ON TYPES OF KNOTTED CURVES.*

By J. W. Alexander and G. B. Briggs.

1. The problem of determining the various possible types of closed, knotted curves in 3-space was originally studied by Gauss, and has since received the attention of a number of other mathematicians.† Very little progress seems to have been made, however, toward finding definite, calculable invariants with which to distinguish one type of knot from another, though classified tables of the more elementary knots have been arrived at by somewhat empirical methods.‡

A number of years ago,§ one of the authors of the present paper pointed out that if the space of a knotted curve be covered by an n-sheeted "Riemann 3-spread" (the three dimensional analogue of a Riemann surface) with a branch curve of order $n - 1$ covering the knot itself, then, the topological invariants of the covering spread will also be invariants of the knot. He further calculated the Betti numbers and coefficients of torsion|| of the covering spreads determined by some of the simpler knots and found these invariants sufficient, in the cases actually examined, to distinguish one type of knot from another. The torsion numbers to which we have just referred have recently been rediscovered by K. Reidemeister¶ who derives them from a study of the group of the space complementary to the knot and then identifies them with the invariants of the Riemann covering spreads.

In this paper, we propose, first of all, to obtain the torsion numbers of a knot by direct, elementary considerations, without appealing to the idea of a Riemann covering spread. Next, we shall prove, with the aid of these invariants alone, that all types of knots of eight or less crossings listed as distinct in the knot tables of Tait and Kirkman actually are

* Received April 28, 1927.

† For a list of references on the subject of knots, the reader is referred to the article on Analysis Situs, by Dehn and Heegaard, Encyclop. der Math. Wiss., III A B 3, pp. 207–213.

‡ References to the knot tables of Tait, Kirkman, and Little will be found in the Dehn-Heegaard article, loc. cit., p. 207.

§ In a paper read before the National Academy, Nov., 1920, cf. Veblen's Cambridge Colloquium Lectures on Analysis Situs (1922), last page.

|| For the definition of these numbers cf., for example, Veblen's Colloquium Lectures, pp. 109, et. seq.

¶ K. Reidemeister. Knoten und Gruppen. Abh. aus d. Math. Semin. der Hamburgischen Universität, 1926, pp. 7–23. See also Elementare Begründung der Knotentheorie, pp. 24–32.

distinct. We shall also prove that among the eighty-two listed types of knots of nine or less crossings three cases arise where knots of presumably distinct types have the same torsion numbers, but that, in all other cases, the torsion numbers are sufficient to distinguish one type of knot from another. Finally, we shall describe, briefly, the method of obtaining the torsion numbers of a knot from its associated Riemann covering spreads, after the manner in which the invariants were originally discovered.

2. To simplify the exposition as much as possible, we shall approach the knot problem from the elementary, combinatorial point of view. Thus, for our purposes, a *knot* will be nothing more than a simple, closed, sensed* polygon in the space of three real variables x, y, z. The knot will be composed of a finite number of straight segments, or *edges*, together with their end points, or *vertices*.

A knot will be subject to certain simple transformations. On any edge AB we may construct a triangle ABC, so drawn that neither the vertex C, the edge AC, the edge CB, nor the plane triangular region bounded by ABC has a point in common with the knot. We may then transform the knot by removing the edge AB and substituting in its place the edges AC and CB, along with the vertex C. We may also perform the reverse operation which consists in replacing a pair of consecutive edges AC and CB, together with their common vertex C, by a single edge AB, provided neither the edge AB nor the plane triangular region bounded by ABC has a point in common with the knot. Each of the transformations here described will be called an *elementary deformation*.

The operation of subdividing an edge AB into two edges AC and CB along which a vertex C may be regarded as a degenerate case of an elementary deformation; so also may the inverse operation: the amalgamation of two consecutive collinear edges and their common vertex into a single edge. By an extension of terminology, these degenerate operations will also be described as elementary deformations. Evidently, a degenerate elementary deformation is the resultant of three suitably chosen non-degenerate ones.

Two knots will be said to be of the same *type* if, and only if, the one is transformable into the other by a finite succession of elementary deformations. A knot will be said to be *unknotted* if, and only if, it is of the same type as a sensed triangle. Any two sensed triangles are, of course, of the same type,

Let K be any knot. Then the knot K^{-1} obtained by reserving the sense of the polygon K will be called the *inverse* of the knot K. More-

* It is, perhaps, more customary not to impose the condition that the polygon be sensed.

over, the knot K' obtained by reflecting the knot K about the xy-plane will be called the *reflection* of the knot K. In general, no two of the three knots K, K^{-1}, and K' need be of the same type.

3. If we project a knot orthogonally upon the xy-plane, we shall, in general, obtain, a singular polygon Π with at most a finite number of crossing points, where each crossing point corresponds to two distinct points on different edges of the knot. In certain exceptional cases, the polygon Π will have singularities of a more complicated sort, but we shall agree to leave such cases aside, once and for all. It will evidently be legitimate to do this, because a knot with an exceptional projection may always be transformed into a neighboring one with a projection of the ordinary sort by making a slight shift in the position of one or more vertices. Moreover, the transformed knot will be of the same type as the original one, since the shifting of a vertex may always be accomplished by means of two properly chosen elementary deformations.

The projection of a knot upon the xy-plane will serve as a schematical picture, or *diagram*, of the knot itself, provided we indicate by some suitable notation which of the two segments at each crossing point is to represent the "upper" branch of the knot and which the "lower". The device ordinarily used is to indicate the upper branch by a solid line and the lower one by a line from which a small segment has been removed in the neighborhood of the crossing point (cf. Fig. 1 a). A better notation for our present purposes will be the following. We imagine an observer standing on the xy-plane and describing the polygon of the diagram once in the positive sense (as determined by the positive sense on the knot), thereby passing twice through each crossing point. Then, as the observer passes through a crossing point on the segment representing an upper branch of the knot, we mark with dots the two corners on his right; as he passes through a crossing point on the segment representing a lower branch, we make no notation of any sort. The resulting system of dots will enable us, at a glance, to distinguish between the segments corresponding to upper and to lower branches of the knot respectively; (cf. Fig. 1 b which represents the same knot as Fig. 1 a, but in the new notation).

If a figure like a diagram be formed without reference to any particular knot in space, by merely drawing a sensed polygon Π in the xy-plane, deciding on the segments to represent the upper and lower branches at each crossing point, and dotting the corners accordingly, there will always be a family of knots in space having the figure in question as their common diagram, provided any point of the polygon Π except a crossing point is allowed to represent a knot vertex. If only the vertices of the polygon Π are allowed to represent knot vertices, it may happen, in certain cases, that there exists no knot with sufficient flexibility to weave over and under itself in the manner precribed by the diagram. To make matters definite, we shall agree that only the vertices of a diagram are to correspond to the vertices of a knot, but we shall not prohibit diagrams with three or more collinear vertices. Thus, every figure of the type described above may be made into a true diagram, if it is not one already, by intoducing a certain number of new vertices at suitable points on the edges of the polygon Π, thereby subdividing the edges in question into sub-edges.

Since the knots corresponding to any giving diagram are all of the same type, as may readily be proved, the problem of classifying knots reduces to that of classifying diagrams. We shall say that two diagrams are of the same *type* if they correspond to knots of the same type; that a diagram D_1 is transformable into a diagram D_2 by an *elementary deformation* if some knot K_1 with the diagram D_1 is transformable into some knot K_2 with the diagram D_2 by an elementary deformation. A necessary and sufficient condition that two diagrams D_1 and D_2 be of the same type is, thus, that one be transformable into the other by a finite succession of elementary deformations.

If a knot K is the inverse of a knot L, the diagram of K will be called the *inverse* of the diagram of L. Similarly, if a knot K is the reflection of a knot L, the diagram of K will be called the *reflection* of the diagram of L.

4. An elementary deformation δ transforms a diagram D into a diagram D' which may be very different in appearance from D. We shall prove, however, that it is always possible to perform the deformation δ in a stepwise manner, in such a way, that, at each step, the change undergone by the diagram on which we operate is comparatively slight. Since the diagram D is transformable into the diagram D' by an elementary deformation, the polygons of the two diagrams must be relate to one another in the following simple manner: A certain edge AB of one polygon corresponds to a vertex C and a pair of edges AC and CB of the other, but the remaining parts of the two polygons are identical. To fix matters, we shall so assign our notation that D is the diagram with the edge AB and D' the diagram with the edges AC and CB. The triangle ABC will be called the *characteristic triangle* both of the deformation δ and of the inverse deformation δ^{-1} transforming the diagram D' into the diagram D. In certain trivial cases which will not require further analysis, the vertex C will lie on the edge AB and the characteristic triangle will be degenerate. We shall measure the degree of simplicity of an elementary deformation δ by the degree of simplicity of the figure formed by that part of the diagram D which is enclosed by the characteristic triangle ABC of δ.

