XLI.

ON KNOTS. PART III.

[*Transactions of the Royal Society of Edinburgh*, 1885.]

(Chapter I. read June 1st, Chapter II. July 20th, 1885. *One change, small but important, was made during printing. It is described at the end of the paper.*)

THE following additional remarks are the outcome of my study of the polyhedral data for tenfold knottiness, which I received from Mr Kirkman on the 26th of last January. My main object was, as in the first chapter of Part II., to determine the number of different types; as well as the number of essentially different forms which each type can assume, as distinguished from mere deformations due to the mode of projection.

This study has been a somewhat protracted one, in consequence (1) of the great number of tenfold knots; (2) of the very considerable number of distortions of several of the types, many of which are essentially distinct while others present themselves in pairs differing by mere *reversion*; and especially (3) of the fact that the polyhedral method often presents some of the distinct forms of one and the same type projected from essentially different points of view (of which, in the present case, there are sometimes twelve in all). Reason (3) depends on the fact that Kirkman's method occasionally builds up various forms of one type on different bases of a lower order, and it really involves additional labour only; but great care is requisite to avoid confusion as regards (2), and in consequence I may not have fully reduced the final number of distinct types. [At the end of this paper I shall give a simple illustration of the nature of this special difficulty.]

The fact that I was dealing with knottiness of an even order induced me to commence the testing of the materials at my command by picking out the Amphicheirals. This led to some new considerations of a very singular nature, which are treated in the first of the following chapters. The second deals with the tenfolds as a whole.

I. *Various Orders and Classes of Amphicheirals.*

1. As one form of check on Kirkman's results, I sought for an independent method of forming all the amphicheirals of a given order. But, as will be seen below, we must be careful in this matter, which is not so simple as I first thought. I therefore commence by recalling the original definition of an amphicheiral.

In § 17 of my first paper I introduced it thus:—

An amphicheiral knot is one which can be deformed into its own perversion.

The word "deformed" was here used in the sense of alteration of form by mere change of point of view, or mode of projection; a process which leaves the number of corners in each mesh, and the *relative* positions of the various meshes, unchanged. This definition implies that the right and left handed meshes are similar in pairs and similarly situated in *congruent* groups; and it will be adhered to for the present, though we shall afterwards find that there are at least three other senses in which a knot may be called amphicheiral, and shall thus be led to speak of different *orders* and *classes* of amphicheirals. The above definition will then be considered to belong to amphicheirals of the *First Order* and *First Class.*

2. Suppose an amphicheiral knot to be constructed in cord, and extended over the surface of a sphere which swells out when necessary so as to keep the cord tight like the netting on a gazogene. Let its various laps be displaced until the several corresponding pairs of right and left handed meshes are made equal as well as similar. Trace its position on the sphere. Now suppose it to become rigid, and move it about on the surface of the sphere. We can again bring it to coincide with its former trace, but in such a way that each left-handed compartment now stands where the corresponding right-handed one was, and each right-handed where its corresponding left-handed was. Now such a displacement, as we know, can always be effected by a finite rotation about a diameter of the sphere as axis.

This axis, of course, cannot terminate (at either end) inside a mesh, else that compartment could not be shifted by the rotation to the original position of the corresponding one of the other kind. Hence either end of the axis must be at a crossing, or midway on the lap of cord passing through two adjoining crossings. A little consideration shows that if one end be at a crossing the other also must be at a crossing, and the whole must be a link. This is easily seen from the fact that, if one end of the axis be at a crossing, the four meshes which meet there must each exactly fit that next it when the whole is turned through a right angle; and the series which immediately surrounds these must possess a similar property, &c., &c. Thus the whole spherical surface must be covered with a pattern which consists of four equal and similar parts, each of which takes the place of the preceding one at every quarter of a rotation about the axis. And four laps of the string must therefore proceed all in the same way from one end of the axis to the other; since, if we can trace one lap of the string continuously from one crossing to the other, exactly the same must be true of the other three. [Of course, if the string cannot be traced from one crossing to the other, there must be two separate strings at least.]

Hence, for a true knot, both ends of the axis must be the middle points of laps; and therefore—

There must be two laps, at least, in every amphicheiral knot, each of which is common to a pair of corresponding right and left handed meshes; and when the whole is symmetrically stretched over a sphere the middle points of these laps are at opposite ends of a diameter.

3. With regard to the middle point of either of these laps, the various pairs of corresponding right and left handed meshes are situated at equal arcual distances measured in opposite directions on the same great circle. Hence if the whole be opened up at the middle point of either of these laps and projected on a plane symmetrically about the middle point of the other, the halves into which the plane figure is divided by any straight line passing through the latter point are congruent figures applied on opposite sides of that line as base; the point being, as it were, a *centre*. There are, thus, at least two ways of opening up any amphicheiral knot so as to exhibit this species of quasi-symmetry.

What precedes is on the supposition that the system of right, or of left, handed meshes can be applied to itself *in one way only*. If there be, as happens in some specially symmetrical cases, more than one way of doing this, there is a corresponding increase in the number of pairs of common laps, as defined in the preceding section.

It has also been assumed above that, on the sphere, the systems of right and left handed meshes are not only similar but *congruent*. The question of the possible existence of knots in which the system of right hand meshes shall be the *reversion* of the left hand system will be considered later.

