JOHN HENRY CONSTANTINE WHITEHEAD

1904-1960

HENRY WHITEHEAD was at the height of his powers and of his mathematical influence when he died suddenly on 8 May 1960 after a heart-attack. In the last months before his death he had written several important papers, which showed how much he had still to contribute, and how keenly alive he was to the value of the newest work of young contemporaries.

His life was a happy one, with little for the biographer to chronicle. He was born in India on 11 November 1904, the son of the Right Reverend Henry Whitehead, for twenty-three years Bishop of Madras and sometime Fellow of Trinity College, Oxford; and of Isobel Whitehead daughter of Canon Duncan of Calne in Wiltshire. She was one of the early mathematical students of Lady Margaret Hall. There were many clerics and teachers among earlier generations of Whiteheads; the philosopher A. N. Whitehead was Bishop Whitehead's brother.

At the age of one and a half the young Henry was sent back to England and saw little of his parents until their retirement to England in 1920. At Eton, and as a Balliol undergraduate, his high spirits and convivial habits and his many-sided successes at games gave little grounds for expecting him to excel on the academic side. He boxed and played billiards for the University, and just missed winning a blue for cricket. Even after his rather easily won First Classes in both Moderations and Finals, it seemed quite natural to his family, and to his College tutor, that he should go off to the City to start on a financial career. But after little more than a year he broke away and returned to Oxford to do more work in mathematics (1928-1929). There he met Professor Oswald Veblen, on leave from Princeton University. This meeting, and the award of a Commonwealth Fellowship, were events of critical importance in his life, for it was in the next three years, spent at Princeton, that the permanence of his interest in mathematics and the reality of his talent were established beyond doubt. Many friendships were begun during that visit; indeed the declared purpose of the Commonwealth Fund, to promote mutual amity and understanding between the English-speaking peoples, can rarely have been more thoroughly and satisfactorily fulfilled.

In 1932 he returned to Oxford and soon afterwards was elected to a Fellowship at Balliol. In 1934 he married Barbara Smyth, a young concert pianist, and they set up house in St Giles, in the heart of Oxford. There were two sons of the marriage. In their house in Oxford, and later at the Manor Farm in Noke, generations of mathematicians, including Whitehead's own students

JHC Whitehead

and many visitors from abroad, enjoyed their delightful hospitality and some serious mathematical conversation.

From 1941 to 1945 he was fully engaged in war-work for various government departments. It was during this time that he was elected a Fellow of the Royal Society (1944).

His election to the Waynflete Chair at Oxford was fortunately timed. With a substantial body of work behind him, including his two big papers on combinatorial homotopy (28) and the Stiefel manifolds (37), he was ready to start building up a school of topology in Oxford. This he set about doing, with the brilliant success that is well known to all mathematicians.

There are few more events to be told. He received many invitations from foreign universities, to deliver lectures or to make prolonged stays. Some he accepted, but on the whole he preferred to stay in Oxford and entertain there his many mathematical friends. He was President of the London Mathematical Society in 1953-1955, but held few other offices. Through a mixture of good fortune and design his single-minded devotion to mathematics was never seriously interrupted, except by the four years of war work.

His mathematical life was spent in a series of long and strenuous campaigns, in which he attacked and wore down the resistance of large and deep problems. The longest of all, which lasted on and off for fifteen years, was the characterization of the homotopy-type of complexes and spaces, of which the first stage, the search for a combinatorial description, was completely successful, and the second stage very nearly so. Among the by-products of this campaign were some of the most valuable tools of modern algebraic and combinatorial topology: the homotopy sequence of a complex and subcomplex, first introduced at the end of paper (37); the 'Whitehead product'; the 'regular neighbourhoods', which have been found to be of fundamental importance in the brilliant advances in combinatorial topology that have been made in the last two years. This revival of geometrical topology made his early death doubly tragic, since there were many signs that it had started him afresh on that blend of geometrical and algebraical thinking which came more naturally to him than the rather rigid formality of his work in the later nineteen fifties.

