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The Characterization of Topological Manifolds of Dimension $n \geqslant 5$

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That rich, unkempt world of wild and tame topology, born in the minds of Antoine and Alexander, recalled from obscurity by Fox, Artin, and Moise, and brought to full bloom by Bing, has spawned a conjecture on the nature of the topological manifold having as one of its minor corollaries the famous double suspension theorem for homology spheres. F. Quinn in the Saturday morning topology seminar of this congress expressed confidence that he has the right conceptual and technical framework to complete the final step in its proof. Whatever the result after Quinn has had opportunity to verify his intuitions, the result is at the very least almost true; and we wish to discuss it. As is often the case, much of the visualization and example which gave the conjecture birth will surely disappear in the powerful application of engulfing, local surgery, etc., which should constitute its final proof. And so, for those of us who have always savored the interplay among point set topology, taming theory, decomposition space theory, and other visual aspects of geometric topology, we record here the milieu in which the conjecture became reasonable and the pressures leading to its formulation.

But first we summarize the conjecture itself and its most recent history. In the early spring of 1977 we conjectured,

Characterization Conjecture. A generalized n-manifold having the disjoint disk property, $n \geqslant 5$, is a topological n-manifold.

A generalized n-manifold M is an ENR satisfying $H_*(M, M-x; Z)=$

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$H_*(E^n, E^n-0; Z)$ for each $x \in M$. The space M satisfies the disjoint 2-disk property if maps $f, g: B^2 \to M$ can be approximated by maps $f', g': B^2 \to M$ having disjoint images.

We proved the conjecture for generalized manifolds having nonmanifold set of trivial dimension $k \leq (n-2)/2$ in the spring of 1977, and now, less than two years later, its proof appears on the verge of completion in two steps:

Resolution Conjecture (to be proved by? Quinn?). A generalized n-manifold of dimension $n \geq 5$ is a cell-like quotient of a topological n-manifold.

A cell-like subset of an ENR is a compactum contractible in each neighborhood of itself. A quotient map $f: M \to N$ of ENR's is cell-like if it is a closed map and each point preimage is cell-like.

Quotient Conjecture (proved by R. D. Edwards, late spring, 1977). A finite-dimensional cell-like quotient of an n-manifold, $n \geq 5$, is a manifold if and only if it has the disjoint disk property.

In addition to our own earlier weak versions of the Resolution and Quotient Theorems, early spring, 1977, Bryant–Hollingsworth and Bryant–Lacher had proved early versions of the Resolution Theorem.

As recently as four years ago no one dreamed that a useful characterization of topological manifolds was possible; all proposals ran afoul of the delicate, fiendishly manifold-like nonmanifolds of R. H. Bing and his school. That the notions of the generalized manifold and disjoint disk property were precisely appropriate for a characterization conjecture appeared only slowly from considerations of the taming and decomposition space theory pioneered by R. H. Bing. We give here an abbreviated exposition of Bing's work relevant to the characterization conjecture.

Bing, examining E. E. Moise's work on the triangulation theorem and Hauptvermutung for 3-manifolds in the early 1950s, was led to a profound study of the embeddings of polyhedra and compacta in the 3-dimensional sphere S^3. Bing set himself the problem of understanding the phenomenon called wildness. While it was clear that a simple closed curve can be knotted in S^3, it is not at all obvious that the Cantor set or 2-sphere can be knotted in S^3. Nevertheless, such knotting, necessarily infinite in nature, does occur and was discovered in the 1920's by M. L. Antoine and J. W. Alexander. An infinitely knotted set is called wild; other more standardly embedded sets are called tame. One of Bing's many beautiful discoveries was that the wildness of a 2-sphere or Cantor set in S^3 can be traced to a simple homotopy theoretic failing in dimension one: the complement of a wild set in S^3 is not 1-ULC; that is, there exist arbitrarily small simple closed curves in the complement of the wild set that are not contractible in small subsets of the complement. Extensions of Bing's results came to be known as taming theory.

Decomposition space theory as developed by Bing obtained its early impetus from the following remarkable theorem of R. L. Moore, Bing's teacher: if $f: S^2 \to X$ is a surjection from the 2-sphere S^2 onto a Hausdorff space X such that, for each $x \in X$, $S^2 - f^{-1}(x)$ is nonempty and connected, then X is also a 2-sphere.

