On the Hauptvermutung

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§1. Introduction.

An abstract simplicial complex K determines a topological space, the polyhedron $|K|$. A **triangulation** (K, f) of a topological space X is a simplicial complex K together with a homeomorphism $f : |K| \longrightarrow X$. A topological space X is **triangulable** if it admits a triangulation (K, f).

The topology of a triangulable space X is determined by the combinatorial topology of the simplicial complex K in any triangulation (K, f) of X.

Hauptvermutung is short for **die Hauptvermutung der kombinatorischen Topologie**, which is German for the main conjecture of combinatorial topology. The conjecture states that the combinatorial topology of a simplicial complex K is determined by the topology of the polyhedron $|K|$. More technically, the conjecture is that triangulations of homeomorphic spaces are combinatorially equivalent, i.e. become isomorphic after subdivision. A triangulable space would then have a canonical class of triangulations. The problem was formulated by Steinitz [44] and Tietze [48] in 1908, and there are statements in Kneser [20] and Alexandroff and Hopf [2, p.152].

The modern version of combinatorial topology is codified in the PL (piecewise linear) category, for which Rourke and Sanderson [35] is the standard reference.

Simplicial Approximation Theorem. *Every continuous map $f : |K| \longrightarrow |L|$ between polyhedra is homotopic to the topological realization $|f'| : |K| = |K'| \longrightarrow |L|$ of a simplicial map $f' : K' \longrightarrow L$, where K' is a simplicial subdivision of K.*

Thus every continuous map of polyhedra is homotopic to a PL map. It does not follow that a homeomorphism of polyhedra is homotopic to a PL homeomorphism.

Hauptvermutung. *Every homeomorphism $f : |K| \longrightarrow |L|$ between polyhedra is homotopic to the topological realization of a simplicial isomorphism $f' : K' \longrightarrow L'$, where K', L' are simplicial subdivisions of K, L, i.e. every homeomorphism of polyhedra is homotopic to a PL homeomorphism.*

This will also be called the **Polyhedral Hauptvermutung**, to distinguish it from the Manifold Hauptvermutung stated below.

The Simplicial Approximation Theorem shows that the homotopy theory of polyhedra is the same as the PL homotopy theory of simplicial complexes. Ever since Seifert and Threlfall [39] standard treatments of algebraic topology have used this correspondence to show that combinatorial homotopy invariants of simplicial complexes (e.g. simplicial homology, the simplicial fundamental group) are in fact homotopy invariants of polyhedra. The Hauptvermutung is not mentioned, allowing the reader to gain the false impression that the topology of polyhedra is the same as the PL topology of simplicial complexes. In fact, the Hauptvermutung has been known for some time to be false, although this knowledge has not yet filtered down to textbook level.

A simplicial complex K is finite if and only if the polyhedron $|K|$ is compact. The Hauptvermutung is only considered here for compact polyhedra. However, the resolution of the conjecture in this case requires an understanding of the difference between the PL and continuous topology of open PL manifolds, which is quantified by the Whitehead group.

The Polyhedral Hauptvermutung is true in low dimensions: it was verified for 2-dimensional manifolds by the classification of surfaces, for all polyhedra of dimension ≤ 2 by Papakyriakopoulos [32], and by Moïse [28] for 3-dimensional manifolds.

Milnor [25] obtained the first counterexamples to the Polyhedral Hauptvermutung in 1961, using Reidemeister torsion and some results on non-compact manifolds of Mazur and Stallings to construct a homeomorphism of compact polyhedra which is not homotopic to a PL homeomorphism. Stallings [43] generalized the construction, showing that any non-trivial h-cobordism determines a counterexample to the Polyhedral Hauptvermutung. These counterexamples arise as homeomorphisms of one-point compactifications of open PL manifolds, and so are non-manifold in nature.

An m-**dimensional combinatorial (or PL) manifold** is a simplicial complex K such that $\mathrm{link}_K(\sigma)$ is a PL $(m-|\sigma|-1)$-sphere for each simplex $\sigma \in K$.

Manifold Hauptvermutung. *Every homeomorphism $f : |K| \longrightarrow |L|$ of the polyhedra of compact m-dimensional PL manifolds is homotopic to a PL homeomorphism.*

Following Milnor's disproof of the Polyhedral Hauptvermutung there was much activity in the 1960's aimed at the Manifold Hauptvermutung – first proving it in special cases, and then disproving it in general.

The Manifold Hauptvermutung is the rel ∂ version of the following conjecture :

Combinatorial Triangulation Conjecture. *Every compact m-dimensional topological manifold M can be triangulated by a PL manifold.*

The Manifold Hauptvermutung and Combinatorial Triangulation Conjecture hold in the low dimensions $m \leq 3$.

The 1963 surgery classification by Kervaire and Milnor of the latter's exotic differentiable spheres led to smoothing theory, which gave a detailed understanding of the relationship between differentiable and PL manifold structures. The subsequent Browder-Novikov-Sullivan-Wall surgery theory of high-dimensional manifolds was initially applied to differentiable and PL manifolds. The theory deals with the homotopy analogues of the Manifold Hauptvermutung and the Combinatorial Triangulation Conjecture, providing the necessary and sufficient algebraic topology to decide whether a homotopy equivalence of m-dimensional PL manifolds $f : K \longrightarrow L$ is homotopic to a PL homeomorphism, and whether an m-dimensional Poincaré duality space is homotopy equivalent to an m-dimensional PL manifold, at least for $m \geq 5$. The 1965 proof by Novikov [31] of the topological invariance of the rational Pontrjagin classes ultimately made it possible to extend the theory to topological manifolds and homeomorphisms, and to resolve the Manifold Hauptvermutung and the Combinatorial Triangulation Conjecture using algebraic K- and L-theory.

In 1969 the surgery classification of PL structures on high-dimensional tori allowed Kirby and Siebenmann to show that the Manifold Hauptvermutung and the Combinatorial Triangulation Conjecture are false in general, and to extend high-dimensional surgery theory to topological manifolds. The book of Kirby and Siebenmann [19] is the definitive account of their work. Some of the late 1960's original work on the Manifold Hauptvermutung was announced at the time, e.g. Sullivan [46], [47], Lashof and Rothenberg [21], Kirby and Siebenmann [18], Siebenmann [42]. However, not all the results obtained have been published. The 1967 papers of Casson [5] and Sullivan [45] are published in this volume, along with the 1968/1972 notes of Armstrong et. al. [3].

Kirby and Siebenmann used the Rochlin invariant to formulate an invariant $\kappa(M) \in H^4(M;\mathbb{Z}_2)$ for any closed topological manifold M, such that, for $\dim(M) \geq 5$, $\kappa(M) = 0$ if and only if M admits a combinatorial triangulation. A homeomorphism $f : |K| \longrightarrow |L|$ of the polyhedra of closed PL manifolds gives rise to an invariant $\kappa(f) \in H^3(L;\mathbb{Z}_2)$ (the rel ∂ combinatorial triangulation obstruction of the mapping cylinder) such that for $\dim(L) \geq 5$ $\kappa(f) = 0$ if and only if f is homotopic to a PL homeomorphism. These obstructions are realized. For $m \geq 5$ and any element $\kappa \in H^3(T^m;\mathbb{Z}_2)$ there exists a combinatorial triangulation (τ^m, f) of T^m with $\kappa(f) = \kappa$, so that for $\kappa \neq 0$ the homeomorphism $f : \tau^m \longrightarrow T^m$ is not homotopic to a PL homeomorphism. For $m \geq 3$, $k \geq 2$ and any $\kappa \in H^4(T^m;\mathbb{Z}_2)$ there exists a closed $(m+k)$-dimensional topological manifold M with a homotopy equivalence $h : M \longrightarrow T^m \times S^k$ such that $\kappa(M) = h^*\kappa$, so that for $\kappa \neq 0$ M does not admit a combinatorial triangulation. Such counterexamples to the Manifold Hauptvermutung and the Combinatorial Triangulation Conjecture in dimensions ≥ 5 can be traced to the 3-dimensional Poincaré homology sphere Σ. See §§3-5 for a more detailed account of the Kirby-Siebenmann obstruction.

§2. The Polyhedral Hauptvermutung.

Theorem. (Milnor [25]) *The Polyhedral Hauptvermutung is false: there exists a homeomorphism $f : |K| \longrightarrow |L|$ of the polyhedra of finite simplicial complexes K, L such that f is not homotopic to a PL homeomorphism.*

The failure of the Polyhedral Hauptvermutung is detected by Whitehead torsion. The construction of the Polyhedral Hauptvermutung counterexamples of Milnor [25] and Stallings [43] will now be recounted, first directly and then using the end obstruction theory of Siebenmann [40]. See Cohen [7] for a textbook account.

Given a topological space X let

$$X^\infty = X \cup \{\infty\}$$

be the one-point compactification. If X is compact then X^∞ is just the union of X and $\{\infty\}$ as topological spaces.

