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A GEOMETRIC PROOF OF ROCHLIN'S THEOREM

MICHAEL FREEDMAN* AND ROBION KIRBY**

0. In 1974 Andrew Casson outlined to us a proof of Rochlin's Theorem (stated below) on the index of a smooth, closed 4-manifold M^4. His proof involved th Arf invariant of a certain quadratic form defined on the first homology group of a surface in M^4 which is dual to the second Stiefel-Whitney class of M^4. Our proof was derived from Casson's; it is the same in principle but differs considerably in detail. After this manuscript was written, we discovered that Rochlin had already in 1971 given a short sketch of this proof; it appears in a paper [R_3] about real algebraic curves in RP^2.

In addition we obtain (Theorem 2) a "stable" converse to the Kervaire-Milnor nonimbedding theorem [K-M], and in §2, by relaxing some orientability assumptions, we prove a new (but unspectacular) nonimbedding theorem (Theorem 4) and find an obstruction to approximating unoriented simplicial 3-chains in a 5-manifold by an immersed 3-manifold.

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1. Let M^4 be a closed, orientable, PL (hence smooth) 4-manifold. It has an intersection form $H_2(M; Z)/_{\text{torsion}} \times H_2(M; Z)/_{\text{torsion}} \to Z$ which is a symmetric, unimodular, integral bilinear form [M_1]; denote this pairing by $x \cdot y$ or xy. Its signature is $\sigma(M) = \text{index } M$.

We say that $\omega \in H_2(M; Z)/_{\text{torsion}}$ is characteristic if its mod 2 reduction $[\omega]_2$ is Poincaré dual to the second Stiefel-Whitney class $w_2 \in H^2(M; Z_2)$. This implies that $\omega \cdot x = x \cdot x$ (mod 2) for all $x \in H_2(M; Z)$. For this congruence follows from the equality $[\omega]_2 \cdot y = y \cdot y$ for all $y \in H_2(M; Z_2)$, which is dual to the equality $w_2 \cup \hat{y} = \hat{y} \cup \hat{y}$ ($\hat{}$ denotes Poincaré dual) which is a definition of the second Steifel-Whitney class w_2 of a 4-manifold.

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It is an easy bit of algebra [M-H, p. 24], that $\omega \cdot \omega = \sigma(M)$ (mod 8). Thus if $w_2 = 0$ so that 0 is characteristic, then $\sigma(M) = 0$ (mod 8). Rochlin improved this by a factor of 2.

THEOREM [R_2]. *If M is closed, orientable,* PL *and* $\omega_2 = 0$, *then* $\sigma(M) = 0$ (mod 16).

Rochlin's Theorem is an anomaly in this sense: in dimensions $4k$, $k > 1$, there are closed, orientable, almost parallelizable PL (not smooth) manifolds P^{4k} with $\sigma(P^{4k}) = 8$ [M_2]. These PL manifolds are missing in dimension 4, and this accounts for the counterexamples to existence and uniqueness of PL structures on manifolds [K-S], [S_2]. Rochlin's Theorem is not known for topological 4-manifolds; the existence of such a topological 4-manifold of index 8 is equivalent to proving topological transversality in codimension 4 [S_1].

Let θ_3 be the group of homology cobordism classes of homology 3-spheres. Then Rochlin's Theorem provides an epimorphism $\theta_3 \to^R Z_2$. If Σ^3 is a representative of θ_3, it bounds a PL, parallelizable 4-manifold Q^4. Then $\sigma(Q^4)/8$ (mod 2) $\in Z_2$ is easily seen to be an invariant of the homology cobordism class of Σ^3 by use of Rochlin's Theorem [R_1].

The usual proof [R_2], [M-K] of Rochlin's Theorem is homotopy theoretic, requiring: the decomposability of Sq^3, the calculation of $J: \pi_3(SO) \to \pi_3^s$, Hirzebruch's identity, $\sigma(M^4) = \frac{1}{3}p_1(\tau_{M^4})\,[M^4]$, and the identification of $p_1(\tau_{M^4})$ with $\pm 2\{$obstruction to extending over M a trivialization of $\tau_{M^4} \mid (M^4\text{-point})\}$. Our proof is geometric, except for the use of the isomorphism $\Omega_4 \to^\sigma Z$ where Ω_4 is the oriented bordism group of oriented 4-manifolds.

Here is some motivation for our proof. From now on, all 4-manifolds are closed, orientable and smooth. First consider the generalization [K-M]: if the characteristic element $\omega \in H_2(M^4; Z)$ is represented by a smooth, imbedded S^2, then $\omega \cdot \omega - \sigma(M^4) = 0$ (16).

In general, ω is represented by an orientable (if $H_1(M; Z/2) = 0$) surface K^2, and $\omega \cdot \omega - \sigma(M^4) = 0$ (8), so it is predictable that there is a Z^2 obstruction associated with surgering K^2 to a 2-sphere. Here is the simplest case where that obstruction occurs.

Let $Q^4 = CP^2 \#\#^8 \overline{CP}^2$ be complex projective space connected sum eight copies reversed orientation. Let α_0 generate $H_2(CP^2; Z)$ and let α_i generate the ith copy of $H_2(\overline{CP}^2; Z)$. Then $\omega = 3\alpha_0 + \alpha_1 + \cdots + \alpha_8$ is characteristic and $\omega \cdot \omega = 1$. Since $\omega \cdot \omega - \sigma(Q^4) = 8$, ω cannot be represented by a smooth imbedded S^2. (To see this directly from Rochlin's Theorem, suppose S^2 represents ω; its normal disk bundle N is the Hopf bundle, since $\omega \cdot \omega = 1$, so $\partial N = S^3$; remove N and sew in B^4; the new manifold has $\omega_2 = 0$ and index -8, a contradiction.) In this case α_0 is represented by any nonsingular cubic, all of which are tori $S^1 \times S^1 \cdot \alpha_i$ is represented by $CP^1 = S^2$, so ω is represented by an $S^1 \times S^1$, and we cannot reduce the genus.

