XI.—On Knots, with a Census of the Amphicheirals with Twelve Crossings. By Mary Gertrude Haseman. *Communicated by* Dr C. G. Knott, *General Secretary.* (With One Plate.)

(MS. received March 28, 1917. Read June 4, 1917. Issued separately February 1, 1918.)

The theory of the knotting of curves, except for a few elementary theorems due to Listing,* was entirely neglected until Tait † was led to a consideration of knots by Sir W. Thomson's (Lord Kelvin's) work on the Theory of Vortex Atoms. He attacked chiefly the problem ‡ of constructing knots with any number of crossings, and obtained a census of the knots of not more than ten crossings. Those knots which exhibit a special kind of symmetry—the amphicheiral knots—offer certain points of interest.

§ 1. Knot Schemes.

Tait has introduced two schemes for representing knots: the alphabetical and compartment symbols.

Alphabetical Symbol.—The alphabetical scheme of a knot is based upon the idea of the sequence of the crossings which exist on the plane projection of the knot. In the case of the alternating knot, the thread passes alternately over and under at the crossings. It is convenient to distinguish between the over and under crossings by means of the signs + and − respectively. Starting with an over crossing $\underset{+}{a}$, the alternate crossings may be denoted by $\underset{+}{b}$, $\underset{+}{c}$, $\underset{+}{d}$, etc. In this way there is obtained a definite sequence of the letters a, b, c, arranged so that those occupying the odd places represent over crossings, while those in the even places denote under crossings. Thus Tait's problem of constructing the plane knots with n crossings reduced itself to a question of the essentially different ways in which the even places of the sequence

$$\underset{+}{a}\ \underset{+}{b}\ \underset{+}{c}\ .\ .\ .\ .\ \underset{+}{n}$$

may be filled in with the same letters so as to form unipartite closed curves. For example, the only arrangement in the case of three crossings is

$$\underset{+}{a}\ \underset{-}{c}\ \underset{+}{b}\ \underset{-}{a}\ \underset{+}{c}\ \underset{-}{b}\ |\ \underset{+}{a}$$

Hence the "trefoil" knot is the only knot of order 3.

Compartment Symbol.—Tait obtained his idea of the compartment symbol from the Listing type-symbol, which depends upon the division of the plane into $n+2$

* Listing, *Vorstudien zur Topologie* (1874).

† Tait, *Trans. Roy. Soc. Edin.*, xxviii (1876–77), pp. 145–191; xxxii (1882–86), pp. 327–342, 493–506. See also *Scientific Papers*, vol. i, pp. 273–347.

‡ The same problem has been considered by Kirkman, *Trans. Roy. Soc. Edin.*, xxxii (1882–86), pp. 281–309; and by Little, *Proc. Conn. Academy*, vii (1885–88), pp. 27–43.

compartments by the projection of the knot of order n. Both LISTING and TAIT showed that, of these compartments, no one contained less than 2 or more than n angles. Following LISTING'S notation, the angle on the left along with its vertical, as a crossing is passed by the upper thread, is denoted by δ and the remaining pair by λ. The various compartments of an alternating knot are monotype; that is to say, the angles are of the same character, as shown in fig. 1. The Listing type-symbol is merely an enumeration of the two sets of compartments, in which an exponent is used to indicate the number of angles in a compartment and a coefficient to represent the number of such compartments. Thus the Listing type-symbol for the knot given in fig. 1 is

$$2\delta^2 + 2\delta^3 + \delta^4 + 2\delta^5$$
$$2\lambda^2 + 2\lambda^3 + 2\lambda^4 + \lambda^6$$

In general, each part of the Listing type-symbol for a knot of order n amounts to

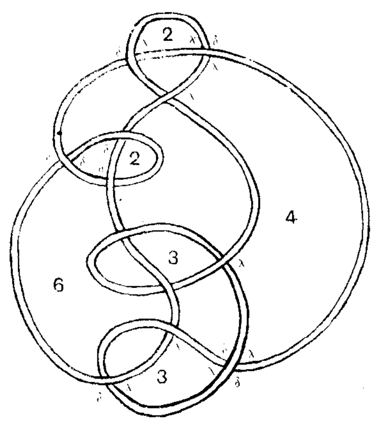


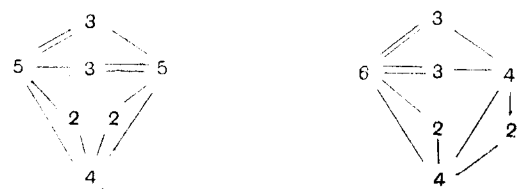
FIG. 1.

nothing more than a set of partitions of the number $2n$, where each member of the partition indicates the presence of a compartment with the same number of angles as there are units in this member. For example, the partitions corresponding to the type-symbol above are:

2 2 3 3 4 5 5
2 2 3 3 4 4 6

But it is not sufficient for the determination of the knot to know simply the number of compartments and the number of angles in each. It is necessary to know the number of joins between the various compartments as well as the arrangement of these joins. The number and arrangement of these joins is expressed by the compartment symbol, in which the joining lines indicate the number as well as the arrangement of the crossings connecting the set of δ compartments and the number as well as the arrangement of the laps of the thread bounding the set of λ compartments, or *vice versa*. We may assume that TAIT recognised the importance of the order of the crossings, for his symbols conform to the above definition of the compartment symbol. LITTLE mentions the fact that the knot is not uniquely determined unless its so-called partition symbol, which is the same as the compartment symbol, as defined above, indicates the sequence of the crossings on the knot.

The compartment symbol of the knot (fig. 1) is



where the first refers to the set of δ compartments and the second to the set of λ compartments. In the following pages these symbols are referred to as primary and secondary. As TAIT showed, the knot may be constructed from either the primary or secondary symbol by connecting the mid-points of the joins with a line which intersects itself only at these points.

§ 2. METHODS OF VARYING A GIVEN KNOT.

Since the positions of the apparent double points of a twisted wire in space vary as the point of view changes, the plane projection is susceptible of two types of variation, which TAIT calls deformation and distortion respectively.

By deformation of the knot TAIT means a projection which leaves unaltered the relative positions of the compartments, as well as the number of angles in each compartment. For instance, any compartment may be made the amplexus, or infinite compartment, by turning the knot into this compartment; or, which is precisely equivalent, by inversion of the knot with respect to a point of this compartment as origin. But the knot scheme is unaltered by deformation, and the two knots are said to be equivalent.

Distortion, on the other hand, is a projection which changes the position of one or more of the crossings, so that, in general, it is impossible to represent the distorted form by the same scheme. In certain cases the number of angles and the arrangement of the joins of the various compartments is unaltered by a distortion. Such a distortion, therefore, reduces to a deformation, and the distorted form is equivalent to the original. As regards the knot in the plane, distortion is the process of shifting a crossing from one lap of the thread to another by a twist through two right angles of a limited portion of the knot. For example, by a rotation through two right angles about an axis in the plane of the paper, downwards through the crossing α, fig. 2 is distorted into fig. 2′.

Effect of Distortion on the Alphabetical Symbol.—If from a limited portion of a knot there emerge two free ends, that is to say, if a single part of the complete thread exhibits a certain number of crossings, then the knot consists of two or more separate knots on the one thread, and is said to be composite.