4. We now obtain a perfectly general, though of course in one sense tentative, method of constructing amphicheirals of any order. Think of the result of § 3 as to the congruency of the halves of the knot when opened at either of the pair of corresponding laps. As a continuous line necessarily cuts the projection of a complete knot in an *even* number of points, the half figure which is to be drawn on one side of the common base must meet it in an *odd* number of points because one lap has been opened. Let these be called, in order, A, B, C, &c. Then, to form the half figure, these points must be joined in pairs, the odd one forming one end of the line whose other terminal is at the broken lap. These joining lines, and that with the free end, must be made to intersect one another in a number of points equal to *half* the knottiness of the amphicheirals sought. Every mode of doing this gives a figure which, when its congruent has been applied on the other side of the base, possesses the amphicheiral quasi-symmetry above described.

5. To ensure that the figure shall be a knot, and not a link or a set of detached figures, the following precautions are necessary. If A′, B′, C′, &c., in the congruent figure correspond to A, B, C, &c., in the original, they will be adjusted to one another as follows. (The case of five is taken as being sufficient to show the general principle.)

A B C D E
E′ D′ C′ B′ A′.

Now if B be joined with D, however the joining line be linked with the others, B will be joined to D′; and these parts will form, together, one closed circuit, so that the figure is not a knot. Similarly if A and E be joined. Similarly if A be joined to B, and *also* D to E. If C be the terminal of the free lap, so will C′; and again we have a figure consisting of more than one string.

It will be observed that the common characteristic of these excepted cases is that each possesses at least partial symmetry in the mode in which points to be joined are selected from the group. Hence the rule for selection is simply to *avoid every trace of symmetry.*

Even when this is done the final result may be a composite knot, *i.e.*, two or more separate knots on the same string. These can be detected and removed at once, so that it is not necessary to lay down rules for preventing their occurrence.

Repetitions of the same form from different points of view form the only really troublesome part of this process. These are inevitable, as we see at once from the fact that there may be several essentially different ways of cutting the complete quasi-symmetrical figure into congruent halves by lines meeting it in the *same* odd number of points. But it may also often be cut by one such line in one odd number of points and by another in a different odd number.

Still, with all these inherent drawbacks, the method is applicable without much labour to the tenfold amphicheirals; and it fully answered my purpose.

6. I had proceeded but a short way with the application of this method when I found that *there may be more than one distinct amphicheiral belonging to the same type.*

One example of this had been already given in § 48 of Part I. while I was dealing with amphicheirals, and again in Part II. in my census of eightfolds (Type V.), but I had carelessly passed it over as a special peculiarity probably due to the fact that the knot in question, though not composite, was constructed of portions each of which possessed, all but complete, the outline of the fourfold amphicheiral. From the point of view taken in § 4 above, however, the reason of the property is evident. For if the half knot, when the extremities of the strings are all held fixed, be capable of a distortion which shall change the relative positions of some of its meshes or the numbers of their corners, the same can of course be done with the congruent half. The whole preserves its type, and is still amphicheiral, but it becomes an essentially distinct form.

It will be seen that there is one type of tenfolds which has four different amphicheiral forms; another contains three; while there are four types each with two forms. The remaining seven amphicheiral types are either unique forms or have no amphicheiral distortion.

7. We are now prepared for one extension of the definition of an amphicheiral given in § 1 above. But we prefer to establish a new and independent definition:—thus

An amphicheiral knot of the First Order and Second Class is one which can be distorted into its own perversion.

Under this definition every distortion of an amphicheiral knot is included, even although it be such that its right and left handed meshes do not correspond to one another in pairs. For, whatever be the distortion, and whatever parts of the knot be affected by it, an exactly similar distortion might have been applied to the congruent parts of the original amphicheiral. These two distorted forms are, of course, capable of being distorted one into the other:—and that other is its perversion.

Every amphicheiral knot of the first order and second class corresponds to, and can be distorted into, at least one of the first class:—but the converse is not necessarily true.

8. Whether there are other classes of amphicheirals of the first order besides these I do not yet know. I have made attempts to construct a specimen of a supposed Third Class which should have the property of being changed into its own perversion by the twisting of a *single*, limited, portion, while the result could *not* be obtained by any simpler method. Such forms, if they exist, must in general be incapable of distortion into amphicheirals of either the first or the second class. This search has been fruitless. Among the requirements which it introduces, is the necessity for an ordinary amphicheiral in which *two* pairs of corresponding right and left hand meshes shall have one common corner; a condition which does not seem to be satisfied except by the simplest (amphicheiral) link, in which indeed it *must* be satisfied, as there are but four compartments in all. But this gives no satisfactory solution.

9. We may now take up the curious question raised in the last paragraph of § 3 above.

A simple method of producing arrangements in which the group of left handed meshes is similar to, but not congruent with, that of the right handed follows at once from the fact that, if one end of a diameter of a sphere trace a figure of any kind, the other end traces a similar and equal but (except in special cases of symmetry) non-congruent figure. These figures can, if we choose, be taken so as together to form one closed curve; and this, *along with a great circle of the sphere,* forms a link of two cords possessing the required property. On the plane we can carry out this construction by describing any figure within a circle, along with its inverse as regards the circle but on the opposite side of the centre; and arranging so that these may join into a continuous curve linked with the circle. But this arrangement *remains a link* when we unite the new curve with the circle by so introducing new meshes as to leave the whole possessed of the required property.

Or, we may trace any curve on a hemisphere, and its image (in the common base) on the other hemisphere. These, together with the great circle separating the hemispheres, give another link solution.