We can try to surger K inside M^4 to get a 2-sphere. Let $A_1, \cdots, A_{2s}$ be imbedded circles representing the generators of a symplectic basis of $H_1(K; Z) = Z^{2s}$. To surger some A_i, we must smoothly imbed a 2-ball D_i in M with $D_i \cap K = A_i$, D_i and K transverse, and so that the normal vector field v to ∂D_i which is tangent to K extends to a normal vector field V to D_i. We can then replace the normal 1-disk bundle to A_i in K by the normal 0-sphere bundle (the boundary of the 1-disk bundle determined by the vector field V) of D_i, thereby reducing the genus of K by 1.

We can always imbed D_i transversely to K with $\partial D_i = A_i$. The obstruction to extending v over D_i is an integer $x \in Z = \pi_1(SO(2))$. The algebraic intersection of int D_i with K, $\pitchfork$(int D_i, K), is another integer d. Also we may spin D_i once around A_i, as in Figure 1, changing x to $x \pm 1$ and d to $d \pm 1$, so that $d - x$ is unchanged. By iteration we can make either d or x zero.

Let S be a smooth imbedded 2-sphere in M^4. Since K is characteristic, $K \cdot S = S \cdot S$ (mod 2); $S \cdot S$ is the Euler class of the normal bundle. Under the connected sum $D_1 \# S$, x changes to $x + S \cdot S$ and d changes to $d + K \cdot S$. Thus $d - x = d + x = d + x + K \cdot S + S \cdot S$ (mod 2) is a possible Z_2 obstruction to surgery on A_i. Associating $d + x$ (mod 2) to each A_i, we obtain a quadratic form $\tilde{q}: H_1(K; Z_2) \to Z_2$. The Arf invariant of $\tilde{q}$, $\phi(M, K)$, is shown in Lemmas 3–5 to be an invariant of the pair (M, K) up to cobordism of such pairs.

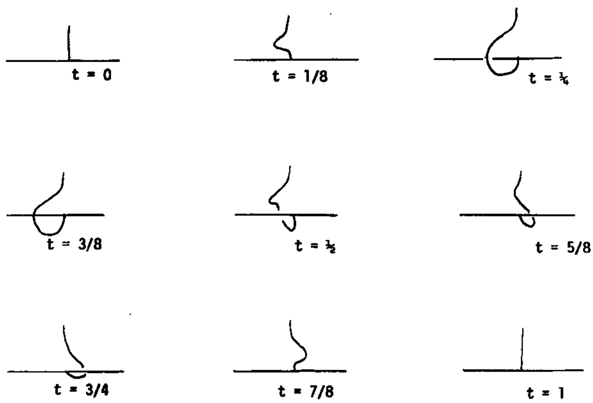


FIGURE 1

This is a movie of D_i spinning around A_i. In particular, we choose an interval of A_i, represented by the time t axis, so that at a fixed time A_i is represented by the center dot. The horizontal line is a slice of K, normal to A_i. The vertical line represents a collar of ∂D_i in D_i. The one point of intersection of K and int D_i occurs at time 3/8.

To be precise, let Ω_4^{char} be the group (under disjoint union) of "characteristic" pairs (M^4, K^2) up to "characteristic" bordism, where M and K are closed and oriented and $[K] \in H_2(M; Z)/_{\text{torsion}}$ is characteristic. Two pairs (M, K) and (M', K') are characteristically bordant if there exist an oriented 5-manifold $\bar{M}$ and an oriented 3-submanifold $\bar{K}^3$ with $[\bar{K}]_2$ dual to $\omega_2(\tau_{\bar{M}})$ and $\partial(\bar{M}, \bar{K}) = (M, K) \cup -(M', K')$.

Thus we have $\phi: \Omega_4^{\text{char}} \to Z_2$.

We show (Lemmas 1 and 2) that $\alpha: \Omega_4^{\text{char}} \to Z \oplus Z$ is an isomorphism where $\alpha(M, K) = (\sigma(M), (K \cdot K - \sigma(M))/8)$, and exhibit generators of Ω_4^{char}. Only here do we need the fact that $\Omega_4 = Z$.

Finally we show that the Z_2-invariant $\theta: \Omega_4^{\text{char}} \to Z_2$, defined by $\theta(M, K) = ((K \cdot K - \sigma(M))/8)(2)$, is equal to ϕ, by showing they are equal on the generators of Ω_4^{char} (Lemma 6, Theorem 1).

LEMMA 1. *$\alpha: \Omega_4^{char} \to Z \oplus Z$ is an injection.*

PROOF. $\alpha(M, K) = (\sigma(M), (K\cdot K - \sigma(M))/8)$. Signature and intersection are both additive with respect to disjoint sum. σ is well known to be an oriented bordism invariant. Also, $K\cdot K$ is invariant, for if $(M, K) = \partial(\bar{M}, \bar{K})$, then

$$K\cdot K = \hat{K} \cup \hat{K}[M] = \hat{\bar{K}} \cup \hat{\bar{K}}(i_*[M]) = \hat{\bar{K}} \cup \hat{\bar{K}}(0) = 0$$

(here we denote Poincaré duals by "^" and $i: M \to \bar{M}$ is inclusion). Therefore α is a well-defined homomorphism.

To show α is injective, suppose a characteristic pair (M, K) satisfies $\sigma(M) = K\cdot K = 0$. M and K can be assumed connected. It is tempting to let $\bar{K}$ be an oriented 3-manifold with $\partial\bar{K} = K$ and $\bar{M} = M \times I \cup_{K\times B^2} \bar{K} \times B^2 \cup W$, where W is a spin 5-manifold whose boundary is the component of $\partial(M \times I \cup \bar{K} \times B^2)$ other than M. Unfortunately $\bar{K}$ may not be dual to $\omega_2(\bar{M})$.

Given a Morse function on M, we can assume any extra critical points of index 0 or 4 have been cancelled. The critical points of index 1 determine descending 1-manifolds which in turn determine a family of disjointly imbedded, oriented circles, which, by general position, are disjoint from K. Since K is dual to $\omega_2(M)$, we can frame the tangent bundle of $M - K$. This determines a framing of the normal bundles of the circles. Let N^5 be obtained from $M \times I$ by adding 2-handles to the circles in $M \times 1$ according to the framing. Then $\partial N = M \times 0 \cup \partial_1 M$ and $\partial_1 N$ is 1-connected. Furthermore $K \times I$ in N is dual to $\omega_2(N)$ because the complement is still framed and $K \times I$ is still the obstruction to extending this framing. The same is true for $\tilde{N} = N \natural k(S^2 \times B^3)$, the boundary connected sum of N along $\partial_1 N - K \times 1$ with k copies of $S^2 \times B^3$.