Bing studied the extent to which Moore's theorem extends to closed surjections $g: S^3 \to Y$. G. T. Whyburn, another Moore student, had already suggested an appropriate condition on point inverses $g^{-1}(y)$, $y \in Y$: $S^3 - g^{-1}(y)$ was to be homeomorphic with $S^3 -$(point) and such a set was to be called pointlike or cellular. (The notion of cell-like set occurring in the Resolution and Quotient Conjectures is a generalization of Whyburn's notion of cellular set slightly more appropriate than cellularity in general.) But even among cellular quotients of S^3 Bing found nonmanifolds, such as his dogbone space—nonmanifolds because they failed to have a certain appropriate 3-dimensional variant of the disjoint disk property.

For the purposes of this paper we shall occasionally call the nonmanifold cell-like quotients of a manifold M wild spaces and the manifold quotients tame. The corresponding decompositions of M into point preimages of the quotient map are called nonshrinkable (wild) or shrinkable (tame) decompositions, respectively, for important technical reasons. Both shrinkable and nonshrinkable decompositions are important for the theory, a nonshrinkable decomposition always yielding a wild space, but an interesting shrinkable decomposition often yielding an unusual wild subspace or wild embedding.

From the middle of the 1950s until the early 1970s mathematicians of the Bing school developed the two theories in parallel. As years passed it became more and more apparent that, especially in high dimensions, wild subspaces and wild spaces were but two aspects of the same phenomenon and that 1-ULC properties on the one hand and variants of the disjoint disk property on the other played analogous and decisive roles. To demonstrate just how closely the theories were related, we will now explain two major areas in which one theory was used to further the other:

The construction of wild examples. B. J. Ball recognized Bing's wild dogbone space as the result of sewing together two subspaces of S^3 bounded by wild 2-spheres. On the other hand, N. Hosay and L. L. Lininger proved that wild 2-spheres in S^3 are always the image of tame 2-spheres under interesting cellular quotient maps from S^3 onto S^3. W. T. Eaton and R. J. Daverman established high dimensional analogues of the Ball and Hosay–Lininger results, respectively. M. A. Stan'ko proved results about codimension-three compacta which, R. D. Edwards pointed out, could be used to prove that wildly embedded codimension-three compacta are images of tame compacta under tame or shrinkable quotients. W. T. Eaton mixed wild compacta to create wild quotients. In other words, interesting examples in each of the theories spawned interesting examples in the other.

The characterization of tame subspaces and tame spaces. M. Brown, R. Kirby, and A. V. Cernavskii all proved various 1-ULC taming theorems for $n-1$ spheres in S^n by decomposition space techniques. On the other hand, W. T. Eaton, R. J. Daverman, and J. W. Cannon proved shrinking theorems for decomposition spaces by using 1-ULC properties, and taming theoretic techniques. Particularly in W. T. Eaton's work, a variant of the disjoint disk property was connected with certain

1-ULC taming properties and was used not just as a method of recognizing non-manifolds but as a tool in recognizing tame quotient spaces. And finally R. D. Edwards began his marvelous work on the double and triple suspension problems. At this point L. C. Glaser should be recognized for popularizing the decomposition space approach to the double suspension problem. Edwards made the key observation that intrinsic to the decomposition spaces associated with the double suspension problem were certain natural finite approximations to wild spheres of Alexander horned sphere type. Edwards had been led to expect such objects in decompositions by his study of M. A. Stan′ko's work on taming compacta.

In addition to the direct aid given one of the theories by the other in the two areas just mentioned (as in others), one noticed a number of parallel and analogous results, connections not well-understood but highly suggestive. Particularly striking were the results obtained upon stabilization (multiplication by some number of lines). A wild embedding $f: S^k \to S = f(S^k) \subset E^n$ into Euclidean n-space became tame in E^{n+1} (that is, $S \subset E^n \subset E^n \times E^1 = E^{n+1}$ is a tame topological k-sphere). This fact may be deduced as a consequence of the various known 1-ULC taming theorems. Furthermore, although the product $S \times E^1 \subset E^n \times E^1$ is wild when S is wild, it has the mildest possible form of wildness in terms of its 1-ULC properties according to R. J. Daverman's analysis of the same. On the other hand, no cell-like quotient Q of E^n was known to fail to be a factor of E^{n+1} ($Q \times E^1$ and $E^n \times E^1$ were almost always known to be homeomorphic). In particular, Bing had shown that his dogbone space was a manifold factor. And largely due to the impetus given the subject by some clever arguments and ideas of L. Rubin, a number of mathematicians began to prove that large and very general classes of manifold quotients were manifold factors—the best results issuing from C. Pixley, W. T. Eaton, R. T. Miller, and R. D. Edwards. J. L. Bryant and J. G. Hollingsworth considered the converse problem: is a manifold factor a manifold quotient? and proved the first weak resolution theorem. Pursuing the analogy between the 1-ULC taming properties and the decomposition space disjoint disk property further we note that though many decompositions failed to have the disjoint disk property, products with a line generally did (and as Daverman noticed in 1977, the product with two lines always had the property).

