SOME RECENT RESULTS ON TOPOLOGICAL MANIFOLDS

REINHARD SCHULTZ, Purdue University

Although topological spaces exist in great variety and can exhibit strikingly unusual properties, the main concern of topology has generally been the study of spaces which are relatively well-behaved. One particularly interesting class of examples is given by those spaces which locally look like Euclidean n-space R^n. Explicitly, a Hausdorff space X is called a **topological** n**-manifold** (without boundary) if each point of X has an open neighborhood which is homeomorphic to an open subset in R^n. Since open sets in R^m and R^n are homeomorphic if and only if $m=n$, the integer n is a homeomorphism invariant of X and is called the **dimension** of X. In this paper all manifolds under consideration are assumed to be second countable.

Topological manifolds arise naturally in several different ways. For example, they are useful in the qualitative study of differential equations inaugurated by Poincaré (compare [1]). Topological manifolds are also a natural generalization of the mathematical systems studied in non-Euclidean and Riemannian geometry. Many interesting results on topological manifolds are generalizations of older theorems originally proved for these and similar mathematical systems.

During the nineteen sixties important advances in the study of topological manifolds yielded a great deal of information on their basic geometric structure. In particular, two long standing conjectures regarding topological manifolds were shown to be systematically false (see Section 4). One of the most useful results on topological manifolds of dimension $\neq 4$, 5—their description in terms of attaching handles—will be discussed in Section 5. This result allows one to take certain theorems which had previously been proved under additional structural assumptions and generalize them to topological manifolds with only minimal changes in the proofs.

I wish to thank R. Kirby for his detailed comments on an earlier version of this paper.

1. Classification of topological manifolds. Before beginning our discussion, it will be useful to generalize the definition of topological n-manifolds to include the possibility of a boundary. Let R^n_+ be the set of points in R^n whose last coordinate is nonnegative. Then a **topological** n**-manifold with boundary** is a Hausdorff space X each point of which has an open neighborhood homeomorphic to an open subset of R^n or R^n_+.

Of course, the set of all points having neighborhoods homeomorphic to open subsets of R^n is a topological n-manifold without boundary as previously defined. It is easy to see that the set of such points is open and dense in M; this subset is called the **interior of** M and written Int M. The complement of Int M is called the **boundary** of M and written ∂M; it follows that ∂M is a topological $(n-1)$-

Reinhard Schultz received his Chicago Ph.D. in 1968, under Richard K. Lashof. His main research interest is differential topology, and he is currently Assistant Professor at Purdue. *Editor.*

manifold without boundary. The following theorem of M. Brown [9] is extremely important in the study of manifolds with boundary:

THEOREM 1.1. (Collar Neighborhood Theorem) *Let M be a manifold with boundary. Then there is an open neighborhood V of ∂M which is homeomorphic to $\partial M \times [0, 1)$ such that $\partial M \subseteq V$ corresponds to $\partial M \times \{0\}$.*

One of the most immediate problems regarding topological manifolds is their classification up to homeomorphism. The techniques of point set topology suffice for the classification of one-dimensional manifolds; this was completed during the second decade of the twentieth century (see [37] or [41]). There are only four different homeomorphism types of connected one-dimensional manifolds: The open interval, the half-open interval, the closed interval, and the circle.

The study of two-dimensional manifolds is somewhat more difficult and requires a systematic investigation of polyhedra in the Euclidean plane (e.g., see [29], [30], or [41]). One of the earliest results was the Jordan Curve Theorem, first proved correctly by Veblen in 1905 [59]. This theorem was augmented by a result of Schoenflies [48], and we may combine the two theorems into the following single statement:

THEOREM 1.2. (Jordan-Schoenflies Theorem). *Let X be a subset of R^2 which is homeomorphic to a circle. Then $R^2 - X$ has two components, one bounded and one unbounded, and X is the point set-theoretic frontier of each component. The homeomorphism from the unit circle to X extends to a homeomorphism from the unit disk to the closure of the bounded component of $R^2 - X$.*

This theorem is the basic result needed for the following theorem of Radó [45]:

THEOREM 1.3. *Any (unbounded) topological two-dimensional manifold M may be triangulated; i.e., there is a countable locally finite covering $\{T_i\}$ of M by compact subspaces satisfying:*

(i) *There are canonical homeomorphisms h_i from T_i to the solid triangle*

$$\{(x, y) \in R^2 \mid x \geqq 0, y \geqq 0, \text{ and } x + y \leqq 1\}.$$

(ii) *Under these homeomorphisms any nonempty intersection $T_i \cap T_j$ corresponds to either a common side or a common vertex.*

The classification of two-dimensional manifolds up to homeomorphism then follows from a study of triangulated manifolds (e.g., see [18], [30], [46]). Results of Moise imply that the classification of three-dimensional manifolds reduces to a study of triangulated three-dimensional manifolds (e.g., [35], [36]), and a classification scheme in the compact case exists modulo Conjecture 1.4 below.

