XL.

ON KNOTS. PART II.

[*Transactions of the Royal Society of Edinburgh,* Vol. XXXII. Read 2nd June, 1884.]

ONE main object of the present brief paper is to take advantage of the results obtained by Kirkman*, and thus to extend my census of distinct forms to knottiness of the 8th and 9th orders; for the carrying out of which, by my own methods, I could not find time. But I employ the opportunity to give, in a more extended form than that in the short abstract in the *Proceedings*, some results connected with the general subject of knots, which were communicated to the Society on January 6, 1879, as well as others communicated at a later date, but not yet printed even in abstract.

I. *Census of* 8-*Fold and of* 9-*Fold Knottiness.*

1. The method devised and employed by Kirkman is undoubtedly much less laborious than the thoroughly exhaustive process (depending on the *Scheme*) which was fully described and illustrated in my former paper†; but it shares, with the *Partition* method, which I described in § 21 of that paper and to which it has some resemblance, the disadvantage of being to a greater or less extent tentative. Not that the rules laid down, either in Kirkman's method or in my partition method, leave any room for mere guessing, but that they are too complex to be always completely kept in view. Thus we cannot be absolutely certain that by means of such processes we have obtained all the essentially different forms which the definition we employ comprehends. This is proved by the fact that, by the partition method, I detected certain omissions in Kirkman's list, which in their turn enabled him to discover others, all of which have now been corrected. And, on this ground, the present census may still err in defect, though such an error is now perhaps not very probable.

* The Enumeration, Description, and Construction of Knots with fewer than Ten Crossings. *Trans. R.S.E.* XXXII.

† No. XXXIX. above.

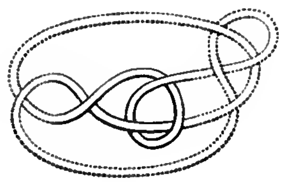
On the other hand, the treatment to which I have subjected Kirkman's collection of forms, in order to group together all mere varieties or transformations of one special form, is undoubtedly still more tentative in its nature; and thus, though I have grouped together many widely different but equivalent forms, I cannot be *absolutely* certain that all those groups are essentially different one from another.

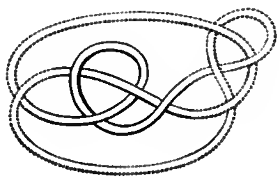
Unfortunately these sources of possible error, though they tend (numerically) in opposite directions, and might thus by chance compensate one another so far as to make the assigned numbers of essentially different forms accurate, cannot in any other sense compensate. In other words, there may still be some fundamental forms omitted, while others may be retained in more than one group of their possible transformations. Both difficulties grow at a fearfully rapid rate as we pass from one order of knottiness to the next above; and thus I have thought it well to make the most I could of the valuable materials placed before me; for the full study of 10-fold and 11-fold knottiness seems to be relegated to the somewhat distant future.

2. The problem which Kirkman has attacked may, from the point of view which I adopt, be thus stated:—"*Form all the essentially distinct polyhedra* (whether solids, quasi-solids, or unsolids) which have three, four, &c., eight, or nine, four-edged solid angles.*" Thus, in his results, there is no fear of encountering two different projections of the same polyhedron; or, in the language of my former paper, no two of his results will give the same scheme. Thus there is no one which can be formed from another by the processes of § 5 of my former paper.

3. But, when a projection of a knot is viewed as a polyhedron, we necessarily lose sight of the changes which may be produced, by *twisting*, in the knot itself when formed of cord or wire; a process which (without introducing nugatory crossings) may alter, often in many ways, the character of the corresponding polyhedron. This subject was treated in §§ 4, 11, 14, &c. of my former paper. But it is so essential in the present application that it is necessary to say something more about it here. It would lead to great detail were I to discuss each example which has presented itself, especially in the 9-folds; but they can all be seen in Plate VI., by comparing together two and two the various members of each of the groups.

The following example, however, though one only of several possible transformations is given, is sufficiently general to show the whole bearing of the remark, so far at least as we at present require it.





* This word is objectionable, on many grounds, in the present connection. But a more suitable one does not occur to me; and the qualification (given in brackets) will prevent any misconception. Of course no projection of a *true* polyhedron can be cut by a straight line in two points only.

It is obvious that either figure may be converted into the other, by merely rotating through two right angles the part drawn in full lines, the dotted part of the cord being held fixed. Also, the numbers of corners or edges in the right and left-handed meshes in these two figures are respectively as below:—

55332	and	64332
443322		433332.

These numbers would necessarily be *identical* if the forms could be represented by the same scheme. As will be seen by the list below, § 6, these are respectively the second, and the sixth, of the group of equivalent forms of number VIII of the ninefold knots. (See Plate VI.)

The characters of the various faces of the representative polyhedra (so far at least as the number of their sides is concerned) are widely different in the two cases. [Mr Kirkman objects to this process that it introduces twisting of the cord or tape *itself*. No doubt it does, or at least seems to do so, but the algebraic sum of all the twists thus introduced is always zero; *i.e.*, by "ironing out" the tape in its new form, all this twist will be removed. I have often used a comparison very analogous to this, to give to students a notion of the nature of the kinematical explanation of the equal quantities of + and − electricity, which are always produced by electrification. If the two ends of a stretched rope, along whose cylindrical surface a generating line is drawn, be fixed, and torsion be applied to the middle by means of a marlinspike passed through it at right angles, one half of the generating line becomes a right-handed, the other an equal left-handed cork-screw. Thus the algebraic sum of the distortions is zero. And, in consequence, if the rope be untwistable (the *Universal Flexure Joint* of § 109 of Thomson and Tait's *Natural Philosophy*) and endless, the turning of the spike merely gives it rotation like that of a vortex-ring. Such considerations are of weighty import in many modern physical theories.]