As a mathematical writer, Whitehead had little time and not much taste for elegance. He gave his readers a rough ride, seldom favouring them with those informal 'Leitfaden' which help to smooth the way. Even the titles of some of his later papers seem designed to discourage the light-minded. The style of his mathematical letters, though terse, was much easier, through the use of just those colloquial touches and geometrical 'cribs' that were missing, or even deliberately taken out, from the published papers. But his preferred medium was mathematical talk, for which he never lost his appetite, and in which he excelled. If he expected concentrated attention, he gave it in full measure himself. At the end of one of these leisurely and searching conversations he would understand one's story thoroughly, and if it had survived without mishap one could feel sure that it was right.

The immediate attractiveness of Henry Whitehead's manner, and his tremendous high spirits, would not alone have brought him the affection of so many mathematicians all over the world if there had not been behind them his devotion to mathematics, and his readiness to interest himself deeply in other people's problems. His unusually acute perception of the thoughts and moods of others enabled him to accept and enjoy every kind of human behaviour, except perhaps a certain kind of conscious dignity, which he heartily disliked. His own easy informality, though it might be disconcerting to hosts and hostesses on formal occasions, when his spirits rose, and he might even break into song, enabled him to talk about mathematics on equal terms with one and all. It did not conceal the natural authority which in his later years was given to him by his mathematical achievement and by his profound belief in the importance of winning the mathematical sieges and battles of which so much of his work seemed to consist; and it made easy of access the generous help which he would give not only to his pupils, but to anyone with a mathematical problem to discuss and get right.

His friends will not soon be consoled for their unexpected loss.

Scientific Work

(This section contains an account of Whitehead's major mathematical enterprises rather than a review of his complete works, which will soon be available in collected form. I am grateful to Dr M. G. Barratt for much material on the later work in algebraic topology.)

When Whitehead reached Princeton in 1929 the 'geometry of paths', based on the theory of linear connexions and their generalizations, was already nearing the end of its purely formal development. He wrote a few papers on the formal side of the theory (1 to 5, 8) but soon turned to more geometrical topics. The first outcome was the joint paper (6) and book (7) with Oswald Veblen on the foundations of differential geometry. Here for the first time an exact definition of a differentiable manifold was given, in the form of 'axioms for differential geometry'. Although somewhat difficult to read now, owing to changes in terminology and to the mixing in of the topology, they contain essentially the definition of a differentiable manifold as a Hausdorff space with a system of coverings by neighbourhoods, each with an associated homeomorphism, ψ_i, on to the closed unit ball in R^n. The *class* of the differentiable structure is the highest order of existing continuous partial derivatives of the maps $\psi_i\psi_j^{-1}$ of subsets of R^n into R^n, in so far as these maps exist. Independence proofs of the axioms were provided in the paper (6), notably an example to demonstrate the rather surprising fact that a locally euclidean space can fail to be a Hausdorff space.

Under the influence of Marston Morse Whitehead now turned to problems of differential geometry in the large. In (9, 10, 11) he showed that simple convex regions of finite extent exist in path-geometries, i.e. regions in which any two points are joined by one and only one path which does not leave the region. The climax of his work in differential geometry was the paper (17, 1935) on the geometry of geodesics (paths of minimal length) in an analytic manifold, F, with a Finsler metric. He determined in great detail the properties of the locus of characteristic points of a given point O, i.e. of the points at which the natural projection, p, of the tangent plane, M, at O on to the space F (straight lines through O on to geodesics) fails to be (1,1); and he established the existence of an open n-cell $\mathcal{E}$ in M such that $p(\bar{\mathcal{E}}) = F$ and $p|\mathcal{E}$ is a homeomorphism.

This paper, his last before turning away to topology, has remained one of the important pioneering works on differential geometry in the large. It is the first of his works in which his quality as a mathematician can be clearly discerned.