If X is any arcwise connected space, then X is said to be **simply connected** if any continuous map from the unit circle in R^2 to X (i.e., a *closed curve* in X) extends to a continuous map of the unit disk. Given this definition, we may state the following conjecture made by Poincaré in 1904 [**44**]:

CONJECTURE 1.4 (Poincaré Conjecture). *Let M be a compact topological 3-manifold without boundary that is simply connected. Then M is homeomorphic to the unit sphere in R^4 (i.e., the 3 dimensional sphere).*

Relatively little is known about four-dimensional manifolds; the direct approach used in lower dimensions becomes increasingly complicated as the dimension increases, and in four dimensions the problems involved becomes forbiddingly difficult. There is a marked change, however, when one considers manifolds of dimension at least five. In this case one has enough space in which to make geometric constructions involving circles and disks almost at will. A particular consequence of this freedom of construction is that no general classification scheme for compact topological manifolds exists in any dimension $\geqq 5$ (compare [**5**, pp. 375–376]); for the freedom in constructing higher dimensional manifolds implies that any classification scheme would yield a solution to the word problem for finitely presented groups (see [**47**, Ch. XII] for a discussion of the latter problem).

2. Generalized Schoenflies and Poincaré Conjectures. The Jordan curve theorem was soon generalized to higher dimensions by Brouwer ([**7**]; also see [**14**, §18] or [**54**]). However, Antoine [**4**] and Alexander [**3**] constructed examples of subspaces X in R^3 that are homeomorphic to the unit sphere in R^3 but are not the frontiers of subspaces homeomorphic to the unit disk; counterexamples similar to Alexander's exist in all higher dimensions. On the other hand, Alexander also proved that X bounds a disk if it is a polyhedron in R^3 [**2**]. Around 1960 B. Mazur [**31**], M. Morse [**38**], and M. Brown [**8**] proved results implying the following generalization of the Jordan-Schoenflies theorem:

THEOREM 2.1. (Generalized Schoenflies Theorem). *Let X be a subset of R^n that is homeomorphic to the unit sphere, and assume that the closure of the bounded component of $R^n - X$ is a topological n-manifold with boundary. Then the homeomorphism from the sphere to X extends to a homeomorphism from the disk to the closure of the bounded component of $R^n - X$.*

About the same time that the Generalized Schoenflies Theorem was proved, Smale [**53**], Stallings [**55**], and Zeeman [**64**] proved a generalization of the Poincaré Conjecture (1.4 above) in all dimensions greater than four; however, their proofs required additional structure on the manifolds under consideration (i.e., they had to be differential or combinatorial manifolds as defined in Section 3). Several years later Newman gave a proof of this result for topological manifolds using his generalization of Stallings' techniques and arguments of E. H. Connell [**42**]. For completeness, we state the result below:

THEOREM 2.2. (Generalized Poincaré Conjecture). *Let M be a compact topo-*

logical n-manifold ($n \geqq 5$) without boundary that is $\frac{1}{2}(n-1)$-connected if n is odd and $\frac{1}{2}n$-connected if n is even. Then M is homeomorphic to the unit sphere in R^{n+1}.

REMARKS 1. A topological space X is said to be **k-connected** if any continuous map from the unit sphere in R^{k-m+1} (for any $m \geqq 0$) extends to a map of the unit disk.

2. We already noted that the three-dimensional case of Theorem 2.2 is unknown; the four-dimensional case is also unknown.

The proof breaks down in dimensions 3 and 4 because in these cases there is not enough room in the manifold to make all the constructions needed in the proof (compare the last paragraph of Section 1).

Smale's proof of the generalized Poincaré Conjecture (most of whose details are independently due to A. H. Wallace [**62**]) was a central technique in the theory of surgery on manifolds developed by Kervaire, Milnor, S. P. Novikov, W. Browder, and C. T. C. Wall (for a definitive account see Wall's book [**61**]). Wall's theory in turn was important in studying the following elaboration of the Generalized Schoenflies Conjecture:

CONJECTURE 2.3. (Annulus Conjecture). *Let $A \subseteq R^{n+1}$ be a compact topological $(n+1)$-manifold whose boundary is homeomorphic to a disjoint union of two copies of the unit sphere in R^{n+1}. Then A is homeomorphic to the closed annulus in R^{n+1} bounded by the spheres of radius* 1 *and* 2.