But a distortion * of a non-composite knot is possible if there emerge from a limited portion of it, four free ends, of which an adjacent pair is crossed. The four free ends indicate that the portion considered is made by a number of crossings of two distinct parts of the complete thread, and hence may be called a "reversible two-thread † tangle"—or, more simply, a "reversible tangle." Denote by x, y, p, q the four free ends, and let the adjacent pair x, y be crossed at a point α. By a rotation through two right angles as above, the two threads x, y are untwisted, and the threads p, q are crossed at a point α'. Hence, as LITTLE points out, there will be no change in the number of compartments involved in the primary and secondary symbols, although the order in which they are joined may be disturbed.

A distortion is of order n if it involves $n+2$ crossings. Denote by D_n^{α} a distortion of order n which operates on the crossing α. The reversible tangle R_n is the portion of the knot which admits the possibility of a distortion D_n^{α}.

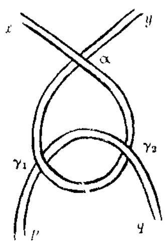


FIG. 2.

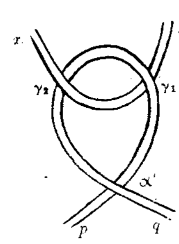


FIG. 2'.

Let the knot be described in such a direction that the reversible tangle R_n is entered by the thread x over the crossing α, and denote by $\gamma_1, \gamma_2, \ldots . \gamma_{n+1}$ the remaining $n+1$ crossings in the order in which they are met. To leave the reversible tangle it is necessary either to return to the crossing α and pass out along the thread y, or to leave by one of the two remaining threads, say p. Accordingly, the alphabetical schemes for the reversible tangle R_n are respectively

$$\underset{-}{x}\ \underset{+}{\alpha}\ \underset{-}{\gamma_1}\ \underset{+}{\gamma_2} \ldots . \gamma_{n+1}\ \alpha\ y \ldots . q\ \gamma_j \ldots . \gamma_i\ p$$

or

$$\underset{-}{x}\ \underset{+}{\alpha}\ \underset{-}{\gamma_1}\ \underset{+}{\gamma_2} \ldots . \gamma_n\quad \alpha\ y \ldots . p\ \gamma_i \ldots . \gamma_j\ q$$

and

$$\underset{-}{x}\ \underset{+}{\alpha}\ \underset{-}{\gamma_1}\ \underset{+}{\gamma_2} \ldots . \gamma_{n+1}\ p \ldots . y\ \alpha\ \gamma_i \ldots . \gamma_j\ q$$

$$\underset{-}{x}\ \underset{+}{\alpha}\ \underset{-}{\gamma_1}\ \underset{+}{\gamma_2} \ldots . \gamma_{n+1}\ p \ldots . q\ \gamma_j \ldots .. \gamma_i\ \alpha\ y$$

where $\gamma_i, \ldots . \gamma_j (i \mp j)$ denotes the γ's in some order. If a crossing $\gamma_\mu (\mu = 1, 2, \ldots . n+1)$ occurs twice on one thread of the reversible tangle R_n, then only the remaining n crossings exist on the second thread. The effect of a distortion on the alphabetical

* TAIT, *Trans. Roy Soc. Edin.*, xxxii (1882–83), p. 328, or *Scientific Papers*, i, p. 320, recognises the possibility of such distortions; to LITTLE, *Proc. Conn. Acad.*, vii (1885), p. 44, § 10, is due the formulation of necessary conditions in the appearance of the knot.

† In the case of the alternating knot the possibility of a distortion is limited to the two-thread tangle; for non-alternating knots there may exist distortions of reversible tangles of more than two threads.

schemes is made clear by a consideration of the representation in figs. 3 and 3′ of a reversible tangle R_n, where the crossings $\gamma_1, \gamma_2, \ldots \gamma_{n+1}$ are supposed to lie within

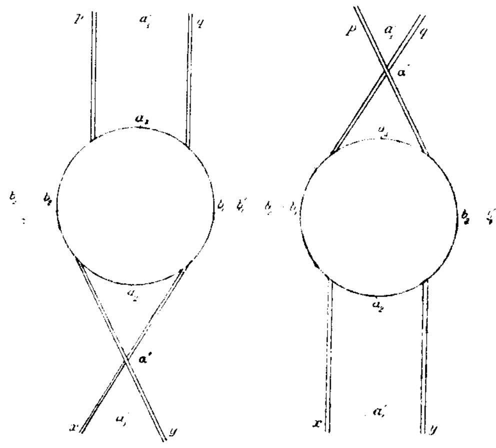


FIG. 3. FIG. 3′.

the circle drawn in both figures. By the application of the distortion D^a_n, the alphabetical scheme becomes in the first case

$$\underset{-}{x}\ \underset{+}{\gamma_1}\ \underset{-}{\gamma_2} \ldots \gamma_{n+1}\ y \ldots p\ a\ \gamma_i \ldots \gamma_j\ a\ q$$

or

$$\underset{-}{x}\ \underset{+}{\gamma_1}\ \underset{-}{\gamma_2} \ldots \gamma_{n+1}\ y \ldots q\ a\ \gamma_j \ldots \gamma_i\ a\ p$$

and in the second case

$$\underset{-}{x}\ \underset{+}{\gamma_1}\ \underset{-}{\gamma_2} \ldots \gamma_{n+1}\ a\ p \ldots y\ \gamma_i \ldots \gamma_j\ a\ q$$

or

$$\underset{-}{x}\ \underset{+}{\gamma_1}\ \underset{-}{\gamma_2} \ldots \gamma_{n+1}\ a\ p \ldots q\ a\ \gamma_j \ldots \gamma_i\ y$$

Within a reversible tangle R_n, may exist the possibility of distortions $D_{n-1}, D_{n-2}, \ldots D_2, D_1, D_0$, which may be applied singly or in combination with others. A knot is invariant under a distortion D^a_0, since by it, the two threads of the reversible tangle R_0 are untwisted at a point a preceding γ_1, to be twisted at a point a' beyond γ_1, and consequently the general arrangement of the crossings is undisturbed. Hence in a consideration of the different forms of a given scheme the distortion of lowest order to be considered is the distortion D_1. The reason that makes the consideration of D_0 unnecessary applies also to a similar distortion D^γ_n on the sequence

$$\ldots \gamma_1\ \gamma_2 \ldots \gamma_{n+2} \ldots \gamma_1\ \gamma_2 \ldots \gamma_{n+2} \ldots$$

or

$$\ldots \gamma_1\ \gamma_2 \ldots \gamma_{n+2} \ldots \gamma_{n+2}\ \gamma_{n+1} \ldots \gamma_1 \ldots$$

Also it is unnecessary to consider a distortion affecting more than one-half of the total number of crossings, since it is equivalent to a distortion applied to the remainder of the knot. Hence the different knots of orders 3, 4, and 5 have but one form.