After two isolated dips into topology (12, and 15 with S. Lefschetz) Whitehead plunged into the midst of the fray with a proof (16, 1934) of the Poincaré hypothesis (that a simply-connected closed* 3-manifold is a topological 3-sphere). The proof contained an error, and the truth of the hypothesis is still an open question (for 3-spaces). But it was the reading and experimentation for the attempted proof, and for his subsequent counter-example for open 3-manifolds, that committed him to topology for the rest of his life.

The counter-example (20, 1935) was based on a geometrical configuration of two polygons which has become so familiar through related examples made by himself and others (notably Bing and his school, and Mazur, 1961) that it seems right to reproduce it here.

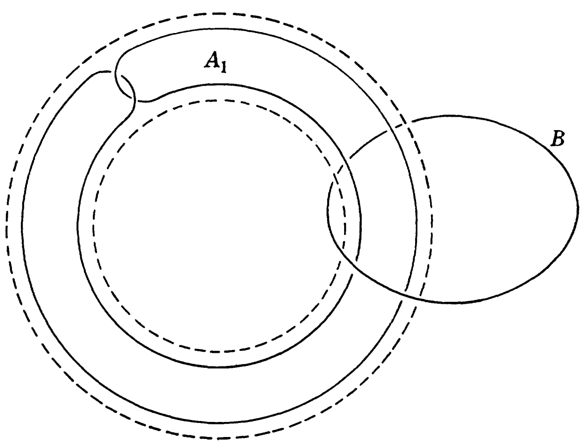

FIGURE 1.

The curve A_1 is clearly null-homotopic (shrinkable to a point, self-intersections allowed) in the open set S^3-B. Moreover, as experiment will show, the positions of A_1 and B can be interchanged by a deformation with-

* Compact and without boundary.

out self-intersections; and so B is null-homotopic in S^3-A_1. But the polygons cannot be pulled away from each other; more precisely, neither is contained in a 3-cell not meeting the other. Now A_1 is embedded in a solid tube not meeting B (indicated by the dotted lines). If A_1 is itself regarded as a thin solid tube, a 'doubled' tube, A_2 can be similarly embedded in it; and A_3, A_4 can be similarly defined. If $X = \bigcap A_m$, the polygon B and, similarly, any other polygon in S^3-X is null-homotopic in S^3-X, i.e. S^3-X is simply connected; and it is easily seen that $H_2(S^3-X) = 0$. But the compact set B is not contained in any 3-cell in the set S^3-X, which cannot therefore be homeomorphic to R^3.*

In the course of this work he became thoroughly familiar with two varieties of combinatorial topology, at that time scarcely connected with each other, which were to be the basis of his work for the next few years. In some studies on knots and linkages (26, 29, 30) arising out of 'the' linkage (figure 1) he became acquainted with the Reidemeister theory of homotopy based on the group-ring for a complex. From this sprang the important papers (33, 34) which will be discussed below.

The other domain which he now entered was the strictly combinatorial kind of topology which had been developed by J. W. Alexander and myself in the years 1925 to 1932. The subject of these theories is the *skeleton complex*, which is a set, V, with a designated set of finite subsets, called *simplexes*, satisfying the conditions that the simplexes cover V, and that all the subsets (faces) of a simplex are simplexes. Two complexes are *combinatorially equivalent*, $K_1 \longleftrightarrow K_2$, if one can be changed into the other by a succession of allowed transformations, or *moves*, of certain prescribed kinds.

A skeleton-complex can be 'realized' as a Euclidean complex by taking as its vertices points of R^k in general position (k being sufficiently large) and replacing each skeleton simplex by the euclidean simplex with the same vertices (i.e. their convex cover). The *locus* $|K|$, of K is then the point-set union of the euclidean simplexes. The aim of such theories is to choose the allowed transformations so that the relation $K_1 \longleftrightarrow K_2$ corresponds as closely as possible to homeomorphism between $|K_1|$ and $|K_2|$. It is now known that a polyhedron which is not a manifold can have two triangulations which are combinatorially inequivalent, under the moves now generally adopted. For manifolds the question (the so-called *Hauptvermutung*) is still open.

