Surveys on Surgery Theory
Volume 2

Papers dedicated to C. T. C. Wall

edited by

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Preface

Surgery theory is now about 40 years old. The 60th birthday (on December 14, 1996) of C. T. C. Wall, one of the leaders of the founding generation of the subject, led us to reflect on the extraordinary accomplishments of surgery theory, and its current enormously varied interactions with algebra, analysis, and geometry. Workers in many of these areas have often lamented the lack of a single source surveying surgery theory and its applications. Indeed, no one person could write such a survey. Therefore we attempted to make a rough division of the areas of current interest, and asked a variety of experts to report on them. This is the second of two volumes which are the result of that collective effort. (The first volume has appeared as *Surveys on Surgery Theory: Volume 1*, Ann. of Math. Studies, vol. 145, Princeton Univ. Press, 2000.) We hope that these volumes prove useful to topologists, to other interested researchers, and to advanced students.

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Surveys on Surgery Theory

Volume 2

Surgery theory today — what it is and where it's going

Jonathan Rosenberg*

Introduction

This paper is an attempt to describe for a general mathematical audience what surgery theory is all about, how it is being used today, and where it might be going in the future. I have not hesitated to express my personal opinions, especially in Sections 1.2 and 4, though I am well aware that many experts would have a somewhat different point of view. Why such a survey now? The main outlines of surgery theory on compact manifolds have been complete for quite some time now, and major changes to this framework seem unlikely, even though better proofs of some of the main theorems and small simplifications here and there are definitely possible. On the other hand, when it comes to applications of surgery theory, there have been many important recent developments in different directions, and as far as I know this is the first attempt to compare and contrast many of them.

To keep this survey within manageable limits, it was necessary to leave out a tremendous amount of very important material. So I needed to come up with selection criteria for deciding what to cover. I eventually settled on the following:

1. My first objective was to get across the major ideas of surgery theory in a non-technical way, even if it meant skipping over many details and definitions, or even oversimplifying the statements of major theorems.

2. My second objective was to give the reader some idea of the many areas in which the theory can be applied.

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3. Finally, in the case of subjects covered elsewhere (and more expertly) in these volumes, I included a pointer to the appropriate article(s) but did not attempt to go into details myself.

I therefore beg the indulgence of the experts for the fact that some topics are covered in reasonable detail and others are barely mentioned at all. I also apologize for the fact that the bibliography is very incomplete, and that I did not attempt to discuss the history of the subject or to give proper credit for the development of many important ideas. To give a complete history and bibliography of surgery would have been a very complicated enterprise and would have required a paper at least three times as long as this one.

I would like to thank Sylvain Cappell, Karsten Grove, Andrew Ranicki, and Shmuel Weinberger for many helpful suggestions about what to include (or not to include) in this survey. But the shortcomings of the exposition should be blamed only on me.

1 What is surgery?

1.1 The basics

Surgery is a procedure for changing one manifold into another (of the same dimension n) by excising a copy of $S^r \times D^{n-r}$ for some r, and replacing it by $D^{r+1} \times S^{n-r-1}$, which has the same boundary, $S^r \times S^{n-r-1}$. This seemingly innocuous operation has spawned a vast industry among topologists. Our aim in this paper is to outline some of the motivations and achievements of surgery theory, and to indicate some potential future developments.

The classification of surfaces is a standard topic in graduate courses, so let us begin there. A surface is a 2-dimensional manifold. The basic result is that compact connected oriented surfaces, without boundary, are classified up to homeomorphism by the genus g (or equivalently, by the Euler characteristic $\chi = 2-2g$). Recall that a surface of genus g is obtained from the sphere S^2 by attaching g *handles*. The effect of a surgery on $S^0 \times D^2$ is to attach a handle, and of a surgery on $S^1 \times D^1$ is to remove a handle. (See the picture on the next page.) Thus, from the surgery theoretic point of view, the genus g is the minimal number of surgeries required either to obtain the surface from a sphere, or else, starting from the given surface, to remove all the handles and reduce to the sphere S^2. There is a similar surgery interpretation of the classification in the nonorientable case, with S^2 replaced by the projective plane $\mathbb{RP}^2$.

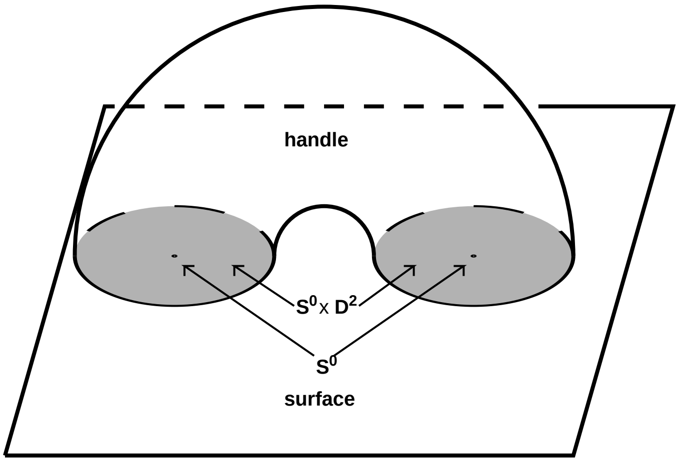

Figure 1. Surgery on an embedded $S^0 \times D^2$.

In dimension $n = 2$, one could also classify manifolds up to homeomorphism by their fundamental groups, with $2g$ the minimal number of generators (in the orientable case). But for every $n \geq 4$, *every* finitely presented group arises as the fundamental group of a compact n-manifold.[1] It is not possible to classify finitely presented groups. Indeed, the problem of determining whether a finite group presentation yields the trivial group or not, is known to be undecidable. Thus there is no hope of a complete classification of *all* n-manifolds for $n \geq 4$. Nevertheless, in many cases it is possible to use surgery to classify the manifolds within a given homotopy type, or even with a fixed fundamental group (such as the trivial group).

Just as for surfaces, high-dimensional manifolds are built out of handles. (In the smooth category, this follows from Morse theory [14]. In the topological category, this is a deep result of Kirby and Siebenmann [11].) Again, each handle attachment or detachment is the result of a surgery. That is why surgery plays such a major role in the classification of manifolds. But since the same manifold may have many quite different handle decompositions, one needs an effective calculus for keeping track of the effect of many surgeries. This is what usually goes under the name of *surgery*

[1] This fact is easy to prove using surgery. Suppose one is given a group presentation $\langle x_1, \ldots, x_k \mid w_1, \ldots, w_s \rangle$. Start with the manifold $M_1 = (S^1 \times S^{n-1}) \# \cdots \# (S^1 \times S^{n-1})$ (k factors), whose fundamental group is a free group on k generators x_i. Then for each word w_j in the generators, represent this word by an embedded circle (this is possible by the [easy] Whitney embedding theorem since $n \geq 3$). This circle has trivial normal bundle since M_1 is orientable, so perform a surgery on a tube $S^1 \times D^{n-1}$ around the circle to kill off w_j. The restriction $n \geq 4$ comes in at this point since it means that the copies of S^{n-2} introduced by the surgeries do not affect π_1. The final result is an n-manifold M with the given fundamental group.

theory.