By the early to mid 1970s the interconnections between taming and decomposition space theory had become so numerous and obvious that we attempted to formalize the interconnections. We began a program to prove that 1) every taming theorem had a decomposition space analogue and 2) every decomposition space theorem had a taming theoretic proof. What emerged was first the realization that if some decomposition space theorems were to admit a taming theoretic proof, then one would have to extend taming theory to allow consideration of general manifold-like objects. We discovered early in 1975 that the properties of the generalized manifold introduced decades before by R. L. Wilder were precisely those amenable to 1-ULC taming theory and as a consequence proved by taming theoretic methods that every generalized $(n-1)$-manifold which embeds in an n-manifold is at least stably a cell-like quotient of a manifold. We were so struck by the discovery that an algebraically defined class of spaces (the generalized manifolds) should have such strong geometric properties that we immediately began to advertise the possibility in private discussions and in lectures that topological characterizations of manifolds, contrary to all appearances, might indeed be possible. When we were able to prove by the same taming techniques in the spring of 1976 that the double suspension of every homology sphere is a cellular quotient of a sphere, a result anticipated a few months by Edwards, we became even more convinced that the generalized manifold was exactly the right candidate for resolution theorems of the type suggested first by Bryant and Hollingsworth. Furthermore, we felt that the completion of Edwards' program of proving the double suspension theorem was at that point assured. A second consequence of our program was that we began consciously to concentrate on the 1-ULC properties of cell-like quotients. The realization came that the possible wildness of the double suspension quotients depended at worst on the 1-ULC wildness of the suspension circle and that, in 1-ULC taming theoretic terms, the wildness of the circle was of the simplest known type. After explanations by Edwards of his double suspension work, we saw that the looseness of the 1-ULC structure allowed us to find, actually embedded in the decomposition space in a very simple way, the wild spheres of Alexander horned sphere type whose finite approximations Edwards had noted. A complete proof of the double suspension theorem followed quickly from results on taming theory. We explained our work to Bing. He was not excited. He found the proof obscure. In frustration we sought the simplest possible conceptual framework encompassing the mildly wild 1-ULC properties of the examples. The appropriate property proved to be the disjoint disk property. Suddenly the connections between the various 1-ULC taming properties and the disjoint disk decomposition properties became clear in our minds and the characterization conjecture immediately took its present form.

Almost immediately after receiving our initial applications of the disjoint disk property, Edwards was able to prove the quotient conjecture. The disjoint disk property allowed one to embed the (infinite) 2-skeleton of the domain of the quotient map in the quotient space, and 1-ULC taming theory for decompositions allowed one to make the quotient map one to one over that infinite skeleton. Edwards noted that the embedding process forced the remaining nondegenerate point preimages of the quotient map to have 1-ULC complement, even low geometric embedding dimension, hence to be essentially tame or untangled in the decomposition space sense. Edwards was able by a clever engulfing type induction to untangle and shrink the remaining elements to points.

On the other hand, again using insights suggested by our use of the disjoint disk property, we, and Bryant and Lacher independently, were able to prove vastly improved versions of the known resolution theorems. Our versions depended on certain 1-ULC taming theorems of Černavskiĭ and Seebeck. Since the results just described were proved early in 1977, Ferry, Quinn, and Chapman have proceeded to generalize and strengthen those 1-ULC taming theorems to generalized manifolds in more and more general settings. Quinn suggests that the appropriate taming problems are exactly suited to the completion of an early dream from his graduate school days of establishing local versions of surgery. The proof of the resolution conjecture, and with it the characterizations of topological n-manifolds, $n \geq 5$, may soon, therefore, be complete.

References

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