Since the problems of homeomorphism appeared at that time to be quite intractable, Hurewicz had in 1935 begun a general change over to an attack on the more hopeful problems of *homotopy-equivalence* ($X \simeq Y$). This he defined to mean that continuous maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ exist such that $gf: X \rightarrow X$ and $fg: Y \rightarrow Y$ are homotopic to the identical maps in X and Y respectively. The simplest kind of homotopy-equivalence is *retraction by deformation* of X on to

* In this brief outline distinctions between combinatorial and point-set equivalence are suppressed. The full result, as here stated, was derived in paper (25) from Whitehead's combinatorial theorem.

a subset, Y, of itself, i.e. a map $f\colon X \times I \to X$ such that $f(x,0) = x$ and $f(x,1) \in Y$ for all $x \in X$, and $f(y,t) = y$ for all $y \in Y$ and $0 \leqslant t \leqslant 1$.

Whitehead now had the brilliant idea of attacking the theory of homotopy-equivalence by the strictly combinatorial method of allowed transformations. This was the subject of his massive paper (28, 1939), 'Simplicial spaces, nuclei and m-groups', one of those on which his reputation as a mathematician will surely rest. As allowed moves he took in the first place the addition of a new simplex, $a\sigma^k$ to a complex K,

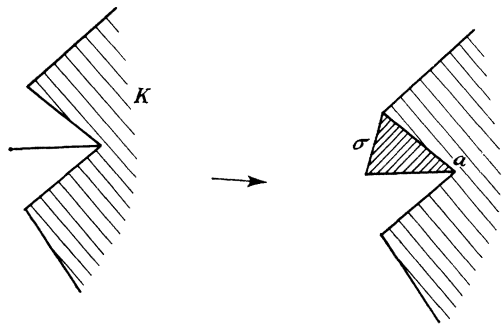

FIGURE 2.

when $a\dot{\sigma}^k$ is a subcomplex of K but $\sigma^k \notin K$.* The transformation $K \to K \cup a\sigma^k$ is then an *elementary expansion*, and $K \cup a\sigma^k \to K$, also allowed, an *elementary contraction*. An *expansion* is a finite series of elementary expansions, and similarly for contractions. A succession of moves of either type is a *formal deformation*, $K_1 \underset{D}{=} K_2$, and two complexes so related have the *same nucleus*.

It is intuitively obvious, and easily proved, that if K_1 is contractible to K_2, $|K_2|$ is a deformation retract of $|K_1|$. It follows that if $K_1 \underset{D}{=} K_2$, $|K_1| \simeq |K_2|$. Simple examples suggest that this implication might be reversible. Whitehead posed this in (28) as a question he was unable to answer, and he therefore introduced another pair of moves, with the help of which he could complete the combinatorial characterization of homotopy type. (This step was later justified when he showed, in (34, 1941), that the answer to his question is 'no': $|K_1| \simeq |K_2|$ does not imply $K_1 \underset{D}{=} K_2$.)

The two further allowed moves were the addition to K, or removal from it, of a simplex, σ^k, of dimension, k, exceeding a fixed number m, when the entire boundary $\dot{\sigma}^k$ is in K. If $K_1 \to K_2$ by a series of moves of any of the four types, K_1 and K_2 have some *m-group*. By means of the *mapping-cylinder*, a device which he first introduced in this paper, he succeeded in giving a thoroughly geometrical treatment of the rather abstract relation of homotopy-equivalence, and he achieved his first main purpose by proving that

* The join $\sigma^h\tau^k$ of two distinct skeleton simplexes, σ^h and τ^k, is the $(h+k+1)$-simplex $\sigma^h \cup \tau^k$; the join KL of two complexes without common vertices is the set of simplexes $\sigma\tau$ ($\sigma \in K$, $\tau \in L$). $\dot{\sigma}$ is the set of all faces of σ except σ itself.

a necessary and sufficient condition for $|K_1| \simeq |K_2|$ *is that* K_1 *and* K_2 *have the same m-group for every m.*

(The somewhat bizarre names 'nucleus' and 'm-groups' for abstract equivalence classes he later (50, 1950) changed to *simple homotopy type* and *m-type* respectively, generalizing the two definitions at the same time. The more familiar name 'simple homotopy type', will be used in this memoir; but as 'm-type' was still later changed to '$(m-1)$-type' it seems best to keep to the unambiguous 'm-group'.)