Let $(W, \partial W)$ be a compact n-dimensional topological manifold with non-empty boundary ∂W, so that the interior

$$\dot{W} = W \backslash \partial W$$

is an open n-dimensional manifold. Since ∂W is collared in W (i.e. the inclusion $\partial W = \partial W \times \{0\} \longrightarrow W$ extends to an embedding $\partial W \times [0,1] \longrightarrow W$) the effect of collapsing the boundary to a point is a compact space

$$K = W/\partial W$$

with a homeomorphism

$$K \cong \dot{W}^\infty$$

sending $[\partial W] \in K$ to ∞. Now suppose that $(W, \partial W)$ is a PL manifold with boundary, so that $\dot{W}$ is an open n-dimensional PL manifold, and K is a compact polyhedron such that there is defined a PL homeomorphism

$$\mathrm{link}_K(\infty) \cong \partial W\ .$$

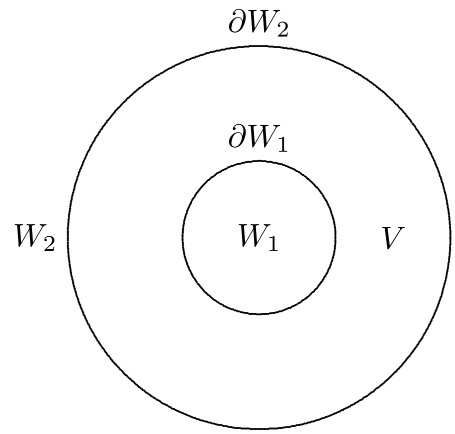
The compact polyhedron K is a closed n-dimensional PL manifold if and only if ∂W is a PL $(n-1)$-sphere. If ∂W is not a PL $(n-1)$-sphere then K is a PL

stratified set with two strata, $\dot{W}$ and $\{\infty\}$.

Suppose given compact n-dimensional PL manifolds with boundary $(W_1, \partial W_1)$, $(W_2, \partial W_2)$ such that

$$W_2 \;=\; W_1 \cup_{\partial W_1} V$$

for an h-cobordism $(V; \partial W_1, \partial W_2)$.



There is defined a PL homeomorphism

$$(V \backslash \partial W_2, \partial W_1) \;\cong\; \partial W_1 \times ([0,1), \{0\})$$

of non-compact n-dimensional PL manifolds with boundary, which is the identity on ∂W_1. The corresponding PL homeomorphism of open n-dimensional PL manifolds $\dot{W}_1 \longrightarrow \dot{W}_2$ compactifies to a homeomorphism of compact polyhedra

$$f \;:\; K_1 \;=\; W_1/\partial W_1 \;=\; \dot{W}_1^\infty \longrightarrow K_2 \;=\; W_2/\partial W_2 \;=\; \dot{W}_2^\infty \;.$$

The homeomorphism f is homotopic to a PL homeomorphism if and only if there exists a PL homeomorphism

$$(V; \partial W_1, \partial W_2) \;\cong\; \partial W_1 \times ([0,1]; \{0\}, \{1\})$$

which is the identity on ∂W_1.

If M is a closed m-dimensional PL manifold then for any $i \geq 1$

$$(W, \partial W) \;=\; M \times (D^i, S^{i-1})$$

is a compact $(m+i)$-dimensional PL manifold with boundary such that

$$\begin{aligned} \dot{W} \;&=\; M \times \mathbb{R}^i \;, \\ W/\partial W \;&=\; \dot{W}^\infty \;=\; M \times D^i / M \times S^{i-1} \;=\; \Sigma^i M^\infty \;. \end{aligned}$$

Milnor [25] applied this construction to obtain the first counterexamples to the Hauptvermutung, using the Reidemeister-Franz-deRham-Whitehead classification of the lens spaces introduced by Tietze [48]. The lens spaces $L(7,1)$, $L(7,2)$ are closed 3-dimensional PL manifolds which are homotopy equivalent but not simple homotopy equivalent, and hence neither PL homeomorphic nor homeomorphic (by the topological invariance of Whitehead torsion). For $i \geq 3$ the compact $(i+3)$-dimensional PL manifolds with boundary

$$\begin{aligned} (W_1, \partial W_1) &= L(7,1) \times (D^i, S^{i-1}) \ , \\ (W_2, \partial W_2) &= L(7,2) \times (D^i, S^{i-1}) \end{aligned}$$

are such that $W_2 = W_1 \cup_{\partial W_1} V$ for an h-cobordism $(V; \partial W_1, \partial W_2)$ with torsion

$$\tau(\partial W_1 \longrightarrow V) \neq 0 \in Wh(\mathbb{Z}_7) \ = \ \mathbb{Z} \oplus \mathbb{Z} \ .$$

(See Milnor [26] for $Wh(\mathbb{Z}_7)$.) The corresponding PL homeomorphism of open $(i+3)$-dimensional PL manifolds

$$\dot{W}_1 \ = \ L(7,1) \times \mathbb{R}^i \longrightarrow \dot{W}_2 \ = \ L(7,2) \times \mathbb{R}^i$$

compactifies to a homeomorphism of compact polyhedra

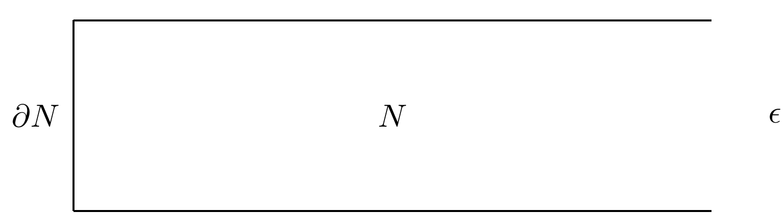
$$f \ : \ K_1 \ = \ \Sigma^i L(7,1)^\infty \longrightarrow K_2 \ = \ \Sigma^i L(7,2)^\infty$$

which is not homotopic to a PL homeomorphism. In fact, f is homotopic to the i-fold suspension of a homotopy equivalence $h : L(7,1) \longrightarrow L(7,2)$ with Whitehead torsion

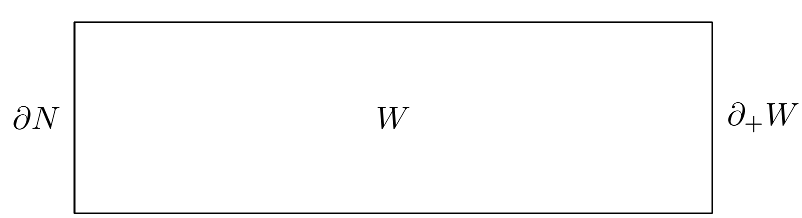
$$\begin{aligned} \tau(h) &= \tau(\partial W_1 \longrightarrow V) + \tau(\partial W_1 \longrightarrow V)^* \\ &= 2\tau(\partial W_1 \longrightarrow V) \neq 0 \in Wh(\mathbb{Z}_7) \ . \end{aligned}$$

The homotopy equivalence h is not homotopic to a homeomorphism (by the topological invariance of Whitehead torsion) let alone a PL homeomorphism.

Let $(N, \partial N)$ be a non-compact n-dimensional PL manifold with a compact boundary ∂N and a tame end ϵ.



A **closure** of the tame end ϵ is a compact n-dimensional PL cobordism $(W; \partial N, \partial_+ W)$



with a PL homeomorphism

$$N \cong W\backslash\partial_+W$$

which is the identity on ∂N, in which case $\pi_1(\partial_+W) = \pi_1(\epsilon)$ and there is defined a homeomorphism

$$W/\partial_+W \cong N^\infty .$$

The **end obstruction** $[\epsilon] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)])$ of Siebenmann [40] is such that $[\epsilon] = 0$ if (and for $n \geq 6$ only if) ϵ admits a closure. The end obstruction has image the Wall finiteness obstruction $[N] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(N)])$, which is such that $[N] = 0$ if and only if N is homotopy equivalent to a compact polyhedron. See Hughes and Ranicki [16] for a recent account of tame ends and the end obstruction.

The closures of high-dimensional tame ends ϵ are classified by the Whitehead group $Wh(\pi_1(\epsilon))$. This is a consequence of:

s-cobordism Theorem. (Barden, Mazur, Stallings)
An n-dimensional PL h-cobordism $(V;U,U')$ with torsion

$$\tau(U \longrightarrow V) = \tau \in Wh(\pi_1(U))$$

is such that $\tau = 0$ if (and for $n \geq 6$ only if) there exists a PL homeomorphism

$$(V;U,U') \cong U \times ([0,1];\{0\},\{1\})$$

which is the identity on U.

Let $(W_1;\partial N,\partial_+W_1)$, $(W_2;\partial N,\partial_+W_2)$ be two closures of an n-dimensional tame end ϵ (as above), so that there are defined PL homeomorphisms

$$N \cong W_1\backslash\partial_+W_1 \cong W_2\backslash\partial_+W_2$$

and a homeomorphism of compact polyhedra

$$f : K_1 = W_1/\partial_+W_1 \longrightarrow K_2 = W_2/\partial_+W_2 .$$

The points

$$\infty_1 = [\partial_+W_1] \in K_1 \quad , \quad \infty_2 = [\partial_+W_2] \in K_2$$

are such that

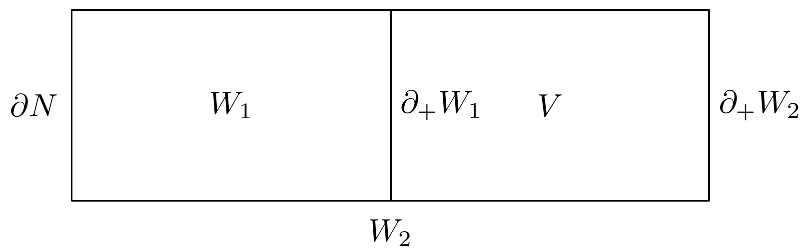
$$\mathrm{link}_{K_1}(\infty_1) = \partial_+W_1 \quad , \quad \mathrm{link}_{K_2}(\infty_2) = \partial_+W_2 .$$

If neither ∂_+W_1 nor ∂_+W_2 is a PL $(n-1)$-sphere then these are the only non-manifold points of K_1, K_2 – any PL homeomorphism $F : K_1 \longrightarrow K_2$ would have to be such that $F(\infty_1) = \infty_2$ and restrict to a PL homeomorphism

$$F : \mathrm{link}_{K_1}(\infty_1) = \partial_+W_1 \longrightarrow \mathrm{link}_{K_2}(\infty_2) = \partial_+W_2 .$$

If ∂_+W_1 is not PL homeomorphic to ∂_+W_2 there will not exist such an F which provides a counterexample to the Hauptvermutung. In any case, for $n \geq 6$ there exists an n-dimensional PL h-cobordism $(V;\partial_+W_1,\partial_+W_2)$ such that up to PL homeomorphism

$$(W_2;\partial N,\partial_+W_2) \;=\; (W_1;\partial N,\partial_+W_1)\cup(V;\partial_+W_1,\partial_+W_2)$$



and the following conditions are equivalent:

(i) the Whitehead torsion

$$\tau \;=\; \tau(\partial_+W_1{\longrightarrow}V)\in Wh(\pi_1(V)) \;=\; Wh(\pi_1(\epsilon))$$

is such that $\tau = 0$,

(ii) there exists a PL homeomorphism

$$(W_1;\partial N,\partial_+W_1) \;\cong\; (W_2;\partial N,\partial_+W_2)$$

which is the identity on ∂N,

(iii) there exists a PL homeomorphism

$$(V;\partial_+W_1,\partial_+W_2) \;\cong\; \partial_+W_1\times([0,1];\{0\},\{1\})$$

which is the identity on ∂_+W_1,

(iv) the homeomorphism $f: K_1{\longrightarrow}K_2$ is homotopic to a PL homeomorphism.