Now, suppose we divide up the triangle ABC into a pair of triangles Δ_1 and Δ_2 by joining one of the vertices A, B, or C to a point K of the edge opposite the vertex chosen. Suppose, moreover, that we select the point K in general position, so that K is not at a crossing point of either of the diagrams D or D', and so that the new edge connecting the point K to the opposite vertex of the triangle ABC passes through no vertex or crossing point of the diagram D.. Then, it is obvious that the deformation δ with the characteristic triangle ABC may be thought of as the resultant of two smaller deformations δ_1 and δ_2 with the characteristic triangles Δ_1 and Δ_2 respectively. Moreover, each of these last may, in turn, be thought of as the resultant of two still smaller deformations obtained by a similar process of factorization, and so on, as often as we please. In other words, the deformation δ may always be regarded as the resultant of a finite succession of deformations with characteristic triangles of arbitrarily small linear dimensions. It is, therefore, possible to carry the factorization of the deformation δ to such a point that each factor δ^* of δ has the following degree of simplicity: If any portion of the diagram D^* on which the deformation δ^* operates is interior to the characteristic triangle Δ^* of the deformation, that portion is connected and includes not more than one point from among the vertices and crossing points of the diagram D^*; (here, we make use of the fact that since the triangles Δ^* corresponding to the different factors δ^* of δ are non-overlapping any portion of a diagram D^* interior to a triangle Δ^* must also be a portion of the original diagram D). When the above state of affairs is realized, we see, by inspection, that each of the factors δ^* must be an elementary deformation of one of four simple categories determined resectively by the following conditions:

(i) No part of the diagram D^* is interior to the triangle Δ^*.

(ii) The part of D^* interior to Δ^* consists of one sub-edge of D^*.

(iii) The part of D^* interior to Δ^* consists of one vertex and two sub-edges meeting at the vertex.

(iv) The part of D^* interior to Δ^* consists of one crossing point and two sub-edges passing through the crossing point.

The analysis may be carried somewhat further in the last three cases. We recall that either the deformation δ^* or its inverse—for our present purposes it makes no difference which—carries an edge A^*B^* of the characteristic triangle Δ^* into the figure consisting of the vertex C^* opposite A^*B^* and the other two edges A^*C^* and B^*C^*. Then, under cases (ii), and (iv), there are several essentially distinct sub-cases to be distinguished according to the location of the points in which the polygon of the diagram D^* crosses the triangle Δ^*. Let us first examine case (ii), where the portion of the polygon D^* interior to the triangle Δ^* consists of a single sub-edge $x_0' x_0''$. There are then three sub-cases to be considered, to which all others may be reduced by a mere change of notation:

(iia) One end of the edge $x_0' x_0''$ is at the vertex A, the other on the side BC. (Fig. 2 II.)

(iib) One end of the edge $x_0' x_0''$ is on the side AB, the other on the side BC.

(iic) One end of the edge $x_0' x_0''$ is on the side AC, the other on the side BC. (Fig. 2 III a.)

However, sub-case (iib) may be eliminated from the discussion since it reduces by factorization to sub-case (iic). To prove this, we join the vertex C of the triangle Δ^* to a point D of the edge AB in such a manner that the new edge CD crosses the segment $x_0' x_0''$. Then, we regard the deformation δ^* with the triangle ABC as the resultant of two elementary deformations with the triangles DBC and ACD respectively. Each of the latter is of the sub-category (iic).

Under case (iii), where the portion of the polygon of D^* interior to the triangle Δ^* consists of two sub-edges $x_0' z$ and $z x_0''$ meeting in a common vertex z, we may dispose of all sub-cases, except the one where the point x_0' lies on the side AC and the point x_0'' on the side BC (Fig. 2 IIIb). Any other deformation of the third category reduces, by an obvious process of factorization, to one of the types just described together with a certain number of deformations of the first and second categories.

Under case (iv), when the portion of the polygon D^* interior to the triangle Δ^* consists of two sub-edges $x_1 x_1'$ and $x_2 x_2'$ crossing in a point x_0 we may dispose of all sub-cases except the one where the points x_1 and x_2 both lie on the side AB while the points x_1' and x_2' lie on the sides AC and BC respectively (Fig. 2 IV). Any other deformation of the fourth category may be factored into one of the types just described together with a certain number of deformations of the first and second categories.

To sum up, the most general type preserving transformation of one diagram into another may be factored into a finite succession of elementary deformations each of which is one of the four types pictured schematically in Fig. 2. Of course, the dotting of the corners in Fig. 2 merely corresponds to one possible type of dotting that may actually arise. The letters R in the figure refer to a matter that we have not yet taken up, hence, they have no significance at the present stage of the discussion. Deformations III a and IIIb, in Fig. 2, are going to play essentially the same role throughout the discussion; this is why we have classified them together.

5. At this point, we shall make a slight digression to remind the reader of certain arithmetical theorems about linear systems. A set of m linearly independent marks

$$(5.1) \qquad x_i, \qquad (i = 1, 2, \cdots, m),$$

is the *base* of a linear system X consisting of all linear combinations of the marks x_i with integer coefficients, where the ordering of the terms in a linear combination is immaterial. Moreover, a set of k arbitrary marks

$$(5.2) \qquad y_s = \sum_{i=1}^{m} \varepsilon_{si}\, x_i. \qquad (s = 1, 2, \cdots, k),$$

belonging to the linear system X is the base of a linear sub-system Y of X. If a mark x of the system X is also a mark of the system Y, we shall indicate the fact by writing the homology

$$x \sim 0 \pmod{Y}.$$

Furthermore, we shall say that two marks x and x' of the system X are homologous, and write

$$x \sim x' \pmod{Y}.$$

if their difference $x - x'$ is a mark of the system Y. From the way in which homologies are defined it follows that any linear combination with integer coefficients of a given set of homologies is itself an homology; also, that the most general homology that can be written is a linear combination with integer coefficients of the fundamental homologies

$$(5.3) \qquad y_s \sim 0 \qquad (s = 1, 2, \cdots, k).$$

determined by the base (5.2) of the system Y. In operating with homologies, it must be borne in mind that if the coefficients of an homology possess a common factor it is not, in general, permissible to simplify the homology by cancelling out the common factor, for if a multiple of a mark x of the system X belongs to the system Y it does not necessarily follow that the mark x itself belongs to Y.

Now, let the marks of the system X be arranged in sets z_i such that two marks belong to the same set if, and only if, they are homologous. Then, if we define the *negative* of a set z_i containing a mark x_s as the set containing the mark $-x_s$ and the *sum* of the sets z_i and z_j containing the marks x_s and x_t respectively as the set containing the mark $x_s + x_t$, we may regard the sets z_i as the elements of a linear domain Z, in which the ordinary linear operations with integer coefficients may be performed. We shall designate an element z_i of the domain Z by means of any one of the marks x_s in the set z_i.

Now, by a well known theorem, if the bases (5.1) and (5.2) of the linear systems Y and X are properly chosen, the fundamental homologies (5.3) will assume the form of monomials in the x's,

$$(5.4) \qquad \alpha_s x_s \sim 0. \qquad (s = 1, 2, \cdots, k' \leqq k),$$

where the coefficients α_1, α_2, $\cdots$, are positive integers with the property that each one after the first is exactly divisible by its immediate predecessor. The coefficients α_s are, in fact, the elementary divisors of the matrix of coefficients ε_{si} in (5.2), which divisors remain invariant when the bases of the systems X and Y undergo a transformation. Moreover, when the fundamental homologies are in the normal form (5.4) the linear domain Z is also in a normal form. For it is easy to see that the elements of Z are now representable, respectively, by the symbols

$$(5.5) \qquad \sum_{s=1}^{k'} a_s x_s + \sum_{s=k'+1}^{m} a_s x_s,$$

where each coefficient a_s of the first sum is a non-negative integer less than the corresponding elementary divisor α_s in (5.4), and where each coefficient a_s of the second sum is wholly arbitrary integer. Thus, the internal structure of the domain Z is completely characterized by the number of terms in the second sum of (5.5) together with the values of all the elementary divisors α_s that are greater than unity. If a divisor α_s is equal to unity, the corresponding term $a_s x_s$ of the first sum of (5.5) must have the coefficient zero; therefore, the presence or absence of this term has no effect upon the internal structure of the domain Z. It will be convenient to characterize the domain Z by a sequence of integers consisting of as many zeros as there are terms in the second sum in (5.5) followed by as many of the elementary divisors α_s as are different from unity. The terms of this sequence will be called the *characteristic invariants* of the system X with reference to its sub-system Y. Among the characteristic invariants, repetitions are, of course, to be expected, since there may be more than one term in the second sum in (5.5), and since two or more of the elementary divisors α_s, $(\alpha_s > 1)$, may be equal to one another.