The second part of this paper contains theorems the importance of which has only recently been fully appreciated, though their use by Mazur, Stallings, Zeeman and others in problems related to the general field of the Poincaré hypothesis. They were the first of those theorems which show that certain dimension-raising processes obliterate fine distinctions between complexes, or between spaces.* A *regular neighbourhood* of a subcomplex K of a combinatorial n-manifold M is a subcomplex U (K,M) of M which (1) is an n-dimensional manifold, (2) contracts into K. If M is suitably subdivided, the closure of the set of all simplexes meeting K is such a neighbourhood. Whitehead proved the fundamental theorem that *any two regular neighbourhoods of K in M are combinatorially equivalent.* Two of the many interesting corollaries of this theorem are:

(A) *if* $|K^n|$ *is an absolute retract rectilinearly embedded in* R^k (i.e. if $|K^n|$ is of the homotopy-type of a point), $U(K^n, R^k)$ *is a k-element if* $k \geqslant 2n + 5$;

(B) (proved in paper 32) *if two 3-manifolds* M_1^3, M_2^3, *(without boundary) have smooth differentiable loci, and the same simple homotopy type, then* $M_1^3 \times \sigma^k \longleftrightarrow M_2^3 \times \sigma^k$ *if* $k \geqslant 5$.

Among the many applications that have been made of (A) was his proof many years later (81, 1957) that a certain n-manifold is a combinatorial n-sphere, but admits an involution whose fixed points form an $(n-1)$-manifold well known not to be (even topologically) an $(n-1)$-sphere. This was the first example of a 'nice' set realizing this possibility.

The change in Whitehead's interests from strongly geometrical to predominantly algebraic topology was not, like the change from differential geometry to topology, an abrupt one, but was brought about gradually by his search for invariants to characterize the homotopy-type of complexes, and for methods of computing their homotopy groups.

His first algebraic attack on the homotopy-equivalence problem was in the paper (34, 1941) in which purely algebraic 'L- and L^*-equivalences' between incidence matrices of complexes, with elements in the Reidemeister group-ring, were shown to be necessary conditions for the complexes to have

* I have not found it possible to state these theorems quite accurately in the absence of definitions too elaborate to be given in this memoir. The 'contractions' of this paragraph are nearly, but not quite, the same as those defined above. A *combinatorial n-element* and $(n-1)$-*sphere* are complexes combinatorially equivalent, respectively, to an n-simplex and its boundary. In a *combinatorial n-manifold* the closure of the simplexes containing any vertex a is the join of a to an $(n-1)$-sphere or $(n-1)$-element.

the same simple homotopy type, and the same homotopy type, respectively. This did not prove a very easy piece of apparatus to handle, but it enabled him to construct the example referred to above, showing that two complexes may have the same homotopy type but different simple homotopy types; and, more important, to give a complete homotopy classification of the (3-dimensional) lens-spaces. A necessary and sufficient condition for the lens-spaces of types (p,q) and (p,q') to be homotopy-equivalent is (he showed) that either qq' or $-qq'$ be a quadratic residue mod p. This result has been the basis of many counter-examples constructed since that time, including Milnor's for the Hauptvermutung (1961).

Towards the end of the war there appeared the paper (37), in which he computed various homotopy groups of the Stiefel manifold $V_{n,m}$, the space of orthogonal m-tuples at the origin in R^n. This paper suffered severely from various stresses and strains of the times. There are obscurities not only in the subject-matter, to which he was unable to give an effective final revision, but in the relation of his work to that of Hurewicz, to whom he ascribes notions not to be found in Hurewicz's published works. The long delay in publication (received 1941, appeared 1944) diminished the direct influence of the paper. Finally some of the groups were found to be wrongly computed. Nevertheless this paper was of great importance since in it there was introduced for the first time the *homotopy sequence* of a space and subspace, which he proved to be exact in the last paragraphs of the paper and from which he there derived the homotopy sequence for fibre-spaces.

