ASPECTS OF QUADRATIC FORMS IN THE WORK OF HIRZEBRUCH AND ATIYAH

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Mission statement

- To provide a guided tour of <u>some</u> of the more elementary aspects of the work of Hirzebruch and Atiyah involving quadratic forms.
- The tour will visit the connections between

  (i) the algebraic theory of quadratic forms,

  (ii) the geometric theory of manifolds and singular spaces,

  (iii) the number theory of Dedekind sums,

  (iv) index theory.

- Somewhat like doing all of Europe in a day!
- Will travel in time and mathematics rather than space, starting in the 19th century.

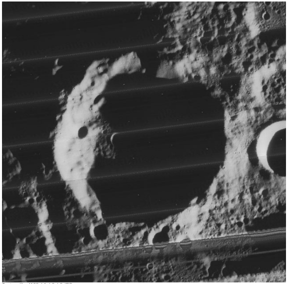
James Joseph Sylvester (1814–1897)

Honorary Fellow of the RSE, 1874

Sylvester's 1852 paper

A DEMONSTRATION OF THE THEOREM THAT EVERY HOMOGENEOUS QUADRATIC POLYNOMIAL IS REDUCIBLE BY REAL ORTHOGONAL SUBSTITUTIONS TO THE FORM OF A SUM OF POSITIVE AND NEGATIVE SQUARES.

- Fundamental insight: the invariance of the numbers of positive and negative eigenvalues of a symmetric matrix S under linear congruence.
- Impact statement: the **Sylvester crater** on the Moon



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Linear congruence, the indices of inertia and the signature

- Symmetric $n \times n$ matrices S, T are **linearly congruent** if $S = A^*TA$ for an invertible $n \times n$ matrix $A = (a_{ij})$ with $A^* = (a_{ji})$ the transpose.
- The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a symmetric matrix S are real.
- The **positive** and **negative index of inertia** of a symmetric $n \times n$ matrix S are

$$\begin{aligned} \sigma_+(S) &= (\text{no. of eigenvalues } \lambda_k > 0) , \\ \sigma_-(S) &= (\text{no. of eigenvalues } \lambda_k < 0) \in \{0, 1, 2, \ldots, n\} . \end{aligned}$$

- The **signature** (= **index of inertia**) of S is the difference

$$\begin{aligned} \sigma(S) &= \sigma_+(S) - \sigma_-(S) \\ &= \sum_{k=1}^{n} \operatorname{sign}(\lambda_k) \in \{-n, \ldots, -1, 0, 1, \ldots, n\} . \end{aligned}$$

- Linearly congruent S, T have the same indices of inertia.

Sylvester's Law of Inertia (1852)

- A symmetric $n \times n$ symmetric matrix S is linearly congruent to the diagonal matrix

$$D(\mathrm{sign}(\lambda_1), \mathrm{sign}(\lambda_2), \ldots, \mathrm{sign}(\lambda_n)) \;=\; \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\sigma_+(S) = p$, $\sigma_-(S) = q$, $\sigma(S) = p - q$.

- **Law of Inertia** Symmetric $n \times n$ matrices S, T are linearly congruent if and only if they have eigenvalues of the same signs, i.e. same indices

$$\sigma_+(S) \;=\; \sigma_+(T) \text{ and } \sigma_-(S) \;=\; \sigma_-(T) \in \{0, 1, \ldots, n\} \ .$$

- **Important special case** Invertible symmetric $n \times n$ matrices S, T are linearly congruent if and only if they have the same signature

$$\sigma(S) = \sigma(T) \in \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n\} \ .$$

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Regular symmetric matrices

- The **principal** $k \times k$ **minor** of an $n \times n$ matrix $S = (s_{ij})_{1 \leqslant i,j \leqslant n}$ is
$$\mu_k(S) \ = \ \det(S_k) \in \mathbb{R}$$
with $S_k = (s_{ij})_{1 \leqslant i,j \leqslant k}$ the principal $k \times k$ submatrix
$$S \ = \ \begin{pmatrix} S_k & \dots \\ \vdots & \ddots \end{pmatrix} .$$
- An $n \times n$ matrix S is **regular** if $\mu_k(S) \neq 0 \in \mathbb{R}$ $(1 \leqslant k \leqslant n)$, that is if each S_k is invertible. In particular, $S_n = S$ is invertible.
- **Theorem** (Sylvester 1852, Gundelfinger 1881, Frobenius 1895) The eigenvalues $\lambda_1(S), \lambda_2(S), \dots, \lambda_n(S)$ of a regular symmetric $n \times n$ matrix S have the same signs as the minor quotients
$$\mathrm{sign}(\lambda_k(S)) \ = \ \mathrm{sign}(\frac{\mu_k(S)}{\mu_{k-1}(S)}) \in \{-1, 1\} \ (k = 1, 2, \dots, n)$$
with $\mu_0(S) = 1$.

