Applications of algebra to a problem in topology

Joint work with

Mike Hill

and

Doug Ravenel

Pontryagin (1930’s)

Pontryagin (1930's)

cobordism group of stably framed k-manifolds $\longleftrightarrow$ $\pi_{n+k} S^n, n \gg 0$

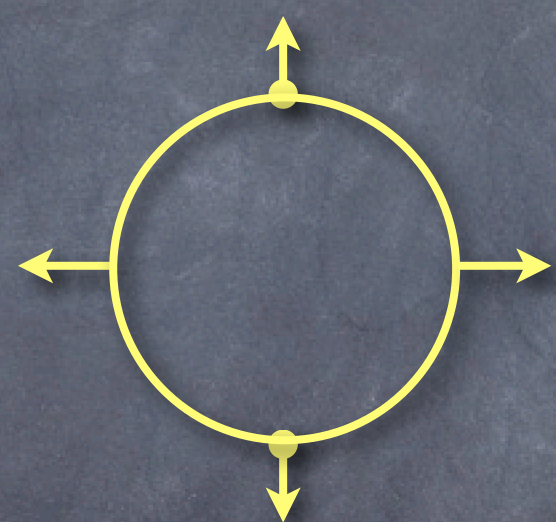
$\pi_k S^0$

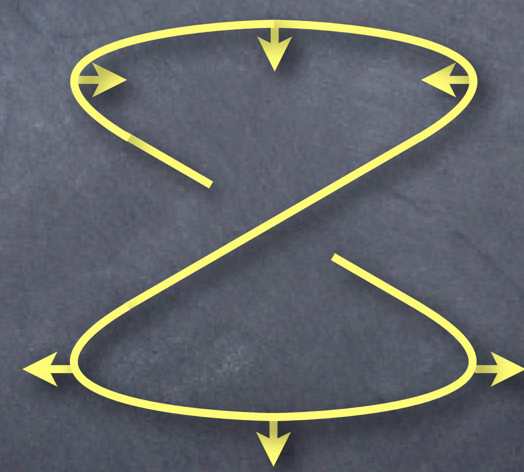
Pontryagin (1930's)