If this conjecture were false for $n=1$ or 2, then an argument of Brown and Gluck [**10**, p. 42] would imply that the compact unbounded topological manifold A' formed from A by gluing together the two components of the boundary of A could not be triangulated. Hence the conjecture is certainly true in these dimensions by *reductio ad absurdum* (more elementary arguments are also possible). In [**19**] Kirby gave an elegant argument which reduced the proof of the annulus conjecture for $n \geqq 4$ to a problem which could be handled by means of Wall's surgery theory. This surgery theoretical problem was solved independently by Wall [**60**] and W.-C. Hsiang and Shaneson ([**15**], [**16**]); thus Conjecture 2.3 is true except possibly in the case $n=3$.

3. Differentiable and Combinatorial Manifolds. In this section we shall describe the kinds of "additional structure" often associated to topological manifolds and mentioned in the previous sections.

The topological manifolds appearing in analysis and differential geometry usually satisfy the conditions appearing in the following definition:

DEFINITION. A topological n-manifold is **smoothable** if there is a collection of pairs $\{(U_\alpha, h_\alpha)\}_{\alpha \in A}$ satisfying:

(i) U_α is an open subset in R^n.

(ii) The map $h: U_\alpha \to M$ is a homeomorphism onto an open subset.

(iii) The functions $h_\beta^{-1}h_\alpha: h_\alpha^{-1}h_\beta(U_\beta) \to h_\beta^{-1}h_\alpha(U_\alpha)$ are functions of class C^r for some $r \geqq 1$.

If U and W are open subsets of Euclidean spaces, recall that a map $f: U \to W$ is a **function of class** C^r if the coordinate functions f^i defined by $f(w) = (f^1(w), \cdots, f^m(w))$ each have all possible partial derivatives of order r and these functions are continuous; a function is C^∞ if it is C^r for every positive integer r. Two collections $\{(U_\alpha, h_\alpha)\}$ and $\{(V_\beta, k_\beta)\}$ satisfying (i)–(iii) are **equivalent** if their union satisfies property (iii); it follows that every collection $\{(U_\alpha, h_\alpha)\}$ is equivalent to a unique maximal collection $\mathfrak{A}$ which is called a **smooth atlas** for M **of class** C^r. A **differential** (or **smooth**) n-**manifold** is a pair $(M, \mathfrak{A})$ consisting of a smoothable n-manifold M and a smooth atlas $\mathfrak{A}$. We shall always assume that the atlas is smooth of class C^∞, since it is known that any C^r atlas corresponds to a unique C^∞ atlas [**40**, Sections 4 and 5].

More generally, if Γ is any reasonable family of continuous functions from open sets in R^n to open sets in R^n (technically a **pseudogroup;** see [22]), then it is possible to define a Γ atlas and a Γ n-manifold. In topological investigations Γ is usually taken to be the C^r functions defined above or the piecewise linear (PL) functions defined below. Thus in order to define a piecewise linear n-manifold, it is only necessary to specify which mappings on open subsets of Euclidean space are piecewise linear; this requires a succession of definitions.

DEFINITION. Let $x_0, \cdots, x_n$ be points in R^m such that $x_1 - x_0, \cdots, x_n - x_0$ are linearly independent. Then the n-dimensional **simplex** (or n-simplex) with vertices $x_0, \cdots, x_n$ is the set of all linear combinations $y = \sum t_i x_i$, where each t_i is nonnegative and $\sum t_i = 1$ (the last condition and linear independence imply that the t_i are unique). The x_i are called the **vertices** of the simplex.

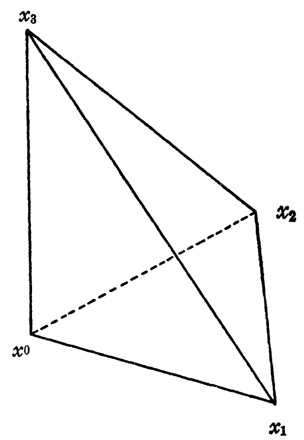

FIG. 1

A simplex is actually a generalized version of a triangle. It is immediate from the definition that a 1-simplex is a line segment and a 2-simplex is a solid triangle. Furthermore, a 3-simplex is a tetrahedron (see Figure 1).