The misfortunes and mistakes of this paper had a profound effect on Whitehead's later work. He was himself convinced that the rather free geometrical style of writing used in it was partly to blame for the troubles, and in the autumn of 1946, which both he and I spent in Princeton, he resolved to adopt a more algebraic form of statement in his future writings, a resolution which he kept, though he never ceased to look for geometrical meanings. It finally led him to undertake what amounted to a complete re-statement of his earlier work on homotopy, in the papers (46, 47 and 50, 1949-1950).

The basis of this revision was the notion of the *CW-complex* (46) which soon came to be regarded as the proper category of objects for homotopy theories. It was suggested to him by the process of attaching and detaching spheres which had played an important part in the proofs of theorems about 'm-groups' in the original treatment in (28).

A *CW* (i.e. 'closure-finite weak-topology') complex is a Hausdorff space presented as the union of disjoint 'cells'. The closure, $\bar{e}^n$, of an n-cell e^n is given as the image of a mapping, f, of the unit ball B^n, and the following conditions are imposed: (1) $f \mid \text{int } B^n$ is a homeomorphism on to e^n; (2) $f(\dot{B}^n)$ meets only a finite number of cells, all of dimension $< n$; (3) (the 'weak topology') a subset of X is closed if and only if it meets each $\bar{e}^n$ in a closed set. A *CW*-complex may be thought of as built up by the addition of successive cells, with singular boundaries, but disjoint and non-singular interiors, the cells of lower dimension coming first. Subject to (2) and (3), the boundary of a new n-cell may be mapped in any way into the union of cells of dimension $< n$ already present. The *CW*-complexes, though far more flexible than polyhedra, have many of their most useful properties. They are locally contractible, and (owing to the weak topology) they have the 'homotopy extension property', that is, every homotopy defined over a subcomplex can be extended over the whole space. This property he used (45) to obtain the important and still often used theorem that two connected *CW*-complexes X, Y, are homotopy-equivalent if, and only if, a map $f: X \to Y$ exists which induces an isomorphism of the homotopy groups. Moreover if the dimensions are finite and $\leqslant n$, the isomorphism of the groups up to π_n suffices.

In the paper (50, 1950) the theory of simple homotopy type based on the Reidemeister invariant (34, above) was recast, with a new definition based on the *torsion* (a generalization of the Reidemeister-Franz torsion). This is a certain function, τ, of homotopy-equivalences, $\phi: K \simeq K'$, between *CW*-complexes, with values in an abelian group. K and K' have now, by definition, the same simple homotopy type if there exists a ϕ with $\tau(\phi) = 0$; and this was shown to be so if and only if $K \to K'$ by transformations which are natural generalizations, for *CW*-complexes, of the 'elementary expansions and contractions' of simplicial complexes. Thus the meaning of 'simple homotopy type' is substantially unchanged. This version of the theory has not been found by others to be much easier to handle than the first; but a modified form of the torsion has lately been used by Milnor to prove (as part of the counter-example for the Hauptvermutung) that $L_1 \times S^{2n}$ and $L_2 \times S^{2n}$ are not homeomorphic if L_1 and L_2 are the (7,1) and (7,2) lens-spaces, and n is large enough.

