XXXIX.

ON KNOTS.

[*Transactions of the Royal Society of Edinburgh*, 1876–7. Revised *May* 11, 1877.]

THE following paper contains, in a compact form, the substance of several somewhat bulky communications laid before the Society during the present session. The gist of each of these separate papers will be easily seen from the abstracts given in the Proceedings. These contain, in fact, many things which I have not reproduced in this digest. Nothing of any importance has been added since the papers were read, but the contents have been very much simplified by the adoption of a different order of arrangement; and long passages of the earlier papers have been displaced in favour of short general statements from the later ones. With the exception of the portion which deals with the main question raised, this paper is fragmentary in the extreme. Want of leisure or press of other work may justly be pleaded as one cause; but there is more than that. The subject is a very much more difficult and intricate one than at first sight one is inclined to think, and I feel that I have not succeeded in catching the key-note. When that is found, the various results here given will no doubt appear in their real connection with one another, perhaps even as immediate consequences of a thoroughly adequate conception of the question.

I was led to the consideration of the forms of knots by Sir W. Thomson's Theory of Vortex Atoms, and consequently the point of view which, at least at first, I adopted was that of classifying knots by the number of their crossings; or, what comes to the same thing, *the investigation of the essentially different modes of joining points in a plane, so as to form single closed plane curves with a given number of double points.*

The enormous numbers of lines in the spectra of certain elementary substances show that, if Thomson's suggestion be correct, the form of the corresponding vortex atoms cannot be regarded as very simple. For though there is, of course, an infinite number of possible modes of vibration for every vortex, the number of modes whose period is within a few octaves of the fundamental mode is small unless the form of the atom be very complex. Hence the difficulty, which may be stated as follows (assuming, of course, that the visible rays emitted by a vortex atom belong to the graver periods):—"What has become of all the simpler vortex atoms?" or "Why have we not a much greater number of elements than those already known to us?" It will be allowed that, from the point of view of the vortex-atom theory, this is almost a vital question.

Two considerations help us to an answer. *First*, however many simpler forms may be geometrically possible, only a very few of these may be forms of kinetic stability, and thus to get the sixty or seventy permanent forms required for the known elements, we may have to go to a very high order of complexity. This leads to a physical question of excessive difficulty. Thomson has briefly treated the subject in his recent paper on "Vortex Statics*," but he cannot be said to have as yet even crossed the threshold. But *secondly*, stable or not, are there after all very many different forms of knots with any given small number of crossings? This is the main question treated in the following paper, and it seems, so far as I can ascertain, to be an entirely novel one.

When I commenced my investigations I was altogether unaware that anything had been written (from a scientific point of view) about knots. No one in Section **A** at the British Association of 1876, when I read a little paper (No. XXXVIII. above) on the subject, could give me any reference; and it was not till after I had sent my second paper to this Society that I obtained, in consequence of a hint from Professor Clerk-Maxwell, a copy of the very remarkable Essay by Listing, *Vorstudien zur Topologie*†, of which (so far as it bears upon my present subject) I have given a full abstract in the Proceedings of the Society for Feb. 3, 1877. Here, as was to be expected, I found many of my results anticipated, but I also obtained one or two hints which, though of the briefest, have since been very useful to me. Listing does not enter upon the determination of the number of distinct forms of knots with a given number of intersections, in fact he gives only a very few forms as examples, and they are curiously enough confined to three, five, and seven crossings only; but he makes several very suggestive remarks about the representation of knots in general, and gives a special notation for the representation of a particular class of "reduced" knots. Though this has absolutely no resemblance to the notation employed by me for the purpose of finding the number of distinct forms of knots, I have found a slight modification of it to be very useful for various purposes of illustration and transformation. This work of Listing's, and an acute remark made by Gauss (which, with some comments on it by Clerk-Maxwell, will be referred to later), seem to be all of any consequence that has been as yet written on the subject. I have acknowledged in the text all the hints I have got from these writers; and the abstract of Listing's work above referred to will show wherein he has anticipated me.

* *Proc. R.S.E.* 1875—6 (p. 59).

† *Göttinger Studien*, 1847.

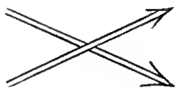
Part I.

The Scheme of a Knot, and the number of distinct Schemes for each degree of Knottiness.

§ 1. My investigations commenced with a recognition of the fact that in any knot or linkage whatever the crossings may be taken throughout alternately over and under. It has been pointed out to me that this seems to have been long known, if we may judge from the ornaments on various Celtic sculptured stones, &c. It was probably suggested by the processes of weaving or plaiting. I am indebted to Mr Dallas for a photograph of a remarkable engraving by Dürer, exhibiting a very complex but symmetrical linkage, in which this alternation is maintained throughout. Formal proofs of the truth of this and some associated properties of knots will be found in the little paper already referred to*. They are direct consequences of the obvious fact that two closed curves in one plane necessarily intersect one another an *even* number of times. It follows as an immediate deduction from this that in going continuously round any closed plane curve whatever, an even number of intersections is always passed on the way from any one intersection to the same again. Hence, of course, if we agree to make a knot of it, and take the crossings (which now correspond to the intersections) over and under alternately, when we come back to any particular crossing we shall have to go *under* if we previously went *over*, and *vice versâ*. This is virtually the foundation of all that follows.

But it is essential to remark that we have thus two alternatives for the crossing with which we start. We may make the branch we begin with cross *under* instead of *over* the other at that crossing. This has the effect of changing any given knot into its own image in a plane mirror—what Listing calls *Perversion*. Unless the form be an *Amphicheiral* one (a term which will be explained later), this perversion makes an essential difference in its character—makes it, in fact, a different knot, incapable of being deformed into its original shape.

Listing speaks of crossings as *dexiotrop* or *laeotrop*. If we think of the edges of a flat tape or india-rubber band twisted about its mesial line, we recognise at once the difference between a right and a left handed crossing. (Plate IV. fig. 1.) Thus the acute angles in the following figure are left handed, the obtuse, right handed; and they retain these characters if the figure be turned over (*i.e.*, about an axis in the plane of the paper):—



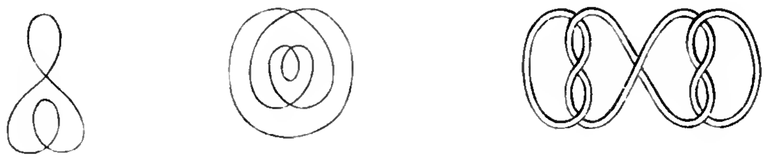
but in its image in a plane mirror these characters are interchanged.

* No. XXXVIII. above.

§ 2. Suppose now a knot of any form whatever to be projected as a shadow cast by a luminous point on a plane. The projection will always necessarily have double points*, and in general the number of these may be increased—though not always diminished—by a change of position of the luminous point, or by a distortion of the wire or cord, which we may suppose to form the knot. This wire or cord must be supposed capable of being bent, extended, or contracted to any amount whatever, subject to the *sole* condition that no lap of it can be pulled through another, *i.e.*, that its continuity cannot be interrupted. There are, therefore, projections of every knot which give a *minimum* number of intersections, and it is to these that our attention must mainly be confined. Later we will consider the question how to determine this minimum number, which we will call *Knottiness*, for any particular knot; but for our present purpose it is sufficient to get rid of what are *necessarily* nugatory intersections, *i.e.*, intersections which no alteration of the mode of crossing can render permanent. These crossings are essentially such that if both branches of the string were cut across at one of them, and their ends reunited crosswise, so as to form two separate closed curves, these separate curves shall not be linked together, however they may individually be knotted, *i.e.*, that if they are knots they are separate from one another, so that one of them may be drawn tight so as to present only a roughness in the string. For in this case the nugatory crossing will thus be made to bound a mere *loop*.

[We may define a necessarily nugatory crossing as one through which a closed, or an infinitely extended, surface may pass without meeting the string anywhere but at the crossing. Or, as will be seen later (§ 20), we may recognise a necessarily nugatory crossing as a point *where a compartment meets itself.*]

In the first two of the sketches subjoined all the crossings are necessarily nugatory; in the third, only the middle one is so.



Now these diagrams, when lettered in the manner forthwith to be explained (see, for instance, Plate V. fig. 1), present respectively the following *schemes*:—

A*A*B*B* | A

A*C*B*B*C*A* | A

A*C*B*D*C*B*D*A*E*G*F*E*G*F* | A.

* Higher multiple points may, of course, occur, but an *infinitesimal* change of position of the luminous point, or of the relative dimensions of the coils of the knot, will remove these by splitting them into a number of double points, so that we need not consider them.

These and similar examples show that in a scheme a crossing is necessarily nugatory, if between the two appearances of the letter denoting that crossing there is a group consisting of any set of letters *each occurring twice.* The set may consist of any number whatever, including zero. For our present purpose it will be found sufficient to consider this last special case alone, *i.e., the same letter twice in succession denotes a necessarily nugatory crossing.*

§ 3. If we affix letters to the various crossings, and, going continuously round the curve, write down the name of each crossing in the order in which we reach it, we have, as will be proved later, the means of drawing without ambiguity the projection of the knot. If, in addition, we are told whether we passed over or under on each occasion of reaching a crossing we can, again without any ambiguity, construct the knot in wire or cord. Passing over is, in what follows, indicated by a + subscribed to the letter denoting the crossing—passing under by a −. Any specification which includes these two pieces of information is necessarily *fully descriptive* of the knot; and when it is given in the particular form now to be explained we shall call it the *Scheme.*

If in accordance with § 1 we make the crossings alternately over and under, it is obvious that the odd places and even places of the scheme will each contain all the crossings. As the choice of letters is at our disposal, we may therefore call the crossings in the odd places A, B, C, &c., in alphabetical order, starting from any crossing we please, and going round the knotted wire in any of the four possible ways, *i.e.,* starting from any crossing by any of the four paths which lead from it, put the successive letters at the first, third, fifth, &c., crossings as we meet them. Then it is obvious that the essential character of the projected knot must depend only upon *the way in which the letters are arranged in the even places of the scheme.* Of course, the nature and reducibility (*i.e.,* capability of being simplified by the removal of nugatory crossings) of the knot itself depend also upon the subscribed signs. [In general there will be four different schemes for any one knot, but in the simpler cases these are often identical, two and two, sometimes all four.]

§ 4. Here we may remark that it is obvious that when the crossings are alternately + and − no reduction is possible, unless there be essentially nugatory crossings, as explained in § 2. For the only way of getting rid of such alternations of + and − along the same cord is by *untwisting*; and this process, except in the essentially nugatory cases, gets rid of a crossing at one place only by introducing it at another. It will be seen later that this process may in certain cases be employed *to change the scheme* of a knot, and thus to show that in these cases there may be more than four different schemes representing the same knot; though, as we have already seen, a scheme is perfectly definite as to the knot it represents. Hence, in the first part of our work, we shall suppose that the crossings are taken alternately + and −, so that no reduction is possible. But it will afterwards be shown that, even when all essentially nugatory crossings are removed, it is not always necessary to have the regular alternation of + and − in order that the knot may not be farther reducible. It is easy to see a reason for this, if we think of a knot made up of different knots on the same string, whether separate from one another or linked together. For the irreducibility of each separate knot depends only upon the alternations of + and − *in itself*, and the two knots may be put together, so that this condition is satisfied in the partial schemes, but not in the whole. As there cannot be a knot with fewer than three crossings, we do not meet with this difficulty till we come to knots with six crossings. And as there can be no linking without at least two crossings, we do not meet with linked knots on the same string till we come to eight crossings at least.

§ 5. We are now prepared to attack our main question.

Given the number of its double points, to find all the essentially different forms which a closed curve can assume.

Going round the curve continuously, call the first, third, &c., intersections A, B, C, &c. In this category we evidently exhaust all the intersections. The complete scheme is then to be formed by properly interpolating the same letters in the even places; and the form of the curve depends solely upon the way in which this is done.

It cannot, however, be done at random. For, *first*, neither A nor B can occur in the second place, B nor C in the fourth, and so on, else we should have necessarily nugatory intersections, as shown in § 2. Thus the number of possible arrangements of n letters (viz., $n\,.\,n-1\ldots2\,.\,1$) is immensely greater than the number which need here be tried. But, *secondly*, even when this is attended to, the scheme may be an impossible one. Thus, the scheme

$$\mathrm{A}\;D\;\mathrm{B}\;E\;\mathrm{C}\;A\;\mathrm{D}\;B\;\mathrm{E}\;C\mid\mathrm{A}$$

is lawful, but

$$\mathrm{A}\;D\;\mathrm{B}\;A\;\mathrm{C}\;E\;\mathrm{D}\;C\;\mathrm{E}\;B\mid\mathrm{A}$$

is not.

