) \to \mathscr{H}^1(B, \mathscr{H})$$

which we would expect.

Let $\Gamma^H$ be a bundle of $H$-groups and $\Gamma^G$ the bundle of $G$-groups it defines. Every principal bundle $E^G$ relative to $\Gamma^G$ induces naturally a principal bundle $E^H$ owing to the inclusion map $\Gamma^G \to \Gamma^H$. This way we have a map which associates $E^H$ to the pair $(E^G, \Gamma^H)$, furthermore these pairs are easily made a groupoid $\mathscr{H}^1_\Sigma(B, \mathscr{G})$ and we have a map

$$i^1 : \mathscr{H}^1_\Sigma(B, \mathscr{G}) \to \mathscr{H}^1(B, \mathscr{H})$$

which is actually a homomorphism of groupoids[4].

If we set $K = H/G$ and if we assume that $G$ has a local section $H$ we get an exact sequence of sheaves

$$(\Sigma) \qquad e \to \mathscr{G} \xrightarrow{i} \mathscr{H} \xrightarrow{j} \mathscr{K} \to e.$$

Also every bundle of $H$-groups $\Gamma^H$ defines a bundle of $K$-groups $\Gamma^K = \Gamma^H/\Gamma^G$ and we easily obtain a homomorphism

$$j^1 : \mathscr{H}^1(B, \mathscr{H}) \to \mathscr{H}^1(B, \mathscr{K}).$$

PROPOSITION 4.1. *The image of $i^1$ coincides with the kernel (inverse image of the set of units of $j^1$).*

In other words the sequence

$$\mathscr{H}^1_\Sigma(B, \mathscr{G}) \xrightarrow{i^1} \mathscr{H}^1(B, \mathscr{H}) \xrightarrow{j^1} \mathscr{H}^1(B, \mathscr{K})$$

is *exact.*

## §5. 0-dimensional cohomology groupoids

These will be associated to the exact sequence $(\Sigma)$.

[3] We will no longer say "generalised bundles" but simply "bundles".

[4] This means: if $x \otimes y$ is defined, so is $i(x) \otimes i(y)$ and $i(x \otimes y) = i(x) \otimes i(y)$.

To every $\Gamma^H$ we associate the corresponding $\Gamma^G$ and $\Gamma^K$. This way we define three sheaves of local sections $\mathscr{E}^H$, $\mathscr{E}^G$ and $\mathscr{E}^K$ and we consider their global sections $\Gamma(B, \mathscr{E}^H)$, $\Gamma(B, \mathscr{E}^G)$ and $\Gamma(B, \mathscr{E}^K)$. The sums of these sets (for all possible choices of units in the groupoid $\mathscr{H}^1_\Sigma(B, G)$) are denoted respectively by $\mathscr{H}^0(B, \mathscr{H})$, $\mathscr{H}^0_\Sigma(B, \mathscr{G})$, $\mathscr{H}^0_\Sigma(B, \mathscr{K})$; we also introduce an object $\mathfrak{e}$ in one-one correspondence with the units of $\mathscr{H}^1_\Sigma(B, G)$

The $\Gamma(B, \mathscr{E}^G)$ and $\Gamma(B, \mathscr{E}^H)$ are naturally groups and this defines an obvious structure of groupoids on their sums $\mathscr{H}^0_\Sigma(B, \mathscr{G})$ and $\mathscr{H}^0(B, \mathscr{H})$. Moreover, there is an injection

$$\mathscr{H}^0_\Sigma(B, \mathscr{G}) \xrightarrow{i_0} \mathscr{H}^0(B, \mathscr{H}).$$

We have a similar structure on $\mathscr{H}^0_\Sigma(B, \mathscr{K})$ but it is an uninteresting one, at least for the present purpose. Let $k : B \to \mathscr{E}^K$ be an element of $\Gamma(B, \mathscr{E}^K) \subset \mathscr{H}^0_\Sigma(B, \mathscr{K})$ and choose an open covering $\mathfrak{U} = (U_i)_{i \in I}$ of $B$ such that there exist sections $h_i : U_i \to \Gamma^H$ such that $j \circ h = k|U_i$. The $h_i h_j^{-1} : U_{ij} \to \Gamma^H$ are in the image of $i$, namely there exist sections $g_{ij} : U_{ij} \to \Gamma^G$ such that $i \circ g_{ij} = h_i h_j^{-1}$. The $g_{ij}$ define a bundle $E$ relative to $\Gamma^G$ and an associated bundle of $G$-groups $\Gamma'^G$ as well as an associated bundle of $H$-groups $\Gamma'^H$ both of structure bundle $\Gamma^G(\Gamma'^G \subset \Gamma'^H)$. (Note that $\Gamma^G$ and $\Gamma'^G$ (also $\Gamma^H$ and $\Gamma'^H$) become isomorphic if given $\Gamma^H$ as structure bundle.)

Now $\mathscr{H}^0_\Sigma(B, \mathscr{K})$ is made a groupoid as follows. A product $k' \otimes k$ of sections $k : B \to \mathscr{E}^K$, $k' : B \to \mathscr{E}'^K$ is defined if and only if $\mathscr{E}'^K$ is the sheaf of sections of $\Gamma'^K$, the quotient of the $\Gamma'^H$ described above by the $\Gamma'^G$ contained in it. In this case there exists a natural isomorphism $\tau : \mathscr{E}'^K \to \mathscr{E}^K$ and by definition $k' \otimes k = \tau k' \cdot k$ where $\cdot$ denotes the multiplication of sections of the sheaf of groups $\mathscr{E}^K$.

The above construction has also exhibited a map

$$\delta^0 : \mathscr{H}^0_\Sigma(B, \mathscr{K}) \to \mathscr{H}^1_\Sigma(B, \mathscr{G})$$

which proves to be homomorphic. There is also a homomorphism

$$j^0 : \mathscr{H}^0(B, \mathscr{H}) \to \mathscr{H}^0_\Sigma(B, \mathscr{K}).$$

THEOREM 5.1. *The following sequence of groupoids and homomorphisms is exact:*

$$\begin{aligned}\mathfrak{e} \to \mathscr{H}^0_\Sigma(B, \mathscr{G}) &\xrightarrow{i^0} \mathscr{H}^0(B, \mathscr{H}) \xrightarrow{j^0} \mathscr{H}^0_\Sigma(B, \mathscr{K}) \xrightarrow{\delta^0} \mathscr{H}^1_\Sigma(B, \mathscr{G}) \\ &\xrightarrow{i^1} \mathscr{H}^1(B, \mathscr{H}) \xrightarrow{j^1} \mathscr{H}^1(B, \mathscr{K}).\end{aligned} \tag{5.1}$$

It is to be observed that some of the homomorphisms of this sequence are *perfect*.[5] This sequence has therefore a similar utility as an ordinary exact sequence of groups.

[5] We say that a groupoid homomorphism $h: \Phi \to \Phi'$ is perfect if for any pair of units $e_1, e_2 \in \Phi$ such that $h(e_1) = h(e_2)$ there exists an element $x \in \Phi$ with $e_1$ and $e_2$ as left and right units such that $h(x) = h(e_1) = h(e_2)$. Then two elements $x, y \in \Phi$ have the same image under $h$ if and only if they are in the relation $n = myn$ for suitable $m, n \in \operatorname{Ker} h$; actually $m$(or $n$) can be chosen arbitrary in Ker $h$ provided $x^{-1} my$ is defined (or $ynx^{-1}$). In case of a non-perfect homomorphism such a criterium exists only if $x$ and $y$ have the same right (or left) unit: then $h(x) = h(y)$ if and only if there exists $m \in \operatorname{Ker} h$ such that $x = my$(or $n \in \operatorname{Ker} h$ such that $x = yn$).

Remark. Previously one considered cohomology sets with a privileged element giving rise to an exact sequence of sets

$$e \to H^0(B, \mathscr{G}) \to H^0(B, \mathscr{H}) \to H^0(B, \mathscr{K}) \overset{\delta_0^0}{\to} H^1(B, \mathscr{G}) \to H^1(B, \mathscr{H}) \to H^1(B, \mathscr{K}).$$

Actually these sets were groups in dimension zero and so was $H^1(B, \mathscr{G})$ if $G$ abelian. As observed by A. Grothendieck $\delta^0$ was not in that case a homomorphism but in a sense a "crossed homomorphism"; this property is generalized here by giving $\mathscr{H}^0_\Sigma(B, \mathscr{K})$ a more elaborate groupoid structure than $\mathscr{H}^0_\Sigma(B, \mathscr{G})$ and $\mathscr{H}^0(B, \mathscr{H})$ so that $\delta^0$ becomes a groupoid homomorphism.

## §6. Non holonomic bundles of groups

We keep the notations of the preceding section and consider a $\Gamma^H$, the corresponding $\Gamma^G$ imbedded in it and $\Gamma^K = \Gamma^H/\Gamma^G$. For each $x \in B$. we have the fibre $\Gamma^H_x$ and its group $I(\Gamma^H_x)$ of interior automorphisms. These span a set $I(\Gamma^H)$ with a natural projection onto $B$ (which allows us to speak of sections of $I(\Gamma^H)$ over a subset of $B$). A continuous section $h : U \to \Gamma^H$ induces a section $\tau h : U \to I(\Gamma^H)$ such that for $x \in U$, $\tau h(x)$ is the interior automorphism of $\Gamma^H_x$ associated to $h(x)$. Such a section of $I(\Gamma^H)$ will be termed *admissible*. In a similar way we define $I(\Gamma^G)$ and $I(\Gamma^K)$; in $I(\Gamma^H)$. The subset induced by interior automorphisms of elements in $\Gamma^G$ will be denoted $J(\Gamma^G)$; we have a natural epimorphism $J(\Gamma^G) \to I(\Gamma^G)$.

Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of $B$ and consider a family $\alpha$ of admissible sections $\alpha_{ij} : U_{ij} \to I(\Gamma^H)$ such that

$$\alpha_{ii} = \text{unit section}, \ \alpha_{ij} = \alpha_{ji}^{-1}, \tag{6.1}$$

$$\alpha_{ij} \cdot \alpha_{jk} \cdot \alpha_{ki} = \text{section of } J(\Gamma^G) \text{ over } U_{ijk}. \tag{6.2}$$

The first conditions mean that $\alpha$ is an alternate cochain and the second that when followed by the map $I(\Gamma^H) \to I(\Gamma^K)$ it becomes a cocycle. We say that $\alpha$ is a *non holonomic bundle of groups* relative to $\mathfrak{U}$, $\Gamma^H$ and $G$ (briefly non holonomic $\mathfrak{U}$-$\Gamma^H$-$G$-bundle of groups). Another non holonomic bundle of this type $\alpha' = (\alpha'_{ij})$ is said to be *equivalent* to $\alpha$ if there exists a family $\eta = (\eta_i)$ of admissible sections $\eta_i = U_i \to I(\Gamma^H)$ such that

$$\alpha'_{ij} \equiv \eta_i \alpha_{ij} \eta_j^{-1} \pmod{J(\Gamma^G)}.$$

Then the corresponding cocyles in $I(\Gamma^K)$ are cohomologous. If we pass to a finer covering $\mathfrak{V} = (V_r)_{r \in R}$ any map $\omega$: $R \to I$ such that $V_r \subset U_{\omega r}$ associates to $\alpha$ a $\mathfrak{V}$-bundle $\omega\alpha$ and another map $\bar{\omega}$: $R \to I$ defines $\bar{\omega}\alpha$ which is equivalent to $\omega\alpha$. We then apply the usual direct limit process which leads to a set $\mathbf{F}(B, \Gamma^H, G)$, the objects of which we call *non holonomic bundles of groups* over $B$ and relative to $\Gamma^H$ and $G$.

This set is obviously in one-one correspondence with the units of $\mathscr{H}^1(B, \mathscr{K})$, at least when $B$ is paracompact. We could also start from another $\Gamma'^H$ and get a set $\mathbf{F}(B, \Gamma'^H, G)$ in one-one correspondence with $\mathbf{F}(B, \Gamma^H, G)$ ($B$ paracompact or not). We could therefore choose $\Gamma^H = B \times H$ and denote the corresponding $\mathbf{F}$ by $\mathbf{F}(B, B \times H, G)$ or $\mathbf{F}(B, \mathscr{H}, \mathscr{G})$; for this choice $\Gamma^G = B \times G$, $\Gamma^K = B \times K$, $\mathscr{E}^H = \mathscr{H}$, $\mathscr{E}^G = \mathscr{G}$ and $\mathscr{E}^K = \mathscr{K}$.

## §7. Non holonomic principal bundles

Returning to the covering $\mathfrak{U}$ we consider pairs $(h, \alpha)$ having the following properties:

(a) $\alpha = (\alpha_{ij})$ is a non holonomic bundle of groups relative to $\mathfrak{U}$, $\Gamma^H$ and $G$;

(b) $h = (h_{ij})$ represents a one-dimensional cochain of $\mathfrak{U}$ in $\mathscr{E}^H$ (i.e., it is defined by continuous sections $h_{ij} : U_{ij} \to \Gamma^H$) such that:

$$h_{ii} = \text{unit section over } U_i,\ h_{ij} = \alpha_{ij}(h_{ji}^{-1}), \tag{7.1}$$

$$h^i_{ij}\, h^{ij}_{jk}\, h^{ijk}_{ki} \text{ is a section of } \Gamma^G \text{ over } U_{ijk} \tag{7.2}$$

$$h^i_{ij} = h_{ij}, \qquad h^{ij}_{jk} = \alpha_{ij}(h_{jk}), \qquad h^{ijk}_{ki} = \alpha_{ij} \circ \alpha_{jk}(h_{ki}).$$

These conditions parallel (6.1) and (6.2); once (a) is satisfied, (b) implies that the 1-cochain $\tau h \cdot \alpha = (\tau h_{ij} \cdot \alpha_{ij})$ is still a non holonomic bundle of groups relative to $\mathfrak{U}$, $\Gamma^H$ and $G$.

Lemma, *If for* $h' = (h'_{ij})$, $h = (h_{ij})$ *we set* $h' \cdot h = (h'_{ij} \cdot h_{ij})$ *and if furthermore* $(h', \alpha')$ *and* $(h, \alpha)$ *are pairs satisfying the conditions* (a), (b) *above and* $\alpha' = \tau h \cdot \alpha$, *then the pair* $(h' \cdot h, \alpha)$ *also satisfies* (a) *and* (b).

First we have

$$\begin{aligned} h'_{ij}h_{ij} &= \alpha'_{ij}(h'^{-1}_{ji})\, \alpha_{ij}(h_{ji}^{-1}) = h_{ij} \cdot \alpha_{ij}(h'^{-1}_{ji}) \cdot h_{ij}^{-1} \cdot \alpha_{ij}(h_{ji}^{-1}) \\ &= \alpha_{ij}(h_{ji}^{-1}) \cdot \alpha_{ij}(h'^{-1}_{ji}) \cdot \alpha_{ij}(h_{ji}) \cdot \alpha_{ij}(h_{ji}^{-1}) \\ &= \alpha_{ij}[(h'_{ji} \cdot h_{ji})^{-1}]. \end{aligned}$$

So (7.1) is satisfied. Next for a pair $(h, \alpha)$ we set

$$(\Delta_\alpha h)_{ijk} = (h^i_{ij} \cdot h^{ij}_{jk} \cdot h^{ijk}_{ki}). \tag{7.3}$$

To prove that (7.2) is satisfied it is sufficient to check the formula

$$\Delta_\alpha(h' \cdot h) = \Delta_{\tau h \cdot \alpha}\, h' \cdot \Delta_\alpha h. \tag{7.4}$$

This goes as follows

$$(\Delta_\alpha(h' \cdot h))_{ijk} = h'_{ij}\, h_{ij} \cdot \alpha_{ij}(h'_{jk}\, h_{jk}) \cdot \alpha_{ij}\alpha_{jk}(h'_{ki}\, h_{ki}).$$

This is equal to

$$\begin{aligned} & h'_{ij} \cdot [h_{ij}\alpha_{ij}(h'_{jk})h_{ij}^{-1}] \cdot [h_{ij}\alpha_{ij}(h_{jk}) \cdot \alpha_{ij}\alpha_{jk}(h'_{ki}) \cdot \alpha_{ij}(h_{jk}^{-1})h_{ij}^{-1}] \cdot h_{ij}\alpha_{ij}(h_{jk})\alpha_{ij}\alpha_{jk}(h_{ki}) \\ &= [h'_{ij} \cdot (\tau h_{ij}\alpha_{ij})(h'_{jk}) \cdot (\tau h_{ij}\alpha_{ij})(\tau h_{jk} \cdot \alpha_{jk})(h'_{ki})]\, h^i_{ij}h^{ij}_{jk}h^{ijk}_{ki} \\ &= [h'^i_{ij} \cdot h'^{ij}_{jk} \cdot h'^{ijk}_{ki}] \cdot [h^i_{ij} \cdot h^{ij}_{jk} \cdot h^{ijk}_{ki}] = (\Delta_{\tau h.\alpha}\, h')_{ijk} \cdot (\Delta_\alpha\, h)_{ijk} \end{aligned}$$

c.q.f.d.

A pair $(h, \alpha)$ satisfying (a) and (b) is called a *non holonomic principal* bundle relative to $\mathfrak{U}$, $\alpha$ and $G$. If $\alpha$ is followed by the map $I(\Gamma^H) \to I(\Gamma^K)$ it defines a bundle of $F$ of $K$-groups. If we similarly compose $h$ with $\Gamma^H \to \Gamma^K$ it turns out that it defines a principal bundle relative to $\Gamma^k$ (see[4]). Let $\mathscr{Z}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G})$ denote[6] the set of these non holonomic principal bundles. The lemma renders it a groupoid if we decide:

$$(h', \alpha') \otimes (h, \alpha) \text{ is defined if } \alpha' = \tau h \cdot \alpha;$$

$$\text{it is then equal to } (h' \cdot h, \alpha).$$

Then the above map

$$\kappa_{\mathfrak{U}} : \mathscr{Z}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G}) \to \mathscr{H}^1(\mathfrak{U}, \mathscr{K})$$

becomes a homomorphism.

Two non holonomic bundles $(h, \alpha)$, $(h', \alpha')$ are said to be equivalent if (1)$\alpha$ and $\alpha'$ are equivalent, which means

$$\alpha'_{ij} \equiv \alpha_i \, \alpha_{ij} \, \alpha_j^{-1} \bmod J(\Gamma^G):$$

(2) if there exist sections $h_i : U \to \Gamma^H$ such that

$$h'_{ij} \equiv h_i \cdot \alpha_i(h_{ij}) \cdot (h^i_j)^{-1} \bmod \Gamma^G, \; h^i_j = \alpha'_{ij}(h_j).$$

Obviously, an equivalence class of non holonomic principal bundles is mapped onto a single element in $\mathscr{H}^1(\mathfrak{U}, \mathscr{K})$. The kernel $\mathscr{N}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G})$ of $\kappa_{\mathfrak{U}}$ is an invariant subgroupoid of $\mathscr{Z}^1$ and we denote the quotient $\mathscr{Z}^1/\mathscr{N}^1$ by $\mathscr{H}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G})$[8]. We have thus a perfectly exact sequence:

$$\mathbf{e} \to \mathscr{N}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G}) \to \mathscr{Z}^1 (\mathfrak{U}; \mathscr{H}, \mathscr{G}) \to \mathscr{H}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G}) \to \mathbf{e}'.$$

There exists a natural homomorphism

$$\kappa^*_{\mathfrak{U}} : \mathscr{H}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G}) \to \mathscr{H}^1(\mathfrak{U}, \mathscr{K}).$$