It is clear, from the essentially limited nature of the spherical surface, that these two methods give the only possible solutions of the problem:—*i.e.*, when the corresponding right and left handed meshes required by the conditions are made equal in pairs, the lines joining similarly situated points in them must either meet in one point (which, of course, must be the centre of the sphere), or they must be parallel.

10. As I did not at once see how to obtain solutions corresponding to *unifilar* knots by means of either of these methods, I asked Mr Kirkman whether he knew of a polyhedron which possessed the requisite property. The first he suggested to me corresponded, as I easily found, to a trifilar which belongs to the results of the first method above:—*i.e.*, one of its cords being taken as the circle, the other two were inverses of one another with regard to it. But, as soon as he mentioned to me that the polyhedron, corresponding to a composite knot consisting of two separate once-beknotted 5-folds on the same string, satisfies the special conditions of the present question (though inadmissible on other grounds), I saw why I had failed in obtaining unifilars by the first of the two methods above. For the purpose of avoiding trifilars from the first I had always made the curve traced by either end of the moving diameter (in the process of § 9 above) *cross* the great circle wherever it met it, so as to join that traced by the other end. No insertions of new meshes could then reduce the whole to a unifilar without depriving it of the property for which it was sought.

11. But if we make the closed curve traced by one end of the moving diameter *touch* the great circle in one point, the point of contact must of course be regarded as a *crossing*, while the circle and the closed curve necessarily fuse into one continuous line. The same happens with the curve traced by the opposite end of the diameter. Thus we may obtain with the greatest ease any number of unifilars satisfying the conditions. And it is clear that, by a slight extension of the definition above, all such knots will be brought under the general term amphicheiral. To make them true knots, *i.e.*, not composites, the curves traced by the ends of the diameter must intersect one another, which implies that they must each cut the great circle in two points at least besides touching it at one or more. Hence the lowest knottiness in which they can possibly occur is 10-fold; *i.e.*, 2 points of contact with the great circle, 4 intersections with it, and 4 intersections of the two branches.

This process fails when applied in connection with the second method of § 9, for it brings in triple points which cannot be opened up into three double ones without depriving the whole figure of the desired property.

12. The 10-fold, whose genesis is described in last section, has the form shown in Plate VII. fig. D, where the great circle is made prominent. It is easily recognised as the ordinary amphicheiral, fig. 31, of Plate VIII. The reason why it figures in both categories is that the arrangement of the right or left handed meshes, being symmetrical, is not changed by reversion. Thus every ordinary amphicheiral, which is in this sense symmetrical, belongs also to the new kind of amphicheirals with which we are now dealing.

Plate VII. fig. A shows a 12-fold knot, which is its own inverse with regard to the part drawn as nearly circular, and which is not amphicheiral in the ordinary sense.

Equal distortions of two corresponding parts give it the new form fig. B, which is also its own inverse with regard to the circular part.

But if, as in fig. C, we perform one of these distortions alone, the form is no longer its own inverse. But it is certainly amphicheiral, in the sense that it can be distorted into its own perversion. This is effected, of course, by undoing the single distortion which produced C from A, and inflicting the other of the pair of distortions which, together, produced B from A.

13. Thus there are at least four different senses in which a knot may be amphicheiral.

A (α) Those in which the systems of right and left hand meshes are similar and congruent.

A (β) Unsymmetrical distortions of any of the preceding, when such exist. [When the distortion is symmetrical the knot remains one of A (α).]

B (α) Those in which the systems of right and left hand meshes are similar but not congruent.

B (β) Unsymmetrical distortions of any of the preceding. [When the distortion is symmetrical the knot remains one of B (α).]

A and B may be spoken of as different *Orders*, the *First* and *Second*; α and β as *Classes*, *First* and *Second*. As already stated, the knot of fig. D belongs to both orders. But no knot can belong either to both classes of one order, or to the first of one order and the second of the other.

14. In fig. (D) the 10-fold (fig. 31) of § 11 is drawn so as to exhibit its symmetry. And we thus see at a glance that there are at least two ways (indicated by *heavier* lines, one continuous, the other dotted) in which we can choose the laps which are to form the circle with regard to which it is its own inverse.

Fig. 38 of the 10-folds, which by reason of its symmetry belongs to both orders of amphicheirals, can have its circles shown as in figs. (E) and (F).

15. But if we take a non-symmetrical knot of the kind B (α), such as fig. A above treated, we obtain some still more striking results as to the number of ways

in which we can choose the laps which form the circular portion. In this figure corresponding right and left handed meshes are marked with the same letter.

Thus, if we throw out the right hand mesh, *d*, from the contents of the circle and take in the left hand *d* instead, the figure (drawn to show the new circle) becomes fig. G.

If we throw out *f*, and take the amplexum instead, we obtain fig. H.

But, if we throw out from the circle *g*, *c*, and *e*, and take instead of them the corresponding external meshes, the figure takes the curious form K. Here the full line is the new boundary between the two halves of the figure. This new boundary, as well as the entire figure, is easily seen to be its own inverse with regard to the part bounded by the heavier portion of the full line. This, however, is only one of four ways in which it might be selected from the full line alone. Such modifications are very curious as well as numerous, but we cannot pursue them here.

16. In the upper rows of Plate VII. I have given the amphicheirals of the first class, up to the tenfolds inclusive. They are drawn on the principle of § 4 above, and the first form in which each presented itself has been preserved. A comparison of these, with the corresponding figures as drawn in Plate VIII. directly from Kirkman's results, is very instructive.

