Pin Structures on Low–dimensional Manifolds

by

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§0. Introduction

Pin structures on vector bundles are the natural generalization of *Spin* structures to the case of non-oriented bundles. $Spin(n)$ is the central $\mathbf{Z}/2\mathbf{Z}$ extension (or double cover) of $SO(n)$ and $Pin^-(n)$ and $Pin^+(n)$ are two different central extensions of $O(n)$, although they are topologically the same. The obstruction to putting a *Spin* structure on a bundle ξ $(= R^n \to E \to B)$ is $w_2(\xi)\epsilon H^2(B;\mathbf{Z}/2\mathbf{Z})$; for Pin^+ it is still $w_2(\xi)$, and for Pin^- it is $w_2(\xi) + w_1^2(\xi)$. In all three cases, the set of structures on ξ is acted on by $H^1(B;\mathbf{Z}/2\mathbf{Z})$ and if we choose a structure, this choice and the action sets up a one–to–one correspondence between the set of structures and the cohomology group.

Perhaps the most useful characterization (Lemma 1.7) of $Pin^\pm$ structures is that Pin^- structures on ξ correspond to *Spin* structures on $\xi \oplus \det\xi$ and Pin^+ to *Spin* structures on $\xi \oplus 3\det\xi$ where $\det\xi$ is the determinant line bundle. This is useful for a variety of "descent" theorems of the type: a $Pin^\pm$ structure on $\xi \oplus \eta$ descends to a Pin^+ (or Pin^- or *Spin*) structure on ξ when $\dim\eta = 1$ or 2 and various conditions on η are satisfied.

For example, if η is a trivialized line bundle, then $Pin^\pm$ structures descend to ξ (Corollary 1.12), which enables us to define $Pin^\pm$ bordism groups. In the *Spin* case, *Spin* structures on two of ξ, η and $\xi \oplus \eta$ determine a *Spin* structure on the third. This fails, for example, for Pin^- structures on η and $\xi \oplus \eta$ and ξ orientable, but versions of it hold in some cases (Corollary 1.15), adding to the intricacies of the subject.

Another kind of descent theorem puts a $Pin^\pm$ structure on a submanifold which is dual to a characteristic class. Thus, if V^{m-1} is dual to $w_1(T_M)$ and M^m is $Pin^\pm$, then $V \pitchfork V$ gets a $Pin^\pm$ structure and we have a homomorphism of bordism groups (Theorem 2.5),

$$[\cap w_1^2] : \Omega_m^{Pin^\pm} \longrightarrow \Omega_{m-2}^{Pin^\mp}$$

that proved useful in [K–T]. Or, if F^{m-2} is the obstruction to extending a Pin^- structure on $M^m - F$ over M, then F gets a Pin^- structure if M is oriented (Lemma 6.2) or M is not orientable but $F \pitchfork V$ has a trivialized normal bundle in V (Theorem 6.9). These results give generalizations of the Guillou-Marin formula [G-M], Theorem 6.3,

$$2\beta(F) \equiv F \cdot F - \text{sign}\, M \qquad (\text{mod } 16)$$

[1] Partially supported by the N.S.F.

to any characterized pair (M^4, F^2) with no condition on $H_1(M^4; \mathbf{Z}/2\mathbf{Z})$.

Here, $\beta(F)$ is the $\mathbf{Z}/8\mathbf{Z}$ Brown invariant of a $\mathbf{Z}/4\mathbf{Z}$ quadratic enhancement of the $\mathbf{Z}/2\mathbf{Z}$ intersection form on $H_1(F; \mathbf{Z}/2\mathbf{Z})$; given a Pin^- structure on F, the enhancement counts half-twists, mod 4, in imbedded circles representing elements of $H_1(F; \mathbf{Z}/2ZZ)$. This is developed in §3, where it is shown that

$$\beta : \Omega_2^{Pin^-} \longrightarrow \mathbf{Z}/8\mathbf{Z}$$

gives the isomorphism in the following table.

$\Omega_1^{Spin} = \mathbf{Z}/2\mathbf{Z}$	$\Omega_2^{Spin} = \mathbf{Z}/2\mathbf{Z}$	$\Omega_3^{Spin} = 0$	$\Omega_4^{Spin} = \mathbf{Z}$
$\Omega_1^{Pin^-} = \mathbf{Z}/2\mathbf{Z}$	$\Omega_2^{Pin^-} = \mathbf{Z}/8\mathbf{Z}$	$\Omega_3^{Pin^-} = 0$	$\Omega_4^{Pin^-} = 0$
$\Omega_1^{Pin^+} = 0$	$\Omega_2^{Pin^+} = \mathbf{Z}/2\mathbf{Z}$	$\Omega_3^{Pin^+} = \mathbf{Z}/2\mathbf{Z}$	$\Omega_4^{Pin^+} = \mathbf{Z}/16\mathbf{Z}$

In §2 we calculate the 1 and 2 dimensional groups and show that the non–zero one dimensional groups are generated by the circle with its Lie group framing, S^1_{Lie}, (note the Möbius band is a Pin^+ boundary for S^1_{Lie}); $\mathbf{RP}^2$ generates $\Omega_2^{Pin^-}$; the Klein bottle, the twisted S^1_{Lie} bundle over S^1, generates $\Omega_2^{Pin^+}$; and T^2_{Lie}, the torus with its Lie group framing generates Ω_2^{Spin}. By §5 enough technique exists to calculate the remaining values and show that $\Omega_3^{Pin^+}$ is generated by the twisted T^2 bundle over S^1 with Lie group framing on the fiber torus; $\Omega_4^{Pin^+}$ is generated by $\mathbf{RP}^4$. The Cappell–Shaneson fake RP^4 represents $\pm 9 \in \mathbf{Z}/16\mathbf{Z}$ [Stolz]; the Kummer surface represents $8 \in \mathbf{Z}/16\mathbf{Z}$ and in fact, a *Spin* 4–manifold bounds a Pin^+ 5-manifold iff its index is zero mod 32. The Kummer surface also generates Ω_4^{Spin}.

Section 4 contains a digression on *Spin* structures on 3–manifolds and a geometric interpretation of Turaev's work [Tu] on trilinear intersection forms

$$H_2\left(M^3; \mathbf{Z}/2\mathbf{Z}\right) \otimes H_2\left(M^3; \mathbf{Z}/2\mathbf{Z}\right) \otimes H_2\left(M^3; \mathbf{Z}/2\mathbf{Z}\right) \longrightarrow \mathbf{Z}/2\mathbf{Z}\ .$$

This is used in calculating the μ–invariant: let $\mu(M, \Theta_1)$ be the μ–invariant of M^3 with *Spin* structure Θ_1. The group $H^1\left(M^3; \mathbf{Z}/2\mathbf{Z}\right)$ acts on *Spin* structures, so let $\alpha \in H^1\left(M^3; \mathbf{Z}/2\mathbf{Z}\right)$ determine Θ_2. Then α is dual to an imbedded surface F^2 in M which gains a Pin^- structure from Θ_1 and

$$\mu(\Theta_2) = \mu(\Theta_1) - 2\beta(F) \qquad (\text{mod } 16)$$

Four dimensional characteristic bordism $\Omega_4^!$ is studied in §6 with generalizations of [F-K] and [G-M]. We calculate, in Theorem 6.5, the μ–invariant of circle bundles over surfaces, $S(\eta)$, whose disk bundle, $D(\eta)$, has orientable total space. Fix a *Spin* structure on $S(\eta)$, Θ. Then

$$\mu(S(\eta), \Theta) = \text{sign }(D(\eta)) - \text{ Euler class}(\eta) + 2 \cdot b(F) \qquad (\text{mod } 16)$$

where $b(F) = 0$ if the *Spin* structure Θ extends across $D(\eta)$ and is β of a Pin^- structure on F induced on F from Θ otherwise.

The characteristic bordism groups are calculated geometrically in §7, in particular,

$$\Omega_4^! = \mathbf{Z}/8\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} .$$

Just as Robertello was able to use Rochlin's Theorem to describe the Arf invariant of a knot [R], so we can use $\beta : \Omega_2^{Pin^-} \longrightarrow \mathbf{Z}/8\mathbf{Z}$ to give a $\mathbf{Z}/8\mathbf{Z}$ invariant to a characterized link L in a *Spin* 3–manifold M with a given set of even longitudes for L (Definition 8.1). This invariant is a concordance invariant (Corollary 8.4), and if each component of L is torsion in $H_1(M;Z)$, then L has a natural choice of even longitudes (Definition 8.5).

Section 9 contains a brief discussion of the topological case of some of our 4-manifold results. In particular, the formula above must now contain the triangulation obstruction $\kappa(M)$ for an oriented, topological 4–manifold M^4:

$$2\beta(F) \equiv F \cdot F - \text{sign}\,(M) + 8\kappa(M) \qquad (\text{mod } 16)$$

(recall that (M,F) is a characterized pair).

§1. *Pin* Structures and generalities on bundles

The purpose of this section is to define the *Pin* groups and to discuss the notion of a *Pin* structure on a bundle.

Recall that rotations of $\mathbf{R}^n$ are products of reflections across $(n-1)$–planes through the origin, an even number for orientation preserving rotations and an odd number for orientation reversing rotations. These $(n-1)$–planes are not oriented so they can equally well be described by either unit normal vector. Indeed, if $\mathbf{u}$ is the unit vector, and if $\mathbf{x}$ is any point in $\mathbf{R}^n$, then the reflection is given by $x - 2(\mathbf{x}\cdot\mathbf{u})\mathbf{u}$. Thus an element of $O(n)$ can be given as $(\pm\mathbf{v}_1)(\pm\mathbf{v}_2)\cdots(\pm\mathbf{v}_k)$ where each $\mathbf{v}_i$ is a unit vector in $\mathbf{R}^n$ and k is even for $SO(n)$. Then elements of $Pin(n)$, a double cover of $O(n)$, are obtained by choosing an orientation for the $(n-1)$–planes or equivalently choosing one of the two unit normals, so that an element of $Pin(n)$ is $\mathbf{v}_1\cdots\mathbf{v}_k$; if k is even we get elements of $Spin(n)$. With this intuitive description as motivation, we proceed more formally to define *Pin* (see [ABS]).

Let V be a real vector space of dimension n with a positive definite inner product, $(\ ,\ )$. The Clifford algebra, $\mathbf{Cliff}^{\pm}(V)$, is the universal algebra generated by V with the relations

$$\mathbf{vw} + \mathbf{wv} = \begin{array}{ll} 2(\mathbf{v},\mathbf{w}) & \text{for } \mathbf{Cliff}^{+}(V) \\ -2(\mathbf{v},\mathbf{w}) & \text{for } \mathbf{Cliff}^{-}(V) \end{array}$$

If $\mathbf{e}_1, \cdots \mathbf{e}_n$ is an orthonormal basis for V, then the relations imply that $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i\mathbf{e}_j$, $i \neq j$ and $\mathbf{e}_i\mathbf{e}_i = \pm 1$ in $\mathbf{Cliff}^{\pm}(V)$. The elements $\mathbf{e}_I = \mathbf{e}_{i_1}\cdots\mathbf{e}_{i_k}$, $I =$

$\{1 \le i_1 < i_2 \cdots < i_k \le n\}$ form a ($\mathbf{e}_I\mathbf{e}_J = 0$, $\mathbf{e}_I\mathbf{e}_I = \pm 1$) basis for $\mathbf{Cliff}^{\pm}(V)$. So $\dim \mathbf{Cliff}^{\pm}(V) = 2^n$; note that as vector spaces, $\mathbf{Cliff}^{\pm}(V)$ is isomorphic to the exterior algebra generated by V, but the multiplications are different, e.g. $\mathbf{e}_i\mathbf{e}_i = \pm 1 \neq 0 = \mathbf{e}_i \wedge \mathbf{e}_i$.

Let $Pin^{\pm}(V)$ be the set of elements of $\mathbf{Cliff}^{\pm}(V)$ which can be written in the form $\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_k$ where each $\mathbf{v}_i$ is a unit vector in V; under multiplication, $Pin^{\pm}(V)$ is a compact Lie group. Those elements $\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_k \in Pin^{\pm}(V)$ for which k is even form $Spin(V)$.

Define a "transpose" $\mathbf{e}_I^t = \mathbf{e}_{i_k}\cdots\mathbf{e}_{i_1} = (-1)^{k-1}\mathbf{e}_I$ and an algebra homomorphism $\alpha(\mathbf{e}_I) = (-1)^k\mathbf{e}_I = (-1)^{|I|}\mathbf{e}_I$ and extend linearly to $\mathbf{Cliff}^{\pm}(V)$. We have a $\mathbf{Z}/2\mathbf{Z}$–grading on $\mathbf{Cliff}^{\pm}(V)$: $\mathbf{Cliff}^{\pm}(V)_0$ is the $+1$ eigenspace of α and $\mathbf{Cliff}^{\pm}(V)_1$ is the -1 eigenspace. For $w \in \mathbf{Cliff}^{\pm}(V)$, define an automorphism $\rho(w)\colon \mathbf{Cliff}^{\pm}(V) \to \mathbf{Cliff}^{\pm}(V)$ by

$$\rho(w)(v) = \begin{cases} wvw^t & \text{for } \mathbf{Cliff}^{-}(V) \\ \alpha(w)vw^t & \text{for } \mathbf{Cliff}^{+}(V) \end{cases}$$

We can define a norm in the Clifford algebra, $N\colon \mathbf{Cliff}^{\pm} \to \mathbf{R}^+$ by $N(x) = \alpha(x)x$ for all $x \in \mathbf{Cliff}^{\pm}(V)$. Then we can define $Pin^{\pm}(V)$ to be $\{w \in \mathbf{Cliff}^{\pm}(V) \mid \rho(w)(V) = V \text{ and } N(w) = 1\}$. Hence if $w \in Pin^{\pm}(v)$, $\rho(w)$ is an automorphism of V so ρ is a representation $\rho\colon Pin^{\pm}(V) \to O(V)$ and by restriction $\rho\colon Spin(V) \to SO(V)$.

It is easy to verify that $\rho(w)$ acts on V by reflection across the hyperplane $w^{\perp}$, e.g. for $Pin^{-}(V)$,

$$\rho(\mathbf{e}_1)\mathbf{e}_i = \mathbf{e}_1\mathbf{e}_i\mathbf{e}_1 = \begin{cases} -\mathbf{e}_1^2\mathbf{e}_i = \mathbf{e}_i & i \neq 1 \\ \mathbf{e}_1^2\mathbf{e}_i = -\mathbf{e}_i & i = 1 \end{cases}$$

If r and I are basepoints in the components of $O(V)$, where r is reflection across $\mathbf{e}_1^{\perp}$, then $\rho^{-1}\{r, I\} = \{\pm\mathbf{e}_1, \pm 1\}$ and

$$\rho^{-1}\{r, I\} \cong \begin{cases} \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \text{for } Pin^{+}(V) \\ \mathbf{Z}/4\mathbf{Z} & \text{for } Pin^{-}(V) \end{cases} .$$

The $\mathbf{Z}/2\mathbf{Z} = \{-1, 1\} \in Pin^{\pm}$ is central and $Pin^{\pm}(V)/\{\pm 1\} = O(V)$. If $n > 1$, this $\mathbf{Z}/2\mathbf{Z}$ is the center of $Pin^{\pm}(V)$ and, since $O(V)$ has a non–trivial center, for $n > 1$, the $\mathbf{Z}/2\mathbf{Z}$ central extensions $Pin^{\pm} \to O(V)$ are non–trivial.

Thus $Pin^{\pm}(V)$ is a double cover of $O(V)$. As spaces, $Pin^{\pm}(V) = Spin(V) \amalg Spin(V)$ but the group structure is different in the two cases. We can think of $-1 \in \rho^{-1}(I)$ as rotation of V (about any axis) by 2π and $+1 \in \rho^{-1}(I)$ as the identity. More precisely, an arc in $Pin^{\pm}(V)$ from 1 to -1 maps by ρ to a loop in $O(V)$ which generates $\pi_1(O(V))$; in fact, for $\theta \in [0, \pi]$, the arc $\theta \to \pm\mathbf{e}_1 \cdot (\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2)$ is one such. Even better, we may think of $Pin^{\pm}$ as scheme for distinguishing an odd number of full twists from an even number.

We use $Pin^{\pm}(n)$ to denote $Pin^{\pm}(V)$ where V is $\mathbf{R}^n$.

Remark. The tangent bundle of $\mathbf{RP}^2$, $T_{\mathbf{RP}^2}$, has a $Pin^-(2)$–structure.

We can "see" the $Pin^-(2)$ structure on $T_{\mathbf{RP}^2}$ as follows: decompose $\mathbf{RP}^2$ into a 2–cell, B^2, and a Möbius band, MB, with core circle $\mathbf{RP}^1$. Then $T_{\mathbf{RP}^2}|_{MB}$ can be described using two coordinate charts, U_1 and U_2, with local trivializations $(\mathbf{e}_1, \mathbf{e}_2)$, in which $\mathbf{e}_1$ is parallel to $\mathbf{RP}^1$ and $\mathbf{e}_2$ is normal, and with transition function $U_1 \cap U_2 \to Pin^-(2)$ which sends the two components of $U_1 \cap U_2$ to 1 and $\mathbf{e}_2$. Then $T_{\mathbf{RP}^2}|_{\partial MB}$ is a trivial $\mathbf{R}^2$–bundle over $S^1 = \partial MB$ which is trivialized by the transition function 1 and $\mathbf{e}_2^2 = -1$. Now $\mathbf{e}_1$ would be tangent to S^1 but the $\mathbf{e}_2^2 = -1$ adds a rotation by 2π as $S^1 = \partial MB$ is traversed. But this trivialization on $T_{\mathbf{RP}^2}|_{S^1}$ is exactly the one which extends over the 2–cell B^2. Thus $\mathbf{RP}^2$ is Pin^-. Note that this process fails if $\mathbf{e}_2^2 = +1$, and, in fact, $\mathbf{RP}^2$ does not support a Pin^+ structure (see Lemma 1.3 below).

We now review the theory of G bundles, for G a topological group, and the theory of H structures on a G bundle. A principal G bundle is a space E with a left G action, $E \times G \to E$ such that no point in E is fixed by any non–identity element of G. We let $B = E/G$ be the orbit space and $p: E \to B$ be the projection. We call B the *base* of the bundle and say that E is a *bundle over* B. We also require a local triviality condition. Explicitly, we require a numerable cover, $\{\mathcal{U}_i\}$, of B and G maps $r_i: \mathcal{U}_i \times G \to E$ such that the composite $\mathcal{U}_i \times G \xrightarrow{r_i} E \xrightarrow{p} B$ is just projection onto $\mathcal{U}_i$ followed by inclusion into B. Such a collection is called an *atlas* for the bundle and it is convenient to describe bundles in terms of some atlas. The functions $r_j^{-1} \circ r_i$ are G maps, $\mathcal{U}_i \cap \mathcal{U}_j \times G \to \mathcal{U}_i \cap \mathcal{U}_j \times G$, which commute with the projection. Hence they can be given as *transition* functions $g_{ij}: \mathcal{U}_i \cap \mathcal{U}_j \to G$. Note $g_{ii} = id$, $g_{ij}^{-1} = g_{ji}$ and $g_{ik} = g_{ij} \circ g_{jk}$ on $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$. Conversely, given any numerable cover of a space B and a set of maps satisfying these three conditions, we can find a principal G bundle and an atlas for it so the base space is B and the transitions functions are our given functions.

Suppose E_0 and E_1 are two G bundles over B_0 and B_1 respectively. Let $f: E_0 \to E_1$ be a map. A bundle map covering f is a G map $F: E_0 \to E_1$ so that $p_1 \circ F = f \circ p_0$, where p_i is the projection in the i-th bundle. We say two bundles over B are *equivalent* iff there exists a bundle map between them covering the identity.

Given a bundle over B, say E, with atlas $\mathcal{U}_i$ and g_{ij}, and a map $f: B_0 \to B$, the *pull–back of E along f* is the bundle over B_0 with numerable cover $f^{-1}(\mathcal{U}_i)$ and transition functions $g_{ij} \circ f$. The pull–backs of equivalent bundles are equivalent. A bundle map between E_0 and E_1 covering $f: B_0 \to B_1$ is equivalent to a bundle equivalence between E_0 and the pull–back of E_1 along f. Hence we mostly discuss the case of bundle equivalence.

Given any atlas for a bundle, say $\mathcal{U}_i; g_{ij}$, and a subcover $\mathcal{V}_\alpha$ of $\mathcal{U}_i$ we can restrict the g_{ij} to get a new family of transition functions $g_{\alpha\beta}$. Clearly these two atlases represent the same bundle. Given two numerable covers, it is possible to find a third numerable cover which refines them both, so it is never any loss of generality when considering two bundles over the same base to assume the transition functions are defined on a common cover.

A bundle equivalence between bundles given by transition functions g_{ij} and g'_{ij} for the same cover is given by maps $h_i: \mathcal{U}_i \to G$ such that, for all i and j and all $u \in \mathcal{U}_i \cap \mathcal{U}_j$, $g'_{ij}(u) = h_i(u)\, g_{ij}(u)\, \big(h_j(u)\big)^{-1}$.

Given a continuous homomorphism $\psi: H \to G$, we can form a principal G bundle from a principal H bundle by applying ψ to any atlas for the H bundle. If $p: E \to B$ is the H bundle, we let $p_\psi: E \times_H G \to B$ denote the associated G bundle. Equivalent H bundles go to equivalent G bundles. We say that a G bundle, $p: E \to B$, had an H structure provided that there exists an H bundle, $p_1: E_1 \to B$ so that the associated G bundle, $(p_1)_\psi: E_1 \times_H G \to B$ is equivalent to the G bundle. More correctly one should say that we have a ψ structure on our G bundle, but we won't. An H *structure* for a G bundle, $p: E \to B$ consists of a pair: an H bundle, $p_1: E_1 \to B$, and a G equivalence, γ from $(p_1)_\psi: E_1 \times_H G \to B$ to the original G bundle, $p: E \to B$. Two structures $p_1: E_1 \to B$, γ_1 and $p_2: E_2 \to B$, γ_2 on $p: E \to B$ are *equivalent* if there exists an equivalence of H bundles $f: E_1 \to E_2$ such that, if f_ψ denotes the corresponding equivalence of G bundles, $\gamma_1 = \gamma_2 \circ f_\psi$.

We assume the reader is familiar with this next result.

Theorem 1.1. *For any topological group, G, there exists a space B_G such that equivalence classes of G bundles over B are in 1–1 correspondence with homotopy classes of maps $B \to B_G$. (A map $B \to B_G$ corresponding to a bundle is called a classifying map for the bundle.) Given $\psi: H \to G$ we get an induced map $B\psi: B_H \to B_G$. If this map is not a fibration, we may make it into one without changing B_G or the homotopy type of B_H, so assume $B\psi$ is a Hurewicz fibration. Given a G bundle with a classifying map $B \to B_G$, H structures on this bundle are in 1–1 correspondence with lifts of the classifying map for the G bundle to B_H.*

Example. Let $p: E \to B$ be a trivial $O(n)$ bundle, and suppose the atlas has one open set, namely B, and one transition function, the identity. One $SO(n)$ structure on this bundle consists of the same transition function but thought of as taking values in $SO(n)$ together with the bundle equivalence which maps B to the identity in $O(n)$. Another $SO(n)$ structure is obtained by using the same transition functions but taking as the bundle equivalence a map B to $O(n)$ which lands in the orientation reversing component of $O(n)$. Indeed any map $B \to O(n)$ gives an $SO(n)$ structure on our bundle. It is not difficult to see that any two maps into the same component of $O(n)$ give equivalent structures and that two maps into different components give structures that are not equivalent as structures. Clearly the $SO(n)$ bundle in all cases is the same. One gets from here to the more traditional notion of orientation for the associated vector bundle as follows. Since the transition functions are in $O(n)$, $O(n)$ acts on the vector space fibre. But for matrices to act on a vector space a basis needs to be chosen. This basis orients the $SO(n)$ bundle: in the first case the equivalence orients the underlying $O(n)$ bundle one way and in the second case the equivalence orients the bundle the other way.

Finally recall that an $O(n)$ bundle has an orientation iff the first Stiefel–Whitney class, w_1 of the bundle vanishes. If there is an $SO(n)$ structure then $H^0(B; \mathbf{Z}/2\mathbf{Z})$ acts in a simply transitive manner on the set of structures.

The Lie group $Spin(n)$ comes equipped with a standard double cover map $Spin(n) \to SO(n)$, and this is the map ψ we mean when we speak of an $SO(n)$ bundle, or an oriented vector bundle, having a *Spin* structure. There is a fibration sequence $B_{Spin(n)} \to B_{SO(n)} \to K(\mathbf{Z}/2\mathbf{Z}, 2)$, so the obstruction to the existence of a *Spin* structure is a 2–dimensional cohomology class which is known to be the second Stiefel–Whitney class w_2. If the set of *Spin* structures is non–empty, then $H^1(B; \mathbf{Z}/2\mathbf{Z})$ acts on it in a simply transitive manner.

The action can be seen explicitly as follows. Fix one *Spin* structure, say g_{ij}. An element in $H^1(B; \mathbf{Z}/2\mathbf{Z})$ can be represented by a Cech cocycle: i.e. a collection of maps $c_{ij}: \mathcal{U}_i \cap \mathcal{U}_j \to \pm 1$ satisfying the same conditions as the transition functions for a bundle. The new *Spin* structure consists of the transition functions $g_{ij} \cdot c_{ij}$ with the same $SO(n)$ bundle equivalence, where we think of ± 1 as a subgroup of $Spin(n)$ and $\cdot$ denotes group multiplication. It is not hard to check that cohomologous cocycles give equivalent structures.

We now explore the relation between *Spin* structures on an oriented vector bundle and framings of that bundle. A framing of a bundle is the same thing as an H structure where H is the trivial subgroup. Hence H is naturally a subgroup of $Spin(n)$ and an equivalence class of framings of a bundle gives rise to an equivalence class of *Spin* structures. Consider first the case $n = 1$. Recall $SO(1)$ is trivial and $Spin(1) = \mathbf{Z}/2\mathbf{Z}$. Hence an $SO(1)$ bundle already has a unique trivialization, and hence a "canonical" *Spin* structure. There are often other *Spin* structures, but, none of these come from framings. In case $n = 2$, $Spin(2) = S^1$, $SO(2) = S^1$ and the map is the double cover. If an $SO(2)$ bundle is trivial, framings are acted on simply transitively by $H^1(B; \mathbf{Z})$. The corresponding *Spin* structures are equivalent iff the class in $H^1(B; \mathbf{Z}/2\mathbf{Z})$ is trivial. If B is a circle the bundle is trivial iff it has a *Spin* structure and both *Spin* structures come from framings. The *Spin* structure determines the framing up to an action by an even element in $\mathbf{Z}$, so we often say that the *Spin* structure determines an *even* framing. If $n > 2$ and B is still a circle, then the bundle is framed iff it has a *Spin* structure and now framings and *Spin* structures are in 1–1 correspondence.