Since $\sigma(\partial_1\tilde{N}) = 0$ and $\pi_1(\partial_1\tilde{N}) = 0$, $\tilde{N}$ has the same intersection form as $X = \# r(S^2 \times S^2) \# \varepsilon(S^2 \tilde{\times} S^2)$ where $S^2 \tilde{\times} S^2$ is the nontrivial S^2 bundle over S^2 and $\varepsilon = 0$ or 1. By Theorem W_1 below, $\partial_1\tilde{N}$ is diffeomorphic, via h, to X for some k. We must choose h correctly.

In X, let $\alpha_i, \beta_i \in H_2((S^2 \times S^2)_i; Z)$ and $\gamma, \delta \in H_2(S^2 \tilde{\times} S^2; Z)$ be the standard bases with intersection forms

$$\begin{array}{c} \\ \alpha_i \\ \beta_i \end{array}\begin{array}{cc} \alpha_i & \beta_i \\ \left[\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right] \end{array} \quad \text{and} \quad \begin{array}{c} \\ \gamma \\ \delta \end{array}\begin{array}{cc} \gamma & \delta \\ \left[\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}\right] \end{array},$$

respectively. Since $k_*[K]$ is dual to $\omega_2(X)$, it follows that $h_*[K]$ is a sum of even multiples of the α_i's, β_i's and δ and an odd multiple of γ. Since $K\cdot K = 0$, by Theorem W_2 below [W_2], we can find an orthogonal automorphism of $H_2(X; Z)$ taking $h_*[K]$ to $2n\alpha_1$ (if $\varepsilon = 0$) or to $(2n + 1)\gamma$ (if $\varepsilon = 1$). By Theorem W_3 below [W_3], this automorphism is realized by a diffeomorphism $h': X \to X$.

Let $Y = \# r(S^2 \times B^3) \# \varepsilon(S^2 \tilde{\times} B^3)$, $\partial Y = X$. Let $\bar{M} = \tilde{N} \cup Y$ where we identify ∂Y with $\partial_1\tilde{N}$ by $h'h$. Let $\bar{\alpha}_1$ and $\bar{\gamma} \in H_3(Y, \partial; Z)$ be the classes represented by B^3 fibers so that $\partial\bar{\alpha}_1 = \alpha_1$ and $\partial\bar{\gamma} = \gamma$. There is an oriented 3-manifold $(J, \partial) \subset (Y, X)$ with $\partial J = (h'h)(K \times 1)$ and $[J, \partial] = 2n\bar{\alpha}_1$ or $(2n + 1)\bar{\gamma}$ as $\varepsilon = 0$ or 1. (This follows from a relative version of the representability of codimension two integral homology classes by oriented submanifolds.) Clearly $[J, \partial]$ is dual to $\omega_2(Y)$.

Now let $\bar{K} = K \times I \cup J$, identified along $\partial J = K \times 1$. $\bar{K}$ is dual to $\omega_2(\bar{M})$ (use the Mayer-Vietoris sequence on $\bar{M} = \tilde{N} \cup Y$), $\bar{K}$ is oriented, and $\partial\bar{K} = K$. Thus we have constructed a null bordism of (M, K) in Ω_4^{char}. □

Note. $B\,\mathrm{spin}^c$ is defined as the pullback

$$\begin{array}{ccc} B\,\mathrm{Spin}^c & \longrightarrow & * \\ \downarrow & & \downarrow \\ BSO & \xrightarrow{\delta w_2} & K(Z,3) \end{array}$$

A spin^c-structure on an oriented 4-manifold is a lifting of the tangent bundle to $B\,\mathrm{spin}^c$. It is known that every oriented smooth manifold has a spin^c-structure, [*H*, *H*]. It follows from Lemma 1 that every oriented smooth 4-manifold, M, with $\sigma(M) = 0$, admits a spin^c-structure which is a spin^c boundary. (Proof. If M is spin let $K \subsetneq M$ satisfy $[K] = 0 \in H_2(M; Z)$. Now $K \cdot K = 0$ and $(M, K) = \partial(\bar{M}, \bar{K})$ with $[\bar{K}] \in H_3(\bar{M}, M; Z)$ dual to $w_2(\tau(\bar{M}))$. So M bounds $\bar{M}$ with $w_2(\tau(M))$ integral; hence there is a spin^c-structure on $\bar{M}$; its restriction is the desired spin^c-structure on M. If M is not spin, by the classification of symmetric-unimodular-odd-bilinear forms there is a basis $\alpha_1, \cdots, \alpha_n, \alpha_{n+1}, \cdots, \alpha_{2n}$ for $H_2(M; Z)$ satisfying $\alpha_i \cdot \alpha_i = +1$ for $1 \leqq i \leqq n$ and $\alpha_i \cdot \alpha_i = -1$ for $n + 1 \leqq i \leqq 2n$, and $\alpha_i \cdot \alpha_j = 0$ for $i \neq j$. Let $K \subsetneq M$ represent $\sum_{i=1}^{2n} \alpha_i$. K is characteristic and $K \cdot K = 0$. Now, as above, M bounds $\bar{M}$ and can be a spin^c-boundary.)

We have used:

THEOREM W_1 [W_1, P. 147]. *If two simply connected, smooth, closed, 4-manifolds, M_1 and M_2, have isomorphic intersection forms, then $M_1 \# k(S^2 \times S^2)$ is diffeomorphic to $M_2 \# k(S^2 \times S^2)$ for some k.*

REMARK. The proof relies on the fact that $\Omega_4^{SO} \to^{\sigma} Z$ is injective.
Only here do we need the calculation of Ω_4^{SO}.

THEOREM W_2 [W_2, THEOREM 4]. *The group of automorphisms of*

$$H_2(M \# 2(S^2 \times S^2) \colon Z)/_{\text{torsion}}$$

which are orthogonal (preserve the intersection form) is transitive on primitive characteristic elements of a given square.

