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Poul Heegaard's 1898 thesis

TRANSLATED BY HANS J. MUNKHOLM

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INTRODUCTION

It is commonly known which growth occurred for the theory of functions of one complex variable when one took into consideration imaginary values for the independent variable and tied the theory to a geometric presentation of the imaginary quantities.

Were one to build a theory for functions of two independent variables, it would therefore be natural to seek a similar presentation. That such a long time has lapsed before one has started to work on this natural extension is due in part to the fact that the investigations for two independent variables are much more difficult than for one. The multitude of possibilities produces phenomena for which no analogues exist for one variable.Thus, Picard notes "On voit, par ce qui précède, les différences profondes qui séparent la théorie des fonctions algébriques d'une variable de la théorie des fonctions algébriques de deux variables indépendentes. L'analogie qui souvent est un guide excellent, peut devenir içi trompeuse.[1]" Of course, one difficulty for a transparent presentation also lies therein that the geometric formations which should play the role of the Riemann surfaces would be 4-dimensional.

Even if the analogies thus may be misleading for our detailed investigations, a general outline of the tools that one has used in the theory for one independent variable will give a good working plan. Let us then consider such an outline.

The theory of functions of one independent variable is intimately tied to the theory of algebraic curves. The geometry of such curves are therefore of fundamental importance.

[1]Translation by HJM: One sees from the preceding the deep differences that separate the theory of algebraic functions of one variable from the theory of algebraic functions of two independent variables. Here, analogy which can often be an excellent guide can become deceitful.

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The investigations have been conducted from very different viewpoints. The most important ones rely on

I. elementary **algebraic** theorems. Here, one may mention

  α) Investigations of *adjoint polynomials*, a theory created by *Brill* and *Nöther* in the treatise: Ueber die algebraische Functionen (M.A. Bd. 7, 1873), [1].[2]

  β) *The Italians'* investigations into *linear point groups*. They have tried to liberate the former theory from its projective form so that they develop it independent of the notions of degree, class etc., cf. a report by *Castelnuovo* and *Enriques* (Sur quelques récent resultats dans la théorie des surface algébriques. M.A. Bd. 48. p. 242, 1897).

  γ) *Enumerative geometrical* investigations, especially a series of treatises by *Zeuthen* (e.g. M.A. Bd. 3 and Bd. 9).

II. the study of the **transcendental** functions associated with the algebraic curve. These investigations which could be said to originate in *Riemann's* pioneering work (1857) are so well known that there is no reason for further mentioning.

III. **topological** investigations of the Riemann surfaces that represent the algebraic curve.
Here, once more, one has followed two somewhat different paths:

  α) Either one determines the connectivity numbers of the surface by means of a theorem that may be considered a *generalization of Euler's polyhedral Theorem* (Riemann, Neumann).

  β) Or one *punctures* the Riemann surface and goes on to continuously deform it into a normal form. As far as we know, this procedure has been carried out only by *Jul. Petersen* (Forelæsninger over funktionsteori, Chapter IV). Certainly, *Listing* employs such a procedure to form the "diagram" of a spatial figure, thereby arriving at an extension of Euler's Theorem (Census Räumlicher Complexe, 1862). And *Betti* (1871) uses such a consideration in his investigations of connectivity numbers for n-dimensional spaces in general. However, both papers have apparently been disregarded for a long time. This method gives a very transparent treatment of the matters at hand.

[2] Translator's note: Heegaard gave his references in the text. We have added a bibliography containing, we hope, full details, including links to relevant current databases where possible.

For the theory of functions of two variables, transformations of algebraic surfaces play an analogous role. There are already some papers – chiefly of late – in which the matter has been treated from viewpoints corresponding to those mentioned.

I. **Elementary algebraic** investigations.

α) *Adjoint polynomials.* This theory originates with *Clebsch* (C.R. Dec. 1868) and *Nöther* (Zur eindeutigen Entsprechen … I II, M.A. Bd. 2, 1869, and Bd. 8, 1874)

β) *Linear curve systems. The Italians* have created a theory of linear curve systems on surfaces, analogous to the formerly mentioned point groups on curves. By means thereof, they determine invariants for surfaces (cf. the report by Castelnuovo and Enriques, mentioned above).

γ) Surface transformations have also been investigated by *enumerative geometry* by *Zeuthen* (Études géométriques …, M.A. Bd. 4).

II. Investigations through **transcendental** functions.
Already in his treatise in M.A. Bd. 2, *Nöther* considers integrals of the form

$$\int\int \frac{Q(x\,y\,z)\,dx\,dy}{f'_z}.$$

Picard introduces integrals of the form

$$\int_{x_0\,y_0\,z_0}^{x\ \,y\ \,z} P\,dx + Q\,dy,$$

where P and Q satisfy the integrability condition (Liouville Journ. 1885 & 86). Finally, Picard, in his award winning treatise: Mémoire sur la théorie des fonctions algébriques de deux variables (Liouv. Journ. 4th series, Bd. 5, 1889), and in the recently published book on the same topic (*Picard et Simard:* Théorie des fonctions algébriques de deux variables) has given a coherent exposition of the whole theory.
Poincaré's treatise: "Sur les residues … (Acta mathematica, Bd. 9, 1887) must also be mentioned in this connection.

III. **Topological** investigations.
In this direction there is not much to be found. Certainly, Picard's works contain something, but nothing thorough since he always prefers the analytic presentation. The difficulty lies in the fact that the Riemann-Betti theory of connectivity numbers is very incomplete and difficult to apply when manifolds of more than 2 dimensions are under discussion. Poincaré has attempted to complete this theory (Analysis Situs, Journ. de l'école polytéchnique, 2nd series,

Cah. 1, 1895), albeit without success in our opinion. Later, Picard has given his contribution; already earlier, W. Dyck has dealt with the question (Beiträge zur Analysis Situs, M.A. Bd. 32 & 37), but nowhere does there exist a completely satisfactory theory. A theory of topological correspondence of manifolds of higher dimension count than 2 must, therefore, precede investigations in this direction. What already exists in this direction must be compared to IIIα). There are no investigations analogous to those mentioned under IIIβ).

The following pages contain no completed whole, only studies. The difficulty of the subject must serve as an excuse for this fact. To place the investigations in the proper light, it will probably be suitable to put forward the train of thoughts that has been guiding my investigations.

By chance, I had noticed that the Zeuthen–Halphen extension of the Genus Theorem (M.A. Bd. 3; Bull. de la Soc. math. de France, Bd. 5) could be proved by a purely *topological* argument; in fact, from the two Riemann surfaces between the points of which there is assumed to be a μ-to-ν correspondence one can construct new Riemannian surfaces, which correspond uniquely to one another. The equation expressing equality of the connectivity numbers of the two surfaces constructed then precisely enunciates [conclusion of the] the Theorem mentioned. Hereby, new light is thrown upon an important fact in enumerative geometry: The extended Genus Theorem can be applied without infinitesimal investigations at the coincidence points (or replacements for those) while such investigations are necessary for other correspondence formulae (cf. Acta mathem., Bd. 1, p. 171). Here, I shall not go into the proof of this theorem, the less so because later I have found it in a treatise by Hurwitz (M. A., Bd. 39). My thought now was that one must be able to form a similar Theorem for algebraic surfaces. But there was first a large task that had to be done: For the algebraic surfaces there had to be created something which would correspond to the Riemann surfaces for the algebraic curves. Then topological criteria for the bijective correspondences among such had to be established. And, as already mentioned, the material that existed beforehand for such an investigation was either insufficient or full of mistakes. The contents of the following pages originate in my attempts to correct these mistakes and to fill these gaps.

FIRST SECTION
On a visually transparent representation of the complex points of an algebraic surface.

§1. A visually transparent representation of a 4-dimensional surface.

To get a general view of the connectivity of the points of an algebraic curve $y = f(x)$, one first chooses a doubly infinite collection of points (a plane or a sphere), in which one interprets the complex values of the independent variable. Above this, one then constructs a Riemann surface, on which the values of dependent variable can be uniquely distributed. Then the connectivity of the algebraic curve is essentially characterized by the number of leaves and the number of branching points on the corresponding Riemann surface, or rather by the genus derived from these numbers.

To carry out an analogous investigation of the connectivity of an algebraic surface $z = f(x, y)$, one must first form a fourfold infinite collection of elements to which one can let correspond all value pairs (x, y) obtained by letting x and y assume all possible complex values independently of one another. By multiply covering this collection and introducing suitable "branching surfaces" and connecting these by 3-dimensional creations through which the different "layers" are connected to one another, one can create a 4-dimensional manifold in which the connectivity of the algebraic surfaces can be studied. Here, the number of "layers" and the properties of the branching surfaces will be of essential importance for the connectivity of the surface. From these constituents one should then attempt to form notions analogous to the genus of the algebraic curves.

The question now becomes which 4-dimensional manifold should one choose. One could use the fourfold infinite collection of lines in space. One might e.g. choose two planes in which one represents the complex values of x, respectively, y, in the usual way; the line connecting the point (x) with the point (y) would then correspond to the value pair (x, y). Or one could use v. Staudt's presentation of the points of an imaginary plane. This would be a very elegant way to treat the matter, but it is hardly practicable - at least not at the present scientific stage.

However, in our investigations where everything depends on a clear visualization of the elements we work with, the above mentioned representations are not advisable. Not only must the element which represents (x, y) be simple; also, the totality of elements which surround a given element, must be as easy to visualize as possible. But it is no doubt almost impossible to visualize clearly the collection of straight lines in space which form the “surroundings” of a straight line. In contrast, it is easy to form clear visualizations of the surroundings of a point in a surface and of a point in space. However, we do not know of any 4-dimensional manifold in which any point is directly an element. In our investigations, we shall use the following representation:

In a horizontal plane, (x), we choose two mutually orthogonal axes, X_1 and X_2; X_1 with the positive direction pointing right, X_2 with positive direction away from us. In this plane we represent the complex values of $z = x_1 + ix_2$ in the usual way. Now, in order to find room for representing both of the numbers y_1 and y_2 that appear in $y = y_1 + iy_2$, we shall consider as elements not just the points in space, but rather such points equipped with a real numerical value.

We erect at the point (x) a line segment of length y_1, orthogonal to the plane (X_1X_2) (positive direction upwards), and the end point of this segment we think of as equipped with the number y_2.

The procedure is similar to the one used by a surveyor when he presents and investigates points in space on a map: the points are replaced by their projections on the picture plane, and the projections are equipped with an elevation number representing the height over, or the depth under, the plane. When the elevation number is zero, we get the picture plane itself.

We can get the totality of elements – or as we shall say, points – by first thinking of space as usual, equipping all its points with the elevation number zero. And then, continuously let all the elevation numbers vary first from 0 to $+\infty$, next from 0 to $-\infty$. We shall denote this collection by $\mathbf{T}$.

By analogy with the surveyor’s considerations, we could then say that points of our 4-dimensional space are represented by projection onto the usual space. Here, by the projection of a point we understand the point we get by deleting the elevation number (or rather: by changing the elevation number to zero).

For reasons of convenience, we shall use a number of names and designations for which the analogy to well known matters are easy to see. The collection of points obtained by equipping a point in the usual space with elevation numbers from $-\infty$ to $+\infty$ is said to form a straight line *perpendicular* to our space. Analogously, one defines a perpendicular plane and a perpendicular planar space by means of a straight line, respectively a plane. When a manifold in $\mathbf{T}$ is projected to a manifold that has one dimension less, but where each point carries infinitely many elevation numbers, we call the original manifold *vertical*. If the dimension number is unchanged, and the elevation number is the same everywhere in the manifold then it is called *horizontal*. Points with positive elevation numbers are said to lie *over* our space, those with negative elevation numbers lie *under*. The projection into our space of a manifold will be called the *carrier* of the manifold.

Below, 4-dimensional manifolds will be designated by boldface Latin letters, the one constructed above by $\mathbf{T}$. Spaces (3-dimensional manifolds) are denoted by capital Greek letters, our usual space by P. Surfaces are denoted by lower case Greek letters (φ, π), lines by capital Greek letters (L) and points by lower case Latin ones (p).

A curve $\mathbf{T}$ is then represented simply by its projection, the carrier curve, the points of which are equipped with elevation numbers.

The carrier for a *surface* is in general a surface. To get an overview of the elevation numbers from the carrier surface, we trace curves through points with constant elevation numbers. The surface in question is completely determined by the system of elevation curves that are traced in this way on the carrier surface.

Just as the projection of a surface onto a plane usually covers parts of the plane several times, and just as these parts are bordered by curves (the contour), in the same way the projection of a *space* onto P will usually be a space that covers parts of P several times, and these parts will be bordered by surfaces (the boundary surfaces). The elevation numbers are determined by elevation surfaces through points with the same elevation number.

By continuous deformations of the forms described one must both observe what happens to the carrier and investigate which displacements occur for the elevation numbers.

Based upon the foundation just given one can easily define what should be understood by a straight line, a plane, and a planar space in $\mathbf{T}$, and one can develop their usual descriptive properties. Spatial collineations can be traced back to central projections, etc. We shall not touch on this. The metric geometry in $\mathbf{T}$ one could for instance base on a definition of the distance between two points

$$d = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (y_1 - \eta_1)^2 + (y_2 - \eta_2)^2}.$$

Or by developing the notion of congruence on the following definition of a rotation around a plane, thereby reducing the question to one that concerns our space. In order to enunciate the above mentioned definition of a rotation we shall first define what should be understood by a circle in a "perpendicular" plane. The projection of it shall be a straight line segment ab in the trace of the plane in P, and the elevation numbers on this segment should be determined by

$$y_2 = c \pm r \sin\theta,$$

where c is a real constant, $2r = ab$, and where $r\cos\theta$ is the distance from the midpoint of ab to the variable point. The midpoint with elevation number c is called the center, r the radius. A point is *rotated* around a "horizontal" plane by letting the point describe the "vertical" circle that passes through it and which projects onto a straight line through the projection of the point, orthogonal to the projection of the horizontal plane. During the rotation the projection runs back and forth on a straight line the endpoints of which lie symmetric with respect to the projection of the rotation plane. The square of its maximal distance to this projection equals the sum of the square of the distance at any given moment and the square of the corresponding elevation number. The point alternately pass "above" and "below" the rotation plane.

A planar space intersects P in a plane, and any figure in it can therefore be embedded in P by rotating it around the trace.

Now it is not difficult to develop the basic metric notions angle, distance, etc.

It is also easily seen how one can bring two symmetric bodies to cover one another by rotation (e.g., by rotation around a symmetry plane). Likewise that a rotation may bring a point that is inside a closed surface in P outside the surface without crossing its boundary.

§ 2. The straight line.

Our presentation of the analytic plane geometry has the advantage that the visual results that one arrives at by sticking to real coordinates occur as special cases of the visual results we reach here. In fact the points with elevation number 0 lying in the $X_1\,Y_1$-plane are the usual real points.
We shall first investigate the collection of points that is given by the equation

$$y = \alpha x.$$

If we let

$$\begin{aligned} x &= x_1 + ix_2 \\ y &= y_1 + iy_2 \\ \alpha &= \alpha_1 + i\alpha_2 \end{aligned}$$

and we split into real and imaginary parts we get

$$y_1 = \alpha_1 x_1 - \alpha_2 x_2 \qquad (1)$$
$$y_2 = \alpha_1 x_2 + \alpha_2 x_1 \qquad (2)$$

(1) is the equation of the carrier surface in an orthogonal coordinate system with X_1, X_2 and Y_1 as axes. It represents *a plane* through the origin. If one constructs the point

$x_1 + iy_1 = 1 : \alpha$ in (X_1, X_2), the line that connects this point with the origin will be orthogonal to the trace of the plane; in this point, y_1's value is 1. Hereby, the position of the plane is easily determined.

(2) determines the projection of the elevation lines onto the (X_1X_2) plane, in that y_2 is viewed as a parameter. Since these projections are orthogonal to the trace of the plane (1) the *elevation lines become fall lines*. The fall line through the origin corresponds to $y_2 = 0$; it contains the point constructed above, determined by $x_1 + ix_2 = 1 : \alpha = \frac{\alpha_1 - i\alpha_2}{\alpha_1^2+\alpha_2^2}$. Besides, the elevation numbers of the different fall lines are proportional to their distance form the elevation line $y_2 = 0$. It therefore suffices to know the elevation number of one more fall line. The particular point in the (X_1X_2)-trace of the plane which has elevation number 1 has the coordinates

$$\xi_1 = \frac{\alpha_2}{\alpha_1^2 + \alpha_2^2}, \quad \eta_1 = \frac{\alpha_1}{\alpha_1^2 + \alpha_2^2}.$$

Thus, according to the above, it is obtained by multiplying the point $1 : \alpha$ by i.

Thus, the plane in $\mathbf{T}$ representing $y = \alpha x$ is determined in the following manner: A perpendicular to the (X_1X_2)-plane of length $+1$ is erected at the point $1 : \alpha$. A plane is placed through this point [i.e. the endpoint of the vector] and a vector obtained by rotating the vector from the origin to the point $1 : \alpha$ through an angle of $+90°$ in the positive direction. The fall line through the origin is given the elevation number 0, and the fall line through the rotated vector's other end point the elevation number 1.