If we pass to a finer covering we get natural maps

$$\mathscr{H}^1(\mathfrak{U}, \mathscr{H}, \mathscr{G}) \to \mathscr{H}^1(\mathfrak{V}, \mathscr{H}, \mathscr{G})$$

and commutative diagrams

$$\begin{array}{ccc} \mathscr{H}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G}) & \to & \mathscr{H}^1(\mathfrak{U}, \mathscr{K}) \\ \downarrow & & \downarrow \\ \mathscr{H}^1(\mathfrak{V}; \mathscr{H}, \mathscr{G}) & \to & \mathscr{H}^1(\mathfrak{V}, \mathscr{K}). \end{array}$$

Let $\mathscr{H}^1(B; \mathscr{H}, \mathscr{G})$ denote the groupoid direct limit of the $\mathscr{H}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G})$. We then have a homomorphism

$$\kappa^* : \mathscr{H}^1(B; \mathscr{H}, \mathscr{G}) \to \mathscr{H}^1(B, \mathscr{K}).$$

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[6] *This notation and the coming ones suppose implicitly that the choice mentioned at the end* of §6 has been made, namely $\Gamma^H = B \times H$. This is the most natural one although any other is good; instead of $Z^1(\mathfrak{U}; \mathscr{H}, \mathscr{G})$ one should then better write $Z^1(\mathfrak{U}; \mathscr{E}^H, \mathscr{E}^G)$.

but it will not be injective if not perfect[7]; in fact, *if the kernel of a groupoid homomorphism is trivial* (i.e., consists only of units), *it is injective if and only if it is perfect.*

## §8. Two-dimensional cohomology

For a covering $\mathfrak{U}$ we consider triples $(\alpha', g, \alpha)$ satisfying the following conditions:

(a) $\alpha = (\alpha_{ij})$, $\alpha' = (\alpha'_{ij})$ are non holonomic bundles of groups relative to $\mathfrak{U}$, $\Gamma^H$ and $G$.

(b) $g$ is a family of sections $g_{ijk} : \mathfrak{U}_{ijk} \to \Gamma^G$.

Such a triple is called a *two-dimensional cochain of* $\mathfrak{U}$ with values in $\mathscr{E}^G$(or $\Gamma^G$). The bundles $\alpha$ and $\alpha'$ are respectively called the *origin* and *target* of the cochain.

The set $C^2_\Sigma(\mathfrak{U}, \mathscr{E}^G)$ of these cochains becomes a groupoid if we decide: a product

$$(\bar{\alpha}', \bar{g}, \bar{\alpha})(\alpha', g, \alpha)$$

is defined if and only if $\bar{\alpha} = \alpha'$ and then is equal to

$$(\bar{\alpha}', \bar{g} \cdot g, \alpha), \quad \bar{g} \cdot g = (\bar{g}_{ijk} \cdot g_{ijk}).$$

Using the notation (7.3) we define a map

$$\Delta : \mathscr{Z}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G}) \to C^2_\Sigma(\mathfrak{U}; \mathscr{E}^G) \text{ by } \Delta(h, \alpha) = (\tau h \cdot \alpha, \Delta_\alpha h, \alpha).$$

Formula (7.4) implies that it is a groupoid homorphism, in fact

$$\begin{aligned} &\Delta\,[(h', \alpha') \otimes (h, \alpha)] = \Delta(h' \cdot h, \alpha) \qquad (\alpha' = \tau h \cdot \alpha) \\ &= (\tau(h' \cdot h) \cdot \alpha, \Delta_\alpha(h' \cdot h), \alpha) = (\tau h' \cdot \alpha', \Delta_{\tau h \cdot \alpha} h' \cdot \Delta_\alpha h, \alpha) \\ &= (\tau h' \cdot \alpha', \Delta_\alpha h', \alpha')(\alpha', \Delta_\alpha h, \alpha) = \Delta\,(h', \alpha') \cdot \Delta(h, \alpha). \end{aligned}$$

A two-dimensional cochain of $\mathfrak{U}$ with values in $\mathscr{E}^G$ will be called a *cocycle* (with respect to the exact sequence $\Sigma$) if it is in the image of $\Delta$. The cocycles form a sub groupoid $\mathscr{Z}^2_\Sigma(\mathfrak{U}, \mathscr{E}^G) \subset C^2_\Sigma(\mathfrak{U}, \mathscr{E}^G)$ and one readily sees that the images of equivalence classes of non holonomic principal bundles form equivalence classes in $\mathscr{Z}^2_\Sigma(\mathfrak{U}, \mathscr{E}^G)$: in other words if two non holonomic principal bundles $(h, \alpha)$, $(h', \alpha')$ have the same coboundary $\Delta(h, \alpha) = \Delta(h', \alpha')$, then the $\Delta$-images of their classes coincide. Every such class of cocycles will be called a *two-dimensional cohomology class with values in* $\mathscr{E}^G$ (with respect to the exact sequence $\Sigma$). The set of these classes will be denoted by $\mathscr{H}^2_\Sigma(\mathfrak{U}, \mathscr{E}^G)$.

Let us denote $\mathscr{B}^2_\Sigma(\mathfrak{U}; \mathscr{E}^G)$ the image in $\mathscr{Z}^2$ of the subgroupoid $\mathscr{N}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G}) \subset \mathscr{Z}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G})$. We know that $\mathscr{N}^1$ is invariant in $\mathscr{Z}^1$ and on the other hand it is clear that if $x \in \mathscr{N}^1$ has an image $\Delta x$ which is a loop in $\mathscr{Z}^2_\Sigma(\mathfrak{U}; \mathscr{E}^G)$, then $x$ is

[7] So that it be so, it is necessary and sufficient that for any pair of units $\varepsilon, \varepsilon' \in \mathscr{H}^1(B; \mathscr{H}, \mathscr{G})$ having the same image by $\kappa^*$ there exist a covering $\mathfrak{U}$, units $\varepsilon_\mathfrak{U}$, $\varepsilon'_\mathfrak{U}$ in $\mathscr{H}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G})$ defining $\varepsilon, \varepsilon'$ and an element $E \in \mathscr{H}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G})$ having left and right units $\varepsilon_\mathfrak{U}, \varepsilon'_\mathfrak{U}$ so that $\kappa^*_\mathfrak{U}(E)$ be a unit in $\mathscr{H}^1(B, \mathscr{K})$.

itself a loop. From this it follows that $\mathscr{B}^2_\Sigma$ is an invariant subgroupoid of $\mathscr{Z}^2_\Sigma$ and the quotient $\mathscr{Z}^2_\Sigma/\mathscr{B}^2_\Sigma = \mathscr{H}^2_\Sigma(\mathfrak{U}, \mathscr{E}^G)$ is therefore a groupoid.[8]

We have the commutative diagram with perfectly exact horizontal lines:

$$\begin{array}{ccccccccc} e \to & \mathscr{N}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G}) & \xrightarrow{i} & \mathscr{Z}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G}) & \xrightarrow{j} & \mathscr{H}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G}) & \to e \\ & \downarrow & & \downarrow & & & \\ e \to & \mathscr{B}^2_\Sigma(\mathfrak{U}; \mathscr{E}^G) & \xrightarrow{i'} & \mathscr{Z}^2_\Sigma(\mathfrak{U}, \mathscr{E}^G) & \xrightarrow{j'} & \mathscr{H}^2_\Sigma(\mathfrak{U}; \mathscr{E}^G) & \to e. \end{array}$$

From which we deduce a homomorphism

$$\delta^1_\mathfrak{U} : \mathscr{H}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G}) \to \mathscr{H}^2_\Sigma(\mathfrak{U}; \mathscr{E}^G)$$

completing the diagram and leaving it commutative.[9]

Passing now to the direct limit we get a groupoid $\mathscr{H}^2_\Sigma(B, \mathscr{E}^G)$ and a homomorphism

$$\delta^1 : \mathscr{H}^1(B; \mathscr{H}, \mathscr{G}) \to \mathscr{H}^2_\Sigma(B, \mathscr{E}^G).$$

Suppose now that $B$ is paracompact; then the homomorphism $\kappa^*$ is onto. Let $E \in \mathscr{H}^1(B, \mathscr{K})$ be a principal fibre bundle such that $E : \tilde{\Gamma}^K \to \tilde{\Gamma}'^K$. We can then find a covering $\mathfrak{U}$ and an element $(\alpha) \in \mathbf{F}(\mathfrak{U} : \Gamma^H, G)$ represented by $\alpha = (\alpha_{ij})$ the image $\beta = (\beta_{ij})$ of which in $I(\Gamma^K)$ represents $\tilde{\Gamma}^K$. In the same way we can find $(h, \alpha) \in \mathscr{Z}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G})$ such that its image $(k, \beta)$ in $\Gamma^K$ represents $E$. We can therefore associate to $E$ the image by $\delta^1_\mathfrak{U}$ of the class $[h, \alpha]$ of $(h, \alpha)$. But this is not canonical because we had first to choose $\alpha$ (or more precisely its class). This forces us to define a $\mathscr{H}^1_\Sigma(B, \mathscr{K})$ instead of $\mathscr{H}^1(B, \mathscr{K})$ and to replace the homomorphism $j_1$ by one

$$j'^1 = \mathscr{H}^1(B, \mathscr{H}) \to \mathscr{H}^1_\Sigma(B, \mathscr{K}).$$

This is done if we put $\mathscr{H}^1_\Sigma(\mathfrak{U}, \mathscr{K}) = \mathscr{H}^1(\mathfrak{U}; \mathscr{H}, \mathscr{G})$ and if we take $j'^1$ to be the natural map.

In $\mathscr{H}^2_\Sigma(\mathfrak{U}, \mathscr{E}^G)$ a class $[h, \alpha]$ will be said to be *neutral* if it contains an element $(h, \alpha)$ which (1)$\alpha$ is a true cocycle, (2)$\Delta(h, \alpha) = (\tau h \cdot \alpha, e, \alpha)$ where $e$ is the family of unit sections $e_{ijk} : U_{ijk} \to \Gamma^G$. *Clearly an element $E \in \mathscr{H}^1_\Sigma(B, \mathscr{K})$ is in the image of $j'_1$ if and only if $\delta^1(E)$ is neutral.* The neutral classes form a subgroupoid of $\mathscr{H}^2_\Sigma(\mathfrak{U}, \mathscr{G})$. This leads to *neutral classes* over $B$ forming a subgroupoid in $\mathscr{H}^2_\Sigma(B, \mathscr{G})$.