The former, in fact, may be treated as the result of superposing two closed (and not self-intersecting) curves, both denoted by the letters A D B E C A, so as to make them cross one another at the points marked B, C, D, E, then cutting them open at A, and joining the free ends so as to make a continuous circuit with a crossing at A.

But in the latter scheme above we have to deal with the curves A D B A and C E C E, and in the last of these we cannot have junctions alternately + and − as required by our fundamental principle. In fact, the scheme would require the point C to lie simultaneously inside and outside the closed circuit A D B A.

Or we may treat A D B A and C E D C as closed curves intersecting one another and yet having only one point, D, in common.

Thus, to test any arrangement, we may strike out from the whole scheme all the letters of any one closed part as A——*A*, and the remaining letters must satisfy the fundamental principle, *i.e.*, that they can be taken with suffixes + and − alternately, or (what comes to the same thing) that an even number of letters intervenes between the two appearances of each of the remaining letters.

Or we may strike out all the letters of any two sets which begin and end similarly, *e.g.*, A...*X*, X...*A*, the two together being treated as one closed curve, and the test must still apply.

More generally, we may take the sides of any closed polygon as A − *X*, X − *Y*, Y − *Z*, Z − *A*, and apply them in the same way. But in this, as in the simpler case just given, the sides must all be taken the same way round in the scheme itself.

A simple mode of applying these tests will be given later, when we are dealing with the question of *Beknottedness.*

It may be well to explain here how a change of the crossing selected as the initial one alters the scheme. Take the simple case of making B the first, and reckoning on from it. Then B becomes A, &c., and the scheme, which may be any whatever, suppose for example

A *F* B *L* C *E* D *H* ...

becomes (by writing for each letter that which alphabetically precedes it)

N *E* A *K* B *D* C *G* ...

or beginning with A, A *K* B *D* C *G* ...

Hence the letters *F*, *L*, *E*, *H*, ...

in the even places of a scheme are equivalent to

K, *D*, *G*, ... *E*,

i.e., we may change each to the preceding letter taken in the cyclical order of the alphabet and put the first to the end, or *vice versâ*, without altering the scheme. An arrangement of this kind is *unique* (reproducing itself) if the letters are in cyclical order; and if the number of letters be a prime, any arrangement is either unique or is reproduced after a number of operations of this kind equal to the number of letters. If it be not prime, arrangements may be found which will reproduce themselves after a number of operations equal to any one of its aliquot parts.

Another lawful change is this:—Begin from the A in the even places and letter as usual, *i.e.*, start from the same crossing as before, and in the same direction round the curve, but not by the same branch of the cord or wire. This will be evident from an example. Beginning at the second A, and lettering alphabetically every second crossing, we have the suffixed letters,

A *D* B *A* C *F* D *B* E *C* F *E* | A
F A B C D E

Now write the same equivalents for the same letters in the odd places, and the scheme in its new lettering is

A F C A D B F C E D B E | A

or the following are equivalents in the even places

D A F B C E

D F E B A C,

and each of these has, of course, five other equivalents found by the first of these two processes.

But we may also start from the same intersection A by either of these paths, but *in the reverse direction round the curve.* To effect this we have only to read the scheme backwards, beginning at either A, and changing the lettering throughout in accordance with our plan. Thus, taking the last example,

A *D* B *A* C *F* D *B* E *C* F *E* | A

F E D C B | A

we keep the terminal A unchanged, and write B, C, &c., for the 2nd, 4th, &c., *preceding* letters. We have thus, as it were, the key for translating from the upper line to the lower. Apply this key to all the letters, and then write the result in the reverse order. Thus we get

A *C* B *E* C *F* D *B* E *A* F *D* | A.

This new scheme has for its even places

C E F B A D

which is equivalent (in this particular case) to the *second* of the two direct schemes just given, viz.:—

D F E B A C.

Finally, if we read this reversed scheme from the A in the even places, its even letters become

E A F C B D

which (in this case) is the same as

D A F B C E

the even letters of the original scheme.

The notation we shall employ is this—*do, de, ro, re,* signifying the even places of the four cases

d o the *direct* scheme, read from A in the *odd* place

d e the *direct* scheme, read from A in the *even* place

r o the *reversed* scheme, read from A in the *odd* place

r e the *reversed* scheme, read from A in the *even* place

and we shall denote by an appended numeral the number of times the operation above has to be performed. Thus, in the example just given it will be found that

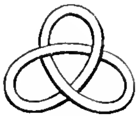
$$r\,o = d\,e\,2$$

$$r\,e = d\,o\,2.$$

§ 6. With one intersection or two only, a *knot* is thus impossible, for the crossings must necessarily be nugatory. Hence we commence with *three*. And here there is but one case, for by our rule we must write A, B, C in the odd places, and *we have no choice* as to what to interpolate in the even ones. Thus the only knot with three intersections has the scheme

A *C* B *A* C *B* | A.

One of its two projections is the "trefoil" knot below.



For *four* intersections our choice in the even places is restricted to C or D for the second, D or A for the fourth, &c., as expressed below,

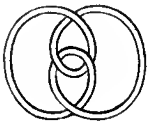
C D A B

D A B C.

Now, if we take C to begin with, we obviously *must* take D next, else we shall not get it at all. Similarly A *must* come third. And if we begin with D, we *must* end with C, so that this case also is determinate. The only possible sets, therefore, are given by these two rows as they are written. But it is obvious that, as they are in cyclical order, the full schemes will be identical if one be read from the beginning, the other from the A in the even places. Thus they represent the same arrangement, and the sole knot with four intersections has the scheme

A *C* B *D* C *A* D *B* | A.

One of its two projections is given by the annexed figure:—



§ 7. When we have *five* intersections, our choice for the even places in order is limited to the following groups of three for each, viz.:—

C D E A B

D E A B C

E A B C D.

This gives the following thirteen arrangements:—

(1)	C D E A B
(2)	C E A B D
(3)	C E B A D
(4)	C A E B D
(5)	D E A B C
(6)	D E B A C
(7)	D E A C B
(8)	D A E B C
(9)	D A E C B
(10)	E D A B C
(11)	E D A C B
(12)	E D B A C
(13)	E A B C D.

Now of these (1), (5), and (13) are unique; (6), (7), (8), and (10) can be obtained from (2) by cyclical alteration of the letters and bringing the last to be the beginning, and by the same process (4), (9), (11), (12) may be deduced from (3).

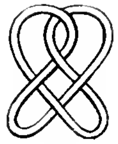
Hence the only possible forms are included in the following arrangements for the letters in the even places:—

C D E A B
C E A B D
C E B A D
D E A B C
E A B C D.

Of these the 1st, 3rd, and 5th violate the conditions laid down in § 5 above. Hence there are but two schemes for five intersections, viz.:—

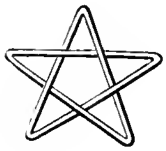
A *C* B *E* C *A* D *B* E *D* | A,

of which this is one of the four forms



and A *D* B *E* C *A* D *B* E *C* | A,

one of the two forms of which is the pentacle or Solomon's seal,



§ 8. The case of six intersections gives the following choice:—

C	D	E	F	A	B
D	E	F	A	B	C
E	F	A	B	C	D
F	A	B	C	D	E.

I found, by trial, that there are 80 possible arrangements included in this form; and that the following 20 alone are distinct. I have appended to each the number of apparently different forms in which it occurs among the 80 arrangements:—

1.	C D E F A B	Unique
2.	C D F B A E	Six forms
3.	C D F A B E	Six forms
4.	C D A F B E	Six forms
5.	C D B F A E	Three forms
6.	C E F B A D	Six forms
7.	C E F A B D	Six forms
8.	C E A F B D	Three forms
9.	D E F A B C	Unique
10.	C F E B A D	Two forms
11.	E F A B C D	Unique
12.	D F A B C E	Six forms
13.	C F A B D E	Six forms
14.	D F A C B E	Six forms
15.	D F B A C E	Three forms
16.	C F B A D E	Six forms
17.	C A F B D E	Six forms
18.	C A B F D E	Three forms
19.	D A F C B E	Two forms
20.	F A B C D E	Unique

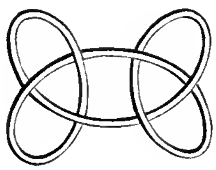
Of these, all but (5), (6), (7), (8), (12), (14), (15), (18), violate the conditions of § 5, and therefore do not correspond to real knots. Of those excepted the schemes agree in pairs when the branch first taken from the starting-point is changed.

Hence there are only *four* forms of 6-fold knottiness. These are as follows:—

(α). (5) and (18) agree in giving the scheme

A *C* B *A* C *B* D *F* E *D* F *E* | A,

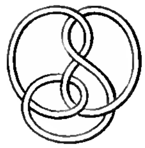
of which one form is the following:—



This form consists of two *separate* trefoil knots.

(β). (6) and (14) give the scheme

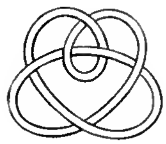
A *C* B *E* C *F* D *B* E *A* F *D* | A, one form of which is as follows:—



(γ). From (7) and (12) we have

A *C* B *E* C *F* D *A* E *B* F *D* | A,

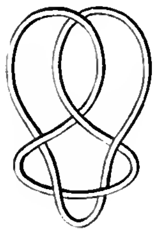
which has as one form



(δ). (8) and (15) give

A *C* B *E* C *A* D *F* E *B* F *D* | A,

of which one form is



§ 9. The case of *seven* intersections is the only other to which I have found leisure to apply this method. As I did not see how otherwise to make certain that I had got all possible forms, I wrote out all the combinations of seven different letters, one from each column (in order) of the scheme—

C	D	E	F	G	A	B
D	E	F	G	A	B	C
E	F	G	A	B	C	D
F	G	A	B	C	D	E
G	A	B	C	D	E	F.

These I thus found to amount to 579. Then, by the help of an improvised arrangement of cardboard, somewhat resembling *Napier's Bones*, I rapidly struck off six of each equivalent set of 7. Thus 87 forms in all were left, viz., one form from each of 82 groups of seven, and 5 unique forms. Here they are—

1. C D E F G A B
2. C D E G A B F
3. C D E A G B F
4.* C D E B G A F
5. C D F B G A E
6. C D F G B A E
7. C D F A G B E
8. C D G F A B E
9. C D G F B A E
10. C D G A B E F
11. C D G B A E F
12. C D A G B E F
13.* C D A B G E F
14.* C D B A G E F
15.* C D B G A E F
16. C E F G A B D
17.* C E F G B A D
18. C E F G A D B
19. C E F A G B D
20.* C E G F B A D
21. C E G F D A B
22.* C E G A B D F
23. C E G A D B F
24.* C E G B A D F
25. C E G B D A F
26.* C E A G B D F
27. C E A G D B F
28. C E A B G D F
29. C E B G A D F
30. C E B G D A F
31. C E B A G D F
32. C F G A B D E
33.* C F G A D B E
34.* C F G A B E D
35. C F G B A E D
36. C F G B A D E
37. C F G B D A E
38.* C F A G B D E
39.* C F A G D B E
40.* C F A G B E D
41. C F A B G D E
42. C F A B G E D
43. C F B G A D E
44. C F B G D A E
45. C F B G A E D
46. C F B A G E D
47. C F B A G D E
48. C G E B A D F
49. C G E B D A F
50. C G F A B D E
51. C G F A B E D
52.* C G F A D B E
53. C G F B A D E
54. C G F B A E D
55. C G A F D B E
56. C G A B D E F
57. C G B A D E F
58. C A F G B D E
59. C A F G B E D
60. C A F G D B E
61. C A F B G E D
62. C A G B D E F
63.* C A B G D E F
64. D E F G A B C
65. D E G A B C F
66. D E G A C B F
67.* D E G B A C F
68. D E G C A B F
69. D E A G B C F
70. D E A G C B F
71.* D F G A B C E
72.* D F G A C B E
73. D F G B A C E
74. D F A G C B E
75. D G A B C E F
76. D G A C B E F
77. D G B A C E F
78. D G B C A E F
79. D A G B C E F
80. D A G C B E F
81.* E F G A B C D
82. E G A B C D F
83.* E G A B D C F
84. E G A C B D F
85.* E G B A D C F
86. F G A B C D E
87. G A B C D E F

On testing these by the rules of § 5, I found that 22 only, viz., those marked with an asterisk, correspond to real knots.