DEFINITION. Let A be a simplex with vertices $a_i (1 \leqq i \leqq n)$, and let V be any real vector space. A function $f: A \to V$ is **affine linear** provided $y \in A$ and $y = \sum t_i a_i$ with $\sum t_i = 1$ imply $f(y) = \sum t_i f(a_i)$.

DEFINITION. Let U and V be any subsets of R^n. A continuous function $f: U \to V$ is **piecewise linear** (or **PL**) if there is a countable locally finite covering $\mathfrak{B}$ of U by simplexes such that f is an affine linear map on each element of $\mathfrak{B}$.

REMARK. Any open subset of R^n_+ has many countable locally finite coverings by simplexes.

EXAMPLES 1. Let $f: R^n \to R^n$ be an affine transformation; i.e., $f(x) = Lx + y$, where L is a linear transformation. Then f is automatically affine linear on every simplex in R^n (compare [6, p. 272]).

2. Let $f: R^2 \to R^2$ be given by $f(x, y) = (x, y)$ if $y \geqq 0$ and $(x, 2y)$ if $y \leqq 0$. Then f is affine linear on any simplex contained in either the upper or lower half plane.

3. Let f be the map which sends the solid regular pentagon $ABCDE$ to the solid irregular pentagon $A'B'C'D'E'$ in Figure 2 by stretching the triangle OXY into $O'X'Y'$.

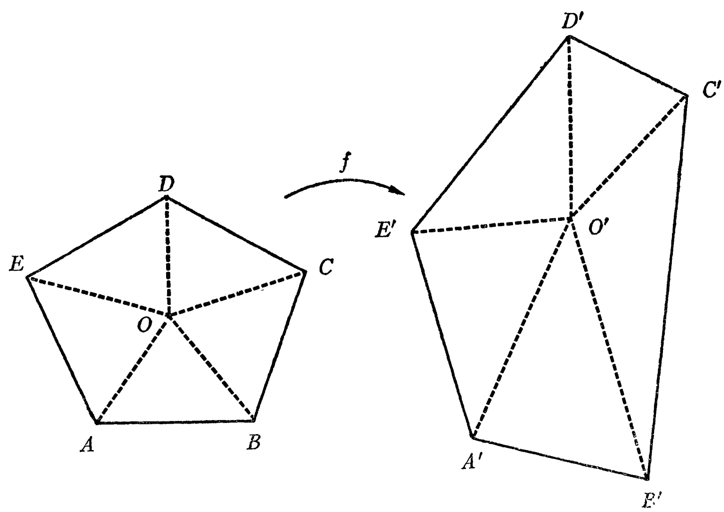

FIG. 2

A fundamental theorem of Cairns and Whitehead states that any smooth manifold determines a basically unique PL manifold ([11], [63], [40, Pt. II]). However, a PL manifold need not be determined by a smooth manifold (a result of Thom [58]), and two distinct smooth manifolds may determine the same PL manifold (a result of Milnor [32]). A comprehensive study of the relationship between smooth and PL manifolds appears in [25].

In the following section we shall discuss the parallel problem regarding the existence and uniqueness of PL manifolds associated to a given topological manifold.

REMARK. For historical reasons the study of PL manifolds and related objects is frequently called **combinatorial topology** and PL manifolds are often called **combinatorial manifolds.**

4. The Triangulation Conjecture and the Hauptvermutung. The following conjectures were formulated (in roughly equivalent form) soon after the establishment of combinatorial topology as a subject in its own right.

TRIANGULATION CONJECTURE. *Any topological n-manifold has a* PL *atlas.*

HAUPTVERMUTUNG FOR MANIFOLDS. *Any two homeomorphic* PL *n-manifolds are equivalent as* PL *manifolds.*

The results quoted in Section 1 imply that the first conjecture is true if $n \leqq 3$. Similarly, the second conjecture is true if $n \leqq 3$ ($n=1$, straightforward; $n=2$, see Papakyriakopoulos [**43**]; $n=3$, see Moise [**35**], [**36**]). The solution of the generalized Poincaré conjecture in higher dimensions implies that the second conjecture is true for PL manifolds homeomorphic to spheres of dimension at least five. A fairly strong version of the *Hauptvermutung* for simply connected manifolds was proved by Lashof and Rothenberg [**26**], and Sullivan ([**56**], [**57**]); in the next paragraph we shall discuss subsequent results which eliminated the simple connectivity assumption (see Theorem 4.2).