Tridiagonal matrices

- The **tridiagonal symmetric** $n \times n$ **matrix** of $\chi = (\chi_1, \chi_2, \ldots, \chi_n) \in \mathbb{R}^n$

$$\mathrm{Tri}(\chi) \;=\; \begin{pmatrix} \chi_1 & 1 & 0 & \ldots & 0 & 0 \\ 1 & \chi_2 & 1 & \ldots & 0 & 0 \\ 0 & 1 & \chi_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \chi_{n-1} & 1 \\ 0 & 0 & 0 & \ldots & 1 & \chi_n \end{pmatrix}$$

- A vector $\chi = (\chi_1, \chi_2, \ldots, \chi_n) \in \mathbb{R}^n$ is **regular** if

$$\chi_k \neq 0 \;,\; \mu_k(\mathrm{Tri}(\chi)) \neq 0 \; (k = 1, 2, \ldots, n)$$

so that the tridiagonal symmetric matrix $\mathrm{Tri}(\chi)$ is regular.

Tridiagonal matrices and continued fractions

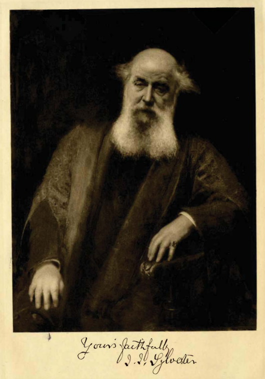
- **Theorem** (Sylvester, 1853) The minor quotients of the tridiagonal matrix Tri(χ) of a regular $\chi \in \mathbb{R}^n$ are continued fractions

$$\begin{aligned}\frac{\mu_k(\mathrm{Tri}(\chi))}{\mu_{k-1}(\mathrm{Tri}(\chi))} &= [\chi_k, \chi_{k-1}, \ldots, \chi_1] \\ &= \chi_k - \cfrac{1}{\chi_{k-1} - \cfrac{1}{\chi_{k-2} - \ddots - \cfrac{1}{\chi_1}}}\end{aligned}$$

- The signature of Tri(χ) is

$$\sigma(\mathrm{Tri}(\chi)) = \sum_{k=1}^{n} \mathrm{sign}([\chi_k, \chi_{k-1}, \ldots, \chi_1]) \in \{-n, -n+1, \ldots, n\} .$$

"Aspiring to these wide generalizations, the analysis of quadratic functions soars to a pitch from whence it may look proudly down on the feeble and vain attempts of geometry proper to rise to its level or to emulate it in its flights." (1850)



Savilian Professor of Geometry, Oxford, 1883-1894

From a 2ℓ-manifold to a $(-)^\ell$-symmetric form

- A $(-)^\ell$-**symmetric form** (K, ϕ) is a vector space K with a $(-)^\ell$-symmetric bilinear pairing $\phi : K \times K \to \mathbb{R}$
$$\phi(x, y) \ = \ (-)^\ell \phi(y, x) \ .$$
For $K = \mathbb{R}^n$ essentially same as $(-)^\ell$-symmetric $n \times n$ matrix $S = (-)^\ell S^*$.
- Will only consider oriented manifolds.
- The **intersection form** of a 2ℓ-manifold with boundary $(M, \partial M)$ is the $(-)^\ell$-symmetric form
$$\phi_M \ : \ H_\ell(M;\mathbb{R}) \times H_\ell(M;\mathbb{R}) \to \mathbb{R} \ ; \ (a[P], b[Q]) \mapsto ab[P \cap Q] \ (a, b \in \mathbb{R})$$
with $[P \cap Q] \in \mathbb{Z}$ the intersection number of transverse ℓ-submanifolds $P^\ell, Q^\ell \subset M$. The adjoint linear map
$$\phi_M \ : \ K = H_\ell(M;\mathbb{R}) \to K^* = \mathrm{Hom}_{\mathbb{R}}(K, \mathbb{R}) \ ; \ x \mapsto (y \mapsto \phi_M(x, y))$$
has $\ker(\phi_M) = \mathrm{im}(H_\ell(\partial M;\mathbb{R})) \subseteq K$. If $\partial M = \emptyset$ the form is nonsingular, with ϕ_M the **Poincaré duality isomorphism**.