6. We are now ready to derive the torsion invariants of a knot. These invariants occur in sets, such that there is one set corresponding to each choice of an integer n greater than unity. We shall determine the set corresponding to a definite, though arbitrary choice of the integer n.

Let ν be the number of crossing points of the knot diagram. Then, by Euler's theorem on polyhedra, the number of connected regions into which the polygon Π of the diagram subdivides the xy-plane must be $\nu + 2$. We shall denote the ν crossing points of the diagram by the symbols

$$x_\alpha, \qquad (\alpha = 1, 2, \cdots, \nu),$$

respectively, the $\nu+2$ connected regions of the plane of the diagram by the symbols

$$R_\sigma, \qquad (\sigma = 1, 2, \cdots, \nu+2),$$

respectively. A crossing point x_α and a region R_σ will be said to be *incident* to one another if, and only if, the point x_α is on the boundary of the region R_σ. If the point x_α and region R_σ are incident, they will be said to be incident *with*, or *without a dot* according as the corner of the region R_σ contiguous to the crossing point x_α is or is not one of the dotted corners of the diagram. In certain exceptional cases, it may happen that a point x_α and a region R_σ are doubly incident in the sense that two different corners of the region R_σ are contiguous to the point x_α. Whenever this occurs, the two corners of R_σ at x_α must, of course, be opposite corners, so that one of them is dotted, the other one not.

To each crossing point of the diagram we shall now assign n marks, thereby obtaining, in all, a set of $n\nu$ marks

$$x_{\alpha a}, \qquad \begin{pmatrix} \alpha = 1, 2, \cdots, \nu \\ a = 1, 2, \cdots, n \end{pmatrix}.$$

These marks will be regarded as the base marks of a linear system x similar to the one described in the last section. For reasons of symmetry, we shall make use of a recursion formula

$$x_{\alpha a} = x_{\alpha(a+n)},$$

and do away with the requirement that the second index a of the mark $x_{\alpha a}$ lie between 0 and $n+1$. The next step will be to assign to each region R_σ of the diagram a set of n homologies of the form

$$y_{\sigma s} = \sum_\beta x_{\beta s} + \sum_\gamma x_{\gamma(s+1)} \sim 0, \qquad (s = 1, 2, \cdots, n), \tag{6.1}$$

where the marks $x_{\beta s}$ in the first sum of the sth homology correspond to the crossing points x_β that are incident without dot to the region R_σ and the marks $x_{\gamma(s+1)}$ in the second sum correspond to the crossing points x_γ that are incident with dot to R_σ. The marks $y_{\sigma s}$ will be the base marks of a certain sub-system Y of X. By definition, the *torsion numbers* of the knot corresponding to the given determination of the integer n will be the characteristic invariants of the system X with reference to its sub-system Y. In § 9, we shall prove that the torsion numbers, as here defined, have a genuine topological significance.

7. We shall now illustrate by an example the actual process of computing the torsion numbers of a knot. Consider the knot pictured in Fig. 1b.

Its diagram D has four crossing points x_α determining $4n$ marks $x_{\alpha a}$, and six regions R_σ determining $6n$ homologies. However, by a general theorem which we shall prove at the very end of the next section, the homologies determined by the outer region R_6 and by the region R_5 adjacent to the outer region are all expressible in terms of the homologies determined by the remaining four regions. Thus, the fundamental homologies reduce to

$$
\begin{aligned}
(7.1)\qquad & x_{2a} \sim -x_{1a}, && \text{(by } R_1\text{)},\\
& x_{3a} \sim -x_{2a} - x_{1(a+1)}, && \text{(by } R_2\text{)},\\
& \quad \sim x_{1a} - x_{1(a+1)}; && \\
& x_{4a} \sim -x_{1a} - x_{2(a+1)}, && \text{(by } R_3\text{)},\\
& \quad \sim -x_{1a} + x_{1(a+1)}; &&
\end{aligned}
$$

and

$$(7.2)\qquad x_{4(a+1)} + x_{3a} + x_{2(a+1)} \sim 0, \qquad \text{(by } R_4\text{)}.$$

Moreover, by means of (7.1) we may eliminate the marks x_{2a}, x_{3a}, and x_{4a} in (7.2) and thereby reduce the fundamental homologies to a set of n only,

$$(7.3)\qquad x_{1(a+2)} - 3x_{1(a+1)} + x_{1a} \sim 0, \qquad (a = 1, 2, \cdots, n),$$

in the n marks x_{1a}.

For $n = 2$, $x_{1(a+3)}$ and x_{1a} are equal, hence the homologies (7.3) become

$$(7.4)\qquad 2x_{12} - 3x_{11} \sim 0, \quad -3x_{12} + 2x_{11} \sim 0.$$

Therefore, since the matrix of the coefficients in (7.4) is of rank 2 and has the elementary divisors 1 and 5, there is a single torsion number equal to 5.

For $n = 3$, the homologies (7.3) become

$$
\begin{aligned}
x_{13} - 3x_{12} + x_{11} &\sim 0,\\
x_{13} + x_{12} - 3x_{11} &\sim 0,\\
-3x_{13} + x_{12} + x_{11} &\sim 0.
\end{aligned}
$$

This time, the rank of the matrix is 3, and the elementary divisors are 1, 4, 4. Therefore, the torsion numbers are 4, 4.

A table of knots with their invariants for $n = 2$ and $n = 3$ will be given in § 11 at the end of the paper.

8. We shall now derive a few general theorems about knot diagrams. In the notation of § 6, let $x_{\alpha a}$ be the marks determined by the crossing points x_α of a diagram D and let

$$\sum_{\beta} x_{\beta s} + \sum_{\gamma} x_{\gamma(s+1)} \sim 0 \tag{8.1}$$

be the fundamental homologies determined by the regions R_σ of D. Then, if we make the change of marks

$$x_{\alpha s} = x'_{\alpha(n-s+1)}, \qquad \begin{pmatrix} \alpha = 1, 2, \cdots, \nu \\ s = 1, 2, \cdots, n \end{pmatrix},$$

which has the effect of reversing the normal cyclical order of the second subscripts s, the homologies (8.1) go over into the homologies

$$\sum_{\gamma} x'_{\gamma t} + \sum_{\beta} x'_{\beta(t+1)} \sim 0, \qquad (t = n - s).$$

But these last are precisely the homologies corresponding to the inverse diagram D^{-1} of D, for to every dotted corner of the diagram D there corresponds an undotted corner of the inverse diagram D^{-1} and to every undotted corner of D, a dotted corner of D^{-1}. We, therefore, have the following theorem.

A knot and its inverse have the same torsion numbers.

We notice, next, that to each region R_σ of the diagram D there corresponds a number, called the *index* of the region, which measures the number of times, algebraically speaking, that the sensed polygon Π of the diagram winds around the region R_σ in a counter clockwise manner. The index of a region is, thus, a certain integer, which may be positive, negative, or zero. Obviously, if two regions meet along an edge, their indices differ by unity. Moreover, if four regions meet at a crossing point x_α and if the one of lowest index is of index $k-1$, two of the others must be of index k and the remaining one of index $k+1$ (cf. Fig. 3). At a crossing point such as x_α the corner of the region of index $k-1$ is always dotted and the corner of the region of index $k+1$ undotted. One, but not both of the corners belonging to the regions of index k is dotted. We shall say that the crossing point x_α is of *index* k (where k is thus the average of the indices of the regions incident to x_α).