When conversely, a point[3] through the origin is given in our space, by this rule it is easy to determine both the slope of the straight line whose carrier it is, and the elevation numbers with which to equip its fall lines. Direction coefficients of constant modulus correspond to planes with the same inclination and the same distance between identically labelled fall lines. The larger the modulus is, the steeper the plane, and the smaller the distance between lines with given elevation numbers. For $\alpha = 0$ we get the X-axis which is thus represented by the X_1X_2-plane (with elevation number 0 everywhere). For $\alpha = \infty$ we get the Y-axis, which is represented by a plane though Y_1 perpendicular to our space P.[4]

It is now easy to see how the line

$$y = \alpha x + q \qquad (q = q_1 + iq_2)$$

[3] Translator's remark: Obviously, this should be a plane rather than a point.

[4] actually, in addition one gets all of space equipped with infinite elevation numbers, cf. §4. Hereby, one obtains the continuous passage from lines with finite direction coefficients in that the carrier plane converges to a plane through the Y-axes into which all fall lines with finite elevation numbers have shrunk while the elevation line $y_2 = \infty$ has spread over all of the remaining plane.

is represented. The equations (1) and (2) change into

$$\begin{aligned} y_1 &= \alpha_1 x_1 - \alpha_2 x_2 + q_1 \\ y_2 &= \alpha_1 x_2 + \alpha_2 x_1 + q_2, \end{aligned}$$

which show that the carrier plane is translated through the segment q_1 and that all the elevation numbers of the fall lines are increased by the quantity q_2.

Straight lines parallel to the X-axis are represented by horizontal planes with constant elevation numbers. Straight lines parallel to the Y-axis are represented by planes that are perpendicular to P[5] and have trace in P parallel to Y_1.

§ 3. Algebraic curves.

Let $F(x, y) = 0$ be the equation of an algebraic curve of the n-th order; it is presupposed to be in general position. If we split into real and imaginary parts, we get two equations of the form

$$\begin{aligned} y_1 &= \varphi(x_1, x_2) && (1) \\ y_2 &= \psi(x_1, x_2), && (2) \end{aligned}$$

where φ and ψ are n-valued functions of x_1 and x_2.

(1) is the equation of the carrier surface. To each point in (X_1X_2) there must correspond n real values of y_1. Therefore, the carrier surface must consist of n real sheets that extend along all of the infinite plane (X_1X_2). Evidently, the surface can have no boundary on (X_1, X_2) whereas two values of y_1 might coincide along a (double-)curve.

The curve system (2) where y_2 is a parameter represents the projections of the elevation lines. Since y is a monogenic function of x one has

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial x_1} &= \frac{\partial\psi}{\partial x_2} \\ \frac{\partial\varphi}{\partial x_2} &= -\frac{\partial\psi}{\partial x_1} \end{aligned}\right\}$$

This shows that the two curve systems (1) and (2) (with y_1, respectively, y_2, as parameters) are orthogonal systems of trajectories. From this it follows that *the elevation lines on are the lines of maximal fall on the carrier surface*.

The equation

$$\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} = 0$$

[5]Translator's remark: Presumably, perpendicularity to the (X_1X_2)-plane is intended.

shows that $\frac{\partial^2\varphi}{\partial x_1^2}\cdot\frac{\partial^2\varphi}{\partial x_2^2}-\left(\frac{\partial^2\varphi}{\partial x_1 x_2}\right)^2$ must be negative. Thus, the curvature of the carrier surface is everywhere *hyperbolic*.

An easy calculation shows that the tangent at an ordinary point of the curve is represented by a plane that is tangent to the carrier surface of the algebraic curve at the point considered; the fall line through the point is tangent to the fall line on the carrier surface - of course with the same elevation number.

Using the rule from page 1009, it becomes easy to determine to which side of the carrier surface the elevation numbers of the fall lines are increasing. Climbing upwards on the surface, one has the ascending elevation numbers to the left.

Singular points. To the substitution

$$\begin{aligned} x &= x' - a \\ y &= y' - b \end{aligned}$$

there corresponds a parallel translation of the axes corner and addition of a constant to the elevation numbers. One may therefore assume that the singular point considered is placed at the origin. The appearance of the carrier surface and the course of the fall lines will be unchanged. A neighborhood of the singular point will be represented by series of the form

$$y = Ax^{\alpha} + Bx^{\beta} + \cdots \quad \left(\alpha = \frac{p}{q}, \beta = \frac{r}{q}, \cdots\right)$$

where the exponents are increasing and all have the same denominator. We concentrate our attention on one complete branch of the curve. Introducing the modulus, ρ, and the argument, φ, of the abscissa, one gets

$$y_1 + iy_2 = A\rho^{\alpha}(\cos\alpha\varphi + i\sin\alpha\varphi) + \rho^{\beta}(\cos\beta\varphi + i\sin\beta\varphi) + \cdots .$$

Letting

$$\begin{aligned} A &= r\,(\cos v \; + i\sin v\;), \\ B &= r_1(\cos v_1 + i\sin v_1), \\ &\text{etc}, \end{aligned}$$

one gets

$$\begin{aligned} y_1 &= r\cdot\rho^{\alpha}\cdot\cos(v+\alpha\varphi) + r_1\cdot\rho^{\beta}\cdot\cos(v_1+\beta\varphi) + \cdots \quad (1) \\ y_2 &= r\cdot\rho^{\alpha}\cdot\sin(v+\alpha\varphi) + r_1\cdot\rho^{\beta}\cdot\sin(v_1+\beta\varphi) + \cdots \quad (2) \end{aligned}$$

(1) is the equation of the carrier surface in semipagan coordinates while (2) determines the projections of the elevation lines onto (X_1X_2). Restricting to the initial term, the

latter system[6] can be obtained by rotating the projections of the level curves through the angle $\frac{\pi}{2\alpha}$ around the origin. We shall consider some special cases.

$$1^o. \quad q = 1, \alpha = 2. \quad y = Ax^2 + Bx^3 + \cdots .$$

The point is an ordinary one with the X-axis as a tangent. The appearance of the carrier surface in a neighborhood of the origin is determined by the equation

$$y_1 = r\rho^2 \cdot \cos(v + 2\varphi).$$

This surface is obtained by constructing the branch of the parabola $y_1 = rx_1^2$ corresponding to $x_1 \geq 0$, and rotating its plane around Y through the angle φ while simultaneously multiplying its ordinates by $\cos(v + 2\varphi)$. All of the surface is then obtained by letting φ assume all values from 0 through 2π. The intersection with a cylinder around Y[7] is a wavy line that passes through (X_1X_2) 4 times. At a point of an algebraic curve where $\frac{dy}{dx} = 0$ (without there being other peculiarities), the carrier surface gets a saddle point with a horizontal tangent plane that intersects the surface in two mutually orthogonal curve branches. The system of elevation lines is of the same type as the system of hyperbolae with common asymptotes. In this latter system, a hyperbola passes through its asymptotes to its conjugate hyperbola. Similarly, fall lines with elevation numbers exceeding that of the point of tangency pass into fall lines with smaller elevation numbers through a curve with a double point. The mutually orthogonal branches of this curve are tangents to the horizontal tangent plane, and they intersect each other under a right angle. One of them runs above the plane, the other one below. The tangents at the double point bisect the angles between the principal tangents of the carrier surface.

$$2^o. \quad q = 2, \alpha = \frac{1}{2}. \quad y = Ax^{\frac{1}{2}} + \cdots .$$

Here one gets

$$\begin{aligned} y_1 &= r\rho^{\frac{1}{2}} \cdot \cos(v + \tfrac{1}{2}\varphi) + \cdots , \\ y_2 &= r\rho^{\frac{1}{2}} \cdot \sin(v + \tfrac{1}{2}\varphi) + \cdots . \end{aligned}$$

By an analogous investigation of the carrier surface, it is seen that the wavy line on the cylinder goes twice around before it closes up after completing one full oscillation. The appearance is reminiscent of a general branching point on a Riemann surface in that there is a double line which has a pinch point in the point considered so that thereafter the double line proceeds isolated and loses importance for us.

[6] Translator's remark: i.e. the system of projections of the elevation lines.
[7] Translator's remark: Clearly, Y_1 was intended

If one considers only the first terms, the projections of the level curves become a system of confocal parabolas. The projections of the fall lines are obtained by rotating the parabolas 180^O around the origin (the focal point). To each projection there corresponds in the two leaves two branches that intersect on the double line. The passage between the two sets of fall lines happens through a vertical parabola.

Hereby, we have described the appearance of the carrier surface in an ordinary point where $\frac{dy}{dx} = \infty$.

$$3^o. \quad q = 1, \alpha = 3. \quad y = Ax^3 + Bx^4 \cdots .$$

An inflection point with horizontal tangent. The wavy line closes up after one revolution with 3 oscillations. The carrier surface consists of one leaf, the elevation line $y_2 = 0$ has a triple point, and the other elevation lines send 3 tongues into the 3 openings between the branches, alternatingly.

A similar behaviour occurs in the case

$$y = Ax^p + Bx^{p+1} + \cdots .$$

$$4^o. \quad q = 2, \alpha = \frac{3}{2}. \quad y = Ax^{\frac{3}{2}} + \cdots . \quad \text{A cusp.}$$

$$\begin{aligned} y_1 &= r\rho^{\frac{3}{2}} \cdot \cos(v + \tfrac{3}{2}\varphi) + \cdots , \\ y_2 &= r\rho^{\frac{3}{2}} \cdot \sin(v + \tfrac{3}{2}\varphi) + \cdots . \end{aligned}$$

The wavy line passes 3 times around and describes 3 oscillations. The surface is described (approximately) by using the wavy line as a base curve when the positive branch of $y_1 = rx_1^{\frac{3}{2}}$ rotates around Y_1. (X_1X_2) is covered 2 times, and one gets 3 branching lines which meet at the singular point forming angles of 120^o. The continuation of these double lines as isolated lines are of no interest to us.

$$5^o.$$

In a similar way an arbitrary complete branch can be investigated. The shape of the surface and the number of branch lines can be obtained by investigating the wavy line of intersection with the cylinder around Y_1. It contains p oscillations and passes q times around the cylinder. There are $p(q-1)$ branching lines emanating from the point, for a segment of the wavy line from a maximum point to a minimum point (or vice versa) will be crossed 1 time by during each of the following $q-1$ revolutions, so that in total there will be $p(q-1)$ intersection points. If the first exponent is reducible, matters are somewhat more complicated. We shall not pursue that case any further.

Conic sections. When an equation for a conic section is split into its real and imaginary parts, one arrives at two equations of the form

$$\begin{aligned} y_2^2 + a_1 y_2 + a_2 &= 0 \\ y_2^2 + b_1 y_2 + b_2 &= 0, \end{aligned}$$

where the coefficients are polynomials in $x_1, x_2,$ and y_1 and where the indices indicate the degrees. By subtraction one arrives at an equation of the form

$$c_1 y_2 + c_2 = 0 \qquad (1)$$

and elimination of y_2 gives

$$c_2^2 - c_1 a_1 c_2 + c_1^2 a_2 = 0. \quad (2)$$

Thus, the carrier surface is an algebraic surface of order 4. (1) assigns a unique value for y_2 to each point of the carrier surface, except when the point lies on the conic section obtained by intersecting the plane $c_1 = 0$ by $d_2 = 0$.[8] The carrier surface has two pinch points corresponding to the two points where $\frac{dy}{dx} = \infty$; at infinity, it essentially coincides with the asymptotes. From all this, it can be deduced that the two leaves of the carriersurface intersect in two double lines which come from infinity and stop at two pinch points, and which are parts of a hyperbola. In particular cases, this hyperbola may dissolve into two intersecting lines. The branch lines of the carrier surface could then be parts of one of these lines or they could consist of a bounded piece of one of the lines and all of the other line which must then intersect the bounded piece.A double point on the branching line will occur for an arbitrary algebraic curve whenever the tangents to two points with the same abscissa are parallel while simultaneously the real parts of their ordinates are equal. This happens, e.g., for a hyperbola when the tangents parallel to the X-axis are imaginary. Examples of this are the curves $y^2 = \pm x^2 \pm 1$ whose carrier surfaces are easily investigated. An example of a curve with asymptote parallel to Y is given by $xy = a$. The projections of the level curves and the elevation lines are two families of circles tangent to X_1 and X_2 at the origin.

If you let the conic section dissolve into two straight lines, the carrier surface will transform into two planes, the branch line becomes the intersection of the two planes, and the two pinch points coalesce into the point corresponding to the intersection of the two lines. By the reverse process, the intersection point will split into two pinch points, and along the segment between these two points, the connection of the leaves interchange.

The carrier surface in its generality. We consider a general algebraic curve of order n, with n' tangents parallel to the Y-axis, with d double points, e cusps, and we assume

[8]Translator's remark: Clearly, $c_2 = 0$ was intended.

that it has no special position relative to the line at infinity or the coordinate system. The course of the n leaves at very great distances is determined by the asymptotes. Thus $\frac{1}{2}n(n-1)$ branch lines go to infinity. We say that the n' branch points corresponding to $\frac{dy}{dx} = \infty$ are of the first kind; from each of them is issued one branch line. To the e cusps there correspond branch points which we call of the second kind; from them 3 branch lines are issued. The double points have no corresponding peculiarities in the carrier surface; they occur on the branch lines, and the only remarkable fact about such a point is that the elevation number is the same in both leaves. In general, double points on the branch line, as described in the above, do not occur, although triple points do where 3 leaves intersect one another.

If we let the surface come infinitely close to dissolving into n straight lines, then it is easy to account for the course of the branch lines: from each of the $n(n-1)$ branch points of the first kind there emanates a branch line which stretches to infinity; and, conversely, any branch line coming from infinity ends at a branch point. There are $\frac{1}{2}n(n-1)(n-2)$ triple points on the branch lines. When a double point is created by a continuous change, this happens by two branch points coalescing. If a cusp is created on the curve, then in the carrier surface, 3 branch points of the second kind will move together to form a branch point of the second kind. It might seem possible to treat the appearance of the carrier surface in general by the "method of continuous change"; however, difficulties do arise. Partly, there is probably nothing to prevent the number of triple points from changing; and partly the course, of the branch lines may change through the creation of double points on them, as described above. Albeit, the latter phenomenon cannot produce any essential change in their course, since one can easily prove that a branch line emanating from a branch point cannot end in another branch point; it must stretch to infinity. Indeed, if a branch line connects two branch points, then the rule on page 1010 about the growth of elevation numbers easily lets one see that the branch line in question must be intersected by another branch line (in order that the connection of the leaves can be interchanged), but when this double point is dissolved, the connection between the two branch points is broken.

§ 4. The plane.

We now go on to investigate how we should construe elements at infinity in **T** in order that it correspond to the projective-geometrically defined plane (XY). (On other viewpoints later). In (XY) we have 1) all straight lines have one point at infinity; 2) parallel straight lines have the same point at infinity; 3) all the points at infinity taken together form a straight line.

The collection of points at infinity in our space, P, we shall denote by U. Each direction in P determines a point of U.

The elements of **T** that correspond to the points at infinity in the plane are obtained, in part from the points at infinity of space by equipping them with arbitrary elevation numbers, and in part by equipping the finite points of space with the infinite elevation number. However, the matter is quite complicated. All points at infinity of a carrier plane, equipped with their elevation numbers, should represent the same point, namely the point at infinity of the corresponding straight line. From this it follows that the point in U determined by the fall lines in parallel planes represents the same point of (XY) no matter with which finite elevation number it is equipped. Conversely, a point in U determined by a direction which is neither horizontal, nor vertical, corresponds to the point at infinity of a straight line the carrier plane of which has fall lines in the direction of the given point, no matter with which finite elevation number it is equipped.

All the points at infinity in the carrier plane lying in a direction different from the fall lines get infinite elevation numbers. Conversely, an arbitrary point of U with infinite elevation number corresponds to a full collection of points at infinity, namely those belonging to lines the carrier planes of which are parallel t the direction in P in which the point of U lies, without the fall lines having that direction.

The points at infinity of the horizontal planes represent the point at infinity of the X-axis if the elevation number is finite, whereas the correspondence is indeterminate if the elevation number is infinite. Points of U determined by the vertical direction, equipped with an arbitrary elevation number correspond to the point at infinity of the Y-axis, and the same is true for points of P at a finite distance with infinite elevation numbers. Evidently, the matters are rather complicated.

For ease of visualization, we shall consider the plane (XY) decomposed into parts that can easily and conveniently be represented separately. Afterwards, one recombines the 4-dimensional manifolds that represent the parts, or one indicates how the points in the boundary spaces correspond.

Spherical spaces. We shall first investigate the 3-dimensional manifold which is determined by the equation

$$x_1^2 + x_2^2 + y_1^2 + y_2^2 = r^2,$$

and which we shall call a spherical space. $y_2 = 0$ determines a sphere which becomes a boundary, in that the part of P which lies inside it, becomes the carrier for a space situated *above* P as well as a space situated *below* P. For both of these spaces, the elevation surfaces become concentric spheres that degenerate to the center for $y_2 = \pm r$.