The signature and index theorems

- Weyl (1923) The **signature** of a $4k$-manifold with boundary $(M, \partial M)$ is the signature of the intersection symmetric matrix S_M
$$\sigma(M) \;=\; \sigma(S_M) \in \mathbb{Z} \;.$$
- Hirzebruch, *On Steenrod's reduced powers, the index of inertia and the Todd genus* (1953)
The signature theorem for a <u>closed</u> $4k$-manifold M states that
$$\sigma(M) \;=\; \int_M \mathcal{L}(M) \in \mathbb{Z} \subset \mathbb{R}$$
with $\mathcal{L}(M) \in H^{4k}(M;\mathbb{Q})$ the $\mathcal{L}$-genus, a rational polynomial in the Pontrjagin classes $p_j(\tau_M) \in H^{4j}(M;\mathbb{Z})$ of the tangent bundle τ_M.
- Atiyah and Singer, *The index of elliptic operators* (1968)
The **index** of an elliptic operator is a K-theoretic generalization of the signature. The A-S theorem for an operator over a closed manifold expressed the index in terms of characteristic classes. For the signature operator recovers the Hirzebruch signature theorem.

The signature defect

- The **signature defect** of a $4k$-manifold with boundary $(M, \partial M)$ measures the extent to which the Hirzebruch signature formula holds
$$\mathrm{def}(M) \ = \ \int_M \mathcal{L}(M, \partial M) - \sigma(M) \in \mathbb{R} \ .$$
This depends on the existence and choice of a cohomology class $\mathcal{L}(M, \partial M) \in H^{4k}(M, \partial M; \mathbb{R})$ with image $\mathcal{L}(M) \in H^{4k}(M; \mathbb{R})$.
- Exotic spheres of Milnor (1956) detected by signature defect.
- Computed by Hirzebruch and Zagier in particular cases (60's,70's).
- Atiyah, Patodi and Singer, *Spectral asymmetry and Riemannian geometry* (1974). Index theorem identifies $\mathrm{def}(M)$ with a spectral invariant $\eta(\partial M)$ which depends on the Riemannian structure of ∂M. Generalization of the Hirzebruch signature theorem for closed manifolds.
- Atiyah, Donnelly and Singer, *η-invariants, signature defects of cusps, and values of L-functions* (1983) Topological proof of Hirzebruch's interpretation of the values of L-functions of totally real number fields. (Continued fractions!)

Realizing matrices by manifolds

- (Milnor 1959, Hirzebruch 1961)
  For $\ell \geqslant 2$ every integer $(-)^\ell$-symmetric $n \times n$ matrix $S = (s_{ij} \in \mathbb{Z})$ is realized as the intersection matrix of a 2ℓ-manifold with boundary $(M, \partial M)$
  $$(H_\ell(M;\mathbb{R}), \phi_M) \ = \ (\mathbb{R}^n, S) \ .$$
  Constructed by the "plumbing" of n ℓ-plane bundles over S^ℓ with Euler numbers $\chi_i = s_{ii} \in \mathbb{Z}$, required to be even for $\ell = 2k$ with $k \neq 1, 2, 4$ (as the Hopf invariant $\neq 1$ in these dimensions).
- The **weighted adjacency graph** of S is the graph with n vertices $1, 2, \dots, n$ and $|s_{ij}|$ edges joining i to j $(i \neq j)$ with weight $s_{ii} \in \mathbb{Z}$ at i.
- If the adjacency graph of S is a tree then
  - for $\ell \geqslant 2$ M is $(\ell - 1)$-connected, with $H_r(M) = 0$ $(1 \leqslant r \leqslant \ell - 1)$,
  - for $\ell \geqslant 3$ M and ∂M are both $(\ell - 1)$-connected.

The realization of a tridiagonal matrix

- The weighted adjacency graph of $\mathrm{Tri}(\chi) = \begin{pmatrix} \chi_1 & 1 & 0 & \dots \\ 1 & \chi_2 & 1 & \dots \\ 0 & 1 & \chi_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ is the **weighted A_n-tree**

$$A_n \;:\; \overset{\chi_1}{\bullet} \;\text{———}\; \overset{\chi_2}{\bullet} \;\text{———}\; \overset{\chi_3}{\bullet} \;\text{———}\; \dots \;\text{———}\; \overset{\chi_{n-1}}{\bullet} \;\text{———}\; \overset{\chi_n}{\bullet}$$

- A regular $\chi = (\chi_1, \chi_2, \dots, \chi_n) \in \mathbb{Z}^n$ is realized by a 4-manifold $(M(\chi), \partial M(\chi))$ obtained by the A_n-plumbing together of n 2-plane bundles over S^2 with Euler numbers χ_i. The symmetric intersection form is
$$(H_2(M(\chi);\mathbb{R}), \phi_{M(\chi)}) \;=\; (\mathbb{R}^n, \mathrm{Tri}(\chi)) \;.$$
- The 4-manifolds $(M(\chi), \partial M(\chi))$ have many interesting geometric properties!