The boundary homomorphism $\delta^1$ is onto and gives the

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[8] A *loop* in a groupoid $\Phi$ is an element the right and left units of which are equal. A subgroupoid $\varphi \subset \Phi$ is *invariant* if for any loop $u \in \varphi$, any element $x \in \Phi$ multiplying $u$ to the left, then the loop $xux^{-1}$ is still in $\varphi$. If $\varphi$ is invariant then the equivalence $x \sim uxv$ ($x \in \Phi$, $u, v \in \varphi$) is compatible with the multiplication in $\Phi$ and the quotient is still a groupoid called the quotient of $\Phi$ by $\varphi$. See [3].

[9] One may observe that this conclusion does not depend on the fact that $i$ and $i'$ are inclusions nor on the fact that $j'$ is perfect; perfectness of $j$ is however essential.

THEOREM. *If the space $B$ is paracompact, the following sequence of groupoids and homomorphisms is exact.*

$$(8.1)\quad \begin{aligned} &e \to \mathscr{H}^0_\Sigma(B, \mathscr{G}) \xrightarrow{i^0} \mathscr{H}^0(B, \mathscr{H}) \xrightarrow{j^0} \mathscr{H}^0_\Sigma(B, \mathscr{K}) \xrightarrow{\delta^0} \\ &\mathscr{H}^1_\Sigma(B, \mathscr{G}) \xrightarrow{j^1} \mathscr{H}^1(B, \mathscr{H}) \xrightarrow{j'^1} \mathscr{H}^1_\Sigma(B, \mathscr{K}) \xrightarrow{\delta^1} \mathscr{H}^2_\Sigma(B, \mathscr{G}) \to e \end{aligned}$$

Of course at $\mathscr{H}^1_\Sigma(B, \mathscr{K})$ exactness means

$$\text{image of } j'^1 = \text{inverse image of neutral subgroupoid.}$$

This property characterizes the image of $j'^1$ and also that of $j^1$. It is sufficient to pass through the inverse image of $E \in \mathscr{H}^1(B, \mathscr{K})$ by $\kappa^*$ (see (7.5)) which must contain some element mapped by $\delta^1$ on a neutral class so that $E \in \operatorname{Im} j^1$.

## §9. Final remarks

1. This machinery could probably be generalized by taking imbeddings of $G$ not only in $H$ but in any group $L$ in which it is normal. One would then obtain more general non holonomic bundles which would still retain a law of composition owing to the possibility of amalgamating two imbeddings of $G$ in $L_1$ and $L_2$ in the amalgamate product of $L_1$ and $L_2$.[10] This would give rise to more two dimensional cohomology classes and so to a larger groupoid $\mathscr{H}^2(B, \mathscr{G})$ (not depending on $\sum$ this time) and a larger groupoid of neutral classes.

Then to define an $\mathscr{H}^2_\Sigma(B, \mathscr{H})$ we should only consider imbedding $H \to L$ such that both $H$ and $G$ are normal, $H$ and $J(H)$ now playing similar roles to $G$ and $J(G)$ and yielding a homomorphism.

$$i^2_2 : \mathscr{H}^2(B, \mathscr{G}) \to \mathscr{H}^2_\Sigma(B, \mathscr{H}).$$

Finally $\mathscr{H}^2(B, \mathscr{K})$ would be defined like $\mathscr{H}^2(B, \mathscr{G})$ just replacing $G, J(\Gamma^G)$ by $K, J(\Gamma^K)$. Out of this an onto-homomorphism

$$j_2 : \mathscr{H}^2_\Sigma(B, \mathscr{H}) \to \mathscr{H}^2(B, \mathscr{K})$$

would appear with a possibility of ending (8.1) by

$$\xrightarrow{\delta^1_1} \mathscr{H}^2(B, \mathscr{G}) \xrightarrow{i^2_2} \mathscr{H}^2_2(B, \mathscr{H}) \xrightarrow{j^2_2} \mathscr{H}^2(B, \mathscr{K}) \to e.$$

2. One notes that the only two-dimensional classes introduced are those which can be killed by a suitable imbedding of the coefficient group. Therefore, in the abelian case we do not get all the classical two-dimensional classes. P. Olum has suggested to me to search a definition of 2-cocycles assuming local acyclicity. [(*Note added in proof.*) A hint to this consists in the following identities satisfied by $(\alpha_{ij}, g_{ijk}, \varepsilon_{ij}) = \Delta(h_{ij}, \varepsilon_{ij})(\varepsilon_{ij} = \text{unit section}, \alpha_{ij} = \tau h_{ij})$:

$$g_{jki} = \alpha_{jk}\,\alpha_{ki}(g_{ijk}), \quad g_{kij} = \alpha_{ki}(g_{ijk}),$$

$$g_{ikj} = g^{-1}_{ijk}, \; g_{kji} = \alpha_{kj}\,\alpha_{ji}(g^{-1}_{ijk}), \quad g_{jik} = \alpha_{ji}(g^{-1}_{ijk}),$$

[10] About amalgamate products, see H. Neumann [8].

(these formulas are substitute for the *alternate* condition on 2-cochains) and

$$g_{ijk} = g^{ili}_{ijl}\, g^{ilj}_{jkl}\, g^{ilk}_{kil}, \quad g^{mni}_{ijk} = \alpha_{mn}\, \alpha_{ni}(g_{ijk})$$

(which reduces to the usual cocycle condition in the abelian case).]

3. Finally we want to justify the terms *non holonomic bundle of groups* and *non holonomic principal bundles* by a geometrical picture.

(a) *non holonomic bundles of groups.* For a covering $\mathfrak{U}$ and the projection $p$: $\Gamma^H \to B$, we put $X_{i_1 \cdots i_k} = p^{-1}(U_{i_1 \cdots i_k})$, $X = \bigcup_i X_i \times \{i\}$, $X^i_{i_1 \cdots i_k} = X_{i_1 \cdots i_k} \times \{i\}$. Let $(\alpha) = (\alpha_{ij})$ be a non holonomic bundle of groups relative to $\mathfrak{U}$, $\Gamma^H$ and $G$. We define a family of maps

$$\psi_{ij} : X^j_{ij} \to X^i_{ij}, \quad \text{by} \quad \psi_{ij}(z, j) = (\alpha_{ij}(z), i)$$

where $p(z) = x$ is in $U_{ij}$ and the value of $\alpha_{ij}$ at $x$ is considered. These involve *non commutative triangles*

$$\begin{array}{ccc} X^i_{ijk} & \xleftarrow{\ \psi_{ij}\ } & X^j_{ijk} \\ & {\scriptstyle \psi_{ki}} \searrow \quad \nearrow {\scriptstyle \psi_{jk}} & \\ & X^k_{ijk} & \end{array}$$

The measure of the non commutativity is the non trivial coboundary $\alpha_{ij}\alpha_{jk}\alpha_{ki}$ of $\alpha$. It is therefore not possible to identify points in $X$ as in the case of usual bundle theory. However, the family of these maps is defined and is transformed in a similar one by passage to a finer covering or passing to an equivalent non holonomic bundle. This last operation is in a certain sense an isomorphism between the families of maps. These maps keep invariant the group structure of the fibres of the sets $X^k_{ij}$; this is why we speak of non holonomic bundles of *groups*.

(b) *non holonomic principal bundles.* For the same covering $\mathfrak{U}$, we consider the pair $(h, \alpha)$ satisfying (a) and (b) of §7. We set $X^{k,l}_{i_1 \ldots i_k} = X_{i_1 \ldots i_k} \times \{k\} \times \{l\}$. Now we set

$$h^k_{ij} = \alpha_{ki}(h_{ij}) \quad (\text{defined in } U_{ijk})$$

$$h^{k,l}_{ij} = \alpha_{kl}\, \alpha_{li}(h_{ij}) \quad (\text{defined in } U_{ijkl})$$

Then condition (7.1) is equivalent to $h^k_{ij} = (h^k_{ij})^{-1}$ for $k = i$ or $j$ (only). The expression. $h^i_{ij} h^{ij}_{jk} h^{ijk}_{ki} = h^i_{ij} h^i_{jk} h^{ij}_{ki}$ in (7.2) may be considered "$i$-component of the $(i, j, k)$-component of the boundary" of $h$. Now we define maps for $k = i$ or $j$:

$$\varphi^k_{ij} : X^{k,j}_{ij} \to X^{k,i}_{ij}, \quad \text{by} \quad \varphi^k_{ij}(z, k, j) = (h^k_{ij}\, z, k, i)$$

and we identify the maps $\psi_{ij}$ above as

$$\psi_{ij} : X^{j,k}_{ij} \to X^{i,k}_{ij}, \qquad \psi_{ij}(z, j, k) = (\alpha_{ij}(z), i, k).$$

Condition (7.1) may also be viewed as meaning that the following diagram is commutative

$$\begin{array}{ccc} X^{l,l}_{ij} & \xrightarrow{\psi_{kl}} & X^{k,l}_{ij} \\ \varphi^{l}_{kl} \Big\downarrow & & \Big\downarrow \varphi^{k}_{kl} \\ X^{l,k}_{ij} & \xrightarrow[\psi_{kl}]{} & X^{k,k}_{ij} \end{array} \qquad (l, k = i, j).$$

These families of maps retain these properties when we pass to a finer covering; also passing to an equivalent object may be interpreted as an isomorphism between the corresponding families.

Now the maps $\psi$ keep invariant the group structure of the sets $X^{k}_{ij}$, or $X^{k,1}_{ij}$; the maps $\varphi$ do not. However, $X^{k}_{ij}$ acts from the right on $X^{k,i}_{ij}$ in a way compatible with these maps, meaning by that

$$\varphi^{k}_{ij}(\bar{z} \cdot z) = \varphi^{k}_{ij}(\bar{z}) \cdot z \quad \text{for} \quad \bar{z} \in X^{k,j}_{ij},\ z \in X^{k}_{ij}$$
$$p(\bar{z}) = p(z),\ k = i, j.$$

In this sense the non holonomic bundles of groups $\alpha$ act from the right on the non holonomic principal bundle $(h, \alpha)$.