As will be seen, by an examination of the latter part of Plate VI., even among the forms of 9-fold knottiness there are several which are capable of more than one different changes of this kind. Some of these I may have failed to notice. But it is worthy of remark that the 8-folds seem, with two exceptions, to resemble the 7-folds in having at most two distinct polyhedral forms for any one knot.

4. Kirkman's results for knottiness 3, 4, 5, 6, 7, when bifilars and composites are excluded, agree exactly with those given in my former paper. I have figured these afresh in Plate VI., in the forms suggested by Kirkman's drawings, omitting only the single 6-fold, and the single 7-fold, which are composite knots.

As will be seen in the Plate, where they are figured in groups, there are but 18 simple forms of 8-fold knottiness. Besides these there are 3 not properly 8-fold, being composite (*i.e.*, made up of two *separate* knots on the same string); either two of the unique 4-fold, or a trefoil with one or other of the two 5-folds. These it was not thought necessary to figure, especially as they may present themselves in a variety of forms.

And the Plate also shows that there are 41 simple forms of 9-fold knottiness. Besides these, and not figured, there are 5 made up of two mere separate knots of lower orders, and one which is made up of three separate trefoils.

5. Thus the distinct forms of each order, from the 3rd to the 9th inclusive, are in number

1, 1, 2, 4, 8, 21, 47;

or, if we exclude combinations of separate knots,

1, 1, 2, 3, 7, 18, 41.

The later and larger of the numbers in these series, however, would be considerably increased if we were to take account of arrangements of sign at the crossings, other than the alternate over and under which has been tacitly assumed; for these are, in certain cases, compatible with non-degradation of the order of knottiness. This raises a question of considerable difficulty, upon which I do not enter at present. Applications to one of the 8-folds and to one of the 9-folds will be found in my former paper, § 42 (1).

Another interesting fact which appears from Plate VI. is, that there are six distinct amphicheiral forms of 8-fold knottiness: at least if we include one, not figured, which consists of two separate 4-folds; in which case we must consider that there are two six-fold amphicheirals, the second being the combination of right and left-handed trefoils, described in § 13 of my former paper. Thus the number of amphicheirals is, in the 4-fold, 6-fold, and 8-fold knots respectively, either 1, 2, 6, or (if we exclude composites), 1, 1, 5. All but two of these 8-fold amphicheirals were treated in my former paper, two having been separately figured, and the other being a mere common case of the general forms of § 47.

Finally, as a curious addition to the paragraphs on the genesis of amphicheiral knots, given in my first paper, I mention the following, which is at once suggested by the amphicheiral 6-fold:—Keeping one end of a string fixed, make a loop on the other; pass the free end through it and across the fixed end; pass the free end again through the external loop last made, then across the fixed end, and so on indefinitely. The second time the fixed end is reached we have the trefoil (if the alternate over and under be adhered to), the third time we have the amphicheiral 6-fold; and, generally, the nth time, a knot of $3(n-1)$ fold knottiness, which is amphicheiral if n is odd. Three of these were, incidentally, given in my former paper.

But, reverting to the main object of my former paper, we now see that the distinctive forms of less than 10-fold knottiness are together more than sufficient (with their perversions, &c.) for the known elements, as on the Vortex Atom Theory.

6. From the point of view of theory, as suggested in §§ 12, 21, of my former paper, it may be well to give here the partitions of $2n$ which correspond to true knots—for the values of n from 3 to 9 inclusive. The various partitions, subject to the proper conditions, are all given, in the order of the number of separate parts in each; those

which have a share in one or more of the true knots, as given in the Plate, are printed in larger type.

$n=3$

- **33**
- **222**

$n=4$

- 44
- 422
- **332**
- 2222

$n=5$

- **55**
- 532
- **442**
- 433
- 4222
- **3322**
- **22222**

$n=6$

- 66
- 642
- 633
- **552**
- **543**
- 444
- 6222
- 5322
- 4422
- **4332**
- 3333

$n=6$ (*contd.*)

- 42222
- **33222**
- 222222

$n=7$

- **77**
- 752
- 743
- **662**
- 653
- **644**
- 554
- 7322
- 6422
- 6332
- **5522**
- **5432**
- **5333**
- 4442
- **4433**
- 62222
- 53222
- **44222**
- **43322**
- **33332**
- 422222
- **332222**
- **2222222**

$n=8$

- 88
- 862
- 853
- 844

$n=8$ (*contd.*)

- **772**
- **763**
- **754**
- 664
- **655**
- 8422
- 8332
- 7522
- 7432
- 7333
- 6622
- **6532**
- 6442
- **6433**
- **5542**
- 5533
- **5443**
- 4444
- 82222
- 73222
- 64222
- 63322
- **55222**
- **54322**
- **53332**
- **44422**
- **44332**
- **43333**
- 622222
- 532222
- 442222
- **433222**
- **333322**
- 4222222
- **3322222**
- 22222222

$n=9$

- **99**
- 972
- 963
- 954
- **882**
- 873
- **864**
- 855
- 774
- 765
- **666**
- 9522
- 9432
- 9333
- 8622
- 8532
- 8442
- 8433
- **7722**
- **7632**
- **7542**
- **7533**
- **7443**
- 6642
- **6633**
- **6552**
- **6543**
- 6444
- **5553**
- **5544**
- 93222
- 84222
- 83322
- 75222
- 74322
- 73332

$n=9$ (*contd.*)

- **66222**
- **65322**
- **64422**
- **64332**
- **63333**
- **55422**
- **55332**
- **54432**
- **54333**
- **44442**
- **44433**
- 822222
- 732222
- 642222
- 633222
- **552222**
- **543222**
- **533322**
- 444222
- **443322**
- **433332**
- **333333**
- 6222222
- 5322222
- **4422222**
- **4332222**
- **3333222**
- 42222222
- **33222222**
- **222222222**

The whole numbers of available partitions are thus in order:—

2, 4, 7, 14, 23, 40, 66.