(Compare with the presentation of a sphere by the projection of its levelcurves onto a horizontal diameter plane).

The spherical space separates **T** into two parts containing those points whose distances to the origin are, less than, respectively, greater than r. The first set of points project onto points inside the boundary, and their elevation numbers range between the two elevation numbers that appear on the spherical space vertically *above* and *below* the projection point.

The intersection line between the spherical space and the plane represented by $y = \alpha x$ is determined by

$$\begin{cases} y_1 = \alpha_1 x_1 - \alpha_2 x_2 \\ y_2 = \alpha_2 x_1 + \alpha_1 x_2 \\ x_1^2 + x_2^2 + y_1^2 + y_2^2 = r^2 \end{cases}$$

The equation for the projection of the carrier curve onto (X_1X_2) is

$$(1 + \alpha_1^2 + \alpha_2^2)(x_1^2 + x_2^2) = r^2.$$

Therefore, the carrier curve is an ellipse; its major axis is the diameter of the boundary sphere which is intercepted on the carrier plane's fall line through the origin; the minor axis is obtained by rotating the projection of the major axis onto (X_1X_2) through an angle of 90^o. The end points of the major axis separate the ellipse into two parts the elevation numbers of which have opposite signs so that part passes *over* P, the other one *under*; the elevation numbers reach their maximal absolute value at the end points of the minor axis.

For α infinitely large, the intersection line becomes the vertical circle

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ y_1^2 + y_2^2 = r^2 \end{cases}$$

Thus, the spherical space separates each straight line through the origin into two simply connected surfaces; one of these contains the line's point at infinity and is determined by the inequality

$$|x| \geq \frac{r}{\sqrt{1 + |\alpha|^2}}$$

(for α infinitely large by $|y| \geq r$).

All of the plane (XY) is described by the line $y = \alpha x$ by letting α assume all complex values. We first cut out the 4-dimensional manifold **A** which lies inside the spherical space, Σ, centered at the origin and with radius r. The remaining part of (XY) which completely contains the plane's line at infinity, is then cut into two parts as follows:

Each line $y = \alpha x$ for which $|\alpha| = 1$ has a surface part lying outside Σ and this part is determined by $|x| \geq r : \sqrt{2}$. All of these surface parts determine a space, T, which separate the considered part of (XY) into two parts which we shall denote by $\mathbf{B}'$ and $\mathbf{C}'$. $\mathbf{B}'$ shall be the part that contains the point at infinity of the X-axis, $\mathbf{C}'$ the part that contains that of the Y-axis. The space T originates from a surface, τ in Σ, described by the boundaries of the surface parts. T is determined by

$$|\alpha| = 1; \quad |x| \geq \frac{r}{\sqrt{2}}.$$

τ is determined by

$$|\alpha| = 1; \quad |x| = \frac{r}{\sqrt{2}},$$

or, since $y = \alpha x$, by

$$|x| = \frac{r}{\sqrt{2}}; \quad |y| = \frac{r}{\sqrt{2}}.$$

The projection onto P is a cylindric surface with axis Y and radius $r : \sqrt{2}$, and bounded by two circles in the contour of Σ. It can be described by vertical circles carried by the generators of the cylinder.

τ cuts Σ into two parts, Σ_1 and Σ_2, through which $\mathbf{A}$ is connected to $\mathbf{B}'$, respectively, $\mathbf{C}'$.

We shall now transform $\mathbf{B}'$ and $\mathbf{C}'$ into the finite realm.

$\mathbf{B}'$ is determined by

$$|\alpha| \leq 1; \quad |x| \geq \frac{r}{\sqrt{1 + |\alpha|^2}} \tag{1}$$

Therefore, α and $1 : x$ are finite everywhere in $\mathbf{B}'$.

In a similar way $\mathbf{C}'$ is determined by

$$|\alpha| \geq 1; \quad |y| \geq \frac{|\alpha| \cdot r}{\sqrt{1 + |\alpha|^2}};$$

or, letting $\beta \cdot \alpha = 1$, by

$$|\beta| \leq 1; \quad |y| \geq \frac{r}{\sqrt{1 + |\beta|^2}} \tag{2}$$

In $\mathbf{C}'$, β and $1 : y$ are everywhere finite.

$\mathbf{B}'$ and $\mathbf{C}'$ are now transformed by

$$\left.\begin{aligned} \xi &= \alpha \\ \eta &= \tfrac{r}{\sqrt{1+|\alpha|^2}} \cdot \tfrac{1}{x} \end{aligned}\right\} \tag{3}$$

respectively by

$$\left.\begin{aligned} \xi' &= \beta \\ \eta' &= \frac{r}{\sqrt{1+|\beta|^2}} \cdot \frac{1}{y} \end{aligned}\right\} \qquad (4)$$

into two manifolds **B** and **C**, defined by

$$|\xi| \leq 1; \quad |\eta| \leq 1,$$

respectively by

$$|\xi'| \leq 1; \quad |\eta'| \leq 1.$$ [9]

Interpreting ξ and η in a 4-dimensional manifold in the usual way, and similarly ξ' and η', we have hereby changed $\mathbf{B}'$ and $\mathbf{C}'$ into two simple 4-manifolds lying completely in the finite realm. The first one is described by vertical discs which have radius 1 and are centered on the disc $|\xi| \leq 1; \eta = 0$, and whose projections onto P are vertical line segments. The projection of the full manifold onto P is a right, circular cylinder. The boundary has two parts. The first one is a vertical space $T(\mathbf{B})$ corresponding to T; it projects onto the curved surface of the cylinder mentioned, and it is determined by

$$|\xi| = 1; |\eta| \leq 1.$$

The second part is a space $\Sigma_1(\mathbf{B})$ corresponding to Σ_1; its projection doubly covers the part of P contained in the cylinder; this space is determined by

$$|\xi| \leq 1; |\eta| = 1.$$

These two parts of the boundary are separated from one another by a surface $\tau(\mathbf{A})$ corresponding to τ, and described by the vertical circles

$$|\xi| = 1; |\eta| = 1.$$

Analogous remarks hold for **C**.[10]

The line at infinity in (XY) is represented by two horizontal discs, one in **B**, the other one in **C**, given by $\eta = 0$, respectively $\eta' = 0$. Their boundaries $|\xi| = 1; \eta = 0$ and $|\xi'| = 1, \eta' = 0$ are the loci of the centers of the vertical circles that produce $T(\mathbf{B})$ and $T(\mathbf{C})$.

We shall finish by reviewing briefly how the boundaries of the three manifolds $\mathbf{A}, \mathbf{B}$ and **C** correspond to one another.

[9] Translator's remark: The original has two = signs, rather than our inequalities.
[10] Translator's remark: The original has **B** rather than **C**.

The formulas (3) and (4) show that **A** and **C** are to be joined in the spaces $T(\mathbf{B})$ and $T(\mathbf{C})$ after the formulae

$$\left.\begin{aligned} \xi \cdot \xi' &= 1 \\ \frac{\eta}{\eta'} &= \alpha \end{aligned}\right\} \qquad (5)$$

The manifold $\mathbf{B}+\mathbf{C}$ is bounded by $\Sigma_1(\mathbf{B})$ and $\Sigma_2(\mathbf{C})$, and it must, from a topological viewpoint, be of the same kind as the spherical space Σ. (We shall return to this point later). The formulae (3) and (4) show the correspondence.

§ 5. Algebraic surfaces.

Let $F(x, y, z) = 0$ be the equation of an algebraic surface of order n. We shall assume that, when considered as a point set, it is a general surface (without double curves, cuspidal edges or multiple points), and that it is in general position relative to the coordinate system and the plane at infinity. Just as an algebraic function of 1 independent variable can be spread out in an n-valent way on a sphere, it must similarly be possible to spread out the algebraic function $z = f(x, y)$ defined by the above equation in an n-valent way on the manifold representing the plane (X, Y). To each point in the plane there corresponds a value of z.[11] These will all be different when the straight line through the point and parallel to the Z-axis intersects the surface in n separate points. The collection of points in **T** for which this does not happen we shall call the *branching surface*. This will be the surface that represents the boundary of the algebraic surface in the plane (XY). The boundary is an algebraic curve, of order $n(n-1)$ and of class $n(n-1)^2$. The number of cusps equals the number of principal tangents to the surface parallel to the Z-axis; this number is $n(n-1)(n-2)$. The number of double points equals the number of double tangents to the surface parallel to the Z-axis which is equal to

$$n(n-1)(n-2)(n-3).$$

Thus, the branching surface, which we shall denote by φ, is formed by $n(n-1)$ *leaves along* (X_1X_2) *with* $n(n-1)^2$ *branch points of the first kind and* $n(n-1)(n-2)$ *branch points of the second kind.*

We now consider each of the parts **A**, **B** and **C** separately.The radius, r, of Σ we shall consider infinitely big, so that **A** contains all the branch points of the branching surface.

[11]Translator's note: Here, the author must mean "at most n values of z," cf the opening plural form in the following sentence.

The part of φ lying in $\mathbf{A}$ we shall denote by $\varphi(\mathbf{A})$; it is bounded by $n(n-1)$ closed curves that can approximately be cut out from Σ by means of the asymptotes of the contour. Without essential restrictions we can assume that these curves are completely contained in the part of Σ that we have denoted Σ_1 - in other words, we may assume that the infinite parts of the contour are completely contained in $\mathbf{B}$. Indeed, the partition of the plane described in §4 will be essentially unchanged if we determine $\mathbf{T}$, not by $|\alpha| = 1$, but rather by $|\alpha| = k$ where k is an arbitrary positive number.

The parts of φ in $\mathbf{B}$ are represented (approximately) by $n(n-1)$ vertical discs

$$\begin{array}{cc} \xi = \alpha_1; & |\eta| \le 1 \\ \xi = \alpha_2; & |\eta| \le 1 \\ \cdots\cdots & \cdots\cdots \\ \xi = \alpha_{n(n-1)}; & |\eta| \le 1, \end{array}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_{n(n-1)}$ are the direction coefficients for the asymptotes of the contour. We shall denote these surfaces by $\varphi_1(\mathbf{B}), \varphi_2(\mathbf{B}), \cdots, \varphi_{n(n-1)}(\mathbf{B})$.

Now, in order to spread the values of z in $\mathbf{A}$, we cut this manifold by a vertical space, Φ, which projects onto $\varphi(\mathbf{A})$ in P, and which is otherwise determined thereby that each point of the projection is equipped with elevation numbers that vary from the elevation number the point should have to belong to φ to the number it should have to belong to the *upper* part of Σ; in other words: The space is the part of $\mathbf{A}$ lying vertically *above* $\varphi(\mathbf{A})$.
2 lines in $\mathbf{A}$, which have the same end points and do not intersect Φ can be connected by a surface which does also not intersect Φ. Indeed, we only have to connect the projections of the lines in P by a surface and equip its points with elevation numbers that (if the surface intersects the projection of φ) are smaller than the smallest elevation number on the intersection line. The boundaries of this surface are easily connected to the given lines by vertical surfaces, and one has constructed a surface with the required property.

In a similar way, one sees that two arbitrary points in $\mathbf{A}$ can always be connected by lines that do not intersect Φ.

At a point a of $\mathbf{A}$ we place one of the n z-values belonging to the point. If we let a move to a point b along two different paths that do not intersect Φ, and we let the z-value vary continously in accordance with the equation $F(x, y, z) = 0$ we shall arrive at the same value in both cases; this follows from the fact that the two paths taken together bound a surface that does not intersect Φ. After being cut along Φ all of the manifold $\mathbf{A}$ can be uniquely equipped with z-values starting from the given one. Proceeding similarly from the $n-1$ other z-values, one gets n sheets over $\mathbf{A}$. Two points that lie across one another in the boundaries towards Φ carry the same n values (but not in the same order); the sheets are glued together accordingly.

The 4-dimensional manifold that is spread over $\mathbf{A}$ in n sheets in this way, is denoted $\mathbf{A}'$. Each surface patch in $\mathbf{A}$ becomes carrier for a patch of a Riemann surface in $\mathbf{A}'$, and the branch points lie in the intersection of the patch and $\varphi(\mathbf{A})$.

Distributing the z-values in $\mathbf{C}$ creates no difficulties, since there are no points from the branching surfaces here; one gets n manifolds $\mathbf{C_1}, \mathbf{C_2}, \cdots, \mathbf{C_n}$ completely analogous to $\mathbf{C}$.

As we noticed earlier, the branching surface sends $n(n-1)$ surface patches into $\mathbf{B}$; the z-values are easily distributed here, e.g., by means of $n(n-1)$ branching spaces emanating from one of $\mathbf{B}$'s vertical discs and ending at $\varphi_1(\mathbf{B}), \cdots, \varphi_{n(n-1}(\mathbf{B})$. The fact that all of this manifold, $\mathbf{B}'$, forms one total whole, follows thereby that the plane at infinity intersects $F(x,y,z) = 0$ in an algebraic curve ψ[12] which under our assumptions is indecomposable. The part, $\psi(\mathbf{B}')$, of that curve which lies in $\mathbf{B}'$ is carried by the disc $|xi| \leq 1; \eta = 0$. The rest consists of n elementary surface patches carried by $|\xi'| \leq 1; \eta' = 0$ in $\mathbf{C}$. All of $\mathbf{B}'$ can be described by vertical discs centered in the Riemann surface patch $\psi(\mathbf{B}')$.

[12]Translator's remark: Here, the original has φ rather than our ψ.

SECOND SECTION
On topological connectivity numbers.

§6. Topology.

The analytic method of Descartes was, admittedly, a universal method for the solution of geometric problems, but as a rule it gave constructions which were inferior in simplicity and elegance to those of the Greeks. During his attempts to penetrate the principles for the peculiar geometric analysis applied by the Greeks, *Leibnitz* produced a series of considerations that he denoted *Analysissitus* or *Geometria situs* (Leibnizens gesammelte Werke herausg. von G. H. Pertz, 1858, mathematischen Schriften Bd. 1, p. 178: De Analysi situs). Occasionally, in this connection one has mentioned Leibnitz as the father of the modern topology, that is, to the branch of mathematics aiming at qualitative properties of the objects without considering quantitative, metric aspects. Leibnitz, in consequence of his theory, remarks somewhere: "Figura in universum praeter quantitatem continet qualitatum seu formam[13]", but in reality his theory has no similarity to what one now understands as topology.

Recently, mathematicians have begun taking an interest in several topological questions. One only has to call to mind the investigations of knots, nets, etc, by *Tait, Simony* and others; graphs; the appearance of graphical curves; the task of painting, and thereby separating, the countries on a map by means of 4 different colours; the stamp folding problem; and, last not least, the extensive attempts at forming a theory of connectivity numbers of n-dimensional manifolds and at generalizing Euler's polyhedron formula – 2 attempts that are closely related. (A good survey of the literature can be found in *W. Dyck:* Beiträge zur Analysis situs, M. A. Bd. 32).

Usage is somewhat wavering but the most reasonable would be to reserve the name Analysis situs for the latter kind of investigations, and on the other hand use the word topology for *all* investigations of a qualitative nature, as *Listing* has proposed (Vorstudien zur Topologie).

§7. Analysis situs.

It will no doubt be very difficult to formulate general theorems about the connectivity of an n-dimensional manifold in a blameless way. At any rate, it will certainly be

[13] Translation into English taken from [**?**]: A figure generally contains besides an extent (quantitas) a nature (qualitas) or form.

fortunate if one would, more than has hitherto happened, study concrete instances instead of going at the matter in its abstract generality. In theory, logic should suffice for mathematical development, but practice shows what a mighty lever is found in the tact that develops when one applies a theory to a large number of concrete cases. Even if a theory looks quite logical and plausible, it easily contains mistakes which come to light only when it is additionally controlled by visual scrutiny.

Riemann and *Betti* were the first ones to attempt a generalization of the theory of connectivity for surfaces which Riemann himself had applied so successfully in the theory of abelian integrals. After the death of Riemann, Betti put forward an article on the subject mentioned (Sugli spazi di un numero qualunque di dimensioni, Ann. di matem. 2nd Series, Vol. 4, 1871). He does not mention any cooperation with Riemann, but judging from those fragments of a theory which Weber collected from notes written down by Riemann (Riemann: Gesammelte mathematische Werke, 2. Aufl. Fragment XXIX), during his stay in Italy (Ges. Werke, p. 555), Riemann has contributed in an essential way to the thoughts expressed in Betti's treatise. In this, one finds for the first time the definition of connectivity numbers for manifolds with dimension count exceeding 2. Later, various authors have tried to improve and complete the theory: *Dyck* in "Beiträge zur Analysis situs (M. A. Bd. 32 and Bd. 37), *Poincar|'e* in the article "Analysis Situs" and *Picard* in his works on functions of two variables, (see p. 1003). Already before I knew the latter works, I had decided to attempt another path than the Riemann-Betti one, namely to attempt generalizing *Jul. Petersens puncture method* which I remembered from lectures (Listing's "Diagram" and "Trema" I only got to know much later, and that also goes for Betti's related investigations which I have known, until recently, only from the brief review in "Fortschritte der Mathematik").

