THE TRIANGULATION OF 3-MANIFOLDS

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Introduction

THIS paper gives another proof of the triangulability of 3-manifolds, which was first solved by Moise (9) in 1951 and then by Bing (2) in 1957. It follows up a suggestion by Kirby and Siebenmann that the methods they used to solve the problem of triangulation in high dimensions (5), (6) could be adapted to the three dimensional case.

Unless otherwise stated, manifolds are taken to be paracompact with or without boundary.

The triangulation problem is closely related to the handle straightening problem (Theorem 1). Those familiar with Kirby and Siebenmann's work will recall that by exploiting a certain ingenious torus unfurling idea they were able to reduce the handle straightening problem in dimensions $\geqslant 5$ to a certain problem in the PL category, namely, deciding the nature of PL homotopy equivalences $W \to B^k \times T^n$ which are homeomorphisms along the boundary (where $B^k = [-1, 1]^k$, $T^n = S^1 \times \ldots \times S^1$ (n times), and W is a $k+n$-dimensional manifold).

In fact they were able, with Wall and Hsiang and Shaneson, to show that for $k+n \geqslant 5$ and $k \neq 3$ the 2^n-fold cover of such a map is homotopic relative boundary to a PL homeomorphism, and hence that non 3-handles in dimensions $\geqslant 5$ were straightenable; more surprisingly Siebenmann found that a non-straightenable 3-handle exists in each dimension $\geqslant 5$.

For $k+n = 3$ it turns out that a similar reduction of the handle straightening problem is possible, and Waldhausen has solved the appropriate problem (Lemma 3) showing that a PL homotopy equivalence $W \to B^k \times T^n$ which is a homeomorphism along the boundary is itself homotopic relative boundary to a PL homeomorphism, but only provided that W is also *irreducible*, that is, every PL 2-sphere in W bounds a PL 3-ball. To achieve the requisite irreducibility the handles of Theorem 1 are supposed from the first to lie inside a chart, i.e. a copy of $\mathbf{R}^3$, the irreducibility of which was established by Alexander (1).

I would like to thank very much my supervisor Dr. G. P. Scott, who spread light where darkness threatened.

1. Results

THEOREM 1 (The 3-dimensional handle straightening theorem). *Given a topological embedding* $h: B^k \times \mathbf{R}^n \to \mathbf{R}^3$ $(k+n = 3)$ *which is PL (piecewise linear) in a neighbourhood of* $\partial(B^k \times \mathbf{R}^n)$, *there exists a topological isotopy* h_t *of* $h = h_0$ *such that*

(a) h_1 *is PL on* $B^k \times B^n$,
(b) $h_t \equiv h$ *on* $\partial(B^k \times \mathbf{R}^n) \cup B^k \times (\mathbf{R}^n - 2B^n)$.

$rB^n = [-r, r]^n$ is a *PL* n-ball neighbourhood of the origin in Euclidean n-space $\mathbf{R}^n$. ∂ denotes boundary. $B^k \times \mathbf{R}^n$ should be regarded as an open k-handle with core $B^k \times \{0\}$, lying inside a chart of the 3-manifold under consideration.

A close consequence of this theorem is:

THEOREM 2 (The triangulation of 3-manifolds).

1. (*Existence*). *Any topological* 3-*manifold* M *has a PL structure.*

2. (*Uniqueness*). *Given two PL structures* Σ, Σ' on M *there exists a topological ambient isotopy* (*appropriately* ε-*isotopy*) *of* M *from the identity to a PL homeomorphism* $M_\Sigma \to M_{\Sigma'}$.

Let d be a metric on M and $\varepsilon: M \to (0, \infty)$ a continuous map. Then an ambient isotopy of M is an ε-*isotopy* if $\mathrm{d}(x, h_t(x)) < \varepsilon(x)$ for all $x \in M$, $t \in [0, 1]$.

Remarks. Theorem 2.1 is equivalent to saying that M is triangulable: see for example Theorem 3.8 of (**12**).

A relative version of Theorem 2.2 also holds—see §3.

2. Proof of Theorem 1

We shall use three lemmas.