[*Added*, Oct. 19, 1885.—Though the general statement in § 11 above is true from the point of view there taken, there is a possibility of evading it. Thus, if we draw a figure like E, Plate VII., but with a four-pointed star inside, we get vii. of the 8-folds; which is thus shown to be an amphicheiral of the Second, as well as of the First, Order. But, if we try a three-pointed star, we get the simplest trifilar locking; as in Part I. § 42 (1), and Part II. § 8.]

II. *Census of Ten-fold Knottiness.*

17. Omitting composites, the number of separate types of 10-fold knottiness is, as shown in Plates VIII., IX., 123. Of these 48 are unique, while the remaining 74 give 315 distinct forms, 364 individuals in all. The largest number of distinct forms for one type is 12; and there are two such groups. One type which furnishes a group of 10, has 4 of them amphicheirals of the first order and first class, the remainder of the second class.

Each of the figures is drawn in the special deformation in which it is presented by the polyhedral method; and, for reference, the corresponding designation of the knot in Kirkman's list is appended to it.

18. Of the 107 partitions of 20, under the limits imposed by the nature of a knot, 52 only are utilised; the rest belonging to links, composites, &c. These are as below; each being followed by a distinctive letter, which will presently be employed (for brevity) in place of it.

For knots with 6 right handed and 6 left handed meshes:—

653222 A	544322 F
643322 B	543332 G
633332 C	533333 H
554222 D	444332 K
553322 E	443333 L

For 5 meshes of one class and 7 of the other:—

77222 *a*	65432 *l*	5522222 α
76322 *b*	65333 *m*	5432222 β
75422 *c*	64442 *n*	5333222 γ
75332 *d*	64433 *p*	4442222 δ
74432 *e*	55532 *q*	4433222 ε
74333 *f*	55442 *r*	4333322 ζ
66422 *g*	55433 *s*	3333332 η
66332 *h*	54443 *t*	
65522 *k*	44444 *u*	

For 4 of one and 8 of the other:—

8732 a	7652 f	43322222 θ
8633 b	7643 g	33332222 κ
8552 c	7544 h	
8543 d	6653 k	
7742 e	6554 l	

And for 3 of one and 9 of the other:—

992 p	965 s	332222222 λ
983 q	875 t	
974 r	776 u	

19. In Part II. of this series I arranged the types of each degree of knottiness in the order in which their respective deformations first appeared in Mr Kirkman's lists. This had the disadvantage of mixing up together types with very different relative numbers of right and left handed meshes. On the present occasion I have taken in the first rank the knots which have an equal number of meshes (six) of each kind, next those which have respectively 5 and 7, 4 and 8, &c. This will considerably simplify the process of seeking for any particular ten-fold in so long a

list. The arrangement of the various types in each rank, however, follows somewhat closely the order of their earliest appearance in the first list which I got from Mr Kirkman, that upon which I commenced the present work.

To identify any 10-fold, all that is necessary is to count the numbers of corners in the respective right and left handed meshes, look out the contracted expressions for the corresponding partitions of 20 in § 18, and then search below for the symbol, or pair of letters so obtained. Their *order*, of course, is immaterial, as it can be altered by a mere change of mode of projection. If the symbol occur more than once, a closer examination must be made, account being now taken of the way in which the right, *or* the left handed, meshes are coupled together. This is easily done as in § 20 of my first paper.

20. The number of distinct forms which I detected as not contained in Mr Kirkman's first list of 10-folds bears a far smaller ratio to the whole than was the case with the nine-folds. I consider that this is due not to my remissness, but to Mr Kirkman's improvements in his methods, *i.e.*, rather to the non-existence than to the non-detection of omissions; and I think it is improbable that any distinct variety of a recognised type has escaped detection. Thus in the present census some types may be omitted (this is more likely to be true of unique types than of others); and I may have, as already indicated, grouped in two or more smaller detachments the varieties of one and the same type. But the possibility of either defect is due to the somewhat tentative nature of the methods employed.

The guarded way in which I spoke (Part II., § 1) of the completeness of the *Census* has been justified by a recent observation made by Mr Kirkman, viz., a 9-fold not included either in his list or in mine. Fortunately this knot, figured as fig. L., Pl. VII., is not a new type but a distinct form of type VI. of the 9-folds as shown in the Plate attached to Part II. My methods ought to have supplied this additional member of a group, of which some forms had been furnished by Kirkman; but I had not, at the time, much readiness in applying them. The labour of the 10-folds has made me much more skilful than before in this matter.

21. In the following list, the order is the same as in the plates. The symbols for each knot are so written that the second, in all cases, corresponds to the group of meshes to which (as the figure happens to be drawn) the amplex belongs.

The various Types of Ten-folds with their distinct Forms.

Six right and six left hand meshes; 24 non-unique types, 14 unique; 133 individual distinct forms in all. Amphicheirals of the first order and first class are marked by a bar over the symbol instead of a repetition.