For example, I found it unfortunate that the n connectivity numbers were not sufficient to characterize a manifold topologically when $n > 2$. When I became acquainted with the latter mentioned articles, especially the one by Poincaré, I began vaccilating in my choice, comparing the elegant method that I met here to the somewhat heavy and clumsy theory that I, myself, worked with. But I believed to discover that the road I had chosen threw lights on matters which did not stand out clearly in the other way. And when, in addition, I came into possession of means to find *sufficient* conditions for the equivalence of n-dimensional manifolds, I decided to continue in spite of the great difficulties I met. Two manifolds being equivalent means that they correspond to one another, point for point, so that when two arbitrary points in one of them approach merging then the corresponding points in the other manifold behave similarly. (Later, we shall return to the relation between Poincaré's notion of homeomorphism, Analysis Situs, §2, and the notion of equivalence defined here).

The question in the foreground is which cuts to place in a closed manifold (variété fermée; Poincaré l. c. § 1) to make it simply connected. For the solution we shall use the following procedure. The manifold is *punctured*, i. e., we excise the elementary manifold which forms the neighbourhood of a point. The puncture creates a boundary which we extend by continuous deformation so that more and more of the given manifold is excised. In this way one continues until some part of the boundary meets another part. In such places, one stops the deformation when the distance between the parts that meet has become infinitely small. Continuing in this way, one ends with a *diagram* formed by a system of manifolds of lower dimension than the given one – or rather: formed by the closest neighbourhood of this system, therefore, by a manifold which is infinitely small in the nth dimension. The system of manifolds of lower dimension which is the limit of the diagram we call its *core.*

This diagram has a double meaning. *First of all, it represents a manifold which is equivalent to the given one after this has been punctured.* If one succeeds in setting up normal forms to which one can reduce the diagrams, then the necessary and sufficient conditions for the equivalence of two closed manifolds will be the identity of this normal forms. Indeed, the elementary manifolds that have been excised are always equivalent. — But *the core of the diagram* indicates *the cuts one has to make in order to make the given manifold simply connected*. Indeed, if one performs the extension process mentioned backwards, the manifold left over when one makes the cuts indicated by the diagram [deformation] retracts to the elementary manifold removed by the puncturing.

Later, we shall return to a comparison between the results found in this way and Riemann-Betti's theory of connectivity numbers.

§ 8. The diagram of Riemann surfaces.

The theory of connectivity numbers for Riemann surfaces is well known and, in addition, it has been treated in a way similar to the one presented here, by *Jul. Petersen* (Funktionsteori, Kap. IV). Nevertheless, for the sake of what follows we shall briefly indicate a development of that theory, especially in order to show that it can be built *solely* on the diagram, that is, without applying the extended Euler Theorem (the theorem on the invariant number, $t-f$, of a surface). We shall consider the Riemann surface carried by a sphere. From a point, a, in the sphere, we place arcs of great circles to the n points that carry the branch points of the surface; we shall assume that these have orders, $k_1 - 1, k_2 - 1, \cdots, k_f - 1$. Above some point of the sphere, we puncture *all* n sheets, and by extension of the punctures, we push their boundaries to the branch points and along the great circle arcs to the point a. After removing some cul-de-sac bands, the remaining surface will consist of $^{1)}\, f$ elementary surface patches around the branch points, $^{2)}\, n$ elementary surface patches over a, and $^{3)}\, k_1 + k_2 + \cdots + k_f$ bands connecting the former elementary surface patches to the latter ones. We cut $n-1$ bands whose boundaries belong to different boundary curves, and the parts are retracted. The surface now represents a diagram corresponding to one puncture. It contains a total of $f+n$ elementary surface patches, connected by $\Sigma k - n + 1$ bands; we use $f+n-1$ of these bands to connect the $f+n$ elementary surface patches so that all the patches and the chosen bands form 1 elementary surface patch. From the boundary of it branch off

$$\Sigma_1^f k - n + 1 - f - n + 1 = \Sigma_1^f (k-1) - 2n + 2$$

handles that can easily be paired off to form double handles. Let m denote the multiplicity of an arbitrary complete curve branch of the algebraic curve that corresponds to the surface. Then $\Sigma(k-1) = \Sigma(m-1) + n'$ (with the sum extending over all curve branches having $m > 1$). Then the number of handles is given by the following number well known from geometry

$$\Sigma(m-1) + n' - 2n + 2 = 2p.$$

Thus, we arrive at a normal form with p double handles, and the necessary and sufficient condition for the topological equivalence of two Riemann surfaces is that they have the same p.

If we want a normal form for the *closed* Riemann surface, at each double handle we let the two entrances to one of the handles slide out to the middle of the other handle and then by widening the first handle we let its entrances absorb the boundaries of the second handle. By finally closing up the surface by an elementary surface patch [for each double handle] we get Klein's normal form: A sphere with p tubular handles.

What we have exposed here can serve as a kind of program for the investigations to be instituted. We should mention yet another method for treating the matter which is well suited for extension to the case of algebraic surfaces. We assume that the algebraic curve φ_n is without singular points (How the state of affairs is in the presence of such points can be found by a limiting procedure from the curve here considered). If the curve is changed continuously, without introducing singular points, and keeping the degree unchanged, then equivalent Riemann surfaces are obtained. We shall let it change into a curve ψ_n which is infinitely close to splitting into n straight lines in general position. To see that this is possible, one considers the linear bundle $\varphi_n + \lambda\psi_n = 0$. The curves get singular points for only a finite set of λ-values, so one can let λ range from 0 to ∞ while avoiding this set. The straight lines are represented by n concentric spheres of the same radius. Since one passes to ψ_n by dissolving the $\frac{1}{2}n(n-1)$ double points, there arise two branch points from each double point. Two such matching branch points are infinitely close to one another, and they are connected by a branch line. Such a pair of branch points is removed from the surface by placing circles around them, each circle in its own leaf, and excising the resulting discs. The excised part is converted into a tube. This procedure is repeated for each of the other pairs, and the $\frac{1}{2}n(n-1)$ tubes are added in the right way at the holes of the n spheres after these have been moved off one another. Then $n-1$ of the tubes are used to convert the spheres into 1 sphere from which there now originate a total of $\frac{1}{2}n(n-1)-(n-1)=\frac{1}{2}(n-1)(n-2)=p$ tubular handles, i.e., once again we have arrived at Klein's normal form.

If a closed surface is unilateral, in addition to the tubular handles its diagram contains a certain number of single handles, each with 1 twist.

§9. The diagram of 3-dimensional manifolds.

Let a closed 3-dimensional manifold be defined by

$$x_i = \theta_i(y_1, y_2, y_3) \qquad (i = 1, 2, \cdots, n)$$

and by a certain number of inequalities of the form

$$\psi(y_1, y_2, y_3) > 0$$

and by analytic continuation of this space (cf. Poincaré, Analysis Situs § 3). Or, to bring the expression more in agreement with the essence of topology: Let it be defined thereby that it shall be created by the joining together of elementary spaces the surfaces of which are subdivided into areas which have to be joined in pairs in a certain way; this should happen so that in the closed manifold, the neighbourhood of any point derived from one of the lines in the net becomes an elementary space. If the manifold defined is to be connected, the given elementary spaces must first be capable of being joined together to 1 elementary space the surface of which satisfies the same conditions. It presents no difficulties for the intuition to follow the expansion of a puncturing.

The diagram core will be formed by surface patches meeting along curve segments; the curves meet at points [to be called] the nodes of the diagram core. We may assume that the surface patches are simply connected since otherwise one may subdivide the patches into simply connected parts by means of curve segments with endpoints on the curve segments of the core. The added curve segments may then be included with the original segments.

We now pass to considering the diagram itself. Each node is surrounded by an elementary manifold which we may take to be a ball. The curve segments connecting these balls are surrounded by thread-shaped spaces that are glued to the balls along small elementary surface patches. Let us call these spaces *threads*. They can be described by a surface patch moving from one of the balls to the other (this, by the way, can happen in two essentially different ways, see § 11). The space that surrounds one of the diagram's elementary surface patches is a plate shaped space which is bounded by two elementary surface patches forming the sides of the plates, and by a surface strip with the connectivity of an annulus. This space we shall call a *plate*. It is sewn along its boundary onto a surface strip on the boundary of the body formed by the nodal balls and the threads. Let the diagram have a total of a nodal balls, b threads, and c plates. Together with $a-1$ threads the a balls form one elementary space which we give the shape of a ball. From this central ball there emanate $b-a+1=p$ threads. On the surface, the connectivity number of which is $2p-1$, there are c closed curves along which the boundary of the plates must be attached.

The necessary and sufficient condition for such a system of threads with attaching curves for plates to form the diagram of a closed space is that the attaching curves are formed by p ring cuts that do not disconnect the surface of the thread system.

Indeed, the necessary and sufficient condition is that the boundary after the addition of the plates be equivalent to the boundary κ of the elementary space removed by the puncturing, i.e.,it should have the connectivity of the sphere.

If the diagram really corresponds to a closed manifold, the two sides of an arbitrary plate will correspond to two simply connected areas on κ. If one removes the plate from the diagram, the connectivity of the boundary of the latter will change since the two sides of the plate are replace by the strip along which the plate was attached and which is equivalent to an annulus. To let κ undergo the same change of connectivity, one cuts away the two corresponding areas and connects them by means of a tube. If one continues in this way until all plates have been removed then κ will be replaced by a sphere equipped with as many handles as there were plates. This surface must be of the same connectivity as the boundary of the thread system. (This *by no means* says that the space it bounds must be of the same connectivity as the thread system itself). From here we can deduce *that the number of plates must be p*. The correspondence between the two surfaces is such that lines on the boundary of the thread system which pass through one of the p attachment strips correspond to lines on the other surface running around one of the p handles (meridian curves). Therefore, the boundary of the thread system stays connected when one removes the attachment strips.

The *sufficiency* of the condition mentioned can be seen by noticing the following. If, in each of two equivalent closed surfaces, one places an arbitrary maximal system of non-disconnecting ring cuts, then the resulting surfaces will be equivalent. The surface remaining when one removes the p attachments strips is, therefore, equivalent to a sphere with $2p$ holes. The boundary of the body obtained by attaching the plates is therefore equivalent to the sphere obtained by closing the $2p$ holes with discs.

§ 10. Oriented manifolds.

Indicatrix. Before we continue the investigation of the diagram it becomes necessary to insert some remarks. Three points a, b, and c on a closed curve determine a positive direction on the curve. In a surface a positive rotation direction can be determined by fixing a postiche direction on a small closed curve without self intersections. Hereby, again, one determines a positive and a negative side of the surface in the part of space immediately surrounding the small closed curve. If one extends such a determination to all of the surface it turns out that one must distinguish between *unilateral* and *bilateral* surfaces. One sees that here is a fields of investigation which must be extended to manifolds of higher dimensions; orientations on curves and positive and negative sides of a surfaces are fundamental topological notions.

The neighbourhood of a point on a curve is a small line segment the endpoints of which we shall denote by 1 and 2, so that 12 indicates the positive direction of the segment. 12 will be called an *indicatrix of the* 1^{st} *order* (cf. *Dyck*, M.A. Bd. 32, p. 473), and we say that a curve equipped with an indicatrix is *oriented.* — The neighbourhood of a point in a surface is a small closed curve; if this curve is oriented the surface patch bounded by it is called an *indicatrix of the* 2^{nd} *order.* — In a space defined in the way indicated, the neighbourhood of a point is a surface of the type of the sphere; if one equips this with an indicatrix of the 2^{nd} order, one gets the space's *indicatrix of the* 3^{rd} *order*. In an entirely similar manner we can form an *indicatrix of the* 4^{th} *order* in $\mathbf{T}$ or in a 4-dimensional manifold obtained by patching together elementary manifolds in $\mathbf{T}$. The definition can be extended to analytically defined manifolds of any dimension, but it becomes difficult to apply in this form when the manifold does not present itself for the intuition. For us, therefore, the investigations are important only when the dimension number is smaller than 5. An *indicatrix of the* n^{th} *order* contains a succession of indicatrices of the order $n-1, n-2, \ldots, 1$; the last one is the segment 12. The given indicatrix can be displaced in the manifold. One can return it to its original position so that the successive indicatrices of orders down to 2 cover one another. There are then two possibilities: *either* 12 can be brought into its old position 12, *or* it can be brought into the position 21. If the first possibility occurs for all possible displacements, the manifold is called *bilateral*, otherwise *unilateral*. A bilateral manifold can be oriented in two different ways. The resulting manifolds are called *opposites.*

I had already worked with these definitions when I saw Poincaré's treatment (Analysis Situs, §4 and §8). I continued to use my own since Poincaré's are essentially meant for topological investigations in analytic form. I have noticed that the definition found in Picard and Simart's book (p. 23) in its content coincides with mine, but here I have preferred to use my original formulation which gives the intuition something tangible to lean on.

The boundary of a bilateral, and oriented, n-dimensional manifold $\mathbf{M}$, can be oriented by means of the indicatrix of the manifold itself. Indeed, the indicatrix can be displaced so that the part of its boundary formed by an indicatrix of order $(n-1)$ is placed in the boundary of $\mathbf{M}$). This part (together with those of lower order) can then be used to orient the boundary. Conversely, in a similar way one may orient the manifold itself by means of an indicatrix of the boundary. In the following we always assume that the boundary is oriented in accordance with the indicatrix of the manifold.

(The usual definition of the positive and negative sides of a surface, based on its positive rotation direction presupposes that space has a given indicatrix or a substitute — the right hand, a clock or something similar).

Oriented corners. Let an m-dimensional manifold be defined as the totality of points

$$(y) \equiv (y_1, y_2, \cdots, y_m),$$

where the real y-values are subject to no conditions in their variation. The point $(0, 0, \cdots, 0)$ we shall call o. Through this point and each of the m points $(a), (b), \cdots, (k)$ we draw the rays $A, B, \cdots, K$ (i.e., the parts of the straight lines that extend in the positive direction from o). We assume that the points are in a sufficiently general position so that the determinant $\Delta = |a, b, \cdots, k|$ is non zero. Letting p be any integer between 1 and m, one can then be sure that no plane manifold of dimension $p-1$ can contain p of the lines. To each ray there corresponds a plane manifold of dimension $m-1$ which contains the other rays. These m plane manifolds, together with the rays, constitute what we shall call an m-dimensional corner with o as its apex, and the rays as its edges. We call is oriented when the order in which the edges are mentioned is stated. Any $m-1$ of the rays determine an $(m-1)$ dimensional corner in the plane manifold in which they lie, and so on. In general: p arbitrarily chosen edges lie in a p-dimensional manifold, and they determine in there a p-dimensional corner, subordinate to the given corner and oriented in accordance with it.

We can easily establish a connection between the order in which the edges are mentioned and the determination of an indicatrix in the manifold (y). As boundary for the indicatrix we can e.g. choose the spherical manifold $\Sigma(y^2) - r^2$. This is intersected by the ray A at a point a'. A neighbourhood of a' is taken as indicatrix of order $m-1$; we shall consider it bounded by the intersection manifold between $\Sigma(y^2) = r^2$ and the plane manifold of dimension $m-1$ passing through $B, \cdots, K$. This manifold is treated in the same way: A neighbourhood of the intersection point, b', between B and the spherical manifold is taken as indicatrix of order $m-2$ and is bounded by means of the plane manifold through $C, \cdots, K$. Continuing in this way, following the order of the rays given by the orientation of the corner, one ends up with a closed curve (the intersection between $\Sigma(y^2) = r^2$ and the plane through the last two rays, I and K) The intersection point of I we call i', and that of K we call 1. The negative extension of K intersects the sphere at a point that we call 2. As indicatrix of 1^{st} order we take the segment $1i'2$. And hereby an indicatrix of order[14] m has been determined. The connection between this indicatrix and the corner can briefly be given as follows: The first edge of the corner meets the boundary of the indicatrix at a point of an indicatrix of order $m-1$; the second edge at a point of an indicatrix of order $m-2$; etc, the $(m-1)^{st}$ edge at a point of an indicatrix of the first order; and finally, the last edge at the point 1, and its negative extension at the point 2.

During continuous displacements of the corner we retain the requirement that p edges shall never lie in any plane manifold of dimension $p-1$. If the apex has been moved to the point (t), and if the edges are then determined by the points $(a'), (b'), \cdots, (k')$, then the determinant[15]

$$\Delta' = |a' - t, b' - t, \cdots, k' - t|$$

will have the same sign as the original Δ.

The corner can be displaced so that the intimal $m-1$ edges cover the corresponding edges of any given corner. The two corners are now said to be of the same kind if the m^{th} edges can be brought to cover each other; if not then the corners are of opposite kinds.

[14]Translator's remark: Here, the original has n rather than m.
[15]Translator's remark: Here, the original has Δ rather than Δ'.