Of course, given any *Spin* structure on a bundle over B, and any map $f: S^1 \to B$, we can pull the bundle back via f and apply the above discussion. Since *Spin* structures on the bundle are in 1–1 correspondence with $H^1(B; \mathbf{Z}/2\mathbf{Z})$, which is detected by mapping in circles, we can recover the *Spin* structure by describing how the bundle is framed when restricted to each circle (with a little care if $n = 1$ or 2). Moreover, if an $SO(n)$ bundle over a CW complex is trivial when restricted to the 2–skeleton, then w_2 vanishes, so the bundle has a *Spin* structure. If $n \neq 2$ and the bundle has a *Spin* structure then, restricted to the 2– skeleton, it is trivial. If $n = 2$ this last remark is false as the tangent bundle to S^2 shows.

Finally, we need to discuss stabilization. All our groups come in families indexed by the natural numbers and there are inclusions of one in the next. An example is the family $O(n)$ with $O(n) \to O(n+1)$ by adding a 1 in the bottom right, and all our other families have similar patterns. This is of course a special case of our general discussion of H structures on G bundles. Given a vector bundle, ξ, and an oriented line bundle, ϵ^1, the $O(n)$ transition functions for ξ extend naturally to a set of $O(n+1)$ transition functions for $\xi \oplus \epsilon^1$ using the above homomorphism, and any of our structures on ξ will extend naturally to a similar structure on $\xi \oplus \epsilon^1$. We call the structure on $\xi \oplus \epsilon^1$ the *stabilization* of the structure on ξ.

A particular case of great interest to us is the relation between tangent bundles in a manifold with boundary. Suppose M is a codimension 0 subset of the boundary of W. We can consider the tangent bundle of W, say T_W, restricted to M. It is naturally identified with $T_M \oplus \nu_{M \subset W}$ where ν denotes the normal bundle. This normal bundle is framed by the "inward–pointing" normal, so we can compare structures on M with structures on W using stabilization.

Since both $Pin^{\pm}(n)$ are Lie groups and have homomorphisms into $O(n)$, the above discussion applies.

Remarks. *With this definition it is clear that, if there is a $Pin^{\pm}$ structure on a bundle ξ over a space B then $H^1(B; \mathbf{Z}/2\mathbf{Z})$ acts on the set of $Pin^{\pm}$ structures in a simply transitive manner. It is also clear that the obstruction to existence of such a structure must be a 2-dimensional cohomology class in $H^2(B_{O(n)}; \mathbf{Z}/2\mathbf{Z})$ that restricts to $w_2 \in H^2(B_{SO(n)}; \mathbf{Z}/2\mathbf{Z})$ and hence is either $w_2(\xi)$ or $w_2(\xi) + w_1^2(\xi)$. Here w_i denotes the i–th Stiefel–Whitney class of the bundle.*

We sort out the obstructions next.

Lemma 1.2. *Let λ be a line bundle over a CW complex B. Then λ has a Pin^+ structure and $\lambda \oplus \lambda \oplus \lambda$ has a Pin^- structure.*

Proof: Since $Pin^+(1) \to O(1)$ is just a projection, $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$, there is a group homomorphism, $O(1) \to Pin^+(1)$, splitting the projection. If we compose transition functions for λ with this homomorphism, we get a set of Pin^+ transition functions for λ. If we have an equivalent $O(1)$ bundle, the two $Pin^+(1)$ bundles are also equivalent.

Transition functions for 3λ are given by taking transition functions for λ and composing with the homomorphism $O(1) \to O(3)$ which sends ± 1 to the matrix $\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$. It is easy to check that this homomorphism lifts through a homomorphism $O(1) \to Pin^-(3)$. If we have an equivalent $O(1)$ bundle, the two $Pin^-(3)$ bundles are also equivalent. ∎

Addendum to 1.2. Notice that we have proved a bit more. The homomorphisms we chose are not unique, but can be chosen once and for all. Hence a line bundle has a "canonical" Pin^+ structure and 3 times a line bundle has a "canonical" Pin^- structure.

Remark. There are two choices for the homomorphisms above. If we choose the other then the two "canonical" Pin^+ structures on a line bundle differ by the action of w_1 of the line bundle, with a similar remark for the Pin^- case.

Lemma 1.3. *The obstruction to lifting an $O(n)$–bundle to a $Pin^+(n)$–bundle is w_2, and to a $Pin^-(n)$–bundle is $w_2 + w_1^2$. If $\xi \oplus \lambda$ = trivial bundle, then ξ has a Pin^- structure iff λ has a Pin^+ structure.*

Proof: A line bundle has a Pin^+ structure by Lemma 1.2, so $w_2 = 0$, but there are examples, e.g. the canonical bundle over $\mathbf{RP}^2$, for which $w_1^2 \neq 0$. Hence w_2 is the obstruction to a bundle having a Pin^+ structure.

For 3 times a line bundle, $w_2 = w_1^2$, so we can find examples, e.g. 3 times the canonical bundle over $\mathbf{RP}^2$, for which $w_2 + w_1^2 = 0$ but $w_2 \neq 0$. Hence $w_2 + w_1^2$ is the obstruction to having a Pin^- structure.

The remaining claim is an easy characteristic class calculation. ∎

The fact that the tangent bundle and normal bundles have different structures can lead to some confusion. In the rest of this paper, when we say a manifold has a $Pin^\pm$ structure, we mean that the *tangent bundle* to the manifold has a $Pin^\pm$ structure. As an example of the possibilities of confusion, the *Pin* bordism theory calculated by Anderson, Brown and Peterson, [ABP2], is Pin^- bordism. They do the calculation by computing the stable homotopy of a Thom spectrum, which as usual is the Thom spectrum for the *normal* bundles of the manifolds. The key fact that makes their calculation work is that w_2 vanishes, but this is w_2 of the normal bundle, so the tangent bundle has a Pin^- structure and we call this Pin^- bordism.

We remark that a $Pin^\pm$ structure is equivalent to a stable $Pin^\pm$ structure and similarly for *Spin*. This can be seen by observing that

$$\begin{array}{ccc} Pin^\pm(n) & \longrightarrow & Pin^\pm(n+1) \\ \downarrow & & \downarrow \\ O(n) & \longrightarrow & O(n+1) \end{array}$$

commutes and is a pull–back of groups, with a similar diagram in the *Spin* case.

In order to be able to carefully discuss structures on bundles, we introduce the following notation and definitions. Given a vector bundle, ξ, let $\mathcal{P}in^\pm(\xi)$ denote the set of $Pin^\pm$ structures on it. If ξ is an oriented vector bundle, let $\mathcal{S}pin(\xi)$ denote the set of *Spin* structures on it. Throughout this paper we will be writing down functions between sets of $Pin^{\pm}$ or *Spin* structures. All these sets, if non-empty are acted on, simply transitively, by $H^1(B;\mathbf{Z}/2\mathbf{Z})$ where B is the base of the bundle.

Definition 1.4. We say that a function between two sets of structures on bundles over bases B_1 and B_2 respectively is *natural* provided there is a homomorphism $H^1(B_1;\mathbf{Z}/2\mathbf{Z}) \to H^1(B_2;\mathbf{Z}/2\mathbf{Z})$ so that the resulting map is equivariant.

One example of this concept is the following construction.

Construction 1.5. Let $\hat{f}: \xi_1 \to \xi_2$ be a bundle map covering $f: B_1 \to B_2$. Given a cover and transition functions for B_2 and ξ_2, we can use f and $\hat{f}$ to construct a cover and transition functions for B_1 and ξ_1. This construction induces a natural function

$$\hat{f}^*: \mathcal{P}in^{\pm}(\xi_2) \to \mathcal{P}in^{\pm}(\xi_1)$$

with a similar map for *Spin* structures if we use $\hat{f}$ to pull back the orientation.

There are two examples of this construction we will use frequently. The first is to consider an open subset $U \subset M$ of a manifold M: here the derivative of the inclusion is a bundle map so Construction 1.5 gives us a natural restriction of structures. The second is to consider a codimension 0 immersion between two manifolds, say $f: N \to M$. Again the derivative is a bundle map so we get a natural restriction of structures.

We can also formally discuss stabilization.

Lemma 1.6. *Let ξ be a vector bundle, and let ϵ^1 be a trivial line bundle, both over a connected space B. There are natural one to one correspondences*

$$\mathcal{S}_r(\xi): \mathcal{P}in^{\pm}(\xi) \to \mathcal{P}in^{\pm}(\xi \oplus \bigoplus_{i=1}^{r} \epsilon^1) \ .$$

If ξ is oriented there is a natural one to one correspondence

$$\mathcal{S}_r^{+}(\xi): \mathcal{S}pin^{\pm}(\xi) \to \mathcal{S}pin^{\pm}(\xi \oplus \bigoplus_{i=1}^{r} \epsilon^1) \ .$$

Given a bundle map $\hat{f}: \xi_1 \to \xi_2$, there is another bundle map $(\widehat{f \oplus \bigoplus_{i=1}^{r} 1}): \xi_1 \oplus \bigoplus_{i=1}^{r} \epsilon^1 \to \xi_2 \oplus \bigoplus_{i=1}^{r} \epsilon^1$. The obvious squares involving these bundle maps and the stabilization maps commute.

We would like a result that relates $Pin^{\pm}$ structures on bundles to the geometry of the bundle restricted over the 1–skeleton mimicking the framing condition for the *Spin* case. We settle for the next result. Let ξ^n be an n–plane bundle over a CW–complex X, and let $\det \xi$ be the determinant bundle of ξ^n.

Lemma 1.7. *There exist natural bijections*

$$\Psi_{4k+1}(\xi): \mathcal{P}in^-(\xi) \to \mathcal{S}pin(\xi \oplus (4k+1)\det\xi)$$
$$\Psi_{4k+3}(\xi): \mathcal{P}in^+(\xi) \to \mathcal{S}pin(\xi \oplus (4k+3)\det\xi)$$
$$\Psi_{4k+2}(\xi): \mathcal{P}in^\pm(\xi) \to \mathcal{P}in^\mp(\xi \oplus (4k+2)\det\xi)$$
$$\Psi_{4k}(\xi): \mathcal{P}in^\pm(\xi) \to \mathcal{P}in^\pm(\xi \oplus (4k)\det\xi)$$

and

$$\Psi^+_{4k}(\xi): \mathcal{S}pin(\xi) \to \mathcal{S}pin(\xi \oplus (4k)\det\xi) \quad .$$

A bundle map $\hat{f}: \xi_1 \to \xi_2$ defines a bundle map $\det\xi_1 \to \det\xi_2$. Using this map between determinant bundles, all the squares involving the Ψ maps commute.

Proof: It follows from Lemma 1.3 that the existence of a structure of the correct sort on ξ is equivalent to the existence of a structure of the correct sort on $\xi \oplus r\det\xi$.

Let us begin by recalling the transition functions for the various bundles. There are homomorphisms $\delta_r: O(n) \to O(n+r)$ defined by sending an $n\times n$ matrix A to the $(n+r)\times(n+r)$ matrix which is A in the first $m\times m$ locations, $\det A$ in the remaining r diagonal locations, and zero elsewhere.

If $\mathcal{U}_i$, $g_{ij}:\mathcal{U}_i \cap \mathcal{U}_j \to O(n)$ is a family of transition functions for ξ, then $\delta_r \circ g_{ij}$ is a family of transition functions for $\xi \oplus r\det\xi$.

Next, we describe a function from the set of structures on ξ to the set of structures on $\xi \oplus \ r\det\xi$.

Begin with the case in which ξ has a Pin^- structure with transition functions $G_{ij}:\mathcal{U}_i \cap \mathcal{U}_j \to Pin^-(n)$ lifting the given set g_{ij} into $O(n)$. Pick an element e in the Clifford algebra for $\mathbf{R}^n \oplus \mathbf{R}^1$ so that $e^2 = -1$ and e maps to reflection through $\mathbf{R}^n$ under the canonical map to $O(n+1)$. There are two such choices but choose one once and for all. Define H_{ij} into $Pin^-(n+1)$ by $H_{ij}(u) = i(G_{ij}(u)) \cdot x_{ij}(u)$ where i denotes the natural inclusion of $Pin^-(n)$ into $Pin^-(n+1)$ and $x_{ij}(u)$ is e if $\det g_{ij}(u) = -1$ and 1 otherwise.

It is clear that the H_{ij} land in $Spin(n+1)$, but what needs to be checked is that they are a set of transition functions for our bundle. Clearly they lift the transition functions for the underlying $SO(n+1)$ bundle, so we need to consider the cocycle relation. This says that $H_{ij}(u)H_{jk}(u)H_{ki}(u) = 1$. If we replace the H's by G's, we do have the relation, so let us compute $H_{ij}(u)H_{jk}(u)H_{ki}(u) = G_{ij}(u)x_{ij}(u)G_{jk}(u)x_{jk}(u)G_{ki}(u)x_{ki}(u)$. Any x commutes past a G if the x associated to the G is 1 and it goes past with a sign switch if the x associated to the G is e. Also note that either none or two of the x's in our product are e. We leave it to the reader to work through the cases to see that the cocycle relation always holds and to note that the key point is that $e^2 = -1$.

Next, consider the case in which ξ has a Pin^+ structure, and let G_{ij} continue to denote the transition functions. Let e_1, e_2 and e_3 denote elements in the Pin^+

Clifford algebra for $\mathbf{R}^n \oplus \mathbf{R}^3$: each e_i covers reflection in a hyperplane perpendicular to one of the three standard basis vectors for the $\mathbf{R}^3$ factor. Define H_{ij} as above except replace e by $e_1e_2e_3$. The proof goes just as before after we note that $(e_1e_2e_3)^2 = -1$.

For the case in which $r = 2$ and ξ may have either a Pin^+ or a Pin^- structure, choose e_1 and e_2; note that $(e_1e_2)^2 = -1$ and proceed as above.

The last natural bijection is also easy. If g_{ij} are transition functions for ξ it is easy to choose the cover so that there are lifts G_{ij} of our functions to $Pin^-(n)$ (or $Pin^+(n)$ if the reader prefers), but the cocycle relation may not be satisfied. We can define new functions H_{ij} into $Spin(4n)$ by just juxtaposing 4 copies of G_{ij} thought of as acting on four copies of the same space. These functions can easily be checked to satisfy the cocycle condition.

Now that we have defined our functions, the results of the theorem are easy. The reader should check that the functions we defined are $H^1(\ ;\mathbf{Z}/2\mathbf{Z})$ equivariant and hence induce 1–1 transformations. ∎

Remark 1.8. We did make a choice in the proof of 1.7. The choice was global and so the lemma holds, but it is interesting to contemplate the effect of making the other choice. It is not too hard to work out that if we continue to use 1, but replace e by $-e$, the new *Spin* structure will differ from the old one by the action of $w_1(\xi)$. The same result holds if we switch an odd number of the e_1, e_2, e_3 in the Pin^+ case or an one of e_1, e_2 in the $r = 2$ case.

For later use, we need a version of Lemma 1.7 in which the line bundles are merely isomorphic to the determinant bundle. To be able to describe the effect of changing our choices, we need the following discussion.

There is a well-known operation on an oriented vector bundle known as "reversing the orientation". Explicitly, suppose that we have transition functions, g_{ij}, defined into $SO(n)$ based on a numerable cover $\{\mathcal{U}_i\}$. Then we choose maps $h_i: \mathcal{U}_i \to O(n) - SO(n)$ and let the bundle with the "opposite orientation" have transition functions $h_i \circ g_{ij} \circ h_j^{-1}$ and use the maps h_i to get the $O(n)$ equivalence with the original bundle. The choice of the h_i is far from unique, but any two choices yield equivalent $SO(n)$ bundles. In the same fashion, given a $Spin(n)$ bundle, we can consider the *opposite Spin structure.* Proceed just as above using $Spin(n)$ for $SO(n)$ and $Pin^+(n)$ or $Pin^-(n)$ for $O(n)$.

Note that a *Spin* structure and its opposite are equivalent Pin^+ or Pin^- structures. Conversely, given a $Pin^\pm$ structure on a vector bundle which happens to be orientable, then there are two compatible *Spin* structures which are the opposites of each other. We summarize the above discussion as

Lemma 1.9. *If ξ is an oriented vector bundle, then there is a natural one to one correspondence, called reversing the spin structure,*

$$\mathcal{R}_\xi : Spin(\xi) \to Spin(-\xi)$$

where $-\xi$ denotes ξ with the orientation reversed. We have that $\mathcal{R}_\xi \circ \mathcal{R}_{-\xi}$ is the identity. Finally, given a bundle map $\hat{f}$ as in Construction 1.5, the obvious square commutes.

Proof: We described the transformation above, and it is not hard to see that it is $H^1(;\mathbf{Z}/2\mathbf{Z})$ equivariant. It is also easy to check that the composition formula holds. ∎

In practice, we can rarely identify our bundles with the accuracy demanded by Lemma 1.7 or Lemma 1.6, so we discuss the effect of a bundle automorphism on the sets of structures. Suppose we have a bundle $\chi = \xi \oplus \bigoplus_{i=1}^{r} \lambda$, where λ is a line bundle. We will study the case λ is trivial (so called "stabilization") and the case λ is isomorphic to $\det \xi$. Let γ be a bundle automorphism of χ which is the sum of the identity on ξ and some automorphism of $\bigoplus_{i=1}^{r} \lambda$. The transition functions for $\bigoplus_{i=1}^{r} \lambda$ are either the identity or minus the identity, both of which are central in $O(r)$ so γ is equivalent to a collection of maps $\gamma: B \to O(r)$, where B is the base of the bundle. The bundle automorphism induces a natural automorphism of $Pin^\pm$ structures on χ, described in the proof of

Lemma 1.10. *Let the base of the bundle, B, be path connected. The map induced by γ on structures, denoted γ^*, is the identity if γ lands in $SO(r)$. Otherwise it reverses the Spin structure in the Spin case and acts via $w_1(\xi)$ in the $Pin^\pm$ case if λ is trivial and by $r \cdot w_1(\xi)$ if λ is isomorphic to* $\det \xi$.

Proof: To fix notation, choose transition functions for a structure on ξ (either *Spin* or $Pin^\pm$). Pick transition functions for λ using the same cover. If λ is trivial, take the identity for the transition functions and if λ is the determinant bundle take the determinant of the transition functions for ξ. The new structure induced by γ has transition functions $\tilde{\gamma}(u)o_{ij}(u)\tilde{\gamma}^{-1}(u)$ where o_{ij} denotes the old transition functions and $\tilde{\gamma}(u)$ denotes a lift of $\gamma(u)$ to $Pin^\pm(r)$ and then into $Pin^\pm(n+r)$ where ξ has dimension n. There may be no continuous choice of $\tilde{\gamma}$, but since the two lifts yield the same conjugation, the new transition functions remain continuous. The element $o_{ij}(u)Pin^\pm(n+r)$ has the form x with x involving only the first n basis vectors in the Clifford algebra if $\det o_{ij}(u) = 1$ or if λ is trivial: otherwise $xe_{n+1}\cdots e_{n+r}$ with x as before.

Recall $\tilde{\gamma}x = (-1)^{\alpha(x)\alpha(\gamma)}x\tilde{\gamma}$ and $\tilde{\gamma}e_{n+1}\cdots e_{n+r} = (-1)^{\alpha(\gamma)(r-1)}e_{n+1}\cdots e_{n+r}\tilde{\gamma}$ where α on $Pin^\pm$ is the restriction of the mod 2 grading from the Clifford algebra and α on $O(r)$ is 1 iff the element is in $SO(r)$. The result now follows for $Pin^\pm$ structures. The result for *Spin* structures is now clear. If γ takes values in $SO(r)$ then the bundle map preserves the orientation and the underlying Pin^- structure, hence the *Spin*structure. If γ takes values in $O(r) - SO(r)$, compose the map induced by γ with the reverse *Spin* structure map. The reverse *Spin* structure map is induced by any constant map $B \to O(r) - SO(r)$. Hence the composite of these two maps is induced by a map $B \to SO(r)$ and hence is the identity. ∎

There are a couple of further compatibility questions involving the functions we have been discussing. Given an $SO(n)$ bundle ξ and an oriented trivial line bundle ϵ^1, we get a natural $SO(n+r)$ bundle $\xi \oplus r\epsilon^1$ and an isomorphism $-\xi \oplus r\epsilon^1 \cong -(\xi \oplus r\epsilon^1)$.

Lemma 1.11. *With the above identifications, stabilization followed by reversing the Spin structure agrees with reversing the Spin structure and then stabilizing: i.e.* $\mathcal{R}_{\xi\oplus r\epsilon}\circ \mathcal{S}_r^+(\xi) = \mathcal{S}_r^+(-\xi)\circ\mathcal{R}(\xi)$.

Proof: Left to the reader. ∎

Let M^m be $Pin^\pm$ and let V^{m-1} be a codimension 1 manifold of M with normal line bundle ν. We wish to apply Lemma 1.7 to the problem of constructing a "natural" structure on V. If there is a natural map from structures on M to structures on V, we say that V *inherits* a structure from the structure on M. Of course, the homomorphism $H^1(M;\mathbf{Z}/2\mathbf{Z}) \to H^1(V;\mathbf{Z}/2\mathbf{Z})$ implicit in the use of "natural" is just the one induced by the inclusion.

Corollary 1.12. *If ν is trivialized then V inherits a $Pin^\pm$ structure from a $Pin^\pm$ structure on M. If M and V are oriented then V inherits a Spin structure from a Spin structure on M.*

Proof: When ν is trivialized the result follows from Lemma 1.6. If M and V are oriented, then we can trivialize (i.e. orient) ν so that the orientation on $T_V \oplus \nu$ agrees with the orientation on $T_M|_V$. ∎

A case much like Corollary 1.12 occurs when M is a manifold with boundary, $V = \partial M$. In this case, the normal bundle, ν, is trivialized by the geometry, namely the preferred direction is inward. Just as in Corollary 1.12, we put ν last getting $T_M|_{\partial M} = T_{\partial M} \oplus \nu$. On orientations this gives the convention "inward normal last" which we adopt for orienting boundaries. Furthermore, a *Spin* or $Pin^\pm$ structure on M now induces one on ∂M, so we have a bordism theory of *Spin* manifolds and of $Pin^\pm$ manifolds.

In the *Spin* case, the inverse in the bordism group is formed by taking the manifold, M, with *Spin* structure on T_M, and reversing the *Spin* structure. In either the Pin^+ or the Pin^- case, the inverse in bordism is formed by acting on the given structure by $w_1(M)$. Having to switch the $Pin^\pm$ structure to form the inverse is what prevents $\Omega_*^{Pin^\pm}$ from being a $\mathbf{Z}/2\mathbf{Z}$ vector space like ordinary unoriented bordism. The explicit formula for the inverse does imply

Corollary 1.13. *The image of $\Omega_r^{Spin}(X)$ in $\Omega_r^{Pin^\pm}(X)$ has exponent 2 for any CW complex X, or even any spectrum.*

The "inward normal last" rule has some consequences. Suppose we have a manifold with boundary M, ∂M, and a structure on $M \times \mathbf{R}^1$. We can first restrict to the boundary, which is $(\partial M) \times \mathbf{R}^1$, and then do the codimension 1 restriction, or else we can do the codimension 1 restriction to M and then restrict to the boundary.

Lemma 1.14. *The two natural functions described above,*

$$\mathcal{P}in^{\pm}(M \times \mathbf{R}^1) \to \mathcal{P}in^{\pm}(\partial M) \ ,$$

differ by the action of $w_1(M)$. The same map between Spin structures reverses the Spin structure.

Proof: By considering restriction maps it is easy to see that it suffices to prove the result for $M = (\partial M) \times [0, \infty)$, and here the functions are bijections. Consider the inverse from structures on ∂M to structures on $\partial M \times \mathbf{R}^1 \times [0, \infty)$. The two different functions differ by a bundle automorphism which interchanges the last two trivial factors. By Lemma 1.10, this has the effect claimed. ∎

In the not necessarily trivial case we also have a "restriction of structure" result.

Corollary 1.15. *If ν is not necessarily trivial, then V inherits a structure from one on M in three of the four cases below:*

	Pin$^+$	*Pin*$^-$
V orientable $\nu = \det T_M$	*Spin*	*None*
V not necessarily orientable $\nu = \det T_V$	*Pin*$^-$	*Pin*$^-$

Proof: In the northwest case, $T_V \oplus \nu = T_M|_V$ has a *Pin*$^+$ structure, so $T_M \overset{3}{\oplus} \det T_M$ has a *Spin* structure. But $T_M \overset{3}{\oplus} \det T_M|_V = T_V \oplus \nu \oplus \overset{3}{\oplus} \det T_M|_V = T_V \overset{4}{\oplus} \det T_M|_V$ so T_V and hence V acquires a *Spin* structure. However, there is a choice in the above equation: we have had to identify ν with $\det T_M|_V$. When we say that the ν and $\det T_M$ are equal, we mean that we have fixed a choice.

A similar argument works in the southeast case: $T_V \oplus \det T_V$ is naturally oriented, so an identification of ν with $\det T_V$ gives $T_V \oplus \nu = T_M|_V$. Since M has a *Pin*$^-$ structure, V gets a *Pin*$^-$structure.

In the southwest case, consider $E \subset M$, a tubular neighborhood of V. Since ν and $\det T_V$ are identified, and since $T_V \oplus \det T_V$ is naturally oriented, E is oriented and hence the *Pin*$^+$ structure reduces uniquely to a *Spin* structure. From here the argument is the same as in the last paragraph.

Lastly, consider the northeast case. If we let $V = \mathbf{RP}^5 \subset \mathbf{RP}^6 = M$, we see that M has a *Pin*$^-$ structure; ν and $\det T_M$ are isomorphic; V is orientable but does not have any *Spin* structures at all. ∎

Remark. If we just assume that the line bundles in the table are isomorphic, which is surely the more usual situation, then we no longer get a well-defined structure. The new structure is obtained from the old one by first reversing orientation in the *Spin* case, and then acting by $w_1(\nu)$. A similar remark applies to Corollary 1.12.

§2. Pin^- structures on low-dimensional manifolds and further generalities.