LEMMA 1 [Whitehead (**15**)]. *Let* M *be a PL manifold of dimension* $n \leqslant 3$, *with no compact unbounded component. Then there exist PL immersions of* M *in* $\mathbf{R}^n$.

LEMMA 2 [Wall (**14**)]. *Let* M *be a PL* 3-*manifold with compact boundary and one simply connected end, and let* A *be a compact subset of* M. *Then* M *contains a compact PL submanifold* K *which*

(a) *contains* A *in its interior,*
(b) *has boundary consisting of* ∂M *and a* 2-*sphere.*

Note that although the result is stated in (**14**) only for manifolds without boundary, it generalises easily to the formulation above, where the boundary is compact.

A map $\phi : M \to N$ of manifolds is *proper* if $\phi^{-1}(\partial N) = \partial M$. A *PL* 3-manifold N is said to be *sufficiently large* if it admits a proper *PL* embedding of a compact surface F not S^2 such that $\pi_1(F)$ maps injectively into $\pi_1(N)$. Let $T^n = S^1 \times \ldots \times S^1$ (n times) be the n-torus. Clearly the 3-manifolds $B^k \times T^n$ ($k = 0, 1, 2$) are sufficiently large.

Lemma 3 [Waldhausen (**13**) or Scott(**10**)]. *Let M, N be connected compact orientable irreducible PL 3-manifolds with N sufficiently large, and let $\phi : M \to N$ be a proper PL homotopy equivalence which is a homeomorphism on the boundary. Then ϕ is homotopic relative boundary to a PL homeomorphism.*

Note that although neither (**13**) nor (**10**) give the above result as a stated theorem, they do actually prove it, as part of the proof of Theorem 6.1 of (**13**), or Theorem 2.1 of (**10**).

Proof of Theorem 1. The proof is based on the construction of the diagram of mappings below [compare Kirby and Siebenmann (**6**)]. Its inspiration was Kirby's observation, used in (**4**) to prove the annulus theorem (the case $k = 0$) for $n \geq 5$, that any homeomorphism $g : T^n \to T^n$ of the n-torus such that $\pi_1(g) =$ identity is covered by a bounded homeomorphism $\tilde{g}$ of $\mathbf{R}^n$; $\tilde{g}$ will then extend by the identity to a homeomorphism of an n-ball D^n for which int $D^n = \mathbf{R}^n$.

Let $T^n - *$ denote T^n with one point removed. Σ_i denote *PL* structures. $\Sigma = h^{-1}$ (standard structure). $\Sigma = \Sigma_1 = \Sigma_2 = \Sigma_3 = \Sigma_4$ on $B^k \times B^n$. All horizontal maps are *PL*. Everything commutes.

$$
\begin{tikzcd}
 & (B^k \times \mathbf{R}^n)_{\Sigma_4} \arrow[r, "G"] & B^k \times \mathbf{R}^n \\
 & (B^k \times \mathbf{R}^n)_{\Sigma_3} \arrow[u, hook, "\gamma"] \arrow[r, "\tilde{g}"] \arrow[d, "e"'] & B^k \times \mathbf{R}^n \arrow[u, hook, "\gamma"'] \arrow[d, "e"] \\
 & (B^k \times T^n)_{\Sigma_2} \arrow[r, "g"] & B^k \times T^n \\
 & (B^k \times T^n - *)_{\Sigma_1} \arrow[u, hook] \arrow[d, "\alpha"'] & \\
(B^k \times 2B^n)_{\Sigma} \arrow[r, hook] \arrow[ur, hook] \arrow[uur, hook] \arrow[uuur, hook] \arrow[uuuur, hook] & (B^k \times \mathbf{R}^n)_{\Sigma} \arrow[r, "h"] & \mathbf{R}^3
\end{tikzcd}
$$

Case $k = 3$.

For $k = 3$ the diagram greatly simplifies:

$$\begin{array}{ccc} (B^3)_\Sigma & \xrightarrow{g} & B^3 \\ \| & & \\ (B^3)_\Sigma & \xrightarrow{h} & \mathbf{R}^3 \end{array}$$

$(B^3)_\Sigma$ is a PL ball since h embeds it in $\mathbf{R}^3$, which is irreducible. An appropriate extension of the identity map $id|_{\partial B^3}$ by coning will thus give rise to a PL homeomorphism g which is the identity on the boundary. Let g_t be the Alexander isotopy from the identity to g. Then

$$hg_0^{-1} = h$$

$$hg_1^{-1} \text{ is } PL \text{ on } B^3$$

$$hg_t^{-1} = h \text{ on } \partial B^3$$

so that $hg_t^{-1} : B^3 \to \mathbf{R}^3$ will be the desired handle-straightening isotopy.

