A note on characteristic classes: Euler, Stiefel-Whitney, Chern and Pontrjagin

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1 Introduction

For any n-dimensional complex vector bundle $\beta : X \to BU(n)$ the Chern classes $c_i(\beta) \in H^{2i}(X)$ have mod 2 reductions the even-dimensional Stiefel-Whitney classes $w_{2i}(\beta^{\mathbb{R}}) \in H^{2i}(X;\mathbb{Z}_2)$ of the underlying $2n$-dimensional real vector bundle $\beta^{\mathbb{R}} : X \to BO(2n)$. This is well-known, being Problem 14-B in Milnor and Stasheff's book *Characteristic classes* (Princeton, 1974). I have written out an explicit proof in the notes below, using the nice derivation of the Stiefel-Whitney and Chern classes from the Euler class in Chapter 17 of Kreck's book *Differential algebraic topology* (AMS, 2011).

2 The Thom class

An n-dimensional real vector bundle

$$\mathbb{R}^n \longrightarrow E(\alpha) \xrightarrow{\ p\ } X$$

is classified by a map $\alpha : X \to BO(n)$. The Thom space of α is the one-point compactification

$$T(\alpha) \ = \ E(\alpha)^\infty$$

(assuming that X is compact). The mod 2 Thom class

$$U_2(\alpha) \in \dot{H}^n(T(\alpha);\mathbb{Z}_2) \ (\dot{H} = \text{reduced})$$

is characterized by the property that for any $x \in X$ the inclusion

$$i_x \ : \ p^{-1}(x)^\infty \ = \ S^n \to T(\alpha)$$

is such that

$$(i_x)^*(U_2(\alpha)) \ = \ 1 \in \dot{H}^n(S^n;\mathbb{Z}_2) \ = \ \mathbb{Z}_2 \ .$$

An oriented n-dimensional real vector bundle $\alpha : X \to BSO(n)$ has an integral Thom class $U(\alpha) \in \dot{H}^n(T(\alpha))$ such that

$$(i_x)^*(U(\alpha)) \ = \ 1 \in \dot{H}^n(S^n) \ = \ \mathbb{Z} \ .$$

3 The Euler class

Given an n-dimensional real vector bundle $\alpha : X \to BO(n)$ let $z : X \to T(\alpha)$ be the zero section, with $z(x) = 0 \in p^{-1}(x)$. The mod 2 Euler class $e_2(\alpha) \in H^n(X;\mathbb{Z}_2)$ is

$$e_2(\alpha) \ = \ z^*U_2(\alpha) \in H^n(X;\mathbb{Z}_2) \ .$$

Product formula: the Whitney sum of real vector bundles $\alpha_1 : X \to BO(n_1)$, $\alpha_2 : X \to BO(n_2)$ is a vector bundle $\alpha_1 \oplus \alpha_2 : X \to BO(n_1 + n_2)$ with mod 2 Euler class the cup product

$$e_2(\alpha_1 \oplus \alpha_2) \ = \ e_2(\alpha_1) \cup e_2(\alpha_2) \in H^{n_1+n_2}(X;\mathbb{Z}_2) \ .$$

The Euler class of an oriented n-dimensional real vector bundle $\alpha : X \to BSO(n)$ is

$$e(\alpha) \ = \ z^*U(\alpha) \in H^n(X)$$

with mod 2 reduction

$$[e(\alpha)]_2 \ = \ e_2(\alpha) \in \dot{H}^n(T(\alpha);\mathbb{Z}_2) \ .$$

The Whitney sum of oriented real vector bundles α_1, α_2 is an oriented real vector bundle $\alpha_1 \oplus \alpha_2$ with Euler class

$$e(\alpha_1 \oplus \alpha_2) \ = \ e(\alpha_1) \cup e(\alpha_2) \in H^{n_1+n_2}(X) \ .$$