We begin this section by recalling some well-known characteristic class formulas. Every 1-dimensional manifold is orientable and has *Spin* and $Pin^{\pm}$ structures. It is easy to parlay this into a proof that $\Omega_0^{Spin} \cong \mathbf{Z}$ and $\Omega_0^{Pin^{\pm}} \cong \mathbf{Z}/2\mathbf{Z}$, with the isomorphism being given by the number of points (for *Spin*) and the number of points mod 2 for $Pin^{\pm}$. Using the Wu relations, [M–S, p. 132], we see that every surface and every 3-manifold has a Pin^- structure, and hence oriented 2 and 3-manifolds have *Spin* structures. We can also say that a 2 or 3-manifold has a Pin^+ structure iff $w_1^2 = 0$. For surfaces this translates into having even Euler characteristic or into being an unoriented boundary.

We next give a more detailed discussion of structures on S^1. The tangent bundle to S^1 is trivial and 1-dimensional, hence a trivialization is the same thing as an orientation. Since $H^1(S^1;\mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$, there are two *Spin* structures on the circle. Since the tangent bundle to S^1 does not extend to a non-zero vector field over the 2-disk, the two *Spin* structures on an oriented S^1 can be described as follows: one of them is the *Spin* structure coming from the framing given by the orientation (this is called the *Lie group framing* or the *Lie group Spin structure*) and the other one is the one induced by the unique *Spin* structure on the 2-disk restricted to S^1.

Theorem 2.1. *The group $\Omega_1^{Spin} \cong \mathbf{Z}/2\mathbf{Z}$, generated by the Lie group Spin structure on the circle; $\Omega_1^{Pin^-} \cong \mathbf{Z}/2\mathbf{Z}$ and the natural map $\Omega_1^{Spin} \to \Omega_1^{Pin^-}$ is an isomorphism; $\Omega_1^{Pin^+} = 0$.*

Proof: Since the 2-disk has an orientation reversing involution, the restriction of this involution to the boundary gives an equivalence between S^1 with Lie group *Spin* structure and S^1 with the orientation reversed and the Lie group *Spin* structure. Hence Ω_1^{Spin} and $\Omega_1^{Pin^{\pm}}$ are each 0 or $\mathbf{Z}/2\mathbf{Z}$. Suppose S^1 is the boundary of an oriented surface $\hat{F}$. It is easy to check that all *Spin* structures on $\hat{F}$ induce the same *Spin* structure on S^1. If we let F denote $\hat{F} \cup B^2$ then F also has a *Spin* structure, and it is easy to see that any *Spin* structure on $\hat{F}$ extends (uniquely) to one on F. In particular, the *Spin* structure induced on S^1 is the one which extends over the 2-disk, so S^1 with the Lie group *Spin* structure does not bound.

The proof for the Pin^- case is identical because any surface has a Pin^- structure.

In the Pin^+ case however, $\mathbf{RP}^2$ does not have a Pin^+ structure. On the other hand, $\mathbf{RP}^2 - \text{int } B^2$ (which is the Möbius band) does have a Pin^+ structure. The induced Pin^+ structure on the boundary must therefore be one which does not extend over the 2–disk, and hence the circle with the Lie group Pin^+ structure does bound. ∎

In dimension 4, the generic manifold supports neither a *Spin* nor a $Pin^\pm$ structure. A substitute which works fairly well is to consider a 4–manifold with a submanifold dual to w_2 or $w_2 + w_1^2$. We will also have need to consider submanifolds dual to w_1. A general discussion of these concepts does not seem out of place here.

Let M be a paracompact manifold, with or without boundary. Let a be a cohomology class in $H^i(M; \mathbf{Z}/2\mathbf{Z})$. We say that a codimension i submanifold of M, say $W \subset M$, is dual to a iff the embedding of W in M is proper and the boundary of M intersects W precisely in the boundary of W. The fundamental class of W is a class in $H^{l.f.}_{n-i}(W, \partial W; \mathbf{Z}/2\mathbf{Z})$, where $H^{l.f.}$ denotes homology with locally finite chains. With the conditions we have imposed on our embedding, this class maps under the inclusion to an element in $H^{l.f.}_{n-i}(M, \partial M; \mathbf{Z}/2\mathbf{Z})$. Under Poincaré duality, $H^{l.f.}_{n-i}(M, \partial M; \mathbf{Z}/2\mathbf{Z})$ is isomorphic to $H^i(M; \mathbf{Z}/2\mathbf{Z})$ and we require that the image of the fundamental class of W map under this isomorphism to a. Specifically, in $H^{l.f.}_{n-i}(M, \partial M; \mathbf{Z}/2\mathbf{Z})$, we have the equation $a \cap [M, \partial M] = i_*[W, \partial W]$.

A cohomology class in $H^n(B; A)$, is given by a homotopy class of maps, $B \to K(A, n)$, where $K(A, n)$ is the Eilenberg–MacLane space with $\pi_n \cong A$. If $TO(n)$ denotes the Thom space of the universal bundle over $BO(n)$, then the Thom class gives a map $TO(n) \to K(\mathbf{Z}/2\mathbf{Z}, n)$. If M is a manifold, the Pontrjagin–Thom construction shows that $a \in H^n(M; \mathbf{Z}/2\mathbf{Z})$ is dual to a submanifold iff the map $M \to K(\mathbf{Z}/2\mathbf{Z}, n)$ representing a lifts to a map $M \to TO(n)$. Similar remarks hold if $A = \mathbf{Z}$ with $BO(n)$ replaces by $BSO(n)$. The submanifold, V, is obtained by transversality, so the normal bundle is identified with the universal bundle over $BO(n)$ or $BSO(n)$ and the Thom class pulls back to a. Hence there is a map $(M, M - V) \to (TO(n), *)$ which is a monomorphism on $H^n(; \mathbf{Z}/2\mathbf{Z})$ by excision. The Thom isomorphism theorem shows $H^n(M, M - V; \mathbf{Z}/2\mathbf{Z}) \cong H^0(V; \mathbf{Z}/2\mathbf{Z})$ so $H^n(M, M - V; \mathbf{Z}/2\mathbf{Z})$ is naturally isomorphic to a direct product of $\mathbf{Z}/2\mathbf{Z}$'s and the Thom class in $H^n(TO(n), *; \mathbf{Z}/2\mathbf{Z})$ restricts to the product of the generators. It follows that a restricted to $M - V$ is 0. It also follows that a restricted to V is the Euler class of the normal bundle.

Since $TO(1) = \mathbf{RP}^\infty = K(\mathbf{Z}/2\mathbf{Z}, 1)$ all 1–dimensional mod 2 cohomology classes have dual submanifolds. Since $TSO(1) = S^1 = K(\mathbf{Z}, 1)$ all 1–dimensional integral homology classes have dual submanifolds with oriented normal bundles. This holds even if M is not orientable, in which case the submanifold need not be orientable either. Since $TSO(2) = \mathbf{CP}^\infty = K(\mathbf{Z}, 2)$, any 2–dimensional integral cohomology class has a dual submanifold with oriented normal bundle. A case of interest to us is $TO(2)$. The map $TO(2) \to K(\mathbf{Z}/2\mathbf{Z}, 2)$ is not an equivalence, and not all 2–dimensional mod 2 cohomology classes have duals. As long as the manifold has dimension ≤ 4, duals can be constructed directly, but these techniques fail in dimensions 5 or more. A more detailed analysis of the map $TO(2) \to K(\mathbf{Z}/2\mathbf{Z},2)$ also shows the same thing: there are no obstructions to doing the lift until one gets to dimension 5 and then there are. It is amusing to note that the obstruction to realizing a class a in a 5–manifold is $Sq^2Sq^1a + aSq^1a \in H^5(M;\mathbf{Z}/2\mathbf{Z})/Sq^1(H^4(M;\mathbf{Z}/2\mathbf{Z}))$: in particular, if M is not orientable, then any class can be realized.

In our case we want to consider duals to w_1, w_2 and $w_2 + w_1^2$. We begin with w_1. This is an example for which the above discussion shows that we always have a dual, say $V^{m-1} \subset M^m$. We want to use the fact that we have a dual to w_1. The first question we want to consider is when is an arbitrary codimension 1 submanifold dual to w_1. The answer is supplied by

Lemma 2.2. *A codimension 1 submanifold $V \subset M$ is dual to $w_1(M)$ iff there exists an orientation on $M - V$ which does not extend across any component of V. The set of such orientations is acted on simply transitively by $H^0(M;\mathbf{Z}/2\mathbf{Z})$.*

Remark. We say that an orientation on $N - X$ does not extend across X if there is no orientation on N which restricts to the given one on $N - X$. We can take $N = (M - V) \cup V_0$ and $X = V_0$, where V_0 is a component of V. By varying V_0 over the path components of V we get a definition of an orientation on $M - V$ which does not extend across any component (= path component) of V. A similar definition applies to the case of a *Spin* or $Pin^{\pm}$ structure on $M - V$ which does not extend across any component of V.

Proof: Suppose that $M - V$ is orientable and fix an orientation. If ν_i denotes the normal bundle to the component V_i of V, let $(D(\nu_i), S(\nu_i))$ represent the disk sphere bundle pair. Each $S(\nu_i)$ is oriented by our fixed orientation on $M - V$ since $M - \amalg D(\nu_i) \subset M - V$ is a codimension 0 submanifold (hence oriented) and $\amalg S(\nu_i)$ can be naturally added as a boundary. Define $b \in H^1(M, M - V;\mathbf{Z}/2\mathbf{Z}) \cong \oplus H^1(D(\nu_i), S(\nu_i);\mathbf{Z}/2\mathbf{Z}) \cong \oplus \mathbf{Z}/2\mathbf{Z}$ on each summand as 1 if the orientation on $S(\nu_i)$ extends across $D(\nu_i)$ and -1 if it does not. The class b hits $w_1(M)$ in $H^1(M;\mathbf{Z}/2\mathbf{Z})$. This can be easily checked by considering any embedded circle in M and making it transverse to the V_i's subject to the further condition that if it intersects V_i at a point then it just enters $S(\nu_i)$ at one point and runs downs a fibre and out the other end. The tangent bundle of M restricted to this circle is oriented iff it crosses the V_i in an even number of points iff $\langle i^*(b), j_*[S^1]\rangle = 1$, where $i^*(b)$ is the image of b in $H^1(M;\mathbf{Z}/2\mathbf{Z})$ and $j_*[S^1]$ is the image of the fundamental class of the circle in $H_1(M;\mathbf{Z}/2\mathbf{Z})$. Since $w_1(M)$ also has this property, $i^*(b) = w_1(M)$ as claimed. If we act on this orientation by $c \in H^0(M - V;\mathbf{Z}/2\mathbf{Z})$, the new element in $H^1(M, M - V;\mathbf{Z}/2\mathbf{Z})$ is just $b + \delta^*(c)$, where $\delta^*(c)$ is the image of c under the coboundary $H^0(M - V;\mathbf{Z}/2\mathbf{Z}) \to H^1(M, M - V;\mathbf{Z}/2\mathbf{Z})$.

Now suppose that $M - V$ has an orientation which does not extend across any component of V. The b for this orientation has a -1 in each summand, and is hence the image of the Thom class. Therefore V is dual to $w_1(M)$.

Next suppose that V is dual to $w_1(M)$. Then $w_1(M)$ restricts 0 to $M - V$, and hence $M - V$ is orientable. Fix one such orientation and consider the corresponding b. Since both b and the image of the Thom class hit w_1, we can find $c \in H^0(M - V; \mathbf{Z}/2\mathbf{Z})$ so that $b + \delta^*(c)$ is the image of the Thom class. If we alter the given orientation on $M - V$ by c, we get a new one which does not extend across any component of V. ∎

There is also a "descent of structure" result here.

Proposition 2.3. *Given M^m, the Poincaré dual to $w_1(M)$ is an orientable $(m-1)$-dimensional manifold V^{m-1}. There is an orientation on $M - V$ which does not extend across any component of V and this orients the boundary of a tubular neighborhood of V. This boundary is a double cover of V and the covering translation is an orientation preserving free involution. In particular, V is oriented. Recall that $\alpha \in H^0(M; \mathbf{Z}/2\mathbf{Z})$ acts simply transitively on the orientations of $M - V$ which do not extend across any component of V. Hence α acts on the set of orientations of V by taking the image of α in $H^0(V; \mathbf{Z}/2\mathbf{Z})$ and letting this class act as it usually does.*

Remark. If V has more components than M, not all orientations on V can arise from this construction.

Proof: Suppose there is a loop λ in V which reverses orientation in V. If the normal line bundle ν to V in M is trivial when restricted to λ, then λ reverses orientation in M also, so $\lambda \bullet V \equiv 1 \pmod 2$; but $\lambda \bullet V = 0$ since ν is trivial over λ, a contradiction. If $\nu|_\lambda$ is nontrivial, then λ preserves orientation in M so $\lambda \bullet V \equiv 0 \pmod 2$; but $\lambda \bullet V = 1$ since ν is nontrivial, again a contradiction. So orientation reversing loops λ cannot exist.

Another proof that V is orientable: As we saw above $w_1(\nu) = i^*(w_1(M))$, where $i: V \subset M$. Since $T_M|_V = T_V \oplus \nu$, it follows easily from the Whitney sum formula that $w_1(V) = 0$.

We now continue with the proof of the proposition. Let E be a tubular neighborhood of V and recall that $H^1(E, \partial E; \mathbf{Z}/2\mathbf{Z})$ is $H^0(V; \mathbf{Z}/2\mathbf{Z})$ by the Thom isomorphism theorem. By Lemma 2.2 each component of ∂E can be oriented so that the orientation does not extend across E. Clearly ∂E is a double cover of V classified by $i^*(w_1(M))$. Since V is orientable, the covering translation must be orientation preserving and we can orient V so that the projection map is degree 1. It is easy to check the effect of changing the orientation on $M - V$ which does not extend across any component of V. ∎

We continue this discussion for the 2–dimensional cohomology classes w_2 and $w_2 + w_1^2$. Again we need a lemma which enables us to tell if a codimension 2 submanifold is dual to one of these classes. We have

Theorem 2.4. *Let M be a paracompact manifold, with or without boundary. Let F be a codimension 2 submanifold of M with finitely many components and with*

$\partial M \cap F = \partial F$. Then F is dual to $w_2 + w_1^2$ iff there is a Pin^- structure on $M - F$ which does not extend across any component of F. Furthermore $H^1(M; \mathbf{Z}/2\mathbf{Z})$ acts simply transitively on the set of Pin^- structures which do not extend across any component of F. There are similar results for Pin^+ structures and $Spin$ structures.

Proof: The proof is rather similar to the proof of the previous result. First, let F be a codimension 2 submanifold of M with $i: F \to M$ denoting the inclusion. Let $\left(D(\nu_i), S(\nu_i)\right)$ denote the disk, sphere bundle tubular neighborhoods to the components of F. Suppose $M - F$ has a Pin^- structure. (The proof for Pin^+ or $Spin$ structures is sufficiently similar that we leave it to the reader.) From Lemma 1.6, each $S(\nu_i)$ inherits a Pin^- structure. Define $b \in H^2(M, M - F; \mathbf{Z}/2\mathbf{Z}) \cong \oplus H^2(D(\nu_i), S(\nu_i); \mathbf{Z}/2\mathbf{Z}) \cong \oplus \mathbf{Z}/2\mathbf{Z}$ on each summand as 1 if the Pin^- structure on $S(\nu_i)$ extends across $D(\nu_i)$ and -1 if it does not. The class b hits $w_2(M)$ in $H^2(M; \mathbf{Z}/2\mathbf{Z})$. To see this, let $j: N \to M$ be an embedded surface which either misses an F_i or hits it in a collection of fibre disks. As before $\langle i^*(b), j_*[N]\rangle$ is 1 if $T_M|_N$ has a Pin^- structure and is -1 if it does not, since a bundle over a surface with a Pin^- structure over $N - \amalg D^2$ such that the Pin^- structure does not extend over the disks has a Pin^- structure iff there are an even number of such disks. Since $w_2(M)$ has the same property, $i^*(b) = w_2(M)$.

Now $H^1(M - F; \mathbf{Z}/2\mathbf{Z})$ acts simply transitively on the Pin^- structures on $M - F$ and, for $c \in H^1(M - F; \mathbf{Z}/2\mathbf{Z})$, the new b one gets is $b + \delta^*(c)$. The proof is now sufficiently close to the finish of the proof of Lemma 2.2 that we leave it to the reader to finish. ∎

There is also a "descent of structure" result in this case, but it is sufficiently complicated that we postpone the discussion until §6.

There are two cases in which we can show a "descent of structure" result for $Pin^\pm$ structures. As above, given M we can find a submanifold V dual to $w_1(M)$. We can then form $V \pitchfork V$ which is the submanifold obtained by making V transverse to itself. If ν denotes the normal bundle to V in M, then the normal bundle to $V \pitchfork V$ in V is naturally identified with $\nu|_{V \pitchfork V}$ and hence the normal bundle to $V \pitchfork V$ in M is naturally identified with $\nu|_{V \pitchfork V} \oplus \nu|_{V \pitchfork V}$. Since V is orientable, 2.3, $\nu|_{V \pitchfork V}$ is isomorphic to $\det T_M|_{V \pitchfork V}$. Hence by Lemma 1.7, a $Pin^\pm$ structure on M induces one on $V \pitchfork V$ after we identify $\nu|_{V \pitchfork V}$ with $\det T_M|_{V \pitchfork V}$. If we choose the other identification, the structure on $V \pitchfork V$ changes by twice $w_1(M)$ restricted to $V \pitchfork V$: i.e. the final structure on $V \pitchfork V$ is independent of the identification.

Theorem 2.5. *The function above*

$$[\cap w_1^2]: \mathcal{P}in^\pm(M) \to \mathcal{P}in^\mp(V \pitchfork V)$$

is a natural function using the map, $H^1(M; \mathbf{Z}/2\mathbf{Z}) \to H^1(V \pitchfork V; \mathbf{Z}/2\mathbf{Z})$, induced by the inclusion. If $V_1 \pitchfork V_1$ is another choice then there is a dual to w_1, $W \subset M \times [0,1]$ which is V at one end and V_1 at the other, so that $W \pitchfork W$ can be constructed *as a $Pin^{\mp}$ bordism between the two $Pin^{\mp}$ structures. The map $[\cap w_1^2]$ induces a homomorphism of bordism theories*

$$[\cap w_1^2]: \Omega_m^{Pin^{\pm}}(X) \to \Omega_{m-2}^{Pin^{\mp}}(X)$$

for any CW complex or spectrum X.

Proof: The naturality result follows easily from the naturality result in Lemma 1.7. The first bordism result follows easily once we recall that $TO(1) \cong K(\mathbf{Z}/2\mathbf{Z}, 1)$ so 1–dimensional cohomology classes in M are the same as codimension 1 submanifolds up to bordism in $M \times [0,1]$. The bordism result is also not hard to prove. ∎

For another example of "descent of structure", we consider the following: given any manifold, M^m, the dual to $w_1(M)$ is a codimension 1 submanifold V^{m-1}. Since V is orientable, Proposition 2.3, we are in the northwest situation of Corollary 1.15 and V receives a pair of *Spin* structures. Let $\left(\Omega_m^{Pin^+}\right)_0$ denote the subgroup of $\Omega_m^{Pin^+}$ consisting of those elements so that the two *Spin* structures on V are bordant. It is not hard to see that if the two structures are bordant for one representative in $\Omega_m^{Pin^+}$, then they are for any representative. Moreover, it is easy to check that the induced map is a homomorphism:

Lemma 2.6. *There is a well–defined homomorphism*

$$[\cap w_1]: \left(\Omega_m^{Pin^+}\right)_0 \to \Omega_{m-1}^{Spin} \quad .$$

Remark. It is not difficult to see that $\left(\Omega_m^{Pin^+}\right)_0$ contains the kernel of the map $[\cap w_1^2]$ since any such element has a representative for which the normal bundle to V is trivial. For such a V, we see a *Spin* bordism of $2 \cdot V$ to zero, so V and $-V$ represent the same element in *Spin* bordism. Moreover, the cohomology class by which we need to change the *Spin* structure is the zero class.

We conclude this section with some results we will need later which state that different ways of inducing structures are the same.

The first relates structures (*Spin* or $Pin^{\pm}$) and immersions. Given an immersion $f: N \to M$ the derivative gives a bundle map between the tangent bundles and so we can use it to pull structures on M back to N. The induced map on structures, denoted f^*, is natural in the technical sense defined earlier. Suppose we have an embedding $M_0 \times \mathbf{R}^1 \subset M$. Let $N_0 = f^{-1}(M_0)$ and note that there is an embedding $N_0 \times \mathbf{R}^1 \subset N$ so that f restricted to $N_0 \times \mathbf{R}^1$ is $g \times$ id where $g: N_0 \to M_0$ is also an immersion.

Lemma 2.7. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{P}in^{\pm}(N) & \xrightarrow{f^*} & \mathcal{P}in^{\pm}(M) \\ \mathcal{S}'_N \downarrow & & \downarrow \mathcal{S}'_M \\ \mathcal{P}in^{\pm}(N_0) & \xrightarrow{g^*} & \mathcal{P}in^{\pm}(M_0) \end{array}$$

where we orient $\mathbf{R}^1$ *and Lemma 1.6 gives us the natural map* $\mathcal{S}'_M$ *as the composite* $\mathcal{P}in^{\pm}(M) \to \mathcal{P}in^{\pm}(M_0 \times \mathbf{R}^1) \xrightarrow{\mathcal{S}} \mathcal{P}in^{\pm}(M_0)$ *with a similar definition for* $\mathcal{S}'_N$*. There is a similar result for Spin structures.*

Proof: We can easily reduce to the case $M = M_0 \times \mathbf{R}^1$. The required result can now be checked by choosing transition functions on M_0 and extending to transition functions for all the other bundles in sight, The two bundle we want to be isomorphic will be identical. ∎

The next result relates double covers and Pin^+ structures. Let M be a manifold with a *Spin* structure, and let $x: \pi_1(M) \to \mathbf{Z}/2\mathbf{Z}$ be a homomorphism (equivalently, $x \in H^1(M; \mathbf{Z}/2\mathbf{Z})$). Let E be the total space of the induced line bundle over M. By Lemma 1.7, E has a natural Pin^+ structure induced from the *Spin* structure on M. Hence ∂E receives a Pin^+ structure. Furthermore, ∂E is orientable and we orient it by requiring the covering map $\pi: \partial E \to M$ to be degree 1. The Pin^+ structure and the orientation give a *Spin* structure on ∂E. We can also use the immersion π to pull the *Spin* structure on M back to one on ∂E.

Lemma 2.8. *The two Spin structures on* ∂E *are the same.*

Proof: Begin with the 1–dimensional case. Here we are discussing *Spin* structures on the circle. Suppose that the line bundle is non–trivial. Thinking of the circle as the boundary of E, we see that it has the Lie *Spin* structure from Theorem 2.1. Thinking of it as the connected double cover we also see that it has the Lie group *Spin* structure, so the result is true in dimension 1. The case in which the line bundle is trivial is even easier.

The proof proceeds by induction on dimension. Suppose we know the result in dimension $m-1$ and let M have dimension $m > 1$. It suffices to show that the two *Spin* structures on ∂E agree when restricted to embedded circles. We can span $H^1(M; \mathbf{Z}/2\mathbf{Z})$ by embedded circles, S^1_i, $i = 1, \cdots, r$, where all the circles except the first lift to disjoint circles in the double cover. The first double covers itself if the line bundle is non–trivial and lifts to disjoint circles otherwise. The group $H_1(\partial E; \mathbf{Z}/2\mathbf{Z})$ is spanned by the collection of connected components of the covers from the circles in M.

Let M_0 be the boundary of the tubular neighborhood of such a circle and let $\tilde{M}_0$ be a connected component of the corresponding double cover. It suffices to prove that the two *Spin* structures on ∂E agree when restricted to $\tilde{M}_0$. We can restrict the line bundle to M_0 and consider the resulting total space E_0. First note that E_0 has trivial normal bundle in E and that it suffices to show that the two *Spin* structures on ∂E agree when restricted to ∂E_0.

Consider first the *Spin* structure induced by the double cover map. This map is an immersion, so Lemma 2.7 shows that inducing the structure on ∂E and then restricting to ∂E_0 is the same as first restricting the structure to M_0 and then inducing via the double cover map $\partial E_0 \to M_0$.

Next consider the *Spin* structure induced by restricting the Pin^+ structure to the boundary. We can restrict the Pin^+ structure on E to E_0 and then restrict to ∂E_0 or else restrict to the boundary and then to ∂E_0. These are not obviously the same: if we let ν_1 be the normal vector to E_0 in E, restricted to ∂E_0, and let ν_2 be the normal bundle to ∂E in E, again restricted to ∂E_0. We have a *Spin* structure on $T_E|_{\partial E_0}$, and in the two cases we identify this bundle with $T_{\partial E_0} \oplus \nu_1 \oplus \nu_2$ in one case and with $T_{\partial E_0} \oplus \nu_2 \oplus \nu_1$ in the other. By Lemma 1.10, these two ways of getting the *Spin* structure via boundaries agree up to a reverse of *Spin* structure. But we are using the orientation of M to keep track of all the other orientations, so the structures turn out to agree.

Our inductive hypothesis applies over M_0 and we conclude that the two *Spin* structures on ∂E_0 agree. ∎

The other result relates double covers and the Ψ_2. Let M be a manifold and let E' be the total space of the bundle $\det T_M \oplus \det T_M$ over M. There is a natural one to one function $\Psi_2 \colon \mathcal{P}in^{\pm}(M) \to \mathcal{P}in^{\mp}(E')$. Let $E \subset E'$ be the total space of the first copy of $\det T_M$: note $\partial E \to M$ is a 2 sheeted cover. The embedding $\partial E \subset E'$ has a normal bundle which we see as two copies of the trivial bundle, which happens to be $\det T_{\partial E}$. This gives a natural function $\Psi_2^{!} \colon \mathcal{P}in^{\mp}(E') \to \mathcal{P}in^{\pm}(\partial E)$.