Case $k = 0, 1, 2$.

Construction up to Σ_1 *level*: Lemma 1 implies that immersions of $T^n - *$ in $\mathbf{R}^n$ exist for each $n \leqslant 3$; let α be the product of the identity on B^k with some such immersion. Arrange α and the inclusion of $B^k \times 2B^n$ in $B^k \times T^n - *$ to commute with the inclusion of $B^k \times 2B^n$ in $B^k \times \mathbf{R}^n$. Let $\Sigma_1 = \alpha^{-1}(\Sigma)$.

Construction up to Σ_2 *level* goes via an intermediate level Σ_{11}. Essentially $(B^k \times T^n)_{\Sigma_{11}}$ is $(B^k \times T^n - *)_{\Sigma_1}$ capped off, and $(B^k \times T^n)_{\Sigma_2}$ is $(B^k \times T^n)_{\Sigma_{11}}$ with the non-irreducible part removed.

Σ_1 coincides with the standard structure near the boundary. Extend Σ_1 along the entire boundary of $B^k \times T^n$ by letting Σ_1 = standard structure on a sufficiently small open collar $N(\partial B^k \times T^n)$ of $B^k \times T^n$.

$(B^k \times T^n - * \cup N(\partial B^k \times T^n))_{\Sigma_1}$ has one simply connected end, and so according to Lemma 2 contains a compact PL submanifold K which

(a) contains $B^k \times 2B^n$ in its interior,

(b) has boundary consisting of $(\partial B^k \times T^n)_{\Sigma_1}$ and a 2-sphere κ.

κ bounds a topological 3-ball in $B^k \times T^n$ by the Schoenflies theorem (3) applied via the universal cover. Extend the identity map $id|_K$ by coning to a homeomorphism $B^k \times T^n \to K \cup \mathscr{C}\kappa$ ($\mathscr{C}$ denotes the cone). This homeomorphism induces a PL structure Σ_{11} on $B^k \times T^n$. Note that $\Sigma_{11} = \Sigma_1$ on $K \supset B^k \times 2B^n$.

If $(B^k \times T^n)_{\Sigma_{11}}$ were irreducible then according to Lemma 3 it would be PL homeomorphic to $B^k \times T^n$. In general however $(B^k \times T^n)_{\Sigma_{11}}$ is a connected sum of manifolds (7) all but one of which, since every 2-sphere in $(B^k \times T^n)_{\Sigma_{11}}$ bounds a topological 3-ball, will be topological spheres. (Note that it is a consequence of this paper that every topological sphere is a PL one, so that $(B^k \times T^n)_{\Sigma_{11}}$ is in fact irreducible and this paragraph is redundant; however we do not know this yet.) Thus $(B^k \times T^n)_{\Sigma_{11}} = A \cup Q$ where Q is topologically a ball, $A \cap Q = \partial Q$ is a 2-sphere σ, and $A \cup \mathscr{C}\sigma$ is indecomposable. Extend the identity map $id|_A$ to a homeomorphism $B^k \times T^n \to A \cup \mathscr{C}\sigma$. This homeomorphism induces a PL structure Σ_2 on $B^k \times T^n$. $(B^k \times T^n)_{\Sigma_2}$ is irreducible being neither $S^2 \times S^1$ nor a connected sum (8). Note that $\Sigma_2 = \Sigma_{11}$ on A. A may be assumed to contain $B^k \times B^n$ as follows.

$k = 0$: $(T^3 - B^3)_{\Sigma_{11}}$ has one simply connected end, so Lemma 2 provides a PL 2-sphere τ in $(2B^3 - B^3)_{\Sigma_{11}} = (2B^3 - B^3)_\Sigma$ which bounds in $(T^3)_{\Sigma_{11}}$ a compact set D containing $(B^3)_\Sigma$. D is a PL ball since h embeds it in $\mathbf{R}^3$, which is irreducible. Choose the connected sum representation of $(T^3)_{\Sigma_{11}}$ in such a way that D lies inside A.