In the series of substantial papers (43, 47 to 50, 52) Whitehead continued to explore the possibility of finding a full algebraic characterization of homotopy type. For n-connected complexes* of dimension $\leqslant n+3$ (the so-called A^n-complexes) he found such a full invariant (43, 49) in the *cohomology system*, a set of cohomology groups and homomorphisms; and he finally came very near to success in the general case with an exact sequence, given in its final form in (57, 1950). This is the sequence

$$(1) \ldots \to \Gamma_n \xrightarrow{i} \pi_n \xrightarrow{j} H_n \xrightarrow{\beta} \Gamma_{n-1} \to \ldots$$

where H_n is the nth singular homology-group of the space X; $\pi_n = \pi_n(X)$; and if X is a *CW*-complex, Γ_n is the group of equivalence-classes of maps $S^n \to X^{n-1}$ for homotopies in X^n.† Since j is the homomorphism used by Hurewicz in his fundamental 1935 paper, and Γ_n is trivial when X is $(n-1)$-connected, the exactness of the sequence generalizes Hurewicz's theorem that the first non-vanishing homotopy and homology groups are isomorphic. Whitehead found an equivalence relation between the sequences (1) of two spaces, which does not quite provide the full classification; but when later

* I.e. $\pi_r(X) = 0$ for $r \leqslant n$.

† X^n = n-section = union of cells of dimension $\leqslant n$. For spaces other than *CW*-complexes the definition of Γ_n is more complicated.

Postnikov found a full system of invariants for homotopy-type, Whitehead was able to show that it could be derived from his sequence.

A problem to which he returned again and again was the computation of the homotopy groups of spheres, one of the principal objects of algebraic topologists' activity in the post-war years. His first direct contribution was the paper (33, 1941), in which he defined the multiplication which later became so generally known as the 'Whitehead product' that he was finally compelled to use the name himself (66). By modifying a geometrical argument of H. Hopf he showed that, if $S^m \vee S^n$ is the union of two spheres with a single common point, $\pi_r\,(S^m \vee S^n)$ is the direct sum

$$\pi_r(S^m) + \pi_r(S^n) + \pi_m.\,\pi_n$$

where $\pi_m.\pi_n$ is the subgroup generated by all the Whitehead products $\alpha.\beta$, $(\alpha \in \pi_m(S^m),\ \beta \in \pi_n(S^n))$, and is (as he also showed) cyclic infinite. The Whitehead product was the principal instrument used by Hilton (1955) in the subsequent calculation of the homotopy groups of a 'bunch' of spheres with a single common point. In the 'Note on suspension' (53, 1950) the problem of computing homotopy groups is attacked in a combinatorial spirit, through a generalization of the Freudenthal suspension theorems.

More distantly related to this work was the series of papers with E. H. Spanier (64, 74, 79, also Whitehead's presidential address to the London Mathematical Society, 78, 1956) in which they were seeking for a duality principle in homotopy, as were many others at that time. The results are formulated in what they called (64) an 'approximation to homotopy theory', or S-theory.

The *suspension*, SX, of a space X is its join to two points, a process which may evidently be iterated to give a space S^rX.

A map $f{:}X \to Y$ determines in an obvious way a map Sf: $SX \to SY$. By a natural generalization of the notion of homotopy, maps $g{:}S^pX \to S^pY$ and $f{:}S^qX \to S^qY$ are 'S-homotopic' if $S^{r-p}g \simeq S^{r-q}h$ in S^rY for some $r \geqslant \max.\,(p,q)$. By using this equivalence as a classification, an abelian group $\{X,\ Y\}$ of 'S-mappings' was defined. If P is a polyhedron embedded in S^n, a polyhedron Q in S^n-P is an *n-dual* of P if it is a deformation retract of S^n-P. The name is justified since the relation is symmetrical. It was shown by Spanier and Whitehead that if (P_1,Q_1) and (P_2,Q_2) are dual pairs, the groups $\{P_1,\ P_2\}$ and $\{Q_2,\ Q_1\}$ are isomorphic. The theory has been applied to the problem of embedding manifolds in spheres.