Of these there are employed for knots proper only

2, 1, 4, 4, 12, 17, 36,

respectively. The remainder give links, or composite knots, or combinations of these. (See *Appendix.*)

To enable the reader to identify, at a glance, any knot of less than 10-fold knottiness, I subjoin the partitions corresponding to each figure in Plate VI. It is to be remembered that (as in § 15 of my former paper) deformations which are compatible with the *same scheme*, however they may change the appearance of a knot, do not alter the partitions. But it is also to be remembered that identity of partitions, alone, does not necessarily secure identity of form.

The 3, 4, 5, and 6-folds may be disposed of in a single line.

$n=3$	$n=4$	$n=5$		$n=6$		
33		442	55		543	552
222	$\overline{332}$	3322 ,	22222	$\overline{4332}$,	33222 ,	33222

Here the bar indicates not only that the right and left-handed partitions are alike in number and value, but also that they are similarly connected, *i.e.*, that the knot is amphicheiral.

For the Sevenfolds, we have

I.			II.			III.		
5333	or	4433	5432	or	5432	5432	or	4433
43322		43322	43322		33332	44222		44222

IV.	V.	VI.	VII.
644	5522	662	77
332222	44222	332222	2222222

For the Eightfolds,

I.	II.					III.		
	54322	or	54322	or	54322	53332	or	44332
$\overline{44332}$	53332		44332		43333	44422		44422

IV.	V.					VI.			VII.
5443	54322	or		or		6532	or	6532	
333322	44332		$\overline{54322}$		$\overline{44332}$	333322		433222	$\overline{43333}$

VIII.			IX.	X.			XI.		
6433	or	5443	5542	54322	or	54322	55222	or	55222
433222		433222	433222	44332		54322	44332		54322

XII.	XIII.	XIV.	XV.	XVI.	XVII.	XVIII.
${ \atop \overline{54322}}$	${6532 \atop 433222}$	${655 \atop 3322222}$	${763 \atop 3322222}$	${754 \atop 3322222}$	${ \atop \overline{55222}}$	${772 \atop 3322222}$

Finally, for the Ninefolds, the list is

I. ${44433 \atop 433332}$

II. ${63333 \atop 533322}$ or ${63333 \atop 443322}$ or ${54333 \atop 533322}$ or ${54333 \atop 443322}$ or ${44433 \atop 533322}$ or ${44433 \atop 443322}$

III. ${54333 \atop 443322}$ or ${44433 \atop 443322}$

IV. ${54432 \atop 533322}$ or ${54432 \atop 533322}$ or ${54432 \atop 443322}$ or ${54432 \atop 443322}$

V. ${44442 \atop 443322}$

VI. ${64332 \atop 443322}$ or ${55332 \atop 443322}$ or ${64332 \atop 443322}$ [or ${55332 \atop 443322}$]*

VII. ${54432 \atop 433332}$ or ${54432 \atop 433332}$

VIII. ${64332 \atop 443322}$ or ${55332 \atop 443322}$ or ${64332 \atop 533322}$ or ${55332 \atop 533322}$ or ${55332 \atop 433332}$ or ${64332 \atop 433332}$

IX. ${54432 \atop 443322}$

X. ${5553 \atop 3333222}$

XI. ${5544 \atop 3333222}$

XII. ${64422 \atop 433332}$ or ${64422 \atop 333333}$ or ${64422 \atop 533322}$ or ${64422 \atop 443322}$

XIII. ${55422 \atop 443322}$ or ${55422 \atop 533322}$ or ${55422 \atop 433332}$

XIV. ${65322 \atop 433332}$ or ${65322 \atop 433332}$ or ${65322 \atop 533322}$ or ${65322 \atop 443322}$

XV. ${65322 \atop 443322}$ or ${55332 \atop 443322}$ or ${55332 \atop 543222}$ or ${65322 \atop 543222}$

XVI. ${7632 \atop 3333222}$ or ${7632 \atop 3333222}$ or ${7632 \atop 4332222}$

XVII. ${64332 \atop 533322}$ or ${64332 \atop 443322}$ or ${54432 \atop 533322}$ or ${54432 \atop 443322}$

XVIII. ${64332 \atop 543222}$ or ${54333 \atop 543222}$ or ${54432 \atop 543222}$

XIX. ${55422 \atop 533322}$ or ${55422 \atop 443322}$

XX. ${55332 \atop 543222}$ or ${54432 \atop 543222}$ or ${54432 \atop 543222}$

* [See Part III. below, § 20, p. 344; and fig. L, pl. VII. 1898.]