k=0

$$\pi_0 S^0 = \mathbb{Z}$$

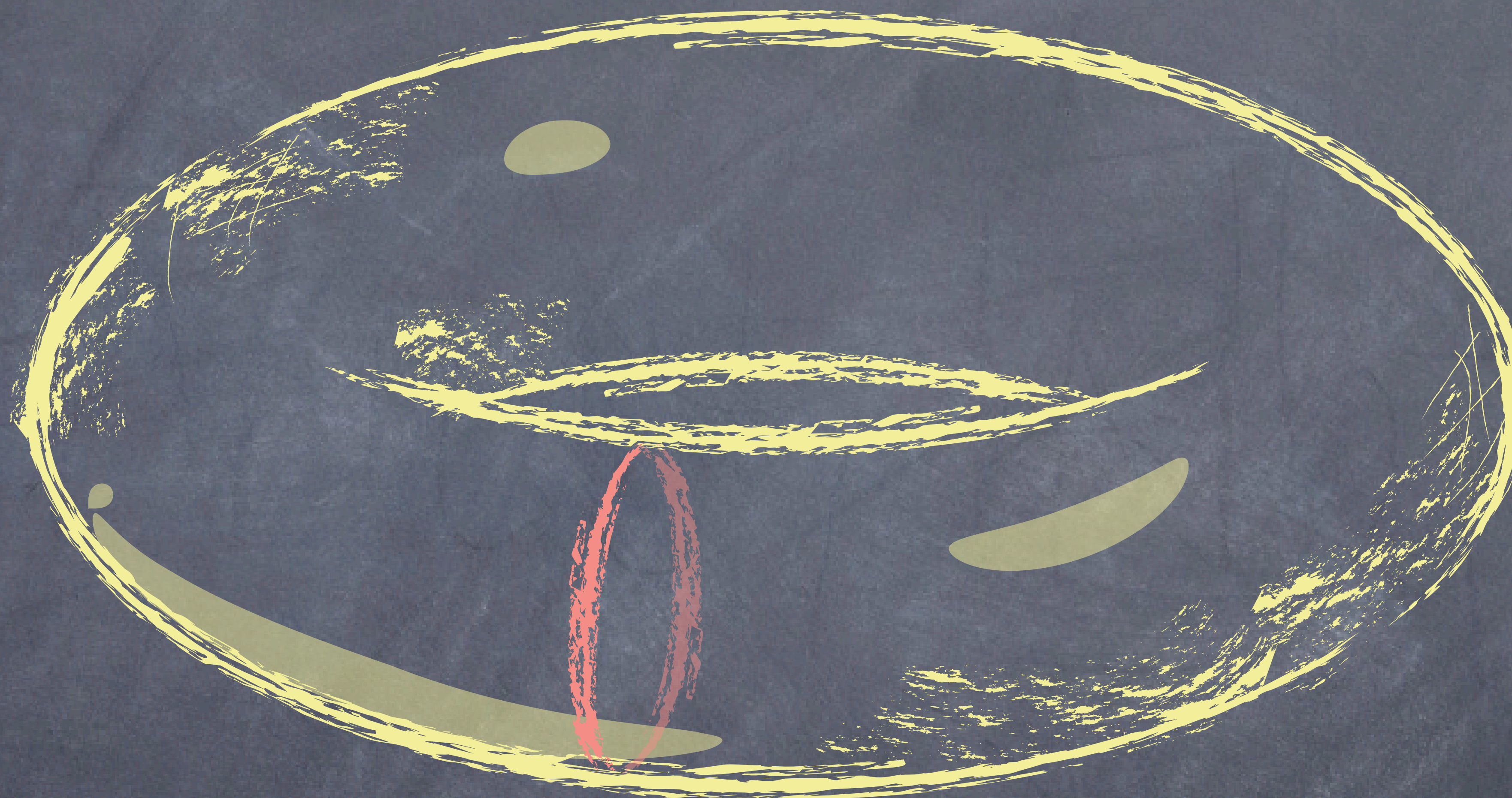
k=1



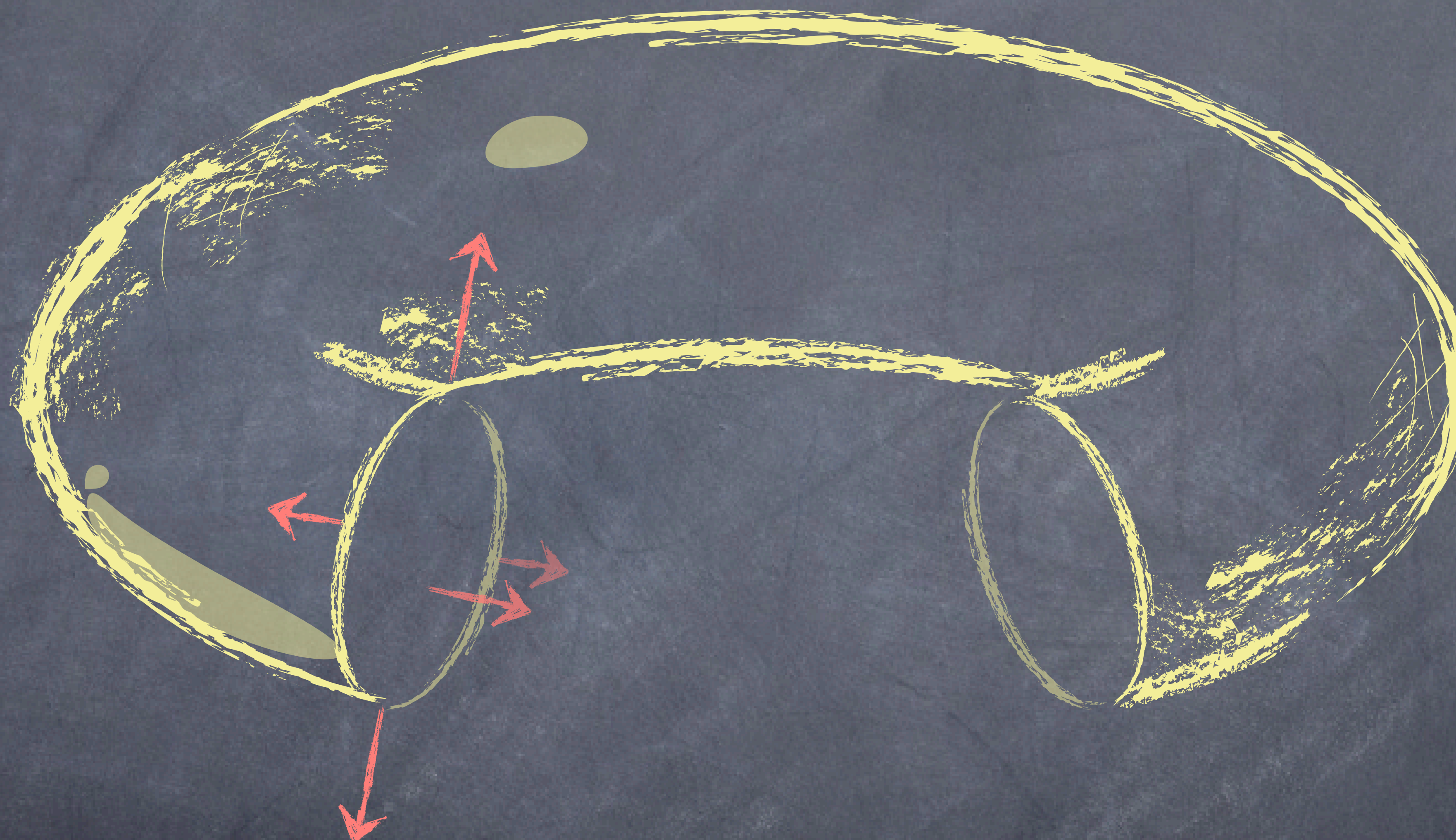


$$\pi_1 S^0 = \mathbb{Z}/2$$

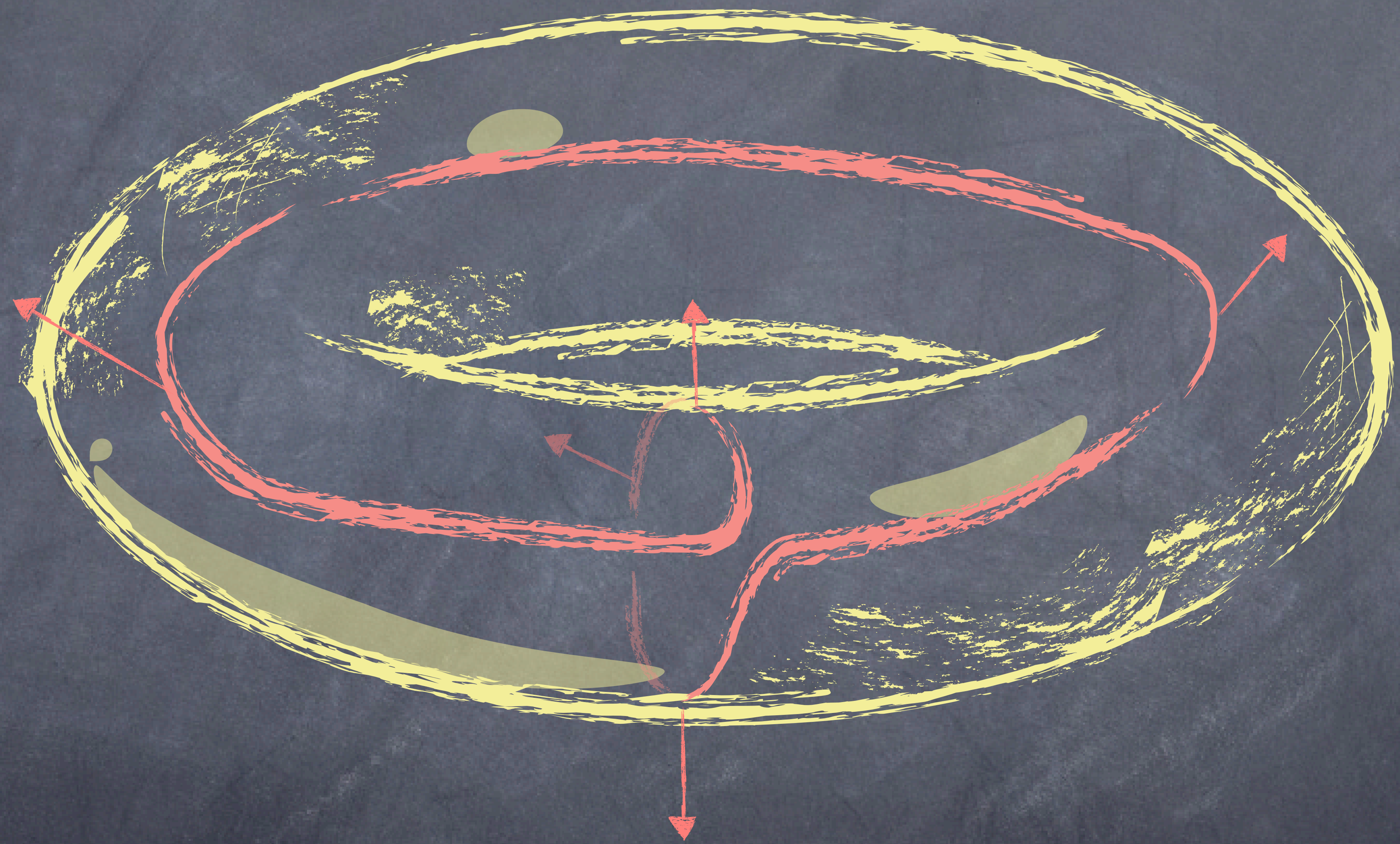
Pontryagin (1930s) k=2



Pontryagin (1930s)   k=2



Pontryagin (1930s)   k=2



Pontryagin (1930s) This defines a fuction

$$\varphi : H_1(M;\mathbb{Z}/2) \to \mathbb{Z}/2$$

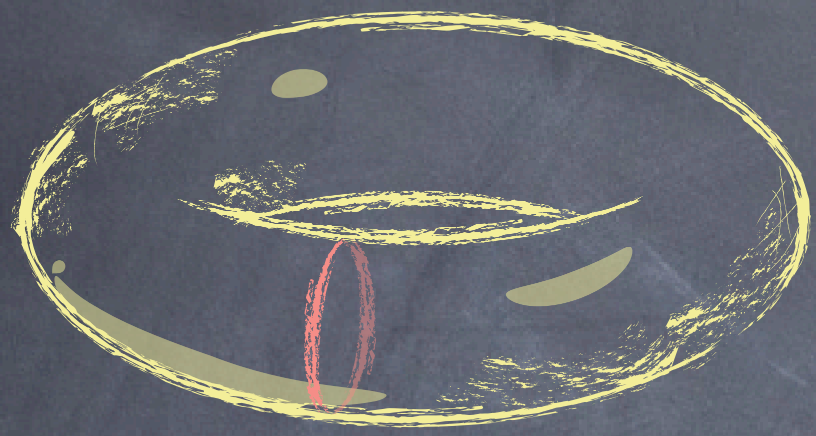
If genus $M > 0$, $\dim H_1(M) > 1$ and so

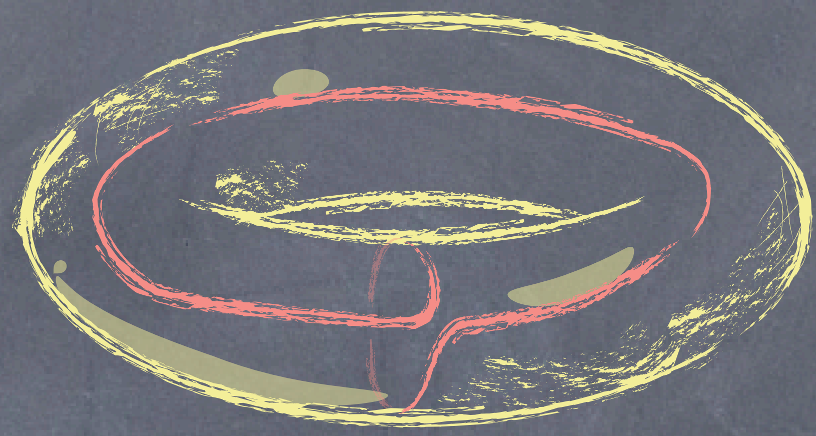
$$\ker \varphi \neq 0$$

You can always lower the genus with surgery

$$\pi_2 S^0 = 0$$

Pontryagin (?)





φ is not linear

it's quadratic and refines
the intersection pairing

Pontryagin (?)

$$\Phi(\Sigma) = Arf(\varphi)$$

$$\boxed{\pi_2 S^0 = \mathbb{Z}/2}$$

Kervaire (1960)

$$M = M^{4k+2}$$ (framed)

defined $\varphi : H^{2k+1}(M;\mathbb{Z}/2) \to \mathbb{Z}/2$

quadratic refinement of the intersection pairing

$$\Phi(M) = Arf(\varphi)$$

showed $\Phi(M^{10}) = 0$

Kervaire (1960)

produced a piecewise linear N^{10}

with $\Phi(N^{10}) \neq 0$

hence N^{10} has no smooth structure

Browder (1969)

$$n \neq 2^j - 1 \qquad \Phi(M^{2n}) = 0$$

$$n = 2^j - 1 \qquad \Phi(M^{2n}) \neq 0$$

$\iff$ there exists $\theta_j \in \pi_{2^{j+2}-2} S^0$

represented by $h_j^2 \in \mathrm{Ext}_A(\mathbb{Z}/2, \mathbb{Z}/2)$

Barratt-Jones-Mahowald (1969, 1984)

The elements θ_j exist for $j = 1, 2, 3, 4, 5$

dimensions $2, 6, 14, 30, 62$

so the first open dimension is **126**

The Kervaire invariant problem

In which dimensions can

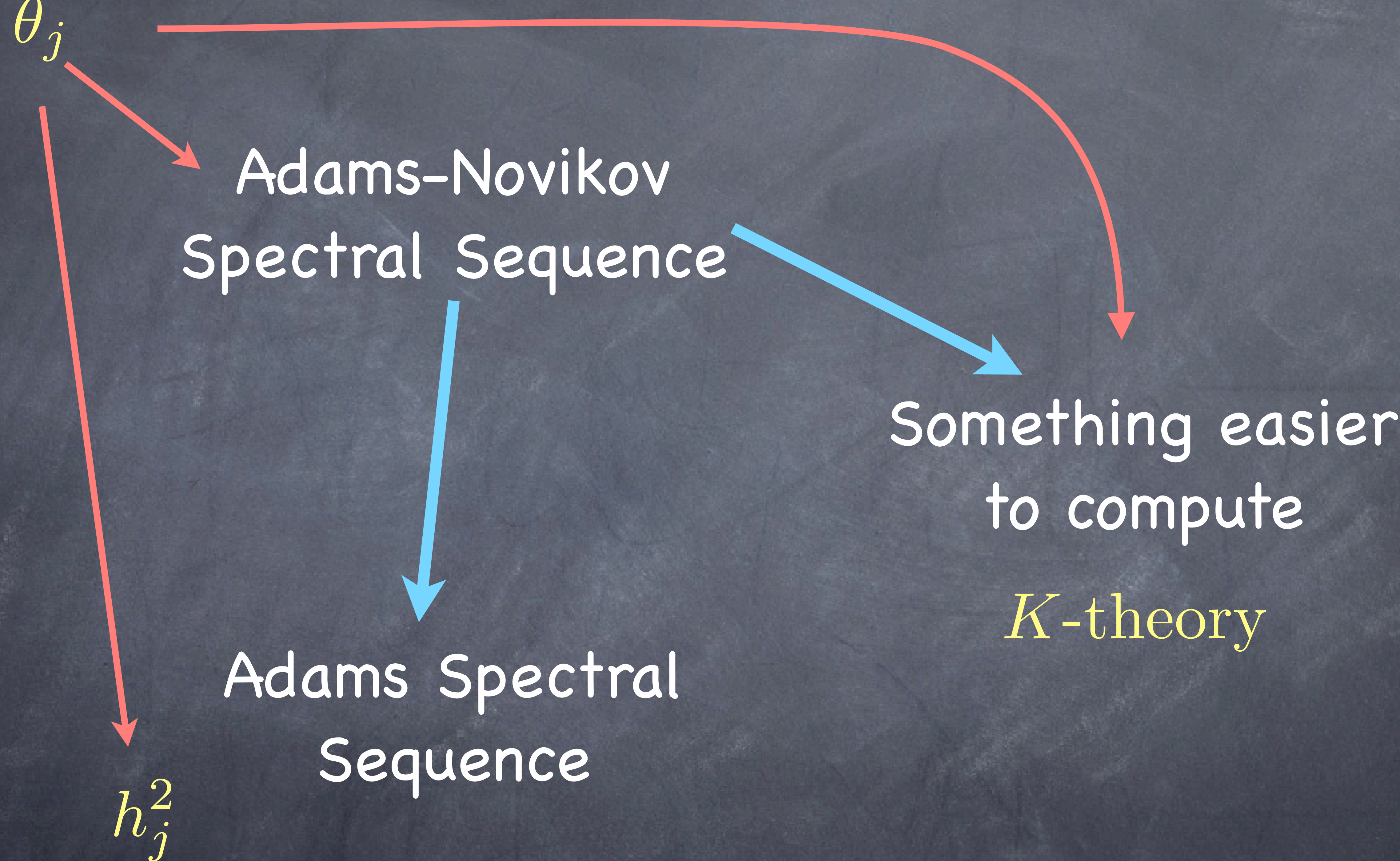
$$\Phi(M)$$

be non-zero?

Doomsday Theorem (Hill, H., Ravenel)

If $\Phi(M^n) \neq 0$ then $n = 2, 6, 14, 30, 62$ or 126

In other words θ_j does not exist for $j \geq 7$


θ_j
Adams-Novikov Spectral Sequence
Something easier to compute
K-theory
Adams Spectral Sequence
h_j^2

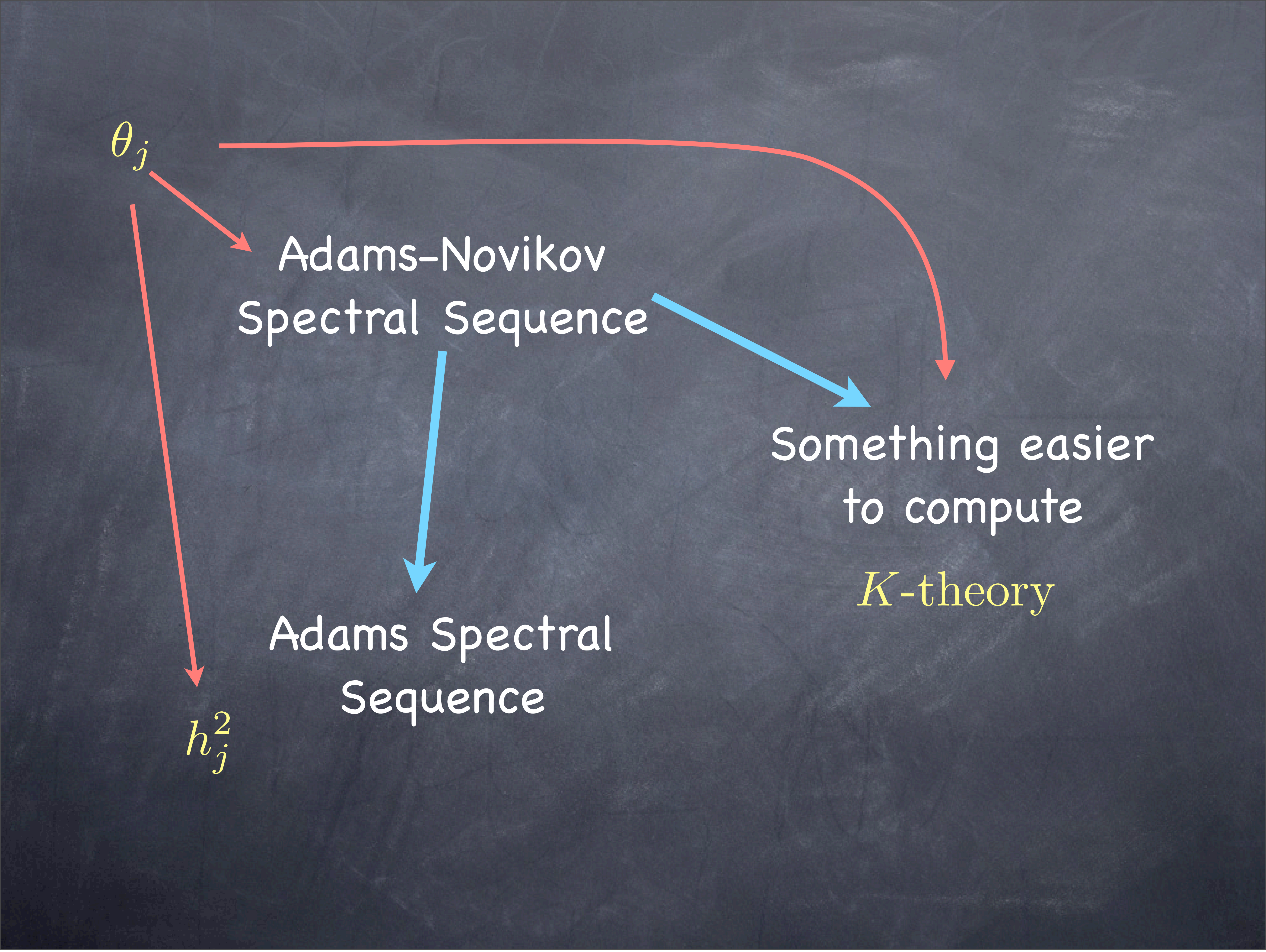
θ_j

$$\mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*) \implies \pi_{t-s}S^0$$