1.2 Successes

Surgery theory has had remarkable successes. Here are some of the highlights:

- the discovery and classification of exotic spheres (see [107] and [94]);
- the characterization of the homotopy types of differentiable manifolds among spaces with Poincaré duality of dimension ≥ 5 (Browder and Novikov; see in particular [33] for an elementary exposition);
- Novikov's proof of the topological invariance of the rational Pontrjagin classes (see [110]);
- the classification of "fake tori" (by Hsiang-Shaneson [82] and by Wall [25]) and of "fake projective spaces" (by Wall [25], also earlier by Rothenberg [unpublished] in the complex case): manifolds homotopy-equivalent to tori and projective spaces;
- the disproof by Siebenmann [11] of the manifold Hauptvermutung, the [false] conjecture that homeomorphic piecewise linear manifolds are PL-homeomorphic [21];
- Kirby's proof of the Annulus conjecture and the work of Kirby and Siebenmann characterizing which topological manifolds (of dimension > 4) admit a piecewise linear structure [11];
- the characterization (work of Wall, Thomas, and Madsen [101]) of those finite groups that can act freely on spheres (the "topological space form problem" — see Section 3.5 below);
- the construction and partial classification (by Cappell, Shaneson, and others) of "nonlinear similarities" (see 3.4.5 below), that is, linear representations of finite groups which are topologically conjugate but not linearly equivalent;
- Freedman's classification of all simply-connected *topological* 4-manifolds, up to homeomorphism [63]. (This includes the 4-dimensional topological Poincaré conjecture, the fact that all 4-dimensional homotopy spheres are homeomorphic to S^4, as a special case.) For a survey of surgery theory as it applies to 4-manifolds, see [90].
- the proof of Farrell and Jones [55] of topological rigidity of compact locally symmetric spaces of non-positive curvature.

The main drawback of surgery theory is that it is necessarily quite complicated. Fortunately, one does not need to know everything about it in order to use it for many applications.

1.3 Dimension restrictions

As we have defined it, surgery is applicable to manifolds of all dimensions, and works quite well in dimension 2. The surgery theory novice is therefore often puzzled by the restriction in many theorems to the case of dimension ≥ 5. In order to do surgery on a manifold, one needs an embedded product of a sphere (usually in a specific homology class) and a disk. By the Tubular Neighborhood Theorem, this is the same as finding an embedded sphere with a trivial normal bundle. The main tool for constructing such spheres is the [strong] Whitney embedding theorem [143], which unfortunately fails for embeddings of surfaces into [smooth] 4-manifolds.[2] This is the main source of the dimensional restrictions. Thus Smale was able to prove the h-cobordism theorem in dimensions ≥ 5, a recognition principle for manifolds, as well as the high-dimensional Poincaré conjecture, by repeated use of Whitney's theorem (and its proof). (See [15] for a nice exposition.) The h-cobordism theorem was later generalized by Barden, Mazur, and Stallings [88] to the s-cobordism theorem for non-simply connected manifolds. This is the main tool, crucial for future developments, for recognizing when two seemingly different homotopy-equivalent manifolds are isomorphic (in the appropriate category, TOP, PL, or DIFF). The s-cobordism theorem is known to fail for 3-manifolds (where the cobordisms involved are 4-dimensional), at least in the category TOP [39], and for 4-manifolds, at least in the category DIFF (by Donaldson or Seiberg-Witten theory). Nevertheless, Freedman ([63], [8]) was able to obtain remarkable results on the *topological* classification of 4-manifolds by proving a version of Whitney's embedding theorem in the 4-dimensional topological category, with some restrictions on the fundamental group. This in turn has led [64] to an s-cobordism theorem for 4-manifolds in TOP, provided that the fundamental groups involved have subexponential growth.

[2] The "easy" Whitney embedding theorem, usually proved in a first course on differential topology, asserts that if M^m is a smooth compact manifold, then embeddings are dense in the space of smooth maps from M into any manifold N^n of dimension $n \geq 2m+1$. The "hard" embedding theorem, which is considerably more delicate, improves this by asserting in addition that any map $M^m \to N^{2m}$ is homotopic to an embedding, provided that $m \neq 2$ and N is simply connected. This fails for smooth manifolds when $m = 2$, since it is a consequence of Donaldson theory that some classes in π_2 of a simply connected smooth 4-manifold may not be represented by smoothly embedded spheres. In fact, the "hard" embedding theorem also fails in the topological locally flat category when $m = 2$.

2 Tools of surgery

2.1 Fundamental group

The first topic one usually learns in algebraic topology is the theory of the fundamental group and covering spaces. In surgery theory, this plays an even bigger role than in most other areas of topology. Proper understanding of manifolds requires taking the fundamental group into account everywhere. As we mentioned before, any finitely presented group is the fundamental group of a closed manifold, but many interesting results of surgery theory only apply to a limited class of fundamental groups.

2.2 Poincaré duality

Any attempt to understand the structure of manifolds must take into account the structure of their homology and cohomology. The main phenomenon here is Poincaré[-Lefschetz] duality. For a compact oriented manifold M^n, possibly with boundary, this asserts that the cap product with the fundamental class $[M, \partial M] \in H_n(M, \partial M; \mathbb{Z})$ gives an isomorphism

$$H^j(M; \mathbb{Z}) \xrightarrow{\cong} H_{n-j}(M, \partial M; \mathbb{Z}) \ . \qquad \text{(eq. 2.1)}$$

This algebraic statement has important geometric content — it tells homologically how submanifolds of M intersect.

For surgery theory, one needs the generalization of Poincaré duality that takes the fundamental group π into account, using homology and cohomology with coefficients in the group ring $\mathbb{Z}\pi$. Or for work with non-orientable manifolds, one needs a still further generalization involving a twist by an orientation character $w : \pi \to \mathbb{Z}/2$. The general form is similar to that in equation (eq. 2.1): one has a fundamental class $[M, \partial M] \in H_n(M, \partial M; \mathbb{Z}, w)$ and an isomorphism

$$H^j(M; \mathbb{Z}\pi) \xrightarrow{\cong} H_{n-j}(M, \partial M; \mathbb{Z}\pi, w) \ . \qquad \text{(eq. 2.2)}$$

2.3 Hands-on geometry

One of Wall's great achievements ([25], Chapter 5), which makes a general theory of non-simply connected surgery possible, is a characterization of when homology classes up to the middle dimension, in a manifold of dimension ≥ 5, can be represented by spheres with trivial normal bundles. This requires several ingredients. First is the Hurewicz theorem, which says that a homology class in the smallest degree where homology is non-trivial comes from the corresponding homotopy group, in other words, is represented by a map from a sphere. The next step is to check that this map is homotopic to an embedding, and this is where [143] comes in. The third step requires keeping track of the normal bundle, and thus leads us to the next major tool:

2.4 Bundle theory

If X is a compact space such as a manifold, the m-dimensional real vector bundles over X are classified up to isomorphism by the homotopy classes of maps from X into $BO(m)$, the limit (as $k \to \infty$) of the Grassmannian of m-dimensional subspaces of $\mathbb{R}^{m+k}$. Identifying bundles which become isomorphic after the addition of trivial bundles gives the classification up to *stable isomorphism*, and amounts to replacing $BO(m)$ by $BO = \varinjlim BO(m)$. This has the advantage that $[X, BO]$, the set of homotopy classes of maps $X \to BO$, is given by $\widetilde{KO}(X)$, a cohomology theory. A basic fact is that if m exceeds the dimension of X, then one is already in the *stable range*, that is, the isomorphism classification of rank-m bundles over X coincides with the stable classification. Furthermore, if X is a manifold, then all embeddings of X into a Euclidean space of sufficiently high dimension are isotopic, by the [easy] Whitney embedding theorem, and so the normal bundle of X (for an *arbitrary* embedding into a Euclidean space or a sphere) is determined up to stable isomorphism. Thus it makes sense to talk about the *stable normal bundle*, which is stably an inverse to the tangent bundle (since the direct sum of the normal and tangent bundles is the restriction to X of the tangent bundle of Euclidean space, which is trivial).

Now consider a sphere S^r embedded in a manifold M^n. If $2r < n$, then the normal bundle of S^r in M^n has dimension $m = n - r > r$ and so is in the stable range, and hence is trivial if and only if it is stably trivial. Furthermore, since the tangent bundle of S^r is stably trivial, this happens exactly when the restriction to S^r of the stable normal bundle of M^n is trivial. If $2r = n$, i.e., we are in the middle dimension, then things are more complicated. If M is oriented, then the Euler class of the normal bundle of S^r becomes relevant.