We shall consider the indicatrix in (y) determined by the coordinate corner, i.e., the corner with apex o and with edges passing through the points

$$\begin{array}{c}(1,0,\cdots,0)\\(0,1,\cdots,0)\\\cdots\cdots\cdots\cdots\\(0,0,\cdots,1)\end{array}$$

Here $\Delta = 1$. Therefore, all corners that are oriented in accordance with the indicatrix of (y) will have Δ' positive, and conversely.

Comparison with Poincaré's theory. (Analysis Situs §8). An elementary manifold determined by (Anal. sit. § 3)[16]

$$x_i = \theta_i(y_1, y_2, \cdots, y_m) \quad (i = 1, 2, \cdots, n)$$

in combination with inequalities of the form

$$\psi(y_1, y_2, \cdots, y_m) > 0$$

may be considered as the image of an elementary manifold in (y) determined by the inequalities $\psi(y) > 0$. We shall orient it by an indicatrix which is the image of the one in (y).

We shall now indicate how one can see that our presentation is in agreement with Poincaré's. The problem is the following:

We are presented with to elementary manifolds of the m^{th} order, v_1 determined by

$$x_i = \theta_i(y_1, y_2, \cdots, y_m), \quad |y_k| < \beta_k, \qquad (1)$$

and v_2 determined by

$$x_i = \theta_i'(z_1, z_2, \cdots, z_m), \quad |z_k| < \gamma_k, \qquad (2)$$

It is assumed that v_1 and v_2 have an elementary manifold v' in common. Poincaré then defines the order of the parameters z by means of the order of the parameters y — or what is essentially the same: the order of the y's from that of the z's — by demanding that the functional determinant

$$\frac{\partial(y_1, y_2, \cdots, y_m)}{\partial(z_1, z_2, \cdots, z_m)}$$

be positive for points of v'.

[16]Translator's remark: Here, the original has $y_i = \theta_i(y_1, y_2, \cdots, y_m) \quad (i = 1, 2, \cdots, m)$, but it seems that our version is compatible with the subsequent development.

We define the order of the z's by demanding that the indicatrices in v' that correspond to the coordinate corners in (y) and (z) should coincide.

We shall show that these two definitions give the same result. Let

$$(y^0) \equiv (y_1^0, y_2^0, \cdots, y_m^0) \quad \text{and} \quad (z^0) \equiv (z_1^0, z_2^0, \cdots, z_m^0)$$

be two points of (y) and (z) corresponding to the same point of v'. We consider the corner of (z) with apex (z^0) and with edges passing through the points

$$\begin{array}{llll} (z_1^0 + dr, & z_2^0 \quad , & \cdots, & z_m^0 \quad), \\ (z_1^0 \quad , & z_2^0 + dr, & \cdots, & z_m^0 \quad), \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots, \\ (z_1^0 \quad , & z_2^0 \quad , & \cdots, & z_m^0 + dr), \end{array}$$

where dr is a positive, infinitely small quantity. Here the determinant Δ' becomes $(dr)^m$, so this corner is of the same kind as the coordinate corner in (z); thus, the indicatrix that it determines is in agreement with the orientation (z). According to *our* definition, the indicatrix in (y) can be determined by using the equations (1) and (2) to map the part of (z) corresponding to v' into (y), while simultaneously mapping (z)'s indicatrix. We are now required to show that a corner corresponding to the indicatrix constructed in that way is of the same kind as the coordinate corner when the latter is oriented after *Poincaré's* definition. The former corner, however, is defined to have its apex at (y^0) and to have its edges pass through the points

$$\begin{array}{llll} \Big(y_1^0 + \big(\frac{\partial y_1}{\partial z_1}\big)_0 dr, & y_2^0 + \big(\frac{\partial y_2}{\partial z_1}\big)_0 dr, & \cdots, & y_m^0 + \big(\frac{\partial y_m}{\partial z_1}\big)_0 dr\Big), \\ \Big(y_1^0 + \big(\frac{\partial y_1}{\partial z_2}\big)_0 dr, & y_2^0 + \big(\frac{\partial y_2}{\partial z_2}\big)_0 dr, & \cdots, & y_m^0 + \big(\frac{\partial y_m}{\partial z_2}\big)_0 dr\Big), \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots \\ \Big(y_1^0 + \big(\frac{\partial y_1}{\partial z_m}\big)_0 dr, & y_2^0 + \big(\frac{\partial y_2}{\partial z_m}\big)_0 dr, & \cdots, & y_m^0 + \big(\frac{\partial y_m}{\partial z_m}\big)_0 dr\Big), \end{array}$$

where the subindices on the partial derivatives indicate evaluation at the point (z^0). For this corner, Δ' has the value[17]

$$\frac{\partial(y_1, y_2, \cdots, y_m)}{\partial(z_1, z_2, \cdots, z_m)} \cdot (dr)^m,$$

which is positive when you follow Poincaré's definition, *q.e.d.*

An interchange of two edges in a corner, and a reversal of the positive direction of an edge, make the corner change its kind.

[17]Translator's remark: Instead of our m, the original has n everywhere in the following formula.

§ 11. Continued investigation of the diagram for 3-dimensional manifolds.

We return to the diagram. The task that ought to be solved was to reduce it to a normal form. I have not succeeded in doing this, but I shall anyway state some observations concerning the solution of the task.

First of all, what is the condition for the closed, 3-dimensional manifold defined by the diagram to be bilateral? Any closed curve in the manifold can initially be deformed to run completely in the diagram, and its course in there can be restricted to the system of threads. The necessary and sufficient condition therefore becomes the following: when one runs along a line in an *arbitrary* thread one returns to the central ball with an unchanged indicatrix. The two entrances of the thread are elementary surface patches on the boundary of the central ball. On the boundary of one of these we fix a positive orientation, e.g., *clockwise* (seen from the outside). If one transports the closed curve along the boundary of the thread, until it covers the boundary of the thread's other entrance then this boundary receives an orientation which will be either clockwise or anti-clockwise (The central ball is always assumed to lie in our space. If the former alternative happens then the attachment of the ends of the thread to the central ball cannot really be carried out in our space, but surely in **T**). *In the former case, the manifold is clearly unilateral. If the second alternative happens for all threads then the manifold is bilateral.*

The thread system of a bilateral manifold can always be assumed to lie in our space. By severing a thread, suitable twisting the result, and regluing correctly, one can obtain a situation where the attachment strips run along the boundary of the thread without twisting around it.

The diagram can be transformed to an equivalent form by continuous deformation; we mention especially

1) The attachment strips can be arbitrarily displaced on the boundary of the thread system as long as different strips never touch one another.

2) Any of the entrances of a thread can be displaced along the surface formed by the central ball and the $p-1$ [other] threads as long as it does not pass any strip during its movement; the strips that run to the strip entrance in question must follow the entrance and be displaced in accordance with 1).

3) In addition to the displacement mentioned in 1), the attachment strips can undergo a displacement along one side of a plate. The result of such a displacement is most easily formed by letting the strip form a loop towards a point of the strip corresponding

to the plate, then disconnect the former strip at a point near the second strip and insert an orbiting along this second strip before closing up again.

By means of such displacements, a great many simplifications of the diagram can be achieved. If, e.g., a strip returns from one entrance of a thread to the same entrance without having passed any other thread in the meantime, then the loop thereby formed by the strip can be removed. If a strip confines itself[18]to having one branch running along one of the threads it traverses, by means of the displacements in 2) and 3), one can arrange that the strip piece runs once over the thread and then backtracks along itself without crossing any other threads, and in addition arrange that all other strip pieces are removed from the thread. If this has been done, the strip can be engulfed in the central ball by means of the corresponding plate, and it is then of no further interest. This sufficient condition for the removal of a strip piece from the diagram, is probably also necessary, but I have not succeeded in giving a fully blameless proof.

One might be tempted to believe it possible to use continuous deformations to separate the strips so that each of them run on its own one among the p threads. This, however, can be proved impossible, and therefore, the problem of reducing the diagram to a normal form is probably very difficult. We shall confine ourselves to considering such cases where the separation is possible. But first we shall briefly mention *the different classes of non separating ring sections* that exist on a torus surface (situated in our space). Here we consider two curves to be in the same class if they can be brought to coincide by continuous displacement.

The first class comprises the *meridian curves* that run around the torus surface; they form the boundaries of simply connected surface patches that are wholly contained in the space inside the torus; no other of the classes considered have this property. Another class is formed by the *longitudinal curves*, defined by their being capable of forming the boundary of a simply connected surface patch situated in the space outside the torus surface. If we draw a meridian curve, λ, and a longitudinal curve, β, (equipped with positive directions), then one may construct new classes of ring sections by going n times around along β in the positive direction and then closing the curve with a line along λ either in the positive or the negative direction. Let us denote such a curve by $[n\beta \pm \lambda]$. In a similar way, $[\beta \pm n\lambda]$ denotes a class represented by a curve which wraps once around along β, and then, before it closes, n times along λ. If the torus is cut along a meridian, and one of the ends is twisted n times in a suitable

[18]Translator's admission: I do not fully comprehend the following argument, and I cannot guarantee the correctness of the translation. In particular, I am uncertain about Heegaard's distinction between a "strip" ("strimmel", in Danish) and a "strip piece" ("strimmelstykke" in Danish).

direction, before the body is reglued in the proper way, then the curve can be reduced to a longitudinal curve; (of course, the original longitude ceases to be longitudinal). The complete classification is rather difficult, and there is no reason to pursue the matter here.

A thread in the diagram can be changed into a torus connected to the central ball by means of a cylinder. On its surface runs the corresponding attachment strip; it follows one of the above mentioned curves. Indeed, under our assumption, each strip can be brought to run alone, each on its own thread. If the strip is *longitudinal*, as mentioned earlier the torus and the plate can be engulfed into the central ball *and hence is of no importance*. The case where all the strips run as *meridian curves* is particularly simple, and matters are quite analogous to what we met in our investigation of surfaces. If we consider the diagram situated in our space, and the latter situated in $\mathbf{T}$, one may easily form a closed space with the given diagram as *its* diagram. If a plate has to be attached to a strip, then we place it in $\mathbf{T}$ *above* our space, P, so that its projection onto P fall in one of the diagram's elementary surface patches which is bounded by the strip. The boundary of the surface is then *bent down* into our space and attached to the diagram along the strip. We now again consider each torus converted into a thread. Then we let the plate become thicker so that the attachment strip completely fills the surface of the thread. The manifold formed in this way is closed up by an elementary manifold the projection of which is the central ball. One has arrived at a form which corresponds completely to *Klein's* normal form for surfaces. The contour in P of the closed 3-dimensional manifold is formed by precisely such a surface, and the rest of the manifold consists of two parts, the projections of which both form the body bounded by the contour, one part situated above, the other one below P. One may say that it arises by placing a certain number of *handles on a spherical space.* Such a handle is formed by removing two elementary manifolds from the spherical space. The boundary of one of these is moved away from the spherical space and then around to cover the boundary of the other one, of course in such a way as to make the manifold bilateral. The neighborhood of a closed curve in $\mathbf{T}$ is bounded by a space of the kind described here.

Every closed surface in $\mathbf{T}$ is surrounded by a 4-dimensional manifold which is bounded by a 3-dimensional manifold, called the *hull* of the surface. (In a similar way one determines the hull of any manifold or a point, situated in a given manifold). Let us determine the diagram for the hull of a torus surface in $\mathbf{T}$. It consists of three threads; each of the three strips runs through two of the threads with two branches in opposite directions. The easiest way to visualize the appearance is to place the central ball at the origin and let the threads follow the three axes which one imagines to be continued through infinity. The intersection lines with the three coordinate planes then indicate the course of the strips. Here one has an example of a diagram where the strips cannot be arranged so that each runs along its own thread. This diagram illustrates the example mentioned in *Picard's* and *Simart's* book, chapter II, no. 18. Similar diagrams, by the way, are obtained for the hulls of closed surfaces in $\mathbf{T}$. However, the multitude of diagrams are not exhausted by these examples.

Earlier, we considered a diagram formed by a torus on the surface of which the attachment line ran as a meridian curve; it could, however, run in infinitely many different ways. We shall only consider one example to which we shall return later, viz. an example where the *attachment line is a curve* $[2\beta + \lambda]$; if the torus is flattened so that the strip becomes its boundary, then one arrives at the wellknown unilateral *Möbius* surface patch.

As already mentioned earlier, the core of the diagram indicates which cuts (along lines and surfaces) should be made to make the punctured manifold simply connected. By the way, to determine these cuts, we could also follow a somewhat different path. Indeed, the diagram is equivalent to the punctured manifold. But to make the diagram simply connected one may first pierce each of the p plates by a canal; the threads we may cut by surfaces which are bounded by meridian curves, and in order for these to become cuts in the diagram, for each intersection point between a meridian curve and a strip, the cut-surface must send an extension into the plate along to the canal piercing the plate. Thus, at such an intersection point, the boundary of the cut-surface runs from the meridian curve, onto the plate, along to the canal, along the boundary of the canal, and back to the meridian curve along the other side of the plate. Of course, this system of cuts can alternatively be seen as a diagram of the original manifold, by imagining the parts that we just removed as the remaining parts. The canals then become threads originating from the manifold removed by the puncturing. And the cut-surfaces become plates. There is a kind of dualistic connection between the two diagrams. The threads in one correspond to the plates in the other and vice versa. As many strips run along a given thread as there are strip segments that bound the corresponding plate. As many plates issue strips along a given thread as there are there are threads with strips from the corresponding plate, etc. The latter method for obtaining the diagram has been used by *Betti*.

As mentioned, if one closes the diagram by connecting its surface to the surface of a ball, one gets a manifold equivalent to the given one. By the way, the parts could be separated in a somewhat different manner, in that the plates – rather than being added to the thread system – could be added to the surface of the ball along their two sides. Thereby, one gets a body with p handles, looking perfectly like the thread system. The surfaces should be joined together. It suffices to know the system of non separating ring cuts on one of them which corresponds to the β curves on the thread surface of the other one, as well as the system which corresponds to the λ curves. As an example, two tori whose surfaces are joined so that meridian curves cover longitudinal curves and vice versa, correspond to a diagram with one thread the strip of which is a longitude. The manifold itself is a spherical space. This agrees with the fact that the space inside a torus surface can be transformed to the space outside the torus surface so that the longitudes become meridian curves and vice versa – all of this under the assumption that the infinitely distant parts of space be considered equivalent to the neighborhood of a point so that everything situated outside a given sphere can be brought inside the sphere by an inversion. As another example, we take two tori the boundaries of which are joined so that longitudes cover longitudes and meridians cover meridians. We then get the manifold which is equivalent to the hull of a closed curve in $\mathbf{T}$. Finally, we choose two tori where longitudes and meridians on one correspond to the curves $[2\beta+\lambda]$ and β on the other; this corresponds to the afore-mentioned diagram with the attachment strip $[2\beta+\lambda]$.

We shall mention yet another circumstance which evidently causes the theory to become more difficult when the dimension count exceeds 2. If, in the diagram of a closed bilateral surface, one cuts a handle, then essentially the following reattachment can happen in only 1 way if the surface is to stay bilateral. Thus, by reattachment one certainly recaptures a diagram equivalent to the original one. If, on the other hand, one has for example a 3-dimensional manifold, like the one forming the boundary of a torus surface in $\mathbf{T}$, then that manifold can be cup along a torus surface (corresponding to a meridian curve on the surface, the hull of which is under investigation). *However, reattachment can happen in infinitely many, essentially different ways:* β and λ in one boundary can be joined to two arbitrary of the above-mentioned ring cuts, as long as these intersect one another in only one point.

§ 12. Comparison to earlier presentations.

In comparing the results we have arrived at with earlier presentations we shall essentially stick to the following works (for which, in the interest of brevity we shall introduce some notation):

1) *Riemann:* Fragment aus der Analysis Situs (Fragment XXIX. Gesammelte mathematische Werke, 2. ed., editor H. Weber; 1892 – to be denoted *R. Fr.*

2) *Betti:* Sugli spazi di un numero qualunque di dimensioni (Annali di matematica; 2. series, vol. 4; 1871) – to be denoted *B. Spz.*

3) *Poincaré:* Analysis Situs (Journal de l'école polytechnique; 2. series, vol. 1; 1895) – to be denoted *Poinc. A.S.*

4) *Picard:* Mémoire sur la théorie des fonctions algébrique de deux variables (Liouville Journal; 4th series, vol. 5; 1889) – to be denoted *Pic. Mém.*

5) *Picard* et *Simart:* Théorie des fonctions algébriques de deux variables indépendentes, 1st vol.; 1897, especially chapter 2 – to be denoted *P. & S.*

W. Dyck's works in M. A. vol. 32 and vol. 37 (Beiträge zur Analysis situs, 1888 and 1890), go in a different direction than the investigations we have made here, so there is no cause for a comparison.

A criticism of Riemann's Fragment would be unfair since it contains only preliminary drafts which have not been intended for publication. Moreover, in relation to mistakes in Riemann's and Betti's works, one may in many instances restrict oneself to a parallelization of the investigations in question to the corresponding amended ones by Picard and Poincaré. At such points, therefore, an exhaustive critique would be superfluous and, in addition, difficult, since the former authors's definitions of the concepts are often very vague.