From a $(2\ell+1)$-manifold with boundary to a lagrangian

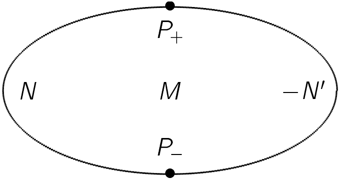
- A **lagrangian** of a $(-)^{\ell}$-symmetric form (K, ϕ) is a subspace $L \subseteq K$ such that $L = L^{\perp}$, i.e.

$$\phi(L, L) \ = \ \{0\} \text{ and } L \ = \ \{x \in K \,|\, \phi(x, y) = 0 \in \mathbb{R} \text{ for all } y \in L\} \ .$$

- 1-symmetric = symmetric, (-1)-symmetric = symplectic.
- A nonsingular symmetric form (K, ϕ) admits a lagrangian if and only if it has signature $\sigma(K, \phi) = 0 \in \mathbb{Z}$, if and only if it is isomorphic to $H_+(\mathbb{R}^n) = (\mathbb{R}^n \oplus \mathbb{R}^n, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix})$ with $n = \dim_{\mathbb{R}}(K)/2$.
- Every nonsingular symplectic form (K, ϕ) admits a lagrangian, and is isomorphic to $H_-(\mathbb{R}^n) = (\mathbb{R}^n \oplus \mathbb{R}^n, \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix})$ with $n = \dim_{\mathbb{R}}(K)/2$.
- A $(2\ell+1)$-manifold with boundary $(M, \partial M)$ determines a lagrangian $L = \ker(H_\ell(\partial M; \mathbb{R}) \to H_\ell(M; \mathbb{R}))$ of the $(-)^{\ell}$-symmetric intersection form $(H_\ell(\partial M; \mathbb{R}), \phi_{\partial M})$.

Cobordism

- An m-**dimensional cobordism** $(M; N, N'; P)$ is an m-manifold M with the boundary decomposed as $\partial M = N \cup_P -N'$ for $(m-1)$-manifolds N, N' with the same boundary $\partial N = \partial N' = P$, and $-N' = N'$ with the opposite orientation. In the diagram $P = P_+ \sqcup P_-$.



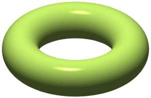
- **Theorem** (Thom 1952 for $P = \emptyset$, Novikov 1967 in general)
  For $m = 4k+1$ the signature is a cobordism invariant:

$$\sigma(N) - \sigma(N') \ = \ \sigma(\partial M) \ = \ 0 \in \mathbb{Z} \ .$$

  The signature of the intersection symmetric form $(H_{2k}(\partial M; \mathbb{R}), \phi_{\partial M})$ is 0 because $L = \ker(H_{2k}(\partial M; \mathbb{R}) \to H_{2k}(M; \mathbb{R}))$ is a lagrangian.

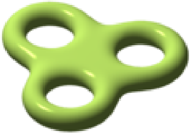
The symplectic group $Sp(2n)$ and automorphisms of the surfaces Σ_n

- The **symplectic group** $Sp(2n) = \mathrm{Aut}(H_-(\mathbb{R}^n))$ $(n \geqslant 1)$ consists of the invertible $2n \times 2n$ matrices A such that $A^* \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
  Similarly for $Sp(2n;\mathbb{Z}) \subset Sp(2n)$.
- The **surface of genus** n is $\Sigma_n = \underset{n}{\#}\, S^1 \times S^1$.



Σ_1

Σ_2



Σ_3

- The **mapping class group** $\Gamma_n = \pi_0(\mathrm{Aut}(\Sigma_n))$ is the group of automorphisms of Σ_n, modulo isotopy. Canonical group morphism

$$\gamma_n \ : \ \Gamma_n \to Sp(2n;\mathbb{Z}) \ ; \ (A : \Sigma_n \to \Sigma_n) \mapsto (A_* : H_1(\Sigma_n) \to H_1(\Sigma_n)) \ .$$

  Isomorphism for $n = 1$. Surjection for $n \geqslant 2$.

The modular group $SL_2(\mathbb{Z})$

- Dedekind, *Erläuterungen zu den vorstehenden Fragmenten*, 1876. Commentary on Riemann's work on elliptic functions.
- The **modular group** $SL_2(\mathbb{Z}) = Sp(2;\mathbb{Z})$ is the group of 2×2 integer matrices
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
such that
$$\det(A) = ad - bc = 1 \in \mathbb{Z} .$$
- Every element $A \in SL_2(\mathbb{Z})$ is induced by an automorphism of the torus
$$A : \Sigma_1 = S^1 \times S^1 \to S^1 \times S^1 ; \ (e^{ix}, e^{iy}) \mapsto (e^{i(ax+by)}, e^{i(cx+dy)}) .$$
- $SL_2(\mathbb{Z}) = \Gamma_1$ is the mapping class group of the torus Σ_1.