Consider the knot given by the alphabetical symbol :—

(1)
$$\begin{array}{cccccccccccccccccccc|c} a & f & b & g & c & j & d & h & e & b & f & a & g & i & h & e & i & c & j & d & a \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & \end{array}$$

The symbol shows the possibility of the following distortions :—

$$\mathrm{D}_0^{a \text{ or } f} : \ldots . \begin{array}{cc} a & f \\ + & - \end{array} \ldots . \begin{array}{cc} f & a \\ + & - \end{array} \ldots .$$

$$\mathrm{D}_0^{f \text{ or } b} : \ldots . \begin{array}{cc} f & b \\ - & + \end{array} \ldots . \begin{array}{cc} b & f \\ - & + \end{array} \ldots .$$

$$\mathrm{D}_0^{c \text{ or } j} ; \ldots . \begin{array}{cc} c & j \\ + & - \end{array} \ldots . \begin{array}{cc} c & j \\ - & + \end{array} \ldots .$$

$$\mathrm{D}_0^{j \text{ or } d} ; \ldots . \begin{array}{cc} j & d \\ - & + \end{array} \ldots . \begin{array}{cc} j & d \\ + & - \end{array} \ldots .$$

$$\mathrm{D}_0^{e \text{ or } h} ; \ldots . \begin{array}{cc} h & e \\ - & + \end{array} \ldots . \begin{array}{cc} h & e \\ + & - \end{array} \ldots .$$

$$\mathrm{D}_1^{i} \; ; \ldots . \begin{array}{cc} h & e \\ - & + \end{array} \ldots . \begin{array}{cccc} i & h & e & i \\ - & + & - & + \end{array} \ldots$$

$$\mathrm{D}_1^{c \text{ or } d} ; \ldots . \begin{array}{ccc} c & j & d \\ + & - & + \end{array} \ldots \begin{array}{ccc} c & j & d \\ - & + & - \end{array} \ldots .$$

$$\mathrm{D}_2^{g} \; ; \ldots . \begin{array}{cccc} a & f & b & g \\ + & - & + & - \end{array} . . \begin{array}{cccc} b & f & a & g \\ - & + & - & + \end{array} \ldots$$

Of these distortions only D_1^i and D_2^g can produce a change of form, since a knot is invariant under a distortion D_0 or a continuation of distortions D_0; that is to say, $\mathrm{D}_1^{c \text{ or } d}$. The application of the distortions D_1^i, D_2^g produce the symbols :—

(2)
$$\begin{array}{cccccccccccccccccccc|c} a & f & b & g & c & j & d & i' & h & e & i' & b & f & a & g & h & e & c & j & d & a \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + \end{array}$$

and

(3)
$$\begin{array}{cccccccccccccccccccc|c} g' & a & f & b & c & j & d & h & e & g' & b & f & a & i & h & e & i & c & j & d & a \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + \end{array}$$

respectively. The distortion $\mathrm{D}_1^i\mathrm{D}_2^g$, that is to say the distortion D_1^i, followed by the distortion D_2^g, gives the symbol

(4)
$$\begin{array}{cccccccccccccccccccc|c} g' & a & f & b & c & j & d & i' & h & e & i' & g' & b & f & a & h & e & c & j & d & g' \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & \end{array}$$

While it is a simple matter to recognise in any alphabetical scheme the existence of reversible tangles, R_n, and the effect thereon of the corresponding distortions, D_n, it is not so easy to say whether the distorted form and the original are the same or different. To meet this difficulty, the alphabetical symbol may be replaced by an equivalent numerical symbol * in which for each letter is substituted a number equal to one-half the number of crossings intervening before the next occurrence of the letter as the knot is described. For some purposes it is convenient to write also in a second row the numbers that arise when the knot is described in the reverse direction; but for a knot of order n, the sum of the numbers immediately above and below any letter is equal to $n-1$. Thus the symbol (1) becomes :—

(1′)
$$\begin{array}{cccccccccccccccccccc} 5 & 4 & 3 & 4 & 6 & 6 & 6 & 3 & 3 & 6 & 5 & 4 & 5 & 1 & 6 & 6 & 8 & 3 & 3 & 3 \\ 4 & 5 & 6 & 5 & 3 & 3 & 3 & 6 & 6 & 3 & 4 & 5 & 4 & 8 & 3 & 3 & 1 & 6 & 6 & 6 \end{array}$$

or more simply

$$5\ 4\ 3\ 4\ 6\ 6\ 6\ 3\ 3\ 6\ 5\ 4\ 5\ 1\ 6\ 6\ 8\ 3\ 3\ 3$$

* Suggested by Professor C. A. Scott, who calls it the intrinsic symbol.

while the variants become

(2′)	6 5 4 5 6 6 6 1 3 3 8 5 4 3 4 6 6 3 3 3
(3′)	4 5 4 3 6 6 6 3 3 5 6 5 4 1 6 6 8 3 3 3
(4′)	5 6 5 4 6 6 6 1 3 3 8 4 5 4 3 6 6 3 3 3

Since the set of numbers (1′) is the same as a set of numbers complementary to (4′), but in the reverse order, the forms (1) and (4) are equivalent. Likewise (2) and (3) represent the same knot.

Effect of Distortion on the Compartment Symbol.—In a consideration of the different forms of a given knot, the compartment symbol is more convenient than the alphabetical symbol, although the statement of the effect of a given distortion is now not quite so simple. It is clear from figs. 4 and 4′ that a reversible tangle R_n exists when a compartment δ_i is joined once, and only once, to a compartment δ_j, which in turn is joined at any number of vertices to a compartment δ_k, and the entire configuration is bounded on the right and left by the compartments λ_r, λ_s respectively, λ_r, λ_s being joined at the same crossing as δ_i, δ_j, let the compartments δ_i, δ_j, δ_k, λ_r, λ_s be joined to the reversible tangle R_n by 0, a_2, a_3, b_1, b_2 crossings, and to the remainder of the knot by a_1', 0, a_3', b_1', b_2' crossings respectively, so that

$$
\begin{aligned}
i &= a_1' + 1 \\
j &= a_2 + 1 \\
k &= a_3 + a_3' \\
r &= b_1 + b_1' + 1 \\
s &= b_2 + b_2' + 1
\end{aligned}
$$

Then the symbol for this portion of the knot becomes

$$
\begin{array}{lcl}
\delta_{a_1'+1} & & \\
| & & \\
\delta_{a_2+1} & \qquad & \lambda_{b_2+b_2'+1} \text{———} \lambda_{b_1+b_1+1} \\
\updownarrow \oplus & & \\
\delta_{a_3+\delta_{a_3'}} & &
\end{array}
$$

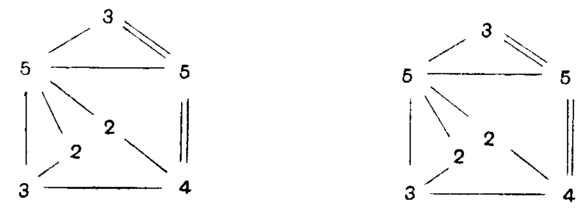
where $\updownarrow$ indicates that the two compartments may be joined directly or indirectly. By the application of the distortion D_n, this symbol becomes

$$
\begin{array}{lcl}
\delta_{a_1'+a_2} & & \\
\updownarrow \oplus & & \\
\delta_{a_3+1} & \qquad & \lambda_{b_1+b_2'+1} \text{———} \lambda_{b_1'+b_2+1} \\
| & & \\
\delta_{a_3'+1} & &
\end{array}
$$

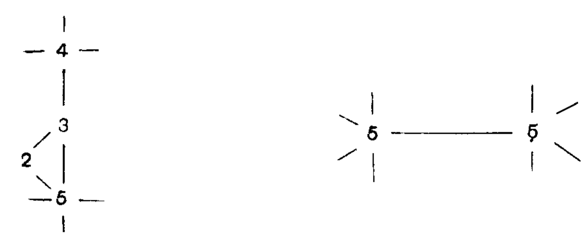
When $a_1' = a_3'$, $a_2 = a_3$, $b_1 = b_2$, the number of compartments and the number of angles in each is undisturbed. If the order in which these compartments are joined is the same as for the original knot, then the distortion reduces to a deformation, since the two forms are equivalent.