This, in particular, holds already for the *definition of the manifolds* to be investigated. In Riemann no definition is explicitly *mentioned*. Betti has realized that one must stand on an analytic foundation in order to define an n-dimensional manifold (B. Spz.; section 1). Thinking suffices to generalize the notion of a function to an arbitrary number of variables, but visual powers are unable to generalize the notion of the 3-dimensional space. The definition, however, is given in terms too general to allow the ensuing investigations to be exact. To see which clarifying additions one must make to the definition, one may compare to the beautiful presentation given by Poincaré (Poinc. A.S. § 1, § 2, § 3, § 4, and § 8) – a presentation, the method of which should be considered an ideal for the treatment of investigations of this sort once they have been worked through to clarity, and their importance for science has been established.

In Riemann, as well as in Betti, the *definition of connectivity numbers* is based on a theorem analogous to the one on which Riemann, himself, bases his definition of the connectivity number of a surface. (R. Fr.: Es seien $a_1, a_2, \cdots, a_n, \cdots$ and B. Spz Section 3: Per giustifiare $\cdots$.) Oddly enough, Poincaré presents the definition of connectivity numbers without such a justification (A.S. § 5 and § 6), but in Picard and Simart the matter is treated in chapter 2, No. 12 and 13. In No. 12, one finds Riemann's lemma in a more precise form (in Poincaré, this thing is essentially found in the form of his claim that homologies can be added just as equations). The reason why the presentation in No. 13 deviates from the Riemann-Betti treatment is that in the latter there is a hole in the proof. In order to really use the method described (B. Spz. section 3: Se t spazi chiusi $\cdots$) to successively replace the A's with the B's, one must be certain that the A's and the B's could be paired together two and two so that each pair constitutes a part of, or all of, the boundary of its individual manifold of the $(m+1)^{st}$ dimension. That this is possible can be proved by using Poincaré's homologies. However, there is no reason to expand on this, since the basic idea in such a proof coincides essentially with the presentation in P. & S.; 2nd chapter No. 13.

Incidentally, it must be noticed that the definition is justified only after it has also been proved that a system of manifolds as described in the definition *really exists*. Here it appears that the modern presentations (P. & S.; 2nd chapter No. 11 and Poinc. A.S. §5 and §6) are not sufficiently clear. The definition of the connectivity number p depends on two things: how one defines the manifolds $V_1, V_2, \cdots$ mentioned, but also which meaning one ascribes to the word "bound" (former frontière). $V_1, V_2, \cdots$ should among other things be *bilateral* and what we call *oriented*, and it is required that the orientations be what we call *in accordance* with the orientation of the manifold they bound (P. & S.; 2nd chapter No. 8). If these requirements are taken literally one will run into difficulties. On a torus surface, a meridian curve λ and a longitude β (both equipped with positive directions) do *not*, even if this is usually assumed, form boundaries with every closed curve on the surface that does bound – not even if one is allowed to β k_1 times and λ k_2 times. . For instance, they do not bound together with $[\beta+\lambda]$, when one sticks to the required dependency between the indicatrices of the surface patch and the boundary. On the whole, it is unclear what to understand by the statement that curves bound on a surface when they intersect. Does a small, closed, plane curve in the shape of the figure eight bound a part of the plane? The answer "Yes" will certainly only be available after artificial addenda. If we draw a circle around the curve, that circle can be given two different positive directions. Which one of these bound together with the given curve? We meet with completely similar difficulties in Poincar ́s homologies (Poinc. A.S. §5). They could no doubt be removed by giving the following definition:

A system of m-dimensional manifolds $V_1, V_2, \cdots$, is said to be homologous to another system of m-dimensional manifolds $W_1, W_2, \cdots$ all situated in a manifold S_n:

$$V_1 + V_2 + \cdots \sim W_1 + W_2 + \cdots ,$$

if, by a continuous deformation in S_n, one can reduce the first system either directly to the second system (with the correct indicatrices) or to a system that can be deduced from the second system by the following two kinds of modifications: Either one places some $(m-1)$-dimensional cuts in the m-dimensional manifolds; the boundaries obtained that way are again removed by joining them (bilaterally) to m-dimensional manifolds that can be partitioned into 2 groups of which the members of one always run infinitely close to the members of the other and are of opposite orientation. Or else one cuts out a neighborhood of some manifolds of dimension less than $m-1$ and the boundaries created thereby are removed again by connecting them with manifolds which are infinitely close to coinciding with manifolds of dimension lower than m. If $V_1, V_2, \cdots$ are homologous to the hull of a point, they are said to homologous to 0: $V_1 + V_2 + \cdots \sim 0$. (The manifolds added will be of no consequence for the corresponding integrals).

The following examples may serve as a further explanation: A sphere which encloses two spheres (all oriented so that the indicatrices for the balls bounded are of the same kind) can be reduced to the latter pair of spheres connected by means of an infinitely thin tube. Conversely, the two interior spheres can be reduced to the first sphere after this has been cut along a meridian circle and the resulting boundaries have been closed up by attaching disks.

Similar relations hold for a sphere enclosing a suitably oriented torus surface.

The curve $[\beta, \lambda]$ on a torus surface can he reduced to the curves $[\beta]$ and $[\lambda]$ after both of these have been cut open and the endpoints have been connected by means of lines that run along one another.

In B. Spz. 4th and 5th section there are investigations which are not found in the works of later authors. Indeed, as we shall see, the contents are not reliable. Most likely, though, the more frequently stated theorem $p_m = p_{n-m}$ (see the following) is inspired by these investigations (cf. B. Spz. section 5: Ora ciascuno degli spazi sarà intersecato da una e da una soltanto delle sezioni trasverse di $n-m$ dimensioni che fanno parte de quelle che rendono R semplicemente connesso).

In section 4, one finds the definition of a transverse section (sezione traversa) in a manifold with boundary; it is required that the boundary of the section fall completely in the boundary of the manifold. Next, there is a presentation showing how a manifold (if it is closed, it must first be punctured) can loose one dimension by a continuous deformation – that is, a presentation of what we call the diagram, although, in Betti, this is considered as an equivalent of the (perhaps punctured) manifold; on the other hand, it is not noticed that the diagram supplies sections which make the manifold simply connected. The following theorem is stated: The connectivity numbers of a closed manifold is unchanged by 1 puncturing, and only the connectivity number of the highest order is changed by $s+1$ puncturings; indeed, it increases by s.

In section 5, the following theorem is stated:

To make a finite n-dimensional space R simply connected by means of sections, it is necessary and sufficient to place p_{n-1} sections of 1 dimension, p_{n-2} of 2 dimensions, p_{n-3} of 3 dimensions, ..., and p_1 of n dimensions, where $p_1+1, p_2+1, \cdots, p_{n-1}+1$ are, respectively, the 1^{st}, 2^{nd},..., $n-1^{st}$ order.

It must here be maintained that by definition, the boundary of any section must be completely contained in the boundary of the *original* manifold, that is it must not fall on the new boundary created by the section that were already placed. This is evidently not an unimportant formalism, for if we do not maintain that requirement, it would be meaningless to call the condition necessary. Indeed, then one could place arbitrarily many sections that make the manifold simply connected. If for example in a ball one makes a canal from one point of the boundary to another, the space can be made simply connected again by a section, which has a part of its boundary running along the surface of the canal. And this operation can be repeated arbitrarily often.

Let us for a moment stick to 3-dimensional manifolds, and let us for example consider the one that we have called a spherical space with p handles. Here $p_1 = p_2 = p$. The diagram consists of threads, each running through its own handle and emanating from the central ball. The plates are attached to the thread along a meridian section, and each separates its handle. The plates can easily be deformed so that the strips slide into the surface of the central ball, and one has really made the manifold simply connected by making p_2 sections of 1 dimension and p_1 sections of 2 dimensions. Let us next take the punctured manifold the diagram of which is one thread on which the attachment strip for the plate runs along the curve $[2\beta+\lambda]$. This can evidently be made simply connected by a certain line section that runs from the puncture back to the puncture, and a surface section whose boundary runs along a line $[2\beta+\lambda]$ in the surface of the line section and the points[19]. But this surface section cannot be made into a transverse section since the attachment strip cannot be carried down to the surface of the puncture. As we shall later mention, in this case $p_1 = 1$ and $p_2 = 0$ (with Betti's conditions; with Picard's latest conditions $p_1 = 2, p_2 = 1$). Thus the theorem does not at all hold here.

It is not difficult to find mistakes in the proof. First of all, it is not substantiated that one can reduce all of the given manifold to the system A (see Betti's notation) connected by means of manifolds of $n-2$ dimensions. This is possible in our first example, but not in the second. Nor is it self-evident that the connectivity numbers of the orders

[19] Translators remark: The phrase "in the surface of the line section and the points" is a literal translation of the original, but it does not seem to make sense.

$n-2, n-3, \cdots$ are unchanged when one uses the way indicated to derive a line section in the original manifold from a puncturing of one of the manifolds A; admittedly, the numbers in question could not decrease, but the line section could possibly make them increase, fx. by preventing a line that used to bound from continuing to do so. Finally, it has been overlooked that the following sections in higher dimensions might get parts of their boundaries running on sections earlier made, so that they are not transverse sections (in the sense of Betti).

We now turn to the frequently mentioned theorem $p_m = p_{n-m}$. It has often been stated and used, but the first published attempt to prove it no doubt is in Poincaré (A.S. p. 33–46). Picard and Simart rightly felt that "peut-être plusieurs points auraient-ils besoin d'être complétés"[20] and restricts his considerations to $m = 1$. However, it appears to us that not even in this special case have they succeeded in proving the theorem, and strangely enough, the faulty assumption in the proof seems to be the same as in Poincaré.

Poincaré first defines a number $N(V, V')$ for each intersection point of two oriented manifolds, V of p dimensions and V' of $h-p$ dimensions, both situated in a manifold U of h dimensions and, like all the following, assumed to be closed and bilateral. Assuming for the present that V is of 1 dimension and $V_1, V_2, \cdots, V_k$ are of $h-1$ dimensions, it is shown that

1. If $\Sigma V_i \sim 0$ then $\Sigma N(V, V_i) \neq 0$,
2. If the homology $\Sigma V_i \sim 0$ does *not* hold, then one can always find a manifold V for which $\Sigma N(V, V_i) \neq 0$.

Next an attempt is made to prove the same when V is of h dimensions and $V_1, V_2, \cdots, V_k$ are of $h-p$ dimensions (A.S. p. 43). For a start, the following assertion is mentioned as an assumption: Quant á $V_1, V_2, \cdots, V_k$ nous les définerons de la manière suivante. Nous pourrons toujours trouver $p-1$ équations

$$\Phi_1 = \Phi_2 = \cdots = \Phi_{p-1} = 0$$

auxquelles satisfont tous les points de $V_1, V_2, \cdots V_k$; pour définir V_i nous adjoindrons une $p^{ième}$ égalité

$$F_i'' = 0.$$[21]

[20]Translation by HJM: Perhaps several points could be in need of supplementation.

[21]Translation by HJM: As for $V_1, V_2, \cdots, V_k$ we shall define them as follows. We can always find $p-1$ equations $\Phi_1 = \Phi_2 = \cdots = \Phi_{p-1} = 0$ satisfied by all the points of $V_1, V_2, \cdots, V_k$; to define V_i, we shall add a p^{th} equation $F_i'' = 0$.

Thus it should be possible to represent each of the manifolds $V_1, V_2, \cdots V_k$ as the complete intersection between p manifolds of $h-1$ dimensions in U. For $h = n, p = n-1$ this assertion coincides with that of Picard and Simart (2nd chapter, No. 24): Nous admettrons, que dans une varieté E_n, on puisse toujours, en vertu de la continuité et de la connexion linéaire, considerer une varieté V_1 comme l'intersection commune unique de $n-1$ varietés V_{n-1} contenues dans E_n.[22]

However, the start of the proof can be carried out without using the assertion in its entirety.

1. If $\Sigma V_i \sim 0$ this means that in U one can place a manifold U' of $h-p+1$ dimensions on which $V_1, V_2, \cdots, V_k$ bound. V intersects U' in a closed curve V' (or a system of such curves).

One has

$$N(V', V_i) = N(V, V_i) \text{ and } \Sigma N(V', V_i) = 0, \text{ thus } \Sigma N(V, V_i) = 0.$$

2. Conversely, when the homology

$$\Sigma V_i \sim 0$$

does not occur, first of all it has not been shown that one can place any $(h-p+1)$-dimensional manifold U' through $V_1, V_2, \cdots, V_k$ (determined by $\Phi_1 - \Phi_2 = \cdots = \Phi_k = 0$); if we assume this to be correct (which it probably is), we do know that in U' one may place a closed curve V' so that

$$\Sigma N(V', V_i) \neq 0,$$

but it is not certain that this curve can be cut out by means of any manifold V. Thus, here one must make use of the assertion mentioned. If, e.g., $p = 2$, then U' would be of dimension $h-1$ and it could perhaps separate U into two parts U_1 and U_2, the boundary of both of which is U'. In this V' runs, and V would then be cut into two parts, each running in its own part of U and having V' as its boundary. But from the outset there is nothing to prevent V' from being exactly the kind of a curve in the boundary that does not bound any surface in U_1 or in U_2.

Therefore, the proof of the theorem $P_p = P_{p-1}$ is not even correct for $h = 1$. Admittedly, the inequality

$$\mu \leq \lambda$$

[22] Translation by HJM: We shall accept that in a manifold E_n by virtue of the continuity and the linear connection (or path-connectedness???), one can always consider a manifold V_1 as the common, unique intersection of $n-1$ manifolds V_{n-1} contained in E_n.

has been shown in this case (cf. the analogous theorem $p_1 \geq p_{n-1}$ in P. & S. 2nd chapter, No. 26); but conversely, the theorem

$$\mu \geq \lambda$$

has really not been proved.

If we now turn to the proof at Picard and Simart (2nd chapter No. 24-27) then as already mentioned we meet the same strange assertion. The proof in No. 26 that $p_1 \geq p_{n-1}$ is – as we have touched upon – undoubtedly correct, but the attempt, in No. 27, at proving the opposite theorem is based upon the assertion mentioned. Indeed, not alone has the theorem not been proved: *It must be untrue*. We have already repeatedly considered a closed, bilateral, 3-dimensional manifold for which $p_1 = 2$ and $p_2 = 1$. However, we shall postpone its treatment to a more convenient time, (cf. p. 1054).

§ 13. Riemann spaces.

In the following we shall consider the infinitely distant in our space as one point. Then a spherical space, Σ, can be converted into our space, P, by means of the following transformation: the part of Σ lying *below* P is bent *up* into P and next converted by inversion with respect to the bounding sphere; next, the remaining part of Σ which lies *above* our space is bent *down* into our space.

We shall investigate the 3-dimensional manifolds that arise when one spreads the function values of an n-valued continuous function of the points of Σ above Σ. Such a manifold is analogous to the Riemann surfaces, and we shall therefore call it a Riemannn space. The n values of the functions are assumed to be different except on certain closed curves (the *branching line*) where two of the values are assumed to coincide. If Σ is transformed into P, these curves are transformed into closed curves which can be assumed to run completely in the finite part of P. By the way, we assume that P_n forms a connected whole – is simple.

We shall now construct an n-sheeted space above P in which the function values can be distributed uniquely. We place a cone with vertex at an arbitrary point and with the branching line F as lead curve. From each generator of the cone, only the part between o and the branching line will be used. The cone gets a double generator if the branching line has an apparent double point when viewed from o; such double generators extend from o to the nearest branch of F. We imagine that P carries n spaces cut open along the cone mentioned (the *branching cut*); they are numbered, and they are called the 1^{st} sheet, the 2^{nd} sheet, ..., the n^{th} sheet. At the n points carried by an arbitrary point of P one places the n values of the function $z_1, z_2, \cdots, z_n$. Starting from these values, the n sheets can now be uniquely covered with values, so that these are continuous in each sheet. The sheets are now glued together in accordance with the z-values sitting right across one another on the cut surfaces. The closed manifold formed in this way we shall denote P_n[23].

Sometimes one glues together cut surfaces that belong to the same sheet. This happens for all the cut surfaces near a point of the branching line except for those two sets which should interchange 2 z-values when one runs once around the branch line near the point considered. At a given point let it be the r^{th} and the s^{th} sheet, whose cuts are alternately connected. One can then demarcate an area on the cone on which the r^{th} and the s^{th} sheets – and only those – are alternately connected. This area is bounded by *an arc ab of the branching line and by two double generators ao and bo*, and into this area there may radiate one or more double generators which do not extend all the way to ab. Of such areas there are as many as there are double generators (h).

At the points a and b, the branching curve is stopped by surfaces each coming from its own branch, A and B, of the branch curve; we say that the line segment ab is stopped by the branches A and B. Each segment ab is labelled by the numbers of the two sheets, r and s, which are alternately connected at the part of the cone surface bounded by the segment. r and s shall be called the *characteristics* of the segment.