The lens spaces

- Tietze, *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten* (1908)
- The **lens spaces** are the closed 3-manifolds

$$L(a,c) \ = \ S^3/\mathbb{Z}_c$$

  defined for coprime $a, c \in \mathbb{Z}$, $c > 0$, with

$$S^3 \ = \ \{(u,v) \in \mathbb{C}^2 \,|\, |u|^2 + |v|^2 = 1\} \ ,$$

$$\mathbb{Z}_c \times \mathbb{C}^2 \to \mathbb{C}^2 \ ; \ (m,(u,v)) \mapsto (\zeta^{am}u, \zeta^m v) \text{ where } \zeta = e^{2\pi i/c} \in \mathbb{C} \ .$$

- $\pi_1(L(c,a)) = \mathbb{Z}_c$, $H_*(L(c,a);\mathbb{R}) = H_*(S^3;\mathbb{R})$.
- The lens space has a genus 1 Heegaard decomposition

$$L(c,a) \ = \ S^1 \times D^2 \cup_A S^1 \times D^2$$

  for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

The Hirzebruch-Jung resolution of cyclic surface singularities I.

- For $A \in SL_2(\mathbb{Z})$ with $c \neq 0$ the Euclidean algorithm gives a regular $\chi \in \mathbb{Z}^n$ with $|\chi_k| > 1$, such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_2 & -1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} \chi_n & -1 \\ 1 & 0 \end{pmatrix} ,$$

$$a/c = [\chi_1, \chi_2, \dots, \chi_n] = \chi_1 - \cfrac{1}{\chi_2 - \cfrac{1}{\chi_3 - \ddots - \cfrac{1}{\chi_n}}} ,$$

$$(H_2(M(\chi); \mathbb{R}), \phi_{M(\chi)}) = (\mathbb{R}^n, \mathrm{Tri}(\chi)) , \ \partial M(\chi) = L(c, a) .$$

- The signature of the A_n-plumbed 4-manifold $M(\chi)$ is

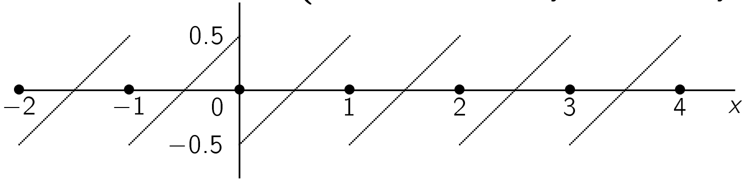
$$\sigma(M(\chi)) = \sigma(\mathrm{Tri}(\chi)) = \sum_{k=1}^{n} \mathrm{sign}([\chi_k, \chi_{k-1}, \dots, \chi_1]) = \sum_{k=1}^{n} \mathrm{sign}(\chi_k) \in \mathbb{Z}.$$

The Hirzebruch-Jung resolution of cyclic surface singularities II.

- The 4-manifold $M(\chi)$ resolves the singularity at $(0,0,0)$ of the 2-dimensional complex space
$$\{(w, z_1, z_2) \in \mathbb{C}^3 \,|\, w^c = z_1(z_2)^{c-a}\} \ .$$
- Jung, *Darstellung der Funktionen eines algebraischen Körpers zweier unabhängigen Veränderlichen* x, y *in der Umgebung* $x = a, y = b$ (1909)
- Hirzebruch, *Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen* (1952).
- Hirzebruch and Mayer, *O(n)-Mannigfaltigkeiten, exotische Sphären und Singularitäten* (1968)
- Hirzebruch, Neumann and Koh, *Differentiable manifolds and quadratic forms* (1971)
- The signature of $M(\chi)$ is closely related to Dedekind sums!

The sawtooth function $((\ ))$

- Used by Dedekind (1876) to count $\pm 2\pi$ jumps in the complex logarithm
$$\log(re^{i\theta}) \;=\; \log(r) + i(\theta + 2n\pi) \in \mathbb{C} \quad (n \in \mathbb{Z})\ .$$
- The **sawtooth function** $((\ )) : \mathbb{R} \to [-1/2, 0)$ is defined by
$$((x)) \;=\; \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$
with $\{x\} \in [0,1)$ the fractional part of $x \in \mathbb{R}$. Nonadditive:
$$((x)) + ((y)) - ((x+y)) \;=\; \begin{cases} -1/2 & \text{if } 0 < \{x\} + \{y\} < 1 \\ 1/2 & \text{if } 1 < \{x\} + \{y\} < 2 \\ 0 & \text{if } x \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \text{ or } x + y \in \mathbb{Z}\ . \end{cases}$$



Dedekind sums and signatures

- The **Dedekind sum** for $a, c \in \mathbb{Z}$ with $c \neq 0$ is

$$s(a,c) \;=\; \sum_{k=1}^{|c|-1} ((\frac{k}{c}))((\frac{ka}{c})) \;=\; \frac{1}{4|c|}\sum_{k=1}^{|c|-1} \cot(\frac{k\pi}{c})\cot(\frac{ka\pi}{c}) \in \mathbb{Q}\ .$$