$$H^s(\mathbb{Z}/2; K_t) \implies KO_{t-s}$$

$$Ext^{s,t}_A(\mathbb{Z}/2, \mathbb{Z}/2) \implies \pi_{t-s}S^0$$

h_j^2


θ_j
Adams-Novikov
Spectral Sequence
Something easier
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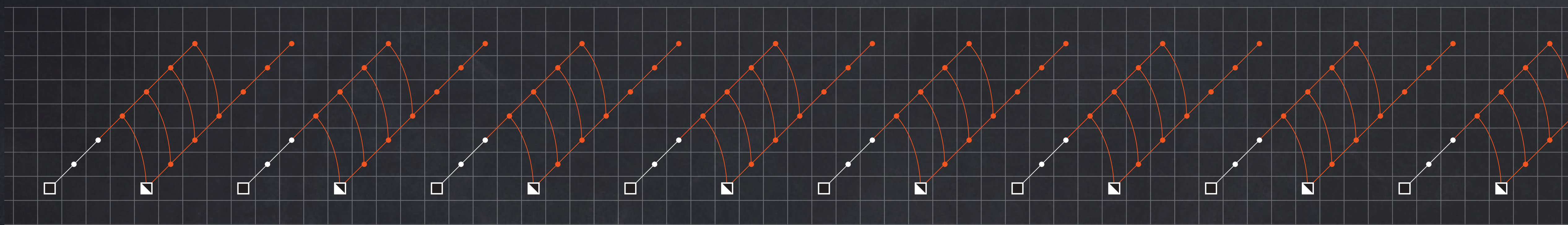
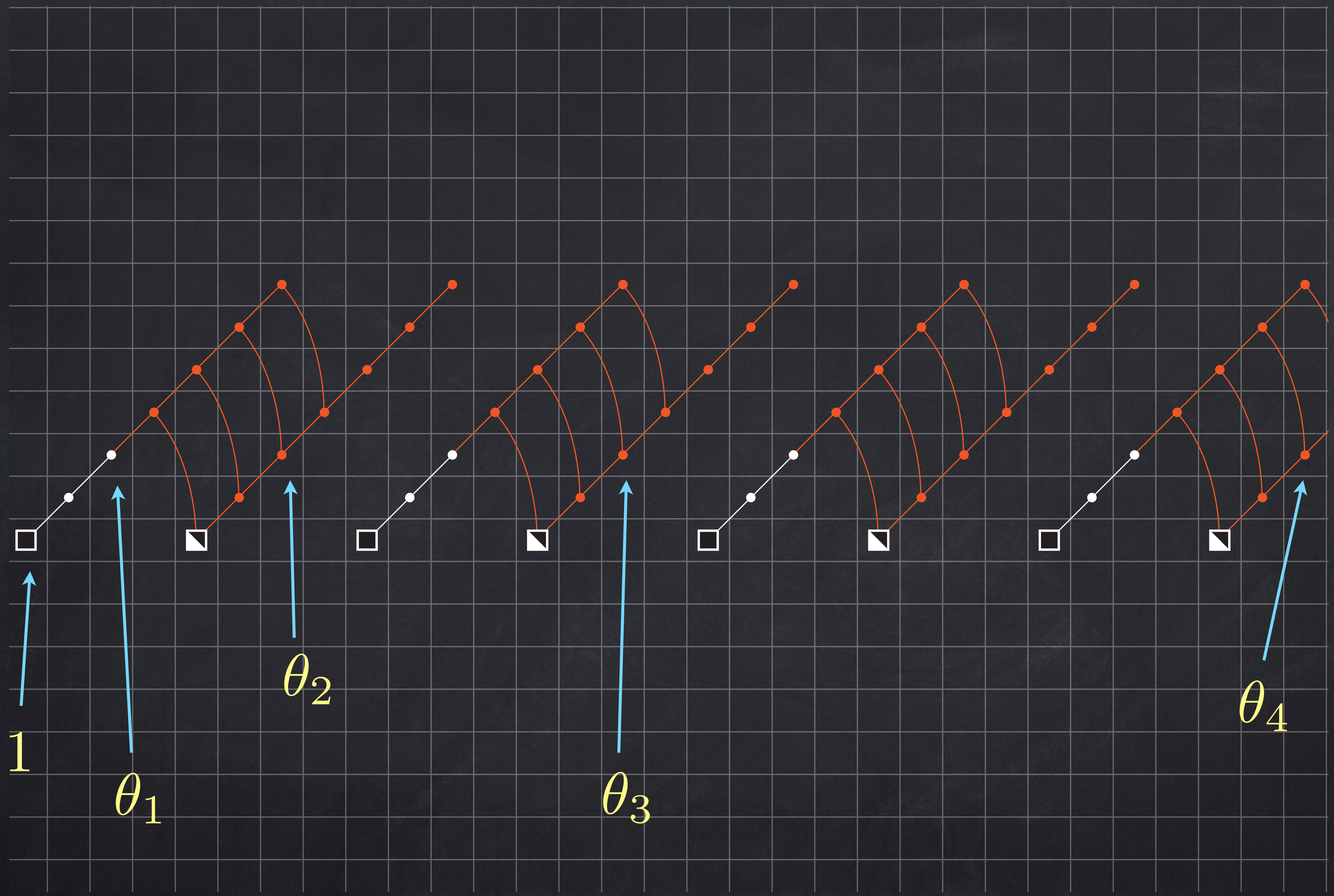
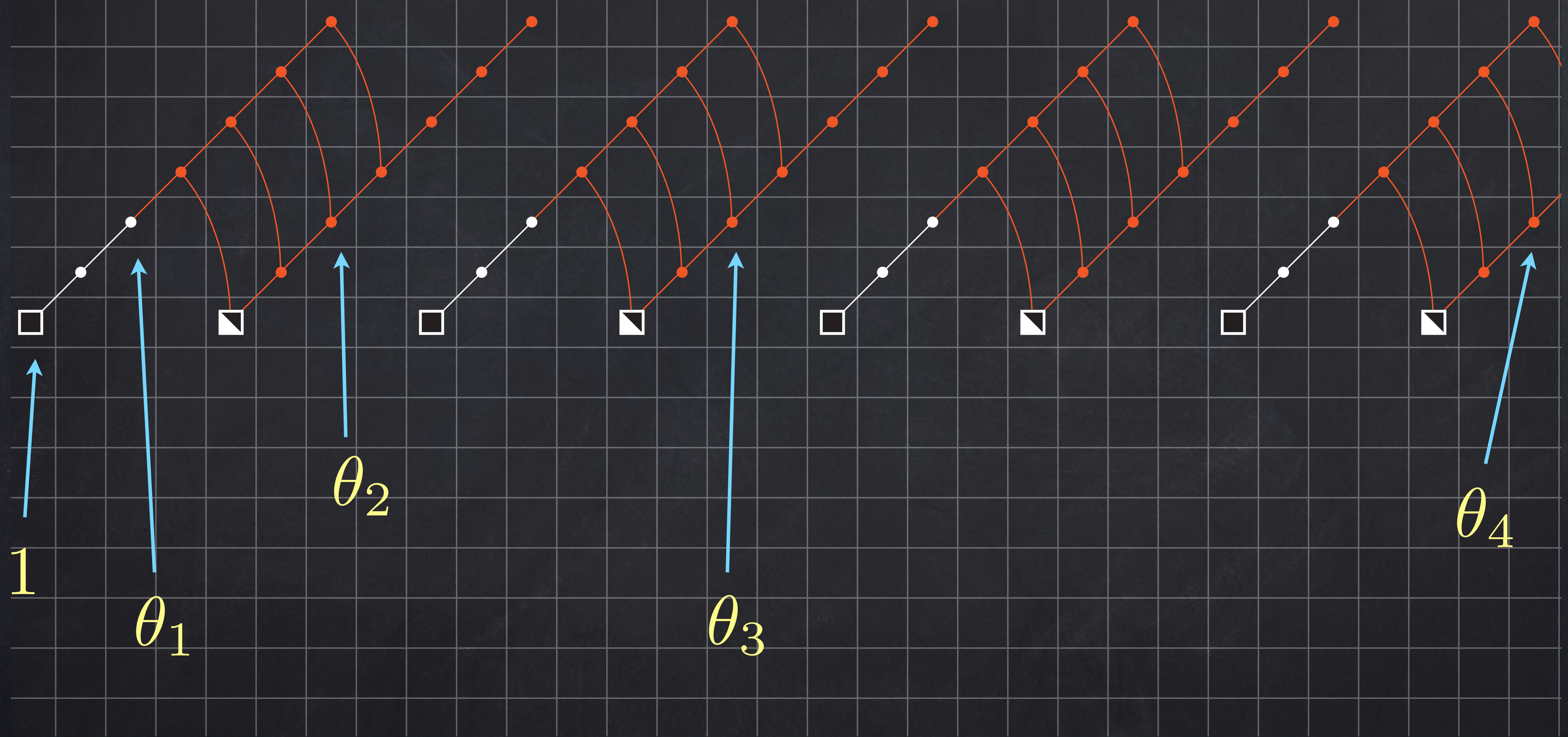
Sequence
h_j^2

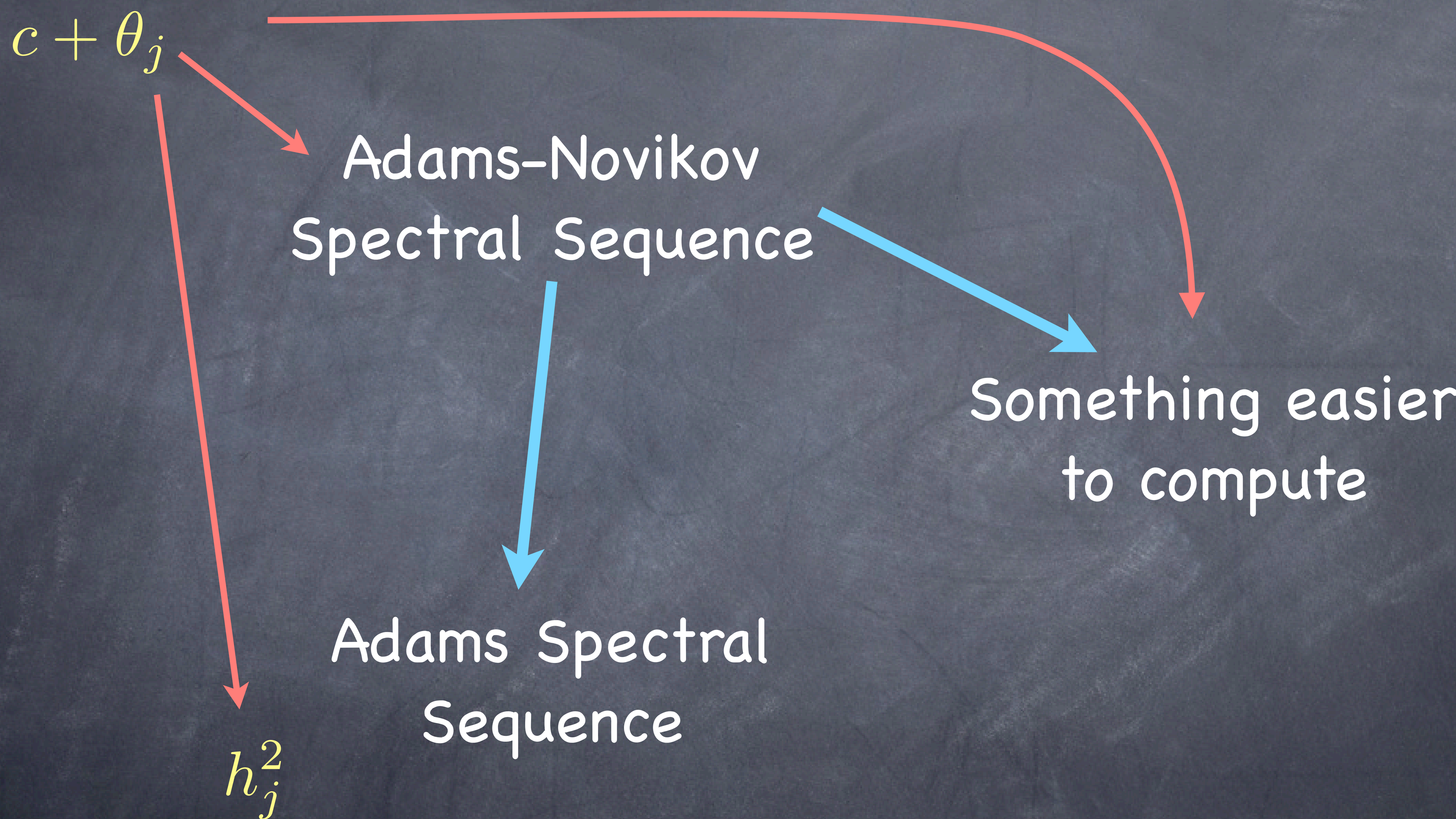
$$H^s(\mathbb{Z}/2; K_t) \implies KO_{t-s}$$