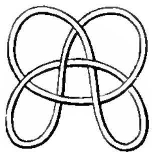
§ 10. When we study these groups by the method of § 5, we find that more than one of them correspond to different readings of the scheme of one and the same knot. Of course that knot will be the least symmetrical which has the greatest number of essentially different schemes. The following grouping has thus been arrived at (the notation is that of § 5 above):—

	d o	*d e*	*r o*	*r e*
I.	(4)	1, (63)	(63)	6, (4)
	(13)	6, (15)	(15)	1, (13)
II.	(17)	3, (83)	5, (83)	2, (17)
III.	(20)	3, (85)	3, (85)	(20)
IV.	(22)	6, (33)	(22)	6, (33)
V.	(24)	(39)	(26)	5, (52)
VI.	(34)	(34)	6, (34)	6, (34)
VII.	(38)	(67)	(67)	(38)
VIII.	(40)	6, (40)	6, (40)	(40)
IX.	(71)	(71)	(71)	(71)
X.	(72)	(72)	5, (72)	5, (72)
XI.	(81)	(81)	(81)	(81)
	(14)	(14)	(14)	(14)

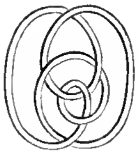
Thus it appears that the knot V., represented by any of the four schemes (24), (26), (39), and (52), is devoid of symmetry, while VI., VIII., IX., X., XI. have the highest symmetry. No number has been in this table affixed to (14), because it is only accidentally a 7-fold knot. It is represented by the third figure in § 2 above, and when the nugatory crossing is removed, it becomes (α) of the 6-fold type, § 8. Also it will be noticed that (4) and (63), although their common scheme differs from that of (13) and (15), are included with them under I. The reason is that the knot represented is a composite one, consisting of a 3-fold and a 4-fold knot, and that either may be slipped along the string or wire into any position whatever relative to the other. But even with this licence it appears that there are only 4 really distinct schemes.

In the second and third rows of figures of Plate IV. projections of each of these classes of 7-fold knottiness are given, with the number of the class attached.

§ 11. But the knots represented by these eleven forms are not all distinct. It will readily be seen that (by the process of inversion of § 15 below) II., when formed of wire, with crossings + and − alternately, may be brought into the form (whose *perversion* will be found in Sir W. Thomson's paper on "Vortex-Motion," *Trans. R. S. E.*, 1867–68, p. 244)



while IV. may be modified into



These are two of the three figures of 7-fold knots given as examples by Listing: and he has stated, though without any explanation, that these two forms are equivalent, *i.e.*, convertible into one another. Hence II. and IV. form but one class of 7-fold knot.

How to effect this transformation has been already hinted in § 4. It is merely the passing of a crossing from one loop of the string to another (which intersects it twice) by a *twist* through two right angles. And the diagrams 5, 6, 7 of Plate IV. show the nature of this transformation, as well as of two others which I have since detected, viz., that of III. into V., and of VI. into VII. Hence there are in reality only *eight* distinct forms of 7-fold knottiness.

Thus, as the result of the last six sections, we have the following table:—

Knottiness,	3,	4,	5,	6,	7.
No. of Forms,	1,	1,	2,	4,	8.

§ 12. I have not attempted the application of the preceding method to forms with more than 7 intersections. Prof. Cayley and Mr Muir kindly sent me general solutions of the problem, "*How many arrangements are there of* n *letters, when A cannot be in the first or second place, B not in the second or third, &c.*" Their papers, which will be found in the *Proceedings R. S. E.*,* of course give the numbers 13, 80, and 579, which I had found by actually writing out the combinations for 5, 6, and 7 letters. But they show that the number for 8 letters is 4738, and that for 9, 43,387; so that the labour of the above-described process for numbers higher than 7 rises at a fearful rate. I cannot spare time to attack the 8-fold knots, but I hope some one will soon do it. There is little chance of anything more than that, at least of an exhaustive character, being done about knots in this direction, until an analytical solution is given of the following problem:—

Form all the distinct arrangements of n *letters, when A cannot be first or second, B not second or third, &c.*

[Arrangements are said to be distinct when no one can be formed from another by cyclic alteration of the letters, at every step bringing the last to the head of the row, as in § 5.] This, I presume, will be found to be a much harder problem than that of merely *finding the number* of such arrangements, which itself presents very grave difficulties, at least where n is a composite number. In fact it is probable that the solution of these and similar problems would be much easier to effect by means

* 1877, p. 338, and p. 382.

of special (not very complex) machinery than by direct analysis. This view of the case deserves careful attention.

In a later section it will be shown how, by a species of *partition*, the various forms of any order of knottiness may be investigated. But we can never be quite sure that we get *all* possible results by a semi-tentative process of this kind. And we have to try an immensely greater number of partitions than there are knots, as the great majority give links of greater or less complexity.

§ 13. But even supposing the processes indicated to have been fully carried out for 8, 9, and 10-fold knottiness, a new difficulty comes in which is not met with, except in a very mild form, in the lower orders. For when a knot is single, *i.e.*, not composite or made up of knots (whether interlinked or not) of lower orders, any deviation from the rule of alternate + and − at the crossings gives it, in general, nugatory crossings, in virtue of which it sinks to a lower order. But when it is composite, and the component knots are separately irreducible, the whole is so. Thus *there are more distinct forms of knots than there are of their plane projections.* For instance, the first species (α) of the 6-fold knots (§ 8) may be made of three essentially different forms, for the separate "trefoil" knots of which it is made may (when neither is nugatory) be both right-handed, both left-handed, or one right and the other left-handed. This species is thus, from the physical point of view, capable of furnishing *three* quite distinct forms of vortex-atom. And it will presently be shown that in each of these forms it is capable of having regular alternations of + and −, or a set of sequences at pleasure.

At least one knot of every even order is *amphicheiral, i.e.*, right or left-handed indifferently, but no knot of an odd order can be so. Hence, as there is but one 3-fold knot form, and one 4-fold, there are two possible 3-fold vortices, right and left-handed, but only one 4-fold. A combination of two trefoil knots gives, as we have seen, three distinct knots; that of two 4-fold knots would give an 8-fold, with only one form. When a 3-fold and a 4-fold are combined, as in Class I. of § 10, there are two distinct vortices, for the trefoil part may be right or left-handed. Thus it appears that though we have shown that there are very few distinct outlines of knots, at least up to the 7-fold order, and though probably only a very small percentage of these would be stable as vortices, yet the double forms of non-amphicheiral knots give more than one distinct knot for each projected form into which they enter as components.

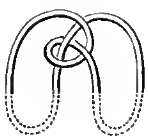
Part II.

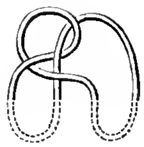
The number of Forms for each Scheme.

§ 14. A possible scheme being made according to the methods just described, with the requisite number of intersections, let it be constructed in cord, with the intersections alternately + and −. Then [since all schemes involving essentially nugatory crossings, like those mentioned in § 2, must be got rid of, as they do not really possess the requisite number of intersections] no deformation which the cord can suffer will reduce, though it may increase, the number of double points. If it *do* increase the number, the added terms will be of the nugatory character presently to be explained. If it do not increase that number, the scheme will in general still represent the altered figure. For, as we have seen, the scheme is a complete and definite statement of the nature of the knot. But, as already stated, in certain cases the knot can be distorted so as no longer to be represented by the same scheme.

All deformations of such a knotted cord or wire may be considered as being effected by bending at a time only a limited portion of the wire, the rest being held fixed. This corresponds to changing the point of view *finitely* with regard to the part altered, and yet *infinitesimally* with regard to all the rest. This, it is clear, can always be done, as the *relative* dimensions of the various coils may be altered to any extent without altering the character of the knot. In general such deformations may be obtained by altering the position of a luminous *point*, and the plane on which it casts a shadow of the knot. Any addition to the normal number of intersections which may be produced by this process is essentially nugatory. As is easily seen, it generally occurs in the form of the avoidable overlapping of two branches, giving *continuations of sign*.

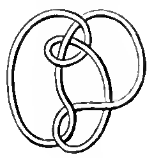
The process pointed out in § 11 gives a species of deformation which it is perhaps hardly fair to class with those just described, though by a slight extension of mathematical language such a classification may be made strictly accurate. It may be well to present, in passing, a somewhat different view of the application of this method. Thus, it is obvious at a glance that the two following figures are mere *distortions* of the second form of the 4-fold knot figured in § 17 below:—

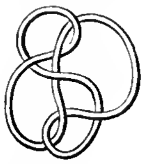




Also it will be seen that by twisting, the dotted parts being held fixed, either of these may be changed into the other, or changed to its own reverse (as from left to right).

We may now substitute what we please for the dotted parts. I give only the particular mode which reproduces the two forms stated by Listing to be equivalent:—





Another mode of viewing the subject, really depending on the same principles, consists in fixing temporarily one or more of the crossings, and considering the impossibility of unlocking in any way what is now virtually two or more *separate* interlacing closed curves, or a single closed curve with full knotting, but with fewer intersections than the original one.

Another depends upon the study of the cases of knots in which one or more crossings can be got rid of. Here, as will be seen in § 33 below, it is proved that *continuations* of sign are in general lost when an intersection is lost; so that, as our system has no continuations of sign, it can lose no intersections.

§ 15. Practical processes for producing graphically all such deformations as are represented by the same scheme are given at once by various simple mechanisms. Thus, taking O any fixed point whatever, let p, a point in the deformed curve, be found from its corresponding point, P, by joining PO and producing it according to any rule such as

$$PO \cdot Op = a^2,$$

or

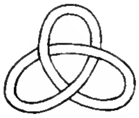
$$PO + Op = a, \text{ \&c., \&c.}$$

The essential thing is that points near O should have images distant from O, and *vice versâ*. And p must be taken in PO *produced*, else the distorted knot is altered from a right-handed to a left-handed one, and *vice versâ*, as will be seen at once by taking the image of the crossing figured in § 1 above.

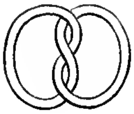
It is obvious, from the mode of formation, that these figures are all represented by the same scheme,—for the scheme tells the order in which the various crossings occur,—and it is easy to show that they give merely different views of the same knot. The simplest way of doing this is to suppose the knot projected on a sphere, and *there* constructed in cord, the eye being at the centre. Arrange so that one closed branch, *e.g.*, A——*A*, forms nearly a great circle. Looking towards the centre of the sphere from opposite sides of the plane of this great circle, the coil presents exactly the two appearances related to one another by the deformation processes given above. What was inside the closed branch from the one point of view is outside it from the other, and *vice versâ*. In fact, because the new figure is represented by the same scheme as the old, the numbers of sides of the various compartments are the same as before, and so also is the way in which they are joined by their corners. The deformation process is, in fact, simply one of *flyping*, an excellent word, very inadequately represented by the nearest equivalent English phrase "turning outside in."

Hence to draw a scheme, select in it any closed circuit, *e.g.*, A...*A*—the more extensive the better, provided it do not include any less extensive one. Draw this, and build upon it the rest of the scheme; commencing always with the common point A, and passing each way from this to the *next occurring* of the junctions named in the closed circuit. [It is sometimes better to construct both parts of the rest of the scheme *inside*, and then invert one of them, as we thus avoid some puzzling ambiguities.] Inversions with respect to various origins will now give all possible forms of the scheme, though not necessarily of the knot.

§ 16. Applying these methods to the "trefoil" knot (§ 6)

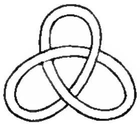


we easily see that if O be external, or be inside the inner *three*-sided compartment, we reproduce (generally with much *distortion*, but that is of no consequence, § 2) the same form; but if O be in any one of the *two*-sided compartments, we have the form



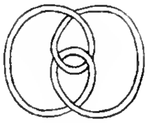
This again is reproduced from itself if O be external, or be within either of the *two*-sided compartments. But it gives the trefoil knot if O be placed inside either of the *three*-sided compartments.

Here notice that the angles of the two-sided compartments are left-handed, and those of the three-sided right-handed in each of the figures. The *perverted* or right-handed form is of course



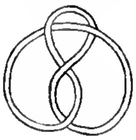
and its solitary deformation is the perversion of the other figure above.

§ 17. When we come to the deformations of the single 4-fold knot



we obtain a very singular result. If we place O external to the figure, we simply reproduce it; but if we put O inside the two-sided compartment in the middle we get the *perversion* of the same figure.