THEOREM W_3 [W_3, THEOREM 2]. *If M is a smooth, simply connected, closed 4-manifold with an indefinite intersection form, then any automorphism of $H_2(M \# S^2 \times S^2, Z)$ can be represented by a diffeomorphism of $M \# S^2 \times S^2$.*

LEMMA 2. *$\alpha\colon \Omega_4^{\text{char}} \to Z \oplus Z$ is onto.*

PROOF. $\alpha\,(CP^2, CP^1) = (1, 0)$ and $\alpha(CP^2 \# \overline{CP}^2, 3\gamma \# \overline{CP}^1) = (0, 1)$, where $\overline{CP}^2$ has the orientation (opposite to CP^2) for which $\overline{CP}^1 \cdot \overline{CP}^1 = -1$ and 3γ is the non-singular complex (elliptic) curve of degree 3 in CP^2 (homologically it is 3 times the generator $[CP^1]$). Since α is additive under disjoint union (or pairwise connected sum), the proof is finished. □

We define a characteristic 2-ad,

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & & \searrow & \\ A & & & & M \\ & \searrow & & \nearrow & \\ & & K & & \end{array}$$

to be a characteristic pair (M, K) together with an imbedding of an unoriented family of circles, A, in K, and an imbedding of an unoriented surface, B, in M, which meets K normally at $A = \partial B$ and transversally at isolated points of $K - A$. (Generally we will suppress inclusions from our notation.) We say that two char-2-ads,

$$\begin{array}{ccc} & B & \\ \nearrow & & \searrow \\ A & & M \\ \searrow & & \nearrow_{i} \\ & K & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & B' & \\ \nearrow & & \searrow \\ A' & & M' \\ \searrow & & \nearrow_{i'} \\ & K' & \end{array}$$

are equivalent (written $\sim$) if (M, K) and (M', K') are characteristically bordant, via $(\bar{M}, \bar{K})$, and $A \cup A' \subset \partial(\bar{K})$ bounds an unoriented surface, $\bar{A} \subset \bar{K}$.

We now define a Z_2 valued invariant, q, of characteristic 2-ads. Let $\nu_{B \subset M}$ and $\nu_{\partial B \subset K}$ denote the normal bundles. We have $\nu_{B \subset M} \,|\, \partial B = \nu_{\partial B \subset K} \oplus \xi$. Since K is orientable the first summand is trivial; so ξ is also trivial (as $\nu_{\partial B \subset K} \oplus \xi$ extends over B). Let F be a framing $\nu_{B \subset M} \,|\, _{\partial B}$, restricting to a framing on each factor. Let $w_2(\nu_{B \subset M} \,|\, _{\partial B}, F) = \chi_2 \in H^2(B, \partial B; Z_2)$ be the relative Stiefel-Whitney class.

(χ_2 is the mod 2 reduction of the obstruction in $H^2(B, \partial B, Z_{\text{twisted}})$ to extending $\nu_{\partial B \subset K}$ to a section of $\nu_{B \subset M}$.) Let $X = \chi_2[B, \partial]_2 = 0$ or 1. We define

$$q\left(\begin{array}{ccc} & B & \\ \nearrow & & \searrow \\ A & & M \\ \searrow & & \nearrow \\ & K & \end{array}\right) = \pitchfork_M(\text{int } B, K) + X \pmod 2.$$

By $\pitchfork(\ \ ,\ \ )$ we mean the number (mod 2) of transverse intersections.

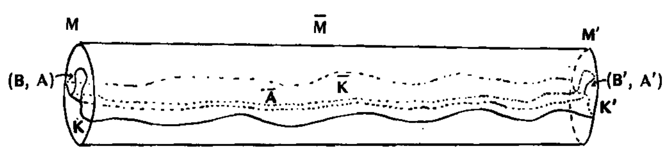
LEMMA 3. *If*

$$\left(\begin{array}{ccc} & B & \\ \nearrow & & \searrow \\ A & & M \\ \searrow & & \nearrow \\ & K & \end{array}\right) \sim \left(\begin{array}{ccc} & B' & \\ \nearrow & & \searrow \\ A' & & M' \\ \searrow & & \nearrow \\ & K' & \end{array}\right)$$

then

$$q\left(\begin{array}{ccc} & B & \\ \nearrow & & \searrow \\ A & & M \\ \searrow & & \nearrow \\ & K & \end{array}\right) = q\left(\begin{array}{ccc} & B' & \\ \nearrow & & \searrow \\ A' & & M' \\ \searrow & & \nearrow \\ & K' & \end{array}\right).$$

PROOF. $Y = \bar{A} \cup B \cup B'$ is a closed unoriented surface.
Diagram:



Let $h: Y \subset M$ be the inclusion

$$Y \cdot \bar{K} = \pitchfork_M(B, K) + \pitchfork_{\bar{M}}(\bar{A}, \bar{K}) + \pitchfork_{M'}(B', K').$$

The middle term is to be interpreted as the obstruction (mod 2) to extending $\nu_{A \amalg A' \subset B \amalg B'}$ to a section of $\nu_{\bar{K} \subset \bar{M}/\bar{A}}$.

On the other hand,

$$\begin{aligned} Y \cdot \bar{K} &= h^*(w_2(\tau(\bar{M})))[Y] = w_2(\tau(\bar{M}) \,|\, _Y)[Y] \\ &= w_2(\nu_{Y \subset \bar{M}}) + w_2(\tau(Y)) + w_1(\nu_{Y \subset \bar{M}}) \cdot w_1(\tau(Y)). \end{aligned}$$

But $\bar{M}$ is oriented, so $0 = w_1(\tau\bar{M}/Y) = w_1(\nu_{Y \subset \bar{M}}) + w_1(\tau(Y))$, so $w_1(\nu_{Y\subset\bar{M}})\cdot w_1(\tau(Y)) = w_1(\tau(Y))^2 = w_2(\tau(Y))$ by the Wu formula, so $w_2(\tau(\bar{M})\,|\,_Y)[Y] = w_2(\nu_{Y\subset\bar{M}})\,[Y]$. (Note that if Y is oriented and $\bar{M}$ unoriented the preceding assertion is still true.)