Kirby's reduction of the Annulus Conjecture, other results appearing in [**19**], and consequences of these results due to Lees [**28**] led directly to initial results on the Triangulation Conjecture due to Lashof [**23**]. These theorems and computations of Casson, Wall, Hsiang, and Shaneson ([**15**], [**16**], [**60**]) in turn led to the following strong results on the Triangulation Conjecture and the *Hauptvermutung* due to Lashof and Rothenberg ([**27**], [**24**]), and Kirby and Siebenmann [**20**]:

THEOREM 4.1. *Let M be a topological manifold of dimension at least six (or five in the unbounded case), and assume that the four-dimensional cohomology group $H^4(M; Z_2)$ is zero. Then M has a* PL *atlas.*

THEOREM 4.2. *Let M be a* PL *manifold satisfying the above dimensional restriction, and assume that the three-dimensional cohomology group $H^3(M; Z_2)$ is zero. Then any* PL *manifold homeomorphic to M is equivalent to M as a* PL *manifold.*

REMARK. For the sake of completeness we shall describe the cohomology groups $H^k(M; Z_2)$ in a geometric manner exploited by Sullivan in his proof of the earlier version of Theorem 4.2; for a more standard description of $H^k(M; Z_2)$ see [**14**, §23] or any algebraic topology text. If X is any topological space, a **smooth k-manifold in X** is a continuous function $f: V \rightarrow X$, where V is a compact smooth k-dimensional manifold. An element in $H^k(X; Z_2)$ is then a function which assigns to each k-manifold in X an element of Z_2 subject to certain consistency conditions which are straightforward but a little too technical to describe here (see [12, §8] or [56] for further discussion; the description does not generalize to odd primes).

The restrictions on cohomology appearing in the above theorems were also shown to be unnecessary if $H^3(\mathrm{Top/PL}; Z_2) = 0$, where Top/PL is a topological space arising from the geometry of the proof of 4.1 and 4.2. However, Siebenmann (first alone and later jointly with Kirby) constructed examples which implied that $H^3(\mathrm{Top/PL}; Z_2)$ is nonzero; it followed quickly that both the Triangulation Conjecture and the *Hauptvermutung* were systematically false in every dimension greater than four.

There are very simple manifolds which yield contradictions to the *Hauptvermutung*. For example, consider the cartesian product $S^3 \times T^2$ of the unit sphere in R^4 with the two-dimensional torus T^2. This product is a smooth manifold and consequently determines a unique PL manifold; results of Shaneson combined with $H^3(\mathrm{Top/PL}; Z_2) \neq 0$ imply the existence of a PL 5-manifold M^5 which is homeomorphic to $S^3 \times T^2$ but inequivalent to $S^3 \times T^2$ as a PL manifold ([49], [50]).

5. Handlebody theory for topological manifolds. In one sense the Kirby-Siebenmann results are disappointing because they disprove two conjectures which would have reduced the study of topological manifolds to combinatorial topology. On the other hand, the results used in the proof of 4.1 and 4.2 yield a convenient method for decomposing topological manifolds of dimension at least six, which will be discussed in this section.

Throughout this section S^p will denote the unit sphere in R^{p+1} and D^{p+1} will denote the unit disk in R^{p+1}. It follows from the definitions that D^{p+1} is a topological $(p+1)$-manifold with boundary, and its boundary is S^p.

DEFINITION. Let V be a topological n-manifold with boundary, and let $f: S^{k-1} \times D^n \to \partial V$ be a one-to-one continuous mapping. Then the manifold W obtained by **attaching a k-handle to V along** f is the disjoint union of V and $D^k \times D^{n-k}$ modulo the identification of $f(S^{k-1} \times D^{n-k}) \subseteq V$, with $S^{k-1} \times D^{n-k} \subseteq D^k \times D^{n-k}$ (see Figure 3 for an illustration).

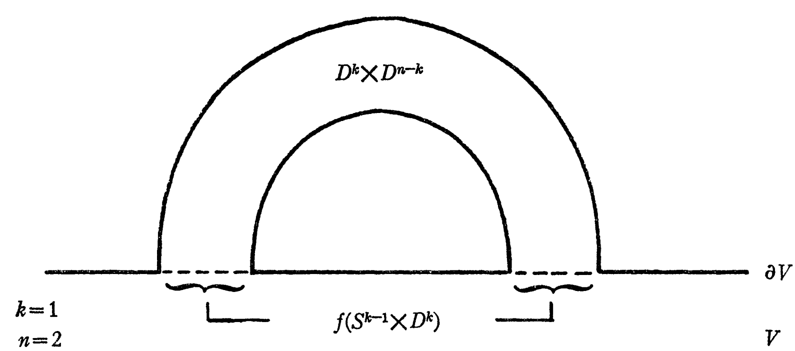

FIG. 3

This construction dates back to the beginnings of the study of manifolds. For example, the classification theorem for compact orientable 2-manifolds may be written as follows (compare [**30**]):