Let us put $\beta_{ij} = \tau(h_{ij})$. The pair $(\beta, \alpha)$ may be considered as a "non holonomic bundle of groups relative to $\alpha$"; instead of condition (7.1) we have $\beta_{ij}\alpha_{ij} = \alpha_{ij}\beta_{ji}^{-1}$. Then if we introduce $\beta^{k}_{ij} = \tau(h^{k}_{ij})$ we have maps

$$\omega^{k}_{ij} : X^{k,j}_{ij} \to X^{k,i}_{ij}, \qquad \omega^{k}_{ij}(z, j, k) = (\beta^{k}_{ij}(z), i, k).$$

To the bundle $\alpha' = \tau(h) \cdot \alpha$ corresponds maps $\psi'_{ij}$ which can be deduced simply from $\omega^{k}_{ij}$ and $\psi_{ij}$. Now we may say that the bundle of groups $\alpha'$ acts from the left on the principal bundle $(h, \alpha)$ since we have

$$\varphi^{k}_{ij}(z \cdot \bar{z}) = \omega^{k}_{ij}(z) \cdot \varphi^{k}_{ij}(z),\ z \in X^{k,j}_{ij},\ \bar{z} \in X^{k,j}_{ij}.$$

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## SPECTRAL SEQUENCES OF CERTAIN MAPS

By I. Fary[1]

### 1. Introduction

We give in this lecture a summary of our papers [3] and [4]. The main result of [3] can be roughly formulated as follows. Given a not too complicated continuous map $f: X \to Y$, a simplicial one, for example, we first define critical sets $Y_0 \supset \cdots \supset Y_q \supset \cdots$ of $f$ in $Y$ ($Y_0 = Y$, $\bigcap Y_q = \varnothing$). Then we construct a spectral sequence $\{E_r\}$ beginning with the direct sum of the cohomology algebras of $Y_q - Y_{q+1}$ ($q = 0, 1, \cdots$) based on local coefficients $\mathscr{F}_q$, and ending up at a graded algebra of the cohomology algebra $H(X)$ of $X$. If $X$ is a fiber space, $f$ is its projection onto the base space $Y$, and $F \approx f^{-1}(y)$ ($y \in Y$) is the fiber, then $Y_1 = \varnothing$, and $\{E_r\}$ is the well known spectral sequence beginning with $H(Y, H(F)_\pi)$, where the fundamental group $\pi$ of $Y$ acts on $H(F)$. The $\mathscr{F}_q$ of the general case is also constructed using cohomology algebras $H(f^{-1}(y))$ (one for each component of $Y_q - Y_{q+1}$) and operator groups (fundamental groups of these components).

Even in the general case, our $\{E_r\}$ is as simple and as easily computable as in the case of fiber spaces. This point is illustrated here by an outline of [4]. This paper aims to give a new version of Lefschetz' classical homology theory of algebraic manifolds (see, for example, [6], [7]). We consider an affine algebraic manifold $V^n$ in the complex affine space $C^t$, and define a map $f: V^n \to C$ onto the Cauchy plane of complex numbers $C$ (example of such an $f$ : $f(x)$ is the last coordinate of $x \in V^n$). The spectral sequence $\{E_r\}$ can be described quite explicitly in this case, and gives relations between the homology of $V^n$, $V^{n-1} = f^{-1}(\xi)$, $\xi \in C$ (non-singular section) and singular points of the series $\{V^{n-1}_\xi\}$.

We have no time to stress the point that our theory can be considered as a generalization of Morse's theory of critical values of a function to the much more complicated case of maps.

To sum up, the theory of $\{E_r\}$ generalizes at the same time Morse's theory of critical values, Lefschetz' theory of homology of algebraic manifolds, as well as the homology theory of fiber spaces. On the other hand, it is a specialization of Leray's theory [8], [9], [10], [11] of spectral sequence of an arbitrary continuous map $f: X \to Y$. However, this is not a trivial, not even an easy specialization, as it is based on our results concerning the possible topological structures of stacks. In the technical sense our papers [1], [3], [4] are thus contributions to Leray's theory [11]: (1) we have initiated a topological structure theory of stacks (see §4 below) based on the notion of critical sets of a stack; (2) we consider what we call homologically simple maps $f$ only, and choose a special filtration in the spectral

[1] While preparing this paper, the author was a fellow of the Summer Research Institute, Kingston, Ont., Canada.

sequence (50.2), (50.3), p. 91, [11], to the effect that its first term be directly connected with the critical values of the map $f$, and be at the same time expressible in terms of classical homology invariants, i.e. cohomology algebras based on local coefficients.

## 2. Notation

We consider *locally compact* spaces $X, Y, \cdots$ only, and continuous maps $f : X \to Y$. We choose once for all a commutative coefficient ring $A$ with unit (in §§6 and 7 this will be either the ring of integers $Z$ or the field of rationals $R$), and we denote by

$$H(X) = \sum_{p=0}^{\infty} H^p(X)$$

the *compact* cohomology algebra of $X$ based on $A$. Let us recall that for a compact space $X$, the groups $H^p(X)$ are isomorphic to the Čech cohomology groups. If $X$ is *locally compact* and $X^*$ denotes its one-point compactification then $H^p(X) = H^p(X^*)$ for $p \geq 1$. If $F$ is a closed sub-space of $X$, the natural homomorphism $H(X) \to H(F)$ is defined; the image of $h \in H(X)$ under this homomorphism will be called restriction of $h$ to $F$ and will be denoted by $Fh$. A characteristic continuity property of the restriction will be formulated below.

## 3. Stacks

The main result of [3] which was mentioned in the introduction does not make explicit use of the notion of stack. However, as this notion is important in other connections, and as it is used in the proofs of our results, we formulate some notions in the general theory of stacks.

We use here the word stack in the sense of [11] and [1], [2], [3]; a similar structure is called "Garbendatum" in [5].

A *stack* $\mathscr{S}$ on a space $X$ is a certain collection of modules and homomorphisms subjected to the axioms (3), (4), (6) below. The collection is

(1) $$\{\mathscr{S}(A); \varphi_{BA}\}$$

such that: (1) for every closed set $A \subset X$, there is a module $\mathscr{S}(A)$; (2) for every inclusion $A \supset B$ for closed sets of $X$, there is a homomorphism

(2) $$\varphi_{BA} : \mathscr{S}(A) \to \mathscr{S}(B)$$

called restriction. These data are subjected to two naturality conditions:

(3) $$\mathscr{S}(\varnothing) = 0$$

(4) $$\text{if } A \supset B \supset C, \quad \text{then} \quad \varphi_{CA} = \varphi_{CB}\varphi_{BA}$$

and the following continuity axiom. Given a closed set $A$ in $X$,

(5) $$\{\mathscr{S}(V), \varphi_{WV}; V, W \textit{ closed neighborhoods of } A\}$$

is a directed system of modules. Hence, we may speak of the direct limit of (5). The continuity axiom requires that

(6) $$\mathscr{S}(A) = \varinjlim \mathscr{S}(V).$$

Let us formulate this rather complicated continuity axiom in a different way. The axiom is equivalent to the conjunction of the following two conditions:

(7) *Given* $a \in \mathscr{S}(A)$, *there is a closed neighborhood* $V$ *of* $A$ *in* $X$ *and a* $v \in \mathscr{S}(V)$ *such that* $a = \varphi_{AV}(v)$.

(8) *Given* $a \in \mathscr{S}(A)$ *and* $B \subset A$ *such that* $\varphi_{BA}(a) = 0$, *there is a closed neighborhood* $W$ *of* $B$ *in* $A$ *such that* $\varphi_{WA}(a) = 0$.

There are several obvious variants of this definition. Instead of stacks of modules $\mathscr{S}(A)$ we may speak about stacks of groups, rings, algebras, etc. Sometimes the modules $\mathscr{S}(A)$ are given by the data of the problem for a special class of closed sets (for example, for the compact sets only), and we define the modules for other closed sets conveniently (setting $\mathscr{S}(A) = 0$ for non compact $A$), or else, we develop the theory by generalizing the notion of stack admitting systems (1) such that $A$ runs over a certain class of closed sets of $X$.

Before giving examples of stacks, let us notice some obvious topological and algebraic operations which can be performed on stacks. Given a locally compact sub-space $Y$ of $X$ we define the *restriction* $\mathscr{S}|Y$: this stack is defined on $Y$ and $(\mathscr{S}|Y)(A) = \mathscr{S}(A)$ for closed $A \subset Y$ (some precautions have to be taken, if $Y$ is not closed in $X$). We say that $\mathscr{T}$ is a sub-stack of $\mathscr{S}$, and we write $\mathscr{T} \subset \mathscr{S}$, if $\mathscr{T}$ is defined on the *same* space $X$ as $\mathscr{S}$, $\mathscr{T}(A)$ is a sub-module of $\mathscr{S}(A)$ for every $A$, and the homomorphisms in $\mathscr{T}$ are the restrictions of the $\varphi_{BA}$. Hence, we may simplify the notation and write $Ba$ instead of $\varphi_{BA}(a)$ in most cases (see the notation $Fh$ in §2). Quotient stacks, homomorphisms of stacks, exact sequences of stacks, tensor product, torsion product, etc., are defined in the same natural way.