Both the primary and secondary symbol admit the possibility of distortions. The simple distortion D on the primary symbol merely changes the position of two joins in the secondary symbol.

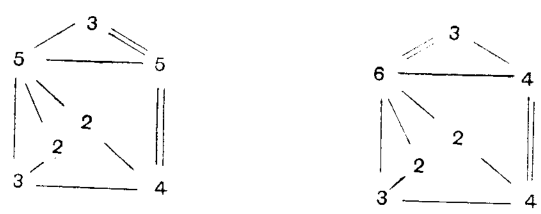
As a simple illustration of the effect of a distortion on the compartment symbol, consider the knot whose primary and secondary symbols are



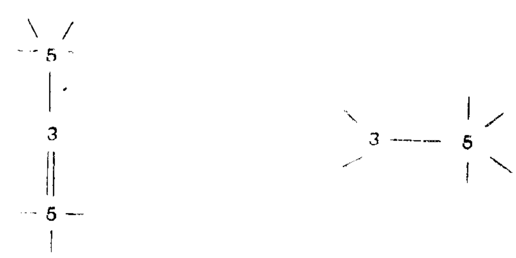
The primary symbol shows the presence of the reversible tangle R_2:



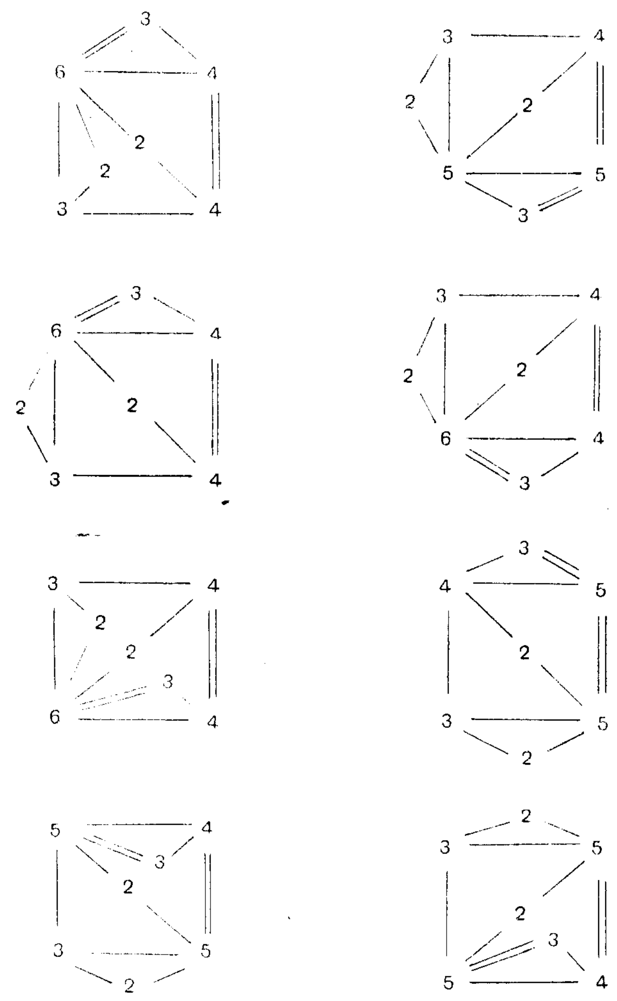
for which $a_1' = b_1' = 3$, $a_2' = a_3 = a_3' = b_2 = b_2' = 2$, $b_1 = 1$. Hence the application of the corresponding distortion D_2 gives the symbol



In addition to the reversible tangle R_2, the primary symbol exhibits the reversible tangle R_1:



Similarly, the secondary symbol shows the presence of the reversible tangles R_1' and R_2'. Hence there exists the possibility of distortions D_1, D_1', D_2, D_2', and all possible combinations of these. It is found that the above form possesses only the four variants—



§ 3. Amphicheiralism.

The perversion of a knot is the form obtained by replacing each crossing with one of the opposite character. The amphicheiral knot was originally defined by Tait as one which can be deformed into its own perversion. From this definition it is to be inferred that to every compartment δ_i there corresponds a similar and similarly placed compartment λ_i; that is to say, a necessary condition for an amphicheiral knot is the identity of the primary and secondary symbols.

Since the primary and secondary symbols of the amphicheiral knot are the same, the plane is divided into an even number of compartments by the plane projection of the knot, and consequently an even number of crossings is involved.

If such a knot is fitted on the surface of a sphere so that the corresponding arcual boundaries are made equal, a spherical compartment δ_i and its corresponding compartment λ_i must either be congruent or symmetrical* (where symmetrical is used in the sense of symmetrical triangles on a sphere). An amphicheiral knot whose corresponding compartments are congruent is called by TAIT an amphicheiral of the first order, as distinguished from one of the second order, in which the corresponding compartments are merely symmetrical.

From the definition of an amphicheiral knot it is seen that every distortion D_n carries with it a conjugate distortion $\overline{D}_n$ such that the product $D_n\overline{D}_n$—that is to say, the simultaneous application of the two distortions—gives an amphicheiral knot. The form obtained by the single operation D_n can be *distorted* into its own perversion by the operator $D_n^{-1}\overline{D}_n$, and is said to be of the second class, while one

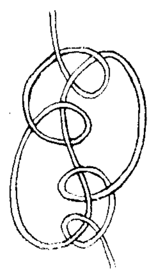


FIG. 4.

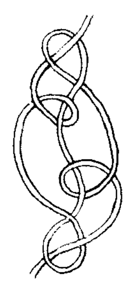


FIG. 4′.

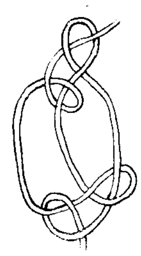


FIG. 4″.

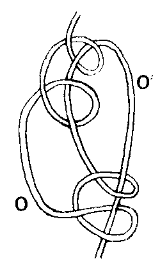
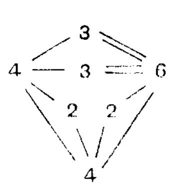
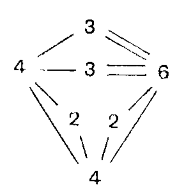


FIG. 4‴.

which can be *deformed* into its own perversion belongs to the first class. Therefore TAIT divides the amphicheirals of each order into those of the first or second class, according as they are the result of operating on the knot with conjugate or non-conjugate distortions.

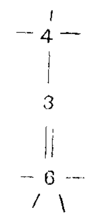
In an investigation of the amphicheiral knots of order 12 it appears that a third classification of amphicheiral knots of the first and second orders is necessary, namely, amphicheirals which are obtained as the product of two or more non-conjugate distortions. For example, consider the amphicheiral knot (fig. 4) whose compartment symbol is



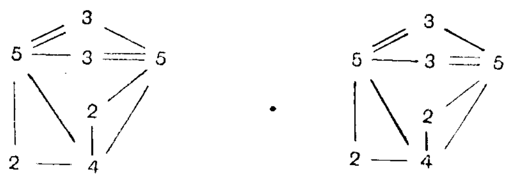


* (*Trans. R.S.E.*, xxxii, p. 494; or *Scientific Papers*, i, p. 336.) In his third paper TAIT deliberately limits himself to this view: but he remarks—" We shall afterwards find that there are at least three other senses in which a knot may be called amphicheiral, and shall thus be led to speak of different *orders* and *classes* of amphicheirals." (See below, § 6.)