$k = 1, 2$: the generalised Dehn's lemma (**11**) provides in $(B^k \times (2B^n - B^n))_\Sigma$ k surfaces, τ, of type $(0, n)$ with boundary $(\partial B^k \times \partial \tfrac{3}{2} B^n)_\Sigma$. $\tau \cup (\partial B^k \times \tfrac{3}{2} B^n)_\Sigma$ forms a PL 2-sphere in $(B^k \times 2B^n)_\Sigma$ bounding a compact set D containing $(B^k \times B^n)_\Sigma$. D is a PL ball since h embeds it in $\mathbf{R}^3$. Choose a connected sum decomposition of $(B^k \times T^n)_{\Sigma_{11}} - D$. If D is reattached along τ to the relevant indecomposable, and hence irreducible, summand, it is not difficult to see that the resulting manifold will also be irreducible. Thus the decomposition of $(B^k \times T^n)_{\Sigma_{11}} - D$ gives rise in a natural way to a decomposition of $(B^k \times T^n)_{\Sigma_{11}}$ in which A contains D.

We have obtained an irreducible manifold $(B^k \times T^n)_{\Sigma_2}$ in which $\Sigma_2 = \Sigma$ on $B^k \times B^n$. The identity map $(B^k \times T^n)_{\Sigma_2} \to B^k \times T^n$ is homotopic relative boundary to a simplicial approximation ϕ, which is a PL homotopy equivalence and a homeomorphism along the boundary. By Lemma 3 ϕ is homotopic relative boundary to a PL homeomorphism g.

Construction up to Σ_3 level: Let e be the universal covering map of $B^k \times T^n$. Let $\Sigma_3 = e^{-1}(\Sigma_2)$. Arrange the inclusion of $B^k \times 2B^n$ in $(B^k \times \mathbf{R}^n)_{\Sigma_3}$ so that everything still commutes. g lifts to a homeomorphism $\tilde{g}$ of $B^k \times \mathbf{R}^n$ which is the identity on the boundary. $\tilde{g}$ is bounded on the $\mathbf{R}^n$ factor since $\pi_1(g)$ is the identity.

Construction up to Σ_4 level: Let $\gamma : B^k \times \mathbf{R}^n \hookrightarrow B^k \times \mathbf{R}^n$ be a PL inclusion which is the identity on $B^k \times B^n$ and which maps $B^k \times \mathbf{R}^n$ onto

$B^k \times 2B^n - \{0\} \times \partial 2B^n$. Let $G = \gamma\tilde{g}\gamma^{-1}$ on $B^k \times 2B^n - \{0\} \times \partial 2B^n$. Since $\tilde{g}$ is bounded on the $\mathbf{R}^n$ factor G will extend by the identity on $\{0\} \times \partial 2B^n$ to a homeomorphism of $B^k \times 2B^n$ which is the identity on the boundary. Extend G by the identity everywhere else to a homeomorphism of $B^k \times \mathbf{R}^n$. Let $\Sigma_4 = G^{-1}$ (standard structure), so that $\Sigma_3 = \Sigma_4$ on $B^k \times B^n$.

Now let $G_t = \begin{cases} \text{Alexander isotopy from the identity to } G \text{ on } B^k \times 2B^n \\ \text{identity on } B^k \times \mathbf{R}^n - B^k \times 2B^n \qquad (t \in [01]). \end{cases}$

Then

$$hG_0^{-1} = h$$
$$hG_1^{-1} \text{ is } PL \text{ on } B^k \times B^n$$
$$hG_t^{-1} = h \text{ on } \partial B^k \times \mathbf{R}^n \cup B^k \times \mathbf{R}^n - 2B^n$$

so that $hG_t^{-1} : B^k \times \mathbf{R}^n \to \mathbf{R}^3$ will be the desired handle-straightening isotopy.

This completes the proof of Theorem 1.

3. Proof of Theorem 2

Proof of Theorem 2.2. First ambient ($\varepsilon-$) isotop M to a homeomorphism which is PL near the boundary, using the classification of surfaces and the uniqueness of collars (**16**; V–20).