A trivial example of stack is the constant stack $\mathscr{C}$ : $\mathscr{C}(A)$ is the same module for every closed set $A$, and $\varphi_{BA}$ is always the identity. A less trivial example is the locally constant stack $\mathscr{L}$ defined by the following conditions. $\mathscr{L}$ is given on a *locally connected* space $X$ (components of open sets of $X$ are open in $X$); there is a covering $\{V_\alpha\}$ of $X$ by connected open sets $V_\alpha$, such that $\mathscr{L}|V_\alpha$ is a constant stack for every $V_\alpha$. Every constant stack (defined on a locally connected space) is locally constant, of course, but the converse is far from being true. In fact, the starting point of our paper [3] was the remark, that in some of the applications of the theory of stacks we can *reduce problems to those concerning the locally constant stacks*. Hence, it is worthwhile to state a few facts concerning these stacks. Let $\mathscr{L}$ be a locally constant stack defined on a space $X$, which is connected, locally connected and locally simply connected by arcs. Then every closed curve based on $x_0 \in X$ determines an endomorphism of $\mathscr{L}(x_0)$ in an obvious way; furthermore, it can be shown that this gives a homomorphism of $\pi_1(X, x_0)$ into the group of endomorphisms of $\mathscr{L}(x_0)$. Now $\mathscr{L}$ is determined by $\mathscr{L}(x_0)$ and this homomorphism, and these data can be chosen arbitrarily. In other words, given a module $M$ as well as a homomorphism of $\pi_1(X, x_0)$ into the group of endomorphisms of $M$, we can construct a locally constant stack $\mathscr{M}$ on $X$, such that $\mathscr{M}(x_0) = M$, and $\pi_1(X, x_0)$ operates in $\mathscr{M}$ in the prescribed way.

Another important stack defined on an arbitrary space $X$ is the cohomology stack $\mathscr{H}$ for which $\mathscr{H}(A)$ is the cohomology algebra $H(A)$ of $A$, and $\varphi_{BA}$ is the restriction homomorphism of cohomology algebras. The axiom (6) or, equivalently, (7) and (8), is satisfied as the compact cohomology satisfies the "continuity axiom". The stack $\mathscr{H}$ is the direct sum of the sub-stacks $\mathscr{H}^p(\mathscr{H}^p(A) = H^p(A))$, and $\mathscr{H}^p(x) = 0$, $p \geqq 1$, for every point $x \in X$.

In what follows, the most important stack will be the so called cohomology stack $\mathscr{F}$ of a given continuous map

$$f : X \to Y. \tag{9}$$

This stack $\mathscr{F}$ is *defined on the space* $Y$: for every closed set $A \subset Y$ define

$$\mathscr{F}(A) = H(f^{-1}A) \qquad (A \textit{ closed in } Y) \tag{10}$$

$$\varphi_{BA} \textit{ is the restriction homomorphism } \quad H(f^{-1}A) \to H(f^{-1}B). \tag{11}$$

Let us notice in particular that

$$\mathscr{F}(y) = H(f^{-1}(y)) \tag{12}$$

is the cohomology of the inverse image of the point $y \in Y$. If, for example, $Y$ is the base of a fiber space $X$, then $f^{-1}(y)$ is homeomorphic to the fiber $F$ and $\mathscr{F}(y)$ is isomorphic to $H(F)$.

Given a space $X$ and a stack $\mathscr{S}$ defined on $X$ we can define the cohomology algebra $H(X, \mathscr{S})$ of $X$ based on $\mathscr{S}$ (see [11], [5]). In what follows, we only use the fact that, in case $\mathscr{L}$ is locally constant, $H(X, \mathscr{L})$ is the classical cohomology algebra based on local system.

Let $\mathscr{M}$ and $\mathscr{N}$ be given stacks on the same space $X$. We say that these stacks are pointwise isomorphic, if there are homomorphisms $\Phi : \mathscr{M} \to \mathscr{N}$ and $\Psi : \mathscr{N} \to \mathscr{M}$ (i.e. collections of homomorphisms $\Phi_A : \mathscr{M}(A) \to \mathscr{N}(A)$, etc. which commute with the homomorphisms of the stacks) such that $\Psi_x\Phi_x$ and $\Phi_x\Psi_x$ are the identity automorphisms of $\mathscr{M}(x)$, $\mathscr{N}(x)$ respectively for every point $x \in X$. Pointwise isomorphic stacks may have fairly different structures (for example, the stack $\mathscr{H}^p$ is pointwise isomorphic to the nul stack if $p \geqq 1$). However, if $\mathscr{M}$ and $\mathscr{N}$ are pointwise isomorphic, $H(X, \mathscr{M})$ and $H(X, \mathscr{N})$ are isomorphic (in fact $\Phi$ and $\Psi$ induce this isomorphism). This fact suggests to introduce a new notion called *sheave* which is the class of stacks pointwise isomorphic to a given stack (see "Garbe" in [5]).

## 4. Critical spaces of a stack

The introduction of this notion (called critical *set* in [3]) is suggested by the properties of the particular stack $\mathscr{F}$ (see (10), (11)), and the classical notion of critical value of a map. Hence, we formulate here our definitions using this stack. It goes without saying that the same definition makes sense for any stack.

In what follows we consider a given continuous map $f : X \to Y$, where $Y$ is supposed to be *locally connected* (this restriction concerning $Y$ could be avoided, but this would complicate the exposition).

It is quite natural to agree that $y_0$ is a non critical value of the map $f$, if $H(f^{-1}y)$, i.e. the cohomology algebra of the inverse image of a point $y \in Y$, does not change, when $y$ runs over a certain neighborhood of $y_0$. Let us notice that every point sufficiently near to $y_0$ is also a non critical value of $f$. In order to formulate this condition in terms of the stack $\mathscr{F}$ defined in (10), (11), let us consider a closed neighborhood $U$ of $y_0$, and let us set

(13) $$\mathscr{F}(U) \underset{\varphi_{yU}}{\longrightarrow} \mathscr{F}(y)$$
$$\ker \varphi_{yU} = N \qquad \operatorname{coker} \varphi_{yU} = Q.$$

DEFINITION 1. *We say that $y_0$ is a non critical point of $\mathscr{F}$ (and a non critical or ordinary value of the map $f$) if there is a closed neighborhood $U$ of $y_0$ such that* $\ker \varphi_{yU}$ *is independent of $y$ and* $\operatorname{coker} \varphi_{yU} = 0$ *when $y$ runs over $U$. If such a $U$ does not exist, $y_0$ is called critical point of $\mathscr{F}$ (and critical value of $f$).*

If $y_0$ is non critical, $\mathscr{F}(U)/N$ is mapped isomorphically onto $\mathscr{F}(y) = H(f^{-1}y)$ by $\varphi_{yU}$, hence the cohomology of the inverse image of a point is invariant in a very strong sense. In other words, our conditions concerning the ordinary points are very restrictive. It is also clear that every point in the interior of $U$ is also an ordinary point; thus the set of ordinary points is an open set. Hence the set of critical points is a closed sub-space of $Y$. Let us also notice that, if $Y$ is not locally connected at $y_0$ (and $X$ is compact) then $y_0$ would necessarily be critical in view of the properties of $H^0(f^{-1}U) \to H^0(f^{-1}y)$; we prefer to exclude this case.

If $X$ is a fiber space with locally connected base space $Y$, and $f$ is the projection, then every point of $Y$ is a non critical point of $\mathscr{F}$ and a non critical value of $f$, as it should be. Another simple instance: $X$, $Y$ are simplicial complexes, $f$ is simplicial, and $y_0$ is an interior point of a principal simplex of $Y$. This example shows that our conditions are not too restrictive.

Returning to the general case, let us denote by $Y_1$ the set of critical points of $\mathscr{F}$. We will call $Y_1$ *first critical space* of $\mathscr{F}$, as we shall define a whole sequence

(14) $$Y_0 \supset Y_1 \supset \cdots \supset Y_\alpha \supset \cdots \supset Y_\Omega$$

of critical spaces by transfinite induction (less generally than in [3]).

Let us set $Y_0 = Y$. $Y_1$ has already been defined; all of its points are critical points of $\mathscr{F}$ and critical values of $f$. If $Y_1$ is not locally connected, (14) contains these two spaces only. If, however, $Y_1$ is locally connected some of its points may be non critical for the stack $\mathscr{F}|Y_1$ and for the map $f|X_1$ where $X_1 = f^{-1}(Y_1)$. In this case we define

(15) $$Y_2 = \textit{set of critical points of } \mathscr{F}|Y_1.$$

The definition of $Y_\alpha$ is then analogous, provided that $\alpha = \beta + 1$. In case of a limit number $\omega$ we set $Y_\omega = \bigcap_{\alpha<\omega} Y_\alpha$. It can be proved then that there is a transfinite number $\Omega$ such that $Y_{\Omega+1} = Y_\Omega$; for every ordinal number $\Omega' \geq \Omega$ we have then $Y_{\Omega'} = Y_\Omega$. By definition, this critical space is the last element of the sequence (14) of critical spaces of $\mathscr{F}$.

In what follows, we consider the most favorable case only, when

$$\bigcap_{q=0}^{\infty} Y_q = \varnothing, \tag{16}$$

i.e. the sequence (14) is finite or else $Y_\omega = \varnothing$ for the first transfinite ordinal. In this case the map $f$ will be called *homologically simple*; it can be proved that every simplicial map $f: X \to Y$ ($X$, $Y$ locally compact simplicial complexes) has this property.

Let us now consider the restrictions $\mathscr{F}|(Y_q - Y_{q+1})$, $q = 0, 1, \cdots$. By the definition of the critical spaces these stacks are free of critical points. It can then be proved that they are pointwise isomorphic to locally constant stacks.

This analysis of the structure of $\mathscr{F}$ will be used in the next section. Let us notice presently that it also could be used in order to compute $H(Y, \mathscr{F})$ in terms of classical invariants. In fact, $H(Y_q - Y_{q+1}, \mathscr{F}_q)$, where $\mathscr{F}_q = \mathscr{F}|(Y_q - Y_{q+1})$, is a classical cohomology algebra, and although these algebras do not determine $H(Y, \mathscr{F})$ completely they form a sort of approximation to it as their direct sum is the first term of a spectral sequence whose last term is (or includes) a graded algebra of $H(Y, \mathscr{F})$.