THEOREM 5.1. *Let M be a compact orientable 2-manifold. Then $M=\partial Q$, where Q is formed by attaching 1-handles to D^3.*

The work of Marston Morse on critical point theory implies that any smooth manifold M may be constructed by successively attaching handles (e.g., see [**33**], [**34**], [**39**]); in terms of the definition below, M has a *handle decomposition.* Standard results of combinatorial topology imply a similar result for PL manifolds [**17**, p. 226].

In the definition below, $X\cup\partial X\times[0, 1]$ will be interpreted to mean the disjoint union modulo the identification of $y\in\partial X$ with $(y, 0)\in\partial X\times\{0\}$. We shall assume the manifold M discussed below is either unbounded or compact in order to simplify the definition.

DEFINITION. Let M be a topological manifold. A **handle decomposition** of M is a (finite or denumerable) sequence of compact subspaces $\{M_j\}_{j\in J}$ (J a well-ordered subset of the integers) satisfying:

(i) $M=\bigcup_{j\in J} M_j$ and each M_j is a compact manifold with boundary.

(ii) For all $j\in J$ we have $M_j\subseteq \operatorname{Int} M_{j+1}$; in fact, M_{j+1} is formed by attaching a k-handle to $M_j\cup\partial M_j\times[0, 1]$ (provided j is not maximal in J).

The following result of Kirby and Siebenmann is a straightforward consequence of the arguments used to prove 4.1 and 4.2 [**21**]:

THEOREM 5.2. *Any topological manifold of dimension greater than five has a handle decomposition.*

This is one case of a general principle implicit in [**21**]; namely, results which work for smooth and PL manifolds in dimensions greater than five also work for topological manifolds in dimensions greater than five. Some particular examples are the theorems of Siebenmann [**51**] and Farrell [**13**] and the surgery theory presented in [**61**].

Since any topological manifold of dimension $\leqq 3$ has a PL atlas, and hence a handle decomposition, the only unknown cases occur in dimensions 4 and 5. The nonvanishing of $H^3(\mathrm{Top/PL}; Z_2)$ implies the following negative result due to Siebenmann [**52**]:

THEOREM 5.3. *For $n=4$ or 5 (possibly both) there exists a compact unbounded topological n-manifold that has no handle decomposition.*

Siebenmann also proves in [**52**] that certain fundamental theorems on smooth and PL manifolds in dimensions greater than five fail somewhere in dimensions three, four, and five. Precise knowledge of where these failures occur would be a useful addition to our relatively meager knowledge of manifolds in these dimensions.

References

1. R. Abraham, Foundations of Mechanics, Benjamin, New York, 1967.

2. J. W. Alexander, On the subdivision of 3-space by a polyhedron, Proc. Nat. Acad. Sci. U.S.A., 10 (1924) 6–8.

3. ———, An example of a simply connected surface bounding a region which is not simply connected, Proc. Nat. Acad. Sci. U.S.A., 10 (1924) 8–10.

4. L. Antoine, Sur l'homéomorphie de deux figures et de leurs voisinages, J. Math. Pures Appl., (8) 4 (1921) 221–325.

5. D. Barden, Simply connected five-manifolds, Ann. of Math., 82 (1965) 365–385.

6. G. Birkhoff and S. MacLane, Survey of Modern Algebra, Third edition, Macmillan, New York, 1965.

7. L. E. J. Brouwer, Beweis des Jordanschen Satzes für den n-dimensionalen Raum, Math. Ann., 71 (1912) 314–319.

8. M. Brown, A proof of the generalized Schoenflies Theorem, Bull. Amer. Math. Soc., 66 (1960) 74–76.

9. ———, Locally flat embeddings of topological manifolds, Ann. of Math. 75 (1962), 331–341.

10. ———, and H. Gluck, Stable structures on manifolds I, II, III, Ann. of Math., 79 (1964) 1–58.

11. S. S. Cairns, Triangulation of the manifold of class one, Bull. Amer. Math. Soc., 41 (1935) 549–552.

12. P. E. Conner and E. E. Floyd, Differentiable Periodic Maps, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F. Bd. 33. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1964.