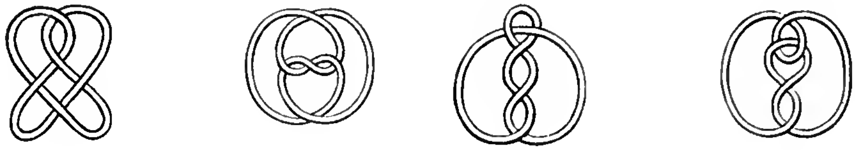
Again, if we place O in either of the *boundary* three-sided compartments we get



but if we place it in either of the *interior* three-sided spaces we get the *perversion* of this last figure.

Thus this 4-fold knot, in each of its forms, *can be deformed into its own perversion.* In what follows all knots possessing this property will be called *Amphicheiral.*

§ 18. The first of the two 5-fold knots (§ 7) has the following forms:—



These I found were long ago given by Listing as reduced forms of a reducible 7-fold knot, and I have now substituted for my former drawing of the second form his more symmetrical one.

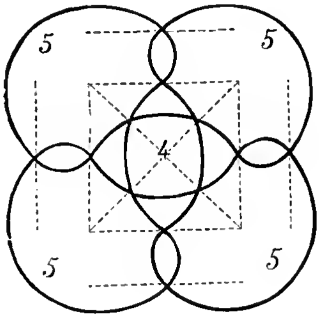
The second of the 5-fold knots has only two forms, viz.:—

§ 19. Plate IV. figs. 2, 3, 4, give various forms of the 6-fold knot distinguished as α in the classification in § 8. It will be seen that in the first of these the crossings are alternately over and under, but that it is not so in the others.

And in fig. 8 we have a collection (not complete) of forms of various species of the 7th order, drawn so as to show their relation to a lower form—the trefoil knot. It will be seen that in none of these is the connection merely *apparent*, the trefoil part having its signs alternately + and − if those of the complete knot have this alternation. But if, for instance, we had drawn the fine line horizontally through the trefoil, so as to divide each of the upper two-cornered compartments into two three-cornered ones, we should have got No. II. of the 7-fold forms, and the original trefoil would have been rendered only *apparent.*

§ 20. In my British Association paper, No. XXXVIII. above, I showed that any closed plane curve, or set of closed plane curves, provided there be nothing higher than double points, divides the plane into spaces which may be coloured black and white alternately, like the squares of a chess-board, or, to take a closer analogy, as the adjacent elevated and depressed regions of a vibrating plate, separated from one another by the nodal lines (Plate IV. figs. 9 and 10). I afterwards found that Listing had employed in his notation for knots, in which the crossings are alternately over and under, a representation which comes practically to the same thing; depending as it does on the fact that in such a knot all the angles in each compartment are either right or left-handed, and that these right and left-handed compartments alternate as do my black and white ones.

I have since employed a method, based on the above proposition, as a mode of symbolising the form of the projections of a knot, altogether independent of its reducibility. I was led to this by finding that Listing's notation, though expressly confined to reduced knots, in which each compartment has all its angles of the same character, is ambiguous: in the sense that a *Type-Symbol*, as he calls it, may in certain cases not only stand for a linkage as well as a knot, but may even stand for two quite different reduced knots incapable of being transformed into one another*. The *scheme*, already described, has no such ambiguity, but it is much less easy to use in the classification of knots. Hence, following Listing, I give the number of corners of each compartment, but, unlike him, only of those which are black or of those which are white. But I connect these in the diagram by lines which show how they fit into one another in the figure of the knot. An inspection of Plate IV. figs. 11 and 12 (species VII. of sevenfold knottiness) will show at once how diagrams are arrived at, either of which fully expresses the projection of the knot in question by means of the black or of the white spaces singly. The connecting lines in the diagrams evidently stand for the crossings in the projection, and thus, of course, either diagram can be formed by mere inspection of the other†, and the rule for drawing the curve when the diagram is given is obvious. Thus the annexed diagram shows the result of the process as applied to a symmetrical symbol.



An inspection of one of these diagrams shows at once

(1) The number of joining lines is the same as the number of crossings. Hence, as each line has two ends, the sum of the numbers representing the number of corners in either the black or the white spaces is twice the number of crossings.

(2) Every additional crossing involves one additional compartment, for the abolition of a crossing runs two compartments into one. But where there is no crossing there are two compartments, the inside and outside (*Amplex*, in Listing's phraseology), of

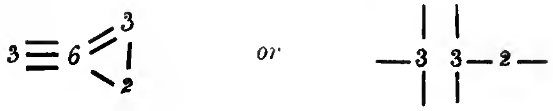
* *Proc. R.S.E.* 1877, p. 310 (footnote), and p. 325.

† Some further illustrations of this will be found in the abstract of my paper on "Links," *Proc. R.S.E.* 1877, p. 321.

what must then be merely a closed oval. Thus when there are n crossings there are $n+2$ compartments.

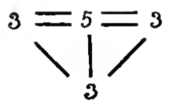
(3) No compartment can have more than n corners. For, as the whole number of corners in the black or white compartments is only $2n$, if one have more than n, the rest must together have less, and thus some of the joining lines in the diagram must *unite the large number to itself, i.e.,* must give essentially nugatory intersections.

As an illustration, let us use this process in giving a second enumeration or delineation of the forms of 7-fold knottiness. The numbering of the various forms is the same as that already employed in §§ 10, 11 above.



I.

The second form of this symbol is particularly interesting as consisting of two parts. This accords with the composite nature of the knot.



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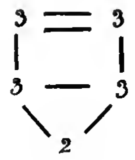
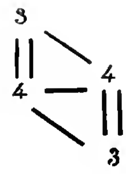
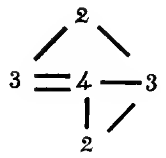


III.



IV.



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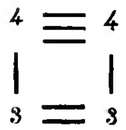

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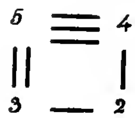
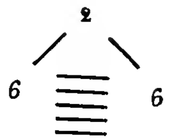
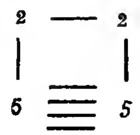


VII.



VIII.



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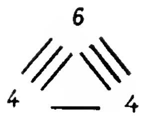
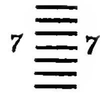
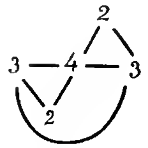


X.



XI.

The relations of equivalence in pairs among six of these forms, which were pointed out in § 11 and in Plate IV. figs. 5, 6, 7, are even more clearly seen as below:—



II.

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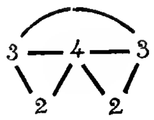

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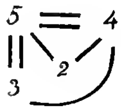

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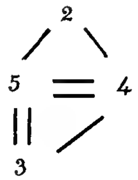

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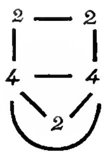

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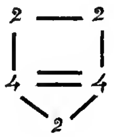


VII.

where the mode of passing from one form to the equivalent one is obvious.

§ 21. A tentative method of drawing all possible systems of closed curves with a given number (n) of double points is thus at once obvious.

Write all the partitions of $2n$, in which no one shall be greater than n and no one less than 2. Join each of these sets of numbers into a group, so that each number has as many lines terminating in it as it contains units. Then join the middle points of these lines (which must not intersect one another) by a continuous line which *intersects* itself at these middle points and there only. When this can be done we have the projection of a *knot*. When more continuous lines than one are required we have the projection of a *linkage*.

To give simple examples of this process, let us limit ourselves to 4 and 5 intersections.

The only partitions of 8, subject to the conditions above, are

(1) 4 4

(2) 4 2 2

(3) 3 3 2

(4) 2 2 2 2

Now the number of black and white compartments together must in this case be $4+2$. Hence there are but four combinations to try, viz., (1) and (4), (2) and (2), (3) and (3), (2) and (3). Of these, the last is impossible; the others are as in Plate V. fig. 16. The third is the amphicheiral knot already spoken of, and the second may for the same reason be called an *amphicheiral link.*

The partitions of 10, subject to our rule, are

5 5

5 3 2

4 4 2

4 3 3

4 2 2 2

3 3 2 2

2 2 2 2 2

and the four figures (Plate V. fig. 17) give the only valid combinations of these. The third and the first are the knots already described (§ 18), the others are links.

§ 22. The spherical projection already mentioned (§ 15) will in general allow us to regard and exhibit any knot as a more or less perfect *plait.* It does so perfectly whenever the coil is *clear, i.e.,* when all the windings of the cord may be regarded as passing in the same direction round a common vertical axis thrust through the knot. When the coil is not clear some of the cords of the plait are doubled back on themselves. Thus by drawing the plait corresponding to a given scheme we can tell at once whether one of its forms is a clear coil or not.

Let us confine our attention for a moment to clear coils. It is easy to see that

If the number of windings is even the number of crossings is odd, and vice versâ.

Various proofs of this may be given, all depending on the fundamental theorem of § 1, but the following one is simple enough, and will be useful in some other applications.

First, in a clear coil of two turns there must be an odd number of intersections. For there must be one intersection, and the two loops thus formed must have their other intersections (if any) in pairs.

Now begin with any point in a clear coil, where the curve intersects itself for the first time. The loop so formed intersects the rest in an even number of points. Hence every turn we take off removes an odd number of intersections. Thus, as two turns give an odd number (or, more simply, as one turn gives none), the proposition is proved.

Thus, to form the symmetrical clear coil of two turns and of any (odd) number of intersections, make the wire into a helix, and bring one end through the axis in the same direction as the helix (not in the opposite direction, as in Ampère's *Solenoids*), then join the ends. [The solenoidal arrangement, regarded from any point of view, has only nugatory intersections.]

§ 23. A very curious illustration of the irreducible clear coils which have two turns only is given by the edges of a long narrow strip of paper. Bend it, without twisting, till the ends meet, and then paste them together. The two edges will form separate non-linked closed curves without crossings.

Give the slip *one half twist* (*i.e.* through 180°) before pasting the ends together. The edges now form one continuous curve—a clear coil of two turns with *one* (nugatory) crossing.

Give *one full twist* before pasting. Each edge forms a closed curve, but there are two crossings. The curves are, in fact, once linked into one another. (See Plate IV. fig. 13.)

Give *three half twists* before joining. The edges now form one continuous clear coil with three intersections.

Two full twists give two separate closed curves with four crossings, *i.e.*, twice linked together. (See Plate V. fig. 12.)

Five half twists give the pentacle of § 7 above. And so on. In all these examples, from the very nature of the case, the crossings are alternately + and −.

§ 24. Now suppose that, in any of the above examples, after the pasting, we cut the slip of paper up the middle throughout its whole length.

The first, with no twist, splits of course into two separate simple circuits.

That which has half a twist, having originally only one edge, and that edge not being cut through in the process of splitting, remains a closed curve. It is, in fact, a clear coil of two turns, which, having only one intersection, may be opened out into a single turn. But in this form it has *two whole* twists, half a twist for each half of the original strip, and a whole twist additional, due to the bending into a closed circuit.

That with one whole twist splits, of course, into two interlinking single coils, each having one whole twist.

That with three half twists gives, when split, the trefoil knot, and when flattened out it has three whole twists.

From two whole twists we get two single coils twice linked, each with two whole twists. This result may be obviously obtained from a continuous strip, *with only half a twist.* One continued cut, which takes off a strip constantly equal to one quarter of the original breadth of the slip, gives a half twist ring of half breadth, intersecting *once* a double twist ring of quarter breadth. A second cut splits the wider ring into one similar to the narrow one, but there is now double linking.

§ 25. A good many of these relations may be exhibited by dipping a wire, forming a two-coil knot, into Plateau's glycerine soap solution, and destroying the film which fills up the clear interior of the coil. Neglecting the surface curvature of the remaining film, it has twists similar to those of the paper strips above treated, and the integral amounts of twist show how far the wire-knot is, if at all, reducible.

This mode of regarding a clear coil of two turns, as, in certain cases, the continuous edge of a strip of paper whose ends are pasted together after any odd number of half twists, is one of many ways in which we are led to study *all clear coils* as specimens of more or less perfect *plaiting,* the number of threads plaited together being the same as the number of turns of the coil. Another mode in which we are led to the same way of regarding them is by supposing a cylinder to be passed through the middle of the (flattened) clear coil, and then to expand so as to draw all the turns tight. As there can be only a finite number of intersections, we have always an infinite choice of generating lines of the cylinder on which no intersection lies. Suppose the whole to be cut along such a line and rolled out flat. It would, of course, be a more or less perfect plait, but with a special characteristic, depending upon the fact that *it is formed from one continuous cord or wire.*

Call the several laps of the cut cord α, β, γ, &c. Then we may arrange the cut ends anyhow as follows:—α to γ, γ to ϵ, ϵ to β, β to δ, δ to α if there be but five; and similarly for any other number, *exhausting all before repeating any one oftener than once.* We may now, after having settled their order, *change their designations,* so as to name them, as they occur, in the natural order of the alphabet. Thus any such plait may be represented by a diagram as in Plate IV. fig. 14, where the dotted parts may cross and recross in any conceivable way, but must begin and end as above.