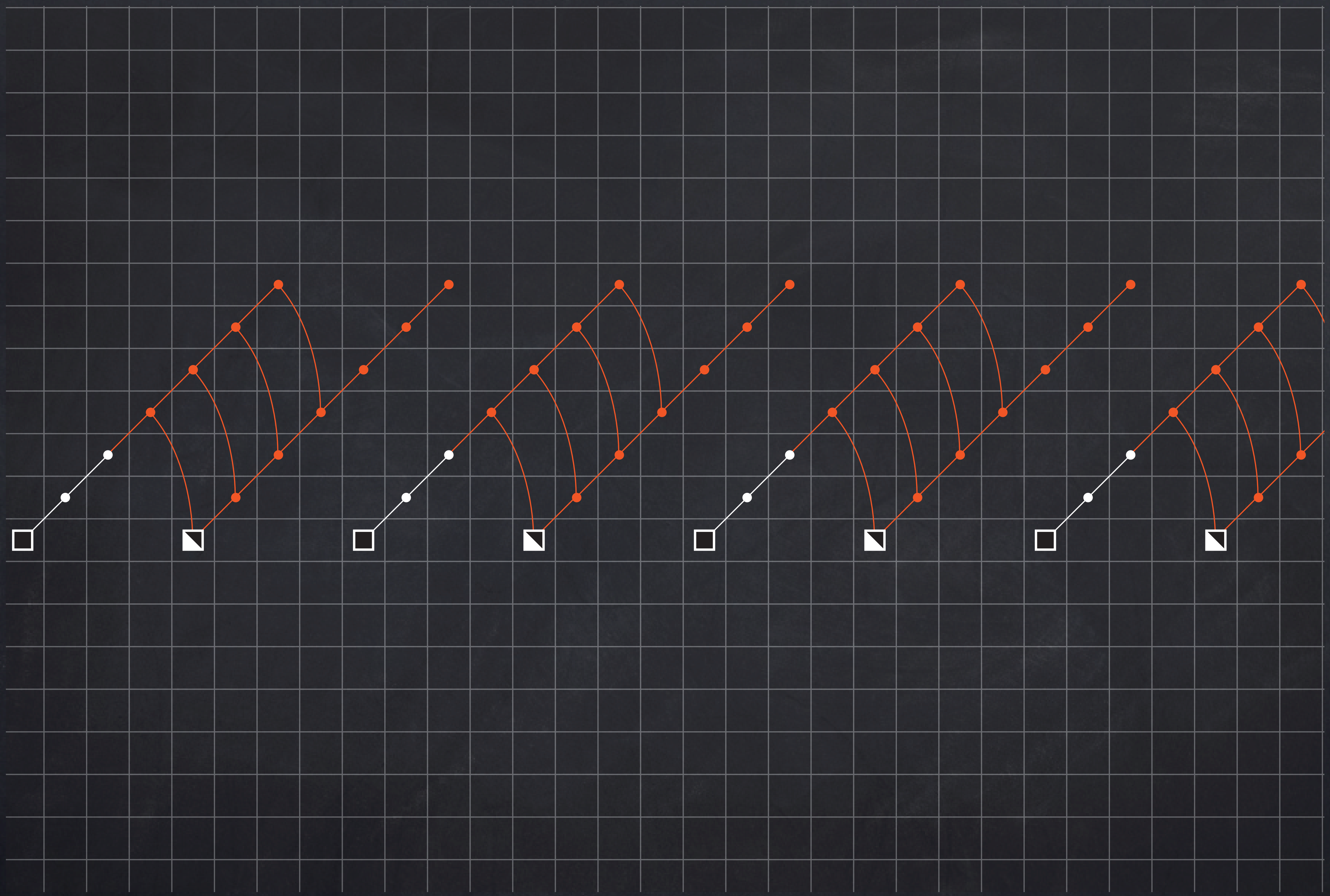


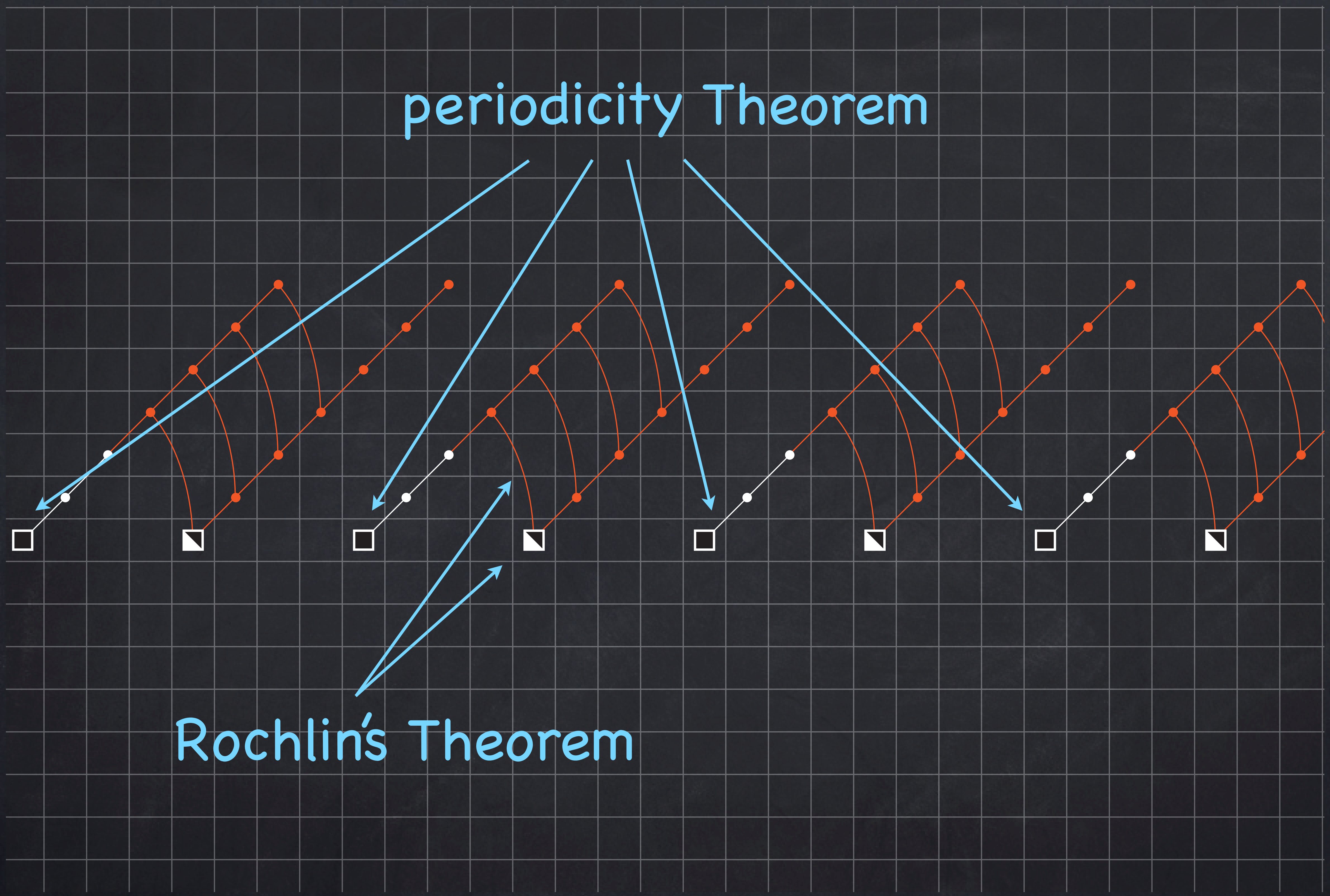

1
θ_1
θ_2
θ_3
θ_4

for $j \geq 2$, θ_j supports a non-zero differential




$c + \theta_j$
Adams-Novikov Spectral Sequence
Something easier to compute
Adams Spectral Sequence
h_j^2




periodicity Theorem
Rochlin's Theorem

K-theory and reality (Atiyah, 1966)