Rounding corners, $\nu_{Y\subset\bar{M}/\bar{A}\amalg A'}$ has a framing, $\bar{F}$, which extends F (restricted to A this is (F, inward normal to B)). We can use this framing to break $w_2(\nu_{Y\subset\bar{M}})$ up into three relative Stiefel-Whitney classes

$$w_2(\nu_{Y\subset\bar{M}})[Y] = \underbrace{w_2(\nu_{B\subset M}, F)[B, \partial]}_{\overset{\text{def}}{=}\, x} + w_2(\nu_{A\subset\bar{M}}, \bar{F})[\bar{A}, \partial] + \underbrace{w_2(\nu_{B'\subset M'}, F')[B', \partial']}_{\overset{\text{def}}{=}\, x'}$$

$$w_2(\nu_{A\subset\bar{M}}, \bar{F}) = w_2(\nu_{\bar{K}\subset\bar{M}}\,|\,_A, \bar{F}_{2,3}) + w_1(\nu_{\bar{K}\subset\bar{M}}, \bar{F}_{2,3})\cdot w_1(\nu_{A\subset\bar{K}}, F_1)$$

where $\bar{F}_{2,3}$ (and $\bar{F}_1$) denote the last two (and the first) vectors of $\bar{F}$. Let $w_1(\nu_{\bar{K}\subset\bar{M}}, F_{2,3}) = u$ and $w_1(\nu_{A\subset\bar{K}}, F_1) = v$.

$$u + v = w_1(\nu_{\bar{A}\subset\bar{M}}, \bar{F}) = w_1(\tau(\bar{A}), \text{framing of } \tau(A \amalg A'))$$

(because $\bar{M}$ is orientable). As in the closed case, $(u + v) \cup x = \mathrm{Sq}^1 x = x^2$ for all $x \in H^1(\bar{A}, \partial; Z_2)$. $v^2 = (u + v) \cup v = uv + v^2$, so $uv = 0$. Therefore, $w_2(\nu_{Y\subset\bar{M}})[Y] = + \pitchfork_M(\bar{A}, \bar{K})$, so

$$\pitchfork_M(B, K) + x = \pitchfork_{M'}(B', K') + x' \quad (\text{mod } 2). \qquad \square$$

Note. The above result holds if the hypothesis that $(\bar{M}; M, M')$ is oriented is replaced by: Y and $\bar{K}$ are oriented.

COROLLARY 1. *If* (M, K) *is a characteristic pair,* q *determines a well-defined function,* $\tilde{q}\colon H_1(K; Z_2) \to Z_2$.

PROOF. We kill $H_1(M; Z_2)$ with a finite number of framed 1-surgeries in the complement of K (call the trace N). To $\beta \in H_1(K; Z_2)$ we associate an unoriented surface $(B, \partial) \subset (M, K)$ with $[\partial B] = \beta$. Define

$$\tilde{q}(\beta) = q\left(\begin{array}{ccccc} & & B & & \\ & \nearrow & & \searrow & \\ \partial B & & & & M \\ & \searrow & & \nearrow & \\ & & K & & \end{array}\right).$$

To check that this procedure is well defined let $\tilde{N}$ be an alternative trace. Let

$$(\bar{M}, \bar{K}) = (N, K\times I)\Big/\bigcup_{M,\,K\times 0}(\tilde{N}, K\times I), \qquad \partial(\bar{M}, \bar{K}) = (M_0, K_0) \cup -(M_1, K_1).$$

Let (B_0, ∂) and (B_1, ∂) denote unoriented surfaces in (M_0, K_0) and (M_1, K_1) with $[\partial B_0] = \beta \in H_1(K_0; Z_2)$, $[\partial B_1] = \beta \in H_1(K_1; Z_2)$. There is an unoriented bordism $\bar{A} \subset \bar{K}$ with $\partial\bar{A} = \partial B_0 \cup \partial B_1$, so by definition

$$\left(\begin{array}{ccccc} & & B_0 & & \\ & \nearrow & & \searrow & \\ \partial B_0 & & & & M_0 \\ & \searrow & & \nearrow & \\ & & K_0 & & \end{array}\right) \sim \left(\begin{array}{ccccc} & & B_1 & & \\ & \nearrow & & \searrow & \\ \partial B_1 & & & & M_1 \\ & \searrow & & \nearrow & \\ & & K_1 & & \end{array}\right).$$

By Lemma 3, $\tilde{q}$ is well defined.

LEMMA 4. $\tilde{q}\colon H_1(K; Z_2) \to Z_2$ *is quadratic, i.e.* $q(\varepsilon + \delta) - q(\varepsilon) - q(\delta) = \varepsilon\cdot\delta$ *for all* $\varepsilon, \delta \in H_1(K; Z_2)$.

PROOF. Let A_ε and A_δ represent ε and δ. Let B_ε and B_δ be as before. Suppose A_ε and A_δ intersect transversally at one point p. Let A_γ be the connected sum (representing $\varepsilon + \delta$) as in Figure 2. Piecewise linearly, we get B_γ from $B_\varepsilon \cup B_\delta \cup T_1 \cup T_2$ where T_1 and T_2 are two curved triangles as shaded in Figure 2; note that $\partial B_\gamma = A_\gamma$. The normal vector fields on B_ε and B_δ must be extended to the new part of A_γ as drawn. Consider the boundary (a circle) of a neighborhood of B in B_γ and push it off itself using the vector field. The two circles link, indicating that the obstruction to extending the vector field over the neighborhood is one. We have verified that $\chi_{\varepsilon+\delta} - \chi_\varepsilon - \chi_\delta = \varepsilon\cdot\delta = 1$.