2.5 Algebra

Poincaré duality, as discussed above in Section 2.2, naturally leads to the study of quadratic forms over the group ring $\mathbb{Z}\pi$ of the fundamental group π. These are the basic building blocks for the definition of the surgery obstruction groups $L_n(\mathbb{Z}\pi)$, which play a role in both the existence problem (when is a space homotopy-equivalent to a manifold?) and the classification problem (when are two manifolds isomorphic?). For calculational purposes, it is useful to define the L-groups more generally, for example, for arbitrary rings with involution, or for certain categories with an involution.

The groups that appear in surgery theory are then important special cases, but are calculated by relating them to the groups for other situations (such as semisimple algebras with involution over a field). In fact the surgery obstruction groups for finite fundamental groups have been completely calculated this way, following a program initiated by Wall (e.g., [136]). For more details on the definition and calculation of the surgery obstruction groups by algebraic methods, see the surveys [118] and [80].

Algebra also enters into the theory in one more way, via Whitehead torsion (see the survey [106]) and algebraic K-theory. The key issue here is distinguishing between homotopy equivalence and *simple* homotopy equivalence, the kind of homotopy equivalence between complexes that can be built out of elementary contractions and expansions. These two notions coincide for simply connected spaces, but in general there is an obstruction to a homotopy equivalence being simple, called the Whitehead torsion, living in the Whitehead group $\mathrm{Wh}(\pi)$ of the fundamental group π of the spaces involved.[3] This plays a basic role in manifold theory, because of the basic fact that if M^n is a manifold with dimension $n \geq 5$ and fundamental group π, then any element of $\mathrm{Wh}(\pi)$ can be realized by an h-cobordism based on M, in other words, by a manifold W^{n+1} with two boundary components, one of which is equal to M, such that the inclusion of either boundary component into W is a homotopy equivalence. In fact, this is just one part of the celebrated *s-cobordism theorem* [88], which also asserts that the h-cobordisms based on M, up to isomorphism (diffeomorphism if one is working with smooth manifolds, homeomorphism if one is working with smooth manifolds), are in bijection with $\mathrm{Wh}(\pi)$ via the Whitehead torsion of the inclusion $M^n \hookrightarrow W^{n+1}$. The identity element of $\mathrm{Wh}(\pi)$ of course corresponds to the cylinder $W = M \times [0, 1]$. By the topological invariance of Whitehead torsion [43], any homeomorphism between manifolds is necessarily a *simple* homotopy equivalence, so $\mathrm{Wh}(\pi)$ is related to the complexity of the family of homeomorphism classes of manifolds homotopy equivalent to M. In addition, the Whitehead group is important for understanding "decorations" on the surgery obstruction groups, a technical issue we won't attempt to describe here at all.

2.6 Homotopy theory

Homotopy theory enters into surgery theory in a number of different ways. For example it enters indirectly via bundle theory, as indicated in Section 2.4 above. More interestingly, it turns out that surgery obstruction groups can be described as the homotopy groups of certain infinite loop spaces,

[3] The Whitehead group $\mathrm{Wh}(\pi)$ is defined to be the abelianization of the general linear group $GL(\mathbb{Z}\pi) = \varinjlim GL(n, \mathbb{Z}\pi)$, divided out by the "uninteresting" part of this group, generated by the units $\pm 1 \in \mathbb{Z}$ and the elements of π.

related to classifying spaces such as G/O, the study of which becomes important in the most comprehensive approaches to the subject. For this point of view, see [13], [19], [21], and [26].

2.7 Analysis on manifolds

While surgery theory in principle provides an algebraic scheme for classifying manifolds, it is rarely sufficiently explicit so that one can begin with pure algebra and deduce interesting geometric consequences. Usually one has to use the correspondence between geometry and algebra in both directions. One way of using the geometry is through analysis, more specifically, the index theory of certain geometrically defined elliptic differential operators, such as the signature operator. For details of how this matches up with surgery theory, see [121] and [120].

2.8 Controlled topology

Another tool which is not needed for the "classical" theory of surgery, but which is playing an increasingly important role in current work, is controlled topology, by which we mean topology in which one keeps track of "how far" things are allowed to move. This idea, introduced into surgery theory by Chapman, Ferry, and Quinn, has played an important role in the work of many surgery theorists, and is especially important in dealing with non-compact manifolds. But as an example of how it can be applied to compact manifolds, suppose one has a homeomorphism $h: M_1^n \to M_2^n$ between compact smooth manifolds, and one wants to know how the smooth invariants (for example, the Pontrjagin classes) of the two manifolds M_1 and M_2 can differ from one another. One way of approaching this, which can be used to prove Novikov's theorem that h^* preserves *rational* Pontrjagin classes, is to observe that we can approximate h as well as we like by a smooth map h'. Now h' will not necessarily be invertible in DIFF (otherwise M_1 and M_2 would be diffeomorphic), but it is a homotopy equivalence. In fact, given $\varepsilon > 0$, we can choose h' and k and homotopies from $k \circ h'$ to id_{M_1} and from $k \circ h'$ to id_{M_1} which move points by no more than ε (with respect to choices of metrics). Or in other words, we can approximate h by a *controlled homotopy equivalence* in the category DIFF. In the other direction, in dimension $n > 4$, Chapman and Ferry showed that any controlled homotopy equivalence is homotopic to a homeomorphism [44]. For more on controlled surgery, see [59], [111], and [112].

3 Areas of application

3.1 Classification of manifolds

The most important application of surgery theory, the one for which the theory was invented, is the classification of manifolds and manifold structures. This begins with the existence problem for manifold structures: when is a given finite complex X homotopy equivalent to a manifold? An obvious prerequisite is that X satisfy Poincaré duality for some dimension n, in the generalized sense of equation (eq. 2.2) above. When this is the case, we call X an n-dimensional *Poincaré space* or *Poincaré complex*. This insures that X has a "homotopy-theoretic stable normal bundle," the Spivak spherical fibration ν. The Browder-Novikov solution to the existence problem, as systematized in [25], then proceeds in two more steps. First one must check if the Spivak fibration is the reduction of a genuine bundle ξ (in the appropriate category, TOP, PL, or DIFF). If it isn't, then X is not homotopy equivalent to a manifold. If it is, then given ξ reducing to ν, one finds by transversality a *degree-one normal map* $(M, \eta) \to (X, \xi)$, in other words, a manifold M with stable normal bundle η, together with a degree-one map $M \to X$ covered by a bundle map $\eta \to \xi$. The gadget $(M, \eta) \to (X, \xi)$ is also called a *surgery problem*. One needs to check whether it is possible to do surgery on M, keeping track of the bundle data as one goes along, in order to convert M to a manifold N (simple) homotopy equivalent to X. Here one needs an important observation of Browder and Novikov (which follows easily from Poincaré duality): for a degree-one map of Poincaré spaces, the induced map on homology is split surjective. So it is enough to try to kill off the homology kernel. This is done working up from the bottom towards the middle dimension, at which point an obstruction appears, the *surgery obstruction* $\sigma((M, \eta) \to (X, \xi))$ of the surgery problem, which lies in the group $L_n(\mathbb{Z}\pi)$, π the fundamental group of X.