Theorem 2.9. *The $Pin^{\pm}$ structure defined above on ∂E is the same as the one induced by the double cover map.*

Proof: We begin by proving that certain diagrams commute. To fix notation, let $M_0 \times \mathbf{R}^1 \subset M$. Let E_0 denote the total space of $\det T_{M_0} \oplus \det T_{M_0}$ and observe that we can embed $E_0 \times \mathbf{R}^1$ in E. We can arrange the embedding so that on 0 sections it is our given embedding, and so that $(\partial E_0) \times \mathbf{R}^1$ is embedded in ∂E. We begin with

$$\begin{array}{ccc}
\mathcal{P}in^{\pm}(M) & \xrightarrow{\Psi_2} & \mathcal{P}^{\mp}(E') \\
{\scriptstyle L_1}\downarrow & & \downarrow{\scriptstyle L_2} \\
\mathcal{P}in^{\pm}(M_0) & \xrightarrow{\Psi_2} & \mathcal{P}^{\mp}(E'_0)
\end{array}$$

where L_1 is just $\mathcal{S}^{-1}$ followed by the restriction map induced by the embedding of $M_0 \times \mathbf{R}^1$ in M and L_2 is defined similarly but using the embedding of $E_0 \times \mathbf{R}^1$ in E. This diagram commutes by Lemma 1.10. We can then restrict this structure to

∂E and then further to $(\partial E_0) \times \mathbf{R}^1$. Since stabilization commutes with restriction we see

$$\begin{array}{ccc} \mathcal{P}in^{\pm}(M) & \longrightarrow & \mathcal{P}in^{\pm}(\partial E) \\ {\scriptstyle L_3}\downarrow & & \downarrow{\scriptstyle L_4} \\ \mathcal{P}in^{\pm}(M_0) & \longrightarrow & \mathcal{P}in^{\pm}(\partial E_0) \end{array}$$

commutes, where L_3 is defined by restricting from M to $M_0 \times \mathbf{R}^1$ followed by the inverse stabilization map and L_4 is defined by restricting from ∂E to $(\partial E_0) \times \mathbf{R}^1$ followed by the inverse stabilization map.

The proof now proceeds much like the last one. First we check the result for S^1. Applying the last diagram to the 2–disk with boundary S^1 shows the result for the structure which bounds. Apply the Pin^+ diagram to the Möbius band to see the result for the Lie Pin^+ structure. The result now holds for any Pin^+ structure on S^1. Hence it holds for *Spin* structures and hence for Pin^- structures.

For M of dimension at least 2 we induct on the dimension. But just like the proof of the preceding result, this follows from the commutativity of our second diagram. ∎

§3. Pin^- structures on surfaces, quadratic forms and Brown's arf invariant.

In this section we want to recall an algebraic way of describing Pin^- structures due to Brown [Br].

Definition 3.1. A function $q\colon H_1(F;\mathbf{Z}/2\mathbf{Z}) \to \mathbf{Z}/4\mathbf{Z}$ is called a *quadratic enhancement* of the intersection form provided it satisfies $q(x+y) = q(x)+q(y)+2\cdot x\bullet y$ for all $x, y \in H_1(F;\mathbf{Z}/2\mathbf{Z})$ (here $2\cdot$ denotes the inclusion $\mathbf{Z}/2\mathbf{Z} \subset \mathbf{Z}/4\mathbf{Z}$ and $\bullet$ denotes intersection number.

The main technical result of this section is

Theorem 3.2. *There is a canonical 1–1 correspondence between Pin^- structures on a surface F and quadratic enhancements of the intersection form.*

Discussion. One sometimes says that there is a 1–1 correspondence between Pin^- structures on F and $H^1(F;\mathbf{Z}/2\mathbf{Z})$, but this is non–canonical. Canonically, there is an action of $H^1(F;\mathbf{Z}/2\mathbf{Z})$ on the set of Pin^- structures which is simply transitive. Once a base point has been selected, the action gives a 1–1 correspondence between $H^1(F;\mathbf{Z}/2\mathbf{Z})$ and the set of Pin^- structures.

Note also that $H^1(F;\mathbf{Z}/2\mathbf{Z})$ acts on the set of quadratic enhancements, by $q \times \gamma$ goes to q_γ defined by

$$q_\gamma(y) = q(y) + 2\cdot\gamma(y) \tag{3.3}$$

and note that this action is simply transitive. The 1–1 correspondence in Theorem 3.2 is equivariant with respect to these actions. Indeed, the proof of Theorem 3.2 will be to fix a Pin^- structure on F and use it to write down a quadratic enhancement. This gives a transformation from the set of Pin^- structures to the set of quadratic enhancements. We will check that it is equivariant for the $H^1(F;\mathbf{Z}/2\mathbf{Z})$ action and this will prove the theorem.

Before describing the enhancement, we prove a lemma that produces enhancements from functions on embeddings. Specifically

Lemma 3.4. *Let $\hat{q}$ be a function which assigns an element in $\mathbf{Z}/4\mathbf{Z}$ to each embedded disjoint union of circles in a surface F subject to the following conditions:*

(a) $\hat{q}$ is additive on disjoint union; if L_1 and L_2 are two embedded collections of circles such that $L_1 \amalg L_2$ is also an embedding then $\hat{q}(L_1 \amalg L_2) = \hat{q}(L_1) + \hat{q}(L_2)$

(b) if L_1 and L_2 are embedded collections of circles which cross transversely at r points, then we can get a third embedded collection, L_3, by replacing each crossing: we require $\hat{q}(L_3) = \hat{q}(L_1) + \hat{q}(L_2) + 2\cdot r$

(c) if L is a single embedded circle which bounds a disk in F, then $\hat{q}(L) = 0$.

Then $\hat{q}(L)$ depends only on the underlying homology class of L, and the induced function $q\colon H_1(F;\mathbf{Z}/2\mathbf{Z}) \to \mathbf{Z}/4\mathbf{Z}$ is a quadratic enhancement.

Proof: The first step is to show how given L, we may replace it with a single embedded circle K such that the L and K represent the same homology class in $H_1(F;\mathbf{Z}/2\mathbf{Z})$ and have the same $\hat{q}$. If L has more than one component, it is possible to draw an arc between two different components. A small regular neighborhood of this arc is a disk, and let K_1 be its boundary circle. By (c), $\hat{q}(K_1) = 0$. The circle K_1 has two pairs of intersection points with L. Apply (b): the new embedding consists of a new collection L_1 which has one fewer components that L, and two small circles K_2 and K_3, each of which bounds a disk. Condition (b) says that $\hat{q}(L_1 \amalg K_2 \amalg K_3) = \hat{q}(L) + \hat{q}(K_1) = \hat{q}(L)$. From (a) and (c) we see that $\hat{q}(L_1 \amalg K_2 \amalg K_3) = \hat{q}(L_1)$, so $\hat{q}(L) = \hat{q}(L_1)$, and L and L_1 represent the same homology class. Continue until there is only one component left.

Next we prove isotopy invariance of $\hat{q}$ in several steps. First, suppose $A \subset F$ is an embedded annulus with boundary $K_0 \amalg K_1$ and core C. We want to show $\hat{q}(K_0) = \hat{q}(K_1) = \hat{q}(C)$. Draw an arc from K_0 to C and let K_3 be a circle bounding a regular neighborhood of this arc. Apply condition (b): the result is two circles, each of which bounds a disk. From conditions (a) and (c) we see $\hat{q}(C) = \hat{q}(K_0)$. A similar proof establishes the rest. We can also show that $\hat{q}(C)$ must be even. Let C_1 be a copy of C pushed off itself in the annular structure. Then $\hat{q}(C) = \hat{q}(C_1)$ since they are both $\hat{q}(K_0)$. Let $L = C \amalg C_1$. Then $\hat{q}(L) = 2\hat{q}(C)$ by (a). On the other hand, just as above, we can use (b) to transform L into a picture with two circles bounding disks, so by (a) and (c) we see $\hat{q}(L) = 0$ and the result follows. Hence any curve in F with trivial normal bundle has even $\hat{q}$. Finally, suppose that C_1 is embedded in A and represents the same element in mod 2 homology as C.

We can find a third curve C_2 which also represents the same element in mod 2 homology and which intersects both C_1 and C transversely. Consider say C_2 and C. Apply (b): r is even as are both $\hat{q}(C)$ and $\hat{q}(C_2)$. Hence $\hat{q}(C) = \hat{q}(C_2)$. Similarly $\hat{q}(C_1) = \hat{q}(C_2)$ and we have our result.

Next suppose that $M \subset F$ is a Möbius band with core C_0. We can push C_0 to get another copy, C_1 intersecting C_0 transversely in one point. We can push off another copy C_2 which intersects C_0 and C_1 transversely in a single point and all three points are distinct. Applying (b) to pairs of these circles, we get $\hat{q}(C_i) + \hat{q}(C_j) = 2$ for $0 \leq i, j \leq 2$, $i \neq j$. Adding all three equations we see $2(\hat{q}(C_0)+\hat{q}(C_1)+\hat{q}(C_2)) = 2$, so at least one $\hat{q}(C_i)$ must be odd. But then returning to the individual equations we see that $\hat{q}(C_0) = \hat{q}(C_1) = \hat{q}(C_2)$, so we see that $\hat{q}(C)$ must be odd whenever the normal bundle to C is non–trivial. Let C_1 be any embedded circle in M which represents the core in mod 2 homology. It is possible to find a third embedded circle, C_2 which also represents the core and intersects C_0 and C_1 transversely. Since $\hat{q}(C_i)$ must be odd, it is not hard to use (b) to show that $\hat{q}(C_0) = \hat{q}(C_1)$.

To show isotopy invariance proceed as follows. Let K be a circle with a neighborhood W. Any isotopy of K will remain for a small interval inside W and the image K_t will continue to represent the core in mod 2 homology. By the above discussion $\hat{q}$ will be constant on K_t, the circle at time t. Hence, the subset of $t \in [0,1]$ for which $\hat{q}(K_t) = \hat{q}(K)$ is an open set. Likewise the set of $t \in [0,1]$ for which $\hat{q}(K_t) \neq \hat{q}(K)$ is an open set, so we have isotopy invariance for a single circle. By part (a), the result for general isotopies follows as above.

Next we prove homology invariance. Suppose L_1 and L_2 represent the same element in homology. By isotopy invariance, we may assume that they intersect transversely. Let L_3 be the result of applying condition (b). $\hat{q}(L_3) = \hat{q}(L_1) + \hat{q}(L_2) + 2 \cdot r$, and L_3 is null–homologous. If we can prove $\hat{q}(L_3) = 0$ then we are done. As we saw above, it is no loss of generality to assume that L_3 is connected, and since it is null–homologous, it has trivial normal bundle, so $\hat{q}(L_3)$ is even. Also, since L_3 is null–homologous, there exists a 2–manifold with boundary a single circle, say W, and an embedding $W \subset F$ so that $\partial W = L_3$. If W is a disk we are done by (c), so we work by induction on the Euler characteristic of W. If W is not a disk then we can write $W = W_1 \cup V$ where $\partial V = \partial_0 V \amalg \partial_1 V = S^1 \amalg S^1$, V is either a twice punctured torus or a punctured Möbius band, and W_1 has larger Euler characteristic than W. We are done if we can show $\hat{q}(\partial_0 V) = \hat{q}(\partial_1 V)$. We begin with the toral case. Using (b) and (c) as usual, we can see that $\hat{q}(\partial_0 V) = \hat{q}(S_a) + \hat{q}(S_b)$ where S_a and S_b are two meridian circles, one on either side of the hole. Likewise $\hat{q}(\partial_1 V) = \hat{q}(S_a) + \hat{q}(S_b)$ so we are done with this case. In the Möbius band case we can again use (b) and (c) and see that $\hat{q}(\partial_0 V) + \hat{q}(\partial_1 V) = 0$. Since they are both even, again they are equal.

This shows that $\hat{q}$ induces a function $q\colon H_1(F; \mathbf{Z}/2\mathbf{Z}) \to \mathbf{Z}/4\mathbf{Z}$, and (b) translates immediately into the relation $q(x + y) = q(x) + q(y) + 2 \cdot x \bullet y$. ∎

Now we describe our function. Let λ be a line bundle over F with $w_1(\lambda) = w_1(F)$ and let $E(\lambda)$ denote its total space. From Lemma 1.7, a *Spin* structure on $E(\lambda)$ gives a Pin^- structure on F. Let K be an embedded circle in F, and let τ denote the tangent bundle of $E(\lambda)$ restricted to K. A *Spin* structure on $E(\lambda)$ yields a trivialization of τ. It is also true that $\tau = T_{S^1} \oplus \nu_{K \subset F} \oplus \nu_{F \subset E(\lambda)}$, where ν denotes normal bundle. Note all three of these bundles are line bundles. Pick a point $p \in K$ and orient each of the line bundles at p so that the orientation on τ agrees with that coming from the *Spin* structure. Since T_{S^1} is trivial, the orientation picks out a trivialization, and hence $\nu_{K \subset F} \oplus \nu_{F \subset E}$ acquires a preferred even framing. (Note that framings of a 2–plane bundle correspond to $\mathbf{Z}$, while those of a 3–plane bundle correspond to $\mathbf{Z}/2\mathbf{Z}$. Hence the framing of the 3–plane bundle picks out a set of framing of the 2–plane bundle, a set we call *even*.)

Definition 3.5. Choose an odd framing on $\nu_{K \subset F} \oplus \nu_{F \subset E}$ and using it, count the number (mod 4) of right half twists that $\nu_{K \subset F}$ makes in a complete traverse of K. This is $\hat{q}(K)$. Given a disjoint union of circles, Lemma 3.4 (a) gives the value of $\hat{q}$ in terms of the individual components.

We first need to check that $\hat{q}$ really only depends on the embedded curve and not on the choice of p or the local orientations made at p or on the choice of odd framing. It is easy to see that the actual choice of framing within its homotopy class is irrelevant because we get the same count in either frame. If we choose a new odd framing the new count of right half twists will change by a multiple of 4, so the specific choice of odd framing is irrelevant. If we move p to a new point, we can move around K in the direction of the orientation and transport the local orientations as we go. If we make these choices at our new point, nothing changes so the choice of point is irrelevant. Since we must keep the same orientation on τ, we are only free to change orientations in pairs. If we keep the same orientation on K, the odd framing on the normal bundle remains the same and so we get the same count. Finally, suppose we switch the orientation on K. We can keep the same framing on the normal bundle provided we switch the order of the two frame vectors. If we do this and traverse K in the old positive direction we get the same count as before, except with a minus sign. Fortunately, we are now required to traverse K in the other direction which introduces another minus sign, so the net result is the same count as before. Hence $\hat{q}$ only depends on the embedded curve.

Since $\hat{q}$ satisfies Lemma 3.4 (a) by definition, we next show that it satisfies conditions (b) and (c) also. We begin with (c). In this case, all three line bundles are trivial, hence framed after our choice of p and the local orientations. However, this stable framing of the circle is the Lie group one, so it is not the stable framing of the circle which extends over the disk, Theorem 2.1. Since the framing from the *Spin* structure does extend over the disk, the framing constructed above is an odd framing, and $\hat{q}$ is clearly 0 for these choices. To show (b), consider a small disk neighborhood of a crossing. It is not hard to check that in the framing coming from that of the disk, we can remove the crossing without changing the count. However, this is the even framing and we are supposed to do the counting using the odd framing. This introduces a full twist, and so we get a contribution of 2 for each crossing. This is (b).

Thanks to Lemma 3.4 we have described a function from the set of Pin^- structures on F to the set of quadratic enhancements on the intersection form on $H_1(F;\mathbf{Z}/2\mathbf{Z})$. Suppose now we change the Pin^- structure by $\gamma \in H^1(F;\mathbf{Z}/2\mathbf{Z})$. The effect of this change is to reverse even and odd framings on K for which $\gamma(K) = -1$ and to leave things alone for K for which $\gamma(K) = 1$. The effect on the resulting q is to add 2 to $q(x)$ if $\gamma(x) = -1$ and add nothing to it if $\gamma(x) = 1$. But this is just q_γ.

This completes the proof of Theorem 3.2.

Next we describe an invariant due to Brown, [Br], associated to any quadratic enhancement q. Given q, form the Gauss sum

$$\Lambda_q = \sum_{x\in H_1(F;\mathbf{Z}/2\mathbf{Z})} e^{2\pi i q(x)/4} \quad .$$

This complex number has absolute value $\sqrt{|H_1(F;\mathbf{Z}/2\mathbf{Z})|}$ and there exists an element $\beta(q) \in \mathbf{Z}/8\mathbf{Z}$ such that $\Lambda_q = \sqrt{|H_1(F;\mathbf{Z}/2\mathbf{Z})|}\, e^{2\pi i\beta(q)/8}$.

Hence we can think of β as a function from Pin^- structures on surfaces to $\mathbf{Z}/8\mathbf{Z}$. It also follows from Brown's work, that β is an invariant of Pin^- bordism: two surfaces with Pin^- structures that are Pin^- bordant have the same β.

Lemma 3.6. *The homomorphism*

$$\beta\colon \Omega_2^{Pin^-} \to \mathbf{Z}/8\mathbf{Z}$$

is an isomorphism. The composite $\Omega_2^{Pin^-} \xrightarrow{\beta} \mathbf{Z}/8\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ *is the* mod 2 *Euler characteristic and hence determines the unoriented bordism class of the surface.*

Proof: Brown proves that β induces an isomorphism between Witt equivalence classes of quadratic forms and $\mathbf{Z}/8\mathbf{Z}$. One homomorphism from the Witt group is the dimension mod 2 of the underlying vector space. Since this is just the mod 2 Euler characteristic of our surface, the second result follows.

Hence, if $\beta(F) = 0$, the manifold is an unoriented boundary, say of W^3. There is an obstruction in $H^2(W,\partial W;\mathbf{Z}/2\mathbf{Z})$ to extending the Pin^- structure on F across W. If this obstruction is 0 we are done, so assume otherwise. There is a dual circle, $K \subset W - F$ and the Pin^- structure on F extends across $W - K$. The boundary of a neighborhood of K is either a torus or a Klein bottle, so if $\beta(F) = 0$, F is Pin^- bordant to a torus or a Klein bottle with β still 0. Moreover, since the Pin^-

structure is not supposed to extend across the neighborhood of K, one of the non-zero classes in H_1 has a non-zero q. For the Klein bottle, two of the non-zero classes have odd square and the other has even square. It is the class with even square that must have a non-trivial q on it to prevent the Pin^- structure from extending across the disk bundle. But the Klein bottle with this sort of enhancement has non-zero β, so the boundary of K must be a torus. For the torus, q must vanish on the remaining classes in H_1 in order to have $\beta = 0$ and it is easy to find a Pin^- boundary for it. ∎

Exercise. Show that $\mathbf{RP}^2$ with its two Pin^- structures has $\beta = \pm 1 \in \mathbf{Z}/8\mathbf{Z}$.

The relation between Pin^- structures and quadratic enhancements is pervasive in low-dimensional topology. In [Ro], [F–K] and [G–M] enhancements were produced on characteristic surfaces in order to generalize Rochlin's theorem. In §6, we will show how to find an enhancement without the use of membranes. This gives some generalizations of the previous work. In the next section we will study surfaces embedded in "spun" 3-manifolds. An interesting theory that we do not pursue is Brown's idea of studying immersions of a surface in $\mathbf{R}^3$. Since $\mathbf{R}^3$ has a unique *Spin* structure, an immersion pulls back a *Spin* structure onto the total space of a line bundle over the surface with oriented total space.

Another point we wish to investigate is the behavior of β under change of Pin^- structure. Hence fix a quadratic form $q\colon V \to \mathbf{Z}/4\mathbf{Z}$: i.e. V is a $\mathbf{Z}/2\mathbf{Z}$-vector space; $q(rx) = r^2 q(x)$ for all $x \in V$ and $r \in \mathbf{Z}$; and $q(x+y) - q(x) - q(y)$ is always even and gives rise to a non-singular bilinear pairing $\lambda\colon V \times V \to \mathbf{Z}/2\mathbf{Z}$.

Given $a \in V$, define q_a by $q_a(x) = q(x) + 2 \cdot \lambda(a, x)$.

Lemma 3.7. *With notation as above,* $\beta(q_a) = \beta(q) + 2 \cdot q(a)$.

Proof: There is a rank 1 form (1) consisting of a $\mathbf{Z}/2\mathbf{Z}$ vector space with one generator, x, for which $q(x) = 1$. There is a similar form (-1). It is easy to check the formula by hand for these two cases. Or, having checked it for (1) and $a = x$ and $a = 0$, argue as follows. Given any form q, there is another form $-q$ defined on the same vector space by $(-q)(x) = -q(x)$. It is easy to check that $\beta(-q) = -\beta(q)$. If the formula holds for q and a, it is easily checked for $-q$ and a after we note $(-q)_a = -(q_a)$.

Given two forms q_1 on V_1 and q_2 on V_2, we can form the *orthogonal sum* $q_1 \perp q_2$ on $V_1 \oplus V_2$ by the formula $(q_1 \perp q_2)(v_1, v_2) = q_1(v_1) + q_2(v_2)$. Brown checks that $\beta(q_1 \perp q_2) = \beta(q_1) + \beta(q_2)$. If $a_i \in V_i$, note $(q_1 \perp q_2)_{(a_1,a_2)} = (q_1)_{a_1} \perp (q_2)_{a_2}$, so if the formula holds for the two pieces, it holds for the orthogonal sum. Moreover, if it holds for the sum and one of the pieces, it holds for the other piece.

Finally, note that if $a = 0$, the formula is true.

Now use Brown, [Br, Theorem 2.2 (viii)] to see that it suffices to prove the formula for a form isomorphic to $m(1) + n(-1)$ and any a and this follows from the above discussion. ∎

Next we present a "geometric" calculation of the *Spin* and Pin^+ bordism groups in dimension 2.

Proposition 3.8. *Any Spin structure induces a unique Pin^- structure, so β is defined just as above for surfaces with a Spin structure. We have β defines an isomorphism $\Omega_2^{Spin} \to \mathbf{Z}/2\mathbf{Z}$. Any surface with odd Euler characteristic with any Pin^- structure is a generator for $\Omega_2^{Pin^-}$ and the 2–torus with the Lie group Spin structure is a generator for Ω_2^{Spin}.*

Proof: The proof is almost identical to that of Lemma 3.6. The surface F bounds an oriented 3–manifold W and by considering the obstruction to extending the *Spin* structure we see that F is *Spin* bordant to a torus with the same *Spin* structure as in the proof of Lemma 3.6. Just note that the boundary constructed there is actually a *Spin* boundary. It is a fact from Brown that β restricted to even forms only takes on the values 0 and 4. The results about the generators are straightforward. ∎

The Pin^+ case is more interesting. We have already seen that the only way a surface can have a Pin^+ structure is for w_1^2 to be 0. Hence the $[\cap w_1^2]$ map must also be 0, so the $[\cap w_1]$ map is defined on all of $\Omega_2^{Pin^+}$.

Proposition 3.9. *The homomorphism $[\cap w_1]: \Omega_2^{Pin^+} \to \Omega_1^{Spin} \cong \mathbf{Z}/2\mathbf{Z}$ is an isomorphism. A generator is given by the Klein bottle in half of its four Pin^+ structures.*

Proof: A surface, F, has a Pin^+ structure iff $w_2(F) = 0$ iff F is an unoriented boundary, say $F = \partial W$. The obstruction to the Pin^+ structure on F extending to W is given by a relative 2–dimensional cohomology class, so its dual is a 1–dimensional absolute homology class. We can assume that it is a single circle, and so F is Pin^+ bordant to either a torus or a Klein bottle, and the Pin^+ structure has the property that it does not extend over the corresponding 2–disk bundle over S^1.

Since S^1 with either Pin^+ structure is a Pin^+ boundary it is not hard to see that the torus with any Pin^+ structure is a Pin^+ boundary. There are two Pin^+ structures on the Klein bottle which do not extend over the disk bundle. If one cuts the Klein bottle open along the dual to w_1 and glues in two copies of the Möbius band, one sees a Pin^+ bordism between these two Pin^+ structures. Hence $\Omega_2^{Pin^+}$ has at most two elements. On the other hand it is not hard to see that the Klein bottle with the Pin^+ structures which do not extend over the disk bundle hit the non–zero element in Ω_1^{Spin} under $[\cap w_1]$. ∎

For future convenience let us discuss another way to "see" structures on the torus and the Klein bottle. We begin with the torus, T^2.

Example 3.10. We can write T^2 as the union of two open sets $U_i = S^1 \times (-1, 1)$ so that $U_1 \cap U_2$ is two disjoint copies of $S^1 \times (-1, 1)$, say $U_1 \cap U_2 = V_{12} \amalg \overline{V}_{12}$.

We can frame $S^1 \times (-1,1)$ using the product structure and the framings of the two 1–dimensional manifolds, S^1 and $(-1,1)$. If we form an $SO(2)$ bundle over T^2 with transition function g_{12} defined by $g_{12}(U_1 \cap U_2) = 1$ then we get the tangent bundle. If we think of 1 as the identity of $Spin(2)$ then the same transition functions give a *Spin* structure on T^2. This *Spin* structure is the Lie group one: clearly the copy of S^1 in the $S^1 \times (-1,1)$'s receives the Lie group structure, and it is not difficult to start with a framing of $(-1,1)$ and transport it around the torus to get the Lie group structure on this circle. If we take as $Spin(2)$ transition functions h_{12} defined by $h_{12}(V_{12}) = 1$ and $h_{12}(\overline{V}_{12}) = -1 \in Spin(2)$, then we get a *Spin* structure whose enhancement is 0 on the obvious S^1 and 2 on the circle formed by gluing the two intervals.

Example 3.11. We can write the Klein bottle, K^2, as the union of two open sets $U_i = S^1 \times (-1,1)$ so that $U_1 \cap U_2$ is two disjoint copies of $S^1 \times (-1,1)$, say $U_1 \cap U_2 = W_{12} \amalg \overline{W}_{12}$. We can frame $S^1 \times (-1,1)$ using the product structure and the framings of the two 1–dimensional manifolds, S^1 and $(-1,1)$. If we form an $O(2)$ bundle over K^2 with transition function g_{12} defined by $g_{12}(W_1) = 1$ and $g_{12}(\overline{W}_{12}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in O(2)$ then we get the tangent bundle (we are writing the S^1 tangent vector first). If we define $h_{12}(W_1) = 1$ and $h_{12}(\overline{W}_{12}) = e_1 \in Pin(2)$, we get a *Pin* structure on the tangent bundle. The copy of S^1 in the $S^1 \times (-1,1)$'s receives the Lie group structure, so if we are describing a Pin^- structure, then we get the bordism generator.

We conclude this section with two amusing results that we will need later.