There exist four reversible tangles R_1, $\bar{R}_1$, R_1', $\bar{R}_1'$ of the type

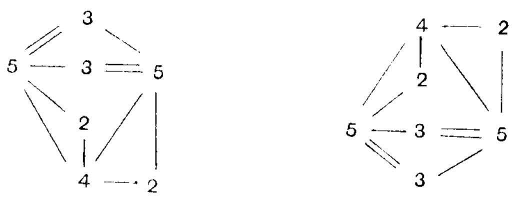


R_1, R_1' being conjugate to $\bar{R}_1$, $\bar{R}_1'$ respectively. The distortions $D_1\bar{D}_1$, $D_1'\bar{D}_1'$ transform the knot into the amphicheiral form



as shown in fig. 4′.

The knot shown in fig. 4″ is obtained from the original by means of the distortion $D_1\bar{D}_1'$. Its compartment symbol is



but the knot is not an amphicheiral knot of the first class.

The distortion $\bar{D}_1\bar{D}_1'$ reproduces the original amphicheiral with amphicheiral centres O, O′, as shown in fig 4‴. In this case the distortion D_1D_1' amounts to a deformation of the knot. Possibly the effect of the above distortion may be accounted for by the peculiar symmetry of the knot.

§ 4. Amphicheirals of the First Order.

A census of the twelvefold amphicheiral knots of the first and second orders is given on pp. 253–255 (shown also on the Plate); in the construction of these the methods of Tait* have been used.

Tait's Method of Construction.—When an amphicheiral knot is fitted on the surface of a sphere, as stated on p. 244, the part of the knot on one hemisphere is congruent to the part on the other. This congruence persists when the knot is subjected to symmetrical deformations by shortening or lengthening corresponding

* *Trans. Roy. Soc. Edin.*, xxxii, pp. 494–497 ; or *Scientific Papers*, i, 336–340.

laps of the thread, and consequently any path drawn across the knot can be made a great circle.

Since the two parts of the knot are congruent, rotation about a certain diameter will bring the first part of the knot into the position originally occupied by the second part. TAIT shows that this diameter must terminate in the mid-points of corresponding laps of the thread. But a rotation about such a diameter necessitates the existence of two pairs of adjacent corresponding compartments, in order that each compartment of the sphere may be rotated into the position of its corresponding compartment. The number of such diameters depends on the number of pairs of adjacent corresponding compartments. Now deform the knot so that the path from O to O′, which meets the knot in the minimum number, p, of points exclusive of O, O′, shall become the arc of a great circle S of the sphere, but in such a way as to keep corresponding crossings at equal arcual distances from the points O, O′. Since

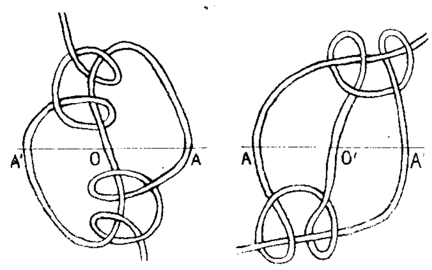


FIG. 5. FIG. 5′.

all great circles through the points O, O′ divide the knot into congruent halves, the projection of the knot from the point O′ on the tangent plane to the sphere at O will be divided into halves by all the straight lines through O, and in particular by the straight line s, which corresponds to the great circle S of the sphere. Of the $2p+2$ points of intersection of the line s and the knot, one, corresponding to the point O′, lies at infinity, and the rest by pairs at equal distances from the amphicheiral centre O. A part of the thread which joins two of these points is a bend. The framework for one half of the knot, that is, the framework on either side of the line s, consists of $p+1$ bends, of which one is infinite, since one point of intersection lies at infinity. Every possible arrangement of the bends must be considered; and in every admissible arrangement the bends are made to intersect so as to exhibit one half of the total number of crossings. The congruent half completes the knot (*cf.* fig. 5).

Inasmuch as two entirely different paths, O, O′, through the knot may give the proper number of intersections, $2p+2$, the knot* may be built upon an entirely different framework, and in such a case the eye may be deceived. The equivalence of the two is immediately detected by means of the compartment or intrinsic symbol.

If the knot is projected from the point O, on the tangent plane at the point O′, the figure exhibits symmetry about the point O′. This projection may be said to be

* TAIT, *Trans. Roy. Soc. Edin.*, xxxii, p. 496; or *Scientific Papers*, i, p. 338.

the complementary form of the knot. For example, the complementary form given by fig. 5 is fig. 5′; the identity of this with fig. 5 may be recognised either from the compartment symbol or from the intrinsic symbol (p. 240).

TAIT points out how, in the structure of the framework, m-filar knots can be avoided, and that composite knots, which are at once detected, must be discarded.

By the index of a knot, with respect to the pair of amphicheiral centres O, O′, is meant the number p which has been defined earlier. For a knot of order n, it may be shown that n is not less than $4p$; hence, for $n=12$, the only values of p to consider are 1, 2, 3. That is, in the construction of the amphicheiral knots of order 12, it is unnecessary to consider frameworks with more than four bends, including the infinite bend.

If a knot has a second pair of amphicheiral centres, not necessarily on the line OO′, the corresponding index may be the same as or different from p, and hence

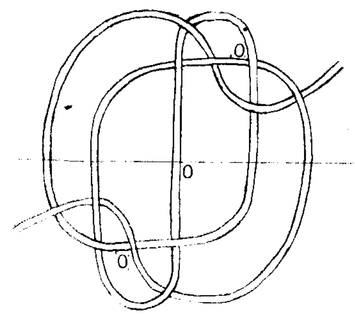


FIG. 6.

the knot may be constructed on a different framework. For example, knot No. 31 on the Plate, shown also in fig. 6, is of index 2 with respect to the pair of amphicheiral centres O, O′, but of index 3 with respect to the pair O_1, O'_1. Also Nos. 34, 35, for which $p=2$, belong to the set $p=3$ (see the Plate).

§ 5. AMPHICHEIRALS OF THE SECOND ORDER.

Tait's Method of Construction.—When an amphicheiral knot of the second order is fitted on the surface of a sphere, a compartment δ_i must either be diametrically opposite to its corresponding compartment λ_i, or it must be the image of the compartment λ_i in a diametral plane.

In the first case the desired arrangement of compartments may be obtained by means of a closed curve on a sphere and its diametrically opposite curve, together with a great circle. Such an arrangement can only lead to a trifilar knot. Further, the knot is bifilar if the curve is taken as its own opposite.

As may be seen by a projection from one pole of the great circle on to the tangent plane to the sphere at the other, the proper correspondence of compartments in a plane is secured by means of a circle, a closed curve, and its inverse as to the circle, but reflected in the origin. If now the closed curve is made to touch the circle in a point, the inverse curve will touch it at the diametrically opposite point, and it is necessary only to replace the contacts by crossings to secure the representation of a unifilar knot which exhibits amphicheiralism.