In the last three years of Whitehead's life there was a great revival of geometrical topology of the kind that he had worked on twenty years before. It was due in the first place to the appearance of a proof by Papakyriakopoulos (1957) of the long outstanding Dehn lemma, and secondly to the 'collar' theorems of Mazur (1958) and Morton Brown (1960) whose methods led on eventually to the proof of the Poincaré hypothesis for $n \geqslant 5$ (Smale, Stallings, Zeeman). The Dehn lemma is a specifically 3-dimensional theorem: *if a* *semi-linear map of a disk into an orientable M^3 has no singularity in the boundary polygon, C, its singularities can be entirely removed*, i.e. C bounds a non-singular 2-element in M^3. Dehn published a faulty proof in 1912; Papakyriakopoulos finally proved the lemma in 1957, and, at the same time, his own related 'sphere theorem'. This says that in an M^3 of a certain general kind, the existence of a *singular* essential 2-sphere implies the existence of a *non-singular* essential 2-sphere.

The Papakyriakopoulos theorems aroused Whitehead's deep interest. He and A. Shapiro soon produced (84) a greatly simplified version of the proof of the Dehn lemma, and Whitehead himself (83) managed to free the sphere theorem from Papakyriakopoulos's rather troublesome restriction on M^3. One of Whitehead's last works (88), a proof that every connected 3-manifold, M^3, which is not compact can be *immersed* in R^3, also derived from his work on the Dehn lemma. (A map f: $M^3 \to R^3$ is an immersion if each point of M^3 has a neighbourhood, U, such that $f|U$ is an embedding.)

But it is the still more recent geometrical theories initiated by the work of Mazur and Morton Brown, that have revived the spirit of the early combinatorial theories, with applications of many of the results of Whitehead's great 1939 paper. His last collaborative paper (90) with R. Penrose and E. C. Zeeman, deals with the problem of embedding a compact n-dimensional manifold in a Euclidean r-space. How small can r be? That R^{2n} is big enough to hold any smooth combinatorial M^n has been known since 1944 (Whitney). It is shown in (90) that the vanishing of the successive homotopy groups allows r to be lowered *pari passu* : *every closed k-connected* n-manifold can be embedded in R^{2n-k} if* $0 \leqslant k \leqslant \frac{1}{2}n-1$.

Everything depends on the lemma that *if an embedding, h, of a polyhedron P^{m-1}, ($m \leqslant \frac{1}{2}n$) as a subcomplex of the interior of an n-manifold is homotopic to a point, there is an n-element, E, such that $P \subseteq int\ E \subseteq E \subseteq int\ M$.* The main steps in the proof are as follows. The existence of the homotopy to a point implies that h can be extended to a mapping of a cone vP^{m-1} into int M. It is first shown that h can be extended to an *embedding* h, of vP^{m-1} into int M. (For cells of dimension less than m this follows easily from $m \leqslant \frac{1}{2}n$, for cells of the top dimension less easily.) Now the 'cone' $h\,(vP^{m-1})$ is collapsible to its vertex $h(v)$. Therefore by Whitehead's theorem (A) above, any regular neighbourhood of $h(vP^{m-1})$ in M is the required n-element. This lemma played an essential part in Stallings's original proof of the Poincaré hypothesis for $n \geqslant 7$.

Simultaneously with his renewed activity in strictly combinatorial topology, Whitehead was working in 1959 on the problems of *transverse k-fields* of an n-manifold, M^n, embedded in R^{n+k} (89). A mapping, ϕ, of M^n into the Grassmann manifold of k-planes through the origin in R^{n+k} defines a transverse k-field if for no three points x, y, z of M sufficiently near to each other is the line xy parallel to the k-plane $\phi(z)$. Whitehead considered particularly whether the existence of a transverse field implies, or is implied by, the existence of a differentiable structure compatible with a given triangulation

* See footnote p. 357.

of M. He showed that the direct implication (existence of field implies differentiable structure) holds if M^n is a combinatorial manifold rectilinearly embedded. The more difficult reversed implication he considered only for $n \leqslant 4$, and showed that it is true only of certain embeddings. From such theorems the existence of differentiable structures can be inferred for rectilinearly embedded combinatorial manifolds. In this paper too, the methods are predominantly geometrical, and even analytical.

The three papers (88, 89, 90) all written within a few months of his death, and published after it, give a vivid impression of the intensity and variety of his mathematical life up to the moment when it was suddenly broken off.

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