13. F. T. Farrell, The obstruction to fibering a manifold over a circle, Bull. Amer. Math. Soc., 73 (1967) 737–740.

14. M. Greenberg, Lectures on Algebraic Topology, Benjamin, New York, 1967.

15. W. C. Hsiang and J. Shaneson, Fake tori, the annulus conjecture, and the conjectures of Kirby, Proc. Nat. Acad. Sci. U.S.A., 63 (1969) 687–691.

16. ———, Fake tori, Topology of Manifolds (Proc. Univ. of Georgia Institute, 1969), 18–51. Markham, Chicago, 1970.

17. J. F. P. Hudson, Piecewise Linear Topology, Benjamin, New York, 1969.

18. B. von Kérékjárto, Vorlesungen über Topologie, Die Grundlehren der mathematischen Wissenschaften, Bd. 1. J. Springer, Berlin, 1923.

19. R. C. Kirby, Stable homeomorphisms and the annulus conjecture, Ann. of Math., 89 (1969) 575–582.

20. ———, and L. C. Siebenmann, On the triangulation of manifolds and the Hauptvermutung, Bull. Amer. Math. Soc., 75 (1969) 742–749.

21. ———, Foundations of Topology, Notices, Amer. Math. Soc., 16 (1969) 848.

22. S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. I., Interscience Tracts in Pure and Applied Mathematics, vol. 15. Interscience, New York, 1963.

23. R. Lashof, Lees' immersion theorem and the triangulation of manifolds, Bull. Amer. Math. Soc., 75 (1969) 535–538.

24. ———, The immersion approach to triangulation, Topology of Manifolds (Proc. Univ. of Georgia Institute, 1969), 52–56. Markham, Chicago, 1970.

25. ———, and M. Rothenberg, Microbundles and smoothings, Topology, 3 (1964/65) 357–388.

26. ———, Hauptvermutung for manifolds, Conference on the Topology of Manifolds (Michigan State, 1967), 81–105. Prindle, Weber, and Schmidt, Boston, 1968.

27. ———, Triangulation of manifolds I, II, Bull. Amer. Math. Soc., 75 (1965) 750–757.

28. J. Lees, Immersions and surgeries of topological manifolds, Bull. Amer. Math. Soc., 75 (1969) 529–534.

29. S. Lefschetz, Introduction to Topology, Princeton Mathematical Series, vol. 11. Princeton University Press, Princeton, 1949.

30. W. S. Massey, Algebraic Topology; an Introduction, Harcourt, Brace, and World, New York, 1967.

31. B. Mazur, On embeddings of spheres, Acta Math., 105 (1961) 1–17.

32. J. Milnor, On manifolds homeomorphic to the 7-sphere, Ann. of Math., 64 (1956) 399–405.

33. J. Milnor, Morse Theory (notes by M. Spivak and R. Wells), Annals of Mathematics Studies No. 51, Princeton University Press, Princeton, 1963.

34. ———, Lectures on the h-cobordism theorem (notes by L. Siebenmann and J. Sondow), Princeton University Press, Princeton, 1965.

35. E. E. Moise, Affine structures in 3-manifolds, V. The triangulation theorem, Ann. of Math., 56 (1952) 96–114.

36. ———, Affine structures in 3-manifolds, VIII. Invariance of knot types; local tame embedding, Ann. of Math., 59 (1954) 159–170.

37. R. L. Moore, Concerning simple continuous curves, Trans. Amer. Math. Soc., 21 (1920) 333–347.

38. M. Morse, A reduction of the Schoenflies extension problem, Bull. Amer. Math. Soc., 66 (1960) 113–115.

39. ———, and S. S. Cairns, Critical Point Theory in Global Analysis and Differential Topology, Pure and Applied Mathematics Series, Vol. 33, Academic Press, New York, 1969.

40. J. Munkres, Elementary Differential Topology (revised edition), Annals of Mathematics Studies No. 54, Princeton University Press, Princeton, 1966.

41. M. H.A. Newman, Elements of the Topology of Plane Sets of Points, Cambridge University Press, Cambridge, 1951.

42. ———, The engulfing theorem for topological manifolds, Ann. of Math., 84 (1966) 555–571.

43. C. D. Papakyriakopoulos, A new proof for the invariance of the homology groups of a complex (Greek), Bull. Soc. Math. Grèce, 22 (1943) 1–154.