The number of ways in which such coils can be exhibited in plaits essentially distinct from one another is therefore, if n be the number of laps, $\overline{n-1}\,\overline{n-2}\ldots\ldots2.1$. All the other possible arrangements, $n-1$ times the last written number, correspond to links or, at all events, to more than one continuous cord.

§ 26. From this point of view another notation for clear coils may be given in the form

$$\begin{matrix}\alpha\,\gamma\,\beta\,\alpha \\ \beta\,\alpha\,\gamma\,\beta\ldots\ldots\end{matrix}$$

Here α, β, γ...... are, as above, the several strings plaited, so that in the coil β is the prolongation of α, γ that of β, &c., and α that of the last of the series. The expression $\begin{matrix}\alpha \\ \beta\end{matrix}$ means that α crosses *over* β. It is sometimes useful to indicate whether a crossing takes place to the right or left. This is done by putting + or − over the symbol. Thus the four crossings above may be more fully written as

$$\begin{array}{cccc} + & - & + & - \\ \alpha & \gamma & \beta & \alpha \\ \beta & \alpha & \gamma & \beta \end{array} \cdots\cdots$$

The properties of this notation were examined in detail in my first paper; but as they are more curious than useful, I merely mention one or two.

Thus the combination just written cannot be simplified in itself; but

$$\begin{array}{cccc} + & - & - & - \\ \alpha & \gamma & \gamma & \alpha \\ \beta & \alpha & \beta & \beta \end{array} = \begin{array}{cc} - & - \\ \gamma & \gamma \\ \beta & \alpha \end{array}, \text{ \&c.}$$

This notation requires care. For instance, the terms

$$\begin{array}{cc} \alpha & \alpha \\ \beta & \beta \end{array}$$

are simply nugatory, and may be cancelled. But, on the other hand, the terms

$$\begin{array}{cc} \alpha & \beta \\ \beta & \alpha \end{array}$$

usually add to the beknottedness of the whole scheme.

When the scheme is not compatible with a clear coil there occur terms of the form

$$\begin{array}{c} \alpha \\ \alpha, \end{array}$$

and the application of this method becomes very troublesome.

§ 27. A question closely connected with plaited clear coils is that of the numbers of possible arrangements of given numbers of intersections in which the *cyclical* order of the letters in the 2nd, 4th, 6th, &c., places of the scheme shall be the same as that in the 1st, 3rd, 5th, &c., *i.e.*, the alphabetical. Instances of such have already been given above. In the first scheme of § 5, for example, the letters in the even places are

D E A B C.

Here the cyclical order of the alphabet is maintained, but A is postponed by two places. It is easy to see that the following statements are true.

Whatever be the number of intersections a postponement of *no* places leads to nugatory results.

A postponement of one place is possible for three and for four intersections only.

Postponement of two places is possible only for (*four*), five, and eight—three for seven and ten—four for nine and fourteen—five for (*eight*), eleven and sixteen,—six for (*ten*), thirteen, and twenty, &c. Generally there are in all cases n postponements for $2n+1$ intersections; and for $3n+2$, or $3n+1$ intersections, according as n is even or odd. The numbers which are italicised and put in brackets above, arise from the fact that a postponement of r places, when there are n intersections, gives the same result as a postponement of $n-r-1$ places. [It will be observed that this cyclical order of the letters in the even places is possible for *any* number of intersections which is not 6 or a multiple of 6.]

When there are n postponements with $2n+1$ intersections the curve is the symmetrical double coil, *i.e.*, the plait is a simple *twist*.

The case with $3n+2$ or $3n+1$ intersections is a clear coil of three turns, corresponding to a regular plait of three strands.

Figures 16, 17 of Plate IV. give the diagrams corresponding to the latter case for $n=2$, 3 respectively; *i.e.*, with 8 and 10 crossings. The diagrams 15 and 18, constructed according to the same plan for 6 and 12 intersections, show why there are no multiples of six in this form of coil. In fact, whenever the number of crossings in this three-ply plait is a multiple of 6, the strands are separate closed curves.

Part III.

Methods of Reduction.

§ 28. Before taking up the question of the complexity of a knot, a word or two must be said about the methods of reducing any given knot to its simplest form. I have not been able as yet to find any general method of doing this, nor have I even discovered, what would probably solve this difficulty, any perfectly general method of pronouncing at once from an inspection of its scheme or otherwise, whether a knot is reducible or not. It is easy to give multitudes of special conformations in which reduction can always be effected; but of these I shall give only a few, with the view of showing their general character.

One very simple case of such reduction has already been given, viz., *where a letter occurs twice in succession.*

For, if we have as part of a scheme, the letters

. . . P Q Q R . . . .

the corresponding part of the coil must have the form shown in Plate IV. fig. 19. Whichever way the crossing at Q is effected, the loop can be at once got rid of, and it is thus nugatory, *because the scheme shows that it is not intersected by any other branch.*

If we put in the signs of the crossings, they must obviously be different for the two Q's; and thus in

. . . P Q Q R . . . .
+ −

we may treat them as $+Q-Q=0$, and obliterate Q altogether.

An immediate consequence of this is, of course, that any group such as

. . . P Q R R Q P . . . .

whatever be the number of letters arranged in this form, may be wholly struck out. Cases corresponding to this have been already figured in § 1.

§ 29. Another useful step in simplification occurs when we have a scheme containing the following terms:—

. . . . P Q . . . . . P Q . . . .
+ + − −

for then both P and Q may be struck out.

[*N.B.*—The *order* of P and Q need not be the same at each occurrence, the essential thing is that they should *twice occur together, and with like signs.* This explanation shows that the process is not confined to clear coils.]

For the corresponding part of the diagram must evidently be of the form shown in Plate IV. fig. 20, since the scheme shows that there are no intersections between P and Q on either branch. Hence, as P and Q have the same sign for each branch, one branch may be slipped off from the other without otherwise altering the coil.

If a single turn of the coil pass across between P and Q, the only ways in which it can prevent the slipping off just described are that shown in Plate IV. fig. 21, and the same looked at from the other side, *i.e.*, with all the signs changed.

Hence in the scheme

. . . P R Q . . . . P S Q . . . . . R S . . . . .
+ + − −

(where the order is again indifferent in each of the groups) we can always leave out P and Q, unless R be negative and S positive, *i.e.*, unless this part of the scheme has in itself the greatest possible number of changes of sign.

But when we *can* thus strike out P and Q, it is necessary to observe that in R S or S R, which *must* occur at some other part of the scheme, the order is to be changed. Thus

. . . . P R Q . . . . . . P S Q . . . . R S . . . .
+ + + − + − − −

is simplified into . . . . R . . . . . S . . . . S R . . . .
+ + − −

§ 30. Such a portion as that figured in Plate IV. fig. 22 evidently goes out of itself, whatever be the character of B; *i.e.*, the whole of it

$$\ldots \underset{-}{A}\underset{+}{B}\underset{+}{C}\;A\underset{-}{B}C\ldots\ldots$$

may be struck out of any scheme. In fact, whichever sign be given to B, § 29 applies and removes two of the intersections. Then § 28 disposes of the remaining one.

This is merely a particular case of the general and obvious theorem, that any portion of a coil which may be treated as a separate coil, and which, if alone, could be reduced, may be reduced *in situ*.

A more general theorem, which includes the preceding, is that, if in

$$\ldots A\,B\,C\ldots\ldots G\,H\,A\ldots\ldots$$

the signs of B, C,... G, H, where they occur between the two A's, are all alike, all these intersections, including A, may be struck out. This is quite obvious, because it indicates a complete turn of the coil entirely above or below the rest. When one or more of B, C, G, H has a different sign from the others, a less amount of simplification is usually still possible.

Along with this we may take the case of fig. 23. Here we have

$$\ldots\ldots \underset{-}{P}\underset{-}{Q}\underset{+}{R}\underset{+}{P}\underset{+}{S}\ldots\ldots\ldots \underset{-}{R}\underset{+}{Q}\underset{-}{S}\ldots\ldots$$

If the sign of P were changed these parts of the scheme would contain alternately + and −. The scheme obviously loses three intersections, and becomes

$$\ldots\ldots \underset{-}{Q}\ldots\ldots \underset{+}{Q}\ldots\ldots$$

If the signs in the complete knot, with the exception of that of P, were all + and − alternately, there will generally be farther reductions possible.

§ 31. A glance shows that the first of the diagrams, 24, 25, Plate IV., can be reduced to the second. Hence in the scheme of a knot

$$\ldots\ldots\ldots \underset{+}{P}\underset{+}{Q}\underset{-}{R}\underset{-}{P}\ldots\ldots\ldots \underset{-}{Q}\underset{+}{R}\ldots\ldots$$

may be simplified into

$$\ldots\ldots\ldots \underset{+}{Q}\underset{-}{R}\ldots\ldots\ldots\ldots \underset{+}{R}\underset{-}{Q}$$

[*N.B.*—The essential point is that P and Q should have the *same* sign, and R the opposite. If Q and R had the same sign they might both be struck out, § 29. But if P and Q have different signs, as also Q and R, no simplification can be effected, though, as has been shown in § 11, a change of scheme is practicable.]

§ 32. The scheme

. . . . A B C . . . . E F G . . . . . . A M N . . . . P Q G . . .
+ + + + + + − −

always admits of striking out A and G. But special consideration is necessary as to what is to take the place of B, C, . . . E, F. Their substitutes will all be positive, and may be called *m*, *n*, . . . *p*, *q*, since they are in number the same as M, N, ... P, Q—irrespective altogether of the number of B, C, ... E, F. In fact, M and *m*, N and *n*, ... &c., lie (as near one another, in pairs, as we please) on the several turns of the coil which intersect the arc A M . . . . Q G. And *m*, *n*, ... &c., are on the *opposite* side of that arc from B, C, ... F.

§ 33. There are numberless other special rules, but those just given are among the simplest, and they are in general sufficient for coils with only a moderate number of intersections. With the present notation it is not easy to classify them, or to show how they may be exhibited as particular cases of more general rules. We will therefore, for the present, employ them only for the simplification (where possible) of a few diagrams of knots. But it must be particularly noticed that the simplifications above are mainly such as *tend to remove continuations of sign from a scheme*, none of them but the first being applicable to a scheme whose signs present no continuations.

§ 34. *Examples.*

I. A E B F C G D A E K F L G D H B K C L H | A
− + + + − + − + − + − + − + − − − + − +

This is, of course, rendered irreducible by changing the sign of B. It is figured Plate V. fig. 1.

[If we were to change the sign of F, L, H, the knot would acquire a great increase of beknottedness, and would consist, in its simplest form, of a pentacle and a trefoil knot linked together, as in Plate V. fig. 25.]

(*a*) Now . . . E B F . . . . E K F . . . . B K . . . .
+ + + − + − − −

become . . . B . . . . K . . . . . K B . . . .
+ + − −

(*b*) Two intersections being thus lost, the knot has now the form, Plate V. fig. 2, with the scheme

A B C G D A K L G D H K B C L H | A
− + − + − + + + − + − − − + − +

Now in . . . . D A K L G . . . . .
+ + +

with G before or D after, we can at once get rid of K, L, if A be put close to G.

(*c*) Hence the scheme becomes

B C A G D A G D H B C H | B
\+ − − + − + − + − − + +

and the knot is as in the figure 3, Plate V.

Now H B. . . . H B. . . . go out (§ 29).
− − + +

(*d*) The scheme is now

C A G D A G D C | C
− − + − + − + +

so that C goes out by § 28, and we have finally

A G D A G D | A
− + − + − +

the trefoil knot.

II. The knot figured in Plate V. fig. 4 has no beknottedness.

III. That in fig. 5 is reducible to the trefoil.

These are left as exercises to the reader.

Part IV.

Beknottedness.