Returning to the construction of Milnor [25], define for any $i\geq 1$ the open $(i+3)$-dimensional PL manifold

$$N \;=\; L(7,1)\times\mathbb{R}^i$$

with a tame end ϵ, which can be closed in the obvious way by

$$(W_1,\partial W_1) \;=\; L(7,1)\times(D^i,S^{i-1})\ .$$

For $i\geq 3$ use the above h-cobordism $(V;\partial W_1,\partial W_2)$ with $\tau(\partial W_1{\longrightarrow}V)\neq 0$ to close ϵ in a non-obvious way, with $W_2 = W_1\cup_{\partial W_1} V$ such that

$$(W_2,\partial W_2) \;=\; L(7,2)\times(D^i,S^{i-1})\ ,$$

and as before there is a homeomorphism of compact polyhedra

$$f\ :\ K_1 \;=\; W_1/\partial W_1 \;=\; \Sigma^iL(7,1)^\infty \longrightarrow K_2 \;=\; W_2/\partial W_2 \;=\; \Sigma^iL(7,2)^\infty$$

which is not homotopic to a PL homeomorphism.

A closed m-dimensional PL manifold M determines a non-compact $(m+1)$-dimensional PL manifold with compact boundary

$$(N,\partial N) \;=\; (M\times[0,1),M\times\{0\})$$

with a tame end ϵ which can be closed, so that

$$[\epsilon] \;=\; 0 \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)]) \;=\; \widetilde{K}_0(\mathbb{Z}[\pi_1(M)]) \;.$$

Assume $m \geq 5$. For any $(m+1)$-dimensional PL h-cobordism $(W;M,M')$ the inclusion $M \subset W\backslash M'$ extends to a PL homeomorphism of open PL manifolds

$$U \;=\; M\times[0,1) \longrightarrow W\backslash M'$$

which compactifies to a homeomorphism of compact polyhedra

$$f \;:\; K \;=\; U^\infty \;=\; M\times[0,1]/M\times\{1\} \;\longrightarrow L \;=\; W/M'$$

such that f is homotopic to a PL homeomorphism if and only if $\tau = 0 \in Wh(\pi_1(M))$. Thus any h-cobordism with $\tau \neq 0$ determines a counterexample to the Polyhedral Hauptvermutung, as was first observed by Stallings [43].

§3. The Rochlin Invariant.

The **intersection form** of a compact oriented $4k$-dimensional manifold with boundary $(M,\partial M)$ is the symmetric form

$$\phi \;:\; H^{2k}(M,\partial M)\times H^{2k}(M,\partial M) \longrightarrow \mathbb{Z} \;;\; (x,y) \;\longmapsto\; \langle x\cup y,[M]\rangle$$

where $[M]\in H_{4k}(M,\partial M)$ is the fundamental class. The **signature** of $(M,\partial M)$ is

$$\sigma(M) \;=\; \text{signature}\,(H^{2k}(M,\partial M),\phi)\in\mathbb{Z} \;.$$

Proposition. *Let M be a closed oriented 4-dimensional topological manifold. For any integral lift $w_2 \in H^2(M)$ of the second Stiefel-Whitney class $w_2(M) \in H^2(M;\mathbb{Z}_2)$*

$$\phi(x,x) \;\equiv\; \phi(x,w_2)\ (\text{mod}\,2) \quad (x\in H^2(M)) \;,$$

and

$$\sigma(M) \;\equiv\; \phi(w_2,w_2)\ (\text{mod}\,8) \;.$$

Proof. See Milnor and Husemoller [27, II.5].

A closed oriented 4-dimensional manifold M is **spin** (i.e. admits a spin structure) if $w_2(M)=0\in H^2(M;\mathbb{Z}_2)$, in which case $\sigma(M)\equiv 0(\text{mod}\,8)$.

Theorem. (Rochlin) *The signature of a closed oriented 4-dimensional PL spin manifold M is such that*

$$\sigma(M)\equiv 0\ (\text{mod}\,16) \;.$$

Proof. See Guillou and Marin [13], Kirby [17, XI].

Definition. (i) The **Rochlin invariant** of a closed oriented 4-dimensional topological spin manifold M is

$$\alpha(M) \ = \ \sigma(M) \in 8\mathbb{Z}/16\mathbb{Z} \ = \ \mathbb{Z}_2 \ .$$

(ii) The **Rochlin invariant** of an oriented 3-dimensional PL homology sphere Σ is defined by

$$\alpha(\Sigma) \ = \ \sigma(W) \in 8\mathbb{Z}/16\mathbb{Z} \ = \ \mathbb{Z}_2 \ ,$$

for any 4-dimensional PL spin manifold $(W, \partial W)$ with boundary $\partial W = \Sigma$.

Proposition. (i) *Let $(M, \partial M)$ be a connected 4-dimensional topological spin manifold with homology 3-sphere boundary $\partial M = \Sigma$. The Rochlin invariant of Σ is expressed in terms of the signature of M and the Kirby-Siebenmann invariant of the stable normal bundle $\nu_M : M \longrightarrow BTOP$ by*

$$\alpha(\Sigma) \ = \ \sigma(M)/8 - \kappa(M) \in H^4(M, \partial M; \mathbb{Z}_2) \ = \ \mathbb{Z}_2 \ .$$

(ii) *Let M be a connected closed oriented 4-dimensional topological spin manifold. The Rochlin invariant of M is the Kirby-Siebenmann invariant of M*

$$\alpha(M) \ = \ \kappa(M) \in H^4(M; \mathbb{Z}_2) \ = \ \mathbb{Z}_2 \ .$$

Proof. (i) By Freedman and Quinn [10, 10.2B], for any 4-dimensional topological spin manifold with boundary $(M, \partial M)$, there exists a 4-dimensional PL spin manifold with boundary $(N, \partial N)$ such that $\partial M = \partial N$, and for any such M, N

$$\tfrac{1}{8}(\sigma(M) - \sigma(N)) \ = \ \kappa(M) \in H^4(M, \partial M; \mathbb{Z}_2) \ = \ \mathbb{Z}_2$$

(ii) Take $\partial M = \emptyset$, $N = \emptyset$ in (i).

Examples. (i) The Poincaré homology 3-sphere

$$\Sigma \ = \ SO(3)/A_5$$

is the boundary of the parallelizable 4-dimensional PL manifold Q obtained by Milnor's E_8-plumbing, such that

$$\sigma(Q) \ = \ 8 \ \in \mathbb{Z} \ , \ \alpha(\Sigma) \ = \ 1 \in \mathbb{Z}_2 \ , \ \kappa(Q) \ = \ 0 \in \mathbb{Z}_2 \ .$$

(ii) Any 3-dimensional topological manifold Σ with the homology of S^3 bounds a contractible topological 4-manifold (Freedman and Quinn [10, 9.3C]). If (Q, Σ) is as in (i) and W is a contractible topological 4-manifold with boundary $\partial W = \Sigma$ there is obtained the Freedman E_8-manifold $M = W \cup_\Sigma Q$, a closed oriented 4-dimensional topological spin manifold such that

$$\sigma(M) \ = \ 8 \ \in \mathbb{Z} \ , \ \alpha(M) \ = \ 1 \in \mathbb{Z}_2 \ , \ \kappa(M) \ = \ 1 \in \mathbb{Z}_2 \ .$$

§4. The Manifold Hauptvermutung.

Theorem. (Kirby-Siebenmann [18], [19])
The Combinatorial Triangulation Conjecture and the Manifold Hauptvermutung are false in each dimension $m \geq 5$: there exist compact m-dimensional topological manifolds without a PL structure (= combinatorial triangulation), and there exist homeomorphisms $f : |K| \longrightarrow |L|$ of the polyhedra of compact m-dimensional PL manifolds K, L which are not homotopic to a PL homeomorphism.

The actual construction of counterexamples required the surgery classification of homotopy tori due to Wall, Hsiang and Shaneson, and Casson using the non-simply-connected surgery theory of Wall [49]. The failure of the Combinatorial Triangulation Conjecture is detected by the Kirby-Siebenmann invariant, which uses the Rochlin invariant to detect the difference between topological and PL bundles. The failure of the Manifold Hauptvermutung is detected by the Casson-Sullivan invariant, which is the rel ∂ version of the Kirby-Siebenmann invariant. For $m \geq 5$ an m-dimensional topological manifold admits a combinatorial triangulation if and only if the stable normal topological bundle admits a PL bundle refinement. A homeomorphism of m-dimensional PL manifolds is homotopic to a PL homeomorphism if and only if it preserves the stable normal PL bundles.