## 5. The spectral sequence of a homologically simple map

In this section we suppose that $f$ is a given homologically simple map, i.e. (16) holds true. Let us set $Y_q - Y_{q+1} = \bigcup Z_{q\alpha}$, where the $Z_{q\alpha}$ are the connected components of these difference spaces. $Z_{q\alpha}$ is open and closed in $Y_q - Y_{q+1}$; it is locally compact, connected and locally connected. $\mathscr{F}_{q\alpha} = \mathscr{F}|Z_{q\alpha}$ is pointwise isomorphic to a locally constant stack. Hence it is determined by $\mathscr{F}(y_{q\alpha}) = H(F_{q\alpha}) = H(f^{-1}(y_{q\alpha}))$ and by the operation of $\pi_1(Z_{q\alpha}, y)$ in this algebra. Thus $H(Z_{q\alpha}, H(F_{q\alpha}))$ is a classical cohomology algebra. Obviously

$$H(Y_q - Y_{q+1}, \mathscr{F}_q) = \textstyle\sum_\alpha H(Z_{q\alpha}, H(F_{q\alpha})_\pi) \tag{17}$$

(direct sum).

We now formulate the main result of [3] concerning the spectral sequence $\{E_r\}$ attached to a given homologically simple map $f$.

The map $f$ determines the critical spaces $Y_q$, $q = 0, 1, \cdots$, such that (16) holds true. It also determines the stacks $\mathscr{F}$, $\mathscr{F}_q = \mathscr{F}|(Y_q - Y_{q+1})$, $\mathscr{F}|Z_{q\alpha}$ hence the classical cohomology algebras (17). The second term of the spectral sequence will be

$$E_2 = \textstyle\sum_{q=0}^{\infty} H(Y_q - Y_{q+1}, \mathscr{F}_q) \tag{18}$$

which is a direct sum of classical cohomology invariants. If $p$ stands for the filtration degree, $n$ for the total degree, and we denote by $E_2^p(n)$ the corresponding sub-module of (18), then

$$E_2^p(n) = \textstyle\sum_{q=0}^{\infty} H^{p+q}(Y_q - Y_{q+1}, \mathscr{F}_q^{\,n-p-q}). \tag{19}$$

(In the theory of fiber spaces the usual notation is $E_2^{p, n-p}$, but $n - p$ has no immediate meaning in this connection; this motivates the change in notation.)

We have the usual rule concerning the differentials:

(20) $$d_2 : E_2^p(n) \to E_2^{p+2}(n+1)$$

(21) $$d_r : E_r^p(n) \to E_r^{p+r}(n+1).$$

The first of these differentials can be rather explicitly given:

(22) $$d_2 = d' + d'', \qquad d'd' = 0, \qquad d''d'' = 0, \qquad d'd'' = -d''d',$$

where

(23) $$d' : H^p(Y_q - Y_{q+1}, \mathscr{F}_q) \to H^{p+1}(Y_{q-1} - Y_q, \mathscr{F})$$

(24) $$d'' : H^p(Y_q - Y_{q+1}, \mathscr{F}_q) \to H^{p+2}(Y_q - Y_{q+1}, \mathscr{F}_q).$$

Here $d'$ is the coboundary operator of the exact sequence belonging to the space $Y_{q-1} - Y_{q+1}$, closed subspace $Y_q - Y_{q+1}$ and open complement $Y_{q-1} - Y_q$ in cohomology based on the stack $\mathscr{F}|(Y_{q-1} - Y_{q+1})$. $d''$ is the second differential of the spectral sequence of the map $f|(X_q - X_{q+1})$, where $X_q = f^{-1}(Y_q)$. As this last map has no critical value, we can compare it to the projection map of a fiber space and say "the homomorphism $d''$ is the second differential in the spectral sequence of a fiber space".

As to the multiplication, $H(Y - Y_1, \mathscr{F})$ has the usual cup-product structure, and all the other products are nul in $E_2$.

The other terms of the spectral sequence cannot be given explicitly in the general case. However, we have the rule (21) for the differentials and the degrees derived from (19).

Finally we have a certain graded algebra $\mathrm{Gr}H(X)$ of $H(X)$, and

(25) $$E_\infty(n) \supset \mathrm{Gr}H^n(X) \qquad (n = 0, 1, \cdots).$$

This inclusion is replaced by the equation

(26) $$E_r(n) = \mathrm{Gr}H^n(X), \quad \textit{if} \quad r > r_0$$

under favorable conditions. A sufficient condition for (26) is: $\dim Y \leqq m$ $(< \infty)$, $Y_s = \varnothing$, and $r_0 = m + s$.

We already mentioned in the introduction that this spectral sequence is obtained by using special filtration in a spectral sequence introduced by J. Leray.

## 6. Homology of algebraic manifolds

In the next section we will describe quite explicitly the spectral sequence $\{E_r\}$ of the preceding section specialized to the map (51), (52) below arising in algebraic geometry. Presently we give the less technical results obtained by this method.

Let $W^n$ be an algebraic manifold of complex dimension $n$ in the complex projective space $P^t$. A point $c \in W^n$ will be called *quadratic singular point* of $W^n$, if local coordinates $\{z_i\}$ having the origin at $c$ can be introduced in a neighborhood

(27) $$B : |z_1|^2 + \cdots + |z_t|^2 < 2$$

of $c$ in $P^t$, such that

(28) $$W^n \cap B : \quad z_1^2 + \cdots + z_{n+1}^2 = 0, \qquad z_i = 0, \qquad i = n+2, \cdots, t;$$

i.e. $W^n$ has the shape of a quadratic cone near $c$.

If we do not state that the manifold has such a *quadratic singular point*, it is understood that the manifold is *free of singularities*.

We say that $W^n$ is a *complete intersection*, if there is a sequence of manifolds (i.e. manifolds having no singularities)

$$W^n \subset W^{n+1} \subset \cdots \subset W^{t-1} \subset P^t, \tag{29}$$

such that $W^{n+i}$ is an element of a linear series $\{W_\xi^{n+i}\}$ on $W^{n+i+1}$ with basis in the hyperplane at infinity $P_\infty^{t-1}$, $W^{n+i}$ being free of singularities, except for $\xi \varepsilon \Gamma_i$, $\Gamma_i$ finite, for which values it has just one quadratic singular point $c$ (see (28)).

We denote by $V^n$ the affine part of $W^n$: $V^n = W^n - (W^n \cap P_\infty^{t-1}) = W^n - W^{n-1}$.

We speak of either the integral cohomology $H(W^n)$, $H(V^n)$ or the rational cohomology (i.e. the ring of coefficients is the field of rational numbers); in the latter case $\Delta H(V^n)$ denotes the dimension of the vector space $H(V^n; R)$.

We shall introduce three sorts of cohomology classes of $W^n$. First, $W^n$ is a closed sub-space of $P^t$, hence we define $H_A(W^n)$ as being the image of the restriction homomorphism $H(P^t) \to H(W^n)$; $H_A(W^n)$ is called the group of algebraic cohomology classes of $W^n$. $H(P^t)$ is multiplicatively generated by an $a \,\varepsilon\, H^2(P^t)$ such that $a^{t+1} = 0$, hence $H(P^t)$ as well as $H_A(W^n)$ is a truncated polynomial ring. The positive generator of $H_A^p(W^n)$ is not divisible by an integer if $p \leqq n - 1$, and is divisible by the degree of $W^n$ if $p \geqq n + 1$. The properties of divisibility in $H^n(W^n)$ are more complicated.

$V^n$ is an open sub-space of $W^n$, hence we consider the injection homomorphism $H(V^n) \to H(W^n)$. The image of this homomorphism will be denoted by $H_F(W^n)$ and called group of *finite* classes as their contain cycles on the "finite part" $V^n$ of $W^n$ (terminology of [13]). It can be proved that $H_F(W^n)$ has a geometrical meaning, i.e. it is independent of the choice of $P_\infty^{t-1}$. The finite classes play a certain rôle in Petrowsky's theory of lacunas for differential equations. (See [13].)

An important theorem of Lefschetz states that

$$H_F^p(W^n) = 0, \quad \text{if} \quad 0 \leqq p \leqq n - 1. \tag{30}$$

In [3] we gave a new proof of this theorem; we even proved that

$$H^p(V^n) = 0, \quad \text{if} \quad 0 \leqq p \leqq n - 1 \tag{31}$$

for non singular $V^n$. (See also Wallace [14]. It is probable that similar theorem holds true for arbitrary $W^n$.)

Petrowsky [13] proved for hypersurfaces $W^n \subset P^{n+1}$ the decomposition

$$H^n(W^n) = H_F^n(W^n) + H_A^n(W^n) \tag{32}$$

(direct sum). We generalized this result to arbitrary non singular $W^n$.

In order to introduce the third kind of cohomology classes, which we call Lefschetz classes, we first consider the affine quadric

$$Q_\theta^n: \ z_1^2 + \cdots + z_{n+1}^2 = e^{i\theta} \qquad (\theta \text{ real}) \tag{33}$$

in the complex affine space $C^{n+1}$. We have

$$H(Q_\theta^n) = H^n(Q_\theta^n) + H^{2n}(Q_\theta^n) \approx Z + Z \tag{34}$$

($Z$: group of integers). A generator of $H^n(Q_0^n)$ is the cohomology class of the sphere

(35) $\quad S^n: \ z_1^2 + \cdots + z_{n+1}^2 = 1, \quad z_i \text{ real}, \quad i = 1, \cdots, n+1.$

This class will be called Lefschetz class $l$ of $Q^n$. Similarly a Lefschetz class $l = l(c, \gamma)$ can be introduced for a $W_\xi^n$ in a linear series which is such that for $\xi = \gamma$ $W_\gamma^n$ has a quadratic singular point $c$ (see (28)), because, if $|\xi - \gamma|$ is small, $\neq 0$, $W^n$ has the shape of a quadric (33) near the singular point $c$. For a general $W^n$ we introduce Lefschetz classes whenever $W^n$ can be embedded in a linear series $\{W_\xi^n\}$ having elements with quadratic singularities. This is possible for a complete intersection. For a given $W^n$ we will denote by $L$ the sub-group of $H^n(W^n)$ generated by certain Lefschetz classes (called "cycles évanouissants" in [6], [7]).