Now, if at each crossing point of index $k (k = 1, 2, \cdots)$, we remove the dot from the corner of one region of index k and place it in the corner of the other region of index k, the effect will be to interchange upper and lower branches at all the crossing points, and, therefore, to transform the diagram D into the reflection of D, (§ 2). On the other hand, if at each crossing point of index $k (k = 1, 2, \cdots)$, we remove the dot from the corner of the region of index $k-1$ and place it in the corner of the region of index $k+1$, the effect will be not only to interchange upper and lower branches but, also, to reverse the positive sense of description of the polygon π. Therefore, the diagram D will be transformed into the inverse of the reflection of D. Incidentally, also, the indices of all regions and crossing points will be changed in sign owing to the change of direction on the polygon π. In view of the above remarks, it will now be easy to prove the following theorem:

A knot and its reflection have the same torsion numbers.

For let k be the index of a region R_σ of D. Then, the corners of the region R_σ may be divided into four classes: (i) Undotted corners x_α at crossing points of index $k+1$; (ii) undotted corners x_β at crossing points of index k; (iii) dotted corners x_γ at crossing points of index k; (iv) dotted corners x_δ at crossing points of index $k-1$. The n homologies determined by the region R_σ are, therefore, of the form

$$\sum_\alpha x_{\alpha s} + \sum_\beta x_{\beta s} + \sum_\gamma x_{\gamma(s+1)} + \sum_\delta x_{\delta(s+1)} \sim 0. \tag{8.2}$$

But, suppose we make the change of marks

$$x_{\alpha i} = x'_{\alpha(i+k_\alpha)},$$

where k_α denotes the index of the crossing point x_α. Then the homologies (8.2) go over into the homologies

$$\sum_\alpha x'_{\alpha(t+1)} + \sum_\beta x'_{\beta t} + \sum_\gamma x'_{\gamma(t+1)} + \sum_\delta x'_{\delta t} \sim 0, \qquad (t = s+k). \tag{8.3}$$

which are precisely the ones we should obtain if we were to remove the dots from the corners of the region R_σ at crossing points of index $k+1$ and dot the corners at crossing points of index $k-1$. That is to say, the homologies (8.3) are precisely the ones determined by the region R_σ in the inverse of the reflection of the diagram D. Hence, in view of the previous theorem, the homologies determined by a diagram are equivalent to the homologies determined by the reflection of the diagram, which completes the argument.

The next theorem is intended, primarily, to shorten the labor involved in computing the torsion numbers of a knot:

The homologies determined by any two regions with consecutive indices k and $k+1$ are consequences of the homologies determined by the remaining ν regions of the diagram.

The proof of the theorem reduces, essentially, to the determination of certain identical relations between the homologies

$$y_{\sigma s} \sim 0 \tag{8.4}$$

corresponding to the different regions R_σ of the diagram. A first set of identities will be the following:

$$(8.5) \qquad H_s = \sum_\sigma \varepsilon_\sigma y_{\sigma s} \sim 0,$$

where each coefficient ε_σ is equal to $+1$ or -1 according as the index of the corresponding region R_σ is even or odd. To verify the identical character of these relations we observe that at each crossing point x_α there are two dotted and two undotted corners, and that one corner of each sort belongs to a region of odd index the other to a region of even index. Because of this fact, the expressions H_s will be sums of groups of terms of the form

$$x_{\alpha s} - x_{\alpha s} + x_{\alpha(s+1)} - x_{\alpha(s+1)} \sim 0,$$

and will, therefore, vanish identically.

A second set of identical relations between the homologies (8.4) will next be determined. Corresponding to each region R_σ of non-negative index k, let us form the sums

$$z_{\sigma s} = y_{\sigma s} + y_{\sigma(s+1)} + \cdots + y_{\sigma(s+k-1)} \sim 0$$

and corresponding to each region R_σ of negative index $-k$, the sums

$$z_{\sigma s} = y_{\sigma(s-k)} + y_{\sigma(s-k+1)} + \cdots + y_{\sigma(s-1)} \sim 0.$$

Then, in terms of the sums $z_{\sigma s}$ we may write the identities

$$(8.6) \qquad G_s = \sum_\sigma \varepsilon_\sigma z_{\sigma s} \sim 0,$$

where the coefficients ε_σ have the same meaning as in (8.5). To prove that the relations (8.6) are identities, let us calculate how the marks corresponding to any crossing point x_α enter into the expressions G_s. We shall suppose, to fix matters, that the index of the crossing point x_α is positive and odd. Then, corresponding to the corner of the region of index $k-1$ ncident at x_α we shall have the set of marks

$$x_{\alpha(s+1)} + x_{\alpha(s+2)} + \cdots + x_{\alpha(s+k-1)},$$

corresponding to the corners of the two regions of indices k the sets of marks

$$-x_{\alpha s} \quad - x_{\alpha(s+1)} - \cdots - x_{\alpha(s+k-1)},$$

and

$$-x_{\alpha(s+1)} - x_{\alpha(s+2)} - \cdots - x_{\alpha(s+k)},$$

respectively, corresponding to the corner of the region of index $k+1$ the set of marks

$$x_{\alpha s}+x_{\alpha(s+1)}+\cdots+x_{\alpha(s+k)}.$$

But the sum of all these marks vanishes, therefore the expressions G_s contain no marks corresponding to crossing points with positive, odd indices. By a similar argument, we may show that the expressions G_s contain no marks corresponding to crossing points with indices that are not positive and odd. Therefore, the expressions G_s must vanish identically.

Now, by means of the identities (8.5) we may express the homologies determined by any region of index zero (or of index k, for that matter) in terms of the homologies determined by the remaining $\nu+1$ regions of the diagram. Morever, by means of the identities (8.6), we may express the homologies determined by any region of index 1 in terms of the homologies determined by the remaining regions of the diagram *of indices greater than zero.* For the identities (8.6) involve none of the marks $y_{\sigma s}$ corresponding to the regions R_σ of index 0, and each of them involves only one mark $y_{\sigma s}$, with coefficient -1, corresponding to each region R_σ of index 1.

Therefore, from the combined relations (8.5) and (8.6) we may express the homologies determined by any one region of index D together with the homologies determined by any one region of index k in terms of the homologies determined by the remaining ν regions of the diagram. This proves the theorem for the special case $k=0$. To prove it for a general value of k, we have only to redefine indices so as to lower all their values by the same constant k and proceed as before.

As a consequence of the last theorem, if we wish to compute the torsion numbers of a knot we may disregard the homologies determined by any pair of regions of the knot diagram that are contiguous along an edge. For, as we have already remarked, the indices of two such regions differ by unity. Thus, we have proved the theorem assumed in § 7.

9. Our next objective will be to prove that the torsion numbers of a knot are topological invariants. To do this, it will be sufficient for us to show that the numbers are unchanged when the knot undergoes an elementary deformation of any one of the four classes pictured in Fig. 2, because we have already proved that the most general type preserving transformation of a knot may be factored into elementary deformations of precisely the four classes in question.

A deformation of Class I obviously leaves the torsion numbers invariant, since it does not disturb the incidence relations between the regions and crossing points of the diagram. A deformation of Class II creates a new crossing point x_0 (Fig. 2 II), corresponding to which we shall have n new marks x_{0i}, It also creates a new region R_0 incident to the crossing point x_0 and to no other, corresponding to which we shall have n new homologies

(9.1) $$x_{0i} \sim 0.$$

If we use the homologies (9.1) to eliminate the new marks x_{0i} we find that the system of marks and homologies determined by the diagram after deformation reduces to the system of marks and homologies determined by it before deformation; therefore, the torsion numbers are invariant under the deformation. A deformation of Class III creates two new crossing points x_0' and x_0'', (Fig. 2 III). We may assume that the branch ACB passes over the branch $x_0' x_0''$, otherwise we could replace the diagram by its reflection, and that the corners are dotted as indicated in the figure, otherwise we could replace the diagram by its inverse. Now, it will be observed that two new regions R' and R'' are created, (among others), such that the first is incident to the crossing point x_0' but not to the crossing point x_0'', whereas the second is incident to x_0'' but not to x_0'. Let