A stable topological bundle η over a compact polyhedron X is classified by the homotopy class of a map

$$\eta \ : \ X \longrightarrow BTOP$$

to a classifying space

$$BTOP \ = \ \varinjlim_{k} BTOP(k) \ .$$

There is a similar result for PL bundles. The classifying spaces $BTOP, BPL$ are related by a fibration sequence

$$TOP/PL \longrightarrow BPL \longrightarrow BTOP \longrightarrow B(TOP/PL) \ .$$

A stable topological bundle $\eta : X \longrightarrow BTOP$ lifts to a stable PL bundle $\widetilde{\eta} : X \longrightarrow BPL$ if and only if the composite

$$X \xrightarrow{\eta} BTOP \longrightarrow B(TOP/PL)$$

is null-homotopic.

Theorem. (Kirby-Siebenmann [18], [19] for $m \geq 5$, Freedman-Quinn [10] for $m = 4$)
(i) *There is a homotopy equivalence*

$$B(TOP/PL) \;\simeq\; K(\mathbb{Z}_2, 4)\ .$$

Given a stable topological bundle $\eta : X \longrightarrow BTOP$ *let*

$$\kappa(\eta) \in [X, B(TOP/PL)] \;=\; H^4(X;\mathbb{Z}_2)$$

be the homotopy class of the composite $X \xrightarrow{\eta} BTOP \longrightarrow B(TOP/PL)$. *The topological bundle* η *lifts to a stable* PL *bundle* $\widetilde{\eta} : X \longrightarrow BPL$ *if and only if* $\kappa(\eta) = 0$.
(ii) *There is a homotopy equivalence*

$$TOP/PL \;\simeq\; K(\mathbb{Z}_2, 3)\ .$$

A topological trivialization $t\ :\ \widetilde{\eta} \simeq \{*\} : X \longrightarrow BTOP$ *of a stable* PL *bundle* $\widetilde{\eta} : X \longrightarrow BPL$ *corresponds to a lift of* $\widetilde{\eta}$ *to a map* $(\widetilde{\eta}, t) : X \longrightarrow TOP/PL$. *It is possible to further refine* t *to a* PL *trivialization if and only if the homotopy class*

$$\kappa(\widetilde{\eta}, t) \in [X, TOP/PL] \;=\; H^3(X;\mathbb{Z}_2)$$

is such that $\kappa(\widetilde{\eta}, t) = 0$.
(iii) *The* **Kirby-Siebenmann invariant** *of a compact* m*-dimensional topological manifold* M *with a* PL *boundary* ∂M *(which may be empty)*

$$\kappa(M) \;=\; \kappa(\nu_M : M \longrightarrow BTOP) \in H^4(M, \partial M;\mathbb{Z}_2)$$

is such that $\kappa(M) = 0 \in H^4(M, \partial M;\mathbb{Z}_2)$ *if and only if there exists a* PL *reduction* $\widetilde{\nu}_M : M \longrightarrow BPL$ *of* $\nu_M : M \longrightarrow BTOP$ *which extends* $\nu_{\partial M} : \partial M \longrightarrow BPL$. *The invariant is such that* $\kappa(M) = 0$ *if (and for* $m \geq 4$ *only if) the* PL *structure on* ∂M *extends to a* PL *structure on* $M \times \mathbb{R}$. *For* $m \geq 5$ *such a* PL *structure on* $M \times \mathbb{R}$ *is determined by a* PL *structure on* M.
(iv) *Let* $f : |K| \longrightarrow |L|$ *be a homeomorphism of the polyhedra of closed* m*-dimensional* PL *manifolds. The mapping cylinder*

$$W \;=\; |K| \times I \cup_f |L|$$

is an $(m+1)$*-dimensional topological manifold with* PL *boundary* $\partial W = |K| \times \{0\} \cup |L|$. *The* **Casson-Sullivan invariant** *of* f *is defined by*

$$\kappa(f) \;=\; \kappa(W) \in H^4(W, \partial W;\mathbb{Z}_2) \;=\; H^3(L;\mathbb{Z}_2)\ .$$

For $m \geq 4$ *the following conditions are equivalent:*

(a) f *is homotopic to a* PL *homeomorphism**
(b) W *has a* PL *structure extending the* PL *structure on* ∂W,
(c) $\kappa(f) = 0 \in H^3(L;\mathbb{Z}_2)$.

---

* *through homeomorphisms* – thanks to Yuli Rudyak for pointing out that this condition was omitted from the original printed edition.

(v) *The Combinatorial Triangulation Conjecture is false for $m \geq 4$: there exist closed m-dimensional topological manifolds M such that $\kappa \neq 0 \in H^4(M;\mathbb{Z}_2)$, which thus do not admit a combinatorial triangulation.*
(vi) *The Manifold Hauptvermutung is false for $m \geq 4$: for every closed m-dimensional PL manifold L and every $\kappa \in H^3(L;\mathbb{Z}_2)$ there exists a closed m-dimensional PL manifold K with a homeomorphism $f : |K| \longrightarrow |L|$ such that $\kappa(f) = \kappa \in H^3(L;\mathbb{Z}_2)$.*

The stable classifying spaces $BPL, BTOP, BG$ for bundles in the PL, topological and homotopy categories are related by a braid of fibrations

$$\begin{tikzcd}
K(\mathbb{Z}_2,3) \arrow[dr] \arrow[rr, bend left] & & BPL \arrow[dr] \arrow[rr, bend left] & & BG \\
& G/PL \arrow[ur] \arrow[dr] & & BTOP \arrow[ur] \arrow[dr, "\kappa"] & \\
G \arrow[ur] \arrow[rr, bend right] & & G/TOP \arrow[ur] \arrow[rr, bend right] & & K(\mathbb{Z}_2,4)
\end{tikzcd}$$

Sullivan determined the homotopy types of the surgery classifying spaces G/PL and G/TOP. See Madsen and Milgram [22] for an account of this determination, and Rudyak [36] for an account of its application to the Manifold Hauptvermutung.

The homotopy groups of G/TOP are the simply-connected surgery obstruction groups

$$\pi_m(G/TOP) = L_m(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 (\operatorname{mod} 4) \\ \mathbb{Z}_2 & \text{if } m \equiv 2 (\operatorname{mod} 4) \\ 0 & \text{if } m \equiv 1,3 (\operatorname{mod} 4) \ . \end{cases}$$

A map $(\eta, t) : S^m \longrightarrow G/TOP$ corresponds to a topological bundle $\eta : S^m \longrightarrow BTOP(k)$ (k large) with a fibre homotopy trivialization $t : \eta \simeq \{*\} : S^m \longrightarrow BG(k)$. Making the degree 1 map $\rho : S^{m+k} \longrightarrow T(\eta)$ topologically transverse regular at $S^m \subset T(\eta)$ gives an m-dimensional normal map by the Browder-Novikov construction

$$(f,b) = \rho| : M^m = \rho^{-1}(S^m) \longrightarrow S^m$$

with $b:\nu_M\longrightarrow\eta$. The homotopy class of (η,t) is the surgery obstruction of (f,b)

$$(\eta,t) \;=\; \sigma_*(f,b)\in\pi_m(G/TOP) \;=\; L_m(\mathbb{Z})\ ,$$

where

$$\sigma_*(f,b) \;=\; \begin{cases} \frac{1}{8}\sigma(M)\in L_{4k}(\mathbb{Z}) \;=\; \mathbb{Z} & \text{if } m=4k \\ c(M)\in L_{4k+2}(\mathbb{Z}) \;=\; \mathbb{Z}_2 & \text{if } m=4k+2 \end{cases}$$

with $c(M)\in\mathbb{Z}_2$ the Kervaire-Arf invariant of the framed $(4k+2)$-dimensional manifold M. Similarly for maps $(\widetilde{\eta},t):S^m\longrightarrow G/PL$.

The low-dimensional homotopy groups of the bundle classifying spaces are given by

$$\pi_m(BPL) \;=\; \pi_m(BO) \;=\; \begin{cases} \mathbb{Z}_2 & \text{if } m=1,2 \\ 0 & \text{if } m=3,5,6,7 \\ \mathbb{Z} & \text{if } m=4 \end{cases}$$

$$\pi_m(BTOP) \;=\; \begin{cases} \mathbb{Z}_2 & \text{if } m=1,2 \\ 0 & \text{if } m=3,5,6,7 \\ \mathbb{Z}\oplus\mathbb{Z}_2 & \text{if } m=4\ . \end{cases}$$

The first Pontrjagin class $p_1(\eta)\in H^4(S^4)=\mathbb{Z}$ and the Kirby-Siebenmann invariant $\kappa(\eta)\in H^4(S^4;\mathbb{Z}_2)=\mathbb{Z}_2$ define isomorphisms

$$\begin{aligned} &\pi_4(BPL) \xrightarrow{\cong} \mathbb{Z}\ ;\ (\widetilde{\eta}:S^4\longrightarrow BPL) \;\longmapsto\; \tfrac{1}{2}p_1(\widetilde{\eta})\ , \\ &\pi_4(BTOP) \xrightarrow{\cong} \mathbb{Z}\oplus\mathbb{Z}_2\ ;\ (\eta:S^4\longrightarrow BTOP) \;\longmapsto\; (\tfrac{1}{2}p_1(\eta),\kappa(\eta))\ . \end{aligned}$$