Uniqueness of manifold structures is handled by the relative version of the same construction. Given a simple homotopy equivalence of n-dimensional manifolds $h\colon M \to X$, one must check if the stable normal bundle of X pulls back under h to the stable normal bundle of M. If it doesn't, h cannot be homotopic to an isomorphism. If it does, M and N are normally cobordant, and one attempts to do surgery on a cobordism W^{n+1} between them in order to convert W to an s-cobordism (a cobordism for which the inclusion of either boundary component is a simple homotopy equivalence). Again a surgery obstruction appears, this time in $L_{n+1}(\mathbb{Z}\pi)$. If the obstruction vanishes and we can convert W to an s-cobordism, the s-cobordism theorem says that the map $M \to X$ is homotopic to an isomorphism (again, in the appropriate category). The upshot of this analysis is best formulated in terms of the *surgery exact sequence* of Sullivan and

Wall,

$$\cdots \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}\pi) \overset{\omega}{\dashrightarrow} \mathcal{S}(X) \longrightarrow \mathcal{N}(X) \xrightarrow{\sigma} L_n(\mathbb{Z}\pi). \qquad \text{(eq. 3.1)}$$

discussed in greater detail in [118] and [33]. This long exact sequence relates three different items:

1. the *structure set* $\mathcal{S}(X)$ of the Poincaré complex X, which measures the number of distinct manifolds (up to the appropriate notion of isomorphism) in the simple homotopy class of X

2. normal data $\mathcal{N}(X)$, essentially measuring the possible characteristic classes of the normal or the tangent bundle of manifolds in the simple homotopy type of X; and

3. the Wall surgery groups $L_n(\mathbb{Z}\pi)$, depending only on the fundamental group π of X and the dimension n (modulo 4) (plus the orientation character w, in the non-orientable case).

The map σ sends a surgery problem to its surgery obstruction. Note incidentally that as $\mathcal{S}(X)$ is simply a set, not a group, the meaning of the exact sequence is that for $x \in \mathcal{N}(X)$, $\sigma(x) = 0$ if and only if $x \in \operatorname{im}(\mathcal{S}(X))$, and ω denotes an *action* of $L_{n+1}(\mathbb{Z}\pi)$ on $\mathcal{S}(X)$ such that if a, $b \in \mathcal{S}(X)$, a and b map to the same element of $\mathcal{N}(X)$ if and only if there is a $c \in L_{n+1}(\mathbb{Z}\pi)$ such that $\omega(c, a) = b$.

3.2 Similarities and differences between categories: TOP, PL, and DIFF

At this point it is necessary to say something about the different categories of manifolds. So far we have implicitly been working in the category DIFF of smooth manifolds, since it is likely to be more familiar to most readers than the categories TOP and PL of topological and piecewise linear manifolds. However, surgery works just as well, and in fact in some ways better, in the other categories. We proceed to make this precise.

In the smooth category, except in low dimensions, most closed manifolds have non-trivial structure sets (or in other words, there are usually plenty of non-diffeomorphic manifolds of the same homotopy type). This phenomenon first showed up in the work of Milnor and Milnor-Kervaire on exotic spheres (see [107], [94]). From the point of view of the surgery exact sequence (eq. 3.1), it is due to the rather complicated nature of the normal data term, $\mathcal{N}(X) = [X, G/O]$, and its relationship with the J-homomorphism $BO \to BG$.

In the piecewise linear category, things tend to be somewhat simpler, as one can already see from looking at homotopy spheres. In the category

DIFF, the homotopy spheres of a given dimension $n > 4$, up to isomorphism, form a finite abelian group Θ_n under the operation of connected sum $\#$, and the order of Θ_n is closely related to the Bernoulli numbers. (See [94] for more details.) But in the PL category, Smale's proof [15] of the h-cobordism theorem shows that all homotopy spheres of a fixed dimension $n > 4$ are PL isomorphic to one another. What accounts for this is the "Alexander trick," the fact that if two disks D^n are glued together by a PL isomorphism of their boundaries, then one can extend the gluing map (by linear rescaling) to all of one of the disks, and thus the resulting homotopy sphere is standard. From the point of view of the exact sequence (eq. 3.1), we can explain this by noting that the normal data term $\mathcal{N}(X) = [X, G/\mathrm{PL}]$ is smaller than in the category DIFF. In fact, after inverting 2, it turns out (a theorem of Sullivan) that G/PL becomes homotopy equivalent to a more familiar space, the classifying space BO for real K-theory [13]. This fact is not obvious, of course; it is itself a consequence of surgery theory.

In the category TOP of topological manifolds, the work of Kirby and Siebenmann [11] makes it possible to carry over everything we have done so far. In fact, their work shows that (in dimensions $\neq 4$), there is very little difference between the categories PL and TOP. What difference there is comes from Rochlin's Theorem in dimension 4, which says that a smooth (or PL) spin manifold of dimension 4 must have signature divisible by 16. (For present purposes, we can define "spin" in the PL and TOP categories to mean that the first two Stiefel-Whitney classes vanish.) In contrast, the work of Freedman [63] shows there are closed spin 4-manifolds in TOP with signature 8. This difference (between 8 and 16) accounts for a single $\mathbb{Z}/2$ difference between the homotopy groups of $B\mathrm{PL}$ and $B\mathrm{TOP}$: $\mathrm{TOP}/\mathrm{PL} \simeq K(\mathbb{Z}/2, 3)$. This turns out to be just enough of a difference to make surgery work even better in TOP than in PL.

To explain this, we need not only the surgery obstruction groups, but also the surgery spectra $\mathbb{L}(\mathbb{Z}\pi)$, of which the surgery obstruction groups are the homotopy groups. These spectra are discussed in detail in [19]; suffice it to say here that they are constructed out of parameterized families of surgery problems. Then the fact that surgery works so well in the category TOP may be summarized by saying that in this category, the normal data term $\mathcal{N}(X)$ is basically just the homology of X with coefficients in $\mathbb{L}(\mathbb{Z})$. Furthermore, obstruction theory gives us a classifying map $X \xrightarrow{c} B\pi$ for the universal cover of X, and the obstruction map σ in the exact sequence (eq. 3.1) is the composite of c_* with the map induced on homotopy groups by an *assembly map* $B\pi_+ \wedge \mathbb{L}(\mathbb{Z}) \to \mathbb{L}(\mathbb{Z}\pi)$. This point of view then makes it possible (when $\mathcal{S}^{\mathrm{TOP}}(X)$ is non-empty) to view the structure set $\mathcal{S}^{\mathrm{TOP}}(X)$ as the zero-th homotopy group of still another spectrum, and thus to put a group structure on $\mathcal{S}^{\mathrm{TOP}}(X)$. (See [19], §18.) When this is done, ω in the exact sequence (eq. 3.1) becomes a group homomorphism, and the whole exact sequence becomes an exact sequence of abelian groups.

3.3 Immediate consequences

This is a good point to give some concrete examples of immediate consequences of the surgery classification of manifolds. Some of these follow from the general form of the theory, and do not require any specific calculations. For example, we have (in any of the three categories DIFF, PL, and TOP):

Proposition 3.1 *Suppose* $f\colon (M,\eta) \to (X,\xi)$ *is a surgery problem, that is, a degree-one normal map, in any of the categories* DIFF, PL, *or* TOP. (*Here* M^n *is a compact manifold and* X^n *is a Poincaré complex. We allow the case where* M *and* X *have boundaries, in which case all constructions are to be done rel boundaries.*) *Then the surgery obstruction of* $f \times \mathrm{id}\colon M \times \mathbb{CP}^2 \to X \times \mathbb{CP}^2$ *is the same as for* f*, and the surgery obstruction of* $f \times \mathrm{id}\colon M \times S^k \to X \times S^k$ *vanishes for* $k > 1$*. In particular, if* $k > 1$ *and* $n + k \geq 5$*, then* $f \times \mathrm{id}\colon M \times S^k \to X \times S^k$ *is normally cobordant to a simple homotopy equivalence, so* $X \times S^k$ *has the simple homotopy type of a compact manifold. And if* $n \geq 5$ *and* $f \times \mathrm{id}\colon M \times \mathbb{CP}^2 \to X \times \mathbb{CP}^2$ *is normally cobordant to a simple homotopy equivalence, then the same is true for* f.