(9.2) $$x_{0i}' + \psi_i' \sim 0$$

and

(9.3) $$x_{0i}'' + \psi_i'' \sim 0$$

be the homologies determined by the regions R' and R'' respectively, where ψ_i' and ψ_i'' are linear combinations of marks determined by the original crossing points. Then, if we use the homologies (9.2) and (9.3) to eliminate the new marks x_{0i}' and x_{0i}'', we may again verify that the system of marks and homologies after deformation reduces to the system of marks and homologies before deformation. Therefore, in this case also, the torsion numbers are invariant. A deformation of Class IV destroys two crossing points x_1 and x_2, and creates two new ones x_1' and x_2', (Fig. 2 IV). Three cases can arise according as the branch AB passes over both, one, or neither of the branches $x_1\ x_1'$ and $x_2 x_2'$. However, the third case reduces to the first by a reflection of the diagram, leaving only the first two to be considered. If the branch AB passes over the two branches $x_1 x_1'$ and $x_2 x_2'$ the branches AC and BC must pass over the branches $x_1 x_1'$ and $x_2 x_2'$ respectively. We may, therefore, assume that the corners at the crossing points x_1, x_1', x_2, and x_2' are dotted in the manner indicated in the figure, otherwise we could replace the diagram by its inverse. There are, however, two essentially different ways in which the corners at the crossing point x_0 may be dotted. If they are dotted in the manner indicated in Fig. 2 IV, the change of marks

$$
\begin{aligned}
x_{0i} &= x'_{1(i+1)} + x'_{2(i+1)}, \\
x_{1i} &= x'_{0(i+1)} + x'_{2i}, \\
x_{2i} &= x'_{0i} \quad\;\; + x'_{1i}
\end{aligned} \tag{9.4}
$$

will transform the marks and homologies before deformation into the marks and homologies after deformation, as may easily be verified. It may happen, however, that the corner of the triangle $x_0 x_1 x_2$ at the crossing point x_0 is not dotted but that the opposite corner at x_0 is. In this case, we must make the change of marks

$$
\begin{aligned}
x_{0i} &= x'_{1i} \quad\;\; + x'_{2i}, \\
x_{1i} &= x_{0(i+1)} + x'_{2i}, \\
x_{2i} &= x'_{0i} \quad\;\; + x'_{1i}
\end{aligned} \tag{9.5}
$$

instead of the change (9.4). There are two other possible ways in which the corners at the crossing point x_0 may be dotted, but each reduces to one of the cases already considered by a mere change of notation.

If the branch AB passes over one, but not both, of the branches $x_1 x'_1$ and $x_2 x'_2$ we may assume that it passes over the branch $x_1 x'_1$, otherwise, we could replace the diagram by its reflection. We may also assume that the positive direction along the edge AB is the direction from B to A, otherwise we could replace the diagram by its inverse. The positive direction along the branch $x_2 x'_2$ (which, in this case, passes over the branches AB, $x_1 x'_1$, and AC) may now be either the direction from x_2 to x'_2 or the direction from x'_2 to x_2. If the positive direction is from x_2 to x'_2, we make the change of marks

$$
\begin{aligned}
x_{0i} &= x'_{1(i+1)} + x'_{2i}, \\
x_{1i} &= x'_{0(i+1)} + x'_{2(i+1)}, \\
x_{2i} &= x'_{i} \quad\;\; + x'_{1i},
\end{aligned} \tag{9.6}
$$

if the positive direction is from x'_2 to x_2 we make the change of marks

$$
\begin{aligned}
x_{0i} &= x'_{1i} + x'_{2i}, \\
x_{1i} &= x'_{0i} + x'_{2i}, \\
x_{2i} &= x'_{0i} + x'_{1(i-1)}.
\end{aligned} \tag{9.7}
$$

The effect, in either case, will be to transform the system of marks and homologies before deformation into the system after deformation. Thus, a deformation of Class IV leaves the torsion numbers invariant. This completes the argument.

10. Before bringing the discussion to a close, we shall indicate how the torsion numbers of a knot K, for any given determination of the integer n, may be interpreted in terms of the Betti numbers and coefficients of torsion

of an n-sheeted covering space J_n with a branch curve of order $n-1$ covering the knot. The covering space J_n will be thought of as spread out upon the closed space J obtained by adding a point at infinity to the space of the knot and treating the point at infinity as a limit point of every unbounded set of points in the space of the knot.

The first step in the argument will be to make a cellular subdivision Σ of the carrier space J with a view to determining, later on, a subdivision of the covering space J_n. To each crossing point x_α of the knot diagram there correspond two points of the knot itself, a point P_α on an upper branch and a point Q_α on a lower branch. We shall choose the combined points P_α and Q_α as the 2ν vertices, or 0-cells, of the subdivision Σ. The edges, or 1-cells, of the subdivision will consist, first, of the ν linear segments $P_\alpha Q_\alpha$ joining the points of corresponding upper and lower branches of the knot, secondly, of the 2ν arcs into which the points P_α and Q_α subdivide the knot. There will, thus, be 3ν edges in all. The 2-cells of the subdivision will be a set of $\nu+1$ properly selected simply connected, regular surfaces S_σ, $(\sigma = 0, 1, \cdots, \nu)$, one associated with each region R_σ of the diagram with the exception of the outer region of all. Each surface S_σ will be so chosen that (i) it projects orthogonally in a point-for-point manner upon the region R_σ with which it is associated and that (ii) its boundary is a simple polygon made up of the portions of the knot which project upon the boundary of the region R_σ together with the edges $P_\alpha Q_\alpha$ which project upon the crossing points x_α on the boundary of R_σ. The subdivision Σ will have a single 3-cell C_3 consisting of all points of the space J that are not on any cell of lower dimensionality.

To the subdivision Σ of the carrier space J there corresponds a subdivision Σ_n of the covering space J_n such that each cell C of the subdivision Σ is covered by n superimposed cells of the subdivision Σ_n unless it so happens that the cell C is made up of points of the knot K. In the latter case, the cell C is covered by a single cell of the subdivision Σ_n, for the knot K determines the branch curve of the space J_n along which the n sheets of J_n merge into one. By a straightforward calculation, we find that the subdivision Σ_n comprises 2ν vertices, $\nu n + 2\nu$ edges, $n\nu + n$ 2-cells, and n 3-cells.

The indices k_σ assigned to the regions R_σ of the diagram (§ 7) have a direct bearing on the incidence relations among the cells of the subdivision Σ_n. Let S_σ be the 2-cell of the subdivision Σ determined by the region R_σ, and let

$$S_{\sigma i}, \qquad (i = 1, 2, \cdots, n),$$

be the n 2-cells of the subdivision Σ_n that cover the 2-cell S_σ. Moreover, let

$$C_i, \qquad (i = 1, 2, \cdots, n),$$

be the n 3-cells of the subdivision Σ. Then, we are to think of C_i and C_{i+k_σ} as the two 3-cells incident to the 2-cell $S_{\sigma i}$. (Here, of course, the index $i+k_\sigma$ is to be reduced, modulo n, to an integer between 0 and $n+1$.) More precisely, we are to imagine that we pass from the cell C_i to the cell C_{i+k_σ} when we travel across the cell $S_{\sigma i}$ on a path leading up to $S_{\sigma i}$ from below.

In order to calculate the topological invariants of the covering space J_n with a minimum of effort, we shall replace the subdivision Σ_n by another composed of fewer cells. Suppose two distinct 3-cells of the subdivision Σ_n are incident to the same 2-cell. Then, we obviously have the right to combine together the 2-cell and the two 3-cells so as to form a single 3-cell. Suppose, moreover, that two distinct 0-cells are incident to the same 1-cell. Then, we also have the right to deform the elements of the subdivision in such a manner that the 1-cell and its two ends shrink up into a single 0-cell, or, what comes to the same thing, to treat the 1-cell and its two ends as a "generalized" 0-cell. The two simplifying operations described above are the duals of one another.

With these facts in mind, let R_0 be a region of the knot diagram which is adjacent along an edge to the very outer region of all and which, therefore, has the index ± 1. Then, corresponding to the region R_0 there will be n 2-cells,

$$S_{0s}, \qquad (s = 1, 2, \cdots, n),$$

of the subdivision Σ_n. As a first simplification, we shall merge the first $n-1$ 2-cells S_{0s} with the n 3-cells C_i to form a single 3-cell C' with a singular boundary. This amalgamation will merely involve $n-1$ repetitions of the first simplifying operation described above. On the boundary of the 3-cell C', the remaining 2-cell S_{0n} will appear twice, once in positive relation to the boundary and once in negative.

Now, let K' denote the circuit composed of the 0-cells and 1-cells of the subdivision Σ_n which cover the points of the knot K. There will then be one definite 1-cell x of the circuit K' which is on the boundary of the 2-cell S_{0n} but not on the boundary of any of the other 2-cells $S_{\sigma s}$ $(\sigma = 1, 2, \cdots, n)$, namely, the 1-cell which projects on the arc separating the region R_0 from the outer region of the diagram. As a second simplification of the subdivision we shall join together all the 0-cells and 1-cells of the circuit K' with the exception of the 1-cell x so as to form a single generalized 0-cell. This will involve $2\nu-2$ applications of the second simplification process.