For any map $(\eta,t):S^4\longrightarrow G/TOP$ with corresponding 4-dimensional normal map $(f,b):M\longrightarrow S^4$

$$\begin{aligned} (\eta,t) \;=\; \sigma_*(f,b) \;=\; \tfrac{1}{8}\sigma(M) \;&=\; -\tfrac{1}{24}p_1(\eta) \\ &\in \pi_4(G/TOP) \;=\; L_4(\mathbb{Z}) \;=\; \mathbb{Z}\ , \end{aligned}$$

by the Hirzebruch signature theorem. In particular, the generator $1\in\pi_4(G/TOP)=\mathbb{Z}$ is represented by a fibre homotopy trivialized topological bundle $\eta:S^4\longrightarrow BTOP$ such that

$$\begin{aligned} p_1(\eta) \;&=\; -24\in H^4(S^4) \;=\; \mathbb{Z}\ , \\ \kappa(\eta) \;&=\; 1\in H^4(S^4;\mathbb{Z}_2). \end{aligned}$$

This corresponds to a normal map $(f,b):M\longrightarrow S^4$ where M is the 4-dimensional Freedman E_8-manifold. For any map $(\widetilde{\eta},t):S^4\longrightarrow G/PL$ and the corresponding 4-dimensional PL normal map $(f,\widetilde{b}):M\longrightarrow S^4$

$$\begin{aligned} &p_1(\widetilde{\eta})\equiv 0(\mathrm{mod}\ 48)\ , \\ &\sigma(M) \;=\; -\tfrac{1}{3}p_1(\widetilde{\eta})\equiv 0(\mathrm{mod}\ 16) \end{aligned}$$

by Rochlin's theorem, so that

$$\begin{aligned}(\widetilde{\eta},t) &= \sigma_*(f,\widetilde{b}) = \tfrac{1}{8}\sigma(M) = -\tfrac{1}{24}p_1(\widetilde{\eta}) \\ &\in \pi_4(G/PL) = 2\mathbb{Z} \subset \pi_4(G/TOP) = \mathbb{Z}\ .\end{aligned}$$

The natural map $G/PL \longrightarrow G/TOP$ induces isomorphisms

$$\pi_m(G/PL) \xrightarrow{\cong} \pi_m(G/TOP) = L_m(\mathbb{Z}) \quad (m \neq 4)$$

and multiplication by 2 in dimension 4

$$2 : \pi_4(G/PL) = \mathbb{Z} \longrightarrow \pi_4(G/TOP) = L_4(\mathbb{Z}) = \mathbb{Z}\ ,$$

and there are isomorphisms

$$\begin{aligned}&\pi_4(G/TOP) \xrightarrow{\cong} \mathbb{Z}\ ;\ ((\eta,t) : S^4 \longrightarrow G/TOP) \longmapsto \tfrac{1}{24}p_1(\eta)\ , \\ &\pi_4(G/PL) \xrightarrow{\cong} \mathbb{Z}\ ;\ ((\widetilde{\eta},t) : S^4 \longrightarrow G/PL) \longmapsto \tfrac{1}{48}p_1(\widetilde{\eta})\ .\end{aligned}$$

See Milgram [24] for a detailed comparison in dimensions ≤ 7 of the homotopy types of $BTOP$, BPL and BG.

By definition, the **structure set** $\mathbb{S}^{TOP}(M)$ of a closed m-dimensional topological manifold M consists of equivalence classes of pairs (L,f) where L is a closed m-dimensional topological manifold and $f : L \longrightarrow M$ is a homotopy equivalence. Equivalence is defined by $(L,f) \sim (L',f')$ if $f'^{-1}f : L \longrightarrow L'$ is homotopic to a homeomorphism. There is a similar definition in the PL category of $\mathbb{S}^{PL}(M)$. The structure sets for $m \geq 5$ (or $m = 4$ and $\pi_1(M)$ good for TOP) fit into a commutative braid of Sullivan-Wall surgery exact sequences of pointed sets

$$\begin{tikzcd}
H^3(M;\mathbb{Z}_2) \arrow[dr] \arrow[rr, bend left] & & {[M,G/PL]} \arrow[dr] \arrow[rr, bend left, "A^{PL}"] & & L_m(\mathbb{Z}[\pi_1(M)]) \\
& \mathbb{S}^{PL}(M) \arrow[ur] \arrow[dr] & & {[M,G/TOP]} \arrow[ur, "A"] \arrow[dr, "\kappa"] & \\
L_{m+1}(\mathbb{Z}[\pi_1(M)]) \arrow[ur] \arrow[rr, bend right] & & \mathbb{S}^{TOP}(M) \arrow[ur] \arrow[rr, bend right] & & H^4(M;\mathbb{Z}_2)
\end{tikzcd}$$

where $L_*(\mathbb{Z}[\pi_1(M)])$ are the surgery obstruction groups and

$$\mathbb{S}^{TOP}(M) \longrightarrow H^4(M;\mathbb{Z}_2) \; ; \; (L,f) \longmapsto (f^{-1})^*\kappa(L) - \kappa(M) \; .$$

The TOP surgery exact sequence was expressed algebraically in Ranicki [33] as an exact sequence of abelian groups

$$\dots \longrightarrow L_{m+1}(\mathbb{Z}[\pi_1(M)]) \longrightarrow \mathbb{S}^{TOP}(M) \longrightarrow H_m(M;\mathbb{L}.) \xrightarrow{A} L_m(\mathbb{Z}[\pi_1(M)])$$

where $\mathbb{L}.$ is the 1-connective quadratic L-spectrum such that

$$\pi_*(\mathbb{L}.) \;=\; L_*(\mathbb{Z}) \;\; (* \geq 1) \; ,$$

the generalized homology groups $H_*(M;\mathbb{L}.)$ are the cobordism groups of sheaves over M of locally quadratic Poincaré complexes over $\mathbb{Z}$, and

$$A \; : \; [M, G/TOP] \;=\; H_m(M;\mathbb{L}.) \longrightarrow L_m(\mathbb{Z}[\pi_1(M)])$$

is the algebraic L-theory assembly map.

Proposition. (Siebenmann [42, §15], Hollingsworth and Morgan [14], Morita [29])
(i) *For any space* M

$$\begin{aligned} &\mathrm{im}(\kappa : [M, BTOP] \longrightarrow H^4(M;\mathbb{Z}_2)) \\ &\quad = \; \mathrm{im}((r_2 \;\, Sq^2) : H^4(M;\mathbb{Z}) \oplus H^2(M;\mathbb{Z}_2) \longrightarrow H^4(M;\mathbb{Z}_2)) \; , \end{aligned}$$

where r_2 *is reduction mod 2.*
(ii) *For a closed* m*-dimensional topological manifold* M *with* $m \geq 5$*, or* $m = 4$ *and* $\pi_1(M)$ *good, the image of the function* $\mathbb{S}^{TOP}(M) \longrightarrow H^4(M;\mathbb{Z}_2)$ *is the subgroup*

$$\mathrm{im}(\kappa : \ker(A) \longrightarrow H^4(M;\mathbb{Z}_2)) \subseteq \mathrm{im}(\kappa : [M, BTOP] \longrightarrow H^4(M;\mathbb{Z}_2)) \; ,$$

with equality if

$$\mathrm{im}(A) \;=\; \mathrm{im}(A^{PL} : [M, G/PL] \longrightarrow L_m(\mathbb{Z}[\pi_1(M)])) \; .$$

Example. (Hsiang and Shaneson [15], Wall [49, 15A], Kirby and Siebenmann [18]). The surgery classification of PL structures on tori is an essential ingredient of the Kirby-Siebenmann structure theory of topological manifolds. The assembly map

$$A \; : \; H_n(T^m;\mathbb{L}.) \longrightarrow L_n(\mathbb{Z}[\mathbb{Z}^m])$$

is an isomorphism for $n \geq m+1$, and a split injection with cokernel $L_0(\mathbb{Z})$ for $n = m$. (This was first obtained geometrically by Shaneson and Wall, and then proved algebraically by Novikov and Ranicki).

The braid of surgery exact sequences of T^m $(m \geq 5)$

$$
\begin{tikzcd}
H^3(T^m;\mathbb{Z}_2) \arrow[dr] \arrow[rr, bend left] & & {[T^m, G/PL]} \arrow[dr] \arrow[rr, bend left, "A^{PL}"] & & L_m(\mathbb{Z}[\mathbb{Z}^m]) \\
& \mathbb{S}^{PL}(T^m) \arrow[ur] \arrow[dr] & & {[T^m, G/TOP]} \arrow[ur, "A"] \arrow[dr, "\kappa"] & \\
L_{m+1}(\mathbb{Z}[\mathbb{Z}^m]) \arrow[ur] \arrow[rr, bend right] & & \mathbb{S}^{TOP}(T^m) \arrow[ur] \arrow[rr, bend right] & & H^4(T^m;\mathbb{Z}_2)
\end{tikzcd}
$$

has

$$\mathbb{S}^{TOP}(T^m) \ = \ 0 \ ,$$

$$\mathbb{S}^{PL}(T^m) \ = \ [T^m, TOP/PL] \ = \ H^3(T^m;\mathbb{Z}_2) \ .$$

Thus every closed m-dimensional topological manifold homotopy equivalent to T^m is homeomorphic to T^m, but does not carry a unique PL structure. A **fake torus** is a closed m-dimensional PL manifold τ^m which is homeomorphic but not PL homeomorphic to T^m. Every element

$$\kappa \neq 0 \in \mathbb{S}^{PL}(T^m) \ = \ H^3(T^m;\mathbb{Z}_2)$$

is represented by a triangulation (τ^m, f) of T^m by a fake torus τ^m such that $\kappa(f) = \kappa$. The homeomorphism $f : \tau^m \longrightarrow T^m$ is not homotopic to a PL homeomorphism, constituting a counterexample to the Manifold Hauptvermutung. The application to topological manifold structure theory makes use of the fact that f lifts to a homeomorphism $\overline{f} : \overline{\tau}^m \longrightarrow \overline{T}^m$ of finite covers which is homotopic to a PL homeomorphism, i.e. every fake torus has a finite cover which is PL homeomorphic to a genuine torus.