1. $\overline{C}$, $\overline{G}$, GC, GG, CG, KC, KG, $\overline{G}$, GK, $\overline{K}$. 2. FC, GF, GF, GF.
3. HB, LB, BC, GB, BG, KB, HF, FC, FG, LF, GF, FK.
4. GE, KE, GE, GE, BK, GB, GB, KB, BG, GB, EK, EG.
5. FF, KB, FB, BF, FF, FK. 6. GE, LE, KE.
7. FE, FG, FB, EB, EG, EE, FE, FG, FB. 8. FK, KF. 9. $\overline{F}$, FF, $\overline{F}$. 10. GF, KF.
11. BF, KE, BB, GF, KG, GB, EF, KB, BE. 12. LF, LF, FH. 13. KG, $\overline{K}$, $\overline{G}$.
14. $\overline{B}$, $\overline{E}$, $\overline{G}$, EB, BG, EG. 15. $\overline{A}$, EA, $\overline{E}$. 16. BB, FB, KB, KF, FB, FF.
17. GF, GF. 18. FD, DB, KD. 19. FA, KA, FA, GA, GA, EA.
20. BA, KA, KA, AF. 21. GA, GA, AF, BA. 22. FB, $\overline{B}$, $\overline{F}$. 23. DB, FD.
24. EA, AA. 25. KG. 26. $\overline{G}$. 27. $\overline{F}$. 28. LK. 29. $\overline{K}$. 30. LG.
31. $\overline{F}$. 32. EF. 33. FE. 34. EF. 35. $\overline{D}$. 36. AD. 37. $\overline{A}$. 38. $\overline{H}$.

Five meshes of one kind, seven of the other. Forty-three non-unique types, twenty-one unique—200 distinct forms in all.

39. *εs, εs, εp, γp, sγ, sγ.* 40. *εm, εm, εf.* 41. *εl, εe, γl, γe, ηe, ηl, ζl, ζe, eζ, lζ.*
42. *ζq, γl, εl, γq, εq, ζl.* 43. *εf, εm, εp, εs, γf, γm, γs, γp, pζ, sζ, ζm, ζf.* 44. *sε, tε.*
45. *ζd, ζh, εh, εh, εd, εd, hζ, dζ.* 46. *ζl, ζl, ηl.* 47. *kγ, εk, ηk, ζk.*
48. *εh, εd, εm, hβ, mβ, dβ.* 49. *ζc, ζc, ηc, γc, ζc, εc.* 50. *gγ, εg, ζg, ζg.*
51. *ζb, ζb, ζb, bε, γb.* 52. *βb, εb, hβ, εh, εh, εb.* 53. *εr, βr, εl, εl, βl, lβ.* 54. *pε, tε.*
55. *sβ, tβ, mβ, sβ, pβ, sβ.* 56. *δf, δp, δs.* 57. *nγ, γn, εn, εn.*
58. *ζr, rγ, εe, γe, ζe, εr.* 59. *δl, δq.* 60. *εe, εγ, lγ, εl, ζl, ζe.*
61. *εe, εl, εt, tβ, eβ, lβ.* 62. *eβ, pβ, lβ, εl, εp, εe.* 63. *εk, kγ, ζk.* 64. *βl, βq, sβ.*
65. *rβ, εc, εr, βc.* 66. *εb, βb, εb.* 67. *dβ, mβ, βl.* 68. *dβ, βl, sβ.*
69. *εg, gγ.* 70. *δd, lδ.* 71. *βr, tβ, eβ.* 72. *lβ, rβ.* 73. *pα, tα, lα, rα.*
74. *εc, rβ, rε, cβ.* 75. *εk, βk.* 76. *βc, εc.* 77. *εa, εa, aβ.* 78. *bβ, hβ.*
79. *βc, εc.* 80. *rα, αc.* 81. *hα, αb.* 82. *ζu.* 83. *sε.* 84. *εt.* 85. *ζt.*
86. *ζr.* 87. *ηs.* 88. *pζ.* 89. *εr.* 90. *ζn.* 91. *εq.* 92. *rε.* 93. *ζs.*
94. *mζ.* 95. *rε.* 96. *εl.* 97. *lε.* 98. *εl.* 99. *εh.* 100. *βk.* 101. *bβ.* 102. *αa.*

Four meshes of one kind, eight of the other. Seven non-unique types and eight unique—twenty-five distinct forms in all.

103. κf, θf. 104. κa, κa, θa. 105. κd, κg, dθ, gθ. 106. κc, θc. 107. θd, lθ.
108. θb, gθ. 109. θl, θh. 110. κk. 111. lκ. 112. κh. 113. κg.
114. θk. 115. θf. 116. θe. 117. aθ.

Three meshes of one kind, nine of the other. Six unique types.

118. λu. 119. λq. 120. λp. 121. λs. 122. λr. 123. λt.

22. The nature of the special difficulty hinted at in the beginning of the paper will be easily seen from the simple case illustrated by the four figures M, Plate VII. They denote various forms of the type 40 of Plate VIII.