X ← space with a $\mathbb{Z}/2$ action

$KR(X)$ ← vector bundles with compatible conjugate-linear action

$$KR(X) \approx KR(X \wedge S^{n,n})$$

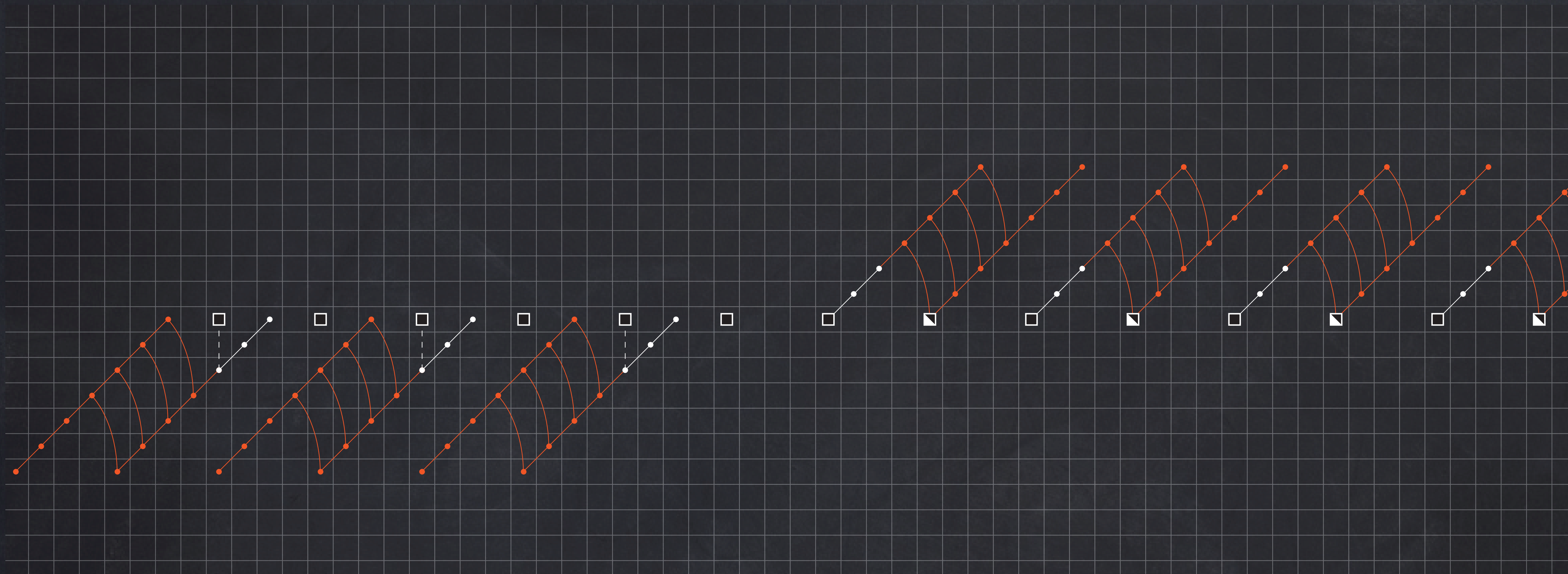
$$S^{n,n} = \overline{\mathbb{C}^n}$$

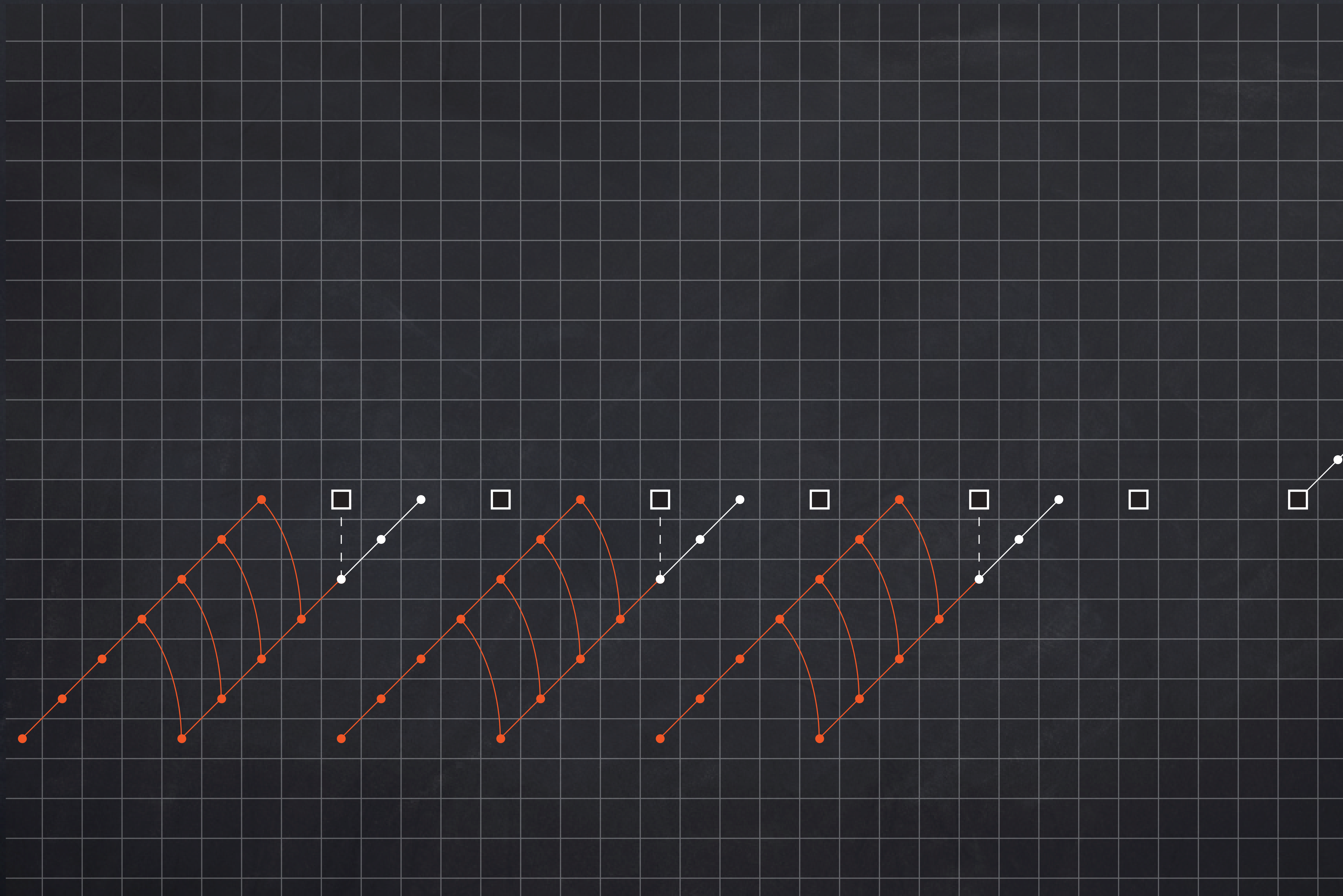
slice filtration

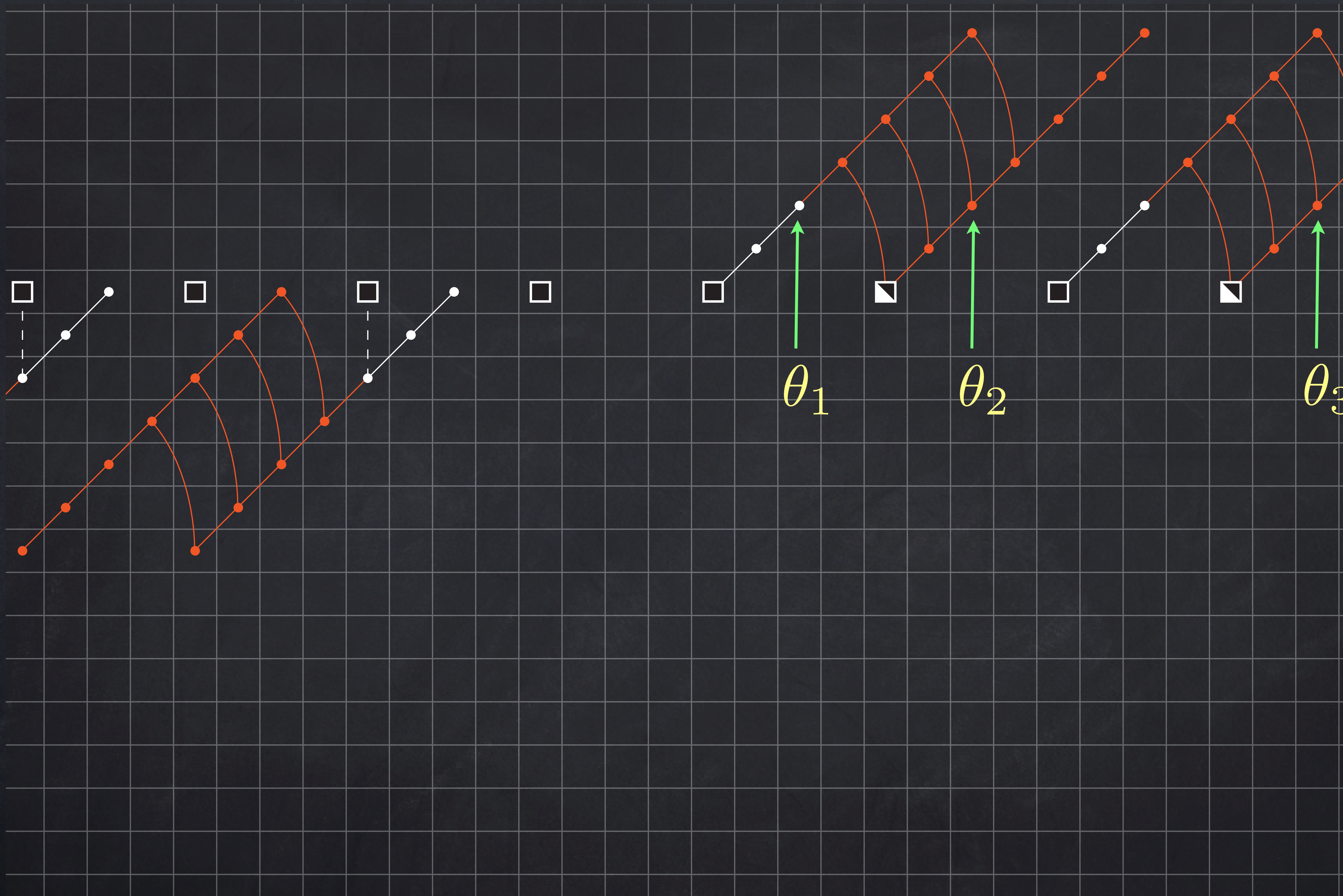
(Dugger, Hu-Kriz, H.-Morel, Voevodsky)

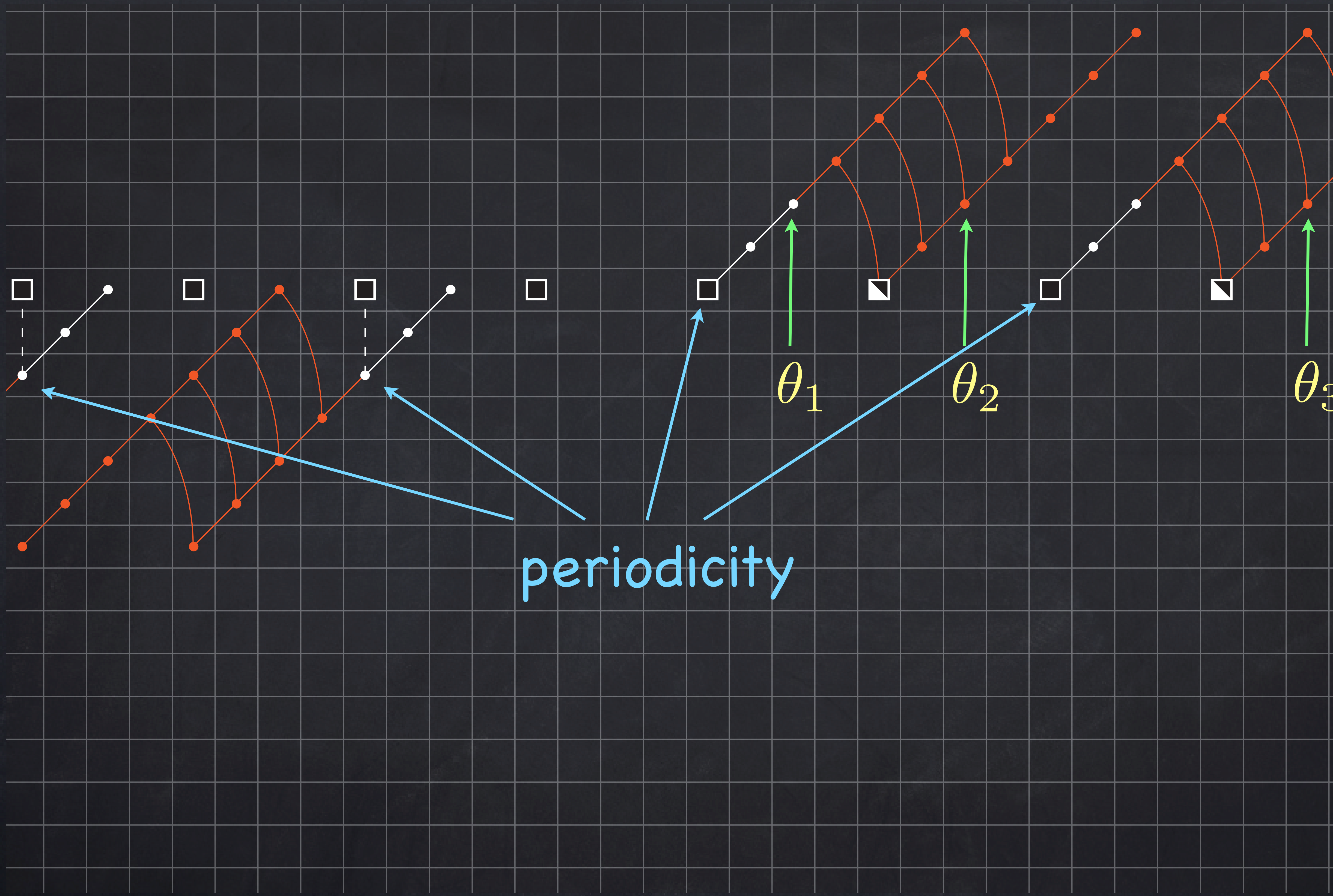
Assemble K-theory
from the equivariant
chains on $S^{n,n}$