Example. The braid of surgery exact sequences of $T^m \times S^k$ $(m+k \geq 5,\ k \geq 2)$

$$\begin{tikzcd}[column sep=small]
H^3(T^m \times S^k;\mathbb{Z}_2) \arrow[dr] \arrow[rr, bend left] & & {[T^m \times S^k, G/PL]} \arrow[dr] \arrow[rr, bend left, "A^{PL}"] & & L_{m+k}(\mathbb{Z}[\mathbb{Z}^m]) \\
& \mathbb{S}^{PL}(T^m \times S^k) \arrow[ur] \arrow[dr] & & {[T^m \times S^k, G/TOP]} \arrow[ur, "A"] \arrow[dr, "\kappa"] & \\
L_{m+k+1}(\mathbb{Z}[\mathbb{Z}^m]) \arrow[ur] \arrow[rr, bend right] & & \mathbb{S}^{TOP}(T^m \times S^k) \arrow[ur] \arrow[rr, bend right] & & H^4(T^m \times S^k;\mathbb{Z}_2)
\end{tikzcd}$$

has

$$\begin{aligned}
\mathbb{S}^{TOP}(T^m \times S^k) &= [T^m, G/TOP]\ , \\
\mathbb{S}^{PL}(T^m \times S^k) &= [T^m, G/PL] \oplus H^{3-k}(T^m;\mathbb{Z}_2)
\end{aligned}$$

with

$$\mathrm{im}(\kappa : \mathbb{S}^{TOP}(T^m \times S^k) \longrightarrow H^4(T^m \times S^k;\mathbb{Z}_2)) = H^4(T^m;\mathbb{Z}_2) \subset H^4(T^m \times S^k;\mathbb{Z}_2)\ .$$

For every element

$$\kappa \in H^4(T^m;\mathbb{Z}_2) \subset H^4(T^m \times S^k;\mathbb{Z}_2)$$

there exists $(L,f) \in \mathbb{S}^{TOP}(T^m \times S^k)$ with $(f^*)^{-1}\kappa(L) = \kappa$. If $\kappa \neq 0$ the closed $(m+k)$-dimensional topological manifold L does not admit a PL structure, constituting a counterexample to the Combinatorial Triangulation Conjecture. (After Freedman $\mathbb{S}^{TOP}(T^m \times S^k) = [T^m, G/TOP]$ also in the case $m+k=4$.) See Siebenmann [42, §2], Kirby and Siebenmann [18, pp. 210-213] for explicit constructions of such high-dimensional torus-related counterexamples to the Hauptvermutung and Combinatorial Triangulation Conjecture, starting from the Milnor E_8-plumbing 4-dimensional PL manifold (Q^4, Σ^3) with boundary the Poincaré homology sphere Σ. See Scharlemann [38] for explicit constructions of manifolds without combinatorial triangulation in the homotopy types of $S^3 \times S^1 \# S^2 \times S^2$, $T^2 \times S^3$ and $\mathbb{CP}^2 \times S^1$.

The original Milnor differentiable exotic spheres arose as the total spaces of PL trivial differentiable sphere bundles over S^4. The original Novikov examples of homotopy equivalences of high-dimensional differentiable manifolds which are not homotopic to diffeomorphisms were constructed using fibre homotopy trivializations of differentiable sphere bundles over S^4. Likewise, fibre homotopy trivial topological sphere bundles over S^4 provided examples of topological manifolds without a combinatorial triangulation:

Example. The structure sets of $S^m \times S^n$ with $m, n \geq 2$ are such that

$$\begin{aligned}\mathbb{S}^{TOP}(S^m \times S^n) &= L_m(\mathbb{Z}) \oplus L_n(\mathbb{Z}) \text{ if } m+n \geq 4 \\ \mathbb{S}^{PL}(S^m \times S^n) &= \widetilde{L}_m(\mathbb{Z}) \oplus \widetilde{L}_n(\mathbb{Z}) \text{ if } m+n \geq 5\end{aligned}$$

where

$$\begin{aligned}\widetilde{L}_m(\mathbb{Z}) &= \pi_m(G/PL) \\ &= \begin{cases} \pi_m(G/TOP) = L_m(\mathbb{Z}) & \text{if } m \neq 4 \\ 2\pi_4(G/TOP) = 2L_4(\mathbb{Z}) & \text{if } m = 4 \end{cases}\end{aligned}$$

(Ranicki [33, 20.4]). For any element $(W, f) \in \mathbb{S}^{TOP}(S^m \times S^n)$ it is possible to make the homotopy equivalence $f : W \longrightarrow S^m \times S^n$ topologically transverse regular at $S^m \times \{*\}$ and $\{*\} \times S^n \subset S^m \times S^n$. The restrictions of f are normal maps

$$\begin{aligned}(f_M, b_M) &= f| : M^m = f^{-1}(S^m \times \{*\}) \longrightarrow S^m , \\ (f_N, b_N) &= f| : N^n = f^{-1}(\{*\} \times S^n) \longrightarrow S^n\end{aligned}$$

such that

$$(W, f) = (\sigma_*(f_M, b_M), \sigma_*(f_N, b_N)) \in \mathbb{S}^{TOP}(S^m \times S^n) = L_m(\mathbb{Z}) \oplus L_n(\mathbb{Z}) .$$

Every element

$$\begin{aligned}x \in L_m(\mathbb{Z}) &= \pi_m(G/TOP) \\ &= \pi_{m+1}(B\widetilde{TOP}(n+1) \longrightarrow BG(n+1)) \quad (n \geq 2)\end{aligned}$$

is realized by a topological block bundle

$$\eta : S^m \longrightarrow B\widetilde{TOP}(n+1)$$

with a fibre homotopy trivial topological sphere bundle

$$S^n \longrightarrow S(\eta) \longrightarrow S^m .$$

Making the degree 1 map $\rho : S^{m+n} \longrightarrow T(\eta)$ topologically transverse regular at $S^m \subset T(\eta)$ gives an m-dimensional normal map

$$(f_M, b_M) = \rho| : M^m = \rho^{-1}(S^m) \longrightarrow S^m$$

with $b_M : \nu_M \longrightarrow \eta$, such that the surgery obstruction is

$$\sigma_*(f_M, b_M) = x \in L_m(\mathbb{Z}) .$$

The closed $(m+n)$-dimensional topological manifold $S(\eta)$ is equipped with a homotopy equivalence $f : S(\eta) \longrightarrow S^m \times S^n$ such that

$$(S(\eta), f) = (x, 0) \in \mathbb{S}^{TOP}(S^m \times S^n) = L_m(\mathbb{Z}) \oplus L_n(\mathbb{Z}) ,$$

where $f^{-1}(S^m \times \{*\}) = M$. The normal bundle of $S(\eta)$ is classified by

$$\nu_{S(\eta)} : S(\eta) \xrightarrow{f} S^m \times S^n \xrightarrow{proj.} S^m \xrightarrow{-\eta} BTOP ,$$

with the Kirby-Siebenmann invariant given by

$$\kappa(S(\eta)) \;=\; \kappa(\eta) \;=\; \begin{cases} x(\mathrm{mod}2) & \text{if } m=4 \\ 0 & \text{if } m \neq 4 \end{cases}$$
$$\in \mathrm{im}(H^4(S^m;\mathbb{Z}_2)\longrightarrow H^4(S(\eta);\mathbb{Z}_2))$$

where

$$\begin{aligned} H^4(S^m;\mathbb{Z}_2) \;&=\; \mathrm{coker}(\widetilde{L}_m(\mathbb{Z})\longrightarrow L_m(\mathbb{Z})) \\ &=\; \pi_{m-1}(TOP/PL) \;=\; \begin{cases} \mathbb{Z}_2 & \text{if } m=4 \\ 0 & \text{if } m\neq 4\ . \end{cases} \end{aligned}$$

The surgery classifying space G/TOP fits into a fibration sequence

$$G/TOP \longrightarrow B\widetilde{TOP}(n) \longrightarrow BG(n)$$

for any $n \geq 3$, by a result of Rourke and Sanderson [34]. The generator

$$\begin{aligned} 1 \in L_4(\mathbb{Z}) \;&=\; \pi_5(B\widetilde{TOP}(3)\longrightarrow BG(3)) \\ &=\; \pi_4(G/TOP) \;=\; \mathbb{Z} \end{aligned}$$

is represented by a map $(\eta,t) : S^4\longrightarrow G/TOP$ corresponding to a topological block bundle $\eta : S^4\longrightarrow B\widetilde{TOP}(3)$ with a fibre homotopy trivialization $t : \eta \simeq \{*\} : S^4\longrightarrow BG(3)$, such that

$$\begin{aligned} p_1(\eta) \;&=\; -24 \in H^4(S^4;\mathbb{Z}) \;=\; \mathbb{Z}\ , \\ \kappa(\eta) \;&=\; 1 \in H^4(S^4;\mathbb{Z}_2) \;=\; \mathbb{Z}_2\ . \end{aligned}$$

For every $n \geq 2$ the closed $(n+4)$-dimensional topological manifold

$$W^{n+4} \;=\; S(\eta \oplus \epsilon^{n-2})$$

is the total space of a fibre homotopy trivial non-PL topological sphere bundle $\eta \oplus \epsilon^{n-2}$

$$S^n \longrightarrow W \longrightarrow S^4\ ,$$

with a homotopy equivalence $f : W\longrightarrow S^4 \times S^n$. The element

$$(W,f) \;=\; (1,0) \neq (0,0) \in \mathbb{S}^{TOP}(S^4 \times S^n) \;=\; L_4(\mathbb{Z}) \oplus L_n(\mathbb{Z})$$

realizes the generator

$$\begin{aligned} x \;=\; 1 \in L_4(\mathbb{Z}) \;&=\; \pi_5(B\widetilde{TOP}(n+1)\longrightarrow BG(n+1)) \\ &=\; \pi_4(G/TOP) \;=\; \mathbb{Z}\ . \end{aligned}$$

The manifold W does not admit a combinatorial triangulation, with

$$\kappa(W) \;=\; 1 \in H^4(W;\mathbb{Z}_2) \;=\; \mathbb{Z}_2$$

and $M^4 = f^{-1}(S^4 \times \{*\}) \subset W$ the 4-dimensional Freedman E_8-manifold.