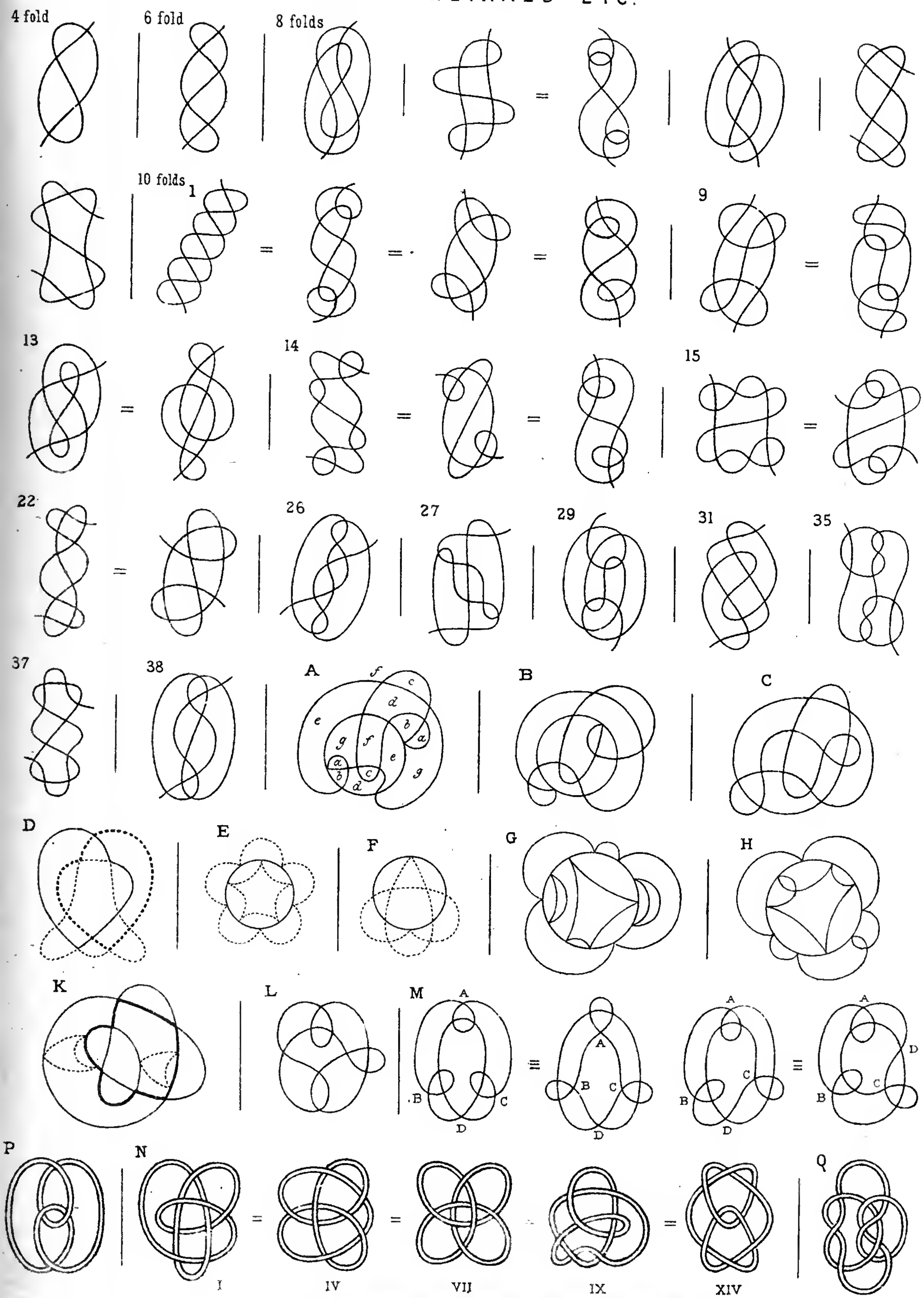
It will be noticed that the crossings A, B, C may, one, two, or all, be changed from one lap of the string to the other, as shown in the second figure. Also D may be transferred to a position between A and B, or between A and C. There are thus two positions for each of A, B, and C; and three positions for D; giving 24 combinations in all. But it is clear that we need not shift D at all, so far as the outline of the figure is concerned; for a mere rotation of the whole in its own plane (as A, B, and C are similar to one another) will effect this. Then a change of B will merely give the *reverse* of the figure obtained by changing C. Again, by inverting the first figure about a point in the inner mesh, we get the second. If we had changed C, and then inverted, we should have got the same figure as by changing simultaneously A and B. By changing C alone in the first, we get the third; but by shifting D in the first we get the fourth; and these two are obviously each the reverse of the other. Thus the 24 figures reduce to the three shown in Plate VIII. As another example, take the third form of the third type of 10-folds as given in Plate VIII. Two of the crossings on its external boundary can be shifted, but each to one other place only. The form itself, and the same with one or both of these crossings shifted, give a set of four; each of which can take five new forms by the shifting of other crossings. But it will be found that the 24 forms thus obtained are identical in pairs;—thus reducing to the 12 given in the Plate.

23. Mr Kirkman informs me that he has nearly completed the enumeration and description of the polyhedra corresponding to the unifilar 11-folds. I hope, therefore, at some future time to lay before the Society the census of 11-fold knottiness. This was the limit to which I ventured to aspire nearly two years ago, in a paper* which, I am happy to think, directed Mr Kirkman's attention to the subject.

24. It must be remembered that, so far as these instalments of the census have gone, we have proceeded on the supposition that in each form the crossings have been taken *over and under alternately*. But, as was shown in § 13 of Part I., as soon as we come to 8-folds we have some knots which may preserve their knottiness even when this condition is not fulfilled. These ought, therefore, to be regarded as proper knots and to be included in the census as new and distinct types. This is a difficulty of a very formidable order. It depends upon the property which I have called *Knotfulness* (Part I. § 35; II. § 6), for whose treatment I have not yet managed to devise any but tentative methods.

To show, by a single case (even though not thoroughly worked out), of how great importance is this consideration, I have appended to Plate VII. the five figures N; with the nature of each crossing indicated. The numbers affixed show the positions they occupied in the census of 8-folds, when the crossings were alternately over and under. *Then* they were all unique knots, incapable of any change of form.