The second method of producing compartments δ_i and λ_i of the desired nature is rejected by TAIT, since it leads to a link solution. While a closed curve and its image in a diametral plane, together with the great circle of the sphere in this plane, will give the desired arrangement of compartments, it is impossible to fuse the curves into a single circuit as in the first case, for the process introduces triple points which cannot be replaced by three dps without destroying the amphicheiral symmetry. Hence, only the simplest trifilar link can result from such an arrangement. If the curve is taken as its own image, a bifilar knot is represented.

Before applying this method of construction, some preliminary considerations are necessary.

A closed curve on a sphere is either* a simple circuit or one member of a twin circuit. The simple circuit, which is its own opposite on the sphere, is met by a great circle in an odd number of pairs of points. In TAIT's method of construction a simple circuit leads always to an m-filar knot.

The twin circuit, which consists of a closed curve and its opposite on the sphere, intersects a great circle in an even number, 2ι, of pairs of points. There are two types of twin circuits which may present themselves. First, if $\iota = 0$, each member of the twin circuit is confined to a single hemisphere. Second, if $\iota \neq 0$, each member exceeds that hemisphere, and therefore the two members may intersect, necessarily in an even number, $2\sigma'$ of points where $\sigma' = 0, 1, 2, \ldots$

The projection, f, of any closed plane curve on to the sphere from its centre gives a twin circuit of the first type. A twin circuit of the second type is obtained by a similar projection of a plane curve of even order, which cannot be projected entirely into the finite part of the plane, as, for example, the Cayley non-singular sextic,† for which $\iota = 6$, $\sigma' = 0$. Every non-singular twin circuit divides the sphere into three regions, in one of which an odd circuit may lie.

Each member of the twin circuit may have σ dps, thus giving rise to σ pairs of crossings in the resulting knot. The only other cause that can produce crossings in the knot is the presence of κ pairs of contacts of the circle and the twin circuit. For a knot of order n, κ may not be greater than $\frac{n}{2}$. Hence for $n = 12$, $\kappa \not> 6$.

The few numerical possibilities for the above numbers to be considered are given in the following table :—

κ	1	1	1	2	2	2	2	3	3	3	4	4	4	5	6
ι	4	2	2	4	2	2	0	2	0	0	2	0	0	0	0
σ	1	3	1	0	0	2	4	1	3	1	0	2	0	1	0
σ'	0	0	2	0	2	0	0	0	0	2	0	0	2	0	0

* MÖBIUS, *Über die Grundformen der Linien der dritten Ordnung*, ii, p. 90.
† CAYLEY, vol. v, op. 361, p. 468.

Of the hundreds of cases arising from the different arrangements of these points, and the different ways of joining them, the greater number lead to composite or *m*-filar knots. Also Nos. 19, 21, 22, 24 ($p=2$) of the amphicheirals of the first order (see Plate) appear among those of the second order, since the arrangement of the δ and λ compartments is symmetrical, and hence unaltered by reversion (TAIT, iii, § 12). Rejection of these reduces the amphicheirals of the second order with twelve crossings to the following two :—

(5) *f j g i b e l d a c h k* $[D_1^c]$

(6) *f k b i a d l e h c g j* $[D_1^b]$

These two knots can be constructed on models involving more than one pair of contacts, and hence may be expected to present themselves several times in the course of the construction. Starting with a given knot of the second order, the different models on which the knot may be constructed are obtained by transferring one or more compartments to the inside of the circle, and therefore their correspondents to the outside of the circle. This amounts merely to a deformation of the knot so as to make any desired path into a circle.

If small letters are used to denote the intersections of the circle and the twin pair, while capital letters indicate the points of contact, that is, the crossings at which a change of thread takes place, then the different models for the above knots may be represented as follows :—

(5′)

(1) A *f b j c* G *l h d i*
(2) I *e* B *j* C *k* H *d*
(3) A *f b* J C G *l h* D I
(4) A F E B *j c* G L K H *d i*
(5) A F B J C G L H D I
(6) A F E B J C G L K H D I

(6′)

(1) A *f d* G *l j*
(2) *a f* D *b c* K *g l* J *h i* E
(3) A *f* B C *k* G *l* H I *e*
(4) A F D *b* C *k* G L J *h i* E
(5) A F D B K G L J H E

§ 6. SKEW AMPHICHEIRALS OF THE SECOND ORDER.

In a note added to his last paper on knots, TAIT gives a special knot* of order 8 which he classes as an amphicheiral of the second order; although, strictly speaking, it does not belong to the second order, since corresponding compartments are not opposite when fitted on a sphere.

In the investigation of amphicheirals with twelve crossings this type assumes

* *Trans. Roy. Soc. Edin.*, xxxii, p. 500; *Scientific Papers*, i, p. 342.

sufficient importance to be worth separate treatment. Knots of this character have, however, so much in common with the regular amphicheirals of the second order, that it seems convenient to call them skew amphicheirals of the second order.

Following TAIT'S construction for the special knot in question, consider on a sphere a non-singular closed curve that has, in common with a great circle, no points except μ points of contact, conveniently placed at the alternate vertices $V_1, V_2, \ldots V_\mu$ of a regular polygon with 2μ sides; and a similar curve on the opposite hemisphere to touch the circle at the remaining vertices $V_1', V_2', \ldots V_\mu'$. Projection from either pole of the great circle shows that this construction may be accomplished in the plane by drawing inside a circle a non-singular closed curve C which touches the circle at the alternate vertices $v_1, v_2, \ldots v_\mu$ of a regular polygon, of 2μ sides and a corresponding curve C′, to touch at the remaining vertices $v_1', v_2', \ldots v_\mu'$. When the points of contact are regarded as crossings, the figure that results possesses the desired amphicheiral symmetry, although it is not necessarily unifilar. The plane projection of such a knot from the mid-point of an arc $| V_1V_1' |$ of the great circle on the tangent plane at the diametrically opposite point exhibits symmetry about a point, as in the case of the amphicheirals of the first order.

In the case when μ is odd, a point of contact v_1 of the curve C is opposite to a point of contact of the curve C′, and the curve C is opposite to the curve C′. Hence the resulting knot is an amphicheiral of the second order.

On the other hand, if μ is even, the point of contact v_1 is opposite to the point $v_{\mu/2}$. Corresponding arcs are no longer opposite as to the circle. Nevertheless the corresponding compartments of the resulting knot are equal and non-congruent, as in the amphicheirals of the second order.

If $\mu = 4$, the peculiar eightfold knot given by TAIT is obtained.

The special case $\mu \equiv 0 \mod 3$ leads always to a trifilar link. For suppose the knot to be described by a point P in a fixed direction, starting from the point v_1 along the arc $| v_1v_2 |$ of the curve C. It leaves this arc at the point v_2 along the circle, only to return to the curve C after the elapse of four vertices of the regular polygon; that is to say, in going once around the circle, every third arc $| v_iv_j |$ of the curve C is described. If therefore the number of such arcs is a multiple of 3, the point P returns to the position v_1 along the arc $| v_1v_2 |$ by which it left, before the complete knot has been described. A second thread of the knot is traversed if the point P starts from the point v_2 along the arc $| v_2v_3 |$. And, starting from the point v_3 along the arc $| v_3v_4 |$, a third thread is obtained, thus completing the description of the knot. On the other hand, $\mu \equiv 0 \mod k$, where k is any other number, must lead to unifilar knots, since it will be necessary for the point P to go around the circle three times before returning to the starting-point along the same arc by which it left. The primary compartment symbol for such a knot contains $2\mu + 1$ compartments, one with μ angles and μ with three angles each. The compartment δ_μ is joined once to each of the compartments δ_3, any two adjacent compartments δ_3 being joined singly.