We can state now some results of [4] concerning the cohomology of $V^n$ and $W^n$. If $V^n$ is a hypersurface in $C^{n+1}$, then

(36) $$H^p(V^n) = 0, \quad \text{if} \quad p \neq n, 2n$$

hence

(37) $$H(V^n) = H^n(V^n) + H^{2n}(V^n)$$

(38) $$\operatorname{Tor} H(V^n) = 0$$

(39) $$\Delta H^n(V^n) = (m-1)^{n+1}$$

where Tor $G$ denotes the torsion sub-group of $G$, and $m$ is the degree of $V^n$.

For projective hypersurfaces $W^n$ in $P^{n+1}$ we have

(40) $$\operatorname{Tor} H(W^n) = 0$$

(41) $$H_F^p(W^n) = 0, \quad \text{if} \quad p \neq n, 2n$$

(42) $$H^p(W^{n-1}) \approx H^p(W^n), \quad \text{if} \quad W^{n-1} = W^n \cap P^n \quad \text{and} \quad p \neq n, n-1$$

(43) $$H^p(W^n) = \begin{cases} Z, & p \text{ even}, \quad 0 \leqq p \leqq 2n, \\ 0, & p \text{ odd} \end{cases} \quad p \neq n$$

(44) $$H_A^p(W^n) = \begin{cases} H^p(W^n), & p \leqq n-1 \\ mH^p(W^n), & n+1 \leqq p \end{cases} \quad p \neq n$$

(45) $$H^n(W^n) = \begin{cases} \{e\} + H_F^n(W^n), & n \text{ even} \\ H_F^n(W^n), & n \text{ odd} \end{cases}$$

where $\{e\} \approx Z$ and a multiple of $e$ belongs to $H_A^n(W^n)$.

(46) $$\Delta H_F^n(W^n) = \textstyle\sum_{j=1}^{n+1} (-1)^{m+1+j}(m-1)^j.$$

These results (partly known for hypersurfaces) were generalized in [4] to complete intersections. Only the equations (39), (46) have to be replaced by the recursion formulæ (47), (48), (49), (50) below, where $\mu_i$ is the number of points of $\Gamma_i$ (see (29) and sequel above). For affine complete intersection $V^n$ in $C^t$ we have

(47) $$\Delta H^{n+1}(V^{n+1}) + \Delta H^n(V^n) = \mu_1$$

hence

(48) $$\Delta H^n(V^n) = \mu_1 - \mu_2 + \cdots \pm \mu_{t-n}$$

(where $V^{n+1} = W^{n+1} \cap C^t$). For projective complete intersection we obtain:

$$\Delta H^n(W^n) = \Delta H^n(V^n) - \Delta H^{n-1}(W^{n-1}) + 1 \tag{49}$$

$$\Delta H^n_F(W^n) = \Delta H^n(V^n) - \Delta H^{n-1}_F(W^{n-1}). \tag{50}$$

$H^n(V^n; R)$ can also be interpreted as the vector space of relations of certain Lefschetz classes of a hyperplane section $V^{n-1}$ of $V^n$.

These results are obtained in [4] by using the spectral sequence of the preceding section. Specifically, let $V^n$ be a given affine manifold and $V^{n-1}$ one of its non singular hyperplane sections. Then we can define a map

$$f : V^n \to C \tag{51}$$

of $V^n$ onto the plane $C$ of complex numbers, which is homologically simple and such that

$$V^{n-1} = f^{-1}(\xi_0) \qquad (\xi_0 \in C). \tag{52}$$

As to the spectral sequence of this map, by (26) we have an equality

$$E_4 = \mathrm{Gr}H(V^n). \tag{53}$$

Computing $E_2$, $d_2$, hence $E_3$, and estimating $d_3$ we arrive at the following relation:

$$\mathrm{Gr}H(V^n) = N_0 + Q_0 + (H(V^{n-1})/L)/(d_3N) \tag{54}$$

where $L$, generated by certain Lefschetz classes, $N$, $N_0$ and $Q_0$ can be computed (or at least estimated) by writing down the other terms of the spectral sequence.

## 7. The spectral sequence of the preceding section

In this section we give some details concerning the spectral sequence $\{E_r\}$ of the map (51), (52). We set:

$$V^{n-1}_\xi = f^{-1}(\xi) \qquad (\xi \in C;\ V^{n-1} = f^{-1}(\xi_0)). \tag{55}$$

We now apply the general theory to this particular map. The corresponding notations in the general and the present particular case are:

$$\begin{matrix} X, & Y, & y, & f : X \to Y, & f^{-1}(y), & Y_1, & Y_2, \\ V^n, & C, & \xi, & f : V^n \to C, & V^{n-1}_\xi, & \Gamma, & \varnothing, \end{matrix} \tag{56}$$

where $\Gamma$ stands for the finite set of points $\gamma$ of $C$ such that:

(57) *if* $\xi \notin \Gamma$, $V^{n-1}_\xi$ *has no singularities*

(58) *if* $\gamma \in \Gamma$, $V^{n-1}_\gamma$ *has just one quadratic singular point* $c_\gamma$.

The last two notations in (56) show that the first critical set of $f$ is $\Gamma$ and that the second one is empty. Hence the spectral sequence $\{E_r\}$ of this map exists.

By the definition of the critical value of a map, (13) is not an isomorphism if $y_0 = \gamma \in \Gamma$. If the neighborhood $U$ of $\gamma$ in $C$ is a retract of $\gamma$, the kernel $N$ and cokernel $Q$ in (13) depend on $\gamma$ only and will be denoted by $N(\gamma)$, $Q(\gamma)$ respectively.

It can be proved that $V_\gamma^{n-1}$ is a retract of $f^{-1}(U)$, hence we have a homomorphism

$$(59) \qquad \begin{array}{ccc} & H(f^{-1}U) & \\ \wr\!\wr & & \searrow \\ H(V_\gamma^{n-1}) & \longrightarrow & H(V_\xi^{n-1}) \\ \ker = N(\gamma) & & \operatorname{coker} = Q(\gamma) \end{array} \qquad (\xi \in U).$$

As we supposed that $V_\gamma^{n-1}$ has just one quadratic singular point, we can prove that

$$(60) \qquad \textit{either} \quad N(\gamma) = 0 \quad \textit{and} \quad Q(\gamma) = R$$

$$(61) \qquad \textit{or} \quad N(\gamma) = R \quad \textit{and} \quad Q(\gamma) = 0$$

hold true in rational cohomology. (The first case is the "usual" one, the second is exceptional. For integral cohomology a less simple result can be proved.)

The first invariant term of the spectral sequence is $E_2$:

$$(62) \qquad E_2 = \textstyle\sum_{\gamma \in \Gamma} H(V_\gamma) + H^1(C - \Gamma, \mathscr{F}_0) + H^2(C - \Gamma, \mathscr{F}_0),$$

where $\mathscr{F}_0$ is a system of local coefficients in $C - \Gamma$, such that $\mathscr{F}_0(\xi) = H(V_\xi^{n-1})$ $(\xi \notin \Gamma)$.

In order to describe another form of $E_2$, we introduce

$$(63) \qquad H^p = \textstyle\sum_{\gamma \in \Gamma} H^p(V^{n-1}), \qquad H = \sum_{p \geqq 0} H^p,$$

which is a direct sum of $\mu$ groups isomorphic to $H^p(V^{n-1})$.

A closed path in $C - \Gamma$, beginning and ending at $\xi_0$, and enclosing only one point $\gamma$ of $\Gamma$, induces an automorphism $\theta_\gamma$ of $H(V^{n-1})$. If $l_\gamma$ denotes the Lefschetz class of $V^{n-1}$ which belongs to the value $\gamma$, it can be proved that

$$(64) \qquad h - \theta_\gamma h = (h, l_\gamma) l_\gamma,$$

where $(h, l_\gamma)$ is the Kronecker index. (This formula of Lefschetz [6] plays an important rôle in the theory; a new proof by J. Leray is published in [4].)

Now we define an endomorphism $\delta$ of $H$ (see (63)). Let $h$ be an element of (63):

$$h = \textstyle\sum_{\gamma \in \Gamma} h_\gamma.$$

Define its image $\delta h$ by

$$\delta h = \textstyle\sum_{\gamma \in \Gamma} (1 - \theta_\gamma) h_\gamma.$$

Then, setting

$$H_0 = \ker \delta,$$

we have:

$$H^1(C - \Gamma, \mathscr{F}_0) \approx H_0^p$$

$$H^2(C - \Gamma, \mathscr{F}_0) \approx H^p(V^{n-1})/L^p,$$

where $L^p$ denotes the sub-group of $H^p(V^{n-1})$ generated by the Lefschetz classes $l_\gamma$ (under our present assumptions $L^p = 0$ for $p \neq n - 1$). This gives us the term $E_2$ in a more explicit form:

$$E_2 = \textstyle\sum_{\gamma \in \Gamma} H(V_\gamma^{n-1}) + H_0 + H(V^{n-1})/L.$$

By (22), (23), (24) we can compute $d_2$ and we find:

$$d_2 : (\sum_{\gamma \in \Gamma} H(V_\gamma^{n-1})) \to H_0$$

$$\ker d_2 = N = \sum_{\gamma \in \Gamma} N(\gamma)$$

$$\operatorname{coker} d_2 = Q_0 = H_0/(H_0 \cap \sum_{\gamma \in \Gamma} I_\gamma) \subset Q = \sum_{\gamma \in \Gamma} Q(\gamma).$$

Thus we arrive at the following form of the third term of the spectral sequence:

$$E_3 = N + Q_0 + H(V^{n-1})/L.$$

Now, taking into account the degree properties of $d_3$, we known that it is a homomorphism

$$d_3 : N \to H(V^{n-1})/L.$$

This gives the final form (54) of the fourth term of the spectral sequence (where $N_0$ is a sub-group of $N$). By the properties of degree of the differentials $d_r$, $r \geqq 4$, $E_4$ is the last term of the spectral sequence, and it is a graded algebra $\mathrm{Gr}H(V^n)$ of $H(V^n)$. This completes the proof of (54). We already mentioned that our results concerning the cohomology of $V^n$, $W^n$ are simply applications of the equation (54).


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