Given an n-dimensional complex vector bundle $\beta : X \to BU(n)$ let $\beta^{\mathbb{R}} : X \to BSO(2n)$ be the underlying oriented $2n$-dimensional real vector bundle. The Euler class of β is defined to be the Euler class of $\beta^{\mathbb{R}}$

$$e(\beta) \ = \ e(\beta^{\mathbb{R}}) \in H^{2n}(X) \ .$$

4 The Chern classes

Use the complex mutiplication action $S^1 \times \mathbb{C} \to \mathbb{C}$ and the free S^1-action on S^∞ to construct the universal complex line bundle $L_{\mathbb{C}}$

$$\mathbb{C} \to E(L_{\mathbb{C}}) \ = \ \mathbb{C} \times_{S^1} S^\infty \to S^\infty/S^1 \ = \ \mathbb{CP}^\infty$$

with classifying map

$$L_{\mathbb{C}} \ = \ 1 \ : \ \mathbb{CP}^\infty = BU(1) \to BU(1) \ .$$

The cohomology of $\mathbb{CP}^\infty$ is the polynomial algebra $H^*(\mathbb{CP}^\infty) = \mathbb{Z}[x]$ with generator the Euler class of $L_{\mathbb{C}}$

$$x \ = \ e(L_{\mathbb{C}}) \ = \ 1 \in H^2(\mathbb{CP}^\infty) \ = \ \mathbb{Z} \ .$$

The external tensor product of an n-dimensional complex vector bundle $\beta : X \to BU(n)$ and $L_{\mathbb{C}}$ is an n-dimensional complex vector bundle

$$\beta \otimes_{\mathbb{C}} L_{\mathbb{C}} \ : \ X \times \mathbb{CP}^\infty \to BU(n) \ .$$

The Euler class of $\beta \otimes_{\mathbb{C}} L_{\mathbb{C}}$ determines the Chern classes $c_i(\beta) \in H^{2i}(X)$ by

$$e(\beta \otimes_{\mathbb{C}} L_{\mathbb{C}}) \ = \ \sum_{i=0}^{n} c_i(\beta)x^{n-i} \in H^{2n}(X \times \mathbb{CP}^\infty) \ = \ \sum_{i=0}^{n} H^{2i}(X) \otimes_{\mathbb{Z}} H^{2(n-i)}(\mathbb{CP}^\infty)$$

5 The Pontrjagin classes

The complexification of an n-dimensional real vector bundle $\alpha : X \to BO(n)$ is the n-dimensional complex vector bundle $\alpha \otimes \mathbb{C} : X \to BU(n)$ with

$$E(\alpha \otimes \mathbb{C}) \ = \ \bigcup_{x \in X} E_x(\alpha) \otimes_{\mathbb{R}} \mathbb{C} \ (E_x(\alpha) = \text{fibre of } \alpha \text{ over } x \in X) \ ,$$

giving a map $BO(n) \to BU(n)$. As in Chapter 15 of Milnor and Stasheff, define the Pontrjagin classes of α to be the Chern classes of $\alpha \otimes \mathbb{C}$ (up to sign)

$$p_i(\alpha) \ = \ (-1)^i c_{2i}(\alpha \otimes \mathbb{C}) \in H^{4i}(X) \ .$$

6 The Stiefel-Whitney classes

The universal real line bundle $L_{\mathbb{R}}$ is constructed by

$$\mathbb{R} \to E(L_{\mathbb{R}}) \ = \ \mathbb{R} \times_{\mathbb{Z}_2} S^{\infty} \to S^{\infty}/\mathbb{Z}_2 \ = \ \mathbb{R}\mathbb{P}^{\infty}$$

with $\mathbb{Z}_2$ acting by the antipodal action $x \mapsto -x$ on both $\mathbb{R}$ and S^{∞}, with classifying map

$$L_{\mathbb{R}} \ = \ 1 \ : \ \mathbb{R}\mathbb{P}^{\infty} = BO(1) \to BO(1) \ .$$

The mod 2 cohomology of $\mathbb{R}\mathbb{P}^{\infty}$ is the polynomial algebra $H^*(\mathbb{R}\mathbb{P}^{\infty};\mathbb{Z}_2) = \mathbb{Z}[y]$ with generator the mod 2 Euler class of $L_{\mathbb{R}}$

$$y \ = \ e(L_{\mathbb{R}}) \ = \ 1 \in H^1(\mathbb{R}\mathbb{P}^{\infty};\mathbb{Z}_2) \ = \ \mathbb{Z}_2 \ .$$