- **Theorem** For any regular sequence $\chi = (\chi_1, \chi_2, \ldots, \chi_n) \in \mathbb{Z}^n$ the signature defect of $(M(\chi), \partial M(\chi))$ is

$$\sigma(\mathrm{Tri}(\chi)) - (\sum_{j=1}^{n} \chi_j)/3 \;=\; \begin{cases} b/3d & \text{if } c = 0 \\ (a+d)/3 - 4\mathrm{sign}(c)s(a,c) & \text{if } c \neq 0\ . \end{cases}$$

- Hirzebruch, *The signature theorem: reminiscences and recreations* (1971) and *Hilbert modular surfaces* (1973)
- Hirzebruch and Zagier, *The Atiyah-Singer theorem and elementary number theory* (1974)
- Kirby and Melvin, *Dedekind sums, μ-invariants and the signature cocycle* (1994)

The tailoring of topological pants

- **Universal cobordism cocycle construction**.
  Geometric key to the signature defect.
- **Input**: Three n-manifolds N_0, N_1, N_2 with $\partial N_0 = \partial N_1 = \partial N_2 = P$.
  Diffeomorphisms $f_j : \partial N_j \to \partial N_{j-1}$ ($j \pmod 3$) satisfy $f_1 f_2 f_3 = \mathrm{Id}$.
- **Output**: The **pair of pants** $(n+1)$-manifold

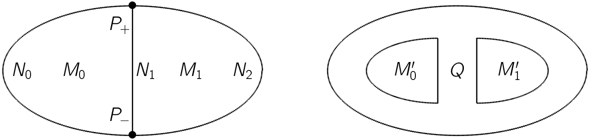
$$Q \ = \ Q(P, N_0, N_1, N_2) \ = \ (N_0 \times I \sqcup N_1 \times I \sqcup N_2 \times I)/\sim \ ,$$

$$(a_j, b_j) \sim (f_j(a_j), 1 - b_j) \ (a_j \in \partial N_j, b_j \in [0, 1/2])$$

  with boundary $\partial Q = (N_0 \cup_P -N_1) \sqcup (N_1 \cup_P -N_2) \sqcup (N_2 \cup_P -N_0)$.
- Standard pair of 2-dimensional pants $(Q(S^0, D^1, D^1, D^1), S^1 \sqcup S^1 \sqcup S^1)$ used in Atiyah, *Topological quantum field theory* (1988).

The Wall non-additivity of the signature I.

- Wall *The non-additivity of the signature* (1969).
  The signature of the union $M = M_0 \cup_{N_1} M_1$ of $4k$-dimensional cobordisms $(M_0; N_0, N_1; P)$, $(M_1; N_1, N_2; P)$ is
  $$\sigma(M) \;=\; \sigma(M_0) + \sigma(M_1) + \sigma(Q) \in \mathbb{Z}$$
  with $Q = Q(P, N_0, N_1, N_2)$ the pair of pants in the middle.



$M = M_0 \cup M_1 = M_0' \cup Q \cup M_1'$

- The **nonadditivity invariant** $\sigma(Q) \in \mathbb{Z}$ is the "triple signature" $\sigma(K, \phi; L_0, L_1, L_2)$ of the nonsingular symplectic form $(K, \phi) = (H_{2k-1}(P; \mathbb{R}), \phi_P)$ with respect to the three lagrangians
  $$L_j \;=\; \ker(K \to H_{2k-1}(N_j; \mathbb{R}))\ (j = 0, 1, 2)\ .$$

The Wall non-additivity of the signature II.

- The **triple signature** $\sigma(K,\phi;L_0,L_1,L_2)=\sigma(V,\psi)\in\mathbb{Z}$ is the signature of the nonsingular symmetric form (V,ψ) defined by

$$V = \frac{\{(x,y,z)\in L_0\oplus L_1\oplus L_2\,|\,x+y+z=0\in K\}}{\{(a-b,b-c,c-a)\,|\,a\in L_2\cap L_0, b\in L_0\cap L_1, c\in L_1\cap L_2\}}~,$$
$$\psi(x,y,z)(x',y',z') = \phi(x,y')\in\mathbb{R}~.$$

- Triple signature = Maslov index (Arnold, Leray, ...).
- **Example** The lagrangians of $H_-(\mathbb{R})$ are the 1-dimensional subspaces

$$L(\theta) = \{(r\cos\theta, r\sin\theta)\,|\,r\in\mathbb{R}\}\subset\mathbb{R}^2~(\theta\in[0,\pi))~.$$

The triple signature jumps by ±1 at $\theta_j-\theta_{j+1}\in\pi\mathbb{Z}$, for $j(\mathrm{mod}~3)$

$$\sigma(H_-(\mathbb{R});L(\theta_0),L(\theta_1),L(\theta_2)) = \mathrm{sign}(\sin(\theta_0-\theta_1)\sin(\theta_1-\theta_2)\sin(\theta_2-\theta_0))$$

The non-multiplicativity of the signature I.