XXI. $\genfrac{}{}{0pt}{}{7443}{4332222}$ or $\genfrac{}{}{0pt}{}{7443}{3333222}$ or $\genfrac{}{}{0pt}{}{6543}{3333222}$ or $\genfrac{}{}{0pt}{}{6543}{4332222}$

XXII. $\genfrac{}{}{0pt}{}{7533}{4332222}$ or $\genfrac{}{}{0pt}{}{6633}{4332222}$ or $\genfrac{}{}{0pt}{}{7533}{3333222}$ or $\genfrac{}{}{0pt}{}{6633}{3333222}$

XXIII. $\genfrac{}{}{0pt}{}{6543}{4332222}$ or $\genfrac{}{}{0pt}{}{5553}{4332222}$

XXIV. $\genfrac{}{}{0pt}{}{6552}{4332222}$ or $\genfrac{}{}{0pt}{}{6552}{3333222}$

XXV. $\genfrac{}{}{0pt}{}{64422}{443322}$ or $\genfrac{}{}{0pt}{}{44442}{543222}$ or $\genfrac{}{}{0pt}{}{44442}{443322}$ or $\genfrac{}{}{0pt}{}{64422}{543222}$

XXVI. $\genfrac{}{}{0pt}{}{66222}{443322}$ or $\genfrac{}{}{0pt}{}{66222}{543222}$ or $\genfrac{}{}{0pt}{}{66222}{443322}$

XXVII. $\genfrac{}{}{0pt}{}{5544}{4422222}$ or $\genfrac{}{}{0pt}{}{6543}{4422222}$

XXVIII. $\genfrac{}{}{0pt}{}{7533}{4332222}$ or $\genfrac{}{}{0pt}{}{6543}{4332222}$

XXIX. $\genfrac{}{}{0pt}{}{64422}{543222}$ or $\genfrac{}{}{0pt}{}{64422}{443322}$

XXX. $\genfrac{}{}{0pt}{}{7542}{4332222}$ or $\genfrac{}{}{0pt}{}{7542}{3333222}$

XXXI. $\genfrac{}{}{0pt}{}{65322}{543222}$ or $\genfrac{}{}{0pt}{}{55332}{543222}$

XXXII. $\genfrac{}{}{0pt}{}{44442}{552222}$ or $\genfrac{}{}{0pt}{}{64422}{552222}$

XXXIII. $\genfrac{}{}{0pt}{}{7632}{4422222}$ or $\genfrac{}{}{0pt}{}{6633}{4422222}$

XXXIV. $\genfrac{}{}{0pt}{}{7542}{4422222}$ or $\genfrac{}{}{0pt}{}{5544}{4422222}$

XXXV. $\genfrac{}{}{0pt}{}{44433}{333333}$

XXXVI. $\genfrac{}{}{0pt}{}{666}{33222222}$

XXXVII. $\genfrac{}{}{0pt}{}{864}{33222222}$

XXXVIII. $\genfrac{}{}{0pt}{}{882}{33222222}$

XXXIX. $\genfrac{}{}{0pt}{}{66222}{552222}$

XL. $\genfrac{}{}{0pt}{}{7722}{4422222}$

XLI. $\genfrac{}{}{0pt}{}{99}{222222222}$

It will be seen that the above list suggests many curious remarks. Thus, in the eightfolds, we have two *different* amphicheirals, each having the partitions $\overline{44332}$. Again, we have $\genfrac{}{}{0pt}{}{54322}{54322}$ for a knot which is not amphicheiral, as well as $\overline{54322}$ for one which is amphicheiral. (See § 47 of my former paper.) And we have $\genfrac{}{}{0pt}{}{54322}{44332}$ standing for two quite distinct knots. All these apparent difficulties, however, are due to the incompleteness of the definition by partitions *merely* (*i.e.*, as by Listing's Type-Symbol). For, in addition to this, it is requisite that we should know the relative grouping of the right-handed or of the left-handed partitions.

In the Plate I have inserted the designations given in my former paper to the various forms of 6-fold and 7-fold knottiness:—and I have also appended to each form the designation of the corresponding figure in Kirkman's drawings.

The Plate contains a great deal of information of a kind not yet alluded to in this paper. It gives, for instance, an excellent set of examples of *Knotfulness.* This term implies (§ 35 of my former paper) "*the number of knots of lower orders (whether interlinked or not) of which a given knot is built up.*" It is to be understood as applied to *simple* forms only; for we have set aside, as *composite* knots, all such as have any one component separable, so that it may be drawn tight without fastening together two laps belonging to one or two of the other components.

Thus, as a few of the examples of 2-fold knotfulness among the 8-folds, we have

VI. and XI. (3-fold and once-beknotted 5-fold);

and II. and V. (each two 4-folds); while

III., IX., and XIV. are different forms of two (linked) 3-folds.

Among the 9-folds we have, for instance,

XXX. and XXXIII. (4-fold and clear coiled 5-fold),

XVI. and XXVI. (3-fold and δ 6-fold),

XIV., XV., XVIII., and XXV. (4-fold and once-beknotted 5-fold).

But we have also

IV., XIII., XXIII., and XXIV. (linked 3-fold and 4-fold),

XX., XXVII. (two 3-folds, linked, and with one kink).

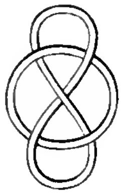
The analysis of self-locked knots, such as IV. and VII. of the 8-folds, and II., IX., X., XIX., &c., of the 9-folds, is considered below.