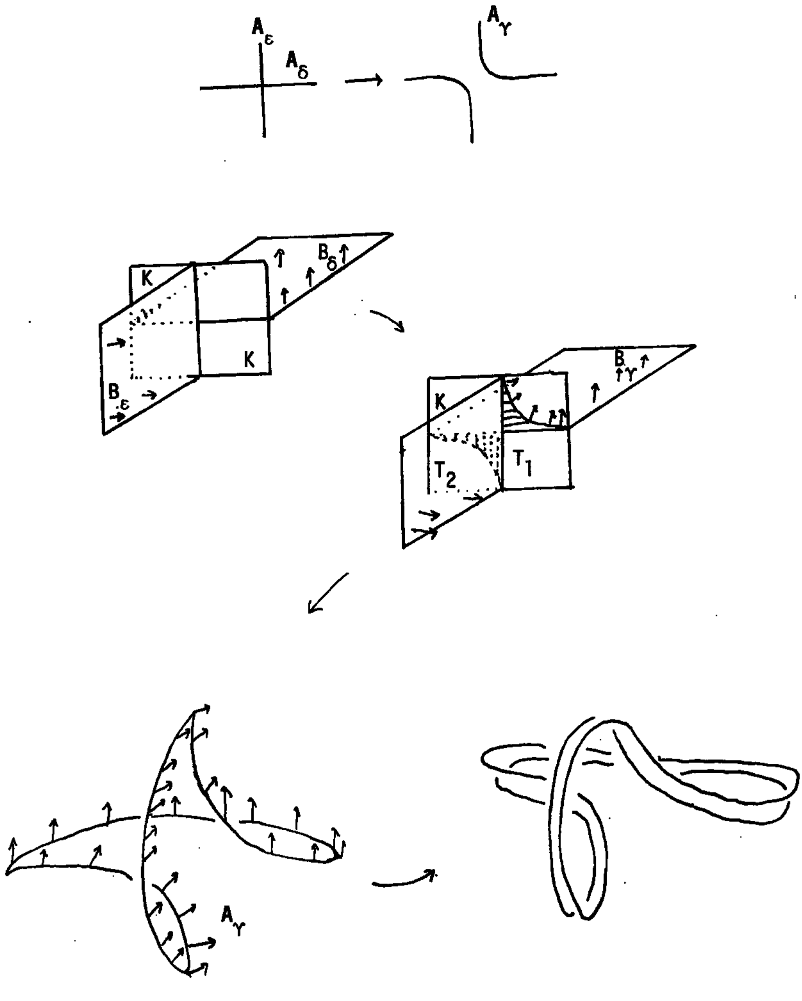


FIGURE 2

If int $B_\varepsilon \cap$ int $B_\delta \neq \varnothing$, then we may push these double points of B_γ off the boundary, adding two points to int $B_\gamma \cap K$. Thus $\pitchfork(B_\varepsilon, K) + \pitchfork(B_\delta, K) = \pitchfork(B_\gamma, K)$ (mod 2). We have shown in this special case that $q(\varepsilon + \delta) - q(\varepsilon) - q(\delta) = \varepsilon\cdot\delta$; the other (easier) cases are left to the reader. $\square$

Let $\phi(M, K)$ be the Arf invariant of $\tilde{q}: H_1(K; Z_2) \to Z_2$. (See the Appendix of [R-S] for a short presentation of the Arf invariant.)

LEMMA 5. *ϕ determines a well-defined homomorphism:* $\Omega_4^{\mathrm{char}} \to^{\phi} Z_2$.

PROOF. Assume that (M, K) and (M', K') are characteristically bordant via $(\bar{M}, \bar{K})$. For the usual reason

$$\dim(V = \mathrm{Ker}(H_1(K \cup K', Z_2) \to H_1(\bar{K}; Z_2))) = \tfrac{1}{2}\dim(H_1(\partial\bar{K}; Z_2)).$$

The intersection pairing on $H_1(K \cup K'; Z_2)$ is identically zero when restricted to V. $\tilde{q}(K \cup K') = \tilde{q}(K) \oplus \tilde{q}(K')$. If viewed properly, Lemma 3 implies that $\tilde{q}(K \cup K') \mid V$ is identically zero. (If $A \in V$ and $\partial\bar{A} = A$, then $\bar{A} \in H_2(\bar{K}, K \cup K'; Z_2)$; one should regard $\partial(\bar{M}, \bar{K}, \bar{A})$ as $(M \cup -M', K \cup -K', A) \cup (\phi, \phi, \phi)$.) Hence $\phi(\tilde{q}(K \cup K')) = 0$, so $\phi(\tilde{q}(K)) = \phi(\tilde{q}(K'))$. □

LEMMA 6. $\phi(CP^2, CP^1) = 0$ *and* $\phi(CP^2 \# \overline{CP}^2, 3\gamma \# \overline{C}P^1) = 1$.

PROOF. CP^1 is S^2, so H_1 is zero and thus $\phi(CP^2, CP^1) = 0$.

$\phi(CP^2 \# \overline{CP}^2, 3\gamma \# \overline{C}P^1) = \phi(CP^2, 3\gamma)$. In CP^2, 3γ is represented by any cubic; it is convenient to pick $x^3 = y^2 z$. In the coordinate charts $x = 1$ or $y = 1$, the solution is nonsingular, but for $z = 1$, $y^2 = x^3$ is the cone on the (2, 3)-torus knot (= trefoil knot). So 3γ is represented by a smooth 2-sphere except for the cone point. If B^4 is centered at the cone point, replace the cone in B^4 by the Seifert surface (Figure 3) of the trefoil knot in $S^3 = \partial B^4$, obtaining a 2-torus T^2 as a representative of 3γ. The circles A_1, A_2 in the Seifert surface generate $H_1(T^2; Z)$. Each A_i is a trivial knot and bounds a 2-ball B_i in B^4. The obstruction to extending $\nu_{A_i \subset T^2}$ to a section of $\nu_{B_i \subset B^4}$ is the linking number of A_i and $s(A_i)$ in S^3, where s is a nonzero section of $\nu_{A_i \subset T^2}$. $L(A_i, s(A_i)) = \pm 1$. Since $\bigcap(\mathrm{int}\, B_i, T^2) = 0$, it follows that $q(A_i) = 1$. Thus $\phi(CP^2, T^2) = 1$ (see Appendix [R-S]). □

Let $\theta: \Omega_4^{\mathrm{char}} \to Z_2$ be the homomorphism given by $\theta(M, K) = (K \cdot K - \sigma(M))/8 \pmod 2$.

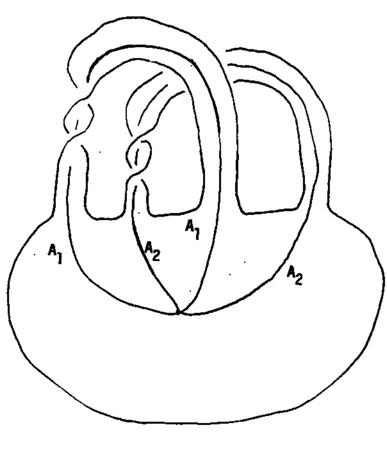


FIGURE 3

THEOREM 1. $\theta(M, K) = \phi(M, K)$.