44. H. Poincaré, Cinquième complément à l'Analysis situs, Rend. Circ. Mat. Palermo, 8 (1904) 45–110.

45. T. Radó, Über den Begriff von Riemannsche Fläche, Acta Sci. Math. (Szeged), 2 (1924/26) 101–120.

46. I. Richards, On the classification of noncompact surfaces, Trans. Amer. Math. Soc., 106 (1963) 259–269.

47. J. Rotman, Theory of Groups: an Introduction, Allyn and Bacon, Boston, 1965.

48. A. Schoenflies, Beiträge zur Theorie der Punktmengen III, Math. Ann., 62 (1906) 286–328.

49. J. Shaneson, Wall's surgery obstruction groups for $G \times Z$, Ann. of Math., 90 (1969) 296–334.

50. ———, Non-simply connected surgery and some results in low-dimensional topology, Comment. Math. Helv., 45 (1970) 333–352.

51. L. C. Siebenmann, The obstruction to finding a boundary for an open manifold of dimension greater than five, Ph.D. Thesis, Princeton University, 1965.

52. ———, Disruption of low-dimensional handlebody theory by Rohlin's theorem, Topology of Manifolds (Proc. Univ. of Georgia Institute, 1969), 57–76. Markham, Chicago, 1970.

53. S. Smale, Generalized Poincaré's conjecture in dimensions greater than four, Ann. of Math., 74 (1961) 391–406.

54. E. H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.

55. J. R. Stallings, Polyhedral homotopy-spheres, Bull. Amer. Math. Soc., 66 (1960) 485–488.

56. D. Sullivan, On the Hauptvermutung for manifolds, Bull. Amer. Math. Soc., 73 (1967) 598–600.

57. ———, Triangulating and smoothing homotopy equivalences and homeomorphisms, Geometric Topology Seminar Notes, mimeographed, Princeton University, 1967.

58. R. Thom, Les classes caractéristiques de Pontrjagin des variétés triangulées, Symposium internacional de topología algebraica (Mexico, 1956), 54–67. Universidad Nacional Autónoma de México y UNESCO, Mexico, 1958.

59. O. Veblen, Theory of plane curves in nonmetrical analysis situs, Trans. Amer. Math. Soc., 6 (1905) 83–98.

60. C. T. C. Wall, On homotopy tori and the annulus theorem I, Bull. London. Math. Soc., 1 (1969) 95–97.

61. ———, Surgery on Compact Manifolds, London Mathematical Society Monographs No. 1, Academic Press, New York, 1970.

62 A. H. Wallace, A geometric method in differential topology, Bull. Amer. Math. Soc., 68 (1962) 533–542.

63. J. H. C. Whitehead, On C^1 complexes, Ann. of Math., 41 (1940) 809–824.

64. E. C. Zeeman, The Poincaré conjecture for $n \geqq 5$, Topology of 3-manifolds and Related Topics (Proc. Univ. of Georgia Institute, 1961), 198–204. Prentice-Hall, Englewood Cliffs, N. J., 1962.

REMINISCENCES OF AN OCTOGENARIAN MATHEMATICIAN

L. J. MORDELL,[1] St. John's College, Cambridge, England

It is customary for the fellows of St. John's College, Cambridge, to dine privately on December 27, the birthday of St. John, the Evangelist. The Master proposes a toast to those fellows who have attained the age of eighty since the preceding December 27, and asks each of them to give a talk. As I became eighty on January 28, 1968, it was my turn to do so.

I started off by saying that this was a really great occasion in my life and that I was very grateful to our College for making it possible. I said that it was not an easy matter to make an appropriate speech on such an occasion. Fortunately it was not too difficult for me to do so, as I have recently been reading a book by the well-known and popular American author Dale Carnegie, entitled *How to Stop Worrying and Start Living.* In this, he makes the cogent remark that no man is so happy as when he is talking about himself. He says nothing about the feelings of his listeners.

There are two reasons why I propose to make myself thoroughly and unashamedly happy by talking about myself. The first is that on several occasions, both in England and America, I have been told that I am a legendary character. As it occurs to me that most legendary characters, for example King Arthur, are dead, I wish to show that I have actually existed and am very much alive, and so I shall give some account of the subject so that there will be no doubt about the matter.

The second reason is that there have been many stories, mostly apocryphal, as to how I, a natural born American, came to study at St. John's College. The reason is a very simple and natural one. I do not mean to be boastful or vainglorious, and I wish to apologize if I seem so and to crave your indulgence.

[1] This talk was presented to the Philadelphia Section of the MAA on Nov. 22, 1969 at Swarthmore College. It was given in part to the Fellows of St. John's College on Dec. 27, 1968 and again to the Adam Society, St. John's College, on March 5, 1969.