After the two simplifications described above, we shall be left with a subdivision Σ_n' consisting of one generalized vertex, $\nu n+1$ edges, $\nu n+1$

2-cells, and one 3-cell. Moreover, each of the edges, when we include its ends, will be in the nature of a closed curve, or 1-circuit, since its two ends coincide at the one vertex present. One of the edges will be the 1-cell x on the boundary of the 2-cell S_{0n}, the others will be the 1-cells covering the segments $P_\alpha Q_\alpha$. We shall denote the latter by the marks $x_{\alpha a}$ respectively.

Now, let us write the homologies (in the sense of analysis situs) which express the fact that the boundary of each 2-cell of the subdivision Σ'_n is a linear combination of the 1-circuits determined by the respective 1-cells of Σ'_n. Then, it is easy to verify that if the 1-cells x_α are suitably sensed, the homology determined by the 2-cell S_{0n} is

$$x + y_{0n} \sim 0 \tag{10.1}$$

and that the homologies determined by the νn 2-cells $S_{\sigma n}$, $(\sigma > 0)$, are

$$y_{\sigma s} \sim 0, \tag{10.2}$$

where the symbols y_{0s} and $y_{\sigma s}$ have precisely the meanings attached to them in § 6. Moreover, since the relation

$$y_{0n} \sim 0$$

follows as a consequence of the relations

$$y_{\sigma n} \sim 0, \qquad (\sigma = 1, 2, \cdots, n),$$

(by a theorem in § 6), we may reduce the homology (10.1) to an homology

$$x \sim 0. \tag{10.3}$$

The geometrical significance of this last homology is that *the branch curve of the space J_n, is always a bounding 1-circuit of J_n*. For the 1-cell x and its "generalized" ends represent the entire branch curve. Finally, if we eliminate the mark x by means of the homology (10.3), we are left with the homologies (10.2) which determine the torsion numbers of the knot. The exact relation between the torsion numbers of the knot and the topological invariants of the covering space J_n may be expressed as follows:

There are exactly as many vanishing torsion numbers as there are 1-circuits of the space J_n linearly independent with respect to homologies. Moreover, the torsion numbers that are greater than zero are the coefficients of torsion of the space J_n.

11. We give below a table of invariants for all types of knots of nine or less crossings listed as distinct by Tait and Kirkman. The symbols appearing in the first column are the distinguishing symbols of the various knots pictured. In each symbol, the large numeral denotes the minimal

number of crossings that the diagram of the knot can have, while the indices a or n indicate whether or not a knot is alternating. The numbers in the second and third columns are the torsion numbers for the cases $n = 2$ and $n = 3$ respectively.

The two members of each of the three pairs of knots marked with the asterisks, crosses and zeros respectively have the same torsion numbers for $n = 2$ and also for $n = 3$. We have found, as a matter of fact, that they also have the same torsion numbers for each value of n so that it is impossible to distinguish them by their torsion numbers. In all other cases considered, two knots were distinguished by their torsion numbers.

Type	$N=2$	$N=3$	Type	$N=2$	$N=3$	Type	$N=2$	$N=3$
3_{1a}	3	2, 2	8_{15a}	33	16, 16	9_{22a}	43	14, 14
4_{1a}	5	4, 4	8_{16a}	35	11, 11	9_{23a}	45	22, 22
5_{1a}	5	none	8_{17a}	37	13, 13	9_{24a}	45	16, 16
5_{2a}	7	5, 5	8_{18a}	3, 15	2, 2, 8, 8	9_{25a}	47	26, 26
6_{1a}	9	7, 7	8_{19n}	5	4, 4	9_{26a}	47	17, 17
6_{2a}	11	5, 5	8_{20n}	9	4, 4	9_{27a}	49	19, 19
6_{3a}	13	7, 7	8_{21n}	15	8, 8	°9_{28a}	51	20, 20
7_{1a}	7	none	9_{1a}	9	none	°9_{29a}	51	20, 20
7_{2a}	11	8, 8	*9_{2a}	15	11, 11	9_{30a}	53	22, 22
7_{3a}	13	4, 4	9_{3a}	19	none	9_{31a}	55	23, 23
*7_{4a}	15	11, 11	9_{4a}	21	7, 7	9_{32a}	59	23, 23
7_{5a}	17	7, 7	9_{5a}	23	17, 17	9_{33a}	61	25, 25
7_{6a}	19	11, 11	9_{6a}	27	4, 4	9_{34a}	69	31, 31
7_{7a}	21	13, 13	9_{7a}	29	13, 13	9_{35a}	3, 9	20, 20
8_{1a}	13	10, 10	×9_{8a}	31	17, 17	9_{36a}	15	2, 2, 4, 4
8_{2a}	17	none	9_{9a}	31	5, 5	9_{37a}	3, 15	28, 28
8_{3a}	17	13, 13	9_{10a}	33	13, 13	9_{38a}	57	28, 28
8_{4a}	19	8, 8	9_{11a}	33	7, 7	9_{39a}	55	32, 32
8_{5a}	21	4, 4	9_{12a}	35	20, 20	9_{40a}	5, 15	4, 4, 8, 8
8_{6a}	23	11, 11	9_{13a}	37	16, 16	9_{41a}	7, 7	28, 28
8_{7a}	23	5, 5	9_{14a}	37	22, 22	9_{42n}	7	2, 2
8_{8a}	25	13, 13	9_{15a}	39	23, 23	9_{43n}	13	2, 2
8_{9a}	25	7, 7	9_{16a}	39	8, 8	9_{44n}	17	10, 10
8_{10a}	27	8, 8	9_{17a}	39	11, 11	9_{45n}	23	14, 14
8_{11a}	27	14, 14	9_{18a}	41	19, 19	9_{46n}	3, 3	7, 7
8_{12a}	29	19, 19	9_{19a}	41	25, 25	9_{47n}	3, 9	5, 5
8_{13a}	29	16, 16	9_{20a}	41	13, 13	9_{48n}	3, 9	17, 17
×8_{14a}	31	17, 17	9_{21a}	43	26, 26	9_{49n}	5, 5	10, 10

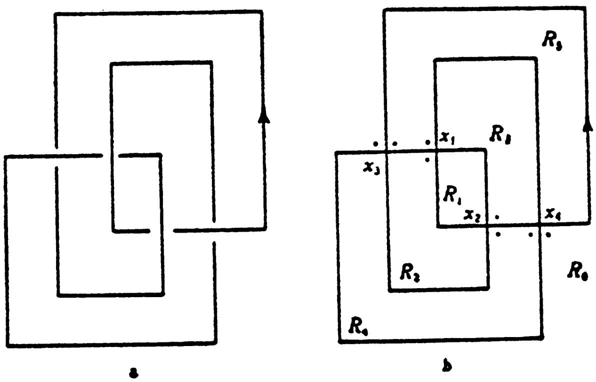

Fig. 1.

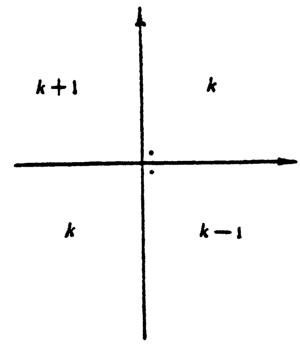

Fig. 3.

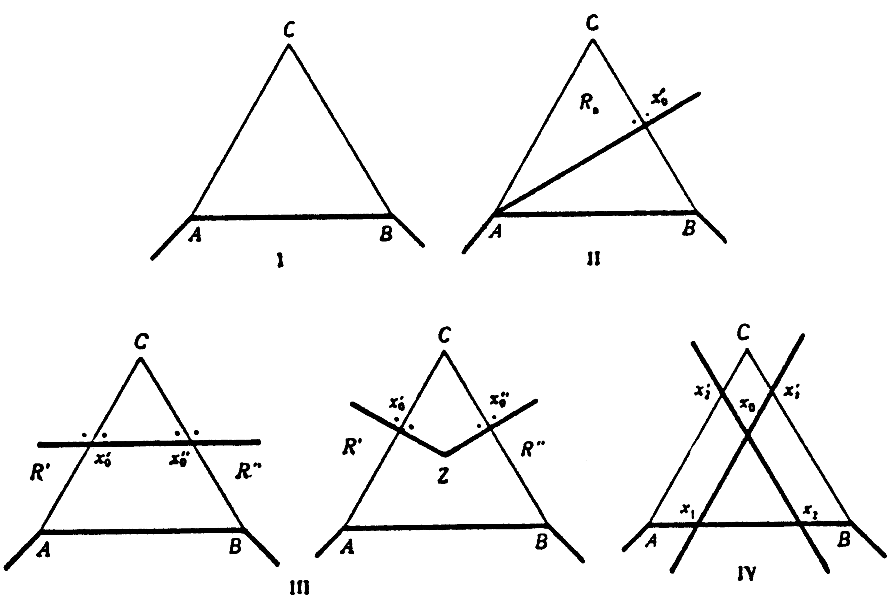

Fig. 2.