§ 35. Recurring to the two species of five-crossing knots discussed in § 18, we easily see that there is less entanglement or complication in the first species than in the second. For if the sign of *either* of the two crossings towards the top of the first figure be changed, it is obvious that it will no longer possess any but nugatory crossings. But if we change the sign of any one crossing in the pentacle, that crossing, and *one* only of the adjacent ones, become nugatory, so that the knot becomes the trefoil with alternating + and −. This, in turn, has all its intersections made nugatory by the change of sign of any one of them. Thus one change of sign removes all the knotting from the first of these knots, but two changes are required for the second.

In what follows the term *Beknottedness* will be used to signify the peculiar property in which knots, even when of the same order of knottiness, may thus differ: and we may define it, at least provisionally, as *the smallest number of changes of sign which will render all the crossings in a given scheme nugatory.* This question is, as we shall soon see, a delicate and difficult one. It is probable that it will not be thoroughly treated until one considers along with it another property, which may be called *Knotfulness*—to indicate the number of knots of lower orders (whether interlinked or not) of which a given knot is in many cases built up. But this term will not be introduced in the present paper.

§ 36. It may be well to premise a few lemmas which will be found useful in examining for our present purpose the plane projection of a knot.

(α) Regarding the projection as a wall dividing the plane into a number of fields, if we walk along the wall and drop a coin into each field as we *reach* it, each field will get as many coins as it has corners, but those fields only will have a coin in each corner whose sides are all described in the same direction round. For we enter by one end of each side and leave by the other. The number of coins is four times the number of intersections; and two coins are in each corner bounded by sides by each of which we enter, none in those bounded by sides by each of which we leave. Hence a mesh, or compartment, which has a coin in each corner has all its sides taken in the same direction round; and we see by fig. 6, Plate V., that this is the case with twists in which the laps of the cord run opposite ways, not if they run the same way. Compare this with the remarks of § 35, as to the two species of 5-fold knottiness.

(β) To make this process give the distinction between crossing *over* and crossing *under*, we may suppose the two coins to be of different kinds,—silver and copper for instance. Let the rule be:—silver to the right when crossing over, to the left when crossing under. Then, however the path be arranged, of the four angles at each crossing, one will have no coins, the vertical or opposite corner will have *two* silver or *two* copper coins, the others *one* copper or *one* silver coin each.

It is easily seen that a reversal of the direction of going round leaves the single coins as they were, but shifts the pair of coins into the angle formerly vacant: also that in all deformed figures the circumstances are exactly the same as in the original. Hence we may divide the crossings into silver and copper ones, according as two silver or two copper coins come together. And the excess of the silver over the copper crossings, or *vice versâ*, furnishes an exceedingly simple and readily applied test (not, however, as will soon be seen, in itself absolutely conclusive of identity, though absolutely conclusive against it), which is of great value in arranging in family groups (those of each family having the same number of silver crossings), the various knots having a given number of intersections.

(γ) Or, still more simply, we may dispense altogether with the copper coins, so that, going round, we pitch a coin into the field to the *right* at each crossing *over*, to the *left* at each crossing *under*. When the coins are in the same angle the crossing is a silver one, when in two vertical angles it is copper. Each of these three processes has its special uses.

§ 37. This process, thus limited, is obviously intimately connected with that required for the estimation of the work necessary to carry a magnetic pole along the curve, the curve being supposed to be traversed by an electric current. Hence it occurred to me that we might possibly obtain a definite measurement of beknottedness in terms of such a physical quantity: as it obviously must be always the same for the same knot, and must vanish when there is no beknottedness. To make the measure complete, we must record the numbers of non-nugatory silver and copper crossings separately, with the number to be deducted as due merely to the *coiling* of the figure. This last is a very important matter, and will be dealt with later.

§ 38. When unit current circulates in a simple circuit, it is known that the work required to carry unit magnetic pole once round any closed curve once linked with the circuit is $\pm 4\pi$. Instead of the current we may substitute a uniformly and normally magnetized surface bounded by the circuit. The potential energy of the pole in any position is measured by the spherical aperture subtended at the pole by the circuit; but its sign depends upon whether the north or south polar side is turned to the pole. Hence the pole has no potential energy when it is situated in the plane of the circuit but external to it, and the potential energy is $\pm 2\pi$ when the pole just reaches the plane of the circuit internally.

In fact the electro-magnetic force exerted by an element $d\alpha$ of a unit current, on a unit north pole placed at the origin of α, is

$$\frac{V\alpha d\alpha}{T\alpha^3}$$

or, as we may write it,

$$V.d\alpha\nabla\frac{1}{T\alpha}.$$

This is identical in form with the expression for the differential whose integral, taken round a closed circuit, is Ampère's *Directrice**.

Hence the element of work done by the closed circuit while the pole describes a vector $\delta\alpha$, is

$$\delta W = -S.\delta\alpha\int\frac{V\alpha d\alpha}{T\alpha^3} = -S.\delta\alpha\int d\alpha\nabla\frac{1}{T\alpha}.$$

But, if $d\Omega$ be the spherical angle subtended at α by a little plane area ds, whose unit normal vector (drawn *towards* the origin of α) is $U\nu$, obviously

$$d\Omega = \frac{S.U\nu U\alpha}{T\alpha^2}ds = -S.U\nu\nabla\frac{1}{T\alpha}ds.$$

Now, in the general formula (No. XIX. above, p. 143)

$$-\int V\sigma d\alpha = \iint ds V.(VU\nu\nabla)\sigma,$$

put

$$\sigma = \nabla\frac{1}{T\alpha}$$

and we have

$$\int\frac{V\alpha d\alpha}{T\alpha^3} = \iint ds\left(U\nu\nabla^2\frac{1}{T\alpha} - \nabla SU\nu\nabla\frac{1}{T\alpha}\right)$$

$$= \iint ds\, U\nu\nabla^2\frac{1}{T\alpha} + \nabla\Omega.$$

Now the double integral always vanishes while $T\alpha$ is finite, and we have therefore

$$\delta W = \int\frac{S.\alpha\delta\alpha d\alpha}{T\alpha^3} = S.\delta\alpha\nabla\Omega = -\delta\Omega.$$

* "Electrodynamics and Magnetism," §§ 5—8, *Anté*, p. 24.

That is, the work done during any infinitesimal displacement of the pole is numerically equal to the change in the value of the spherical angle subtended by the circuit. The angle is, of course, a discontinuous function, its values differing by 4π at points indefinitely near to one another, but lying on opposite sides of the uniformly and normally magnetized surface whose edge is the circuit. There is, however, no discontinuity in the value of the work, for the element of the double integral is finite, and equal to 4π, when $T\alpha = 0$.

Gauss* says (with date January 22, 1833):—"Eine Hauptaufgabe aus dem *Grenzgebiet* der *Geometria Situs* und der *Geometria Magnitudinis* wird die sein, die Umschlingungen zweier geschlossener oder unendlicher Linien zu zählen." And he adds that the integral

$$\iint \frac{(x'-x)(dydz'-dzdy')+(y'-y)(dzdx'-dxdz')+(z'-z)(dxdy'-dydx')}{((x'-x)^2+(y'-y)^2+(z'-z)^2)^{\frac{3}{2}}},$$

extended over both curves, has the value

$$4m\pi,$$

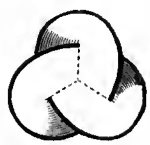
where m is the number of linkings (Umschlingungen). This is obviously the same as the integral of $-\delta W$ above, viz.:—

$$-\iint \frac{S\,.\,\alpha d\alpha \delta\alpha}{T\alpha^3},$$

extended round each of two closed curves, of which $d\alpha$ and $\delta\alpha$ are elements.

§ 39. A very excellent investigation, by means of Cartesian co-ordinates, will be found in Clerk-Maxwell's *Electricity and Magnetism*, §§ 417—422. It is there shown that the above integral may vanish, even when the circuits are inseparably linked together. In fact m may vanish either because there is no real linking at all, or because the number of linkings for which the electro-magnetic work is negative is the same as that for which it is positive. For our present application this is of very great consequence, because it shows that the electro-magnetic work, under the circumstances with which we are dealing, cannot in all cases measure the amount of beknottedness. In fact the processes, soon to be described, enable us, without trouble for any given *linkage*, to find the value of m in Gauss' formula; but there are special ambiguities when we try to apply the process to knots.

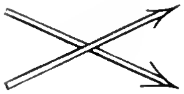
§ 40. To construct the magnetized surface which shall exert the same action on a pole as a current in any given closed circuit does, we may either suppose a cylinder extending to infinity in one direction (say for definiteness, upwards from the plane of the paper), and having the circuit for its edge; or we may form, as in the figure, a finite autotomic surface of one sheet, having the circuit for its edge. In dealing with the *two* curves of Gauss' proposition, our procedure is perfectly definite; but when one and the same curve is to be the current and also



* *Werke*, Göttingen, 1867, v. p. 605.

the path of the pole, there is an ambiguity in estimating the electro-magnetic work. To clear this away we require a definite statement of how the pole moves along the curve itself. For if its path screw round the curve $\pm 4\pi$ must be added to the work for each complete turn. As an illustration, suppose we bend, as in the figure, an india-rubber band coloured black on one side, so that the black is always the concave surface, and so that one loop is the perversion of the other, we find on pulling it out straight that it has no twist. If both loops be made by *over*laying, when pulled out it becomes twisted through two whole turns. This illustrates the kinematical principle that spiral springs act by torsion. An excellent instance of its connection with knots is to be seen in the process employed in § 11. For if we have portions of a cord, as in the diagram (Plate V. fig. 7), the pulling out of the loop in the upper cord changes the arrangement, as shown in the second figure.

A practical rule, which completely meets the Gaussian problem, may easily be given from the consideration of the cylindrical magnetized surface above mentioned. Go round the curve, marking an arrow-head after each crossing to show the direction in which you passed it. Then a junction like the following gives $+4\pi$ for the upper branch, and nothing for the lower (which, on this supposition, does not pass through the magnetic sheet). Change the crossing from *over* to *under*, and this quantity changes sign. The junction figured above would, in our first illustration, be a silver one. But a still simpler process is to go round, as in § 36 (γ), putting a dot to the *right* after each crossing *over*, and *vice versâ*.



§ 41. Now, in order that our rule when applied to *knots* may give no work where there is no beknottedness, we must make the required expression such as to vanish whenever all the intersections are nugatory. Those which are nugatory only in consequence of their signs are in pairs, silver and copper, and will take care of themselves, as we see by special examples like the following. Hence the only part to correct for is that depending on the number of whole turns, and the sketch of the india-rubber band above shows that the work at the vertex of each such partial closed circuit is simply not to be counted, *i.e.*, that the 4π, which would be reckoned for each such crossing by our rule (positively or negatively as the case may be), is to be considered as made up for by the corresponding screwing of the pole round the curve.

§ 42. There must be some very simple method of determining the amount of beknottedness for any given knot; but I have not hit upon it. I shall therefore content myself with a few remarks on the subject, some of which are general, others applicable only to certain classes of forms. There seems to be little doubt that the difficulty will be solved with ease when the true method of attacking amphicheiral forms is found.

1. To form from a given projection the knot with the greatest amount of beknottedness, it is clear that we must in general so arrange the crossings over and

under as to make *all* the crossings simultaneously silver or copper ones. And when this is done, a projection will give greater beknottedness for the same number of crossings the smaller is the number of crossings which have to be left out of account. Thus the simple *twists* (or clear coils with two turns) are the forms which, with a given amount of knottiness, can have the greatest beknottedness. For in them (see § 41) only one crossing has to be left out of the reckoning. Even a regular plait if of more than two strands cannot have so much beknottedness as it would acquire with the same amount of knottiness if two of its strands were first twisted together, then a third round these, and so on. And thus also entirely nugatory forms like the two first cuts in § 1 can have no beknottedness, for *all* their crossings have to be left out of the reckoning.

As an illustration, take the figure (Plate V. fig. 8) where the supposed number of loops may be any whatever. The free ends must, of course, be joined externally.

If we make the crossings alternately + and − it will be seen at a glance that a change of *one* sign (*i.e.*, that of the extreme crossing at either end) removes the whole knotting; so that there is but one degree of beknottedness. The crossings in this figure are in three rows. Those in the upper row are all copper (the last, of course, becomes silver when its sign is changed), and their number is n the number of loops. Each of the other rows has $n-1$, and all of them are silver. Thus when the one sign is changed there are $n-1$ copper crossings, and $2n-1$ silver. By pulling out the right-hand loop we change n to $n-1$, so that one copper and two silver crossings are lost. After $n-1$ operations like this there remains only one (silver) crossing. It is easy to see from this that the crossings to be omitted in the reckoning of beknottedness (as in § 41) must be the lower row. To prove that it is so, study the beknottedness when the crossings are made so that the upper row are copper, silver, copper, &c., alternately, and those of the two other rows, silver, copper, silver, &c., alternately. It will be easily seen that with five loops there are two degrees of beknottedness, &c., and thus that our rule is correct. It is a curious problem to investigate the torsional and flexural rigidities of a wire bent in this form.