Theorem 3.12. *Let F be a surface with a Spin structure. Let $q\colon H_1(F;\mathbf{Z}/2\mathbf{Z}) \to \mathbf{Z}/2\mathbf{Z}$ denote the induced quadratic enhancement. Let $x \in H^1(F;\mathbf{Z}/2\mathbf{Z})$. Corresponding to x there is a double cover of F, $\tilde{F}$ which has an induced Spin structure. There is also a dual homology class a and*

$$[\tilde{F}] = q(a) \in \mathbf{Z}/2\mathbf{Z} \quad .$$

Proof: We can write F as $T^2 \# F_1$ where T^2 is a 2 torus and a is contained in T^2. Then $\tilde{F} = T_1^2 \# F_1 \# F_1$, where T_1^2 is a double cover of T^2 given by $x \in H^1(T^2;\mathbf{Z}/2\mathbf{Z})$. Note $\langle x, a\rangle = 1$ not -1,so a lifts to 2 disjoint parallel circles. Moreover, $H_1(T_1^2;\mathbf{Z}/2\mathbf{Z})$ is generated by one component of the cover of a, say $\tilde{a}$, and another circle, say $\tilde{b}$ which double covers a circle, say b in T^2.

Note $[\tilde{F}] = [T_1^2] + 2[F_1]$, so $[\tilde{F}] = [T_1^2]$. The enhancement $\tilde{q}\colon H_1(T_1^2;\mathbf{Z}/2\mathbf{Z})$ satisfies $\tilde{q}(\tilde{a}) = q(a)$ and $\tilde{q}(\tilde{b}) = -1$. Hence the *Spin* bordism class of T_1^2 in $\mathbf{Z}/2\mathbf{Z}$ is given by $q(a)$. ∎

The second result is the following. Given any surface, F, we can take the orientation cover, $\tilde{F}$, and orient $\tilde{F}$ so that the orientation does not extend across any component of the total space of the associated line bundle. Given a $Pin^{\pm}$ structure on F, we can induce a $Spin$ structure on $\tilde{F}$.

Lemma 3.13. *The orientation double cover map induces homomorphisms*

$$\Omega_2^{Pin^{\pm}} \to \Omega_2^{Spin}$$

which are independent of the orientation on the double cover. The Pin^- map is trivial, and the Pin^+ map is an isomorphism.

Proof: If we switch to orientation on $\tilde{F}$, we get the reverse of the $Spin$ structure we originally had. Since $\Omega_2^{Spin} \cong \mathbf{Z}/2\mathbf{Z}$ this shows that the answer is independent of orientation. By applying the construction to a bordism between two surfaces we see that the maps are well-defined on the bordism groups. Since addition is disjoint union, the maps are clearly homomorphisms.

In the Pin^- case, $\mathbf{RP}^2$ is a generator of the bordism group. The oriented cover is S^2 which has a unique $Spin$ structure and is a $Spin$ boundary. This shows the Pin^- map is trivial.

In the Pin^+ case, a generator is given by the Klein bottle. Consider the transition functions that we gave for this Pin^+ structure in Example 3.11. This give us a set of transition functions for the torus which double covers the Klein bottle. We get 4 open sets, but it is not difficult to amalgamate three of the cylinders into one. The new transition function, h_{12}, takes the value 1 on one component of the overlap and the value e_1^2 on the other. Since $e_1 \in Pin^+(2)$, $e_1^2 = 1$ so we get the Lie group structure on T^2 by Example 3.10. ∎

Remark. If we started with a non-bounding Pin^- structure on the Klein bottle, then the above proof would show that the double cover has $Spin$ transition functions given by 1 on one component of the overlap and -1 on the other, and, as we saw, this $Spin$ structure bounds (as Lemma 3.13 requires).

§4. *Spin* structures on 3-manifolds.

Let M^3 be a closed 3-manifold with a given $Spin$ structure. We begin by generalizing some of the basic ideas in the calculus of framed links in S^3.

Given any embedded circle $k: S^1 \to M^3$, the normal bundle is trivial, and therefore has a countable number of framings. If the homology class represented by k is torsion, we can give a somewhat more geometric description of these framings. Recall that there is a non-singular linking form

$$\ell: tor H_1(M; \mathbf{Z}) \otimes tor H_1(M; \mathbf{Z}) \to \mathbf{Q}/\mathbf{Z}\,.$$

Let $x \in H_1(M; \mathbf{Z})$ be the class represented by k, and assume that x is torsion.

Lemma 4.1. *The framings on the normal bundle to k are in one–to–one correspondence with rational numbers q such that the class of q in $\mathbf{Q}/\mathbf{Z}$ is $\ell(x,x)$.*

Proof: We describe the correspondence. A framing on the normal bundle of k is equivalent to a choice of longitude in the torus which bounds a tubular neighborhood of k. Suppose $r \in \mathbf{Z}$ is chosen so that $r \cdot x = 0$ in $H_1(M;\mathbf{Z})$. Take r copies of the longitude in the boundary torus and let F be an oriented surface which bounds these r circles. Count the intersection of F and k with signs as usual. If one gets $p \in \mathbf{Z}$, then assign the rational number $\frac{p}{r}$ to this framing. It is a standard argument that $\frac{p}{r}$ is well–defined once the framing is fixed. It is also easy to see that $\frac{p}{r}$ mod $\mathbf{Z}$ is $\ell(x,x)$, and that if we choose a new framing which turns through t full right twists with respect to our original framing, then the new rational number that we get is $\frac{p}{r}+t$. ∎

A *Spin* structure on M gives a *Spin* structure on the normal bundle to k as follows. Restriction gives a *Spin* structure on the tangent bundle to S^1 plus the normal bundle. Choose the *Spin* structure on the normal bundle so that this *Spin* structure plus the one on S^1 which makes S^1 into a *Spin* boundary gives the restricted *Spin* structure.

Definition 4.2. We call the above framings *even.*

If x as above is torsion and M is spun, then the *Spin* structure picks out half of the rational numbers for which the longitude gives a framing compatible with the *Spin* structure on the normal bundle. Given one of these rational numbers, say q, the remaining ones are of the form $q+2t$ for t an integer. Hence we can define a new element in $\mathbf{Q}/\mathbf{Z}$, namely $\frac{q}{2}$. This gives a map

$$\gamma\colon tor H_1(M;\mathbf{Z}) \to \mathbf{Q}/\mathbf{Z}$$

which is a *quadratic enhancement of the linking form*:i.e.

$$\begin{aligned}\gamma(x+y) &= \gamma(x)+\gamma(y)+\ell(x,y)\\ \gamma(rx) &= r^2\cdot\gamma(x)\ \text{ for any integer } r\ .\end{aligned}$$

Suppose now that x is zero in $H_1(M;\mathbf{Z}/2\mathbf{Z})$, but not necessarily torsion in $H_1(M;\mathbf{Z})$. Then any *Spin* structure on M induces the same *Spin* structure in a neighborhood of k, and hence the notion of even framing is independent of *Spin* structure for these classes.

Theorem 4.3. *A knot k which is* mod 2 *trivial as above, bounds a surface which does not intersect k. This surface selects a longitude for the normal bundle to k, and this longitude represents an even framing.*

Proof: Let E be a tubular neighborhood for k with boundary T^2. (This T^2 is often called the *peripheral* torus.) We can select a basis for $H_1\left(T^2;\mathbf{Z}/2\mathbf{Z}\right)$ as follows. One element, the *meridian*, is the unique non-trivial element in the kernel of the map $H_1(T^2;\mathbf{Z}/2\mathbf{Z}) \to H_1(E;\mathbf{Z}/2\mathbf{Z})$. One calculates that the sequence $H_1(T^2;\mathbf{Z}/2\mathbf{Z}) \to H_1(M-k;\mathbf{Z}/2\mathbf{Z}) \to H_1(M;\mathbf{Z}/2\mathbf{Z})$ is exact, and that the image of $H_1(T^2;\mathbf{Z}/2\mathbf{Z})$ in $H_1(M-k;\mathbf{Z}/2\mathbf{Z})$ is 1-dimensional and generated by the meridian. Hence there is a unique non-trivial element, the mod 2 *longitude*, in the kernel of $H_1(T^2;\mathbf{Z}/2\mathbf{Z}) \to H_1(M-k;\mathbf{Z}/2\mathbf{Z})$. An even longitude for k is an element $\ell \in H_1(T^2;\mathbf{Z})$ which reduces in mod 2 homology to the mod 2 longitude.

Fix an even longitude, ℓ. It follows that there is an embedded surface, $F^2 \subset M$ such that $\partial F = k$. This surface can be chosen to intersect T^2 transversely in the even longitude. The southeast corner of Corollary 1.15 assigns a Pin^- structure to F. Restricted to k, the normal bundle to F in M is trivial, so the surface frames the normal bundle to k in M. Hence the *Spin* structure on M restricted to k is seen as the *Spin* structure on the circle coming from the restriction of the Pin^- structure on F plus the *Spin* structure on the normal bundle coming from the framing. We saw in the proof of Theorem 2.1 that, regardless of the Pin^- structure on F, the boundary circle receives the non–Lie structure. This is the definition of the even framing. ∎

Remarks 4.4.

(i) In S^3 with its unique *Spin* structure, the framing on k designated by an even number in the framed link calculus is an even framing in the above sense.

(ii) If the class x has odd order, then $\ell(x,x) = \frac{p}{r}$ with r odd. There are then two sorts of representatives in $\mathbf{Q}$ for $\ell(x,x)$: the p is even for half the representatives and odd for the other half. The framings that the *Spin* structure will call even are the ones with even numerator.

(iii) If we change the *Spin* structure on M by a class $\alpha \in H^1(M;\mathbf{Z}/2\mathbf{Z})$ the even framings on a circle change iff α evaluates non-trivially on the fundamental class of the circle.

(iv) If we attach a handle to a knot in a 3-manifold, M^3, we get a 4-manifold W with $H_2(W,M;\mathbf{Z}) = \mathbf{Z}$. If our knot in M^3 is torsion, we get a unique (up to sign) class $x \in H_2(W;\mathbf{Q})$ which hits our relative class. If we attach a handle with framing $q \in \mathbf{Q}$ from Lemma 4.1, then x intersects itself with a value of q. Hence the signature of W is sign (q), where sign $(q) = 1$ if $q > 0$; -1 if $q < 0$ and 0 if $q = 0$.

By Corollary 1.15 the surface F we used in the proof of Theorem 4.3 inherits a Pin^- structure from one on M. This suggests trying to define a knot invariant in this situation. Indeed, for knots in S^3, this is one way to define Robertello's Arf invariant, [R]. The situation in general is more complicated and needs results from §6, so we carry out the discussion in §8.

An invariant of a 3–manifold with a *Spin* structure is the μ–invariant. We discuss in Theorem 5.1 the classical result that $\Omega_3^{Spin} = 0$. It follows that any

3-manifold, M^3, is the boundary of a *Spin* 4–manifold, W.

Definition 4.5. The signature of W, reduced mod 16, is the *μ–invariant* of the manifold M with its *Spin* structure. It follows from Rochlin's theorem that $\mu(M)$ is well-defined once the *Spin* structure on M is fixed.

Remark. Some authors stick to $\mathbf{Z}/2\mathbf{Z}$ homology spheres so that there is a unique *Spin* structure and hence a μ invariant that depends only on the manifold.

We now turn to a geometric interpretation of some work of Turaev [Tu]. Intersection defines a symmetric trilinear product

$$\tau: H_2(M;\mathbf{Z}/2\mathbf{Z}) \times H_2(M;\mathbf{Z}/2\mathbf{Z}) \times H_2(M;\mathbf{Z}/2\mathbf{Z}) \rightarrow \mathbf{Z}/2\mathbf{Z}$$

We introduce a symmetric bilinear form

$$\lambda: H_2(M;\mathbf{Z}/2\mathbf{Z}) \times H_2(M;\mathbf{Z}/2\mathbf{Z}) \rightarrow \mathbf{Z}/2\mathbf{Z}$$

which is defined as follows. Let F_x and F_y be embedded surfaces representing two classes x and y in $H_2(M;\mathbf{Z}/2\mathbf{Z})$. To define $\lambda(x,y)$ put the two surfaces in general position. The intersection will be a collection of embedded circles. The normal bundle of each circle in M has a sub–line bundle, ξ_x, given by the inward normal to the surface F_x. Define $\lambda(x,y)$ to be the number of circles with non–trivial ξ_x.

Here is an equivalent definition of λ. Any codimension 1 submanifold of a manifold is mod 2 dual to a 1–dimensional cohomology class in the manifold. If this cohomology class is pulled–back to the submanifold, it becomes w_1 of the normal bundle to the embedding. Hence, if x^* and y^* are the Poincaré duals to x and y, $\lambda(x,y) = x^* \cup x^* \cup y^*[M]$, where $[M]$ is the fundamental class of the 3–manifold. This follows because $x^* \cup y^* \cap [M]$ is the homology class represented by the intersection circles, and to count the number with non–trivial ξx we just evaluate w_1 of the normal bundle on these circles. But $w_1 = x^*$ so we are done. We can also prove symmetry using this definition. Since M is orientable, $0 = w_1(M)x^*y^* = Sq^1(x^*y^*) = (x^*)^2y^* + x^*(y^*)^2$.

Yet another definition of λ is

$$\lambda(x,y) = \tau(x,x,y) \ .$$

Hence λ is symmetric and bilinear.

Given a *Spin* structure on M, we can enhance λ to a function

$$f: H_2(M;\mathbf{Z}/2\mathbf{Z}) \times H_2(M;\mathbf{Z}/2\mathbf{Z}) \rightarrow \mathbf{Z}/4\mathbf{Z} \ .$$

To begin, we define f on embedded surfaces F_x and F_y in M as above, but now use the *Spin* structure to put even framings on the intersection circles and then count the number of half twists in each ξ_x. (Since the collection of circles is embedded, there is no correction term needed to account for intersections.) Note if we defined ξ_y in the obvious manner and counted half twists in it instead of in ξ_x, we would get the same number, so f is symmetric.

Here is another description of $f(F_x, F_y)$. In M^3, F_y is dual to a cohomology class, $\alpha \in H^1(M; \mathbf{Z}/2\mathbf{Z})$, and we could take α and restrict it to F_x, getting $\alpha_x \in H^1(F_x; \mathbf{Z}/2\mathbf{Z})$. The Poincaré dual of α_x in F_x is just the class represented by our collection of circles, which we will denote by $\hat{y}$. Associated to our Pin^- structure on F_x, there is a quadratic enhancement ψ_x. Note

$$f(F_x, F_y) = \psi_x(\hat{y}) \ . \tag{4.6}$$

In particular, note $f(F_x, F_y)$ only depends on the homology class of F_y, and hence by symmetry also only on the homology class of F_x.

Once we see the pairing is well–defined, it is easy to see that $f(x, 0) = f(0, x) = 0$ for all $x \in H_2(M; \mathbf{Z}/2\mathbf{Z})$. We have lost bilinearity and gained

$$f(x, y+z) = f(x, y) + f(x, z) + 2\tau(x, y, z) \ . \tag{4.7}$$

Proof: With notation as above, we apply formula 4.6. We need to show $\psi_x(\widehat{y+x}) = \psi_x(\hat{y}) + \psi_x(\hat{z}) + 2\tau(x, y, z)$, which is just the quadratic enhancement property of ψ_x and the identification of $\hat{y}\bullet\hat{z}$ in F_x with $\tau(x, y, z)$. ∎

If we change the *Spin* structure on M by $\alpha \in H^1(M; \mathbf{Z}/2\mathbf{Z})$, then we change f as follows. Let f_α denote the new pairing and let $a \in H_2(M; \mathbf{Z}/2\mathbf{Z})$ be the Poincaré dual to α. Then

$$f_\alpha(x, y) = f(x, y) + 2\tau(x, y, a) \ ,$$

or

$$f_\alpha(x, y) = f(x, y + a) - f(x, a) \ .$$

Proof: We prove the first formula. Using 4.6 we see that the first formula is equivalent to $\psi_\alpha(\hat{y}) = \psi(\hat{y}) + 2\tau(x, y, a)$, which follows easily from formula 3.3. ∎

Finally, we have a function

$$\beta\colon H_2(M; \mathbf{Z}/2\mathbf{Z}) \to \mathbf{Z}/8\mathbf{Z} \ . \tag{4.8}$$

We define β by taking an embedded surface representing x, using the *Spin* structure on M to get a Pin^- structure on F_x, taking the underlying Pin^- bordism class, and using our explicit identification of this group with $\mathbf{Z}/8\mathbf{Z}$.

We need to see why this is independent of the choice of embedded surface. Given two such surfaces, there is a bordism in $M \times [0,1]$ between them. Let $W \subset M \times [0,1]$ be a 3–manifold with the two boundary components representing the same element in $H_2(M; \mathbf{Z}/2\mathbf{Z})$. Since $M \times [0,1]$ is spun, we get a Pin^- structure on W which is our given Pin^- structure at the two ends. Since Brown's $\mathbf{Z}/8\mathbf{Z}$ is a Pin^- bordism invariant, we are done. It further follows that $\beta(0) = 0$.

Reduced mod 2 $\beta(x)$ is just the mod 2 Euler class of an embedded surface representing x, and hence β is additive mod 2. We have

$$\beta(x+y) = \beta(x) + \beta(y) + 2f(x,y) \quad . \tag{4.9}$$

which we will prove in a minute. It follows that $f(x,x) = -\beta(x)$ reduced mod 4. Note that, mod 4, $\beta(x+y) = \beta(x) + \beta(y) + 2\tau(x,x,y)$.

How does β change when we change the *Spin* structure by $\alpha \in H^1(M;\mathbf{Z}/2\mathbf{Z})$? The principle is easy. Given a surface, F, restrict α to F and consider it to be a change in Pin^- structure on F. Compute the Brown invariant for this new Pin^- structure, and this is the value of the new β on F. It follows from Lemma 3.7 that

$$\beta_\alpha(x) = \beta(x) + 2f(x,a) \tag{4.10}$$

with notation as above.

Given the theorem below, we now prove formula 4.9. From this theorem we get: $u - u_\alpha = 2\beta(a)$ and $u - u_{\alpha_1} = 2\beta(a_1)$. Also $u_\alpha - u_{\alpha_1} = 2\beta_\alpha(a_1 - a)$. Hence $\beta_\alpha(a_1 - a) = \beta(a_1) - \beta(a)$. Set $a_1 = x + a$ and use formula 4.10. ∎

The main result concerning β is

Theorem 4.11. *Let M be a spun 3–manifold with resulting function β and μ–invariant u in $\mathbf{Z}/16\mathbf{Z}$. Let $\alpha \in H^1(M;\mathbf{Z}/2\mathbf{Z})$ be used to change the Spin structure, and let u_α be the new μ–invariant. Then*

$$u - u_\alpha = 2\beta(a) \qquad \pmod{16}$$

where $a \in H_2(M;\mathbf{Z}/2\mathbf{Z})$ is the Poincaré dual to α.

Proof: The proof is just the Guillou–Marin formula, [G–M, Theoreme, p. 98], or our discussion of it in §6, 6.4. On $M \times [0,1]$ put the original *Spin* structure on $M \times 0$ and put the altered one on $M \times 1$. We can cap this off to a closed 4–manifold by adding *Spin* manifolds that the two copies of M bound to either end. The resulting 4–manifold has index $u_\alpha - u$. Let F be a surface in M representing a. Then $F \times 1/2$ is a dual to w_2 for the 4–manifold. Since F is in a product, $F \bullet F = 0$ and the enhancement used in the Guillou–Marin formula is the same as the one we put on F to calculate β. By formula 6.4, $u - u_\alpha = 2\beta(a)$. ∎

As a corollary we get a result of Turaev, [Tu]

Corollary 4.12. *The quadratic enhancement of the linking form gives the μ–invariant* mod 8 *via the Milgram Gauss sum formula.*

Proof: This was proved in [Ta] for rational homology spheres. Pick a basis for the torsion free part of H_1 and do surgery on this basis. The resulting bordism, W, has signature 0; both boundary components have isomorphic torsion subgroups of H_1; and the top boundary component has no torsion free part. Put a *Spin* structure on the bordism, which puts a *Spin* structure at both ends. The two enhancements on the linking forms are equal, and they stay equal if we change both *Spin* structures by an element in $H^1(W;\mathbf{Z}/2\mathbf{Z})$. Any *Spin* structure on M can be obtained from our initial one by acting on it by an element of the form $x+y$, where x comes from $H^1(W;\mathbf{Z}/2\mathbf{Z})$ and y comes from $H^1(M;\mathbf{Z})$. But acting by this second sort of element does not change the mod 8 μ–invariant or the quadratic enhancement of the linking form. ∎

§5. Geometric calculations of $\Omega_{3,4}^{Pin^\pm}$.

We begin this section with a calculation for the 3–dimensional *Spin*, Pin^- and Pin^+ bordism groups.

Theorem 5.1. *$\Omega_3^{Spin} \cong 0$; $\Omega_3^{Pin^-} \cong 0$ and $[\cap w_1]\colon \Omega_3^{Pin^+} \to \Omega_2^{Spin} \cong \mathbf{Z}/2\mathbf{Z}$ is an isomorphism.*

Proof: The *Spin* bordism result is classical: [ABP1], [Ka] or [Ki].

Given a non–orientable $Pin^\pm$ manifold M^3, we will try to find a $Pin^\pm$ bordism to an orientable manifold which then $Pin^\pm$ bounds by the *Spin* case. The dual to $w_1(M)$ is an orientable surface F by Proposition 2.3. The first step is to reduce to the case when F has trivial normal bundle. If not, consider F intersected transversely with itself. It can be arranged that this is a single circle C, which is dual in F to $w_1(M)$ pulled back to F. The normal bundle to C in M is $\nu_{F\subset M}|_C \oplus \nu_{F\subset M}|_C$ which is also $\nu_{C\subset F} \oplus \nu_{C\subset F}$ which is trivialized. Hence the $Pin^\pm$ structure on M induces a $Pin^\mp$ structure on C. Suppose C with this structure bounds Y^2; let E denote the total space of $\zeta \oplus \zeta$ over Y, where ζ is the determinant line bundle for Y. Note that inside ∂E there is a copy of $(\partial Y) \times B^2$, and E has a $Pin^\pm$ structure extending the one on $(\partial Y) \times B^2$. We can form $M \times [0,1] \cup E$ by gluing $(\partial Y^2) \times B^2$ to $C \times B^2 \times 1$ where $C \times B^2$ is the trivialized disk bundle to C above. Clearly the $Pin^\pm$ structure extends across the bordism, and the "top" is a new $Pin^\pm$ manifold M_1 with a new dual surface F_1 with trivial normal bundle.

In the Pin^- case, C has a Pin^+ structure which bounds ($\Omega_1^{Pin^+} = 0$, Theorem 2.1) so we have achieved the (M_1, F_1) case. In the Pin^+ case an argument is needed to see that we never get C representing the non–zero element in $\Omega_1^{Pin^-} = \mathbf{Z}/2\mathbf{Z}$, i.e. C does not get the Lie group *Spin* structure.

To show this, let V be a dual to w_1 and let E be a tubular neighborhood of V. By the discussion just before Lemma 2.7, since E as a Pin^+ structure, there is an inherited *Spin* structure on V (in fact there are two which differ by the action of $x \in H^1(V;\mathbf{Z}/2\mathbf{Z})$, where x denotes the restriction of w_1 to V). Note x also describes the double cover $\partial E \to V$. The boundary, ∂E, also inherits a Pin^+ structure and we saw, Lemma 2.7, that, if we orient ∂E and V so that the covering map is degree 1, the *Spin* structure on ∂E is the same as the one induced by the covering map. The *Spin* structure on ∂E bounds the *Spin* manifold which is the closure of $M - E$, so if C is the dual to x and q is the quadratic enhancement on $H_1(V;\mathbf{Z}/2\mathbf{Z})$, $q(C) = 0$ by Theorem 3.12. Recall that the normal bundle to V in M, when restricted to C is trivial. Hence the framing on C as a circle in V is the same as the Pin^- structure on C as V intersect V in a Pin^+ manifold. Hence C has the non-Lie group *Spin* structure and hence represents 0 in Ω_1^{Spin}.

Hence we may now assume that F has trivial normal bundle in M. Therefore F inherits a $Pin^\pm$ structure from the one on M, and hence, after choosing an orientation, F has a *Spin* structure. If the *Spin* structure on F is a boundary then it is easy as above to construct a $Pin^\pm$ bordism to an oriented manifold. In the Pin^+ case we are entitled to assume that the surface bounds because that is what the invariant $[\cap w_1]$ is measuring. In the Pin^- case, the Klein bottle $\times S^1$ with the Lie group framing is an example for which the F has the non-bounding *Spin* structure. But if we add this manifold to our original M, for the new manifold, F will bound and we are done.

We have now proved that $[\cap w_1]$ is injective in the Pin^+ case and that $\Omega_3^{Pin^-}$ is generated by $K \times S^1$, where K is the Klein bottle and the Pin^- structure comes from some structure on the surface and the Lie group *Spin* structure on S^1. In some Pin^- structures, K bounds and hence so does $K \times S^1$. In the others, K is Pin^- bordant to two copies of $\mathbf{RP}^2$, so $K \times S^1$ is bordant to two copies of $\mathbf{RP}^2 \times S^1$. Hence, if we can prove that $[\cap w_1]$ is onto and that $\mathbf{RP}^2 \times S^1$ bounds, we are done.

If we take the generator of $\Omega_2^{Pin^+}$ and cross it with S^1 with the Lie group *Spin* structure, we get a 3–manifold with $[\cap w_1]$ being the 2–torus with Lie group *Spin* structure so by Proposition 3.8, $[\cap w_1]$ is onto.

Consider $\mathbf{RP}^2$ in $\mathbf{RP}^4$: it is the dual to $w_1^2 + w_2$ so there is a Pin^- structure on $\mathbf{RP}^4 - \mathbf{RP}^2$ which restricts to the Lie group structure on the normal circle to $\mathbf{RP}^2$. An easy calculation of Stiefel–Whitney classes shows that the normal bundle ν of $\mathbf{RP}^2$ in $\mathbf{RP}^4$ is orientable but $w_2(\nu) \neq 0$. So we take the pairwise connected sum $(\mathbf{RP}^4,\ \mathbf{RP}^2)\#(\mathbf{CP}^2,\ \mathbf{CP}^1)$ and then the normal bundle of $\mathbf{RP}^2 = \mathbf{RP}^2\#\mathbf{CP}^1$ in $\mathbf{RP}^4\#\mathbf{CP}^2$ has $w_1 = w_2 = 0$. For a bundle over $\mathbf{RP}^2$ this means that the bundle is trivial, so its normal circle bundle is $\mathbf{RP}^2 \times S^1$. The two Pin^- structures on $\mathbf{RP}^4\#\mathbf{CP}^2 - \mathbf{RP}^2$ bound two Pin^- structures on $\mathbf{RP}^2 \times S^1$ which have the Lie group structure on S^1. Since this is all the Pin^- structures that there are with the Lie group *Spin* structure on the S^1, we are done. ∎

Next we turn to the 4–dimensional case. The result is

Theorem 5.2. *The group* $\Omega_4^{Spin} \cong \mathbf{Z}$ *generated by the Kummer surface;* $\Omega_4^{Pin^-} = 0$*; and the group* $\Omega_4^{Pin^+} \cong \mathbf{Z}/16\mathbf{Z}$ *generated by* $\mathbf{RP}^4$.