Proof. The first statement is the geometric meaning of the periodicity of the surgery obstruction groups. The second statement is a special case of a *product formula* for surgery obstructions, in view of the fact that all signature invariants vanish for a sphere. But it can also be proved directly, using surgery on f below the middle dimension and the fact that a sphere has no homology except in dimension 0 and in the top dimension. (See [33], §1, proofs of Propositions 1.2 and 1.4, for the trick.) □

Remark. The statement of Proposition 3.1 is *false* if we replace S^k, $k > 1$, by S^1. The reason is that taking a product with S^1 has the effect[4] of replacing the fundamental group π by $\pi \times \mathbb{Z}$, and simply shifting the original surgery obstruction up by one in dimension.

Other simple examples of applications of the classification theory that make use of a few elementary facts about G/O, *etc.*, are the following:

Theorem 3.2 *Let* CAT *be* DIFF *or* PL *and let* M^n *be a closed* CAT *manifold of dimension* $n \geq 5$*. Then there are only finitely many* CAT *isomorphism classes of manifolds homeomorphic to* M^n.

[4] modulo a "decoration" nuance, which we're ignoring here

Remark. This is definitely false in dimension 4, as follows from Donaldson theory or Seiberg-Witten theory.

Proof. The issue here is to look at the commutative diagram of exact surgery sequences (the top sequence only of pointed sets, the bottom one, as we've explained in §3.2, of groups)

$$\begin{tikzcd}
\mathcal{N}^{\mathrm{CAT}}(M \times I;\, \partial) \arrow[r, "\sigma"] \arrow[d] & L_{n+1}(\mathbb{Z}\pi) \arrow[r, dotted, "\omega"] \arrow[d, equal] & \mathcal{S}^{\mathrm{CAT}}(M) \arrow[r] \arrow[d] & \mathcal{N}^{\mathrm{CAT}}(M) \arrow[r, "\sigma"] \arrow[d] & L_n(\mathbb{Z}\pi) \arrow[d, equal] \\
\mathcal{N}^{\mathrm{TOP}}(M \times I;\, \partial) \arrow[r, "\sigma"] & L_{n+1}(\mathbb{Z}\pi) \arrow[r, "\omega"] & \mathcal{S}^{\mathrm{TOP}}(M) \arrow[r] & \mathcal{N}^{\mathrm{TOP}}(M) \arrow[r, "\sigma"] & L_n(\mathbb{Z}\pi),
\end{tikzcd}$$

and to show that the preimage in $\mathcal{S}^{\mathrm{CAT}}(M)$ of the basepoint in $\mathcal{S}^{\mathrm{TOP}}(M)$ is finite. But the maps $\mathcal{N}^{\mathrm{CAT}}(M) \to \mathcal{N}^{\mathrm{TOP}}(M)$ and $\mathcal{N}^{\mathrm{CAT}}(M \times I;\, \partial) \to \mathcal{N}^{\mathrm{TOP}}(M \times I;\, \partial)$ are finite-to-one since M has finite homotopy type and since the homotopy groups of TOP/CAT are finite (see §3.2 above). So the result follows from diagram chasing. □

Proposition 3.3 *For any $n \geq 3$, there are infinitely many distinct manifolds with the homotopy type of $\mathbb{CP}^n$ (in any of the categories* DIFF*,* PL*, or* TOP*).*

Proof. Fix a category CAT, one of DIFF, PL, or TOP. We need to show that $\mathcal{S}(\mathbb{CP}^n)$ is infinite. Now $L_k(\mathbb{Z})$ is $\mathbb{Z}$ in dimensions divisible by 4, $\mathbb{Z}/2$ in dimensions 2 mod 4, 0 in odd dimensions. Since the dimension k of $\mathbb{CP}^n$ is even, $L_{k+1}(\mathbb{Z}) = 0$ and $\mathcal{S}(\mathbb{CP}^n)$ can be identified with the kernel (in the sense of maps of pointed sets) of the surgery obstruction map $\sigma\colon \mathcal{N}(\mathbb{CP}^n) \to L_k(\mathbb{Z})$. Now (see §3.2 above for the topological category and [139] for the argument needed to make this work smoothly as well) modulo finite ambiguities, $\mathcal{N}(\mathbb{CP}^n)$ is just $\widetilde{KO}(\mathbb{CP}^n)$, which has rank $\left[\frac{n}{2}\right]$, and σ is given by the formula for the signature coming from the Hirzebruch signature formula. If $n \geq 4$, then $\widetilde{KO}(\mathbb{CP}^n)$ has rank bigger than 1, and if $n = 3$, then $L_k(\mathbb{Z}) = \mathbb{Z}/2$ is finite. So in either case, the kernel of σ is infinite. □

3.4 Classification of group actions

Surgery theory is particularly useful in classifying and studying group actions on manifolds. Depending on what hypotheses one wants to impose, one obtains various generalizations of the fundamental exact sequence (eq. 3.1) in the context of G-manifolds, G some compact Lie group. A few key references are [34], [4], [17], [24], and Parts II and III of [26]. Many of the early references may also be found in §4.6 of [33]. While there is no room here to go into great detail, we will discuss a few cases:

3.4.1 Free actions

If a compact Lie group G acts freely on a connected manifold M, then the quotient space $N = M/G$ is itself a manifold, and there is a fibration

$$G \to M \to N.$$

Thus the fundamental group π of N fits into an exact sequence

$$\pi_1(G) \to \pi_1(M) \to \pi \to \pi_0(G) \to 1. \qquad \text{(eq. 3.2)}$$

Say for simplicity that we take G to be finite, so $G = \pi_0(G)$ and $\pi_1(G) = 1$. One can attempt to classify the free actions of G on M by classifying such group extensions (eq. 3.2), and then, for a fixed such extension, classifying those manifolds N having M as the covering space corresponding to the map $\pi \twoheadrightarrow G$. Note that in this context there is a *transfer map* $\mathcal{S}(N) \to \mathcal{S}(M)$ defined by lifting to the covering space. It will often happen that there are many manifolds homotopy equivalent to N, but non-isomorphic to it, that also have M as the covering space corresponding to G. Such manifolds give elements of the kernel of this transfer map. In section 3.5 below, we shall have more to say about the important special case where M is a sphere.

3.4.2 Semi-free actions

After free actions, the simplest actions of a compact group G on a manifold M are those which are *semi-free*, that is, trivial on a submanifold M^G and free on $M \setminus M^G$. For such an action, the quotient space M/G is naturally a stratified space with two manifold strata, the closed stratum M^G and the open stratum $\left(M \setminus M^G\right)/G$. A discussion of this case from the point of view of stratified spaces may be found in [26], §13.6. Here is a sample result ([138], Theorem A) about semi-free actions of a finite group G: A PL locally flat submanifold Σ^n of S^{n+k} for $k > 2$ is the fixed set of an orientation-preserving semifree PL locally linear G-action on S^{n+k} if and only if Σ is a $\mathbb{Z}/|G|$ homology sphere, $\mathbb{R}^k$ has a free linear representation of G, and certain purely algebraically describable conditions hold for the torsion in the homology of Σ.

3.4.3 Gap conditions

An annoying but sometimes important part of equivariant surgery theory involves what are called *gap conditions*. When a compact group G acts on a manifold M, these are lower bounds on the possible values of the codimension of M^K in M^H, for subgroups $H \subset K$ of G for which $M^K \neq M^H$. Let's specialize now to the case where G is finite. Roughly speaking, there are three kinds of equivariant surgery theory:

1. Surgery without any gap conditions. This is very complicated and not much is known about it.

2. Surgery with a "small" gap condition, the condition that no fixed set component be of codimension < 3 in another. Such a condition is designed to eliminate some fundamental group problems, due to the fact that if M^K has codimension 2 in M^H, there is no way to control the fundamental group of the complement $M^H \setminus M^K$. When M^K can have codimension 1 in M^H, then things are even worse, since one can't even control the number of components of $M^H \setminus M^K$.

3. Surgery with a "large" gap condition, the condition that each fixed set have more than twice the dimension of any smaller fixed set.