In the construction of these amphicheirals it is not, however, necessary that the curve used be non-singular; it is possible to obtain a figure which exhibits the

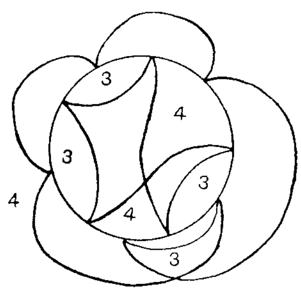
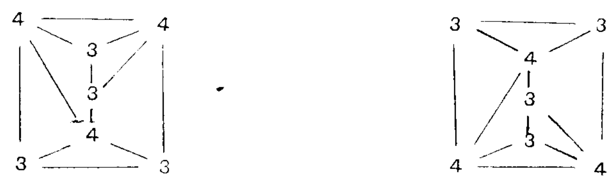


FIG. 7.

amphicheiral property by means of two singular curves arranged as above. For example, fig. 7 represents the amphicheiral knot whose compartment symbol is



Corresponding compartments are not opposite on the sphere; nevertheless the primary and secondary symbols exhibit the identity as to the number and arrangement of the joins, differentiated only by the right- and left-handed property peculiar to an amphicheiral of the second order. By reversal of the one set of compartments the amphicheiralism is undisturbed, and this knot is found to be No. 22 ($p=2$) of the amphicheirals of the first order. However, the skew amphicheiral constructed as shown in No. 61 in the plate of knots is not an amphicheiral of the first class as defined by TAIT. It is equivalent to the knot shown in fig. 4‴, p. 244, which is the result of applying to the amphicheiral knot shown in fig. 4, p. 244, two non-conjugate distortions. Consequently TAIT would call it an amphicheiral of the first order and second class. But its compartment symbol

shows the particular character that belongs to an amphicheiral of the second order. It is prevented from being classed as an amphicheiral of the second order by the fact that it is not obtained by means of a twin circuit in contact with a circle; that is to say, corresponding compartments are not opposite.

Thus it is seen that TAIT * is not justified in stating that there are but two possible ways in which corresponding spherical compartments of a knot may be equal. Further, contrary to TAIT,† it is possible for an amphicheiral knot to belong to the first class of one order and to the second class of another; as, for example, the skew amphicheiral (No. 61), which belongs to the second class of the amphicheirals of the first order, and first class of the skew amphicheirals of the second order.

The explanation of the difficulty is that the set of λ compartments cannot be moved on the sphere without affecting the set of δ compartments; hence it is not always possible to place the whole knot in one of the two ways considered by TAIT. Inasmuch as all possible cases of amphicheiralism do exhibit themselves in the compartment symbol, it seems highly advisable to use this in the definition, which may then be formulated as follows :—

Definition.—An amphicheiral knot of the first class is one whose primary and secondary symbols are identical as to numbers and arrangement, but with rotation in the same sense for those of the first order, in the opposite sense for those of the second order. Any form obtained from an amphicheiral of the first class by non-conjugate distortions ‡ is an amphicheiral of that same order, but of the second class. It is, however, possible for a knot to belong to the first class of one order and to the second class of the other. Since the two symbols are alike, it is possible by a deformation to replace the amplexus δ_i with the corresponding compartment λ_i, and the perversion is obtained.

With one exception (No. 61) the skew amphicheiral with twelve crossings turn out to be amphicheirals of the first order. This overlapping of the different divisions has been detected by TAIT for knots with ten crossings, where there are no amphicheirals of the second order that are not also of the first order. As shown here for twelve crossings, there are some of the second order not included under those of the first order; presumably with a greater number of crossings there may exist amphicheirals of the second order that escape any of the other divisions.

This completes the census of the twelvefold amphicheirals, of which there are sixty-one, as compared with one fourfold, one sixfold, five eightfold, and thirteen tenfold amphicheirals.

* TAIT, *Trans. Roy. Soc. Edin.*, xxxii, p. 498 ; or *Scientific Papers*, i, p. 340.

† TAIT, *Trans. Roy. Soc. Edin.*, xxxii, p. 499 ; or *Scientific Papers*, i, p. 341.

‡ It must be remembered that if a knot is amphicheiral of the first order, with more than one pair of centres, distortions that are non-conjugate for one pair may be conjugate for another pair.

§ 7. Census of Amphicheirals with Twelve Crossings.

In the census of the amphicheiral knots of the first order with twelve crossings the alternate crossings which occur as the knot is described in a fixed direction from the amphicheiral centre at infinity are denoted by a, b, c, . . . The first crossing a in the amphicheirals of the second order has been assigned arbitrarily. Only the letters which occupy the even places in the sequence of the alphabetical symbol of the knot are given, although the distortions D_n, given in brackets, are detected only in the complete scheme. The number of different forms of a given knot is indicated by the number of distinct distortions D_n in the brackets following the knot scheme; the different forms are obtained by the product of a distortion D_n and its conjugate D_n, which occurs at the same distance from the amphicheiral centres as D_n.

In the determination of the pairs of amphicheiral centres of a knot the intrinsic symbol is very convenient. It may be shown that the sum of the numbers at equal distances from an amphicheiral centre is equal to $n-1$, where n is the order of the knot. From the reduced alphabetical symbols given, it is a simple matter to write down the complete alphabetical and therefore the intrinsic symbols of the knots; hence the pairs of amphicheiral centres are known, and the knot may be constructed.

There exist the following amphicheirals with twelve crossings:—

I. Amphicheirals of the First Order:

$p=1$, Nos. 1–18; $p=2$, Nos. 19–54; $p=3$, Nos. 55–58.

(1) *i h g l j k c b a f e d*

(2) *h i g k l j a b c d e f*

(3) *i g h l j k c a b f d e*

(4) *h i g k l j b a c d f e* $[D_2^k,\ D_2^kD_3^d]$

(5) *i g h l j k c b a f e d* $[D_2^b]$

(6) *h a g c l j d b f i e k* $[D_1^a]$

(7) *d h g a k j b c l e f i* $[D_2^g,\ D_3^d]$

(8) *d h b g k i a c l e j f* $[D_1^b,\ D_1^d,\ D_3^a,\ D_1^bD_3^a,\ D_1^dD_3^a]$

(9) *g h b a k l d c f e j i* $[D_1^b,\ D_2^d,\ D_3^g,\ D_4^a,\ D_1^bD_3^g,\ D_2^dD_3^g,\ D_3^gD_4^a,\ D_2^dD_3^gD_4^a,\ D_1^bD_2^dD_3^g]$

(10) *c h a g k i d b f l e k* $[D_1^c,\ D_1^a,\ D_3^d,\ D_1^cD_3^d,\ D_3^dD_4^g]$

(11) *i h f l k c j b a g e d* $[D_3^f]$

(12) *c h i j k d a b g l d f* $[D_1^c,\ D_2^i,\ D_2^a,\ D_1^cD_2^i,\ D_1^cD_3^a,\ D_1^cD_2^iD_4^d]$