II. *Beknottedness.*

7. The question of *Beknottedness* (on which I have occasionally made short communications to the Society since my papers of 1876–7 were printed in a brief condensed form) has been again forcibly impressed on me while endeavouring to recognise identities among Kirkman's groups. I still consider that its proper measure is *the smallest number of changes of sign which will remove all knottiness.* But, shortly after my former paper was published, I was led to modify some ideas on the subject, which were at least partially given there. I had been so much impressed by the very singular fact of the existence of amphicheiral forms, that I fancied their properties might in great measure explain the inherent difficulties of this part of the subject. I have since come to see that this notion was to some extent based on an imperfect analogy, due to the properties of the 4-fold amphicheiral, and that the true difficulty is connected with *Locking.*

8. The existence and nature of this third method of entangling cords were first made clear to me by one of the random sketches which I drew to illustrate Sir W. Thomson's paper on *Vortex-Motion* [*Trans. R. S. E.*, 1867–8]. I had not then even imagined that the crossings in any knot or linkage could *always* be taken alternately over and under, though I found that I could make them so in all these sketches. The particular figure above referred to again presented itself, among others possessing a similar character, while I was studying the peculiar group of plaited knots

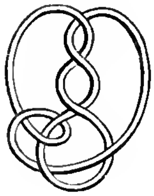
whose schemes contain the lettering in alphabetical order in the even as well as in the odd places. (See §§ 27, 42, of my former paper.) But I soon saw that, though I had first detected locking in those members of the group of plaits where *three* separate strings are involved, essentially the same sort of thing occurs in the other members of the group, though they are also proper knots in the sense of being each formed



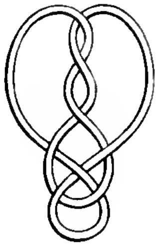
with a *single* continuous and endless string. And, as the above very simple example sufficiently shows, we can have locking, independent of either knotting or linking, with *two* separate strings. For it is clear that the irreducibility of this combination depends solely upon the sign of the *central* crossing. There is no real linking of the two cords, and there is obviously no knotting. But if the sign of any one of the crossings, except the central one, be changed, the whole becomes the simple amphicheiral link, the linking having been *introduced* by the change of sign. [This, as will be seen in § 14 below, is an excellent example of a case in which the key-crossing of a locking is also a root-crossing of a fundamental loop.]

9. We may therefore define, as one degree of locking, any arrangement, or independent part of an arrangement, analogous to that above (whether it be made of one, two, or three separate strings), the criterion being that the change of one sign unlocks the whole. But it is well to notice, again, that if, in the above figure, we change the sign of any crossing except the central one, we have one degree of linking left, and that this has in reality been *introduced* by the change of sign. This remark extends, with few exceptions, to more complex cases.

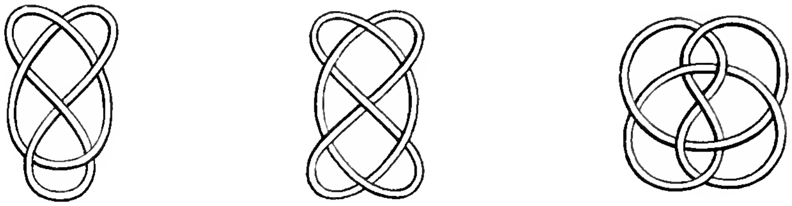
10. Thus, though the following 8-fold knot (which I reproduce from No. XXXIX. above, § 47, p. 314) does not, at first sight, appear to depend on locking, we have only to



make a simple transformation (as *ante*, § 3) to reduce it to the symmetrical form in which the single degree of locking is at once evident. It was by considering this knot, with its (quite unexpected) single degree of beknottedness, that I first saw the true bearing of locking in the present subject. (It is given as x. of the 8-folds in Plate VI.)

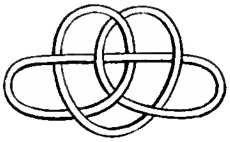


Other excellent instances of the same difficulty are the following. The first of these is completely resolved, the second changed to the 3-fold, while the third becomes apparently two linked trefoils, all by the change of the single crossing in the middle of the lock. But with the 9-fold knot (which is merely a different projection of Plate VI., fig. XXXV.) the trefoils are so linked after this operation, that the change of sign of one crossing of either resolves the whole. This is, however, much more easily seen



by at once changing the signs of the middle and of the lower (or the upper) crossing, for the whole is thus resolved. [This course is at once pointed out by the process of § 13 below, if we choose as *fundamental* crossings the three highest in the figure.] Hence the beknottedness is 1, 2, 2 in the last three figures respectively.

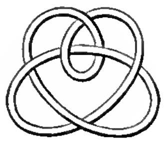
11. Another instructive example is afforded by the 8-fold knot below, which is figured as IV. on Plate VI.:—

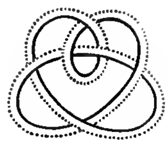


At a first glance it appears to be made of two once-linked trefoils, and therefore to have three degrees of beknottedness. But a little consideration shows that neither the trefoils nor the link have alternations of signs (*i.e.*, there is neither knotting nor linking), but that the whole is kept from resolution solely by the lap of cord which has been drawn as a straight line in the figure. This forms, as it were, the tail of a Rupert's drop; break it, and the whole falls to pieces. A change of sign of either of the interior crossings on that lap *makes* one trefoil; of either of the 4 lateral external crossings, the 6-fold amphicheiral; of the upper crossing, the 4-fold amphicheiral; and of the lower axial crossing, the 5-fold of one degree of beknottedness. All these modes of resolution lead to the result that the knot is of 2-fold beknottedness.