For each of cases (2) and (3), there are analogues of the major concepts of non-equivariant surgery theory: normal cobordism, surgery obstruction groups, and a surgery exact sequence. However, there are several ways to set things up, depending on whether one considers equivariant maps (as in most references) or *isovariant* maps (equivariant maps that preserve isotropy groups) as in [34], and depending on whether one tries to do surgery up to equivariant homotopy equivalence (as in [3]) or only up to *pseudoequivalence* (as in [2]). (A map is defined to be a pseudoequivalence if it is equivariant and if, *non-equivariantly* it is a homotopy equivalence.) The big advantage of the large gap condition is that when this condition is satisfied, then one can show [46] that any equivariant homotopy equivalence can be homotoped equivariantly to an isovariant one. For a detailed study of the differences between gap conditions (2) and (3), see [140].

3.4.4 Differences between categories

In the context of group actions, the differences between different categories of manifolds become more pronounced than in the non-equivariant situation studied in §3.2 above. Aside from the smooth and PL categories, the most studied category is that of *topological locally linear actions*, meaning actions on topological manifolds M for which each point $x \in M$ has a G_x-invariant neighborhood equivariantly homeomorphic to a linear action of G_x on $\mathbb{R}^n$. If one studies topological actions with no extra conditions at all, then actions can be very pathological, and the fixed set for a subgroup can be a completely arbitrary compact metrizable space of finite dimension. In particular, it need not be a manifold, and need not even have finite homotopy type. Various properties of the smooth, PL, and topological locally linear categories are discussed in [4], [24], [26], and [83]; they are too complicated to discuss here.

3.4.5 Nonlinear similarity

One of the most dramatic applications of surgery to equivariant topology (even though the original work on this problem only uses *non*-equivariant surgery theory) is to the *nonlinear similarity problem.* This goes back to an old question of de Rham: if G is a finite group and if $\rho_1\colon G \to O(n)$, $\rho_2\colon G \to O(n)$ are two linear (orthogonal) representations of G on Euclidean n-spaces V_1 and V_2, respectively, does a topological conjugacy between ρ_1 and ρ_2 imply that the two representations are linearly equivalent? Here a *topological conjugacy* means a homeomorphism $h\colon V_1 \to V_2$ conjugating ρ_1 to ρ_2. If such a homeomorphism exists, it restricts to a homeomorphism $V_1^{\rho_1(G)} \to V_2^{\rho_2(G)}$, so these two subspaces must have the same dimension. Since we may compose with translation in V_2 by $h(0) \in V_2^{\rho_2(G)}$, there is no loss of generality in assuming that $h(0) = 0$. Now if such an h were to exist and be a diffeomorphism, then the differential of h at the origin would be an invertible linear intertwining operator between ρ_1 and ρ_2, so this problem is only interesting if we allow h to be non-smooth.

One special case is worthy of note: if G is cyclic, if $\mathbb{R}^n$ carries a G-invariant complex structure, and if G acts freely on the complement of the origin, then $S^{n-1}/\rho_1(G)$ and $S^{n-1}/\rho_2(G)$ are lens spaces, and so the question essentially comes down to the issue of whether homeomorphic lens spaces must be diffeomorphic. The answer is "yes," as can be shown using the topological invariance of simple homotopy type [43] together with Reidemeister torsion [106]. The next important progress was made by Schultz [125] and Sullivan (independently) and then by Hsiang-Pardon [81] and Madsen-Rothenberg [100] (again independently). The upshot of this work is that if $|G|$ is of odd order, then topological conjugacy implies linear conjugacy. Then in [38], Cappell and Shaneson showed that for G cyclic of order divisible by 4, there are indeed examples of topological conjugacy between linearly inequivalent representations. This work has been refined over the last two decades, and a summary of some of the most recent work may be found in [40].

While it would be impractical to go into much detail, we can at least sketch some of the key ideas that go into these results. First let's consider the theorems that give *constraints* on existence of nonlinear similarities. Suppose $h\colon V_1 \to V_2$ is a topological conjugacy between representations ρ_1 and ρ_2, say with $h(0) = 0$ (no loss of generality). Then we glue together V_1 and $(V_2 \setminus \{0\}) \cup \{\infty\}$, using h to identify $V_1 \setminus \{0\} \subset V_1$ with $V_2 \setminus \{0\} \subset (V_2 \setminus \{0\}) \cup \{\infty\}$. The result is a copy of S^n equipped with a topological (locally linear) action of G and with ρ_1 as the isotropy representation at one fixed point (0 in V_1), and ρ_2^*, the contragredient of ρ_2, as the isotropy representation at another fixed point (the point at infinity in V_2). Results such as those of Hsiang-Pardon and Madsen-Rothenberg can then be deduced from a suitable version of the G-signature theorem applied to this situation. (Of course, the classical G-signature theorem doesn't apply here, since the group action is not smooth, so proving such a G-signature theorem is not so easy.) One convenient formulation of what comes out, sufficient to give the Hsiang-Pardon and Madsen-Rothenberg results and much more, is the following:

Theorem 3.4 ([123], Theorem 3.3) *Let ρ be a finite-dimensional representation of a finite group G, and let $\gamma \in G$ be of order k. Define the "renormalized Atiyah-Bott number" $AB(\gamma, \rho)$ to be 0 if -1 is an eigenvalue of $\rho(\gamma)$. If this is not the case, suppose that after discarding the $+1$-eigenspace of $\rho(\gamma)$, $\rho(\gamma)$ splits as a direct sum of n_j copies of counterclockwise rotation by $2\pi j/k$, $0 < j < k$, and define in this case*

$$AB(\gamma, \rho) = \prod_{0<j<k} \left(\frac{\zeta^j + 1}{\zeta^j - 1} \right)^{n_j},$$

where $\zeta = e^{2\pi i/k}$. Then the numbers $AB(\gamma, \rho)$, $\gamma \in G$, are oriented topological conjugacy invariants of ρ, and up to sign are topological conjugacy invariants (even if one doesn't require orientation to be preserved).

Next we'll give a rough idea of how Cappell and Shaneson constructed non-trivial nonlinear similarities between representations ρ_1 and ρ_2 of a cyclic group G of order $4q$ with generator γ_0, in the case where $\rho_j(\gamma_0)$ has eigenvalue -1 with multiplicity 1 and all its other eigenvalues are primitive $4q$th roots of unity. Let V_j be the representation space on which ρ_j acts. Then V_j has odd dimension $2k + 1$, and we may write it as

$$\begin{aligned} \mathbb{R}^{2k+1} \quad \cong \quad & \{0\} \cup \left(S^{2k-1} \times [-1,1] \times (0,\infty)\right) \bigcup_{S^{2k-1}\times\{\pm 1\}\times(0,\infty)} \\ & \left(D^{2k} \times \{-1,1\} \times (0,\infty)\right), \end{aligned}$$

where the factor $(0, \infty)$ at the end represents the radial coordinate. Here ρ_j acts by a free linear representation on S^{2k-1}, for which the quotient is a lens space L_j with fundamental group of order $4q$, γ_0 acts by multiplication by -1 on $[-1, 1]$, and γ_0 acts trivially on $(0, \infty)$. So the idea is to choose L_1 and L_2 so that they are homotopy equivalent but not diffeomorphic, but so that their non-trivial double covers $\widetilde{L}_j$, which are lens spaces with fundamental group of order $2q$, are isomorphic to one another. (This is possible using the known classification theorems for lens spaces, as found in [106] for example.) Then if E_j denotes the non-trivial $[-1, 1]$-bundle over L_j (obtained by dividing $S^{2k-1} \times [-1,1]$ by the group action), one can arrange for E_1 and E_2 to be h-cobordant. (This takes a pretty complicated calculation. First one needs to make them normally cobordant, and then one needs to show that the surgery obstruction to converting a normal cobordism to an h-cobordism vanishes.) Then it turns out that E_1 and E_2 become homeomorphic after crossing with $(0,\infty)$. Lifting back to the universal covers, one gets equivariant homeomorphisms

$$S^{2k-1} \times [-1,1] \times (0,\infty) \to S^{2k-1} \times [-1,1] \times (0,\infty)$$

and

$$D^{2k} \times \{-1,1\} \times (0,\infty) \to D^{2k} \times \{-1,1\} \times (0,\infty),$$

which patch together to give the desired nonlinear similarity.