- The multiplicativity $\sigma(X \times F) = \sigma(X)\sigma(F)$ was a key ingredient of the 1953 proof of the Hirzebruch signature theorem.
- Chern, Hirzebruch and Serre, *On the index of a fibered manifold* (1957).

$$\sigma(M) \;=\; \sigma(X)\sigma(F)$$

  for any fibre bundle $F \to M^{4k} \to X$ with $\pi_1(X)$ acting trivially on $H_*(F;\mathbb{R})$.
- Kodaira, *A certain type of irregular algebraic surfaces* (1967)
  First examples of fibre bundles $F^2 \to M^4 \to X^2$ with

$$\sigma(M) - \sigma(X)\sigma(F) \neq 0 \in \mathbb{Z} \; .$$

- Hirzebruch, *The signature of ramified coverings* (1969)
  Analysis of non-multiplicativity using the signature of branched covers, and the Atiyah-Singer G-signature theorem.

The non-multiplicativity of the signature II.

- Atiyah, *The signature of fibre-bundles* (1969) A characteristic class formula for the signature of a fibre bundle $F^{2\ell} \to M^{4k} \to X^{4k-2\ell}$

$$\sigma(M) \;=\; \langle \mathrm{ch}(\mathrm{Sign}) \cup \widetilde{\mathcal{L}}(X), [X] \rangle \in \mathbb{Z} \subset \mathbb{R}$$

with $\mathrm{Sign} = \{\sigma_K(H_\ell(F_x;\mathbb{C}), \phi_{F_x}) \,|\, x \in X\}$ the virtual bundle of the topological K-theory signatures of hermitian forms, such that

$$(H_*(M;\mathbb{C}), \phi_M) \;=\; (H_*(X;\mathrm{Sign}), \phi_X)$$

with $\mathrm{ch}(\mathrm{Sign}) \in H^{2*}(X;\mathbb{C})$ the Chern character, and $\widetilde{\mathcal{L}}(X) \in H^{4*}(X;\mathbb{Q})$ a modified $\mathcal{L}$-genus.
- All the examples are multiplicative mod 4. In fact: Hambleton, Korzeniewski and Ranicki, *The signature of a fibre bundle is multiplicative mod 4* (2007) For any fibre bundle $F \to M^{4k} \to X$

$$\sigma(M) \;\equiv\; \sigma(X)\sigma(F) \pmod 4 \;.$$

The Meyer signature cocycle

- Let $(K, \phi) = H_-(\mathbb{R}^n)$. For $A, B \in \mathrm{Aut}(K, \phi) = Sp(2n)$ let

  $$\sigma(A, B) \ = \ \sigma(K \oplus K, \phi \oplus -\phi; (1 \oplus A)\Delta_K, (1 \oplus B)\Delta_K, (1 \oplus AB)\Delta_K) \in \mathbb{Z} \, .$$

  For $A, B : \Sigma_n \to \Sigma_n$ $\sigma(A, B) = \sigma(M)$ is the signature of the Σ_n-bundle $\Sigma_n \to M^4 \to Q^2$ over a standard pair of pants Q, with ∂M the union of the mapping tori of A, B, AB.
- W.Meyer *Die Signatur von lokalen Koeffizientensystem und Faserbündeln* (1972) The triple signature function

  $$c_n \ : \ Sp(2n) \times Sp(2n) \to \mathbb{Z} \ ; \ (A, B) \mapsto \sigma(A, B)$$

  is a cocycle for a group cohomology class $[c_n] \in H^2(Sp(2n); \mathbb{Z})$ classifying the **signature** central extension $\mathbb{Z} \to \widehat{Sp}(2n) \to Sp(2n)$.
- The signature of the total space of a Σ_n-bundle $\Sigma_n \to M^4 \to X^2$ with canonical map $\gamma : \pi_1(X) \to \mathrm{Aut}(H_-(\mathbb{R}^n)) = Sp(2n)$ is

  $$\sigma(M) \ = \ \langle \gamma^*[c_n], [X] \rangle \in \mathbb{Z} \ .$$

The Atiyah signature cocycle I.

- Atiyah, *The logarithm of the Dedekind η-function* (1987) Geometric interpretation of the Meyer cocycle, connection with Dedekind η-function as well as the Atiyah-Patodi-Singer η-invariant.
- The characteristic class formula for the signature of a fibre bundle was generalized by Lusztig (1972): for any surface with boundary (X, Y) and group morphism

$$\gamma \;:\; \pi_1(X) \to U(p,q) = \mathrm{Aut}(\mathbb{C}^{p+q}, \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix})$$

  there is a **signature with local coefficients**

$$\sigma(X,\gamma) \;=\; \sigma(H_1(X;\gamma),\phi_X) \in \mathbb{Z}\ .$$

- Let Q^2 be the standard 2-dimensional pair of pants, with boundary $\partial Q = S^1 \sqcup S^1 \sqcup S^1$. The morphisms

$$\gamma \;:\; \pi_1(Q) \;=\; \mathbb{Z} * \mathbb{Z} \to U(p,q)$$

  are in one-one correspondence with $A, B \in U(p,q)$.