12. It is now obvious why, in consequence of locking and not of amphicheiralism as I first thought, the electro-magnetic test fails in certain classes of cases to indicate properly the amount of beknottedness. For it is clear that in pure locking there is no electro-magnetic work along the locked part of any one of the three courses involved. Hence, for the part of a knot or link which is locked, the electro-magnetic test necessarily gives an incorrect indication of beknottedness. Perhaps it may be said that, in such cases, beknottedness is not the proper name for this numerical feature of a knot:—but it is obviously correct *if defined as in* § 7 *above.*

13. A simple but thoroughly practical improvement on the methods given in my first paper for the graphical solution of Gauss' problem (extended) is as follows:—Draw the knot or link, as below, with a double line, like the edges of an untwisted tape, and dot (or go over with a coloured crayon) one of the two lines. Now it

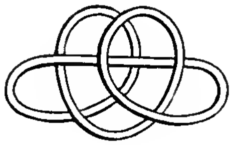


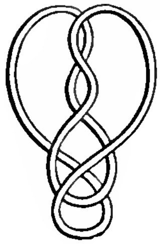


is easy to see that, of the four angles at a crossing, one angle is bounded by full lines, and its vertical angle by dotted lines. These will be called the *symmetrical* angles. Also it is clear that the electro-magnetic work has one sign for the crossings when the symmetrical angles are right-handed, and the opposite sign when they are left-handed. Thus we can at once mark each crossing as *r* or *l*, silver or copper, at pleasure. If the figure be a knot, and if we cut it along a line dividing a symmetrical angle, re-uniting the pairs of ends on either side of that line, the whole remains a knot (still with alternations of over and under if the original was so), but of knottedness at least one degree lower. When the line divides an unsymmetrical angle, the whole becomes (after re-uniting the ends, as before) two separate closed curves, in general linked and, it may be, individually knotted. [When we treat a *link* in this way at any of the linkings (*i.e.*, where two *different* strings cross one another), it becomes a knot. It is curious that by this process a knot is equally likely to be changed into a knot or into a link, while a link *always* becomes a knot.] This method has the farther advantage of showing at a glance the various sets of crossings which we may choose for omission (in the electro-magnetic reckoning), as due merely to the *coiling* of the figure, not to knotting, linking, or locking. For each such crossing must belong to a simple loop, which, for reference, we will call *fundamental.* Such a loop is detected immediately by its having (throughout) the full line or the dotted line for its external boundary, and therefore is necessarily closed at a symmetrical angle. If we now erase these fundamental loops in succession, till no crossings are left, the crossings at their bases form one of the groups which may be tried. When part of the knot has locking, it is sometimes necessary to try more than one of these groups before we arrive at the true measure of beknottedness. As this is a matter of importance, it may be well to discuss it a little farther.

14. When there is no beknottedness (whether true, or depending on linking or locking), the electro-magnetic work, with the proper correction for mere coiling, is certainly *nil.* But this *proper* correction requires to be found, and where there is locking its discovery sometimes presents a little difficulty. When there is no locking, all we need do is to draw the knot afresh, beginning at a point external to each of the fundamental loops, and making each crossing positive *when we first reach it.* It is evident that the fundamental loops or coils will now be simply laid on one another. The signs of *all* the crossings on any one loop may be changed, while that of the base of the loop is immaterial, and this process may be carried out with some or all of the other fundamental loops in any order. Compare the various signs in any state thus produced with those (alternate or not) of the original knot, so as to find the smallest number of changes necessary for its full resolution. The sign of the crossing at the base of each fundamental loop is simply to be disregarded. Another mode of going to work is to alter the signs at pairs of points where two fundamental loops cross, so as to diminish as far as possible the necessary number of real changes of sign. But we must be very careful in using this process, to see *that it does not introduce locking.*

15. When there is locking in part of the knot, the real difficulty is met with *only* if the crossing or crossings, which form as it were the key of the locked part, *must* also be taken as the base or bases of fundamental loops. In this case we commence the fresh drawing of the knot at a point exterior to the locking, but on the fundamental loop of which one of the key crossings forms the base. This ensures that the completion of the fundamental loop is effected by the *last* of the operations on the locked part. But the application of the method can be learned far more easily from an example or two than from any rules which could be laid down. Thus the following drawings represent the results of this method as applied to two of the knots already figured. In the first of these the two lower external crossings are taken





for the fundamental loops, and we see that the knot (if originally over and under alternately) requires for its full resolution only the change of sign of each of the two crossings which lie in its axis of symmetry. But, if we had chosen the crossings last mentioned as bases of fundamental loops, we should at once have felt the difficulty due to locking.

In the second, all four crossings in the axis of symmetry close fundamental loops; but the change of the sign of the *lowest* of these, alone (which is the key of the locked part), is required for the full resolution.

APPENDIX.

Note on a Problem in Partitions.

(Read July 7, 1884.)

In the partition method of constructing knots of any order, n, of knottiness, we have to select from the group of partitions of $2n$ those only in which no part is greater than n, and no part less than 2.

Thus, as given in the text, § 6, we have for sevenfold knottiness the series of partitions of 14;—but they are now arranged below in classes according to the value of the largest partition.

77	662	554	4442	33332	2222222
752	653	5522	4433	332222	
743	644	5432	44222		
7322	6422	5333	43322		
	6332	53222	422222		
	62222				

It is an interesting inquiry to find how many there are in each class, for any value of n. The number of classes is obviously $n-1$; and, if we remove from each the first partition (*i.e.*, that which is not inferior to any of the others), the remainders form a new set of classes of partitions which we may designate as

$$p_n^n,\ p_{n+1}^{n-1},\ p_{n+2}^{n-2},\ \dots\ p_{2n-2}^2$$

respectively;—where p_s^r is defined as the number of partitions of s, in which no partition is greater than r, and none less than 2.