3.5 The topological space form problem

As we mentioned above, one of the successes of surgery theory is the classification of those finite groups G that can act freely on spheres. This subject begins with the observation that if G acts freely on S^n, then the (Tate) cohomology groups of G must be periodic with period $n+1$ ([42], Chapter XVI, §9, Application 4). One of the great classical theorems on cohomology of finite groups ([42], Chapter XII, Theorem 11.6) then says that this happens (for some n) if and only if every abelian subgroup of G is cyclic, or equivalently, if and only if every Sylow subgroup of G is either cyclic or else a generalized quaternion group.

This then raises an obvious question. Suppose G has periodic cohomology. Then does G act freely on a finite CW complex X with the homotopy type of S^n, and if so, can this space X be chosen to be S^n itself? The first part of this question was answered by Swan [131], who showed that, yes, G acts freely on a finite CW complex X with the homotopy type of S^n. The argument for this has nothing to do with surgery; rather, it requires showing that the trivial G-module $\mathbb{Z}$ has a periodic resolution by finitely generated free $\mathbb{Z}G$-modules. (Initially, one only gets such a resolution by finitely generated *projective* $\mathbb{Z}G$-modules, so that it would appear that a finiteness obstruction comes in (see [60]), but one can kill off the obstruction at the expense of possibly increasing the period of the resolution.)

Then one has to determine if X can be taken to be a sphere. One case is classical: if every subgroup of G of order pq (p and q primes) is cyclic, then it is known by a theorem of Zassenhaus that G acts freely and *linearly* (and thus certainly smoothly) on some sphere [146]. Milnor [105] showed, however, that in order for G to act freely on a manifold which is a homology sphere (even in the topological category), it is necessary that all subgroups of order $2p$, p an odd prime, be cyclic rather than dihedral. The argument for this is remarkably elementary, and again doesn't use surgery. An alternative argument using equivariant bordism and equivariant semicharacteristics, again fairly elementary, was given by R. Lee [98]. The point of Lee's proof is basically to show that if a closed oriented manifold M^{2n+1} has as fundamental group the dihedral group $G = D_{2p}$ of order p (p an odd prime), then the formal sum, in an appropriate Grothendieck group, of the G-modules $H_{2i}(M^{2n+1}; FG)$, F a suitable finite field of characteristic 2, if non-zero, has to involve the non-abelian representations of G. This clearly gives a contradiction if the universal cover of M is a homology sphere, since then $H_*(M^{2n+1}; FG)$ is identified with the homology of a sphere, which is only non-zero in bottom and top degree, and the action of G has to be trivial.

The papers [134] and [101] then showed that the condition of Milnor is sufficient as well as necessary for G to act smoothly and freely on a sphere. The method of proof is to go back to Swan's argument in [131] and show that there is a simple Poincaré space with fundamental group G for which the universal cover is (homotopy-theoretically) a sphere, and that its Spivak fibration admits a PL bundle reduction [134], in fact, a smooth bundle reduction [101]. Finally [101], the full power of Wall's surgery theory is used to show that the surgery obstruction vanishes, and thus that there is a manifold with fundamental group G whose universal cover is a homotopy sphere.

3.6 Algebraic theory of quadratic forms

While the main idea of surgery theory is usually to reduce manifold theory to algebra, there are cases where it can be used in the opposite direction, to obtain information about the theory of quadratic forms from geometry. We give just one example. When π is an infinite group with some "non-positive curvature" properties, for example, the fundamental group of a hyperbolic manifold, then various geometrical or analytical techniques can be used to prove the Novikov conjecture or sometimes even the Borel rigidity conjecture for π. (See [61], [45], and [129] for surveys of the literature, which is quite extensive.) This in turn, from the surgery exact sequence (eq. 3.1), implies significant information about the stable classification of quadratic forms over $\mathbb{Z}\pi$.

3.7 Submanifolds, fibrations, and embeddings

Surgery theory can deal not only with individual manifolds, but also with questions concerning how one manifold can embed in another. There is an extensive literature on such problems, but we will only mention a few examples. For instance, suppose M is a manifold, and suppose that from a homotopy point of view, M looks like the total space of a fibration $F \to M \to B$. Then can M be made into the total space of a genuine manifold fiber bundle with base and fiber homotopy equivalent to B and F? Or suppose $X \hookrightarrow Y$ is a Poincaré embedding. That means that Y is a Poincaré complex, say of dimension n, X is a Poincaré complex of dimension $n - q$ with a mapping to Y, and we have subspaces $S \subset C \subset Y$ with the following properties:

1. There is a spherical fibration $S^{q-1} \to S \xrightarrow{p} X$, with S a Poincaré complex of dimension $n-1$. (S is the homotopy analogue of the boundary of a tubular neighborhood of a submanifold X in Y of codimension q. The map p corresponds to the retraction of this tubular neighborhood onto the submanifold X.)

2. (C, S) is a Poincaré pair of dimension n. That is, C has the Poincaré duality properties of an n-dimensional manifold with boundary S. The diagram

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, "p"] & C \arrow[r, hook] & Y \\ X \arrow[urr] & & \end{tikzcd}$$

   is homotopy commutative.

3. Up to simple homotopy equivalence, Y is the union of C and the mapping cylinder of p, joined along S. (The mapping cylinder of p is the homotopy analogue of the closed tubular neighborhood of the submanifold. This says that C plays the role of the complement of an open tubular neighborhood of X in Y.)

Now suppose M is a manifold and $h\colon M \to Y$ is a homotopy equivalence. Then can h be homotoped to a map h' so that ${h'}^{-1}(X)$ is a genuine submanifold N of M (of codimension q) and h' restricted to N is a homotopy equivalence $N \to X$? When this is the case, h is said to be *splittable* along X.

There is an extensive literature on questions such as these but we content ourselves here with a few representative examples.

For the fiber bundle problem, the first case to be studied, but still one of the most important, is whether a certain manifold fibers over S^1. In other words, one is given a compact manifold M^n and a map $f\colon M^n \to S^1$ with f_* surjective on π_1, and one wants to know if one can change f within its homotopy class to the projection map p of a fiber bundle $N^{n-1} \to M^n \xrightarrow{p} S^1$. The map f defines an infinite cyclic covering $\widetilde{M} \to M$, and if the desired fiber bundle exists, then $\widetilde{M}$ must be isomorphic (in the appropriate category) to $N \times \mathbb{R}$ with N a compact manifold. So first one must check if the finiteness obstruction vanishes (so that $\widetilde{M}$ is homotopically finite and is equivalent to a finite Poincaré complex), and then one must solve a surgery problem to see if $\widetilde{M}$ can be realized as cylinder $N \times \mathbb{R}$. The solution to the problem was found by Farrell ([50], [51]) (following earlier work by Browder and Levine in the simply connected case), who found that if $\widetilde{M}$ is indeed homotopically finite, the necessary and sufficient condition for a positive solution to fiber bundle problem is the vanishing of a Whitehead torsion obstruction in the Whitehead group of $\pi_1(M)$.

For the splitting problem, there are essentially three cases.