Proof: The *Spin* result may be found in [Ki, p. 64, Corollary]. Our first lemma determines the image of Ω_4^{Spin} in the $Pin^\pm$ bordism groups.

Lemma 5.3. *The Kummer surface bounds a Pin^- manifold hence so does any 4–dimensional Spin manifold. Twice the Kummer surface bounds a Pin^+ manifold, but the Kummer surface itself does not. Hence a 4–dimensional Spin manifold Pin^+ bounds iff its signature is divisible by 32.*

Proof: The Enriques surface, E, [Ha], is a complex surface with $\pi_1(E) \cong \mathbf{Z}/2\mathbf{Z}$ with $w_2(E) \neq 0$. Habegger shows that $H^2(M;\mathbf{Z}) \cong \mathbf{Z}^{10} \oplus \mathbf{Z}/2\mathbf{Z}$ and $w_2(M)$ is the image of the non-zero torsion class in $H^2(M;\mathbf{Z})$, see paragraph 2 after the Proposition on p. 23 of [Ha]. If $y \in H^1(E;\mathbf{Z}/2\mathbf{Z})$ is a generator, then from the universal coefficient theorem, $y^2 = w_2(W)$. If L is the total space of the line bundle over E with $w_1 = y$, then it is easy to calculate that L is Pin^- (but not Pin^+), and ∂E is the Kummer surface. This proves the Kummer surface bounds a Pin^- manifold. Since $\Omega_4^{Spin} \cong \mathbf{Z}$ generated by the Kummer surface, this proves any *Spin* 4–manifold bounds as a Pin^- manifold.

Let M^4 is a *Spin* manifold and let W^5 be a Pin^- manifold with $\partial W = M$ as Pin^- manifolds. Consider the obstruction to putting a Pin^+ structure on W extending the one on M^4. The obstruction is $w_2(W) = w_1^2(W)$, so the dual class is represented by a 3–manifold formed as the intersection to a dual to w_1 pushed off itself. As usual, this 3–manifold has a natural Pin^+ structure and it is easy to see that we get a well–defined element in $\Omega_3^{Pin^+} \cong \mathbf{Z}/2\mathbf{Z}$. If this element is 0, then we can glue on the trivializing bordism and extend its normal bundle to get a new Pin^- manifold W_1 which still bounds M and has no obstruction to extending the *Spin* structure on the boundary to a Pin^+ structure on the interior. Hence, if our element in $\Omega_3^{Pin^+}$ is 0, M bounds. From this it is easy to see that twice the Kummer surface bounds. Hence any 4–dimensional *Spin* manifold with index divisible by 32 bounds a Pin^+ manifold.

Suppose that W is a Pin^+ manifold with $\partial W = M$ orientable. Let $V \subset W$ be a dual to w_1 contained in the interior of W. Let E be a tubular neighborhood of V with boundary ∂E. As usual, ∂E is orientable and the covering translation is orientation preserving. Since V is orientable with a normal line bundle, if we fix an orientation, *Spin* structures on V correspond to Pin^+ structures on E. Since W is a Pin^+ manifold, E has an induced Pin^+ structure and V acquires an induced *Spin* structure. The bordism between M and ∂E is an oriented Pin^+ bordism, so M and ∂E have the same signature. But ∂E is the double cover of V so has signature twice the signature of V. Since V is *Spin*, the signature of V is divisible by 16, so the signature of M is divisible by 32. This shows that the Kummer surface does not bound a Pin^+ manifold and indeed that any 4–dimensional *Spin* manifold of index congruent to 16 mod 32 does not bound a Pin^+ manifold. ∎

Since $\Omega_4^{Spin} \cong \mathbf{Z}$ generated by the Kummer surface this lemma calculates the image of Ω_4^{Spin} in $\Omega_4^{Pin^\pm}$ and our next goal is to try to produce a $Pin^\pm$ bordism from any $Pin^\pm$ manifold to an orientable one.

To this end let M be a 4–manifold with V^3 a dual to w_1. Consider the dual to w_1 intersected with itself. It is a surface $F \subset M$ and the normal bundle is two copies of the same line bundle. Indeed, the transversality condition gives an isomorphism between the two bundles. This line bundle is also abstractly isomorphic to the determinant line bundle for F. A $Pin^\pm$ structure on F gives rise to a $Pin^\mp$ structure on the total space of the normal bundle of F in M by Lemma 1.7. Hence we can use the $Pin^\pm$ structure on M to put a $Pin^\mp$ structure on F and it is not hard to check that we get a homomorphism $\Omega_4^{Pin^\pm} \to \Omega_2^{Pin^\mp}$. If F bounds in this structure, one can easily see a $Pin^\pm$ bordism to an new 4–manifold M_1 in which the dual to w_1 has trivial normal bundle. This puts a $Pin^\pm$ structure on V_1. By orienting V_1 we get a $Spin$ manifold and if V_1 bounds in this $Spin$ structure, M_1 $Pin^\pm$ bounds an orientable manifold.

Consider the Pin^- case. Any element in the kernel of the map $[\cap w_1^2]\colon \Omega_4^{Pin^-} \to \Omega_2^{Pin^+}$ is Pin^- bordant to a Pin^- manifold whose dual to w_1, say V, has trivial normal bundle. Orienting this normal bundle gives a Pin^- structure on V, and since $\Omega_3^{Spin} = 0$, we can further Pin^- bord our element to an orientable representative. It then follows from Lemma 5.3 that the map $[\cap w_1^2]$ is injective.

To show that this map is trivial, which proves $\Omega_4^{Pin^-} = 0$, proceed as follows. Let $V \subset M$ be a dual to $w_1(M)$ and let F^2 denote the transverse intersection of V with itself. Since the normal bundle to F in M is 2 copies of the determinant line bundle for F, F acquires a Pin^+ structure from the Pin^- structure on M. Let $E \subset V$ be a tubular neighborhood for F in V. Theorem 2.9 applies to this situation to show that the Pin^+ structure on ∂E induced by the double cover map $\partial E \to F$ is the same as the Pin^+ structure induced on $\partial E \subset M$ from the fact that its normal bundle is exhibited as the sum of 2 copies of its determinant line bundle. Since the normal bundle to V in M is trivial on $V - F$, $V - F$ has a $Spin$ structure which restricts to the given one on ∂E. By Lemma 3.13, the oriented cover map $\Omega_2^{Pin^+} \to \Omega_2^{Spin}$ is an isomorphism, so F is a Pin^+ boundary, which finishes the Pin^- case.

So consider the Pin^+ case. This time our homomorphism goes from $\Omega_4^{Pin^+}$ to $\Omega_2^{Pin^-} \cong \mathbf{Z}/8\mathbf{Z}$ and the example of $\mathbf{RP}^4$ shows that it is onto. Just as in the Pin^- case, any element in the kernel of this homomorphism is Pin^+ bordant to an orientable manifold. This together with Lemma 5.3 shows that $0 \to \mathbf{Z}/2\mathbf{Z} \to \Omega_4^{Pin^+} \to \mathbf{Z}/8\mathbf{Z} \to 0$ is exact.

To settle the extension requires more work. Given a Pin^+ structure on a 4–manifold M, we can choose a dual to w_1, say $V \subset M$, and an orientation on $M - V$ which does not extend across any component of V. We can consider the bordism group of such structures, say G_4. There is an epimorphism $G_4 \to \Omega_4^{Pin^+}$ defined by just forgetting the dual to w_1 and the orientation. There is another homomorphism $G_4 \to \mathbf{Q}/32\mathbf{Z}$ defined as follows. Let E be a tubular neighborhood of V with boundary ∂E. The covering translation on ∂E is orientation preserving, so V is also oriented. The normal bundle to ∂E in M is a trivial line bundle, oriented by inward normal last, where inward is with respect to the associated disk bundle. Hence ∂E acquires a *Spin* structure, and hence a μ invariant in $\mathbf{Z}/16\mathbf{Z}$. The manifold ∂E is a 3–manifold with an orientation preserving free involution on it, hence there is an associated Atiyah–Singer α invariant, $\alpha(\partial E) \in \mathbf{Q}$. Define $\psi(M, V) = \sigma(M - \text{int } V) + \alpha(\partial E) - 2\mu(V) \in \mathbf{Q}/32\mathbf{Z}$. It is not hard to check that ψ depends only on the class of (M, V) in G_4 and defines a homomorphism. We can make choices so that $\psi(\mathbf{RP}^4, \mathbf{RP}^3) = +2$. Hence $\psi(8(\mathbf{RP}^4, \mathbf{RP}^3)) = 16$ with these choices. The Pin^+ bordism of 8 copies of $\mathbf{RP}^4$ to an oriented manifold is seen to extend to a bordism preserving the dual to w_1 and orientation data. This oriented, hence *Spin* manifold has index congruent to 16 mod 32, and so we have constructed a Pin^+ bordism (with some extra structure which we ignore) from 8 copies of $\mathbf{RP}^4$ to a *Spin* manifold which is Pin^+ bordant to the Kummer surface. This shows $\Omega_4^{Pin^+} \cong \mathbf{Z}/16\mathbf{Z}$. ∎

§6. 4–dimensional characteristic bordism.

The purpose of this section is to study the relations between 4–manifolds and embedded surfaces dual to $w_2 + w_1^2$.

Definition 6.1. A pair (M, F) with the embedding of F in M proper and the boundary of M intersecting F precisely in the boundary of F is called a *characteristic pair* if F is dual to $w_2 + w_1^2$. A characteristic pair is called *characterized* provided we have fixed a Pin^- structure on $M - F$ which does not extend across any component of F. The characterizations of a characteristic pair are in one to one correspondence with $H^1(M; \mathbf{Z}/2\mathbf{Z})$.

We begin by discussing the oriented case.

Lemma 6.2. *Let M be an oriented manifold with a codimension 2 submanifold F which is dual to w_2. There exists a function*

$$\mathcal{C}har(M, F) \to \mathcal{P}in^-(F) \ .$$

The group $H^1(M; \mathbf{Z}/2\mathbf{Z})$ acts on $\mathcal{C}har(M, F)$, the group $H^1(F; \mathbf{Z}/2\mathbf{Z})$ acts on $\mathcal{P}in^-(F)$ and the map is equivariant with respect to the map induced on $H^1(;\mathbf{Z}/2\mathbf{Z})$ by the inclusion $F \subset M$.

Remark. Later in this section we will define this function in a more general situation.

Proof: There is an obvious restriction map from characteristic structures on (M, F) to those on (E, F), where E is the total space of the normal bundle to F in M, denoted ν. Hence it suffices to do the case $M = E$. In this case we expect our function to be a bijection. After restricting to the case $M = E$ it is no further restriction to assume that F is connected since we may work one component at a time.

We begin with the case that F has the homotopy type of a circle. In this case ν has a section, so choose one and write $\nu = \lambda \oplus \epsilon^1$. Orient ϵ^1 and use it to embed F in ∂E. The normal bundle to ∂E in E is oriented; E is oriented; so ∂E is oriented. The normal bundle to the embedding of F in ∂E is λ so the orientation on E plus the orientation of ϵ^1 pick out a preferred isomorphism between λ and $\det T_F$. From Corollary 1.15, there is a Pin^- structure on F induced from the one on ∂E.

We want to see that this Pin^- structure is independent of the section we chose. It is not difficult to work out the effect of reorienting the section: there is none.

Suppose the bundle is trivial. We divide into two cases depending on the dimension of E. In the 1–dimensional case, we may proceed as follows. The manifold F is a circle and since the bundle has oriented total space, it must be trivial. Hence $\partial E = T^2$ and $H_1(T^2; \mathbf{Z}/2\mathbf{Z})$ has one preferred generator, the image of the fibre, otherwise known as a meridian, denoted m. Let x denote another generator. Since the *Spin* structure is not to extend over the disk, the enhancement associated to the *Spin* structure on T^2, say q, satisfies $q(m) = 2$. The *Spin* structure on the embedded base is determined by q of the image, which is either x or $x + m$. Check $q(x) = q(x + m)$.

In the higher dimensional case, there is an S^1 embedded in F and the normal bundle to this embedding is trivial. Over the S^1 in F there is an embedded T^2 in ∂E and the bundle projection, p, identifies the normal bundle to T^2 in ∂E with the normal bundle to S^1 in F. Fix a *Spin* structure on one of these normal bundles and use p to put a *Spin* structure on the other. The *Spin* structure on ∂E restricts to one on T^2 and it is not hard to check that the Pin^- structure we want to put on F using the section is determined by using the section over S^1 and checking what happens in T^2. We saw this was independent of section so we are done with the trivial case.

Now we turn to the non–trivial case, still assuming that F is the total space of a bundle over S^1. The minimal dimension for such an F is 2 since the bundle, ν, is non–trivial. In this case F is just a Möbius band. Since E is oriented, the bundle we have over F is isomorphic to $\det \nu \oplus \epsilon^1$. Sitting over our copy of S^1 in F is the Klein bottle, K^2, and the normal bundle to K^2 in ∂E is just the pull–back of ν. One can sort out orientations and check that there is an induced Pin^- structure on K^2 so that the Pin^- structure that we want to put on F is determined by the enhancement of the section applied to S^1 as a longitude of K^2. This calculation is just like the torus case. In the higher dimensional case, ν is a non–trivial line bundle plus a trivial bundle so we can reduce to the dimension 2 case just as above.

Now we turn to the case of a general F.

Since we have done the circle case, we may as well assume that the dimension of F is at least 2. If the dimension of F is 2, then we can find a section of our bundle over $F - pt$. The embedding of $F - pt$ in ∂E gives a Pin^- structure on $F - pt$ and this extends uniquely to a Pin^- structure on F. This argument even works if F

has a boundary and we take as the function on the boundary the function we have already defined. Now if we restrict this structure on F to a neighborhood of an embedded circle, we get our previous structure. Since this structure is independent of the section, the structure on all of F is also independent of the section since Pin^- structures can be detected by restricting to circles.

The higher dimensional case is a bit more complicated. We can define our function by choosing a set of disjointly embedded circles and taking a tubular neighborhood to get U, with $H_1(U;\mathbf{Z}/2\mathbf{Z}) \to H_1(F;\mathbf{Z}/2\mathbf{Z})$ an isomorphism. We then use our initial results to put a Pin^- structure on U and then extend it uniquely to all of F. Now let V be a tubular neighborhood of a circle in F. We can restrict the Pin^- structure on F to V, or we can use our "choose a section, embed in ∂E and induce" technique. There is an embedded surface, W^2, in F which has the core circle for V as one boundary component and some of the cores of U as the others. Let X be a tubular neighborhood of W in F. The bundle restricted to X has a section so we can induce a Pin^- structure on X using the section. This shows that the two Pin^- structures defined above on V agree. It is not hard from this result to see that the Pin^- structure on F is independent of the choice of U. ∎

Remarks. Notice that the proof shows that the Pin^- structure on a codimension 0 subset of F, say X, only depends on the Pin^- structure on the circle bundle lying over X. It is not hard to check that our function commutes with taking boundary, we get a well-defined homomorphism, β, from the rth Guillou–Marin bordism group to $\Omega_{r-2}^{Pin^-}$.

Theorem 6.3. *Let M^4 be an oriented 4–manifold, and suppose we have a characteristic structure on the pair (M,F). The following formula holds:*

$$2\cdot\beta(F) = F\bullet F - \mathrm{sign}(M) \qquad (\mathrm{mod}\ 16) \tag{6.4}$$

where the Pin^- structure on F is the one induced by the characteristic structure on (M,F) via 6.2.

Proof: By the Guillou–Marin calculation, their bordism group in dimension 4 is $\mathbf{Z}\oplus\mathbf{Z}$, generated by $(S^4,\mathbf{RP}^2)$ and $(\mathbf{CP}^2,S^2)$. The formula is trivial to verify for $(\mathbf{CP}^2,S^2)$. For $(S^4,\mathbf{RP}^2)$ we must verify that $\mathbf{RP}^2\bullet\mathbf{RP}^2 = 2$ implies that the resulting q is 1 on the generator. Now $\mathbf{RP}^2$ has two sorts of embeddings in S^4. There is a "right–handed" one, which has $\mathbf{RP}^2\bullet\mathbf{RP}^2 = 2$, and a "left–hand" one which has $\mathbf{RP}^2\bullet\mathbf{RP}^2 = -2$. The "right–handed" one can be constructed by taking a 'right–handed" Möbius strip in the equatorial S^3 and capping it off with a ball in the northern hemisphere. For our vector field, use the north–pointing normal. The "even" framing on the bundle to ν_k, the core of the Möbius band, is the one given by the 0–framing in S^3. Hence we may count half twists in S^3, where the right–hand Möbius band half twists once. ∎

It would be nice to check that the Pin^- structure we put on the characterized surface agrees with those of Guillou–Marin and Freedman–Kirby. For the Freedman–Kirby case we take an embedded curve k in F and cap it off by an orientable surface, B, in M. We start B off in the same direction as our normal vector field, so then the normal bundle to B in M, when restricted to the boundary circle, will be the 2–plane bundle around k we are to consider. The Guillou–Marin case is similar except that B need not be orientable. Since B is a punctured surface, the normal bundle to B in M splits off a trivial line bundle and so is a trivial bundle plus the determinant line bundle for the tangent bundle. Having chosen one section, the others are classified by $H^1(B; \mathbf{Z}^{w_1})$, where $\mathbf{Z}^{w_1}$ denotes $\mathbf{Z}$ coefficients twisted by w_1 of the normal bundle. When restricted to the boundary circle, this gives a well–defined "even" framing of the normal bundle.

If B does not intersect F except along ∂B, Theorem 4.3 shows that the framing on ∂B is the even one in the sense of Definition 4.2. We can assume in general that B intersects F transversally away from ∂B. The surface $\hat{B} = B - \amalg D^2$ lies in $M - F$ and each circle from the transverse intersection has the non–bounding *Spin* structure. Hence, in general, the framing on ∂B is even iff the mod 2 intersection number of F and B is even. Moreover, the number of half right twists mod 4 is just the obstruction to extending the section given by the normal to k in F over all of B. This shows that our enhancement and those of Freedman–Kirby and Guillou–Marin agree when both are defined.

The enhancement above is defined more generally since we do not need the membranes to select the Pin^- structure and hence do not need the condition that $H_1(F; \mathbf{Z}/2\mathbf{Z}) \to H_1(M; \mathbf{Z}/2\mathbf{Z})$ should be 0. One nice application of this is to compute the μ–invariant of circle bundles over surfaces when the associated disk bundle is orientable.

Any $O(2)$–bundle, η, over a 2 complex, X, is determined by $w_1(\eta)$ and the Euler class, $\chi(\eta) \in H^2(X; \mathbf{Z}^{w_1})$, where $\mathbf{Z}^{w_1}$ denotes $\mathbf{Z}$ coefficients twisted by $w_1(\eta)$. In our case, X is a surface which we will denote by F; the bundle η has the same w_1 as the surface; and the Euler class is in $H^2(F; \mathbf{Z}^{w_1}) \cong \mathbf{Z}$. Let $S(\eta)$ denote the circle bundle. One way to fix the isomorphism is to orient the total space of η and then $F \bullet F = \chi(\eta)$. The signature of the disk bundle is also easy to compute. We denote it by $\sigma(\eta)$ since we will see it depends only on η; indeed it can be computed from $w_1(\eta)$ and $\chi(\eta)$. If $w_1(\eta) = 0$ then $\sigma(\eta) = \text{sign } \chi(\eta)$ (± 1 or 0 depending on $\chi(\eta)$): if $w_1(\eta) \neq 0$ then $\sigma(\eta) = 0$. By Lemma 6.2, *Spin* structures on $S(\eta)$ which do not extend across the disk bundle are in 1–1 correspondence with Pin^- structures on F.

Theorem 6.5. *With notation as above fix a Spin structure on $S(\eta)$. Let $b(F) = 0$ if this structure extends across the disc bundle and let $b(F) = \beta(F)$ if it does not and the Pin^- structure on F is induced via the function in Lemma 6.2. We have*

$$\mu(S(\eta)) = \sigma(\eta) - \chi(\eta) + 2 \cdot b(F) \qquad \pmod{16} \quad . \tag{6.6}$$

Proof: The result follows easily from 6.4. ∎

We want to describe a homomorphism from various characteristic bordism groups into the Pin^- bordism group in two dimensions less. Roughly the homomorphism is described as follows. We have a characteristic pair (M, F) and we will see that, with certain hypotheses, F is a Pin^- manifold. We then use the characterization of the pair to pick out a Pin^- structure on F. The homomorphism then just sends (M, F) to the Pin^- bordism class of F.

To describe our hypotheses, consider the following commutative square

$$\begin{array}{ccc} F & \longrightarrow & B_{O(2)} \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathrm{TO}(2) \end{array}$$

Let $U \in H^2(\mathrm{TO}(2); \mathbf{Z}/2\mathbf{Z})$ denote the Thom class and recall that U pulls back to w_2 in $H^2(B_{O(2)}; \mathbf{Z}/2\mathbf{Z})$. The 2–plane bundle classified by ν is just the normal bundle to the embedding $i\colon F \subset M$, and $f^*(U) \in H^2(M; \mathbf{Z}/2\mathbf{Z})$ is the class dual to F. Let a denote the class dual to F. Then we see that $i^*(a) = w_2(\nu_{F\subset M})$, where $\nu_{F\subset M}$ is the normal bundle to the embedding. Let us apply this last equation to our characteristic situation. The class a is $w_2(M) + w_1^2(M)$ and we have the bundle equation $i^*(T_M) = T_F \oplus \nu_{F\subset M}$. Now $i^*w_1(M) = w_1(F) + w_1(\nu)$ and $i^*w_2(M) = w_2(F) + w_2(\nu) + w_1(F) \cdot w_1(\nu)$. Hence $i^*(w_2(M) + w_1^2(M)) = w_2(F) + w_2(\nu) + w_1(F) \cdot w_1(\nu) + w_1^2(F) + w_1^2(\nu)$ and using our equation for $w_2(\nu)$ we see that $w_2(F) + w_1^2(F) = w_1(\nu) \cdot i^*w_1(M)$. Hence F is Pin^- iff the right hand product vanishes or

Lemma 6.7. *The surface F has a Pin^- structure iff*

$$(w_1(F) + w_1(\eta)) \cup w_1(\eta) = 0 \ .$$

To study $w_1(\nu) \cdot i^*w_1(M)$ we may equally study $w_1(\nu) \cap (i^*w_1(M) \cap [F, \partial F])$. The term $i^*w_1(M) \cap [F, \partial F]$ can be described as the image of the fundamental class of the manifold obtained by transversally intersecting F and a manifold V in M dual to w_1. Hence, the product $w_1(\nu) \cdot i^*w_1(M)$ vanishes if the normal bundle to $F \cap V \subset V$ is orientable. This suggests studying the following situation.

Definition 6.8 . Let M be a manifold with a proper, codimension 2 submanifold F (proper means that $\partial M \cap F = \partial F$ and that every compact set in M meets F in a compact set). A *characteristic structure* on the pair (M, F) is a collection consisting of

a) a proper submanifold V dual to $w_1(M)$ which intersects F transversely

b) an orientation on $M - V$ which does not extend across any component of V

c) a Pin^- structure on $M - F$ that does not extend across any component of F (so F is dual to $w_2 + w_1^2$)

d) an orientation for the normal bundle of $V \cap F$ in V.

Let $Char^-(M, F)$ be the set of characteristic structures on (M, F).

The next goal of this section is to prove a "reduction of structure " result, the Pin^- Structure Correspondence Theorem.

Theorem 6.9. *There exists a function*

$$\Psi: Char^-(M, F) \to \mathcal{P}in^-(F)$$

which is natural in the following sense. If we change the Pin^- structure on $M - F$ which does not extend across any component of F by acting on it with $a \in H^1(M; \mathbf{Z}/2\mathbf{Z})$, then we change Ψ of the structure by acting on it with $i^*(a) \in H^1(F; \mathbf{Z}/2\mathbf{Z})$, where $i: F \subset M$ is the inclusion. If X denotes a collection of components of $F \cap V$, then the dual to X is a class in $x \in H^1(F; \mathbf{Z}/2\mathbf{Z})$. If we switch the orientation to the normal bundle of $F \cap V$ in F over X and not over the other components, then we alter Ψ by acting with x. If we change the orientation on $M - V$ which does not extend across any component of V, we do not change Ψ of the Pin^- structure. Finally, if $M_1 \subset M$ is a codimension 1 submanifold with trivialized normal bundle such that F and V intersect M_1 transversely (including the case $M_1 = \partial M$), then the characteristic structure on M restricts to one on M_1. The Pin^- structure we get on $F_1 = M_1 \cap F$ is the restriction of the one we got on F.

Remark. The observation that characteristic structures restrict to boundaries allows us to define bordism groups: let $\Omega_r^!$ denote the bordism group of characteristic structures.

Reduction 6.10. Given a closed manifold M with a characteristic structure, let $E \subset M$ denote the total space of the normal bundle of F in M. The associated circle bundle, ∂E, is embedded in M with trivial normal bundle and without loss of generality we may assume that V intersects ∂E transversally. Hence E acquires the above data by restriction.

This reduces the general case to the following local problem. We may deal with one component at a time now and so we must describe how to put a Pin^- structure on a connected Pin^- manifold F, given that we have a 2–disc bundle over F with total space E; a Pin^- structure on ∂E which does not extend to all of E; a codimension 1 submanifold V which is dual to $w_1(E)$ and intersects F transversally; an orientation on $E - V$ which does not extend across any component of V; and an orientation for the normal bundle of $F \cap V$ in V. We must also check that the Pin^-

structure that we get on F is independent of our choice of tubular neighborhood. Note for reassurance that Pin^- structures on F are in one to one correspondence with Pin^- structures on ∂E which do not extend to E.