The Stiefel-Whitney classes $w_j(\alpha) \in H^j(X;\mathbb{Z}_2)$ of an n-dimensional real vector bundle $\alpha : X \to BO(n)$ are determined by the mod 2 Euler class of the external tensor product of α and $L_{\mathbb{R}}$

$$\alpha \otimes_{\mathbb{R}} L_{\mathbb{R}} \ : \ X \times \mathbb{R}\mathbb{P}^{\infty} \to BO(n) \ ,$$

by

$$e_2(\alpha \otimes_{\mathbb{R}} L_{\mathbb{R}}) \ = \ \sum_{j=0}^{n} w_j(\alpha) y^{n-j} \in H^j(X;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^{n-j}(\mathbb{R}\mathbb{P}^{\infty};\mathbb{Z}_2) \ .$$

For any base point $y \in \mathbb{R}\mathbb{P}^{\infty}$ define the inclusion

$$i_y \ : \ X \to X \times \mathbb{R}\mathbb{P}^{\infty} \ ; \ x \mapsto (x,y) \ .$$

Now

$$\begin{aligned} i_y^*(\alpha \otimes_{\mathbb{R}} L_{\mathbb{R}}) \ &= \ \alpha \ : \ X \to BO(n) \ , \\ i_y^* \ &= \ \text{projection} \ : \ H^n(X \times \mathbb{R}\mathbb{P}^{\infty};\mathbb{Z}_2) \to H^n(X;\mathbb{Z}_2) \ , \end{aligned}$$

so that

$$e_2(\alpha) \ = \ i_y^* e_2(\alpha \otimes_{\mathbb{R}} L_{\mathbb{R}}) \ = \ w_n(\alpha) \in H^n(X;\mathbb{Z}_2) \ ,$$

so the top Stiefel-Whitney class is the mod 2 Euler class (Milnor-Stasheff, 9.5).
[Background. A real n-dimensional real vector bundle $\alpha : X \to BO(n)$ is orientable if and only if

$$w_1(\alpha) \ = \ 0 \in H^1(X;\mathbb{Z}_2) \ .$$

In particular, $L_{\mathbb{R}}$ is nonorientable, with nonzero first mod 2 Stiefel-Whitney class

$$w_1(L_{\mathbb{R}}) \ = \ y \in H^1(\mathbb{R}\mathbb{P}^\infty;\mathbb{Z}_2) \ = \ \mathbb{Z}_2 \ .$$

For any $\alpha : X \to BO(n)$ the vector bundle $\alpha \otimes_{\mathbb{R}} L_{\mathbb{R}} : X \times \mathbb{R}\mathbb{P}^\infty \to BO(n)$ is nonorientable, since

$$\begin{array}{l} w_1(\alpha \otimes_{\mathbb{R}} L_{\mathbb{R}}) \ = \ (w_1(\alpha)\otimes 1, 1\otimes y) \neq 0 \\ \in H^1(X\times \mathbb{R}\mathbb{P}^\infty;\mathbb{Z}_2) \ = \ H^1(X;\mathbb{Z}_2)\otimes_{\mathbb{Z}_2} H^0(\mathbb{R}\mathbb{P}^\infty;\mathbb{Z}_2) \oplus H^0(X;\mathbb{Z}_2)\otimes_{\mathbb{Z}_2} H^1(\mathbb{R}\mathbb{P}^\infty;\mathbb{Z}_2) \end{array}$$

It can be shown that for oriented $\alpha : X \to BSO(n)$

$$w_{2k+1}(\alpha) \ = \ 0 \in H^{2k+1}(X;\mathbb{Z}_2) \ \ (k \geqslant 0)$$

and to define integral lifts $w_{2k}(\alpha) \in H^{2k}(X)$ of the Stiefel-Whitney classes.]