§5. Homology Manifolds.

An m**-dimensional homology manifold** is a space X such that the local homology groups at each $x \in X$ are given by

$$\begin{aligned} H_r(X, X\backslash\{x\}) &= H_r(\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \\ &= \begin{cases} \mathbb{Z} & \text{if } r = m \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

An m-dimensional topological manifold is an m-dimensional homology manifold.

The local homology groups of the polyhedron $|K|$ of a simplicial complex K at $x \in |K|$ are such that

$$H_*(|K|, |K|\backslash\{x\}) = \widetilde{H}_{*-|\sigma|-1}(\mathrm{link}_K(\sigma))$$

if $x \in \mathrm{int}(\sigma)$ for a simplex $\sigma \in K$.

An m**-dimensional combinatorial homology manifold** is a simplicial complex K such that for each $\sigma \in K$

$$H_*(\mathrm{link}_K(\sigma)) = H_*(S^{m-|\sigma|-1}) .$$

Similarly for a **combinatorial homotopy manifold**.

A PL manifold is a combinatorial homotopy manifold. A combinatorial homotopy manifold is a combinatorial homology manifold. The polyhedron of a simplicial complex K is an m-dimensional homology manifold $|K|$ if and only if K is an m-dimensional combinatorial homology manifold.

Example. For $m \geq 5$ the double suspension of any $(m-2)$-dimensional combinatorial homology sphere Σ is an m-dimensional combinatorial homology manifold K such that the polyhedron $|K|$ is a topological manifold homeomorphic to S^m (Edwards, Cannon). The combinatorial homology manifold K is a combinatorial homotopy manifold if and only if Σ is simply-connected.

More generally:

Theorem. (Edwards [8]) *For $m \geq 5$ the polyhedron of an m-dimensional combinatorial homology manifold K is an m-dimensional topological manifold $|K|$ if and only if* $\mathrm{link}_K(\sigma)$ *is simply-connected for each vertex $\sigma \in K$.*

This includes as a special case the result of Siebenmann [41] that for $m \geq 5$ the polyhedron of an m-dimensional combinatorial homotopy manifold K is an m-dimensional topological manifold $|K|$.

Triangulation Conjecture. *Every compact m-dimensional topological manifold can be triangulated by a combinatorial homology manifold.*

The triangulation need not be combinatorial, i.e. the combinatorial homology manifold need not be a PL manifold.

It follows from the properties of Casson's invariant for oriented homology 3-spheres that the 4-dimensional Freedman E_8-manifold cannot be triangulated (Akbulut and McCarthy [1, p.xvi]), so that:

Theorem. *The Triangulation Conjecture is false for* $m = 4$.

The Triangulation Conjecture is unresolved for $m \geq 5$. The Kirby-Siebenmann examples of topological manifolds without a combinatorial triangulation are triangulable.

Definition. A **manifold homology resolution** (M, f) of a space X is a compact m-dimensional topological manifold M with a surjective map $f : M \longrightarrow X$ such that the point inverses $f^{-1}(x)$ $(x \in X)$ are acyclic. Similarly for **manifold homotopy resolution**, with contractible point inverses.

A space X which admits an m-dimensional manifold homology resolution is an m-dimensional homology manifold. Bryant, Ferry, Mio and Weinberger [4] have constructed compact ANR homology manifolds in dimensions $m \geq 5$ which do not admit a manifold homotopy resolution.

Let θ^H_3 (resp. θ^h_3) be the Kervaire-Milnor cobordism group of oriented 3-dimensional combinatorial homology (resp. homotopy) spheres modulo those which bound acyclic (resp. contractible) 4-dimensional PL manifolds, with addition given by connected sum.

Given a finite m-dimensional combinatorial homology manifold K define

$$c^H(K) \;=\; \sum_{\sigma \in K^{(m-4)}} [\mathrm{link}_K(\sigma)]\sigma \in H_{m-4}(K; \theta^H_3) \;=\; H^4(K; \theta^H_3) \;.$$

Similarly, given a finite m-dimensional combinatorial homotopy manifold K define

$$c^h(K) \;=\; \sum_{\sigma \in K^{(m-4)}} [\mathrm{link}_K(\sigma)]\sigma \in H_{m-4}(K; \theta^h_3) \;=\; H^4(K; \theta^h_3) \;.$$

Theorem. (Cohen [6], Sato [37], Sullivan [47, pp. 63–65])
(i) *An m-dimensional combinatorial homology manifold K is such that*

$$c^H(K) \;=\; 0 \in H^4(K; \theta^H_3)$$

if (and for $m \geq 5$ only if) K has a PL manifold homology resolution.
(ii) *An m-dimensional combinatorial homotopy manifold K is such that*

$$c^h(K) \;=\; 0 \in H^4(K; \theta^h_3)$$

if (and for $m \geq 5$ only if) K has a PL manifold homotopy resolution.

The natural map $\theta^h_3 \longrightarrow \theta^H_3$ is such that for a finite m-dimensional combinatorial homotopy manifold K

$$H_{m-4}(K;\theta^h_3) \longrightarrow H_{m-4}(K;\theta^H_3) \ ; \ c^h(K) \longmapsto c^H(K) \ .$$

Every oriented 3-dimensional combinatorial homology sphere Σ bounds a parallelizable 4-dimensional PL manifold W, allowing the Rochlin invariant of Σ to be defined by

$$\alpha(\Sigma) \ = \ \sigma(W) \in 8\mathbb{Z}/16\mathbb{Z} \ = \ \mathbb{Z}_2$$

as in §3 above. The Rochlin invariant defines a surjection

$$\alpha \ : \ \theta^H_3 \longrightarrow \mathbb{Z}_2 \ ; \ \Sigma \longmapsto \alpha(\Sigma) \ ,$$

with $\alpha(\Sigma) = 1 \in \mathbb{Z}_2$ for the Poincaré homology 3-sphere Σ.

Remarks. (i) Fintushel and Stern [9] applied Donaldson theory to show that the kernel of $\alpha : \theta^H_3 \longrightarrow \mathbb{Z}_2$ is infinitely generated.
(ii) The composite

$$\theta^h_3 \longrightarrow \theta^H_3 \xrightarrow{\alpha} \mathbb{Z}_2$$

is 0, by a result of Casson (Akbulut and McCarthy [1, p.xv]).

The exact sequence of coefficient groups

$$0 \longrightarrow \ker(\alpha) \longrightarrow \theta^H_3 \xrightarrow{\alpha} \mathbb{Z}_2 \longrightarrow 0$$

induces a cohomology exact sequence

$$\begin{aligned} \dots \longrightarrow H^n(M;\ker(\alpha)) \longrightarrow H^n(M;\theta^H_3) &\xrightarrow{\alpha} H^n(M;\mathbb{Z}_2) \\ &\xrightarrow{\delta} H^{n+1}(M;\ker(\alpha)) \longrightarrow \dots \end{aligned}$$

for any space M.

Theorem. (Galewski-Stern [11], [12], Matumoto [23])
(i) *The Kirby-Siebenmann invariant $\kappa(M) \in H^4(M;\mathbb{Z}_2)$ of a compact m-dimensional topological manifold M is such that*

$$\delta\kappa(M) \ = \ 0 \in H^5(M;\ker(\alpha))$$

if (and for $m \geq 5$ only if) M is triangulable. If M is triangulable then for any triangulation $(K, f : |K| \longrightarrow M)$

$$\begin{aligned} \kappa(M) \ &= \ f_*\alpha(c^H(K)) \in \mathrm{im}(\alpha : H_{m-4}(M;\theta^H_3) \longrightarrow H_{m-4}(M;\mathbb{Z}_2)) \\ &= \ \mathrm{im}(\alpha : H^4(M;\theta^H_3) \longrightarrow H^4(M;\mathbb{Z}_2)) \\ &= \ \ker(\delta : H^4(M;\mathbb{Z}_2) \longrightarrow H^5(M;\ker(\alpha))) \ . \end{aligned}$$

(ii) *Every finite m-dimensional combinatorial homology manifold K for $m \geq 5$ admits a topological manifold homotopy resolution $(M, f : |K| \longrightarrow M)$ such that M is a triangulable m-dimensional topological manifold with*

$$\begin{aligned}\kappa(M) \;&=\; f_*\alpha(c^H(K)) \in \mathrm{im}(\alpha : H_{m-4}(M;\theta_3^H) \longrightarrow H_{m-4}(M;\mathbb{Z}_2)) \\ &=\; \mathrm{im}(\alpha : H^4(M;\theta_3^H) \longrightarrow H^4(M;\mathbb{Z}_2))\ .\end{aligned}$$