Let us consider the following situation. We have a circle bundle $p\colon \partial E \to F$ over F with associated disc bundle ξ. We let E denote the total space of ξ. We have a codimension 1 submanifold, V, of E which is dual to $w_1(E)$ and which intersects F transversally. We are given an orientation on $E - V$ which does not extend across any component of V and we are given an orientation of the normal bundle to $F \cap V$ in V. We are going to describe a one to one correspondence between Pin^- structures on F and Pin^- structures on ∂E which do not extend across E. Furthermore, suppose that $U \subset F$ is a submanifold with trivialized normal bundle. Suppose that U intersects V transversally and let E_U denote the total space of the disk bundle for ξ restricted to U. Then over U we have our data. Notice that any Pin^- structure on F restricts to one on U, and any Pin^- structure on ∂E restricts to one on ∂E_U. Let $\mathcal{P}in^-(F,U)$ denote the set of Pin^- structures on F which restrict to a fixed one on U. Define $\mathcal{P}in^-(\partial E, \partial E_U)$ similarly except we require that the Pin^- structures do not extend across the disk bundles. Below we will define a 1–1 map $\Psi\colon \mathcal{P}in^-(\partial E, \emptyset) \to \mathcal{P}in^-(F, \emptyset)$. If we fix a Pin^- structure on U, which comes from one on F, and use Ψ for U to pick out a Pin^- structure on ∂E_U, then we also get a 1–1 map

$$\Psi\colon \mathcal{P}in^-(\partial E, \partial E_U) \to \mathcal{P}in^-(F,U) \ .$$

There is an isomorphism, $p^*\colon H^1(F,U;\mathbf{Z}/2\mathbf{Z}) \to H^1(\partial E, \partial E_U \cup S^1; \mathbf{Z}/2\mathbf{Z})$, induced by the projection map, $p\colon \partial E \to F$, where S^1 denotes a fibre of the bundle (if $U \neq \emptyset$ then $\partial E_U \cup S^1 = \partial E_U$). The group $H^1(\partial E, \partial E_U \cup S^1; \mathbf{Z}/2\mathbf{Z})$ acts in a simply transitive fashion on $\mathcal{P}in^-(\partial E, \partial E_U)$ and the group $H^1(F,U;\mathbf{Z}/2\mathbf{Z})$ acts in a simply transitive fashion on $\mathcal{P}in^-(F,U)$. The map Ψ is equivariant with respect to these actions and p^*.

The relative version of the Pin^- Structure Correspondence gives the uniqueness result needed in Reduction 6.10 since any two choices are related by a picture with our data over $E \times I$ with structure fixed over $E \times 0$ and $E \times 1$.

Note first that F has a Pin^- structure by the calculations above.

Recall that there is a sub–bundle of $T_{\partial E}$, namely the bundle along the fibres, η. This is a line bundle which is tangent to the fibre circle at each point in ∂E. The quotient bundle, ρ, is naturally isomorphic to T_F, via the projection map, p. Our first task is to use our given data to describe an isomorphism between $\eta \oplus \det(T_{\partial E})$ and $\det(\rho) \oplus \epsilon^1$. To fix notation, let N be a tubular neighborhood of V in ∂E and fix an isomorphism between $\rho \oplus \eta$ and $T_{\partial E}$.

On $\partial E - V$ we have an orientation of $T_{\partial E}$. This describes an isomorphism between $\det(T_{\partial E})$ and ϵ^1. Furthermore, the orientation picks out an isomorphism between η and $\det(\rho)$ as follows. These two line bundles are isomorphic since they have the same w_1, and there are two distinct isomorphisms over each component of $\partial E - V$. Pick a point in each component of $\partial E - V$, and orient η at those points. The orientation of $T_{\partial E}$ picks out an orientation of ρ, and hence $\det(\rho)$, at each point. We choose the isomorphism between η and $\det(\rho)$ which preserves the orientations at each point. It is easy to check that if we reverse the orientation at a point for η, we reverse the orientation for $\det(\rho)$ and hence get the same isomorphism between these two bundles. The isomorphism between $\eta \oplus \det(T_{\partial E})$ and $\det(\rho) \oplus \epsilon^1$ is just the sum of the above two isomorphisms.

We turn our attention to the situation over N. Over $F \cap V$, ξ is the normal bundle to $F \cap V$ in V, and hence it is oriented. Hence so is $p^*(\xi)$ in ∂E, and $p^*(\xi)$ is isomorphic to $\eta \oplus \epsilon^1$. The outward normal to ∂E in E orients the ϵ^1, and hence η is oriented over $p^{-1}(F \cap V)$, and hence over N. This time $\det(\rho)$ and $\det(T_{\partial E})$ are abstractly isomorphic, and we can choose an isomorphism by choosing a local orientation. Since η is oriented and $0 \to \eta \to T_{\partial E} \to \rho \to 0$ is exact, there is a natural correspondence between orientations of $T_{\partial E}$ at a point and orientations of ρ at the same point. As before, if we switch the orientation on $T_{\partial E}$, we still get the same isomorphism between $\det(\rho)$ and $\det(T_{\partial E})$. As before, the orientation for η defines an isomorphism between η and ϵ^1, but this time we take the isomorphism which reverses the orientations. We take the sum of these two isomorphisms as our preferred isomorphism between $\eta \oplus \det(T_{\partial E})$ and $\det(\rho) \oplus \epsilon^1$.

Now over $N - V$, we have two isomorphisms between $\eta \oplus \det(T_{\partial E})$ and $\det(\rho) \oplus \epsilon^1$. If we restrict attention to a neighborhood of ∂N both bundles are the sum of two trivial bundles, and our two isomorphisms differ by composition with the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Parameterize a neighborhood of ∂N in N by $\partial N \times [0, \pi/2]$ and twist one bundle isomorphism by the matrix $\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$. We can now glue our two isomorphisms together to get an isomorphism between $\eta \oplus \det(T_{\partial E})$ and $\det(\rho) \oplus \epsilon^1$ over all of ∂E.

Finally, we can describe our correspondence between Pin^- structures. Suppose that we have a Pin^- structure on F. This is a *Spin* structure on $T_F \oplus \det(T_F)$. Since ρ is isomorphic via p to T_F, we get a *Spin* structure on $\rho \oplus \det(\rho)$, and hence on $\rho \oplus \det(\rho) \oplus \epsilon^1$. Using our constructed isomorphism, this gives a *Spin* structure on $\rho \oplus \eta \oplus \det(T_{\partial E})$. Choose a splitting of the short exact sequence $0 \to \eta \to T_{\partial E} \to \rho \to 0$, and we get a *Spin* structure on $T_{\partial E} \oplus \det(T_{\partial E})$.

If we choose a different splitting, we get an automorphism of $T_{\partial E}$ and hence an automorphism of $T_{\partial E} \oplus \det(T_{\partial E})$ which takes one *Spin* structure to the other. But this automorphism is homotopic through bundle automorphisms to the identity, and so the *Spin* structure does not change.

Finally, let us consider the Pin^- structure induced on a fibre S^1. We will look at this situation for a fibre over a point in F where we have an orientation of $T_{\partial E}$. Restricted to S^1, the bundle $T_{\partial E}$ splits as η plus the normal bundle of S^1 in ∂E, so η is naturally identified as the tangent bundle of S^1 and the normal bundle of S^1 in ∂E is trivialized using the bundle map p. The trivialization of the normal bundle of S^1 in ∂E plus the $Spin$ structure on $T_{\partial E} \oplus \det(T_{\partial E})$ yields a trivialization of $\eta|_{S^1}$, which then yields a trivialization of the tangent bundle of S^1. Since $SO(1)$ is a point, any oriented 1–plane bundle has a unique framing, which in the case of the tangent bundle to the circle is the Lie group framing. The Pin^- structure that results from a framing of the tangent bundle of S^1 is therefore the one that does not extend across the disk, so our Pin^- structure on ∂E does not extend across E.

Recall that Pin^- structures on ∂E that do not extend across E are acted on by $H^1(F; \mathbf{Z}/2\mathbf{Z})$ in a simply–transitive manner by letting $p^*(x) \in H^1(\partial E; \mathbf{Z}/2\mathbf{Z})$ act as usual on Pin^- structures on ∂E. If we change Pin^- structures on F by $x \in H^1(F; \mathbf{Z}/2\mathbf{Z})$, we change the Pin^- structure that we get on ∂E by the $p^*(x)$ in $H^1(\partial E; \mathbf{Z}/2\mathbf{Z})$ so our procedure induces a one to one correspondence between Pin^- structures on F and Pin^- structures on ∂E which do not extend across E.

Next, we consider the effects of changing our orientations. We wish to study how the choices of orientations on $\partial E - V$ and on ξ effect the resulting map between Pin^- structures on F and Pin^- structures on ∂E which do not extend across E. Let us begin by considering the effect of changing the orientation on ξ. This switches the orientation on η and so our bundle map remains the same over $\partial E - N$ and over N it is multiplied by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. This has the effect of putting s full twists into the framing around any circle that intersects $F \cap V$ geometrically t times where $s \equiv t \pmod 2$. Hence the class in $H^1(F; \mathbf{Z}/2\mathbf{Z})$ that measures the change in Pin^- structure is just the class dual to $F \cap V$. If $F \cap V$ has several components and we switch the orientation of ξ over only one of them then the class in $H^1(F; \mathbf{Z}/2\mathbf{Z})$ that measures the change in Pin^- structure is just the class dual to that component of $F \cap V$.

Now suppose that we switch the orientation on ξ and on $M - V$. This time the two bundle maps differ over all of ∂E by multiplication by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The effect of this is to change the Pin^- structure on F via $w_1(F)$. This follows from Lemma 1.6.

From the two results above the reader can work out the effect of the other possible changes of orientations. Finally, the diligent reader should work through the relative version.

This ends our description of the Pin^- Structure Correspondence. ∎

As an application of the Pin^- Structure Correspondence and Reduction 6.10 we present

Theorem 6.11. *There exists a homomorphism* $R\colon \Omega^!_r \to \Omega^{Pin^-}_{r-2}(B_{O(2)})$. *Given an object,* $x \in \Omega^!_r$, *let* F *denote the submanifold dual to* $w_2 + w_1^2$. *This manifold has a map* $F \to B_{O(2)}$ *classifying the normal bundle. Use the above construction to put a* Pin^- *structure on* F*:* $R(x)$ *is the bordism class of this Pin structure on* F.

Variants of this map enter into the discussions below.

Corollary 6.12. *If* MFK_r *denotes the* r*–th bordism group of Freedman–Kirby, then there exists a long exact sequence*

$$\cdots \to \Omega^{Spin}_r \xrightarrow{i} MFK_r \xrightarrow{R} \Omega^{Spin}_{r-2}(B_{SO(2)}) \xrightarrow{a} \Omega^{Spin}_{r-1} \to \cdots$$

where R *takes the Spin bordism class of the classifying map for the normal bundle to* F *in* M*, and* a *takes the Spin structure we put on the total space of the associated circle bundle. The* V *we always take is the empty set.*

Remark 6.13. There are definitely non–trivial extensions in this sequence.

Remark 6.14. The Freedman–Kirby bordism theory is equivalent to the bordism theory $Spin^c$, the theory of oriented manifolds with a specific reduction of w_2 to an integral cohomology class. This bordism theory has been computed, e.g. Stong [Stong], and is determined by Stiefel–Whitney numbers, Pontrjagin numbers, and rational numbers formed from products of Pontrjagin numbers and powers of the chosen integralization of w_2.

Remark 6.15. There are versions of this sequence for the bordism theory studied by Guillou–Marin and for our bordism theory. In both of these cases we replace Ω^{Spin} by the Pin^- bordism groups Ω^{Pin^-}. We also replace $\Omega^{Spin}_{r-2}(B_{SO(2)})$ by the bordism groups of $O(2)$–bundles over Pin^- manifolds with some extra structure. The bordism groups of $O(2)$–bundles over Pin^-manifolds can be identified with the homotopy groups of the Thom spectrum formed from $B_{Pin^-} \times B_{O(2)}$ using the universal bundle over B_{Pin^-} and the trivial bundle over $B_{O(2)}$. The associated bordism groups are denoted $\Omega^{Pin^-}_{r-2}(B_{O(2)})$. In the Guillou–Marin case we define BGM as the fibre of the map $B_{Pin^-} \times B_{O(2)} \to K(\mathbf{Z}/2\mathbf{Z}, 1)$ where the map is the sum of w_1 of the universal bundle over B_{Pin^-} and w_1 of the universal bundle over $B_{O(2)}$. In our case we let BE be the fibre of the map $B_{Pin^-} \times B_{O(2)} \to K(\mathbf{Z}/2\mathbf{Z}, 2)$ where the map is the product of two 1–dimensional cohomology classes: namely w_1 of the universal bundle over B_{Pin^-} and w_1 of the universal bundle over $B_{O(2)}$. Over either BGM or BE we can pull back the universal bundle over B_{Pin^-} plus the trivial bundle over $B_{O(2)}$ and form the associated Thom spectrum. The homotopy groups of these spectra fit into the analogous exact sequences for the bordism theory studied by Guillou–Marin and by us.

Remark 6.16. All the bordism groups defined in Theorem 6.11, Corollary 6.12 and its two other versions are naturally modules over the *Spin* bordism ring, and all the maps defined above are maps of Ω^{Spin}_*–modules.

§7. Geometric calculations of characteristic bordism.

In this section we will calculate the characteristic bordism introduced in the last section up through dimension 4.

The first remark is that any manifold M of dimension less than or equal to 4 has a characteristic structure. Hence !–bordism is onto unoriented bordism through dimension 4. We show next that

Theorem 7.1. *The forgetful map*

$$\Omega_r^! \to \Omega_r^O$$

is an isomorphism for $r = 0$, 1, *and* 2. *Hence* $\Omega_0^! \cong \Omega_2^! \cong \mathbf{Z}/2\mathbf{Z}$ *and* $\Omega_1^! \cong 0$.

Proof: Since the forgetful map is onto, it is merely necessary to show that the !–bordism groups are abstractly isomorphic to $\mathbf{Z}/2\mathbf{Z}$ or 0. We begin in dimension 0. The only connected manifold is the point and it has a unique characteristic structure: F and V are empty. Hence $\Omega_0^!$ is a quotient of $\mathbf{Z}$. It is easy to find a characteristic structure on $[0,1]$ which has 2 times the oriented point as its boundary: F is empty and $V = \{1/2\}$. Hence $\Omega_0^! \cong \mathbf{Z}/2\mathbf{Z}$ given by the number of points mod 2.

In dimensions at least 1, it is easy to add 1–handles to show any object is bordant to a connected one. Hence in dimension 1, the only objects we need to consider are characteristic structures on S^1. Here F is still empty, and V is an even number of points. The circle bounds B^2, the 2–disk, and it is easy to extend V to a collection of arcs in B^2 and to extend the orientation on $S^1 - V$. The Pin^- structure on the circle either bounds a 2–disk, in which case extend it over B^2, or it does not, in which case take F to be a point in B^2 which misses the arcs and extend the Pin^- structure over $B^2 - pt$. Hence $\Omega_1^! \cong 0$.

In dimension 2 we can assume that M is connected and that it bounds as an unoriented manifold. The goal is to prove that it bounds as a characteristic structure. Note V is a disjoint union of circles, and F is a finite set of points with $F \cap V$ being empty. Since every surface has a Pin^- structure, F is an even number of points. Let W be a collection of embedded arcs in $M \times [0,1]$ which miss $M \times 1$ and have boundary F. Since W is a dual to $w_2 + w_1^2$, there is a Pin^- structure on $M \times [0,1] - W$ which extends across no component of W. This induces such a structure on $M \times 0$. Since $H^1(M; \mathbf{Z}/2\mathbf{Z})$ acts on such structures, it is easy to adjust to get a Pin^- structure on $M \times [0,1] - W$ which extends across no component of W and which is our original Pin^- structure on $M \times 0$. Given $V \subset M \times 0$ we can extend to an embedding $V \times [0,1]$ in $M \times [0,1]$. The orientation on $M - V$ extends to one on $M \times [0,1] - V \times [0,1]$. Clearly this orientation extends across no component of $V \times [0,1]$, so this submanifold is dual to w_1. Hence we may assume our surface has empty F with no loss of generality: i.e. M has a fixed Pin^- structure.

Let E_K^3 denote the total space of the non–trivial 2–disk bundle over the circle. The boundary of E_K^3 is K^2, the Klein bottle and $H_1(K; \mathbf{Z}/2\mathbf{Z})$ is spanned by a fibre circle, C_f, and a choice of circle which maps non-trivially to the base, C_ℓ. Consider the Pin^- structure on K^2 whose quadratic enhancement satisfies $q(C_f) = 2$ and $q(C_\ell) = 1$. This structure does not extend across E_K so let F be the core circle in E_K. Let V be a fibre 2–disk. Orient the normal bundle to $V \cap F$ in F any way one likes. It is easy to check that this gives a characteristic structure on E^3_K extending the one on K^2 which does not bound as a Pin^- manifold. By adding copies of this structure on K^2 to M, we can assume that M is a Pin^- boundary, so let W^3 be a Pin^- boundary for M.

Inside W we find a dual to w_1, say X^2, which extends V in M. There is some orientation on $W - X$ which extends across no component of X and this structure restricts to such a structure on $M - V$. Since M is connected, there are only two such structures and both can be obtained from such a structure on $W - X$. Hence our original characteristic structure is a characteristic boundary assuming nothing more than that it was an unoriented boundary. ∎

The results in dimensions 3 and 4 are more complicated. We begin with the 3–dimensional result.

Theorem 7.2. *The homomorphism R of Theorem 6.11, followed by forgetting the map to $B_{O(2)}$ yields an isomorphism*

$$\hat{R}\colon \Omega^!_3 \to \Omega^{Pin^-}_1 \cong \mathbf{Z}/2\mathbf{Z} \ .$$

Proof: We first show that $\hat{R}$ is onto and then that it is injective.

Let E^3_K denote the disk bundle with boundary the Klein bottle as in the last proof. The Pin^- structure received by F in this structure is seen to be the Lie group Pin^- structure. There is a similar story for the torus, T^2. There is a 2–disk bundle over a circle, E^3_T, and a Pin^- structure on the torus which does not extend across the disk bundle so that the core circle receives the Lie group Pin^- structure. Indeed, E^3_T is just a double cover of E^3_K. If we take two copies of K^2 with its Pin^- structure and one copy of T^2 with its Pin^- structure, the resulting disjoint union bounds in $\Omega^{Pin^-}_2$. Let W^3 denote such a bordism. Let $M^3 = \overset{2}{\amalg} E^3_K \amalg E^3_T \amalg W^3$ with the boundaries identified. Let F be the disjoint union of the three core circles, and note F is a dual to $w_2 + w_1^2$ since the complement has a Pin^- structure which does not extend across any of the cores. Let V be a dual to w_1 and arrange it to meet F transversely. Indeed, with a little care one can arrange it so that $V \cap F$ consists of 2 points, one on each core circle in a E^3_K. This is our characteristic structure on M. Our homomorphism applied to M is onto the generator of $\Omega^{Pin^-}_1$.

It remains to show monicity. Let M be a characterized 3–manifold. By adding 1–handles, we may assume that M is connected. First we want to fix it so that $V \cap F$ is empty. In general, $V \cap F$ is dual to $w_2 w_1 + w_1^3$ and, for a 3–manifold, this vanishes. Hence $V \cap F$ consists of an even number of points. We explain how to remove a pair of such points.

Pick two points, p_0 and p_1, in $V \cap F$. Each point in F has an oriented normal bundle. The normal bundle to each point in V is also trivial and V is oriented, so the normal bundle to each point in V is oriented. Attach a 1–handle, $H = (B^1 \times [0,1]) \times B^2$ so as to preserve the orientations at p_0 and p_1. Let W^4 denote the resulting bordism. Inside W^4, we have embedded bordisms, V_1^3 and F_1^2 beginning at V and F in M. Notice that at the "top" of the bordism, the "top" of V_1 and the "top" of F_1 intersect in 2 fewer points. Moreover, the orientation of the normal bundle of $V \cap F$ in F clearly extends to an orientation of the normal bundle of $V_1 \cap F_1$ in F_1.

Since F_1 is a codimension 2 submanifold of W, it is dual to some 2–dimensional cohomology class. Since $H^*(W, M; \mathbf{Z}/2\mathbf{Z})$ is 0 except when $* = 1$ (in which case it is $\mathbf{Z}/2\mathbf{Z}$), this class is determined by its restriction to $H^2(M; \mathbf{Z}/2\mathbf{Z})$. Hence F_1 is dual to $w_2 + w_1^2$, so choose a Pin^- structure on $W - F_1$ which extends across no component of F_1. This restricts to a similar structure on M, and since $H^1(W; \mathbf{Z}/2\mathbf{Z}) \to H^1(M; \mathbf{Z}/2\mathbf{Z})$ is onto, we can adjust the Pin^- structure until it extends the given one on $M - F$.

The above argument does not quite work for V_1, but it is easy in this case to see that $W - V_1$ has an orientation extending the one on $M - V$. Any such orientation can not extend over any components of V_1. Hence we have a characteristic bordism as required.

We may now assume that $V \cap F$ is empty. Since F is a union of circles and $V \cap F = \emptyset$, F has a trivial normal bundle in M. If our homomorphism vanishes on our element, F is a Pin^- boundary, which, in this dimension, means that it is a *Spin* boundary: i.e. F bounds Q^2, an orientable Pin^- manifold. Glue $Q^2 \times B^2$ to $M \times [0,1]$ along $F \times B^2 \subset M \times 1$ to get a bordism X^4. Since Q is orientable, $V \times [0,1]$ is still dual to w_1, and it is not hard to extend the Pin^- structure on $M - F$ to one on $X - Q$ which extends across no component of Q. Since Q and $V \times [0,1]$ remain disjoint, the "top" of X is a new characteristic pair for which the dual to $w_2 + w_1^2$ is empty: i.e. the "top", say N^3, has a Pin^- structure. Since $\Omega_3^{Pin^-} = 0$, N^3 bounds a Pin^- manifold, Y^4. Since M was connected, so is N and there is no obstruction to extending the dual to w_1 in N, say V_1, to a dual to w_1 in Y, say U, and extending the orientation on $N - V_1$ to an orientation on $Y - U$ which extends across no component of U. The union of X^4 and Y^4 along N^3 is a characteristic bordism from M^3 to 0. ∎

The last goal of the section is to compute $\Omega_4^!$. Since the group is non–zero, we begin by describing the invariants which detect it. Given an element in $\Omega_4^!$, we get an associated surface F^2 with a Pin^- structure, and hence a quadratic enhancement, q. We may also consider η, the normal bundle to F in our original 4–manifold. We describe three homomorphisms. The first is $\beta\colon \Omega_4^! \to \mathbf{Z}/8\mathbf{Z}$ which just takes the Brown invariant of the enhancement q. The second homomorphism is $\Psi\colon \Omega_4^! \to \mathbf{Z}/4\mathbf{Z}$ given by the element $q(w_1(\eta)) \in \mathbf{Z}/4\mathbf{Z}$. The third homomorphism is $w_2\colon \Omega_4^! \to \mathbf{Z}/2\mathbf{Z}$ given by $\langle w_2(\eta), [F]\rangle \in \mathbf{Z}/2\mathbf{Z}$. We leave it to the reader to check that these three maps really are homomorphisms out of the bordism group, $\Omega_4^!$.

Theorem 7.3. *The sum of the homomorphisms*

$$\beta \oplus \Psi \oplus w_2\colon \Omega_4^! \to \mathbf{Z}/8\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$$

is an isomorphism.

Proof: First we prove the map is onto and then we prove it is 1–1. Recall from Lemma 6.7 that a surface, M, with a Pin^- structure and a 2–plane bundle, η, can be completed to a characteristic bordism element iff $(w_1(M) + w_1(\eta)) \cup w_1(\eta) = 0$. Notice that this equation is always satisfied since cupping with $w_1(M)$ and squaring are the same. Hence we will only describe the surface with its Pin^- structure and the 2–plane bundle.

First note that $\mathbf{RP}^2$ with the trivial 2–plane bundle generates the $\mathbf{Z}/8\mathbf{Z}$ and maps trivially to the $\mathbf{Z}/4\mathbf{Z}$ and the $\mathbf{Z}/2\mathbf{Z}$.

The Hopf bundle over the 2–sphere maps trivially into the $\mathbf{Z}/8\mathbf{Z}$ and the $\mathbf{Z}/4\mathbf{Z}$ since S^2 is a Pin^- boundary and Ψ vanishes whenever the 2–plane bundle has trivial w_1. However, S^2 and the Hopf bundle maps non–trivially to the $\mathbf{Z}/2\mathbf{Z}$.

Let K^2 denote the Klein bottle, and fix a Pin^- structure for which K^2 is a Pin^- boundary. Let η be the 2–plane bundle coming from the line bundle with w_1 being the class in $H^1\left(K^2; \mathbf{Z}/2\mathbf{Z}\right)$ with non–zero square. Since K^2 is a Pin^- boundary, $\beta(K^2) = 0$. Since η comes from a line bundle, $w_2(\eta) = 0$. However, $q(w_1(\eta))$ is an element in $\mathbf{Z}/4\mathbf{Z}$ of odd order and is hence a generator.

This shows that our map is onto. Before showing that our map is 1–1, we need a lemma.

Lemma 7.4. *There exists a 2–disk bundle B_{2n} over the punctured $S^1 \times S^2$, $S^1 \times S^2 -$ int B^3, whose restriction to the boundary S^2 has Euler class $2n$, $n \in \mathbf{Z}$.*

Proof: Start with the 2–disk bundle $\tilde{B}_n$ over S^2 with Euler number n and pull it back over the product $S^2 \times I$. Now add a 1–handle to $S^2 \times I$, forming $S^1 \times S^2 -$int B^3, and extend the bundle $\tilde{B}_n$ over the 1–handle so as to create a non–orientable bundle B_{2n}. Then $\chi(B_{2n}|_{S^2}) = 2n$. ∎

Suppose M^4, V^3, F^2, η^2 is a representative of an element of $\Omega_4^!$ and that $\beta(F^2) = 0$, $\Psi(w_1(\eta)) = 0$, and $w_2(\eta) = 0$. We need to construct a !–bordism to $\emptyset$.

Since we may assume that F, M and V are connected, there is a connected 1–manifold, an S^1, which is Poincaré dual to $w_1(\eta)$; then the normal vector to S^1 in F makes an even number of full twists in the Pin^- structure on F as S^1 is traversed. It follows that we can form a !–bordism by adding to F a $B^2 \times B^1$

where $S^1 \times B^1$ is attached to the dual S^1 to $w_1(\eta)$ and its normal B^1 bundle. Clearly the Pin^- structure on F extends across the bordism. Since the dual to S^1 has self-intersection zero in F, η restricted to S^1 is orientable, so η extends over $B^2 \times B^1$.

Since $w_2(\eta) = 0$, it follows that $\chi(\eta)[F] = 2n$ for some $n \in \mathbf{Z}$. By Lemma 7.4 there is a bundle B_{-2n} over a punctured $S^1 \times S^2$ with $\chi(B_{-2n}|_{S^2}) = -2n$. We form a 5-dimensional bordism to the boundary connected sum, i.e. in $M^4 \times 1 \subset M^4 \times I$, choose a 4-ball of the form $B^2 \times B^2$ where $B^2 \times 0 \subset F^2 - (V \cap F)$ and $p \times B^2$ is a normal plane of η over p, and identify $B^2 \times B^2$ with $B_{-2n}|_{S^2_-}$ where S^2_- is a hemisphere of S^2.