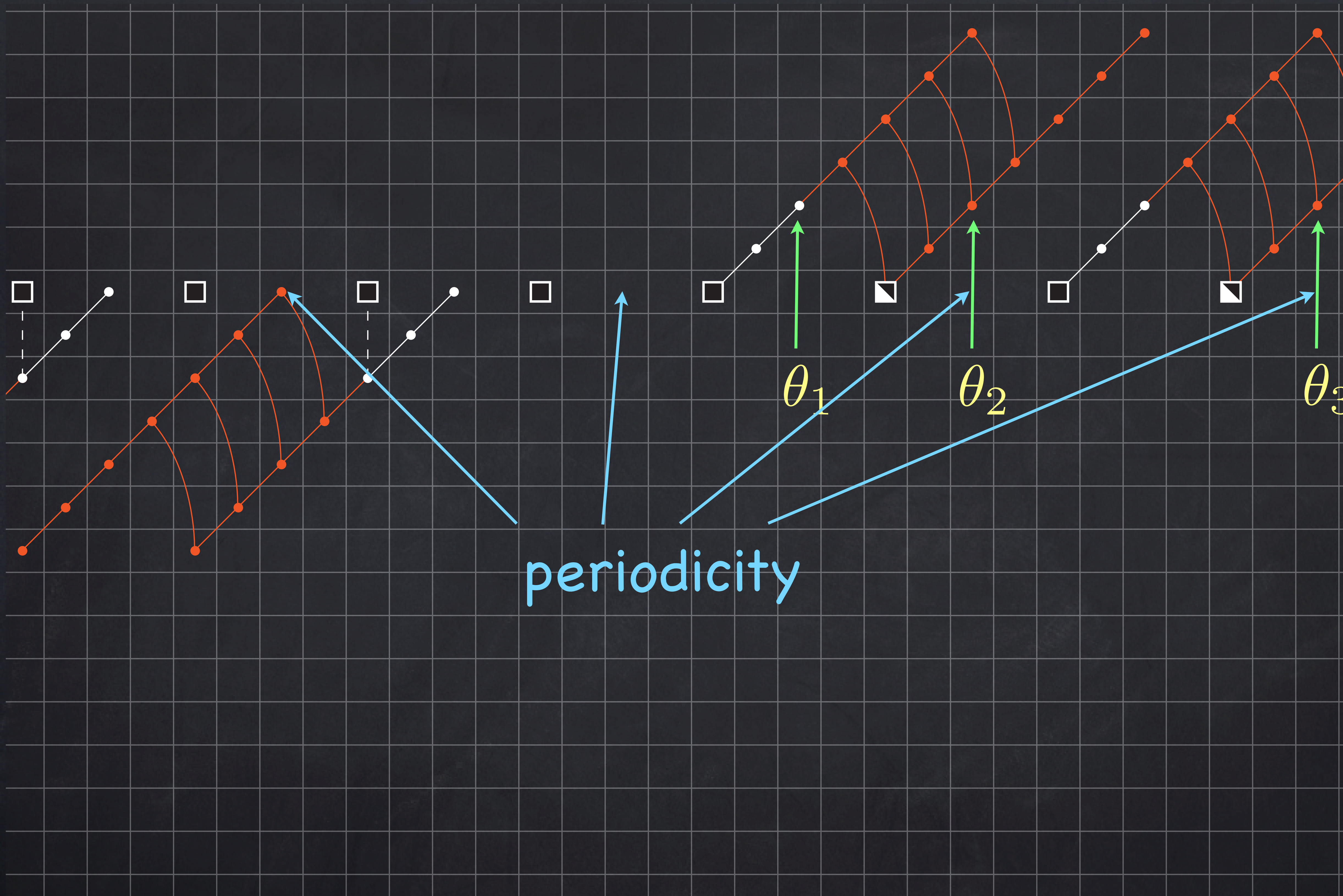
slice filtration






θ_1
θ_2
θ_3

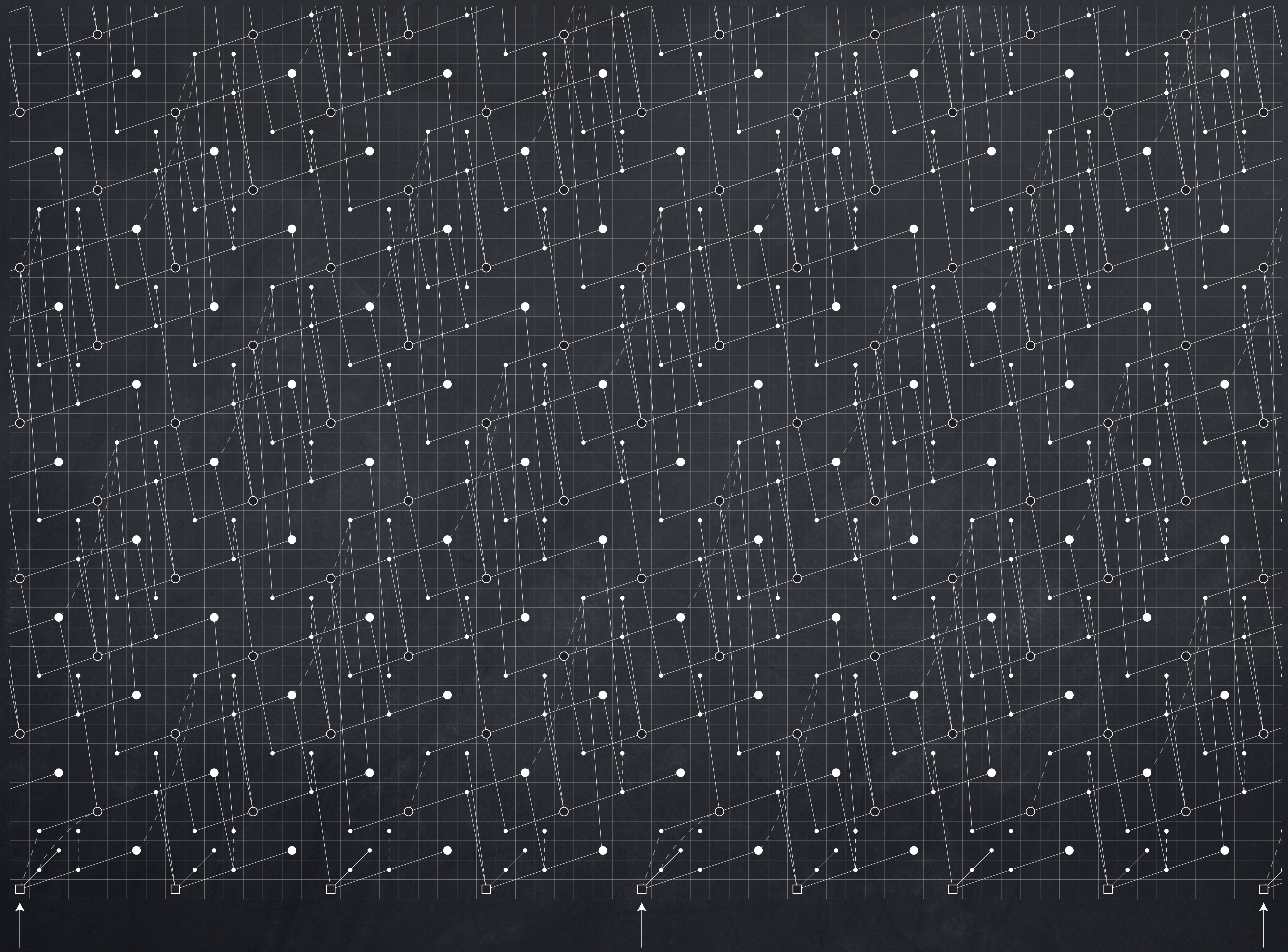

θ_1
θ_2
θ_3
periodicity

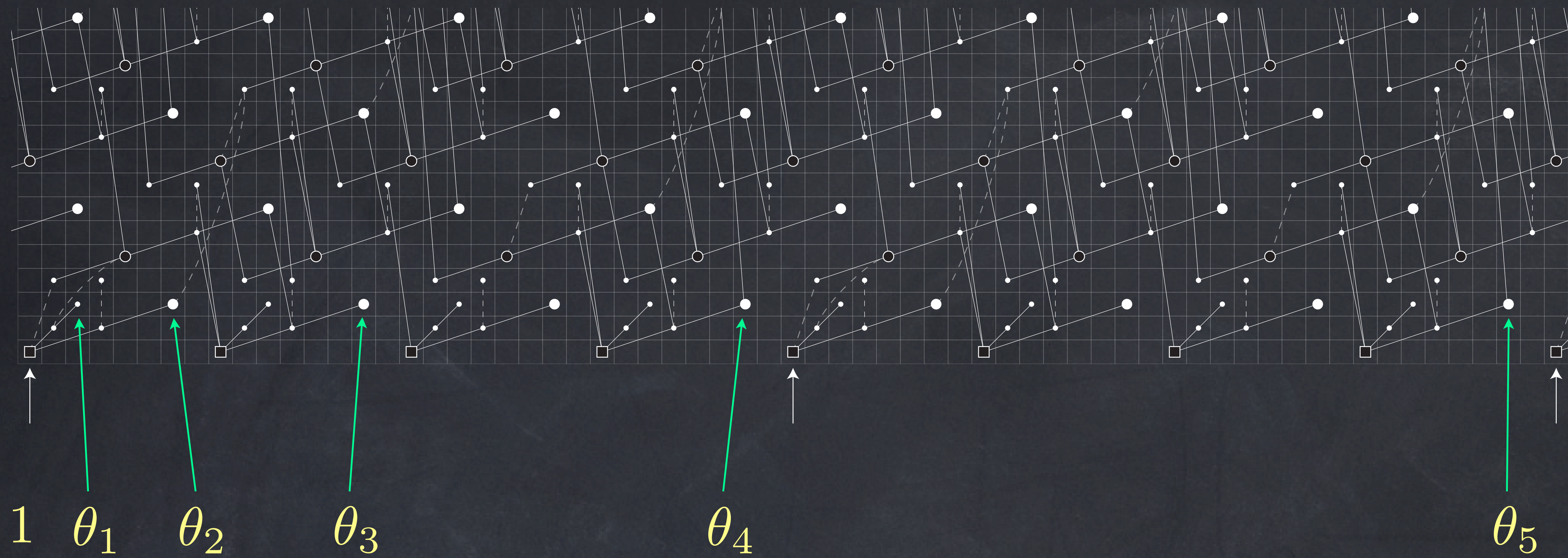


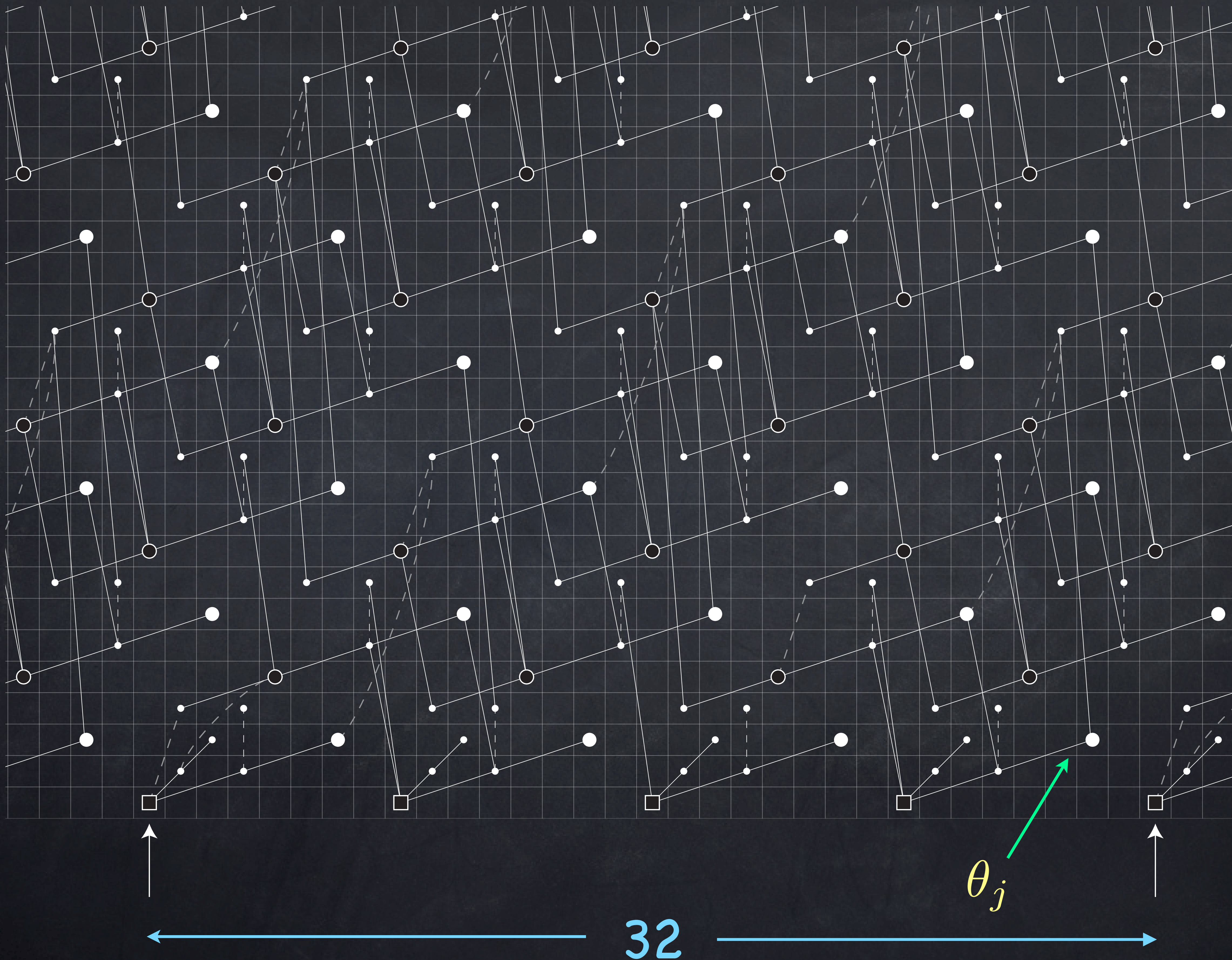
θ_1
θ_2
θ_3
periodicity

level 5 topological modular forms

Like KR with $\mathbb{Z}/4$ instead of $\mathbb{Z}/2$




1
θ_1
θ_2
θ_3
θ_4
θ_5

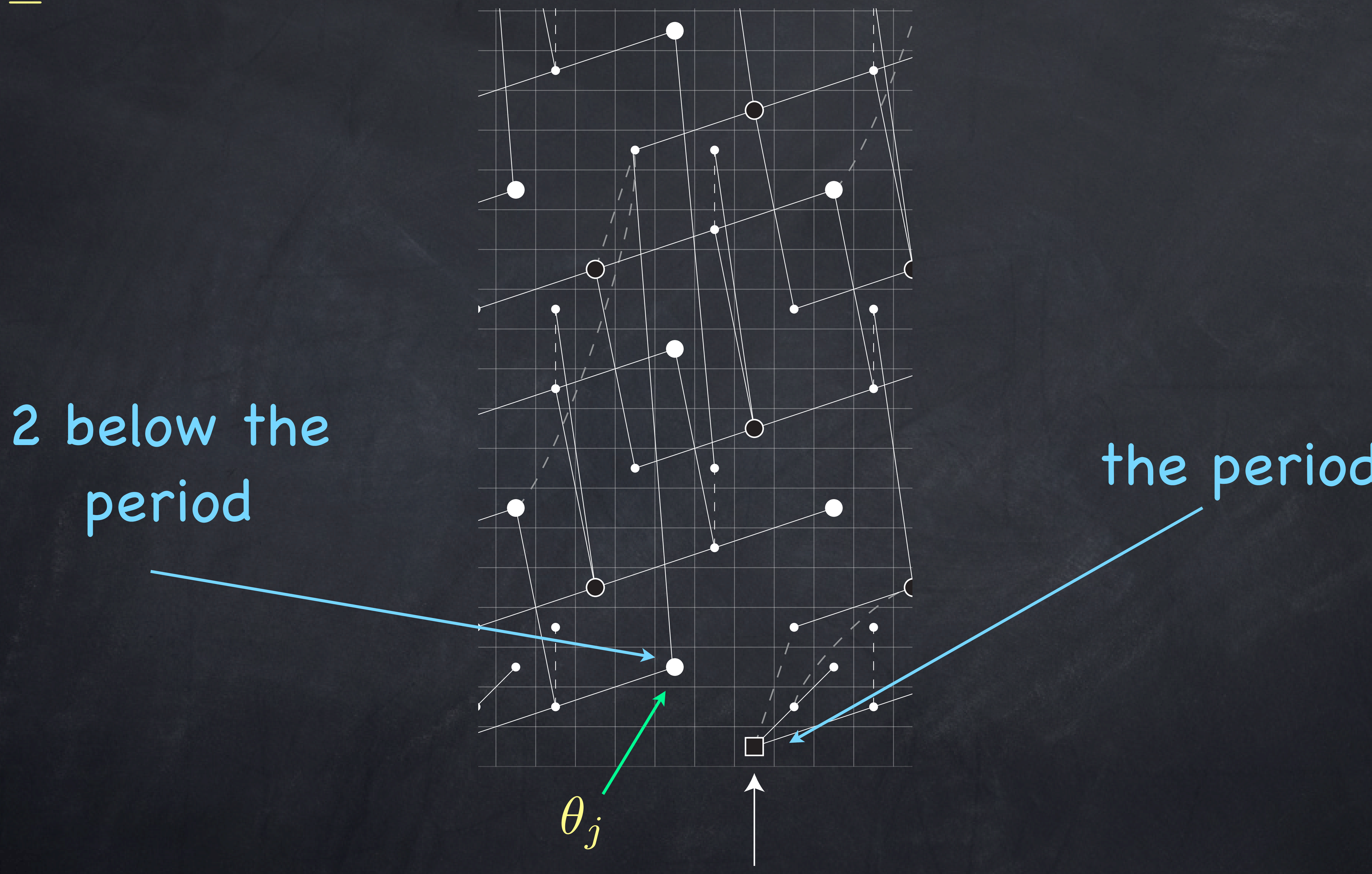


θ_j
32

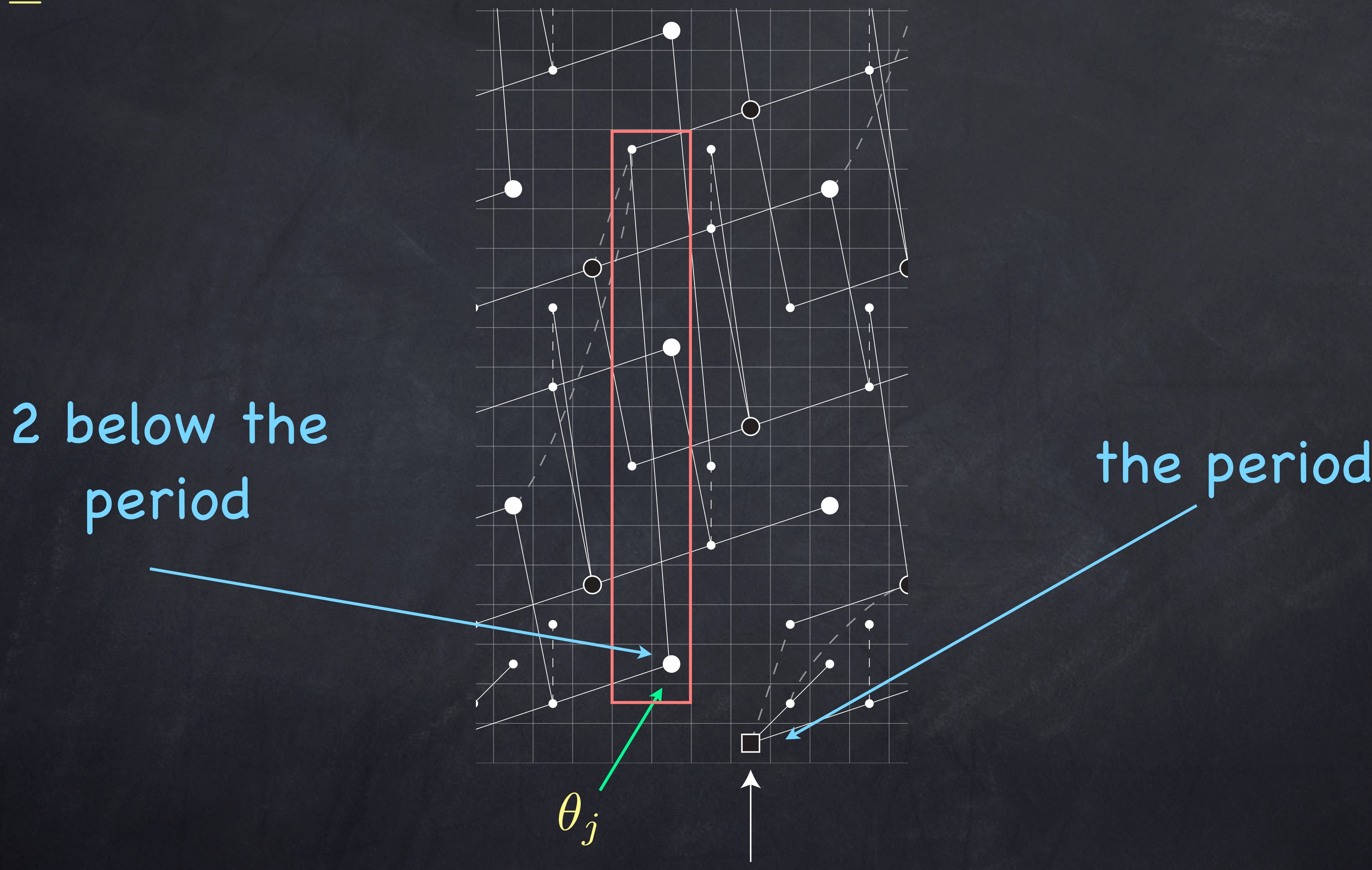
θ_j
32

$j \geq 4$

$j \geq 4$