To give the greatest beknottedness to a knot with the same projection, it is obvious that all we have to do is to make the copper crossings into silver ones, *i.e.*, change the sign of each of the upper row of crossings. This gives fig. 9. With five loops it has four degrees of beknottedness.

Another excellent illustration is given by the coils of the class figured in Plate IV. figs. 16 and 17, which have been already described (§ 27). A full investigation of the higher knottinesses of this class (especially when fully beknotted) would well repay the trouble it would involve.

As they are all amphicheiral, and in each case the crossings are divisible into two sets, those of each set being in all respects alike, while those of different sets differ only as to silver or copper, it is no matter (so far as testing beknottedness is concerned) which crossing we suppose to have its sign changed.

In the 8-fold amphicheiral of fig. 16 the change of any one sign reduces the whole to the irreducible trefoil knot (§ 16), right or left-handed according as we have changed one of the four outer, or of the four inner, crossings in the figure. Hence it has *two* degrees of beknottedness. But if we change the signs of one set of crossings (Plate V. fig. 24) so as to make all the crossings alike silver (or copper), we find the knot irreducible, though with continuations of sign; but with *three* degrees of beknottedness. And it is easy to see that it can now be analysed into two right-handed trefoil knots linked together as shown in the other part of the figure. But the linking is *left-handed*. Had it been right-handed we should have had + and − alternately, and thus we could not have transformed back to the form with continuations of sign (§ 4).

Similar remarks apply to the 10-fold amphicheiral plait (Plate IV. fig. 17). Change of any one sign reduces it to the third form of 6-fold knottiness (γ, § 8), which has only one degree of beknottedness. Hence the 10-fold plait has but *two* degrees of beknottedness when its signs are alternate. If we make all its crossings silver (or copper), as in Plate V. fig. 25, it has *four* degrees of beknottedness; and the reason is obvious from the other half of the figure, where it is seen to be made up of a pair of irreducibles—a pentacle and a trefoil, once linked together. There is one degree of beknottedness for the trefoil, one for the link, and two for the pentacle. The trefoil and pentacle are right-handed, the link left-handed, else we should not have had the continuations of sign which the figure must show.

A very curious illustration of this is to be found in the excepted cases, where the number of crossings is a multiple of six. From the two figured (Plate IV. figs. 15, 18) it is obvious that all of these are formed by three unknotted closed curves, *no two of which are linked together*, yet the whole is irreducible, having alternate signs. Hence we require a *third* term to complete our descriptions—knotting, linking, locking (?).

To give the greatest amount of belinkedness to these figures, let us suppose the ovals taken all the same way round, and arrange so that all the crossings shall be silver. Then we have continuations of sign (Plate V. fig. 26) as in the knots of the same series. But whereas Plate IV. fig. 15, if made of wire, is particularly stiff, the new figure is eminently flexible. This seems to have been practically known to the makers of chain armour.

The 9-fold knot of Plate V. fig. 15 has its crossings so drawn as to be all copper. Three must be left out of reckoning for the coiling, so it has *three* degrees of beknottedness.

But if we made the crossings alternately + and − we should find zero for the corrected electro-magnetic work—three copper and three silver crossings remaining. Change, then, the sign of any one of the three outer or inner crossings, and the whole reduces to the 4-fold knot. Hence it has *two* degrees of beknottedness.

If the crossing whose sign is changed be neither an outer nor an inner one, the result is a very singular 8-fold knot (irreducible, though having continuations of sign), differing from that of fig. 24, Plate V., in the fact that its component trefoil knots are *unsymmetrically* linked together. And it has but *one* degree of beknottedness, while that of fig. 24 has *three.*

I have called attention to this example because of its bearings on the question of *the numbers of different irreducible knots having the same projection*, which we meet with as soon as we reach 8-fold knottiness*.

2. To remove all beknottedness from a projection it is only necessary to make every crossing in its scheme + (or −) when it is first met with, reading from any point whatever. For then the several laps of the coil are, as it were, paid out in succession one over the other. When the beknottedness of a scheme so marked is calculated (as in § 41), it will be found that there is always at least one choice of a set of crossings such that, when these are omitted from the count, the electro-magnetic work is zero.

As an illustration take the very simplest form, the trefoil knot, with the suffixed signs determined by this rule. The scheme is

$$\begin{array}{cccccc|c} - & + & - & - & + & - & \\ A & C & B & A & C & B & A. \\ + & + & + & - & - & - & + \end{array}$$

Now, by § 41 we are entitled to leave out of count either A, B, or C. Leaving out either A or B gives zero for the electro-magnetic work, as it ought to be; but leaving out C gives -8π.

3. The only way in which we can have the intersections + and − alternately while every letter is + on its first appearance, *i.e.*, when there is no beknottedness, is obviously the wholly nugatory scheme

$$\begin{array}{cccc} A & A & B & B, \&c. \\ + & - & + & - \end{array}$$

§ 43. To illustrate these methods let us take again the 5-fold knots (as in § 18) whose schemes are

$$\begin{array}{cccccccccc|c} + & + & + & + & + & + & + & + & + & + & \\ A & D & B & E & C & A & D & C & E & B & A, \\ - & + & - & + & - & + & - & + & - & + & \end{array}$$

$$\begin{array}{cccccccccc|c} - & - & - & - & - & - & - & - & - & - & \\ A & D & B & E & C & A & D & B & E & C & A. \\ - & + & - & + & - & + & - & + & - & + & \end{array}$$

The lower signs refer to over or under, the upper to the electro-magnetic work, or to the silver-copper distinction.

Hence to determine the electro-magnetic work we must divide each scheme into independent circuits, no one of which includes a less extensive one; and omit from the reckoning the work for the terminal of each such circuit, and for each of the intersections which is not included in any one of the separate circuits. There are usually more ways than one of doing this. Sometimes these agree in their results;

* [This very interesting question has since been discussed, for 8-fold and for 9-fold knottiness, by Prof. C. N. Little (*Trans. R. S. E.* xxxv., 1889). 1898.]

but the rule for choosing which to omit is to take them such that *with their proper signs*, and the rest with any signs whatever, they may be capable of making each letter positive on its first appearance. But there are cases even when the knot is not amphicheiral in which this process cannot be carried out. These occur specially when a part of the knot forms a lower knot with which the string is again linked.

In the first of the two schemes above there is but one independent non-autotomic circuit, which may be taken as

A D B E C A.

In this all the intersections are included, so that the whole work is to be found by leaving out that for A only; *i.e.*, it is -16π.

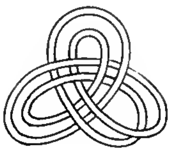
But in the second scheme we may take the two circuits

B A D B and C A D C,

and E is not included in either. Hence we must leave out of count the work for B, C, and E. This is found to satisfy our test, and thus the whole work is only -8π.

This is an instance in which the estimate by the electro-magnetic process exactly agrees with the result of simpler considerations, as given in § 35 above.

§ 44. It will be found that the alteration of five signs is sufficient to remove the knotting from the annexed figure, and the stages of operation of the various modes of reduction show that this form can be regarded as made up of simpler knots intersecting one another on the same string. These separate knots are virtually independent, and to change *all* the signs in any one of them does not in cases like this necessarily simplify the knot. Uncorrected the work is $-13 \times 4\pi$. Corrected it is $-10 \times 4\pi$, which agrees with the removal of the beknottedness by change of *five* signs only.



If the sign of the one unsymmetrical crossing be altered, four changes of sign will suffice; for the uncorrected work is $-11 \times 4\pi$; corrected it is $-8 \times 4\pi$, corresponding to four changes of sign.

§ 45. It is clear from what precedes that the Gaussian integral does not, except in certain classes of cases, express the measure of what may be called, by analogy with § 35, *Belinkedness*. It may be well to examine a simple form of link with all its possible arrangements of sign to see what the integral really gives in each of these. Let us choose for this purpose two lemniscates having four mutual crossings, suggested by the edges of the band shown in fig. 13, Plate IV.

If we suppose the signs to be made alternately + and −, as in Plate V. fig. 10, the form is a six crossing one, and irreducible. The silver or copper character of the *self* crossings does not depend upon the directions in which we suppose the lemniscates to be described, that of the *mutual* crossings does. We thus have, from another point of view than that of § 41, a proof that these are to be left out of account in the reckoning.

The four crossings of the *two* curves are copper, if these curves are supposed to be described in the same way round; those of the separate curves (which do not count) are silver. Hence the work is -16π, or two degrees of belinkedness.

Change the sign of any *one* of P, Q, R, S, that and the adjacent one slip off, U and V become nugatory. The linkage is the simplest possible, and the integral is 8π.

Change the sign of either or both of U and V. In either of these three cases both become nugatory, and the whole takes the form of two doubly-linked ovals, with the integral $= -16\pi$. (Plate V. figs. 12, 13.)

If the signs of both R and S be changed the value of the integral is obviously $4(2-2)\pi$, for R and S have become silver, while P and Q remain copper.

If in addition the signs of U and V be both, or neither, changed, only one crossing is got rid of, and the link may be put in the form (Plate V. fig. 14). It cannot be farther reduced, because the crossings are alternately over and under.

But if the sign of one only of U, V be changed, it will be seen that there is no linking (Plate V. fig. 11). Here the integral vanishes because there is really *no work*, not as in the last case, where there are *equal amounts of positive and negative work*.

§ 46. This gives a hint as to the reckoning of beknottedness from the silver and copper crossings in the cases where we have found a difficulty. After omitting from the reckoning the crossings which belong merely to the *outline* of the figure, there must remain an *even* number of crossings (§ 22). Hence, whatever numbers be silver and copper respectively, the excess of the one of these over the other must be an even number (zero included). In general, *half this number is the beknottedness.* But when the knot, or even part of it, is amphicheiral there is usually more beknottedness than this rule would give. And, in particular, there may be beknottedness when the number is zero. In this case the number of silver (and of copper) crossings is even, and is double the degree of beknottedness.

As I have already stated, I have not fully investigated this point, and therefore for the present I content myself with giving two instructive examples from the six-fold knots. The observations which will be made on these contain at least the germ of the complete solution.

The form γ (of § 8) is not amphicheiral. As there drawn, it has four copper and two silver crossings, the latter being the intersections of the loop with the trefoil; but the scheme shows that two copper crossings must be omitted from the reckoning, one of these being necessarily that which is uppermost in the figure. If the sign of this last be changed, the knot opens out, so that it has but one degree of beknottedness. Hence, in this case, the two copper and two silver crossings correspond to one degree of beknottedness only. But if we change the sign of *any one* of the other three copper junctions the knot sinks to the 4-fold amphicheiral, retaining its one degree of beknottedness.

In the amphicheiral form β (of § 8) there are three silver and three copper crossings. As the figure is drawn, these are to the right and left of the figure respectively; and either crossing at the end of the lower coil may be left out, along with any one of the three on the other side. Thus there remain, as in the former case, two silver and two copper ones. This corresponds to one degree of beknottedness, as in the last case, for the change of sign of *either* crossing at the end of the lower coil unlooses the knot. But if any one of the other four crossings (alone) have its sign changed, the whole becomes a right or left-handed trefoil knot, retaining, as in the former example, its one degree of beknottedness.

To give the greatest beknottedness to these forms, we must alter two signs in (γ) and three in (β). In each case one crossing is lost, and the form becomes the pentacle (§ 7) with its two degrees of beknottedness.

PART V.

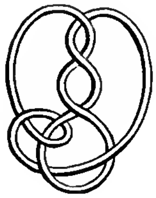
Amphicheiral Forms.

§ 47. These have been defined in § 17, and several examples have been given, not only of knots, but of links, which possess the peculiar property of being transformable into their own perversions.