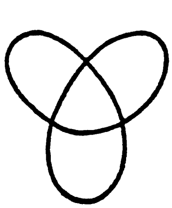

Fig. 3 1a.

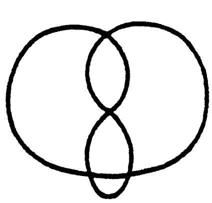

Fig. 4 1a.

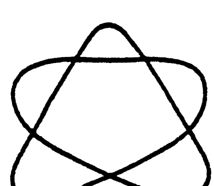

Fig. 5 1a.

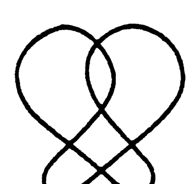

Fig. 5 2a.

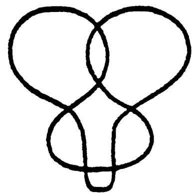
Fig. 6_1a.

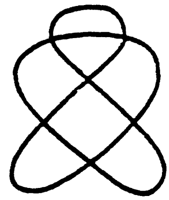
Fig. 6_2a.

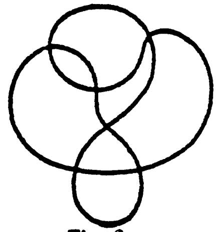
Fig. 6_3a.

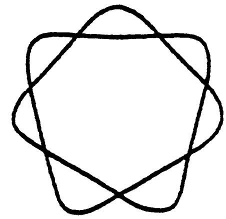
Fig. 7_1a.

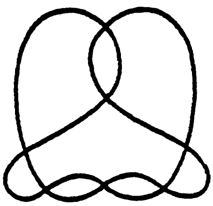
Fig. 7_2a.

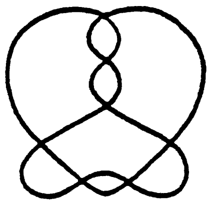
Fig. 7_3a.

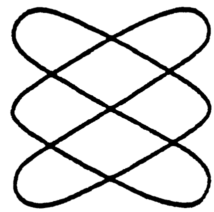
Fig. 7_4a.

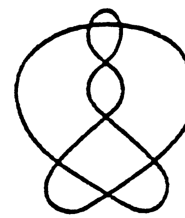
Fig. 7_5a.

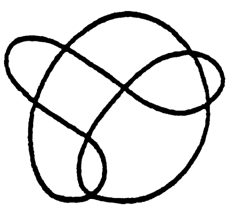
Fig. 7_6a.

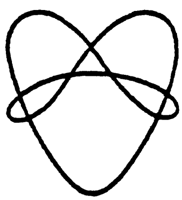
Fig. 7_7a.

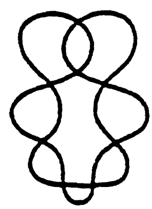
Fig. 8_1a.

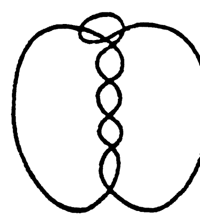
Fig. 8_2a.

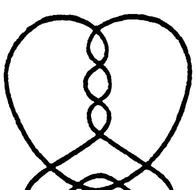
Fig. 8_3a.

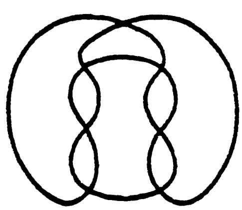
Fig. 8_4a.

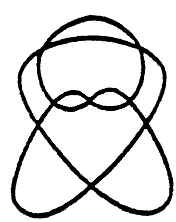
Fig. 8_5a.

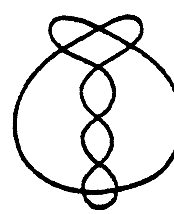
Fig. 8_6a.

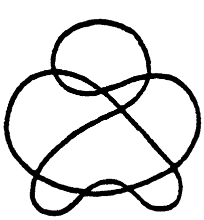
Fig. 8_7a.

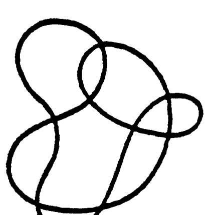
Fig. 8_8a.

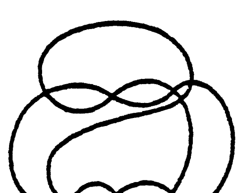
Fig. 8_9a.

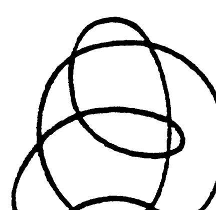
Fig. 8_{10}a.

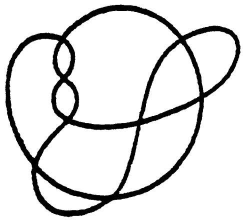

Fig. $8_{11}a$.

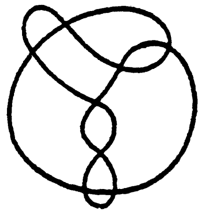

Fig. $8_{12}a$.

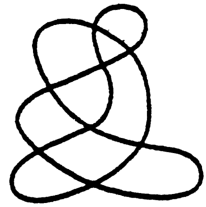

Fig. $8_{13}a$.

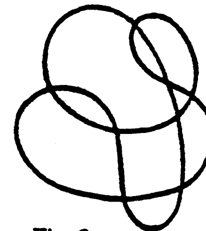

Fig. $8_{14}a$.

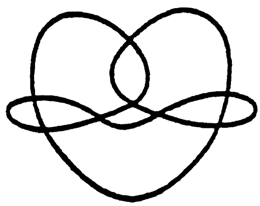

Fig. $8_{15}a$.

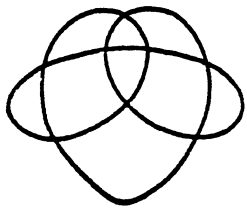

Fig. $8_{16}a$.

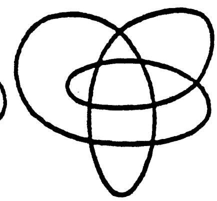

Fig. $8_{17}a$.

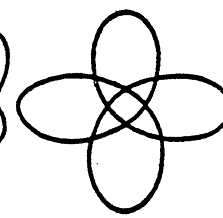

Fig. $8_{18}a$.

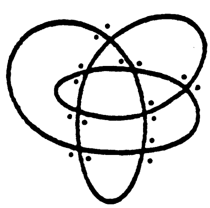

Fig. $8_{19}n$.

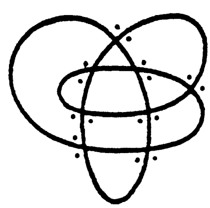

Fig. $8_{20}n$.

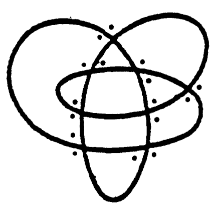

Fig. $8_{21}n$.

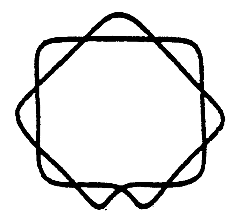

Fig. $9_{1}a$

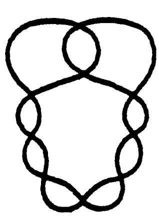

Fig. $9_{2}a$.

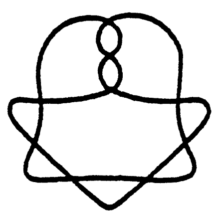

Fig. $9_{3}a$.

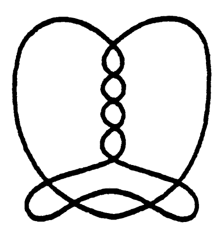

Fig. $9_{4}a$.

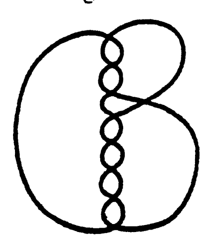

Fig. $9_{5}a$.

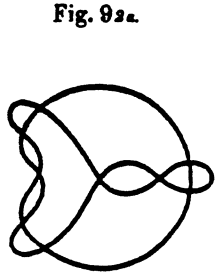

Fig. $9_{6}a$.