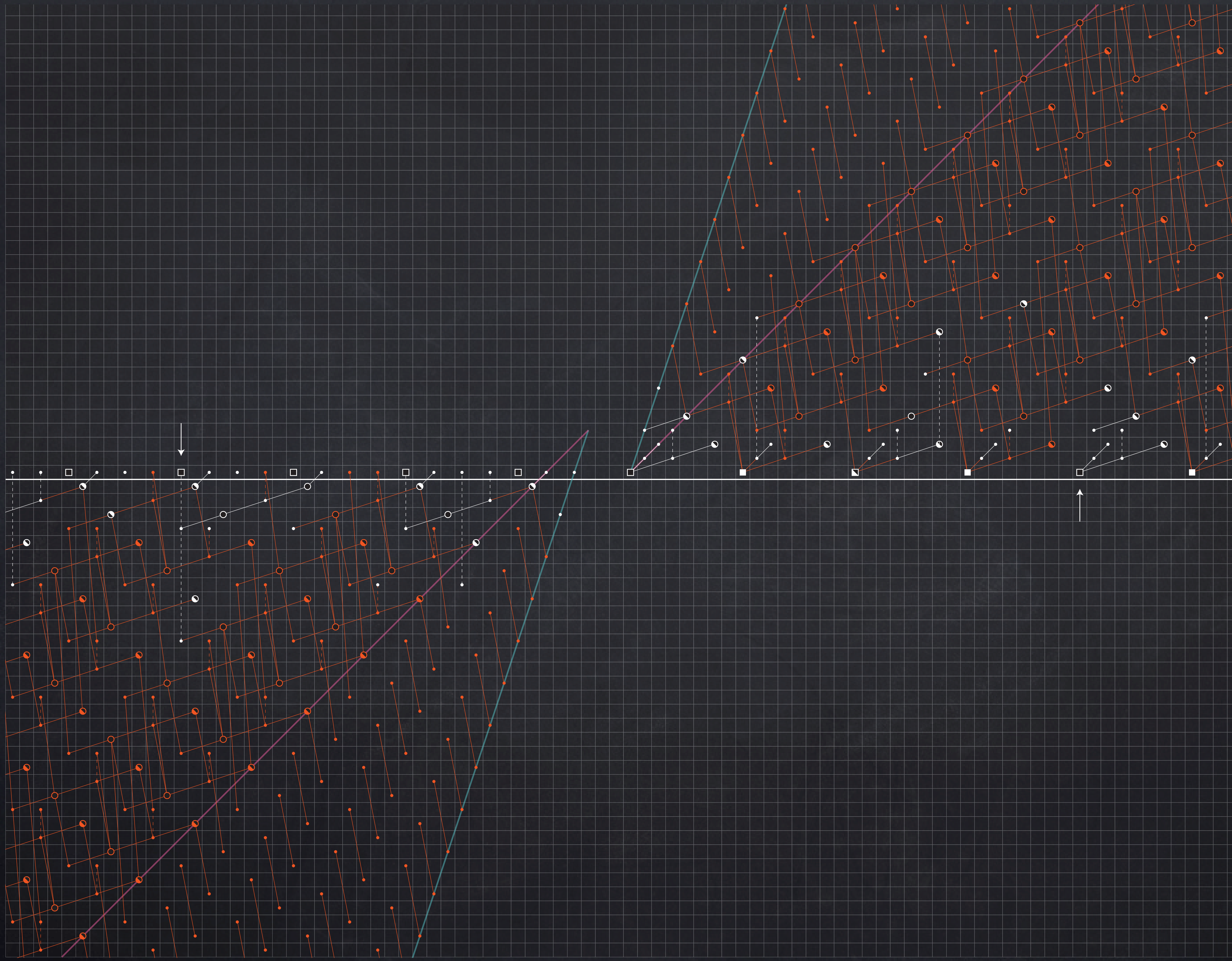
$j \geq 4$

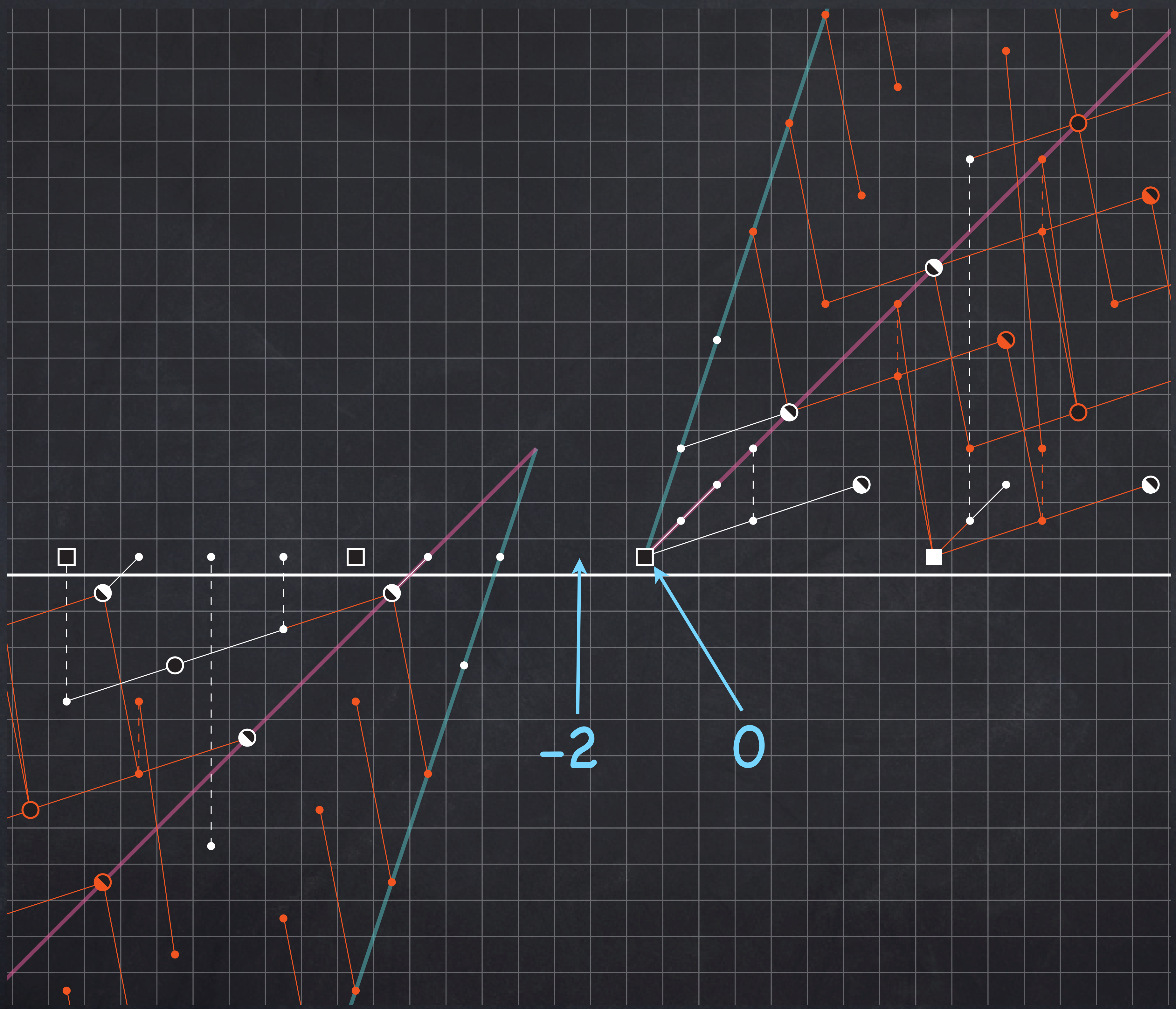


slice filtration

Assemble tmf(5) from
the equivariant chains
on $S^{m\rho_4}$

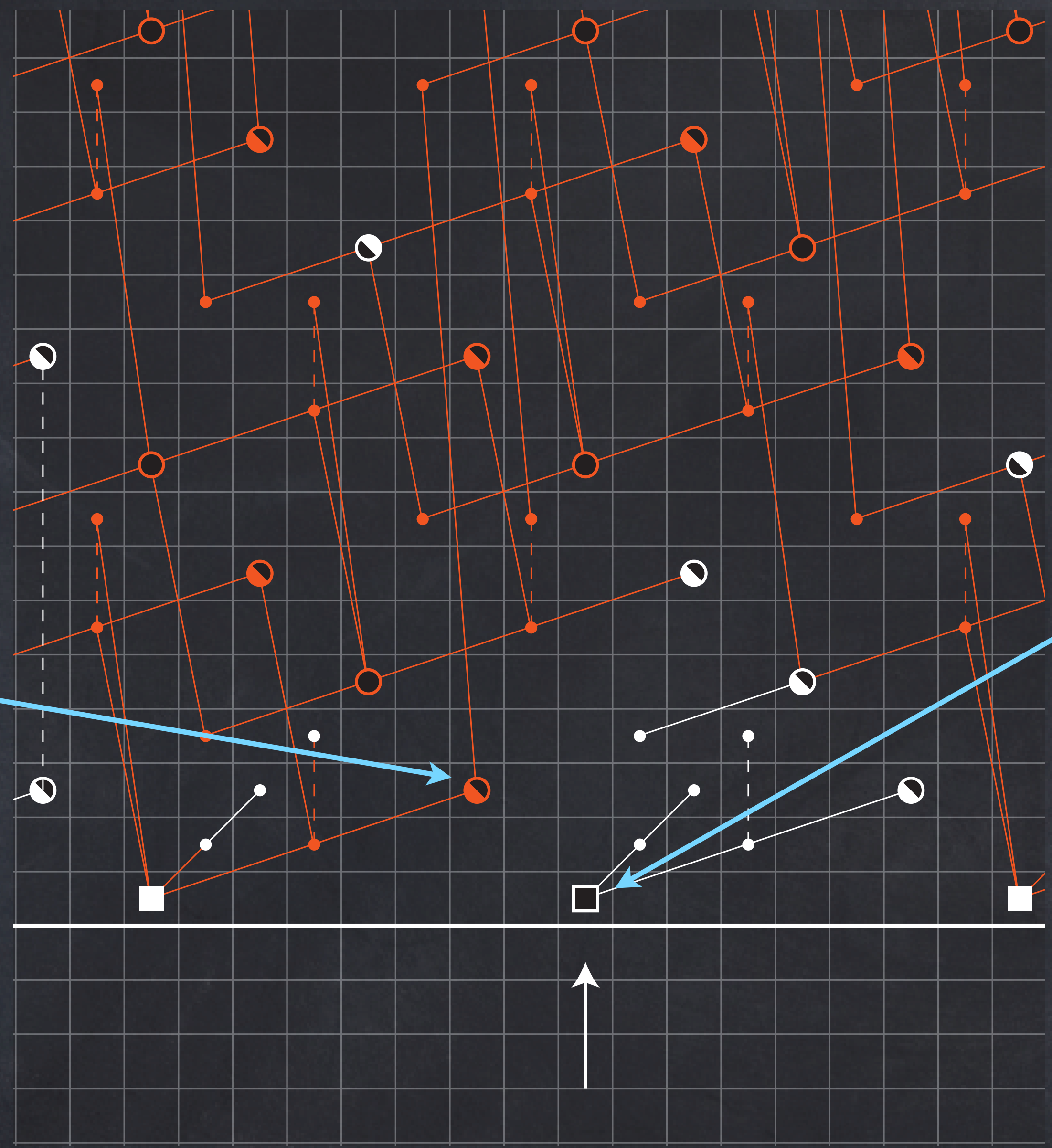
ρ_4 the 4 dimensional real
regular representation of $\mathbb{Z}/4$



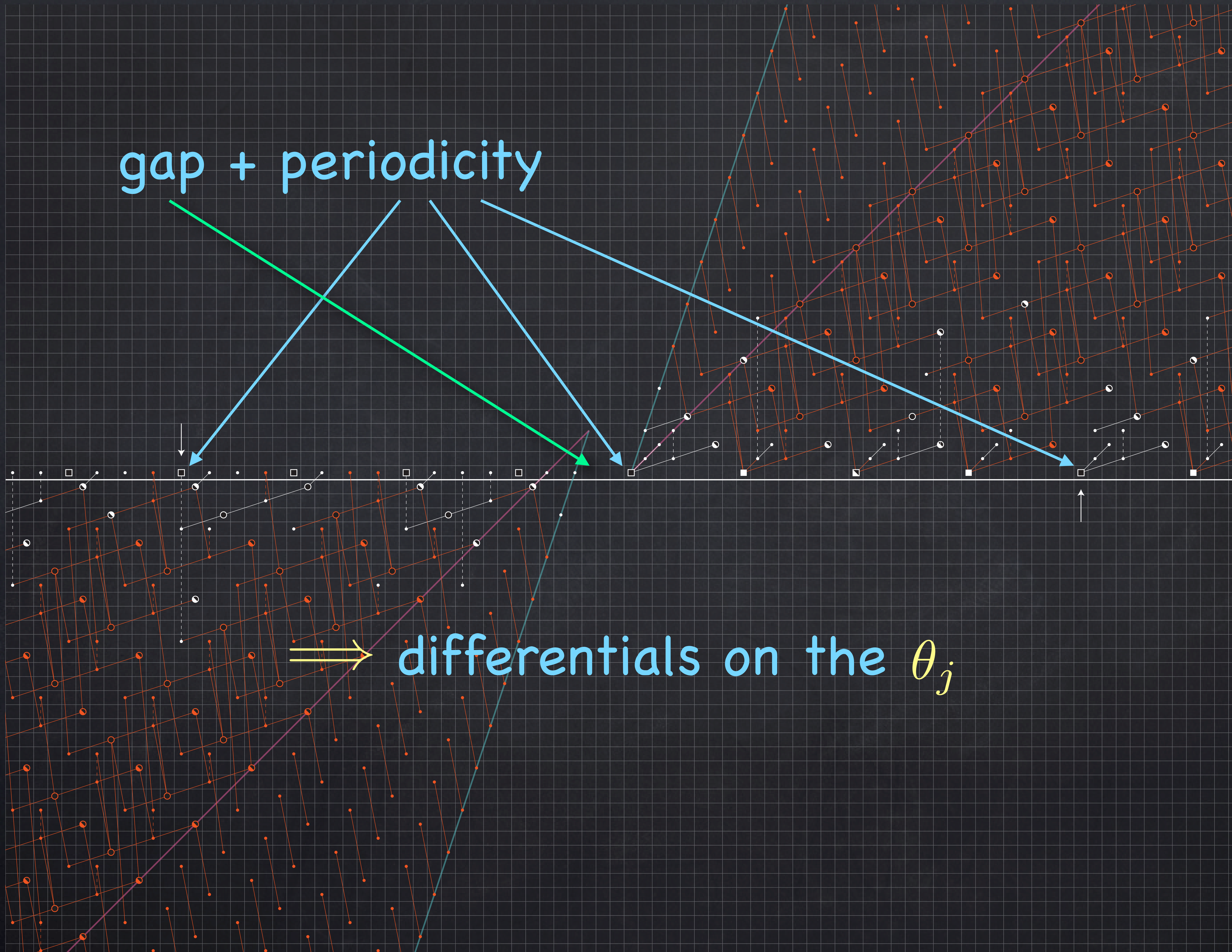


-2
0

2 below the period



the period


gap + periodicity
⟹ differentials on the θ_j

The actual proof

Step 1: Use $\mathbb{Z}/8$ and an appropriate cohomology theory

Step 2: Show that all the choices of θ_j are distinguished

Step 3: Prove a gap theorem (easy)

Step 4: Prove a periodicity theorem (of period 256)

Relation to Geometry/Physics?

4 dimensional field theory?

generalization of Clifford algebras with periodicity of $2^8\,3^3\,5 = 34,560$

(maybe twice that)

Question

Given a *real* manifold M^{2d} whose fixed point space N bounds an unoriented manifold, find a cobordism invariant of M which, when $N = \emptyset$ is

$$\int_{M/(\mathbb{Z}/2)} w_1^{2d}$$