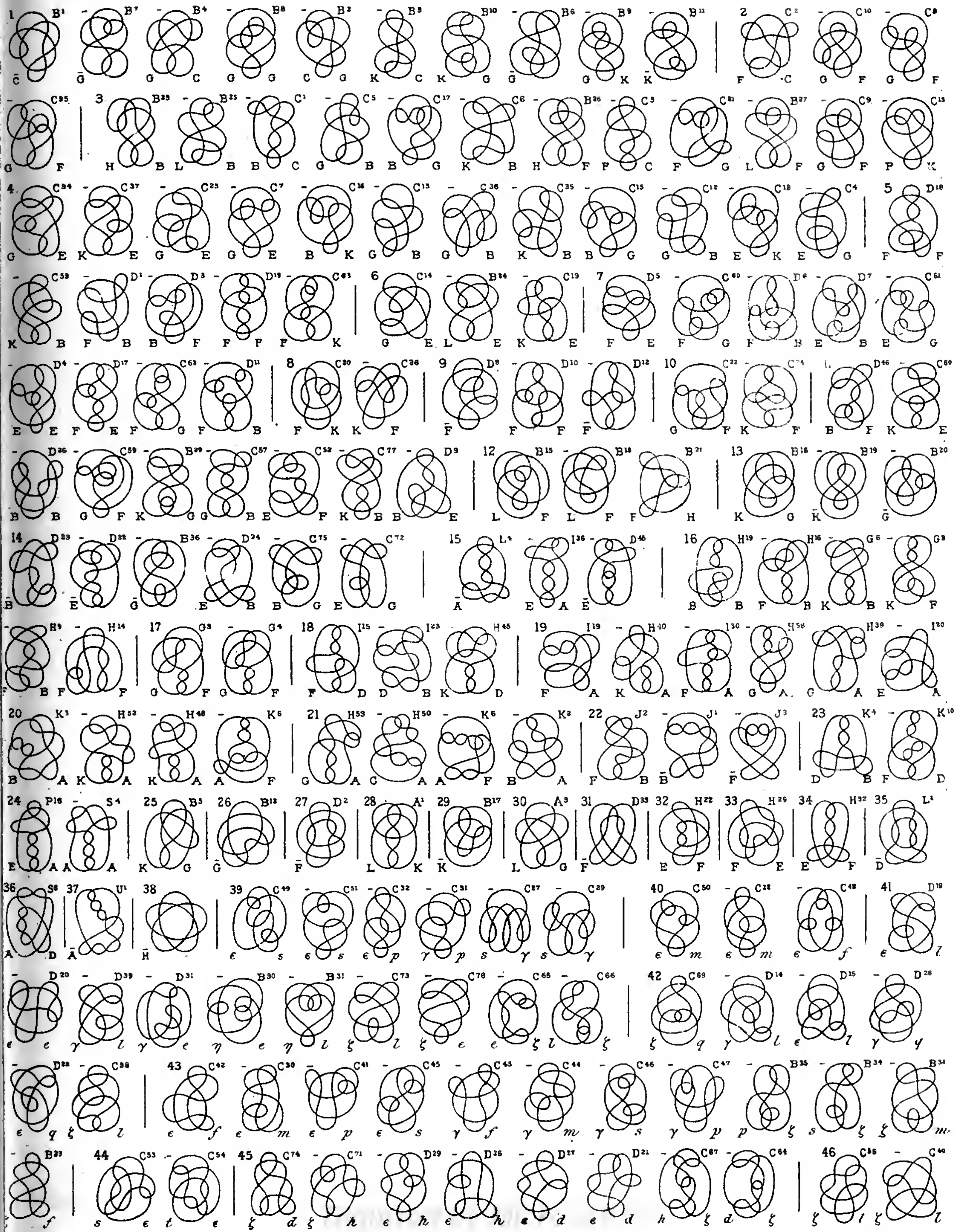
* "Listing's Topologie," § 22, *Phil. Mag.*, Jan. 1884. [To be reprinted below. 1898.]



P G Tait del

To face p. 346.