(13) *e a g b c j d l f h i k* $[D_1^a,\ D_2^c,\ D_1^aD_2^c]$

(14) *e d g a b j c l k f h i* $[D_1^d,\ D_2^d]$

(15) *e g b a d k c l h f j i* $[D_1^d,\ D_1^b,\ D_2^b,\ D_1^dD_1^b,\ D_1^dD_2^b]$

(16) *i d f l b c j a k g h e*

(17) *j e l f b d i k g a h c* $[D_1^i, D_1^g]$

(18) *i f e l c b k j a h g d* $[D_1^f, D_1^e]$

(19) *d h f a k c j b l g e i*

(20) *f d h a j b l c k e g i* $[D_1^d]$

(21) *j e l g b i d k f a h c*

(22) *f e h g j i l k b a d c*

(23) *i e h l j b d k f a g c*

(24) *f j k g b i l d e a h c*

(25) *e i g k a j b l c d f h*

(26) *j h l g k i b d e a f c*

(27) *f a i j b d l c g e h k* $[D_1^a]$

(28) *i a g l c j d b f h e k* $[D_1^a]$

(29) *f h i j k b l c a e g d*

(30) *i f h l j c k b a g e d*

(31) *e f h g j i k l b a d c*

(32) *j f l g b i k d e a h c*

(33) *i f a l b e k g c d h j* $[D_1^e]$

(34) *e h f a k c j l b g d i*

(35) *d i f k a b j c l e g h*

(36) *f i b k a d l c g e j h* $[D_1^b]$

(37) *k l h c j b d a f i g e* $[D_1^e]$

(38) *e f g a b j k l c d h i*

(39) *e f g b a j k l c d i h* $[D_2^k]$

(40) *e f h a j c k l b g d i*

(41) *e j k a b c d l f g h i*

(42) *e j k b a c d l f g i h*

(43) *f j k h i c l a b g d e*

(44) *j f l g a i k c b e d h*

(45) *j i l k a b d c f e g h*

(46) *i f g l b j k a c d h e*

(47) *f i h k j b l c a e g d*

(48) *i j k l a c e f b g d h*

(49) *j i l k a c e f b g d h*

(50) *c f a g b i k d e l h j* $[D_1^e, D_1^a]$

(51) *d f b h i c k a l g j e* $[D_1^b, D_1^d]$

(52) *e g h a j k c l b f d i*

(53) *j g l a b k c d e f h i*

(54) *j g l h i k d a f b c e*

(55) *k f g b l j k a c d i e*

(56) *h j k b l c d a f g i e*

(57) *h j k a l c b d e g f i*

(58) *d j k h i c b a l g f e*

II. Amphicheirals of the Second Order :—

(59) $f\ j\ g\ i\ b\ e\ l\ d\ a\ c\ h\ k$ $[D_1^e]$

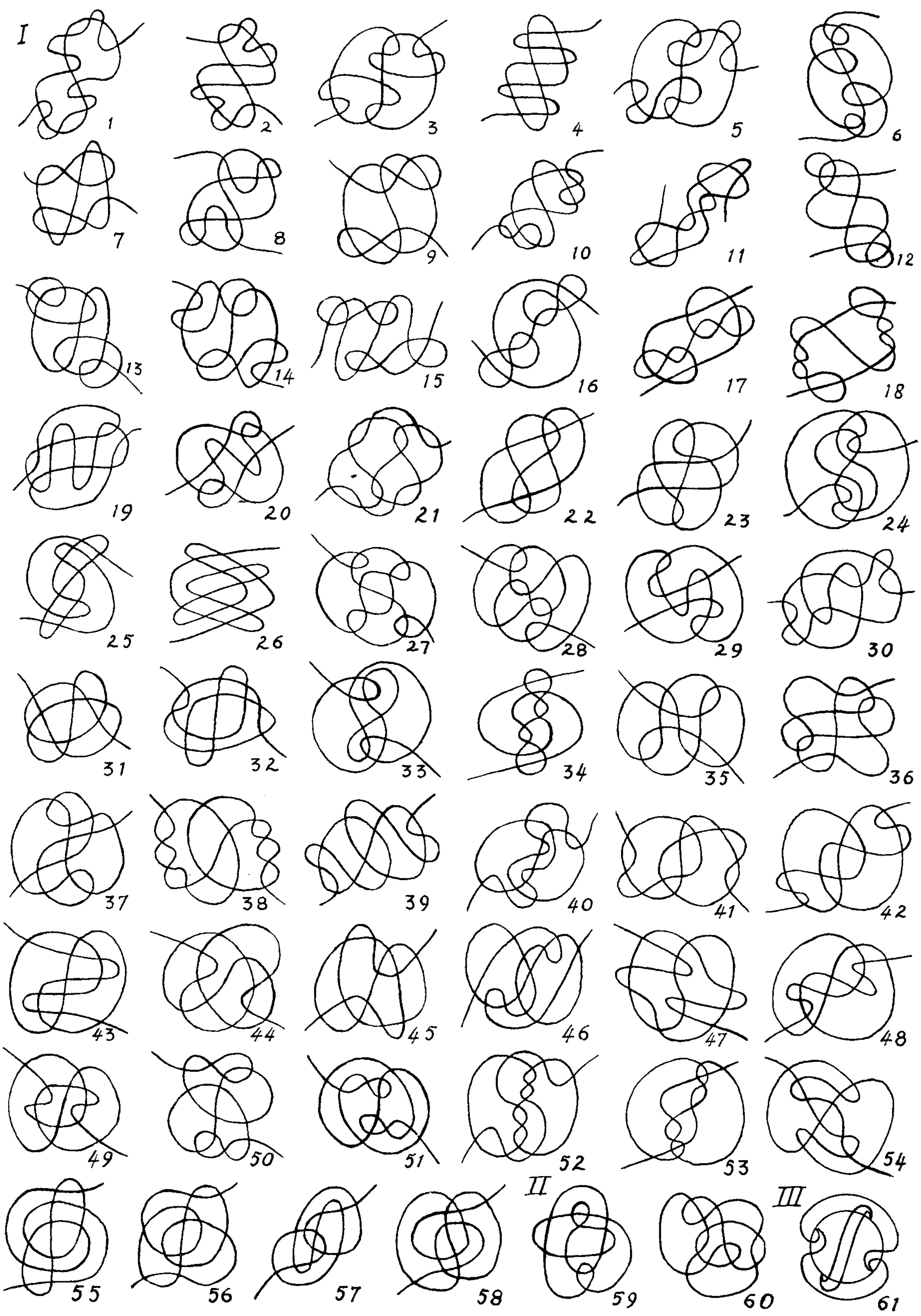
(60) $f\ k\ b\ i\ a\ d\ l\ e\ h\ c\ g\ j$ $[D_1^b]$

III. Skew Amphicheirals of the Second Order :—

(61) $i\ g\ b\ l\ j\ e\ c\ a\ h\ f\ d\ k$

I am indebted to Dr J. R. Conner and Professor Huff of Bryn Mawr College for their interest and encouragement; particularly to Dr Conner for his helpful criticisms. I am especially glad to have this opportunity of expressing to Professor Scott my sincere gratitude for her valuable help and unfailing encouragement during the writing of this dissertation as well as throughout my graduate course.

Mary G. Haseman: Amphicheiral Knots of Twelve Crossings.



J. Ritchie & Son, Edin^r