7 The relation between the Chern and the Stiefel-Whitney classes

Proposition (Problem 14-B in Milnor-Stasheff) The Chern classes $c_i(\beta) \in H^{2i}(X)$ of an n-dimensional complex vector bundle $\beta : X \to BU(n)$ are integral lifts of the Stiefel-Whitney classes $w_{2i}(\beta^{\mathbb{R}}) \in H^{2i}(X;\mathbb{Z}_2)$ of the underlying oriented $2n$-dimensional real vector bundle $\beta^{\mathbb{R}} : X \to BSO(2n)$, that is

$$[c_i(\beta)]_2 \ = \ w_{2i}(\beta^{\mathbb{R}}) \in H^{2i}(X;\mathbb{Z}_2) \ .$$

Proof Let $j : \mathbb{R}\mathbb{P}^\infty \subset \mathbb{C}\mathbb{P}^\infty$ be the inclusion. The restriction of $L_{\mathbb{C}}$ to $\mathbb{R}\mathbb{P}^\infty$ is a complex line bundle such that the underlying oriented 2-dimensional real vector bundle splits as a sum of two copies of the universal real line bundle

$$(j^*L_{\mathbb{C}})^{\mathbb{R}} \ = \ L_{\mathbb{R}} \oplus L_{\mathbb{R}} \ : \ \mathbb{R}\mathbb{P}^\infty \to BO(2) \ .$$

The induced morphism $j^* : H^{2i}(\mathbb{C}\mathbb{P}^\infty) \to H^{2i}(\mathbb{R}\mathbb{P}^\infty)$ is an isomorphism sending the generator

$$e(\bigoplus_i L_{\mathbb{C}}) \ = \ x^i \ = \ 1 \in H^{2i}(\mathbb{C}\mathbb{P}^\infty) \ = \ \mathbb{Z}$$

to an integral lift of the generator

$$e_2(\bigoplus_{2i} L_{\mathbb{R}}) \ = \ y^{2i} \ = \ 1 \in H^{2i}(\mathbb{R}\mathbb{P}^\infty;\mathbb{Z}_2) \ = \ \mathbb{Z}_2 \ ,$$

by a double application of the product formula for the Euler class of a Whitney sum. It now follows from

$$\beta^{\mathbb{R}} \otimes_{\mathbb{R}} L_{\mathbb{R}} \ : \ X \times \mathbb{R}\mathbb{P}^\infty \xrightarrow{1\times j} X \times \mathbb{C}\mathbb{P}^\infty \xrightarrow{\beta\otimes_{\mathbb{C}} L_{\mathbb{C}}} BU(n) \longrightarrow BSO(2g)$$

that $(1 \times j)^* : H^{2n}(X \times \mathbb{C}\,\mathbb{P}^\infty) \to H^{2n}(X \times \mathbb{R}\,\mathbb{P}^\infty)$ sends the integral Euler class

$$\begin{aligned} e((\beta \otimes_{\mathbb{C}} L_{\mathbb{C}})^{\mathbb{R}}) &= \sum_{i=0}^{n} c_i(\beta) x^{n-i} \\ \in H^{2n}(X \times \mathbb{C}\,\mathbb{P}^\infty) &= \sum_{i=0}^{n} H^{2i}(X) \otimes_{\mathbb{Z}} H^{2(n-i)}(\mathbb{C}\,\mathbb{P}^\infty) \end{aligned}$$

to an integral lift of

$$\begin{aligned} e_2(\beta^{\mathbb{R}} \otimes_{\mathbb{R}} L_{\mathbb{R}}) &= \sum_{i=0}^{n} w_{2i}(\beta^{\mathbb{R}}) y^{2(n-i)} \\ \in H^{2n}(X \times \mathbb{R}\,\mathbb{P}^\infty; \mathbb{Z}_2) &= \sum_{j=0}^{2n} H^j(X; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^{2n-j}(\mathbb{R}\,\mathbb{P}^\infty; \mathbb{Z}_2) \ , \end{aligned}$$

and hence that $c_i(\beta) \in H^{2i}(X)$ is an integral lift of $w_{2i}(\beta^{\mathbb{R}}) \in H^{2i}(X; \mathbb{Z}_2)$.