The Atiyah signature cocycle II.

- The function defined for $A, B \in U(p,q)$ by
$$c_{p,q}(A,B) \;=\; \sigma(H_1(Q;\gamma(A,B)),\phi_Q) \in \mathbb{Z}$$
is a cocycle for a group cohomology **signature class**
$$[c_{p,q}] \in H^2(U(p,q);\mathbb{Z}) \;.$$
such that for any surface X and $\gamma : \pi_1(X) \to U(p,q)$
$$\sigma(X,\gamma) \;=\; \langle \gamma^*(c_{p,q}),[X]\rangle \in \mathbb{Z} \;.$$
- The signature class $[c_{p,q}] \in H^2(U(p,q);\mathbb{Z}) = \mathrm{Hom}(\pi_1(U(p,q)),\mathbb{Z})$ is given by
$$\pi_1(U(p,q)) \;=\; \pi_1(U(p))\times\pi_1(U(q)) \;=\; \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\;;\;(x,y) \mapsto 2x-2y \;.$$
- $c_{n,n}$ restricts on $Sp(2n) \subset U(n,n)$ to the Meyer cocycle c_n.

The Atiyah signature cocycle III.

- $c_{1,0}$ is the cocycle on $U(1,0) = U(1) = S^1$

  $c_{1,0} \;:\; S^1 \times S^1 \to \mathbb{Z}\;;\; (e^{2\pi ix}, e^{2\pi iy}) \mapsto$

  $\sigma(H_-(\mathbb{R}); L(0), L(2\pi x), L(2\pi(x+y))) = 2(((x)) + ((y)) - ((x+y)))$

  with $L(\theta) = \{(r\cos\theta, r\sin\theta) \mid r \in \mathbb{R}\} \subset \mathbb{R}^2$ the lagrangian of $H_-(\mathbb{R})$ determined by $e^{i\theta/2} \in S^1$.
- $c_{1,0}$ classifies the **signature** central extension $\mathbb{Z} \to \mathbb{R} \times \mathbb{Z}_2 \to S^1$

$$\mathbb{Z} \to \mathbb{R} \times \mathbb{Z}_2 \;;\; m \mapsto (m/2, m(\bmod 2))\;,$$
$$\mathbb{R} \times \mathbb{Z}_2 \to S^1 \;;\; (x, r) \mapsto e^{2\pi i(x-r/2)}\;(r = 0, 1)\;.$$

- $c_{1,0} = d\eta$ is the coboundary of the function

$$\eta \;:\; S^1 \to \mathbb{R}\;;\; e^{2\pi ix} \mapsto -2((x)) = \begin{cases} 1 - 2\{x\} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}\;. \end{cases}$$

  This is the simplest evaluation of the Atiyah-Patodi-Singer η-invariant.

2-framings and the signature extension

- Atiyah, *On framings of 3-manifolds* (1990) Every closed 3-manifold N has a canonical 2-framing, i.e. a trivialization of $\tau_N \oplus \tau_N$, characterized by the property that for any 4-manifold M with $\partial M = N$ the signature defect is $\mathrm{def}(M) = 0$.
- Recall the mapping class group $\Gamma_n = \pi_0(\mathrm{Aut}(\Sigma_n))$ and the canonical morphism $\gamma_n : \Gamma_n \to Sp(2n) \subset U(n,n)$. The pullback $\gamma_n^*(c_n) \in H^2(\Gamma_n; \mathbb{Z})$ classifies the **signature extension** $\mathbb{Z} \to \widehat{\Gamma}_n \to \Gamma_n$.
- Geometric interpretation of $\widehat{\Gamma}_n$ in terms of the canonical 2-framing.
- The case $n = 1$, $\Gamma_1 = SL_2(\mathbb{Z})$ of particular importance in string theory and Jones-Witten theory.
- 2-framings and the signature extension have many applications to knots and links.

Conclusion

- The full range of the work of Hirzebruch and Atiyah has been a major influence on the mathematics of the 20th and 21st centuries, with roots in the 19th century.
- In his 1998 lecture in Warsaw for Hirzebruch's 70th birthday (1997) Atiyah posed a problem for the following generation:

Find a successor to Fritz Hirzebruch!

- This is of course only half the problem. The full problem is:

Find successors to Fritz Hirzebruch and Michael Atiyah!