Without explicitly introducing finite differences or generating functions it is easy to calculate the values of the quantity p_s^r;—and to put them in a table of double entry which can be developed to any desired extent by the simplest arithmetical processes. The method is similar to one which I employed some years ago for the solution of a problem in Arrangements (No. XXVII. above).

In the first place we see at once that if $r > s$

$$p_s^r = p_s^s.$$

Thus, if r denote the column, and s the row, of the table in which p_s^r occurs, all numbers in the row following p_s^s are equal to it. Thus the values of p_s^s enable us to fill up half the table. In the remaining half r is less than s; and by a dissection of this class of partitions, similar to that which was given above, we see that

$$p_s^r = p_{s-r}^r + p_{s-r+1}^{r-1} + \ \dots\dots + p_{s-2}^2 + p_{s-1}^1 + p_s^0,$$

where the two last terms obviously vanish; and the first term is obviously 1 in the case of $r = s$, unless $r < 2$, when it vanishes.

Hence, if the following be a portion of the table, the crosses being placed for the various values of p_s^r, *nil* or not, Values of r.

Values of s.	0	1	2	3	4	5	6	7	8
0	+	+	+	+	+	+	+	H	+
1	+	+	+	+	+	+	G	+	+
2	+	+	+	+	+	F	+	+	+
3	+	+	+	+	E	+	+	+	+
4	+	+	+	D	+	+	+	+	+
5	+	+	C	+	+	+	+	+	+
6	+	B	+	+	+	+	+	+	+
7	A	+	+	+	K	+	+	L	L

it will be seen at a glance that the above equation tells us to add the numbers A, B, C, D, E together, to find the number at K. This is quite general, so that L, in the second last column, is the sum of A, B, ..., H; and all the numbers beyond it, in the same row, are equal to it. In the table on next page, each number corresponding to the *first* L is printed in heavier type, and its repetitions are taken for granted.

Thus it is clear that simple addition will enable us to construct the table, row by row, provided we know the numbers in the first row and those in the first column. Those in the first and second columns are all obviously zero, as above. The rest of the first row consists of units. These are the values of p_0^r, *i.e.*, the first term of the expression above for p_r^s. Hence we have the table on the following page, which is completed only to $r = 17$, with the corresponding sub-groups.

From the table we see that $p_9^9 = 8$. Hence the partitions of 18, subject to the conditions, are in number

$$8 + 11 + 11 + 14 + 10 + 8 + 3 + 1 = 66,$$

which agrees with the detailed list in § 7 above.

[The rule is to look out the number p_n^n, and add it to all those which lie in the diagonal line drawn from it *downwards* towards the left. But the construction of the table shows us that this is the same as to look out p_{2n}^n at once.]

Similarly we verify the other numbers of partitions given in the text.

And it is to be remembered that p_n^n is the number of required partitions in which n occurs, and that *every one* of the class p_{n+r}^{n-r} has for its largest constituent $n - r$. Thus, looking in the table for p_7^7 and the numbers in the corresponding downward left-handed diagonal, we find the series

$$4 \quad 6 \quad 5 \quad 5 \quad 2 \quad 1,$$

which will be seen at once to represent the dissection of the partitions of 14 given above.

The investigation above was limited by the restriction, imposed by the theory of knots, that no partition should be less than 2. But it is obvious that the method of this note is applicable to partitions, whether unrestricted, or with other restrictions than that above. The only difficulty lies in the *bordering* of the table of double-entry. Thus, if we wish to include unit partitions, all we have to do is to put unit instead of zero at the place $r = 1$, $s = 0$, and develop as before. Or, what will come to the same thing, sum all the columns of the above table downwards from the top, and write each partial sum instead of the last quantity added, putting unit at every place in the second column.

Similarly, we may easily form the corresponding tables when it is required that the partitions shall be all even, or all odd.