PROOF. Since both θ and ϕ are homomorphisms (Lemma 5), it is sufficient to check the equality on the generators of Ω_4^{char}, (CP^2, CP^1) and $(CP^2 \# \overline{CP}^2, 3\gamma \# \overline{CP}^1)$. But we have seen (Lemmas 2 and 6), that θ and ϕ are both zero on the first generator, as on the second. □

COROLLARY 2 [K-M, THEOREM 1]). *If (M, K) is a characteristic pair, and K is a 2-sphere, then* $\theta(M, K) = 0$.

COROLLARY 3 (ROCHLIN'S THEOREM [$\mathbf{R}_2$]). *If (M, ϕ) is a characteristic pair, then* $\theta(M, \phi) = 0$.

PROOFS. $H_1(K, Z_2) = 0$ for $K = S^2$ or $\emptyset$, therefore $\phi(M, K) = 0$. By Theorem 1, $\theta(M, K) = 0$.

Note. Suppose $K \subsetneq M$ is a PL imbedding with nonlocally flat points $p_1, \cdots, p_r$ at which K is the cone on knots $S_1, \cdots, S_r$, Let $\text{Arf}(S_i)$ be the Arf invariant of S_i (see [$\mathbf{R}_2$]). Then if we define

$$\phi(M,K) = \phi(\tilde{q}(H_1(K; Z_2))) + \sum_{i=1}^{r} \text{Arf}(S_i) \quad (\text{mod } 2),$$

we may still conclude that $\theta(M, K) = \phi(M, K)$.

We now show that "stably" ϕ is the only obstruction to surgering K to a 2-sphere. Let $M_s = M \# s(S^2 \times S^2)$, and let j_s be the composition

$$H_2(M; Z) \xrightarrow{\text{inc}_*^{-1}} H_2(M - D^4; Z) \xrightarrow{\text{inc}_*} H_2(M_s; Z).$$

THEOREM 2. *Suppose $\pi_1(M^4) = 0$. Then $\phi(M,K) = 0$ iff for some s, $\exists\, (M_s, K')$ such that K' is a 2-sphere and $i'_*[K'] = j(i_*[K])$.*

Note. Larry Taylor has independently obtained this result.

PROOF. The if direction follows from Theorem 1 [K-M] and our Theorem 1. The argument for the only if direction will be quite liberal with copies of $S^2 \times S^2$. Since $\phi(M, K) = 0$, there is a subspace $V \subset H_1(K; Z_2)$ such that (1) $\dim(V) = \frac{1}{2}\dim(H_1(K; Z_2))$, (2) $v_1 \cdot v_2 = 0$ for $v_1, v_2 \in V$, and (3) $q(v) = 0$ for $v \in V$. Let $A_1, \cdots, A_p$ be circles disjointly imbedded in K representing a basis for V. Let $\tilde{A}_1, \cdots, \tilde{A}_p$ denote copies of $A_1, \cdots, A_p$ pushed off in a normal direction to K so that the linking numbers $L(K, \tilde{A}_i) = 0$. Since K is characteristic, $\tau(M) \mid \tilde{A}_i$ is canonically framed. Since $\pi_1(M) = 0$, framed surgery on $\{\tilde{A}_1, \cdots, \tilde{A}_p\}$ replaces M with M_p. In M_p, there are disjointly imbedded 2-disks, $B_1, \cdots, B_p$ with $\partial B_i = A_i$ and $B_i \cap K = A_i$. There is an Euler class obstruction $\chi_i \in H^2(B_i, \partial; Z)$ to extending $\nu_{A_i \subsetneq K}$ to a section of $\nu_{B_i \subsetneq M_p}$, but $0 = q[A_i] \equiv \chi_i[B_i, \partial] \ (\text{mod } 2)$.

Consider the diagonal imbedding $\Delta: S^2 \subsetneq S^2 \times S^2$. $\chi(\nu_{S^2 \subsetneq^{\Delta} S^2 \times S^2})[S^2] = 2$. $\chi_i[B_i, \partial]$ may be altered by ± 2 by taking a connected sum of pairs $(M_p, B_i) \# (S^2 \times S^2, \pm\Delta(S^2))$. In this way it is possible to alter M_p and B_i so that $\chi_i[B_i, \partial]$ becomes zero and M_p becomes M_{p+r}. χ_i is the only obstruction to ambient surgery on K along B_i. The result of surgery on $B_1, \cdots, B_p$ is a smooth submanifold, K', with $H_1(K'; Z_2) = 0$; so K' is a 2-sphere. To verify $i'_*[K'] = j(i_*[K])$, recall that $L(K, \tilde{A}_i) = 0$. □

Note. A similar argument shows: If $\pi_1(M^4) = 0$ and $\xi \in H_2(M^4; Z)$, then for some s, $j_s(\xi)$ is represented by an imbedded torus.

2. In §1, M, $\bar{M}$, K, $\bar{K}$ were taken to be oriented, while B and Y were unoriented (and possibly unorientable). Orienting M, $\bar{M}$, K, $\bar{K}$ was convenient in that it enabled us to calculate the corresponding bordism group, Ω_4^{char}. We chose B (and Y) unoriented because we were defining a quadratic form on $H_1(K; Z_2)$ and orientations would be superfluous.

In the following application we unorient K.

THEOREM 3. *Although* $(CP^2, \gamma) - (CP^2, 3\gamma) = \partial(CP^2 \times I$, *unoriented simplicial 3-chain*), $(CP^2, \gamma) - (CP^2, 3\gamma)$ *cannot be written as* $\partial(CP \times I$, *immersed unoriented 3-manifold*).

PROOF. This is an exercise from the proof of Lemma 3. $\phi(CP^2, \gamma) \neq \phi(CP^2, 3\gamma)$, but a manifold bordism (even an immersed one) $\bar{K}$ from γ to 3γ would force ϕ to assume equal values at each end. □

REMARK 1. There is an old and elegant procedure for approximating a Z_2-simplicial-cycle of dimension 2 in a triangulated 3-manifold by an unoriented, triangulated submanifold. This procedure generalizes for 2-dimensional cycles in any manifold, and for $(n - 1)$-dimensional cycles in any n-manifold. The first open case would be: Can you approximate a 3-dimensional Z_2-simplicial-chain in a triangulated 5-manifold by an unoriented, triangulated submanifold? Theorem 3 provides an example (in the relative case) where the answer is no.