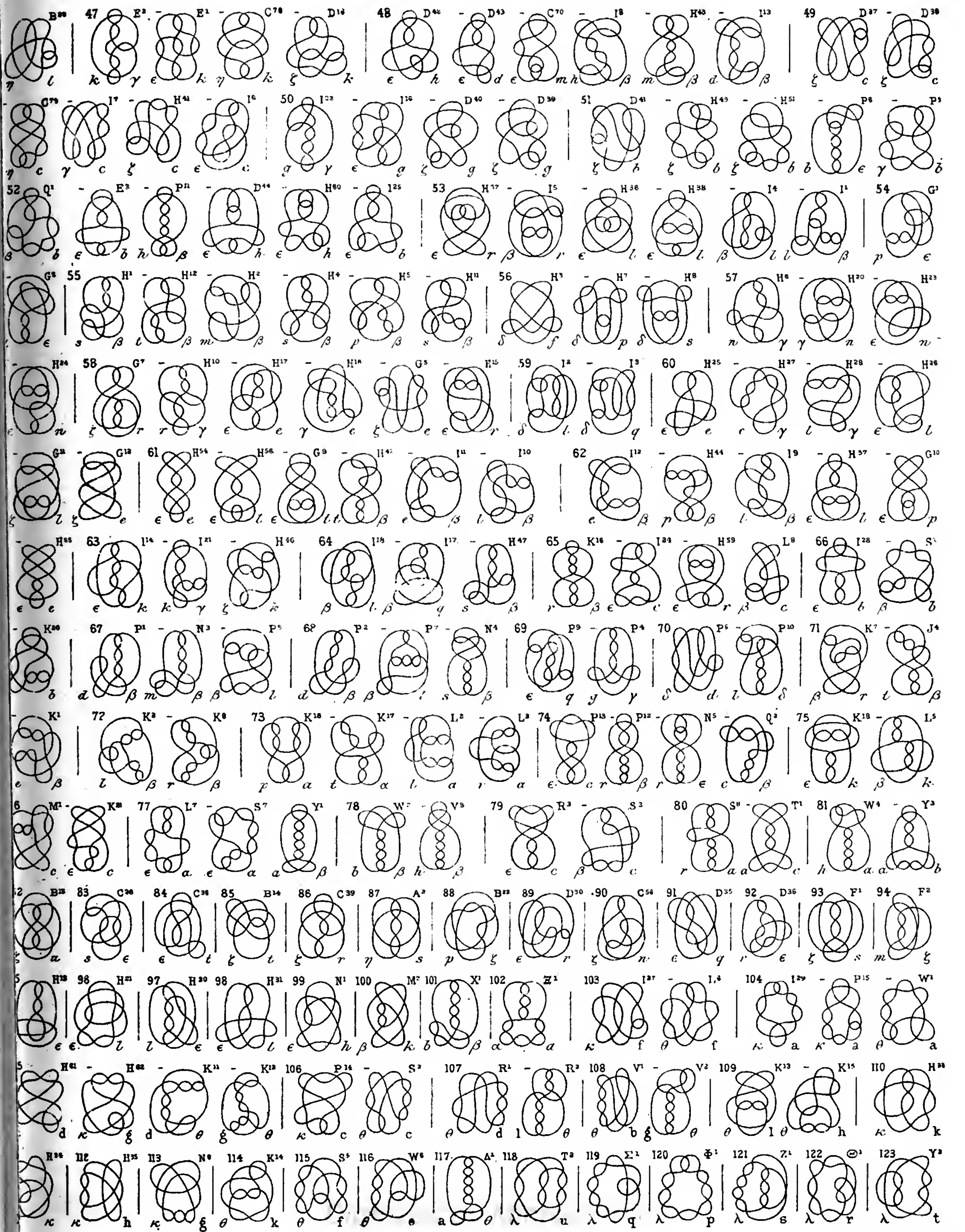
TENFOLD KNOTTINESS.



P G Tait del.

To follow Plate VII.

TENFOLD KNOTTINESS.



Tait del.

To follow Plate VIII

Now they are capable of being changed into one another. The linked trefoils in N, xiv. are perversions of one another. But we may have them of the same kind, and the link such that there shall be continuations of sign. This was briefly treated in Part I. § 42, 1. How many new types may by this process be added to the census, I have not yet made out with certainty even for the 8-folds.

P.S.—I may introduce here, as a note on Part I. of this series of papers, a remark or two with reference to the three-ply plaits treated there; in § 27 as fully knotted, and in § 42, 1, as fully beknotted. First, it is obvious that the 4-fold, as first drawn in § 17, should have been repeated in Plate V., at the head of the series of figures 15, 16, 17, &c. It is the case of $3n+1$ of § 27, with $n=1$. Secondly, with its crossings arranged as in fig. P, Plate VII. of the present paper, it should have come in before figs. 24 and 25 of Plate VI., Part I., in a form reducible to the ordinary trefoil. Fig. 25 of that Plate puzzled me much at the time when I drew it, for I could not account for the production of a 3-fold and a 5-fold (linked) from a figure possessing a peculiar kind of (cyclonic?) symmetry round an axis. The figure is *accurate*, but I now see that it gives an erroneous impression of the true nature of the knotfulness. The correct idea is at once obtained from Plate VII., fig. Q, of the present paper. The knot is an irreducible trefoil, with a second of the same character tied *twice* through one of its three-cornered meshes.

(*Added, September* 3, 1885.)

Three days ago I received from Mr Lockyer a copy of a most interesting pamphlet "On Knots, with a Census for Order Ten," a reprint from the *Trans. Connecticut Acad.*, vol. VII., 1885. The author, Prof. Little of the State University, Nebraska, has made an independent census of 10-fold knots; employing the partition method, with some new special rules analogous to those in Mr Kirkman's recent paper. So far as I can judge from a first hasty comparison of the mere number of types and forms in each class, there are important discrepancies between this census and my own. One of these, at least, is due to a slip on my part; and, as my paper was not printed off when I detected it, I have taken the opportunity of correcting it both in the text and in the corresponding Plate. I had failed to notice that the two forms which now appear under No. 109 really belong to one type. Hence I have had to reduce by one the number of the distinct 10-fold types which was originally given in my paper. I hope in time to make a full comparison of the two versions of the census. Meanwhile I may note that there is one omission, and also one duplicate, in Class VI. of Mr Little's version. This duplicate has led him to insert one type too many.

More than a month ago I received from Mr Kirkman the full polyhedral data for the census of 11-folds, which I hope soon to undertake. The number of forms is so great, and the time I can spare for the work so limited, that I cannot promise it at an early date. [This arduous work was kindly undertaken by Prof. Little, who, in 1890, gave the 357 types in Plates I., II., *Trans. R.S.E.*, vol. XXXVI., 1898.]