The relative version of Theorem 2.2 which follows will now complete the proof.

THEOREM 2.2 (Relative version). *Suppose the topological homeomorphism $h: M_\Sigma \to M_{\Sigma'}$ is PL on a neighbourhood $N(K)$ of some closed subset $K \supseteq \partial M$ of M. Then there is an ambient ($\varepsilon-$) isotopy of M relative K from h to a PL homeomorphism.*

Remark. The condition $K \supseteq \partial M$ may be removed once the initial statement of this section is itself made relative.

Proof. Subdivide some $\Sigma-$triangulation of $M - K$ so finely that every simplex is contained in some $\Sigma'-$chart of M. Such a triangulation gives rise to a handle decomposition of $M - K$ in which each handle, being taken sufficiently narrow, may be supposed to lie inside a $\Sigma'-$chart of M. Apply the handle straightening theorem simultaneously to all those 0-handles which have non-empty intersection with $M - N(K)$ (the others are already straight). The resulting ambient isotopy of M remains the identity on smaller neighbourhood $N_0(K)$ of K. Now simultaneously straighten all those 1-handles which have non-empty intersection with $M - N_0(K)$, to obtain an ambient isotopy of M which remains the identity on another neighbourhood $N_1(K) \subseteq N_0(K)$ of K. Straightening thus successively the handles of increasing index, one eventually attains an ambient isotopy of the desired character.

It becomes an ε-isotopy if the triangulation of $M-K$ is taken sufficiently fine.

Proof of Theorem 2.1. Any topological manifold M certainly has local PL structures: the idea is to isotop these together.

From the classification of surfaces and the existence of collars (**16**; V–13) a PL structure may be assumed to have been defined on a collar of M.

Let $\mathscr{U} = \{U_i;\ \phi_i\}_{i=1,2\ldots}$ be a locally finite (and therefore countable) cover by charts of M. The subset $\mathscr{U}_0$ of $\mathscr{U}$ consisting of the boundary charts may be assumed to be PL compatible, as above. Renumber the elements of $\mathscr{U}_0$, as $U_0, U_{-1}, U_{-2}, \ldots$ and those of $\mathscr{U}-\mathscr{U}_0$ as $U_1, U_2, \ldots$, and suppose inductively that a PL structure has been defined on $\bigcup_{i=-\infty}^{r} U_i$. ϕ_{r+1} induces a PL structure on U_{r+1}. Let U be the manifold $U_{r+1} \cap \bigcup_{-\infty}^{r} U_i$ with the PL structure inherited from $\bigcup_{-\infty}^{r} U_i$, U' the same manifold with the PL structure inherited from U_{r+1}. U_{r+1} intersects a finite subset $\{U_i\}_{i\epsilon I}$ of $\mathscr{U}$. Refine $\mathscr{U}$ to an open covering $\mathscr{V} = \{V_i\}_{i\epsilon z}$ in which

$$\bar{V}_i \subset U_i \quad \text{if} \quad i \in I \qquad (\text{N.B. } r+1 \in I)$$

$$V_i = U_i \quad \text{if} \quad i \notin I.$$

Triangulate U and let K be the (finite) union of closed 3-simplexes of U which have non-empty intersection with $\bar{V}_{r+1} \cap \bigcup_{I \cap \{-\infty \ldots r\}} \bar{V}_i$, which is compact. The triangulation gives rise to a handle decomposition of U: apply the handle straightening theorem to the handles corresponding to K, the 0-handles first, then the 1-handles, and so on, to obtain a homeomorphism $h: U \to U'$ which is PL on K and the identity outside some compact subset $N(K)$ say. Then $\bigcup_{-\infty}^{r+1} V_i$ has a well-defined PL structure inherited from $\bigcup_{-\infty}^{r} U_i$ on $\bigcup_{-\infty}^{r} V_i$, from U_{r+1} on $V_{r+1} - N(K)$, and induced by $\phi_{r+1}h$ on $V_{r+1} \cap U$. Taking the induction to its limit, which the local finiteness condition makes feasible, finally gives rise to a PL structure on all M, as desired.

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