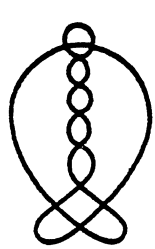

Fig. $9_{7}a$.

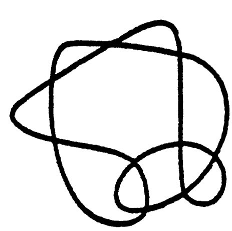

Fig. $9_{8}a$.

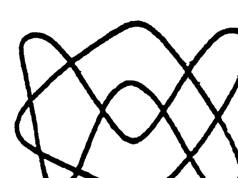

Fig. $9_{9}a$.

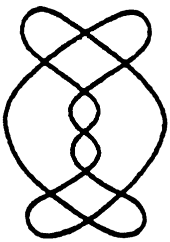
Fig. 910a.

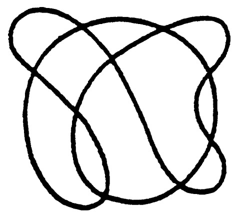
Fig. 911a.

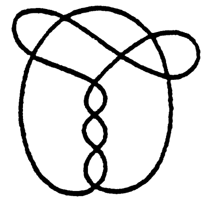
Fig. 912a.

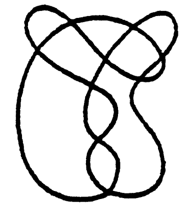
Fig. 913a.

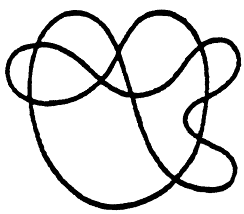
Fig. 914a.

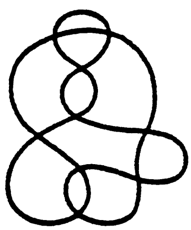
Fig. 915a

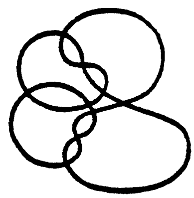
Fig. 916a.

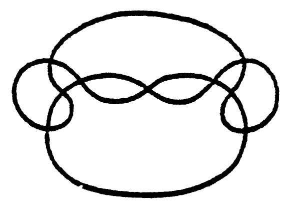
Fig. 917a.

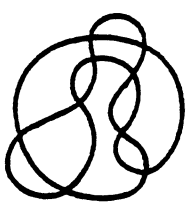
Fig. 918a.

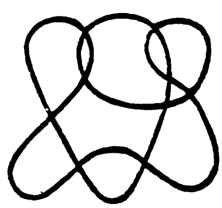
Fig. 919a.

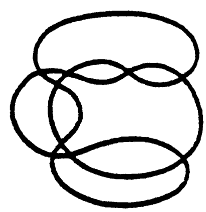
Fig. 920a.

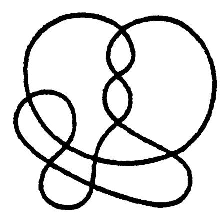
Fig. 921a.

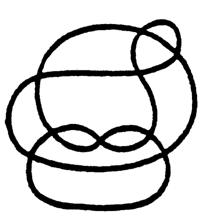
Fig. 922a.

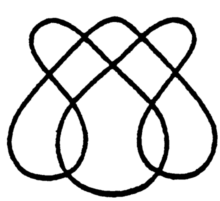
Fig. 923a.

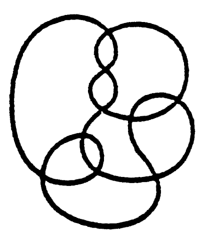
Fig. 924a.

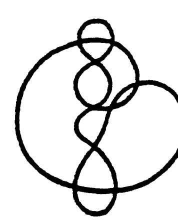
Fig. 925a.

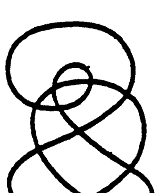
Fig. 926a.

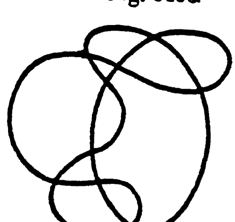
Fig. 927a.

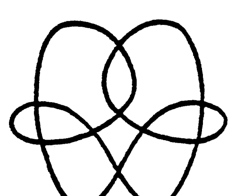
Fig. 928a

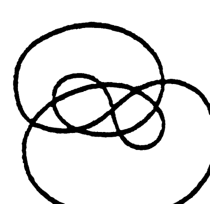
Fig. 929a.

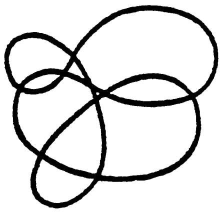
Fig. 930a.

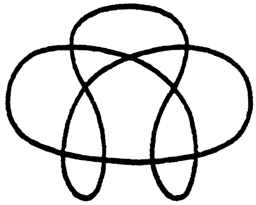
Fig. 931a.

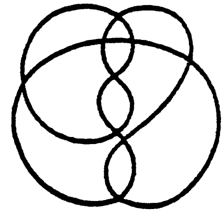
Fig. 932a.

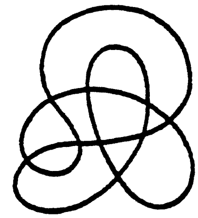
Fig. 933a.

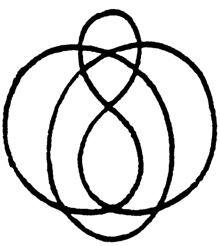
Fig. 934a.

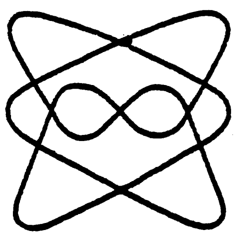
Fig. 935a.

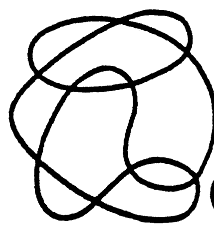
Fig. 936a.

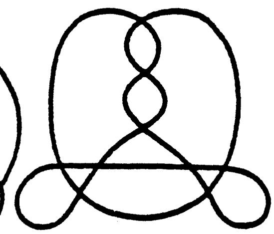
Fig. 937a.

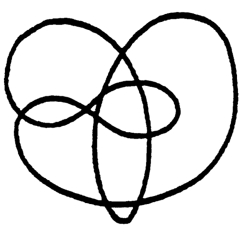
Fig. 938a.

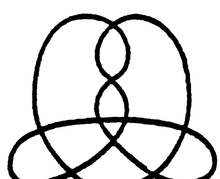
Fig. 939a.

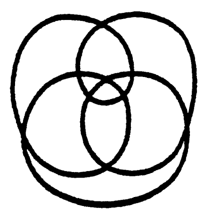
Fig. 940a.

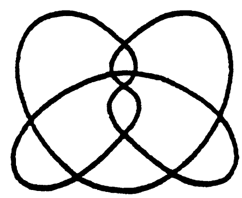
Fig. 941a.

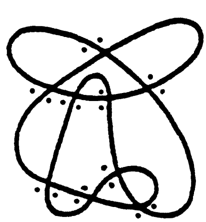
Fig. 942n.

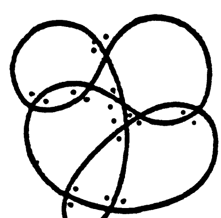
Fig. 943n.

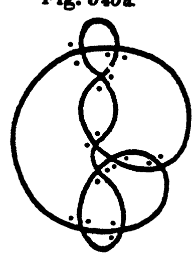
Fig. 944n.

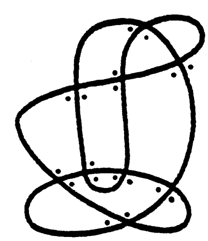
Fig. 945n.

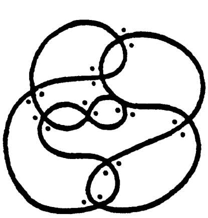
Fig. 946n.

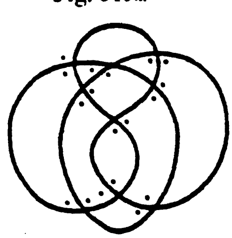
Fig. 947n.

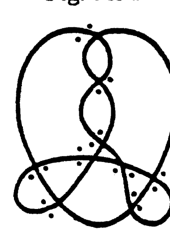
Fig. 948n.

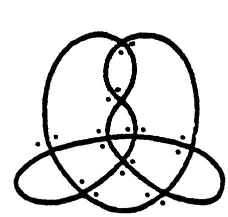
Fig. 949n.