(iii) *The Triangulation Conjecture is true for every $m \geq 5$ if and only if the surjection $\alpha : \theta_3^H \longrightarrow \mathbb{Z}_2$ splits, i.e. if and only if there exists a 3-dimensional combinatorial homology sphere Σ such that $\alpha(\Sigma) = 1 \in \mathbb{Z}_2$ and $2(\Sigma) = 0 \in \theta_3^H$.*
(iv) *The stable classifying spaces $BPL, BTOP, BH$ for the bundle theories associated to PL, topological and combinatorial homology manifolds are related by a braid of fibrations*

$$\begin{tikzcd}
K(\theta_3^H,3) \arrow[dr, "\alpha"] \arrow[rr, bend left] & & BPL \arrow[dr] \arrow[rr, bend left] & & BTOP \arrow[dr, "\kappa"] \arrow[rr, bend left] & & K(\ker(\alpha),5) \\
& K(\mathbb{Z}_2,3) \arrow[ur] \arrow[dr, "\delta"] & & BH \arrow[ur] \arrow[dr, "c^H"] & & K(\mathbb{Z}_2,4) \arrow[ur, "\delta"] & \\
& & K(\ker(\alpha),4) \arrow[ur] \arrow[rr, bend right] & & K(\theta_3^H,4) \arrow[ur, "\alpha"] & &
\end{tikzcd}$$

The Cohen-Sato-Sullivan PL manifold homology resolution obstruction $c^H(K) \in H^4(M;\theta_3^H)$ of an m-dimensional combinatorial homology manifold K is the homotopy class of the composite

$$K \xrightarrow{\nu_K} BH \xrightarrow{c^H} K(\theta_3^H, 4)\ .$$

The Galewski-Matumoto-Stern triangulation obstruction $\delta\kappa(M) \in H^5(M;\ker(\alpha))$ of an m-dimensional topological manifold M is the homotopy class of the composite

$$M \xrightarrow{\nu_M} BTOP \longrightarrow K(\ker(\alpha), 5)\ .$$

Example. Let (Q^4, Σ) be the Milnor 4-dimensional PL E_8-manifold with boundary the Poincaré 3-dimensional homology sphere, such that $\sigma(Q) = 8$, $\kappa(Q) = 0$, $\alpha(\Sigma) = 1$. Coning off the boundary of Q defines a 4-dimensional combinatorial homology manifold $K = Q \cup_\Sigma c\Sigma$ such that

$$c^H(K) \;=\; [\Sigma] \neq 0 \in H^4(K;\theta_3^H) \;=\; \theta_3^H\ ,$$

so that K does not have a PL manifold homology resolution. Let (W, Σ) be the contractible Freedman 4-dimensional topological E_8-manifold, such that $\sigma(W) =$

8, $\kappa(W) = 1$ (as at the end of §3). The polyhedron $|K|$ admits a manifold homotopy resolution $f : M = Q \cup_\Sigma W \longrightarrow |K|$, with the non-triangulable closed 4-dimensional topological manifold M such that

$$\kappa(M) \;=\; [f^* c^H(K)] \;\neq\; 0 \in \mathrm{im}(\alpha : H^4(M;\theta_3^H) \longrightarrow H^4(M;\mathbb{Z}_2)) \;.$$

The product $|K| \times S^1$ is a 5-dimensional topological manifold, with $f \times 1 : M \times S^1 \longrightarrow |K| \times S^1$ homotopic to a homeomorphism triangulating $M \times S^1$ by $K \times S^1$. The Kirby-Siebenmann invariant of $M \times S^1$ is

$$\kappa(M \times S^1) \;=\; p^*\kappa(M) \;\neq\; 0 \in \mathrm{im}(\alpha : H^4(M \times S^1;\theta_3^H) \longrightarrow H^4(M \times S^1;\mathbb{Z}_2))$$

with $p : M \times S^1 \longrightarrow M$ the projection, so that $M \times S^1$ is a triangulable 5-dimensional topological manifold without a combinatorial triangulation. In fact, $M \times S^1$ is not even homotopy equivalent to a 5-dimensional PL manifold.

The rel ∂ version of the Cohen-Sato-Sullivan PL manifold resolution obstruction theory applies to the problem of deforming a PL map of PL manifolds with acyclic (resp. contractible) point inverses to a PL homeomorphism, and the rel ∂ version of the Galewski-Matumoto-Stern triangulation obstruction theory applies to the problem of deforming a homeomorphism of PL manifolds to a PL map with acyclic point inverses, as follows.

Let $f : K \longrightarrow L$ be a PL map of compact m-dimensional PL manifolds, with acyclic (resp. contractible) point inverses $f^{-1}(x)$ $(x \in L)$. The mapping cylinder $W = K \times I \cup_f L$ is an $(m+1)$-dimensional combinatorial homology (resp. homotopy) manifold with PL manifold boundary and a PL map

$$(g; f, 1) \;:\; (W; K, L) \longrightarrow L \times (I; \{0\}, \{1\})$$

with acyclic (resp. contractible) point inverses. For each simplex $\sigma \in L$ let $D(\sigma, L)$ be the dual cell in the barycentric subdivision L' of L, such that there is a PL homeomorphism

$$(D(\sigma, L), \partial D(\sigma, L)) \;\cong\; (D^{m-|\sigma|}, S^{m-|\sigma|-1}) \;.$$

The combinatorial $(m+1-|\sigma|)$-dimensional homology (resp. homotopy) manifold

$$(W_\sigma, \partial W_\sigma) \;=\; g^{-1}(D(\sigma, L) \times I, \partial(D(\sigma, L) \times I))$$

is such that the restriction

$$g| \;:\; (W_\sigma, \partial W_\sigma) \longrightarrow (D(\sigma, L) \times I, \partial(D(\sigma, L) \times I)) \;\cong\; (D^{m+1-|\sigma|}, S^{m-|\sigma|})$$

is a homology (resp. homotopy) equivalence, with

$$\partial W_\sigma \;=\; f^{-1}D(\sigma, L) \cup g^{-1}(\partial D(\sigma, L) \times I) \cup D(\sigma, L) \;.$$

If f has acyclic point inverses define

$$\begin{aligned} c^H(f) \;&=\; c^H_\partial(W; K, L) \\ &=\; \sum_{\sigma \in L^{(m-3)}} [\partial W_\sigma]\sigma \in H_{m-3}(L;\theta_3^H) \;=\; H^3(L;\theta_3^H) \;, \end{aligned}$$

and if f has contractible point inverses define

$$\begin{aligned} c^h(f) &= c^h_\partial(W;K,L) \\ &= \sum_{\sigma\in L^{(m-3)}} [\partial W_\sigma]\sigma \in H_{m-3}(L;\theta^h_3) \;=\; H^3(L;\theta^h_3)\ . \end{aligned}$$

Proposition. *A PL map of compact m-dimensional PL manifolds* $f : K\longrightarrow L$ *with acyclic (resp. contractible) point inverses is such that* $c^H(f) = 0$ *(resp.* $c^h(f) = 0$*) if (and for* $m \geq 5$ *only if)* f *is concordant to a PL homeomorphism, i.e. homotopic to a PL homeomorphism through PL maps with acyclic (resp. contractible) point inverses.*

Remark. Cohen [6] actually proved that for $m \geq 5$ a PL map $f : K\longrightarrow L$ of m-dimensional combinatorial homotopy manifolds with contractible point inverses is homotopic through PL maps with contractible point inverses to a PL map $F : K\longrightarrow L$ which is a homeomorphism. If K, L are PL manifolds then F can be chosen to be a PL homeomorphism, so that $c^h(f) = 0$.

Returning to the Manifold Hauptvermutung, we have:

Proposition. *Let* $f : |K|\longrightarrow|L|$ *be a homeomorphism of the polyhedra of compact m-dimensional PL manifolds* K, L. *The Casson-Sullivan invariant* $\kappa(f) \in H^3(L;\mathbb{Z}_2)$ *is such that*

$$\delta\kappa(f) \;=\; 0 \in H^4(L;\ker(\alpha))$$

if (and for $m \geq 5$ *only) if* f *is homotopic* to a PL map* $F : K\longrightarrow L$ *with acyclic point inverses, in which case* $\kappa(f)$ *is the image under* α *of the Cohen-Sato-Sullivan invariant* $c^H(F)$

$$\begin{aligned} \kappa(f) &= \alpha(c^H(F)) \in \mathrm{im}(\alpha : H^3(L;\theta^H_3)\longrightarrow H^3(L;\mathbb{Z}_2)) \\ &= \ker(\delta : H^3(L;\mathbb{Z}_2)\longrightarrow H^4(L;\ker(\alpha)))\ . \end{aligned}$$

Proof. The mapping cylinder $W = |K| \times I \cup_f |L|$ of f is an $(m + 1)$-dimensional topological manifold with PL boundary $\partial W = |K| \times \{0\} \cup |L|$, and with a homeomorphism

$$(g; f, 1) \;:\; (W; |K|, |L|) \longrightarrow |L| \times (I; \{0\}, \{1\})\ .$$

By the rel ∂ version of the Galewski-Matumoto-Stern obstruction theory $\kappa(f) \in \mathrm{im}(\alpha)$ if (and for $m \geq 5$ only if) the triangulation of ∂W extends to a triangulation of W, in which case it is possible to approximate g by a PL map G such that the restriction $G| = F : K\longrightarrow L$ is a PL map homotopic to f with acyclic point inverses and $\kappa(f) = \alpha(c^H(F))$.

Corollary. *If the Triangulation Conjecture is true for every* $m \geq 5$ *(i.e. if* α :

* *through homeomorphisms*

$\theta_3^H \longrightarrow \mathbb{Z}_2$ splits) every homeomorphism of compact m-dimensional PL manifolds is homotopic to a PL map with acyclic point inverses.

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