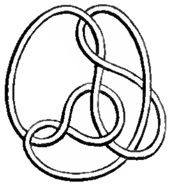
The partition method (§ 21) suggests the following mode of getting amphicheirals:—Since the right-handed and left-handed compartments must agree one by one, and since (§ 20) the whole number of compartments is greater by two than the number of crossings, the number of crossings must be even. Let it be $2n$, and let $p_1, p_2, \ldots p_{n+1}$ be the partitions. Then our selection must be made from the numbers which satisfy

$$p_1 + p_2 + \ldots\ldots + p_{n+1} = 4n,$$

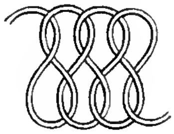
no one being greater than the sum of the others. If a set of such can be grouped as in § 20 so that the other set for the complete scheme shall be the same numbers *with the same grouping*, we have an amphicheiral form. The words in italics are necessary, as the following example shows; for here the black and white compartments have the same set of partitions but not the same grouping, and the knot is not amphicheiral:—



But a different grouping of the *same* set of partitions gives the amphicheiral form below

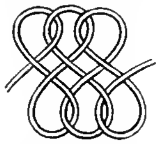


But an easier mode of procedure, though even more purely tentative, is the following:—If a cord be knotted, any number of times, according to the pattern below,



it is obviously *perverted* by simple *inversion*. Hence, when the free ends are joined it is an amphicheiral knot. Its simplest form is that of 4-fold knottiness. All its forms have knottiness expressible as $4n$.

The following pattern gives amphicheiral knottiness $2 + 6n$:—



And a little consideration shows that on the following pattern may be formed amphicheiral knots of all the orders included in $6n$ and $4 + 6n$:—

Among them these forms contain all the even numbers, so that *there is at least one amphicheiral form of every even order.*

Many more complex forms may easily be given. See, for instance, Plate V. figs. 18, 19, 20. Some are closely connected with knitting, &c.

An excessively simple mode of obtaining such to any desired extent is to start with an amphicheiral, whether knot or link, and insert additional crossings. These must, of course, be inserted symmetrically in pairs, each in the original figure being accompanied by another which will take its place in the perversion or image.

Thus, taking the simplest of all amphicheirals, the single link (Plate V. first of figures 27), we may operate on it by successive steps as in the succeeding figures.

The second, third, and fourth are formed from the first by adding, the fifth and sixth from the fourth by removing, pairs of crossings. The third, like the first, is a link; the others are knots.

Figures 28, Plate V., give another series, of which the genesis is obvious. The protuberances put in the first figure, for instance, show how it becomes the second. The fifth of fig. 27, and the second and fourth of fig. 28, all alike represent the amphicheiral form (β) of § 8. But we need not pursue this subject.

§ 48. It will be seen at a glance that the first pattern in last section gives for two loops (*i.e.*, four crossings) the knot of § 6; while the third pattern as drawn is simply β of § 8. In this form of the knot, the two dominant crossings (§ 46) are those in the middle, and mere inspection of the figures shows that the whole knotting becomes nugatory if the sign of either of these be changed.

It might appear at first sight that amphicheirals of the same knottiness, formed on such apparently different patterns as the two first of last section, would be necessarily different. But the very simplest case serves to refute this notion. For the lowest integers which make

$$4n = 2 + 6n'$$

give 8 as the value of either side. Figs. 22, 23, Plate V., represent the corresponding amphicheirals, apparently very different, but really transformable into one another by the processes of § 11. Fig. 21, Plate V., represents another 8-fold amphicheiral form, suggested by a somewhat similar pattern. I hope to return to the consideration of this very curious part of the subject, and at the same time to develop a method of treating knots altogether different from anything here given, which I recently described to the Society*.

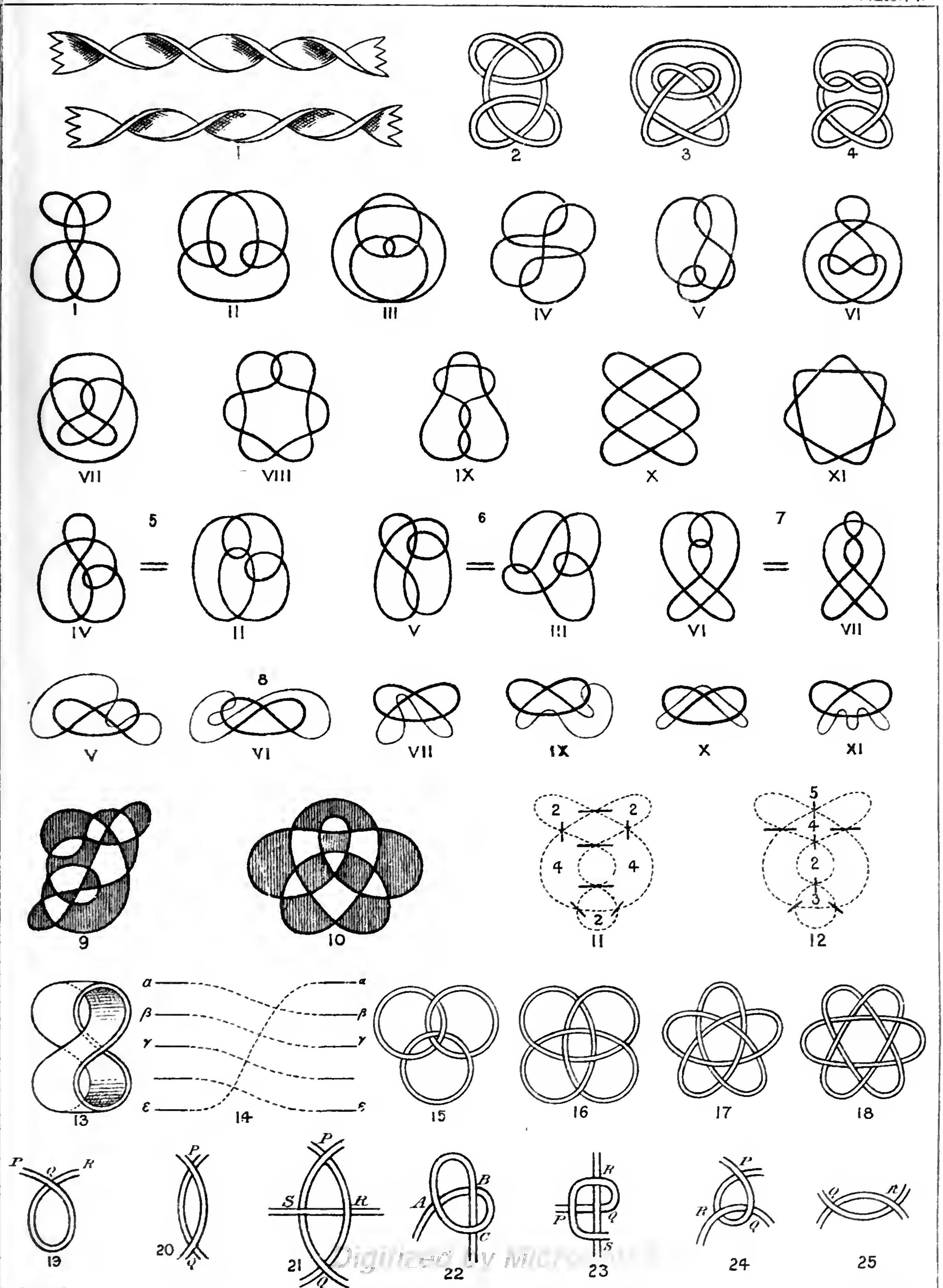
After the papers, of which the foregoing is a digest, had been read, I obtained from Professors Listing and Klein a few references to the literature of the subject of knots. It is very scanty, and has scarcely any bearing upon the main question which I have treated above. Considering that Listing's Essay was published thirty years ago, and that it seems to be pretty well known in Germany, this is a curious fact. From Listing's letter (*Proc. R. S. E.* 1877, p. 316), it is clear that he has published only a small part of the results of his investigations. Klein† himself has made the very singular discovery that *in space of four dimensions there cannot be knots.*

The value of Gauss' integral has been discussed at considerable length by Boeddicker (by the help of the usual co-ordinates for potentials) in an Inaugural Dissertation, with the title, *Beitrag zur Theorie des Winkels*, Göttingen, 1876‡.

* *Proceedings R.S.E.*, May 7th, 1877.

† *Mathematische Annalen*, IX. 478.

‡ Professor Fischer has just shown me an enlarged copy of Boeddicker's pamphlet above mentioned. Twenty pages are now added, mainly referring to the connection of knots with Riemann's surfaces, and the title is altered to *Erweiterung der Gauss'schen Theorie der Verschlingungen*, Stuttgart, 1876.



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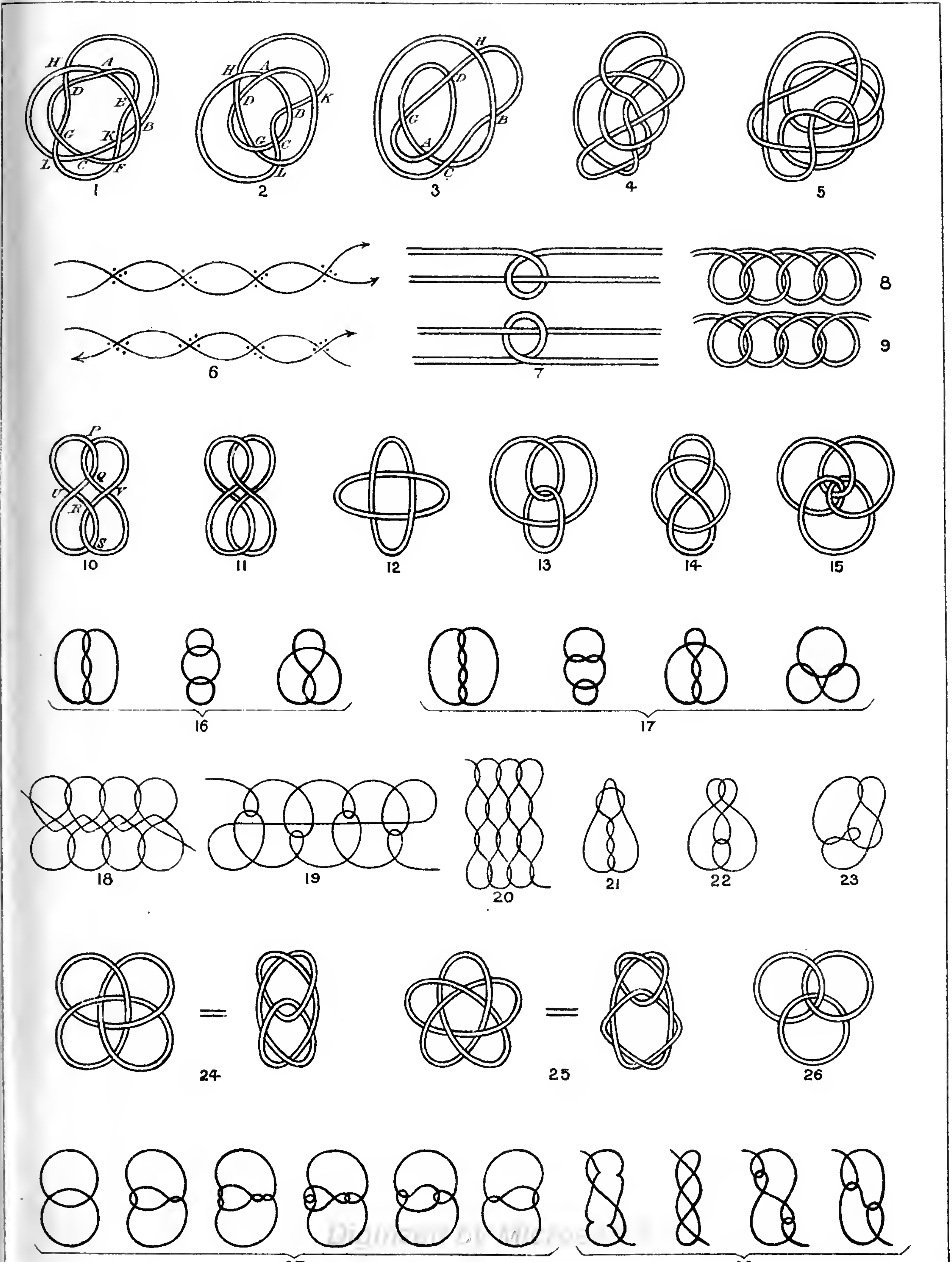
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An Inaugural Dissertation by Weith, *Topologische Untersuchung der Kurven-Verschlingung*, Zürich, 1876, is professedly based on Listing's Essay. It contains a proof that there is an infinite number of different forms of knots! The author points out what he (erroneously) supposes to be mistakes in Listing's Essay; and, in consequence, gives as something quite new an illustration of the obvious fact that there can be irreducible knots in which the crossings are not alternately over and under. The rest of this paper is devoted to the relations of knots to Riemann's surfaces.