REMARK 2. A key calculation occurs in the proof of Lemma 3 in which we show

$$h^* w_2(\tau(\bar{M}))[Y] = w_2(\nu_{Y \subsetneq^h \bar{M}})[Y].$$

We observe that this equality still holds if we replace the assumption (1) $\bar{M}$ is oriented and Y is unoriented, with (2) $\bar{M}$ is unoriented and Y is oriented.

Suppose M, M' are unoriented with $K \subsetneq M$ and $K' \subsetneq M'$ oriented surfaces dual to $w_2(\tau(M))$ and $w_2(\tau(M'))$, respectively, and that image$(H_1(K; Z)) \subset H_1(M; Z)$ and image$(H_1(K'; Z)) \subset H_1(M'; Z)$ are zero. Then if $\bar{M}$ is unoriented and $\bar{K}$ is oriented, with $\bar{K}$ dual to $w_2(\tau(\bar{M}))$, $\tilde{q}(H_1(K; Z_2))$ and $\tilde{q}(H_1(K'; Z_2))$ will be defined and

$$\phi(M, K) = \text{Arf}(\tilde{q}(H_1(K; Z_2))) = \text{Arf}(\tilde{q}(H_1(K'; Z_2))) = \phi(M, K')$$

This is because, with the above assumptions, we will be able to choose B and $\bar{A}$ (see notation preceding Lemma 3) to be oriented compatibly so that Y also is oriented. Now the proof of Lemma 5 goes through using our second set of orientability assumptions. This leads to the following nonimbedding theorem.

THEOREM 4. *Let M be an orientable* PL *4-manifold and let* $\alpha \in H_2(M; Z)$ *satisfy* $(\alpha \cdot \alpha - \sigma(M))/8 \equiv 1 \pmod 2$. *Let N be an unorientable* PL *4-manifold with* $w_2(\tau(N)) = 0$ (*or it is sufficient to assume* $w_2(\tau(N))$ *is represented by a smoothly imbedded oriented surface* $A_2 \subsetneq^{\text{inc}} N$ *with* $\text{inc}_*(H_1(A; Z)) = 0 \in H_1(N; Z)$ and $\phi(N, A) = 0$; *note that ϕ is defined for the pair despite the fact that N is unoriented*). *Then* $\alpha' =$ *image*$(\alpha) \in H_2(M \# N; Z)$ *is not represented by a smooth imbedding* $S^2 \subsetneq^h M \# N$.

LEMMA 7. *A Z-oriented simplicial cycle,* β^3, *of dimension 3 in an unoriented triangulated 5-manifold may be approximated by a Z-oriented imbedded, triangulated submanifold.*

PROOF. By a modification in a spindle neighborhood of the 2-simplexes of β^3 (see Figure 4), we may assume β^3 is a manifold away from its 1 skeleton.

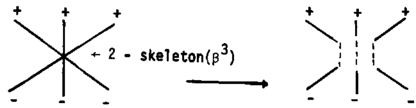


FIGURE 4

Let V be a tubular neighborhood of the 1-skeleton $\partial V = j(S^1 \times S^3) \# K(S^1 \tilde{\times} S^3)$ ($S^1 \tilde{\times} S^3$ denotes the twisted product). $\partial V \cap \beta^3 \stackrel{\text{def}}{=} L$ is an oriented surface imbedded in ∂V (oriented, because L is imbedded with trivial normal bundle in the $(\beta^3 - 1)$-skeleton). We will show that there is an oriented 3-manifold $\bar{L} \subsetneq V$ with $\partial \bar{L} = L$.

There are $j + k$ normal 4-disks $(D^4, \partial)_i \subset (V, \partial)$ such that V_j "cut" along $\bigcup_i D^4_i$ is a closed 4-disk, X. For each i, there are 2-copies of D^4_i, $D^4_{i,1}$ and $D^4_{i,2}$ included in ∂X. We may assume L meets each D_i in a link L_i. Because L was oriented, the associated links $L_{i,1} \subsetneq \partial D^4_{i,1}$ and $L_{i,2} \subsetneq \partial D^4_{i,2}$ are oriented oppositely (comparing orientations by regluing $D^4_{i,1}$ and $D^4_{i,2}$ to form D^4_i). Let $J_i \subsetneq D^4_i$ be an orientable surface with $\partial J_i = L_i$. Again let $J_{i,1} \subsetneq D^4_{i,1}$ and $J_{i,2} \subsetneq D^4_{i,2}$ be the corresponding copies of J_i in ∂X oriented so that $\partial(J_{i,1}) = -L_{i,1}$ and $\partial(J_{i,2}) = -L_{i,2}$.

$$W \stackrel{\text{def}}{=} \left(L - \bigcup_i L_i\right) \cup_i J_{i,1} \cup_i J_{i,2} \subsetneq \partial X$$

is an oriented surface. Let $Z \subsetneq X$ be an oriented 3-manifold with $\partial Z = W$. Because the orientations on $L_{i,1}$ and $L_{i,2}$ are opposite, so are the orientations on $J_{i,1}$ and $J_{i,2}$. As a result, if we reglue X to form V, the image of Z (which we will call $\bar{L}$) is an orientable 3-manifold contained in V with $\partial \bar{L} = L$.

Now we can approximate β^3 by $(\beta^3 - V) \cup -\bar{L}$, an oriented submanifold. □

PROOF OF THEOREM 4. We only consider the case: $w_2(\tau(N)) = 0$. (As an example N might be $S^1 \tilde{\times} S^3$.) By Theorem 1, α is represented by a smoothly imbedded surface $K \subsetneq^{i} M$ with $\phi(M, K) = 1$. If we consider $K \subsetneq^{i'} M \# N$ representing α', $\phi(M \# N, K)$ is defined and equal to 1. If α' were represented by a smooth imbedding, $S^3 \subsetneq^{h} M \# N$, $\phi(M \# N, S^2)$ would be defined and equal 0.

But by a relative form of Lemma 7, there is a smooth oriented 3-manifold, T, with $\partial T = K - S^2$ and a smooth imbedding

$$(T; K, S^2) \xrightarrow{j} (M \# N \times I, N \# N \times 0, M \# N \times 1)$$

restricting to the imbeddings $i' \times 0$ and $h \times 1$ on the boundary.

Remark 2 shows that the existence of $j: T \subsetneq M \# N \times I$ implies $\phi(M \# N, K) = \phi(M \# N, S^2)$, contradicting the above. □

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