The new boundary to our !-bordism, which we shall denote (M, V, F, η) now has a trivial normal bundle η.

Since $\beta(F^2) = 0$, F Pin^- bounds a 3-manifold N^3, so we add $N^3 \times B^2$ to $M \times 1$ along the normal bundle η to F, $F \times B^2$, where it does not matter how we trivialize η. The Pin^- structure on $M - F$ extends over the complement of N (using the Pin^- Correspondence Theorem, 6.9, and the Pin^- structure on N), so the new boundary to our !-bordism consists of a Pin^- manifold M with empty F^2. Since 4-dimensional Pin^- bordism, $\Omega_4^{Pin^-}$, is zero, we can complete our !-bordism by gluing on to $M \times 1$ a 5-dimensional Pin^- manifold. ∎

Remark. *It is worth comparing this argument with the argument in [F–K] showing that if* (M^4, F^2) *is a characteristic pair with* M^4 *and* F^2 *orientable and with* $\text{sign}(M^4) = 0$ *and* $F \bullet F = 0$, *then* (M, F) *is characteristically bordant to zero. The arguments would have been formally identical if we had also assumed that the Spin structure on* F, *obtained from the* Pin^- *Correspondence Theorem, bounded in 2-dimensional Spin bordism,* $\Omega_2^{Spin} = \mathbf{Z}/2\mathbf{Z}$ *(corresponding to* $\beta(F) = 0$ *above). However, it is possible to show that* $\Omega_4^{\text{char}} = \mathbf{Z} \oplus \mathbf{Z}$ *without the extra assumption on* F, *and this* $\mathbf{Z}/2\mathbf{Z}$ *improvement leads to Rochlin's Theorem (see [F–K], [Ki], ...).*

Further Remark. The image of the Guillou–Marin bordism in this theory can be determined as follows. The group is $Z \oplus Z$ generated by (S^4, RP^2) and (CP^2, S^2). Both β and Ψ vanish on (CP^2, S^2), but w_2 is non-zero. On (S^4, RP^2), w_2 evaluates 0 (the normal bundle comes from a line bundle): β is either 1 or -1 depending on which embedding one chooses. Moreover, Ψ is either 1 or -1 (the same sign as β).

§8. New knot invariants.

The goal here is to describe some generalizations of the usual Arf invariant of a knot (or some links) due to Robertello, [R].

We fix the following data. We have a 3-manifold M^3 with a fixed *Spin* structure and a link $L\colon \coprod_i S^1 \to M^3$. Since M is *Spin* , $w_2(M) = 0$ and we require that $[L] \in H_1(M; \mathbf{Z}/2\mathbf{Z})$ is also 0, hence dual to $w_2(M)$. We next require a characterization of the pair, (M, L): i.e. a *Spin* structure on $M - L$ which extends across no component of L. We call such a characterization *even* iff the Pin^- structure induced on each component of L by Lemma 6.2 is the structure which bounds. We say the link is *even* iff it has an even characterization.

One way to check if a link is even is the following. Each component of L has a normal bundle, and the even framing of this normal bundle picks out a mod 2 longitude on the peripheral torus. The link is even iff the sum of these even longitudes is 0 in $H_1(M - L; \mathbf{Z}/2\mathbf{Z})$

Remark. Not all links which represent 0 are even: the Hopf link in S^3 is an example where any structure which extends across no component of L induces the Lie group *Spin* structure on the two circles. We shall see later that a necessary and sufficient condition for a link in S^3 to be even is that each component of the link should link the other components evenly. This is Robertello's condition, [R].

Definition. A link, L, in M^3 with a fixed *Spin* structure on M and a fixed *Spin* structure on $M - L$ which extends across no component of L and induces the bounding Pin^- structure on each component of L is called a *characterized* link.

Given a characterized link, (M, L), we define a class $\gamma \in H^1(M - L; \mathbf{Z}/2\mathbf{Z})$: γ is the class which acts on the fixed *Spin* structure on $M - L$ to get the one which is the restriction of the one on M. The class γ is defined by the characterization and conversely a characterization is defined by a choice of class $\gamma \in H^1(M - L; \mathbf{Z}/2\mathbf{Z})$ so that, under the coboundary map, the image of γ in $H^2(M, M - L; \mathbf{Z}/2\mathbf{Z})$ hits each generator. (Recall that by the Thom isomorphism theorem, $H^2(M, M - L; \mathbf{Z}/2\mathbf{Z})$ is a sum of $\mathbf{Z}/2\mathbf{Z}$'s, one for each component of L.)

Let E be the total space of an open disk bundle for the normal bundle of L, and let S be the total space of the corresponding sphere bundle. Note S is a disjoint union of a peripheral torus for each component of L. The class γ is dual to an embedded surface $F \subset M - E$ and $\partial F \cap S$ is a longitude in the peripheral torus of each component of L. Let ℓ denote this set of longitudes. We will call ℓ a set of *even longitudes*. We will call F a *spanning surface* for the characterized link.

The set of even longitudes is not well–defined from just the characterized link. It is clear that two surfaces dual to the same γ must induce the same mod 2 longitudes. But if we act on one component of L by an even integer, we can find a new surface dual to γ which has the same longitudes on the other components and the new longitude on our given component differs from the old one via action by this even integer. Hence the characteristic structure only picks out the mod 2 longitudes and any set of integral classes which are longitudes and which reduce correctly mod 2 can be a set of even longitudes. Moreover, any set of even longitudes is induced by an embedded surface.

Since M is oriented, the normal bundle to any embedded surface, F, is isomorphic to the determinant bundle associated to the tangent bundle of F. The total space of the determinant bundle to the tangent bundle is naturally oriented. The total space to the normal bundle to F is M is oriented by the orientation on M. Choose the isomorphism between the normal bundle to F in M and the determinant bundle to the tangent bundle of F so that, under the induced diffeomorphism between the total spaces, the two orientations agree. Under these identifications, Corollary 1.15 picks out a Pin^- structure on F from the $Spin$ structure on M. We apply this to an F which is a spanning surface for our link. Of course we could apply the same result but use the $Spin$ structure on $M - L$. It is not hard to check that the two structures on F differ under the action of $w_1(F)$ since this is the restriction of γ to F. Hence it is not too crucial which structure we use but to fix things we use the structure on M.

We can restrict this structure on F to a component of L. If we put the $Spin$ structure on F coming from that on $M - L$ it is easy to see that we get the bounding Pin^- structure on each component of L. Hence this also holds for the Pin^- structure on F coming from the one on M. Hence, a spanning surface for a characterized link has an induced Pin^- structure which extends to the corresponding closed surface uniquely.

Our link invariant is a mod 8 integer which depends on the characterized link and the set of even longitudes.

Definition 8.1. Given a characterized link, (M, L), and a set of even longitudes, ℓ, pick a spanning surface F for L which induces the given set of longitudes. Then define

$$\beta(L, \ell, M) = \beta(\overline{F})$$

where $\overline{F}$ is F with a disk added to each component of L; the Pin^- structure is extended over each disk; and β is the usual Brown invariant applied to a closed surface with a Pin^- structure.

Remarks.

i) Notice that unlike Robertello's invariant, our invariant does not require that the link be oriented.

ii) It follows from the proof of Theorem 4.3 that a knot is even iff it is mod 2 trivial.

iii) If each component of L represents 0 in $H_1(M; \mathbf{Z}/2\mathbf{Z})$ then the mod 2 linking number of a component of L with the rest of the link is defined. If F is an embedded surface in M with boundary L, the longitude picked out for a component of L is even iff the mod 2 linking number of that component of L with the rest of the link is 0.

iv) If M is an oriented $\mathbf{Z}/2\mathbf{Z}$ homology 3 sphere, then it has a unique $Spin$ structure and there is a unique way to characterize an even link L.

v) Let M be an integral homology 3 sphere containing a link L. Orient each component of the link. Let ℓ_i be the linking number of the ith component of L

with the rest of the link. Each component of L has a preferred longitude, the one with self-linking 0, so ℓ_i also denotes a longitude. The link L is even iff each ℓ_i is even. Robertello's Arf invariant is equal to $\beta(L,-\ell,M)$, where the *Spin* structure and characterization are unique and ℓ is the set of longitudes obtained by using $-\ell_i$ on each component. Notice that ℓ_i depends on how the link is oriented.

It is not yet clear that our invariant really only depends on the characterizations and the even longitudes.

Theorem 8.2. *Let L be a link in a 3–manifold M. Suppose M has a Spin structure and that L is characterized. Let ℓ be a collection of even longitudes. Then $\beta(L,\ell,M)$ is well–defined. Let W^4 be an oriented bordism between M_1 and M_2. Let $L_i \subset M_i$, $i = 1,2$ be characterized links. Let $F \subset W$ be a properly embedded surface with $F \cap M_i = L_i$. Suppose $W - F$ has a Spin structure which extends across no component of F and which gives a Spin bordism between the two structures on $M_i - L_i$, $i = 1,2$, given by the characterizations.*

The normal bundle to F in W has a section over every non–closed component of F so pick one. This choice selects a longitude for each component of each link. Suppose the longitudes picked out for each L_i, say ℓ_i, are even. The surface F receives a Pin^- structure by Lemma 6.2. With this structure, each component of ∂F bounds and hence F has a β invariant. If we orient W so that M_1 receives the reverse Spin structure then the following formula holds.

$$\beta(L_2,\ell_2,M_2) - \beta(L_1,\ell_1,M_1) = -\beta(F) - \operatorname{sign}(W) - \mu(M_2) + \mu(M_1) \quad .$$

Proof: We begin by discussing some constructions and results involving a *Spin* 3–manifold N and a spanning surface, V^2 for a characterized link, L. To begin, given $e\colon V^2 \subset N^3$, define $\hat{V} \subset N \times [0,1]$ as the image of $e \times f$, where $f\colon V \to [0,1/2]$ is any map with $f^{-1}(0) = \partial V$. If N has a *Spin* structure, $N \times [0,1]$ receives one. The class represented by $[\hat{V}, L]$ in $H_2\left(N \times [0,1], N \times 0 \amalg N \times 1; \mathbf{Z}/2\mathbf{Z}\right) \cong H_1\left(N \times 0; \mathbf{Z}/2\mathbf{Z}\right)$ is the same as that represented by $[L]$ in $H_1\left(N \times 0; \mathbf{Z}/2\mathbf{Z}\right)$. Hence it represents 0. Since $w_2(N \times [0,1])$ is also trivial, there is a *Spin* structure on $N \times [0,1] - \hat{V}$ which does not extend across any component of $\hat{V}$. Such structures are acted on simply transitively by $H^1\left(N; \mathbf{Z}/2\mathbf{Z}\right)$, so it is easy to construct a unique such *Spin* structure which restricts to the initial one on $N \times 1$.

We proceed to identify the *Spin* structure induced on $N \times 0 - L$. Let $X = V \times [0,1]$ and embed two copies of V in the boundary so that $\partial X = V \cup V$ where the union is along ∂V thought of as $\partial V \times 1/2$. First observe that we can embed X in $N \times [0,1]$ so that ∂X is $V \subset N \times 0$ union $V \times 1 = \hat{V}$. Since X has codimension 1, the Poincaré dual to W is a 1–dimensional cohomology class

$x \in H^1(N \times [0,1] - V; \mathbf{Z}/2\mathbf{Z})$. Suppose we take the *Spin* structure on $N \times [0,1]$ and restrict it to $N \times [0,1] - V$ and then act on it by x. This is a *Spin* structure on $N \times [0,1] - V$ which extends across no component of V and which is the original one on $N \times 1$. On $N \times 0 - L$ it can be described as the one obtained by taking the given *Spin* structure on $N \times 0$, restricting it, and then acting on it by the restriction of x. But the restriction of x is just the Poincaré dual of $F \subset N \times 0$ and so it is the *Spin* structure which characterizes the link. By Lemma 6.2, there is a preferred Pin^- structure on V, which is easily checked to be the same as the one we put on it in §4. The above *Spin* structure on $N \times [0,1] - \hat{V}$ will be called the *standard characterization* of the pair $(N \times [0,1], \hat{V})$.

With this general discussion behind us, let us turn to the situation described in the second part of the theorem. Recall W^4 is an oriented bordism between M_1 and M_2; $L_1 \subset M_1$ and $L_2 \subset M_2$ are characterized links; $F^2 \subset W$ be a properly embedded surface with $F \cap M_i = L_i$; and $W - F$ has a *Spin* structure which extends across no component of F and which gives a *Spin* bordism between the structures on $M_i - L_i$. Define sets of even longitudes ℓ_i as in the statement of the theorem.

Let $F_i \subset M_i$ be a spanning surface for L_i. Inside $\overline{W} = M_1 \times [-1,0] \cup W \cup M_2 \times [0,1]$ embed $\overline{F} = \hat{F_1} \cup F \cup \hat{F_2}$, where $\hat{F_1}$ is defined with function $f\colon F_1 \to [-1/2, 0]$ and still $f^{-1}(0) = \partial F_1$. There is a *Spin* structure on $\overline{W} - \overline{F}$ which extends across no component of $\overline{F}$. It is just the union of the standard characterization of $M_1 \times [-1,0], \hat{F_1}$, the given *Spin* structure on $W - F$ and the standard characterization of $M_2 \times [0,1], \hat{F_2}$.

By Lemma 6.2 again, there is a preferred Pin^- structure on $\overline{F}$, which agrees with the usual ones on F_1 and F_2. In particular, F also receives a Pin^- structure which only depends on W, not on the choice of F_1 or F_2. However, from F_1 and F_2, we see that the Pin^- structure induced on each component of each link is the bounding one. Moreover, $\beta(\overline{F}) = \beta(F) + \beta(F_2) - \beta(F_1)$.

By construction, $\overline{F}\bullet\overline{F}$ is 0, so 6.4 says that

$$\beta(F_2) - \beta(F_1) = -\big(\beta(F) + \mathrm{sign}(W) + \mu(M_2) - \mu(M_1)\big)$$

where the μ invariants arise because 6.4 only applies to closed manifolds.

Apply this to the case $W = M \times [0,1]$, $F = L \times [0,1]$ embedded as a product. Since we may use different spanning surfaces at the top and bottom, this shows β is well–defined. The formula in the theorem now follows from the formula immediately above. ∎

The next thing we wish to discuss is how our invariant depends on the longitudes. Given two different sets of even longitudes, ℓ and ℓ', for a characterized link $L \subset M^3$, there is a set of integers, one for each component of L defined as follows. The integer for the ith component acts on the longitude for ℓ to give the longitude for ℓ'. Since both these longitudes are even, so is this integer.

Theorem 8.3. *Let $L \subset M^3$ be a characterized link with two sets of even longitudes ℓ and ℓ'. Let $2r$ be the sum of the integers which act on the longitudes ℓ to give the longitudes ℓ'. Then*

$$\beta(L, \ell', M) = \beta(L, \ell, M) + r \qquad (\mathrm{mod}\ 8)\ .$$

Proof: Given F_1, a spanning surface for the longitude ℓ, we can construct a spanning surface for ℓ' as follows. Take a neighborhood of the peripheral torus, which will have the form $W = T^2 \times [0,1]$. Inside W embed a surface V which intersects $T^2 \times 0$ in the longitude ℓ, which intersects $T^2 \times 1$ in the longitude ℓ', which has no boundary in the interior of W; and which induces the zero map $H_2(V, \partial V; \mathbf{Z}/2\mathbf{Z}) \rightarrow H_2(W, \partial W; \mathbf{Z}/2\mathbf{Z})$. The *Spin* structure on M restricts to one on W which is easily described: it is the stabilization of one on T^2 and this can be described as the one which has enhancement 0 on the longitude and 0 on the meridian. Since the Pin^- structure induced from Corollary 1.15 is local, we see that $F_2 = V \cup F_1$ has invariant the invariant for F_1 plus the invariant for V. We further see that the invariant for V only depends on the surface and the *Spin* structure in W. But these are independent of the link and so we can calculate the difference of the β's using the unknot.

Furthermore, we see that the effect of successive changes is additive, so we only need to see how to go from the 0 longitude to the 2 longitude, and the 2 longitude is given by the Möbius band, which inherits a Pin^- structure. This Pin^- structure extends uniquely to one on $\mathbf{RP}^2$ and this $\mathbf{RP}^2$ has β invariant $+1$.

Remark. Even in the case of links in S^3, the longitudes used enter into the answer. It is just in this case that there is a unique set of longitudes given by using an orientable spanning surface.

Unfortunately, in general there is no natural choice of longitudes so it seems simplest to incorporate them into the definition. The drawback comes in discussing notions like link concordance. In order to assert that our invariant is a link concordance invariant, we need to describe to what extent a link concordance allows us to transport our structure for one link to another. Recall that a link concordance between $L_0 \subset M$ and $L_1 \subset M$ is an embedding of $(\amalg\ S^1) \times [0,1] \subset M \times [0,1]$ with is $(\amalg\ S^1) \times i$ being L_i for $i = 0, 1$. Suppose L_0 is an even link with ℓ_0 a set of even longitudes. There is a unique way to extend this framing of the normal bundle to L_0 in M to a framing of the normal bundle of $(\amalg\ S^1) \times [0,1]$ in $M \times [0,1]$. Hence the concordance picks out a set of longitudes for L_1 which we will denote by ℓ_1. There is a unique way to extend a characterization of L_0 to a *Spin* structure on $M \times [0,1] - (\amalg\ S^1) \times [0,1]$ and hence to $M - L_1$.

Corollary 8.4. *Let L_0 and L_1 be concordant links in M. Suppose L_0 is characterized and that ℓ_0 is a set of even framings. Then the transport of framings and*

Spin structures described above gives a characterization of L_1 and ℓ_1 is a set of even framings. Furthermore $\beta(L_0, \ell_0, M) = \beta(L_1, \ell_1, M)$.

Proof: The proof follows immediately from Theorem 8.2 and the fact that $(\amalg S^1) \times [0,1]$, when capped off with disks, is a union of S^2's and so has β invariant 0. ∎

We do know one scheme to remove the longitudes which works in many cases. Suppose that each component of the link represents a torsion class in $H_1(M; \mathbf{Z})$. Each component has a self-linking and by Lemma 4.1 the framings, hence longitudes are in one to one correspondence with rational numbers whose equivalence class in $\mathbf{Q}/\mathbf{Z}$ is the self-linking number. There is a unique such number, say q_i for the ith component, so that q_i represents an even framing and $0 \le q_i < 2$. We say that this is the *minimal* even longitude. To calculate linking numbers it is necessary to orient the two elements one wants to link, but the answer for self-linking is independent of orientation.

Definition 8.5. Let L be a link in M so that each component of L represents a torsion class in $H_1(M; \mathbf{Z})$. Suppose L is characterized. Define

$$\hat{\beta}(L, M) = \beta(L, \ell, M)$$

where ℓ is the set of even longitudes such that each one is minimal.

Remark. It is not hard to check that $\hat{\beta}$ is a concordance invariant.

As we remarked above, β and $\hat{\beta}$ (if it is defined) do not depend on the orientation of the link. If we reverse the orientation of M, and also reverse the *Spin* structure on M and on $M - L$, it is not hard to check that the new Pin^- structure on F is the old one acted on by $w_1(F)$ so the new invariant is minus the old one.

The remaining point to ponder is the dependence on the two *Spin* structures. To do this properly would require a relative version of the β function 4.8. It does not seem worth the trouble.

Remark. We leave it to the reader to work out the details of starting with a characteristic structure on M^3 with the link as a dual to $w_2 + w_1^2$ (i.e. represents 0 in $H_1(M; \mathbf{Z}/2\mathbf{Z})$).

§9. Topological versions.

There is a topological version of this entire theory. Just as $Spin(n)$ is the double cover of $SO(n)$ and $Pin^{\pm}(n)$ are the double covers of $O(n)$, we can consider the double covers of $STop(n)$ and $Top(n)$. We get a group $TopSpin(n)$ and two groups $TopPin^{\pm}(n)$. A $Top(n)$ bundle with a $TopPin^{\pm}(n)$ structure and an $O(n)$ structure is equivalent to a $Pin^{\pm}(n)$ bundle.

Any manifold of dimension ≤ 3 has a unique smooth structure, so there is no difference between the smooth and the toplogical theory in dimensions 3 and less. The 3–dimensional bordism groups might be different because the bounding objects are 4–dimensional, but we shall see that even in bordism there is no difference.

We turn to dimension 4. First recall that the triangulation obstruction (strictly speaking, the stable triangulation obstruction) is a 4–dimensional cohomology class so evaluation gives a homomorphism, which we will denote κ, from any topological bordism group to $\mathbf{Z}/2\mathbf{Z}$. Since every 3–manifold has a unique smooth structure, the triangulation obstruction is also defined for 4–manifolds with boundary. Every connected 4–manifold M^4 has a smooth structure on $M - pt$, and any two such structures extend to a smoothing of $M \times [0,1] - pt \times [0,1]$.

Some of our constructions require us to study submanifolds of M. In particular, the definition of characteristic requires a submanifold dual to w_1 and a submanifold dual to $w_2 + w_1^2$. We require that these submanifolds be locally–flat and hence, by [Q], these submanifolds have normal vector bundles. Of course we continue to require that they intersect transversely. Hence we can smooth a neighborhood of these submanifolds. The complement of these smooth neighborhoods, say U, is a manifold with boundary, which may not be smooth. If we remove a point from the interior of each component of U, we can smooth the result. With this trick, it is not difficult to construct topological versions of all our "descent of structure" theorems. In particular, the $[\cap w_1^2]$, $[\cap w_1]$ and R maps we defined into low–dimensional $Pin^{\pm}$ bordism all factor through the corresponding topological bordism theories.

Theorem 9.1. *Let Smooth–bordism$_*$ denote Ω_*^{Spin}, $\Omega_*^{Pin^{\pm}}$, $\Omega_*^{!}$, or the Freedman–Kirby or Guillou–Marin bordism theories. Let Top – bordism$_*$ denote the topological version. The natural map*

$$Smooth - bordism_3 \to Top - bordism_3$$

is an isomorphism.

$$Smooth - bordism_4 \to Top - bordism_4 \xrightarrow{\kappa} \mathbf{Z}/2\mathbf{Z} \to 0$$

is exact.

Proof: The E_8 manifold, [F], is a *Spin* manifold with non–trivial triangulation obstruction. Suppose M^3 is a 3–manifold with one of our structures which is a topological boundary. Let W^4 be a boundary with the necessary structure. Smooth neighborhoods of any submanifolds that are part of the structure. This gives a new 4–manifold with boundary U^4. If the triangulation obstruction for a component of U is non–zero, we may form the connected sum with the E_8 manifold. Hence we may assume that U has vanishing triangulation obstruction. By [L–S] we can add some $S^2 \times S^2$'s to U and actually smooth it. The manifold W can now be smoothed so that all submanifolds that are part of the structure are smooth. Hence M^3 is already a smooth boundary.

The E_8 manifold has any of our structures, so the map $Top\text{-}bordism_4 \to \mathbf{Z}/2\mathbf{Z}$ given by the triangulation obstruction is onto.

Suppose that it vanishes. We can smooth neighborhoods of any submanifolds, so let U be the complement. Each component of U has a triangulation obstruction and the sum of all of them is 0. We can add E_8's and $-E_8$'s so that each component has vanishing triangulation obstruction and the new manifold is bordant to the old. Now we can add some $S^2 \times S^2$'s to each component of U to get a smooth manifold with smooth submanifolds bordant to our original one. ∎

Theorem 9.2. *The topological bordism groups have the following values.* $\Omega_4^{TopSpin} \cong \mathbf{Z}$; $\Omega_4^{TopPin^-} \cong \mathbf{Z}/2\mathbf{Z}$; $\Omega_4^{TopPin^+} \cong \mathbf{Z}/8\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$; *and* $\Omega_4^{Top-!} \cong \mathbf{Z}/8\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. *The triangulation obstruction map is split in all cases except the Spin case: the smooth to topological forgetful map is monic in all cases except the TopPin$^+$ case where it has kernel* $\mathbf{Z}/2\mathbf{Z}$. *The triangulation obstruction map is split onto for the topological versions of the Freedman–Kirby and Guillou–Marin theories and the smooth versions inject.*

Proof: The Top*Pin*$^-$ case is easy from the exact sequence above. The Top*Spin* case is well-known but also easy. The E_8 manifold has non–trivial triangulation obstruction and twice it has index 16 and hence generates Ω_4^{Spin}.

There is a $[\cap w_1^2]$ homomorphism from $\Omega_4^{TopPin^+}$ to $\Omega_2^{Pin^-} \cong \mathbf{Z}/8\mathbf{Z}$ which is onto. Consider the manifold $M = E_8 \# S^2 \times \mathbf{RP}^2$. The oriented double cover of M is *Spin* and has index 16, hence is bordant to a generator of the smooth *Spin* bordism group. It is not hard to see that the total space of the non–trivial line bundle over M has a *Pin*$^+$ structure, so the Kummer surface is a *TopPin*$^+$ boundary. Hence there is a $\mathbf{Z}/2\mathbf{Z}$ in the kernel of the forgetful map and the $[\cap w_1^2]$ map shows that this is all of the kernel. Furthermore, E_8 represents an element of order 2 with non–trivial triangulation obstruction.

The homomorphisms used to compute $\Omega_4^!$ factor through $\Omega_4^{Top-!}$, so $\Omega_4^{Top-!} \cong \Omega_4^! \oplus \mathbf{Z}/2\mathbf{Z}$.

Likewise, the homomorphisms we use to compute smooth Freedman–Kirby or Guillou–Marin bordism factor through the topological versions. ∎

Corollary 9.3. *Let M^4 be an oriented topological 4–manifold, and suppose we have a characteristic structure on the pair (M, F). The following formula holds:*

$$2 \cdot \beta(F) = F \bullet F - \mathrm{sign}(M) + 8 \cdot \kappa(M) \qquad (\mathrm{mod}\ 16)$$

where the Pin$^-$ structure on F is the one induced by the characteristic structure on (M, F) via the topological version of the Pin$^-$ Structure Correspondence, 6.2.

Proof: Generators for the topological Guillou–Marin group consist of the smooth generators, for which the formula holds, and the E_8 manifold, for which the formula is easily checked. ∎

Remark. The above formula shows that the generator of $H_2(\,;\mathbf{Z})$ of Freedman's Chern manifold, [F, p. 378], is not the image of a locally–flat embedded S^2.

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