Schematically, we shall indicate the conditions at a double line by drawing on paper two mutually orthogonal lines which should represent two branches of F; one is considered as being above the other, and o is situated behind the apparent intersection point of the lines. The branching cut extends down under the paper so that the lower line's branching cut penetrates that of the upper line. Thus, the latter stops both branches of the former. Let the characteristics of the upper line be r and s. Then 3 cases are possible:

1^{st} *type:* One of the branches of the lower line has the same characteristics, r and s.

2^{nd} *type:* It has r as one characteristic, but the other one, t, is different from s.

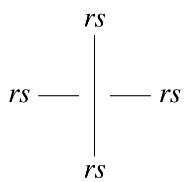
3^{rd} *type:* Both characteristics, t and u, are different from r and s.

We shall determine the characteristics of the other branch of the lower line. A revolution around that branch can be changed so that one first penetrates the branching cut of the upper line, then the branching cut of the first branch and finally that of the upper line

[23] The potential function for an electric current offers a good deal of resemblance to the functions mentioned, only, there, n is infinitely large. The branching cut is formed by the magnetic sheet which can replace the electric current.

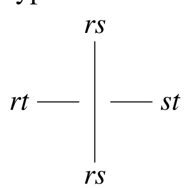
in the reverse direction. If one starts in the r^{th} sheet and if the point is of the 1^{st} type, one will gradually enter the s^{th} sheet, then the r^{th}, and finish in the s^{th}. In this way the characteristics are determined as shown in the following tables.

1^{st} type:



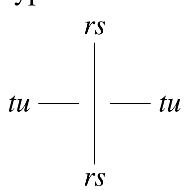
begins at	runs through	ends at	characteristics
r	sr	s	
s	rs	r	rs
t	tt	t	

2^{nd} type:



begins at	runs through	ends at	characteristics
r	ss	r	
s	rt	t	st
t	tr	s	
u	uu	u	

3^{rd} type:



begins at	runs through	ends at	characteristics
r	ss	r	
s	rr	s	tu
t	tu	t	
u	ut	u	

We place a small sphere κ around o; it intersects the branching line in a curve G which is, thus, the central projection of F into κ, and which separates κ into ω fields; it has h double points[24]. The surface system in P_n carried by κ must consist of n simple, pairwise separated spheres. On n spheres of the same radius as κ we draw lines congruent to G; in a set of n analogous fields on these spheres we write the numbers $1, 2, \cdots, n$, and in accordance herewith, we denote the spheres by

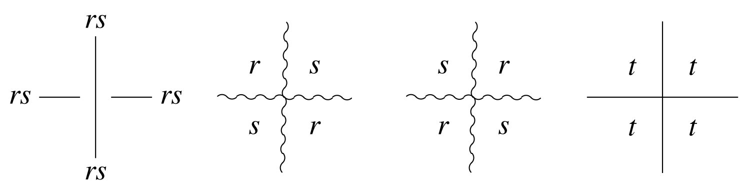
$$\kappa_1, \kappa_2, \cdots, \kappa_n,$$

and the lines by $G_1, G_2, \cdots, G_n.$. The characteristics on F are carried down to the corresponding arcs of G. We now imagine that the n balls in P_n lying above o are removed, and each is carried onto one of the balls $\kappa_1, \kappa_2, \cdots, \kappa_n$. The numbering can be carried out so that κ_i is precisely the ball in which the field labelled i occurs in the

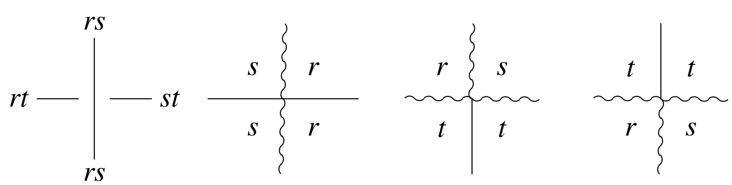
[24] Since the number of segments ab is $2h$ when the branching line contains only one curve, in this case Euler's theorem gives $\omega = h + 2$.

i'th sheet of P_n. By means of the characteristics it is now easy to label all fields of all the spheres with numbers that indicate to which sheet in P_n the field in question belongs. Those segments of $G_1, G_2, \cdots G_n$ which separate fields with different labels will be *drawn as wiggly lines*; the numbers occurring in the fields that they separate are then the characteristics. If the n balls are brought to coincide, the wiggly lines will trace the line G twice.

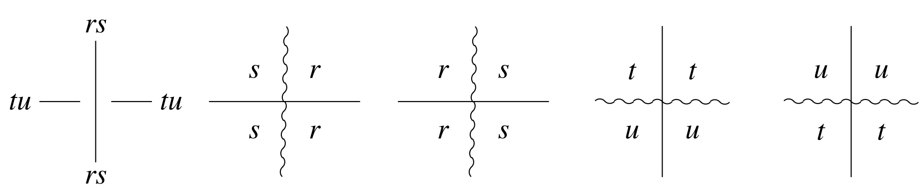
Let us for each sphere investigate the boundary of those 4 fields which meet at the point corresponding to one particular double point on G. From the data in the preceding tables of characteristics it is seen that if the point is of the 1^{st} *type*, then on two of the balls one will alternate between the r^{th} and the s^{th} sheet. On the other balls one will stay in the same sheets.



If the point is of the 2^{nd} *type*, then in $n-3$ balls one stays in the same sheet; the remaining ones display the following picture.



Finally, if the point is of the 3^{rd} *type* then in $n-4$ balls one will stay in the same sheet while the remaining ones display the following picture.



Considering the sequence of double points corresponding to one particular double point on κ there will always be *two and only two for which the line corresponding to the lower line on F is wiggly*[25]. Through each of the remaining $n-2$ double points, either

[25]Translator's remark: That is, for each of the three types in the above, there are precisely two horizontal wiggly lines.

there is no wiggly line or there is one such which corresponds to the upper part of the branch line.

After these preparations, we shall attempt to build the diagram of P_n. For a start, we puncture *all* n sheets, e.g. in their points at infinity. P_n is now bounded by n spheres with big radii; we imagine that these are modified so that all of them are carried by the same surface in P. This carrying surface can be modified to a surface constructed in the following way: in the fields of κ one places closed curves which follow the branches of G closely and at double points switch from one branch to the other one. Thereby, κ will be cut into ω simply connected surface patches and a strip enclosing G. Next, we enclose the branch cut in a surface that emanates from the closed curves and surrounds the branch cut like a hood as it bends around the boundary of the it. The surface mentioned is then constructed by closing the hood by means of the surface patches on κ. The course of the surface at a double line can for example be visualized by standing a quarto (book) on a table with its back up, and on the two sides of it standing identical octavos (books) also with their backs up and perpendicular to the quarto so that one appears as the continuation of the other. The table top then corresponds to a part of κ.

After the n puncturings, P_n can be reduced to manifold which is carried by the part of P bounded by the surface mentioned. From this manifold we have already excised n balls which we have transported to the positions $\kappa_1, \kappa_2, \cdots, \kappa_n$. We now have to bring the remaining part into a simple form, and attach it to these balls.

Around each double line we determine a prism-like body in P. In the above example, it is the body cut out from the quarto book when one imagines the octavo books continued through it. If the curve G has a total of h double points, the space, Λ, which is bounded by the hood surfaces and κ's system of strips will contain h such prisms. The remaining part of Λ consists of h plate shaped bodies which should, for one thing, be attached to the strip system, for another, be attached up and down along prisms, and finally be bounded by two prisms while the remaining part of the boundary runs freely along a part of F.

The manifold in P_n carried by one of the prisms consists of $n-2$ congruent, mutually separated prisms, and 1 doubly covered with a branch line through it. This latter one is also easily converted to a simply covered prism. These bodies should now be attached to the balls. In order to conveniently find the points to which they should be attached, on each side of G next to the original characteristics we shall add two new ones indicating the curves on those two balls on which the arc is wiggly; we shall call these the *derived characteristics.* To determine them on a particular branch, one walks from a point on the branch along some path into the field which was used in the numbering of the ball. One starts the walk carrying along the characteristics of the branch, and whenever one passes a branch where a characteristic coincides with one of the two numbers currently carried, that number is changed into the other characteristic on the branch. The derived characteristics are then the two numbers with which one arrives at the basic field. According to the statement emphasized on page 1048, at a double point, the branches corresponding to the lower branch of F must have the same derived characteristics. We now take the $(n-2)+1$ prisms corresponding to a particular double line; they should be attached to the balls $\kappa_1, \kappa_2, \cdots, \kappa_n$ at the n double points that correspond to the double line in question. The particular prism which used to be doubly covered should have its two ends attached to the two balls numbered by the derived characteristics corresponding to the lower branch of F. The remaining $n-2$ prisms are attached, each at one end, at the double points of the remaining spheres.

We now only have to discuss the manifolds in P_n carried by the h plates. Each plate carries on the one hand $n-2$ congruent, mutually separated plates, on the other hand 1 doubly covered with a branch, ab of F for its branch line. The latter can also be changed into a simply covered plate. The first group of plates shall be attached to the balls along strips that are not wiggly and also to the prisms; since each of them will get a *free boundary* they can be completely absorbed and they play no further role. The plates in the second group, altogether h of them, should along their boundaries be attached 1) along the wiggly branches, 2) up and down prisms with a free end point, 3) along the prisms that connect two balls; they get no free boundaries. The prisms mentioned in 2) can be absorbed into the plater.

To form the diagram (provisionally corresponding to n puncturings) one can then proceed in the following way: First one notes those h segments of G each of which is labelled by fixed characteristics. They run from a double point where the corresponding branch of F is lower to the double point where this happens next. On the n balls $\kappa_1, \kappa_2, \cdots, \kappa_n$ one draws the wiggly lines based upon the derived characteristics. (Compare with the schematic drawings on page 1048 and the statement emphasized on page 1048). These two characteristics indicate the numbers of the two balls on which those double points occur to which the corresponding thread should be attached; we call these double points the *junction points* belong to the double point. The schematic drawings show the course of the wiggly lines at such junction points, depending on the type of the double point. To each line segment ab there corresponds a plate, and it is easy to determine the line on the balls and the threads along which the boundary of the plate must be attached. On ab we choose a positive direction from a to b, and this determines a positive direction on each of the wiggly lines which combine to form ab counted twice. We now start from of one of the junction points corresponding to a

and run through the branch corresponding to ab; we follow that branch – always in the positive direction – until we meet a junction point corresponding to b, *but* whenever we meet an intermediate junction point on our way, we run along the thread over to the other ball and continue there. At the junction point corresponding to b, se run along the thread to the other junction point, and now in a similar way we run along the remaining section in the negative direction until we arrive at the other junction point corresponding to a. By running along the thread, the curve is finally closed up.

To get the diagram corresponding to a single puncturing one only has to pierce, and then remove, $n-1$ suitably chosen plates; the diagram which now consists of n calls connected with h threads and $h-n+1$ plates can be reduced in the usual way to one with 1 central ball, h-n+1 threads , and $h-n+1$ plates.

Until this point the investigation could be carried out completely analogous to the one for Riemann surfaces (beginning on page 1025). For the latter, the connectivity number could be determined on the basis of the material found, but something similar is not possible here. One of the reasons has already been mentioned on page 1038. In addition, we note that while the connectivity number for Riemann surfaces depends only on the number of leaves and the number of branching lines, the connectivity numbers for Riemann spaces depend not only on the number of sheets, the number of closed curves constituting the branch lines, and their windings and knottings, but *also* on the characteristics which can usually be assigned values within certain bounds independent of the former numbers. Therefore, we shall not pursue the matter further in its generality, but restrict ourselves to the consideration of some simple examples.

It will be convenient to imagine that branching line is infinitely close to being plana. This one can do because the connectivity of the manifold is unchanged by continuous displacements of the branching line as long as no parts of it are brought to intersect one another. One profits by first removing all windings and knottings which can be removed. It will also be convenient to imagine the n balls converted into circular thickened disks situated under one another below the plane in which the boundary line runs.For the curves $G_1, G_2, \cdots, G_n$ we choose the projections of F into the upper surface of the slice. At a double point of the 3^{rd} type it does not matter which branch is considered to be the upper one.

§ 14. Applications.

1. If $n=2$, and if F is formed as a simple closed line without knots, then the diagram becomes a point. P_2 is equivalent to a spherical space.

2. The branch line consists of two simple, closed lines without knots and unlinked; $n = 2$. For a start one gets two tube shaped bodies connecting the two central thickened discs; (if one runs from a point on a central thickened disc *inside* the attaching line for the tube, one arrives at the other thickened disc *outside* the attachment line for the tube). One of the tubes has to be pierced and is thereby reduced to a thread. The diagram is a thread to which the plate is attached along a meridian.

3. The branch line consists of ν simple, closed curves, without knots and unlinked. The space is assumed to be n-sheeted, and the characteristics of the branch lines are chosen so as to make the space simple. The diagram consists of n thickened discs connected by $\nu - n + 1$ tubes in a similar way to the preceding example and with $n - 1$ threads. Since the surface is equivalent to a sphere one can make the threads run so that the first one goes from κ_1 to κ_2, the next one from κ_3 to $\kappa_4, \cdots$, the last one from κ_{n-1} to κ_n. The threads and the thickened discs are then deformed into a central ball. It is not difficult to see that each tube can be transformed into a thread with a plate attached along a meridian. The given space is *equivalent to a spherical space equipped with* $\nu - n + 1$ *handles.*

4. The branch line is a closed curve forming a simple knot; $n = 3$. The projection will look like a 4^{th} order curve with 3 double points without bows; all 3 will be of the 2^{nd} type. The construction of the diagram is a simple application of the preceding theories, and we restrict ourselves to mentioning the result: One gets one thread with a plate attached along a latitude. Thus, the diagram boils down to a central ball, and the manifold considered is *equivalent to a spherical space.*

5. The same branch line as in 4, but $n = 2$. The diagram is *a thread with a plate attached along a curve* $[3\beta + \lambda]$.

6. The branch line consists of two simply linked, closed curves without knots; $n = 2$ The projection can be formed by two circles intersecting one another, and both double points are of the 2^{nd} kind. The diagram is *a thread with a plate attached along a curve* $[2\beta + \lambda]$.

We shall present some applications of the results stated to the theory of algebraic surfaces. The neighborhood of a general point of the surface is bounded by a space equivalent to a spherical space. We shall determine the kind of boundary that occurs when the point is singular. As the boundary of a neighborhood of a point in **T** one may take a spherical space but it will often be convenient to determine the neighborhood as the 4-dimensional manifold described by a ball in P when it is equipped with elevation numbers from a number α to a number β.

7. *The neighborhood if a general point on the branch line on an algebraic surface is equivalent to a spherical space.* The boundary is formed by the Riemann space mentioned in example 1.

8. Let the point be a *branch point of the first kind on the branch line.* The neighborhood is bounded by a 3-sheeted Riemann space; the branch line is a closed curve (page 1012) making two evolutions on the surface before it closes up, but it forms no knot; it can therefore be transformed into a simple closed curve. The above example 1 shows that *the boundary is equivalent to a spherical space.*

9. Let the point be a *branch point of the* 2^{nd} *kind.* The branch line forms a simple knot (page 1013). Indeed, the point arises because a principal tangent to the algebraic surface becomes parallel to the Z axis (page 1020). Since such a tangent intersects the surface in 3 coinciding points one gets a 3-sheeted Riemann space. The boundary is formed by a Riemann space as in example 4, and hence *equivalent to a spherical space.*

10. We shall assume that the algebraic surface has an *isolated double point* where the tangent cone is an irreducible cone surface of order 2. To investigate the neighborhood of such a point it suffices to investigate the neighborhood of the apex on an irreducible cone surface of the 2^{nd} order. Its contour in the XY planes formed by two straight lines intersecting at the apex. In **T** they are represented by two planes which intersect a spherical space Σ centered at the origin in two closed curves. If we convert the spherical space into P in the way indicated earlier (page 1045), it is easily seen that these curves become two simple, simply linked curves (example 6). Indeed, the carrier curves in Σ are two ellipses of which he one with the larger value for the modulus of the direction coefficient encircles the other one because the projections onto the X_1Y_1 plane are concentric circles (page 1017). The two halves of the ellipses which carry negative elevation numbers are inverted, but thereby the branch that use to be the inner one becomes the out one, q. e. d.

If one wants this manifold analytically represented one only has to split the equation

$$z^2 = x^2 - y^2$$

into its real and imaginary parts

$$\begin{aligned} z_1^2 - z_2^2 &= x_1^2 - x_2^2 - y_1^2 + y_2^2 \\ z_1 z_2 &= x_1 x_2 - y_1 y_2 \end{aligned}$$

and add the equation

$$x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1.$$

This shows that the manifold belongs to the kind for which the Poincaré-Picardian theory applies, a fact that also follows from our earlier investigations.