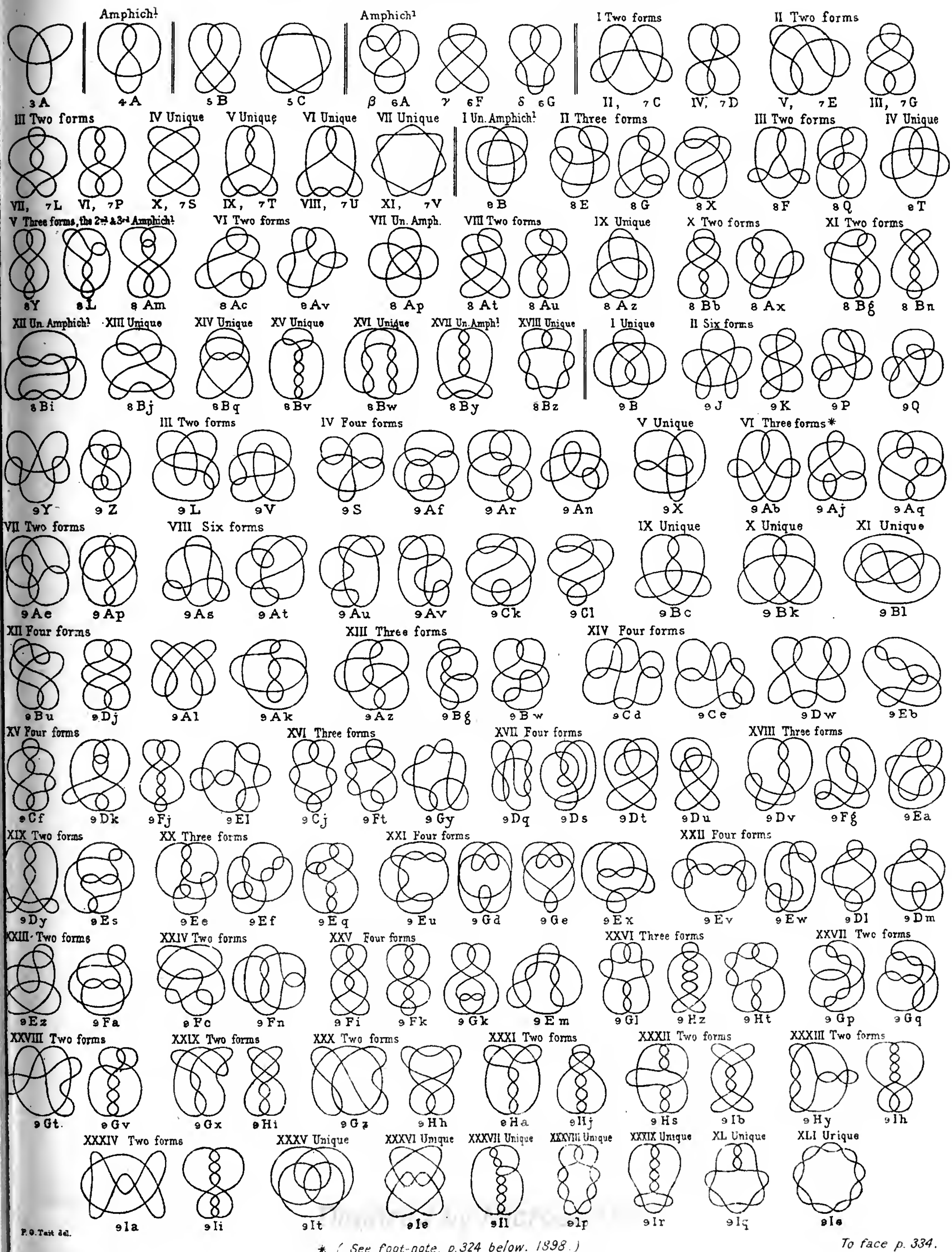
Table of the values of p_s^r; the number of partitions of s in which no one is less than 2, nor greater than r.

(*The values of* r *are in the first row, those of* s *in the first column.*)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	0	**1**	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1	0	**0**	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	0	0	**1**	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	0	0	0	**1**	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	0	0	1	1	**2**	.	.	.	.	.	.	.	.	.	.	.	.	.
5	0	0	0	1	1	**2**	.	.	.	.	.	.	.	.	.	.	.	.
6	0	0	1	2	3	3	**4**	.	.	.	.	.	.	.	.	.	.	.
7	0	0	0	1	2	3	3	**4**	.	.	.	.	.	.	.	.	.	.
8	0	0	1	2	4	5	6	6	**7**	.	.	.	.	.	.	.	.	.
9	0	0	0	2	3	5	6	7	7	**8**	.	.	.	.	.	.	.	.
10	0	0	1	2	5	7	9	10	11	11	**12**	.	.	.	.	.	.	.
11	0	0	0	2	4	7	9	11	12	13	13	**14**	.	.	.	.	.	.
12	0	0	1	3	7	10	14	16	18	19	20	20	**21**	.	.	.	.	.
13	0	0	0	2	5	10	13	17	19	21	22	23	23	**24**	.	.	.	.
14	0	0	1	3	8	13	19	23	27	29	31	32	33	33	**34**	.	.	.
15	0	0	0	3	7	14	20	26	30	34	36	38	39	40	40	**41**	.	.
16	0	0	1	3	10	17	26	33	40	44	48	50	52	53	54	54	**55**	.
17	0	0	0	3	8	18	27	37	44	51	55	59	61	63	64	65	65	**66**
18	0	0	1	4	12	22	36	47	58	66	73	77	81	83	85	86	87	.
19	0	0	0	3	10	23	36	52	64	75	83	90	94	98	100	102	.	.
20	0	0	1	4	14	28	47	64	82	95	107	115	122	126	130	.	.	.
21	0	0	0	4	12	29	49	72	91	110	123	135	143	150	.	.	.	.
22	0	0	1	4	16	34	60	86	113	134	154	168	180	.	.	.	.	.
23	0	0	0	4	14	36	63	96	126	155	177	197	.	.	.	.	.	.
24	0	0	1	5	19	42	78	115	155	189	220	.	.	.	.	.	.	.
25	0	0	0	4	16	44	80	127	171	215	.	.	.	.	.	.	.	.
26	0	0	1	5	21	50	97	149	207	.	.	.	.	.	.	.	.	.
27	0	0	0	5	19	53	102	166	.	.	.	.	.	.	.	.	.	.
28	0	0	1	5	24	60	120	.	.	.	.	.	.	.	.	.	.	.
29	0	0	0	5	21	63	.	.	.	.	.	.	.	.	.	.	.	.
30	0	0	1	6	27	.	.	.	.	.	.	.	.	.	.	.	.	.
31	0	0	0	5	.	.	.	.	.	.	.	.	.	.	.	.	.	.
32	0	0	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

From what has been stated in the previous pages, it is easy to see how to extend this table; forming the successive terms of each row by adding step by step upwards to the right along a diagonal, thence upwards to the top, zig-zag along the row of heavier type as soon as it is reached.

THE FIRST SEVEN ORDERS OF KNOTTINESS.



P. G. Tait del.

* (See foot-note, p. 324 below. 1898.)

To face p. 334.