The neighbourhood of any one of the points mentioned in examples 7, 8, and 9 is an elementary 4-dimensional manifold. This can for example be seen by constructing a 3-dimensional pencil of curves emanating from the point to be investigated such that through each point of the manifold there is one and only one line. If a curve meets the branch line it must be completely contained therein, and similarly if it meets a doubling of the branching surface. Then, when the boundary of the neighborhood is transformed into a spherical space, Σ, in $\mathbf{T}$, the neighborhood can be transformed to the part of $\mathbf{T}$ lying inside Σ, by letting the curves of the pencil mentioned correspond to the radii out to the points of Σ determined by the end points of curves. By the way, the result also follows from the fact that one can move the surface relative to the coordinate system and thereby obtain a situation where the point no longer lies on the branch surface.

This is not possible for an isolated multiple point, and in example 10 we have already seen that the neighborhood of an isolated double point with irreducible tangent cone was not an elementary manifold. The boundary was equivalent to P doubly covered, since the branch line was formed by two simply linked circles. It is the same manifold that we have met repeatedly (pages 1037, 1038, 1045). It could also be made by gluing two tori, letting latitudes and meridians on one be joined to the curves $[\beta]$ and $[2\beta+\lambda]$ on the other. The diagram is formed by a thread with the plate attached along $[2\beta+\lambda]$.

As promised, we shall now prove that *this manifold has connectivity numbers* $p_1 = 2$, $p_2 = 1$.

To find the closed curves that cannot be contracted to a point, we notice that any closed curve in the manifold can be brought to run completely in the thread of the diagram. One may assume that it always runs in the same direction and makes no knots. A curve that runs twice through the thread can be deformed to run in the attachment strip and therefore be deformed to a point through the plate. If the curve runs around $2p$ times, one may split off 2 revolutions at a time, p times in succession. Thus, the curve can be contracted to a point. It there are $2p+1$ revolutions, one ends with a curve with one revolution. *This cannot be contracted to a point.* Indeed, let us assume that the curve would bound a bilateral surface patch in the manifold. We drill a tube shaped canal along the curve so that the manifold gets a boundary which is a torus surface. When we consider the manifold defined by gluing two torus surfaces in the way described above (page 1054), this canal will run will run once around in one torus, and the surface patch will be bounded by a latitude in the surface of the canal. After the canal has been drilled, one can let the manifold deform to a torus (lying in P). In this torus one finds the surface patch mentioned, bounded by a latitude on the torus surface. This latitude runs once around the axis of the torus , but simultaneously it bounds a surface patch in the torus, that is a surface patch which does not intersect the axis. But this is impossible. *Hence* $p_1 = 2$.

All closed surfaces in the manifold (both bilateral and unilateral ones) can be brought completely into the diagram and those parts of such a surface which occur in the plate can be altered into one or more simple elementary surface patches with their boundary/ries in the surface boundary, and hence connected to the remaining part of the surface through the attachment strip. If the surface has only one such elementary surface patch in the plate, its boundary will be a curve $[2\beta + \lambda]$ on the surface of the thread; thus, this curve will bound a surface patch situated in the plate; but no *bilateral* surface in the thread can be bounded by this curve, since it goes twice around the torus axis in the same direction. However, it can be closed by means of M´obius's *unilateral* surface patch. If, next, the surface has several elementary surface patches in the plate, one may puncture two successive such patches and connect the boundaries with a tube; if the original surface does not bound, nor will the new one. If there is an *even* number of surface patches in the plate then all the patches[26] could undergo this procedure pairwise after which they can be pushed off the plate, so that the whole surface ends up completely in the thread. If there is an *odd* number then in the same way one can remove them all except for one; and one is then back at the case first mentioned where we saw that the surface must be unilateral. Altogether, if there is to be a non bounding, closed, bilateral surface in our manifold, then it must appear in the thread. But any closed, bilateral surface in the thread can be deformed to a point or to a point from which there emanates curves that return to the point, that is, all such surfaces are homologous to 0. *Hence* $p_2 = 1$.

From here we see that *the manifold representing an algebraic surface with isolated multiple points contains points for which the neighborhood is not an elementary manifold.* Such points we shall call *topological singularity points.*

But the manifolds that Picard and Poincaré define contain only points for which the neighborhood is simply connected (P. & S., 2^{nd} chapter, No. 2 and No. 3. Poincaré A. S. § 2 and § 3), that is, no topological singularity points. How, then, is the position as regards the theorem "Any algebraic surface with arbitrary singularities can be brought in birational correspondence with a surface that has no other singularities than a double curve with triple points, since these singularities are the most general of their kind"? (P. & S., 4^{th} chapter, No. 8). In the former case there can be topological singularities,

[26]Translator's remark: here the original writes "plates" rather than "patches".

but, as is easily seen, in the latter case none. The explanation for this is easily discovered when one follows the transformations used to bring the surface in the form mentioned. An isolated multiple point becomes an exceptional point for the transformation, so that it corresponds to an algebraic curve that does not pass through multiple points. The boundary of the neighborhood of the double point is then equivalent to the hull of the corresponding algebraic curve. Thus, in the Picard theory, curves are freed from their topological singularities.

Thus, with regard to topology there is an essential difference between Riemann surfaces and the manifolds representing algebraic surfaces. Indeed, in the former the neighborhood of *any* point is an elementary manifold (also the helicoid around a branch point), in the latter this is not always the case. For the coordinates in the neighborhood of each point of an algebraic curve one can easily form expansions as holomorphic series in a parameter. On the other hand, no one has succeeded in representing the coordinates in the neighborhood of every point of an algebraic surface by such series in two parameters. And this apparently is connected to the existence of such singularity points. Corresponding to the theorem that a finite number of such series suffices to represent the full neighborhood of any point on the surface, one has the fact that the neighborhood of a topological singularity point can be obtained by joining a finite number of elementary manifolds; this follows from the fact that the boundary can be obtained by joining a finite number of elementary manifolds. (Cutting along the "cone" with the point as its apex and with the diagram of the boundary as its directrix, will make the neighbourhood simply connected).

Denoting the original surface by f, and denoting by F the surface obtained from f by freeing it of its topological singularity points, one may set oneself the task of constructing the manifold representing f when one knows the one representing F. One looks up the closed surfaces that correspond to each isolated multiple point and to the other fundamental points of f. Neighborhoods of these closed surfaces are excised, and the manifold is again closed up with manifolds equivalent to the surroundings of the corresponding points of f. The exceptional points of F are treated in the opposite way.

In the example considered before we succeeded in finding the connectivity numbers from the diagram. In general this task is apparently difficult. However, the diagram may serve to find *upper limits for the connectivity numbers*, but the difficulty lies in deciding whether the manifolds which one has found from the diagram and which one conjectures to be homologically independent really are so. *Here, integration theory may serve as a tool* (cf P. & S., 4^{th} chapter No. 9), *but not an exhaustive one.* Indeed, the general theorem stated by Poincaré and Picard that the number of linearly independent periods of (more specifically defined) integrals taken along closed m-dimensional manifolds is $p_m - 1$ (Poinc. A. S., § 7, closing; P. & S., 2^{nd} chapter, No. 16), is not reliable. Indeed, in the often mentioned example there exists a closed curve which is not a boundary. *But all the integrals mentioned vanish along this curve*. In fact if one runs through the curve twice, one gets a curve that is contractible to a point, so the integral must be 0. One sees from here *that one cannot always deduce the homology* $V \sim 0$ *from the homology* $2V \sim 0$. (Poinc. A. S., p. 19, l. 4).

CLOSING REMARKS

One might think it hopeless to attempt a topological theory of the connectivity of algebraic surfaces, even more so since one has not even succeeded in forming a general theory for the case of 3-dimensional manifolds. It must, however, be noted that a 4-dimensional manifold representing an algebraic surface is *special* and one can expect it to be relatively simple form a topological viewpoint. Picard's theorem that any closed curve bounds in an algebraic surface without multiple points, points in that direction. (P. & S., 4^{th} chapter, III).

I have not yet succeeded in reaching any complete result. I shall close with some scattered remarks. What has decided me to publish these preliminary studies has *partly* been the feeling that I had to get to the bottom of the other wiewpoints, of a transcendental, an algebraic, and an enumerative geometric nature, from which one has studied the algebraic surfaces, *partly* the appearance of Picard and Simart's book, both because of the parts that agree with my own investigations and because of those that are in conflict with them.

Concerning the connectivity numbers of *the plane*, as for the infinitely distant elements we have taken the projective geometrical viewpoint, so *we cannot come to the same result as Picard* (P. & S., 4^{th} chapter, No. 9). According to his definitions the line at infinity consists of 1) a collection of points $x = \infty,\ y =$ arbitrary finite, 2) a collection of points $y = \infty,\ x =$ arbitrary finite, 3) the point $x = \infty,\ y = \infty$. *We* consider the first two collections each as one point (The point at infinity of the X-axis and the Y-axis). And by taking into account the different values of $\lim(y : x)$ for $x = \infty,\ y = \infty$ we form a whole collection from the point 3). Based on the results from §4, we easily determine the diagram for the projectively defined plane. We puncture the manifold $\mathbf{A}$. It can be completely removed whereby $\Sigma_1(\mathbf{B})$ and $\Sigma_2(\mathbf{C})$ become free boundaries for $\mathbf{B}$ and $\mathbf{C}$. These can be deformed to the neighborhood of the closed surface which represents the line at infinity. Thus, here is the diagram. One therefore gets

$$p_1 = 1,\ \ p_2 = 2,\ \ p_3 = 1.$$

In order that $\mathbf{B} + \mathbf{C}$ be bounded by a space equivalent to a spherical space, one has to join $\mathbf{B}$ and $\mathbf{C}$ is a special way, viz. the way described by the equations (5) in §4. – It is the spaces $\Sigma_1(\mathbf{B})$ and $\Sigma_2(\mathbf{C})$ that have to be joined together; they are bounded by $\tau(\mathbf{B})$ and $\tau(\mathbf{C})$; these are joined so that meridians on the former cover meridians on the latter, while latitudes on the former ($\eta =$ constant) correspond to curves of the type $[\beta + \lambda]$ on the latter. One also easily sees that the diagram for $\Sigma_1(\mathbf{B}) + \Sigma_2(\mathbf{C})$ (joined according to this rule) becomes a point. If one had joined after the formulae $\xi = \xi'$ and

$\eta = \eta'$, the boundary would have received the connectivity numbers $p_1 = 2,\ p_2 = 2$ (cf page 1038), and would not have been equivalent to a spherical space.

Two closed surfaces, for example of genus 0, lying in a 4-dimensional manifold are always equivalent, but, as we see, their *surroundings* are not necessarily equivalent.From here one sees the difference between what we call equivalence, and what Poincaré calls homéomorphisme (Poinc. A. S. §2).

For the surface of the 2^{nd} *order,* F_2*,* I have found that

$$p_1 = 1,\ p_2 = 3,\ p_3 = 1$$

by forming the diagram supported by the exposition in §5. The core of the diagram consists of two closed surfaces of genus 0, intersecting each other in a point. The same result is easily found in the following way: A variable generator in one generating system is determined by the cross ratio x to 3 fixed generators; similarly a generator in the other system is determined by a cross ration y. Every point on the surface is then determined by the x- and y-values of the generators passing through the point, *and vice versa*. (∞, ∞) is only one point; (∞, y) and (x, ∞) correspond to two generators intersecting each other in the point (∞, ∞). According to Picard's view point, the plane thus becomes what we call a surface of the 2^{nd} order after our projective understanding. If one punctures the 4-dimensional manifold (x, y) at the point $(0, 0)$, the puncture can be extended until the boundary meets the two closed, mutually intersecting surfaces corresponding to (∞, y) and (x, ∞). One then arrives at the same result as before, cf. also P. & S., 4^{th} chapter, No. 9, 2^{nd} chapter, No. 16.

The connectivity numbers found for the plane and for the surface of the 2^{nd} order agree with the number of fundamental points in *Chasles's* well known bijective transformation.

I have attempted to investigate *the surface of the* 3^{rd} *order*, F_3, in the same way as F_2, supported by the investigations in §5. However, here I met with difficulties from the double liens in the projection onto P of the branch surface, and from their triple points. I then sought to find the connectivity numbers by transformation. For one, I transformed it into a surface of the 2^{nd} order, F_2. To do so, pick two non intersecting lines on F_3,[27], say A and B. The two systems of planes through A, respectively B, determine by intersection with F_3 two systems of conic sections. We let these correspond bijectively to the two systems of generators of F_2. This transformation has

[27] Translator's remark: Here, the original has "F_2", apparently by misprint. The same applies to our next occurrence of "F_3". Our interpretation agrees with the translation into French, [2].

5 fundamental points in F_2,[28] namely those 5 points which correspond to the five pairs of planes through A and B which intersect one another in 1 of the 5 lines (among a total of 27 straight lines on F_3) which intersect A and B.
In order to transform F_3 into a plane, I used a transformation where the line connecting two points that correspond to one another intersects two fixed lines A and B as above. In the plane one gets, first, 5 fundamental points at the intersections with those 5 straight lines on F_3 which intersect A and B. In addition, one gets two fundamental points at the intersections, a and b, with A and B; these correspond to two conic sections φ and ψ, on F_3 determined by the planes (Ab) and (Ba). The intersection point of φ and ψ (ab's third intersection point with F_3) becomes a fundamental point on F_3 in that it corresponds to the line ab in the plane.
More easily, one may apply the usual birational transformation, which can be found for example in Salmon (Geometry of three dimensions; 4 ed., p. 556); there are 6 fundamental points in the plane, corresponding to a double six on F_3.

These transformations lead to the connectivity numbers

$$p_1 = 1,\ p_2 = 8,\ p_3 = 1.$$

It is not especially difficult to prove Picard's theorem, that $p_1 = p_3 = 1$,[29] in a topological way when the surface has no multiple points. To find p_2, I have attempted an approach like the one at the end of § 8, in that the surface is considered infinitely close to dissolving into n planes; it is particularly interesting to investigate $n = 2$ and $n = 3$. I hope some day to bring these investigations into a sufficiently complete shape so that they can be published as a continuation of the present studies. One should also clear up the connection between the topological connectivity numbers p_1, p_2, p_3 on one side and the various invariants $p_g, p_n, p^{(1)}, p^{(2)}$, etc on the other.

As is seen there ia a wide field for future investigations - investigations that are impeded by the intricate character of the subject as well as the broad basis for the methods of investigation.

[28] Translator's remark: The original talks of fundamental points *in the plane*. Our interpretation agrees with the one in [2].
[29] Translator's remark: Here the original has "$p_1 = p_2 = 1$", obviously a misprint.

THESES.

1. In spite of the revision by *Poincaré and Picard*, the *Riemann-Betti* theory of connectivity numbers still needs improvement (cf § 12). The theorem that

$$p_r = p_{n-r}$$

for closed manifolds, is incorrect. The 4-dimensional manifolds representing the complex points of algebraic surfaces do not belong to those defined by Poincaré and Picard when the surfaces contain isolated multiple points.

2. *Cauchy's proof* related to the 9^{th} definition in *Euclid*'s 11^{th} book (cf Journ. de l'école polyt. cah. 16 and *Ramus* Geometri, p. 228) is not satisfactory.
3. The theory of total differentials could and should be developed via the theory of functions of one real, independent variable.
4. Operations which form a group are numerals (cf *Thiele:* Om Definitionerne af Tallet, o.s.v.[30])
5. Many theorems from the theory of functions are more easily proved when one uses complex values for the independent variables. This is partially due to the fact that the validity of some fundamental theorems becomes completely general, but also to the fact that one restricts oneself to the consideration of monogenic functions. Hence, for applications, the corresponding investigations where one sticks to real variables, might be more valuable (Example: The proofs for Fourier's series).
6. The proof, known already from antiquity (*Democritus, Epicurus, Lucretius*), that the space in which we exist, is infinite (cf *Lange,* Gesch. d. Materialism. p. 105), is false since it operates with the very conceptions that we have to prove. We can not at all know anything about the properties of space in this respect, and mathematical investigations in this direction are purely formal.
7. The physical law about the preservation of inertia cannot be deduced solely on the basis of the law of causation, among other things because the physicists understand the notion of force to mean a reason for change of movement *caused by the presence of certain bodies in the surroundings* (cf *Kroman:* Vor naturerkendelse, p. 292).
8. The development of the mathematical science depends upon a sympathetic cooperation between those scientists whose abilities go especially in the formal direction, and those whose talents especially lead them to add to mathematics

[30]Translation by HJM: On the definitions of numbers, etc

new ideas and new subjects, but not in a systematic way (cf *Klein:* Gött. Nachr. Gesch. Mittheil. 1895, issue 2).

9. The hypotheses of natural science should *scientifically* not be construed as attempts at explaining the real nature of the surrounding world, but only as statements that are capable, in a few formulae, of depicting a large group of properties of the things under consideration, so that – in a purely formal way – we can form a general view of them and control their mutual relations.

References

[1] **A Brill**, **M Noether**, *On algebraic functions and their applicaiton to geometry. (Ueber die algebraischen Funktionen und ihre Anwendbarkeit in der Geometrie.)*, ?? (1873)

[2] **P Heegaard**, *Sur l'"Analysis situs"*, Bull. Soc. Math. France 44 (1916) 161–242

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