1. When X is of codimension one, the issues involved are somewhat similar to what arises in the problem of fibering over a circle, and the key result (in the categories TOP and PL) is due to Cappell [36]. A special case of this concerns the following question. Suppose M^n is a closed manifold that looks homotopy-theoretically like a connected sum. (Since we are assuming $n > 2$, that means in particular that $\pi_1(M)$ must be the free product of the fundamental groups of the prospective summands.) Then does M have a splitting of the form $M \cong M_1 \# M_2$? (This corresponds to the case where $X = S^{n-1}$ and is "two-sided" and separating in Y.) Cappell discovered that when $n \geq 5$, the answer to this question is not always "yes," but that the only obstruction to a positive answer is an algebraic one related to the fundamental groups involved.[5] The obstruction group vanishes when $\pi_1(M)$ has no 2-torsion, so in this case one indeed has a splitting $M \cong M_1 \# M_2$. Incidentally, the dimension restriction is necessary, for it follows from Donaldson theory that there are many simply connected PL 4-manifolds (a K3 surface, for example) which are homotopy theoretically connected sums, but which do not split as connected sums in the PL category. (In dimension 4, the PL and DIFF categories are equivalent.)

2. When X is of codimension two, the splitting problem is closely related to the classification of knots; see [18], §§7.8–7.9, [22], and the survey [96] in this collection for more details.

3. When X is of codimension 3 or more and Y is of dimension 5 or more, the splitting problem always has a positive answer in the PL category, provided that the obvious necessary condition (that X have the simple homotopy type of a PL manifold) is satisfied, as shown by Wall in [25], Corollary 11.3.1. In the smooth case one needs a little more, since the spherical fibration $S \xrightarrow{p} X$ must come from a rank q vector bundle, but the "expected results" are still true.

[5] In the category DIFF, the result is the same as long as one allows generalized connected sums along a (possibly exotic) separating homotopy sphere.

3.8 Detection on submanifolds

The study of submanifolds can also be turned around, and we can ask to what extent surgery obstructions are determined by what happens on submanifolds. This in turn is related to another problem: How much of the Wall surgery groups $L_n(\mathbb{Z}\pi)$ arises from surgery obstructions of degree-one normal maps between closed manifolds? To be more precise, the issue is basically how much information about submanifolds of M is needed either

1. to compute the surgery obstruction of a surgery problem $(N^n, \eta) \to (M^n, \xi)$, or how to best understand *geometrically* the obstruction map $\sigma : \mathcal{N}(M) \to L_n(\mathbb{Z}\pi)$; or
2. to determine when a class in $\mathcal{S}(M^n)$ is trivial, or in other words, when a homotopy equivalence $N \xrightarrow{h} M$ is homotopic to an isomorphism.

These issues often go under the general name of "oozing," which is supposed to suggest how simply connected surgery obstructions on submanifolds "ooze up" to give obstructions on a larger manifold, not usually simply connected.

The first major result along these lines was the characteristic variety theorem of Sullivan (see Sullivan's 1967 notes, republished in [21], pp. 69–103). It says (*roughly* that the answer to question (2) can be formulated in terms of simply connected surgery obstructions (signatures and Arf-Kervaire invariants) to splitting along certain (possibly singular) submanifolds of M. This theorem is also related to various formulas for characteristic classes found in [109].

As far as question (1) is concerned, the basic question was whether surgery obstructions can be computed from simply connected splitting obstructions on submanifolds of bounded codimension. For general fundamental groups this is certainly not the case (see [137]), but it was expected by the experts (the "oozing conjecture") that this would be the case for manifolds with finite fundamental group. This issue has now been settled. Codimension 2 manifolds *do not* suffice; Cappell and Shaneson [37] showed that if M^3 is the usual quaternionic lens space (the quotient of S^3 by the linear action of the quaternion group Q_8 of order 8) and $K^{4k+2} \xrightarrow{\kappa} S^{4k+2}$ is the "Kervaire problem" (a simply connected surgery problem representing the generator of $L_{4k+2}(\mathbb{Z}) \cong \mathbb{Z}/2$), then

$$\sigma(M^3 \times K^{4k+2} \xrightarrow{\mathrm{id}\times\kappa} M^3 \times S^{4k+2}) \neq 0 \text{ in } L_{4k+5}(\mathbb{Z}Q_8),$$

even though the obstruction here comes from the Arf invariant on the codimension 3 manifold K^{4k+2}. But codimension 3 manifolds *do* suffice ([79], [103]).[6]

[6]For those who know what this means: at least for the h decoration.

3.9 Differential geometry

Surgery theory becomes especially interesting when applied to certain problems in differential geometry. We begin with Riemannian geometry. Recall that a *Riemannian metric* on a manifold is a smoothly varying choice of inner products on tangent spaces. This makes it possible to measure lengths of curves, and thus to define geodesics (curves which locally minimize length), and also to measure angles between intersecting curves. The most important intrinsic geometric invariants of a Riemannian manifold are those having to do with curvature. The *sectional curvature* of a Riemannian manifold M (also called the Gaussian curvature if $\dim M = 2$) at a point $p \in M$ in the direction of some 2-plane P in the tangent space T_pM through p measures how the sum of the angles of a small geodesic triangle differs from π, if one vertex of the triangle is at p, and the incident sides there lie in the plane P. The *Ricci curvature* (which is a tensor) and *scalar curvature* (the trace of the Ricci curvature) at p are then obtained from various averages of the sectional curvatures there. As such, the standard curvature invariants are defined locally, but global bounds on curvature (for a closed or complete manifold) have implications for global topology. We mention just a few prominent examples: the Gauss-Bonnet Theorem for closed surfaces M, which says that $\int_M K\, dA = 2\pi\, \chi(M)$, where K is the Gaussian curvature, $\chi(M)$ is the Euler characteristic, and dA is the Riemannian area measure; Myers' Theorem, that any complete manifold of Ricci curvature $\geq c > 0$ is closed and has finite fundamental group; the Cartan-Hadamard Theorem, that any complete manifold of non-positive sectional curvature is aspherical[7], with universal cover diffeomorphic to Euclidean space (and with covering map the exponential map from the tangent space at a basepoint); and the generalized Gauss-Bonnet Theorem of Chern and Allendoerfer, expressing the Euler characteristic as a multiple of the integral of the Pfaffian of the curvature form. Global consequences of positivity of the scalar curvature are discussed in [122].

3.9.1 Rigidity theorems for Riemannian manifolds

One place where surgery can be of particular help in Riemannian geometry is in the study of *rigidity theorems*, results that say that two Riemannian manifolds sharing a very specific geometric property must be homeomorphic, diffeomorphic, etc. Such theorems abound in Riemannian geometry. Classical examples (proved without using surgery) are *sphere theorems*, such as the fact that a complete simply connected manifold with sectional curvature K satisfying $\frac{1}{4} < K \leq 1$ must be the union of two disks glued together via a diffeomorphism of their boundaries, and thus a homotopy sphere. Another famous examples is the Mostow Rigidity Theorem, which

[7] That is, all its higher homotopy groups π_j, $j > 1$, vanish.

says that two irreducible locally symmetric spaces of dimension ≥ 3 and non-compact type, with finite volume and with isomorphic fundamental groups, must be isometric to one another. Mostow's Theorem helped to motivate the Borel Conjecture, that two compact aspherical manifolds[8] with isomorphic fundamental groups are homeomorphic.

Here is a brief [very incomplete] list of a number of rigidity theorems proved using a combination of surgery theory and Riemannian geometry:

1. the work of Farrell and Hsiang [52] on the Novikov Conjecture. This was very influential in its time but has now been superseded by the work of Farrell and Jones cited below.

2. Kasparov's proof ([85], [86]) of the Novikov Conjecture for arbitrary discrete subgroups of Lie groups. This has been improved by Kasparov and Skandalis [87] to give the Novikov Conjecture for "bolic" groups, by weakening nonpositive curvature in Riemannian geometry to a rough substitute in the geometry of metric spaces.

3. the work of Farrell and Jones ([5], [6]) on topological rigidity of manifolds of nonpositive curvature. This includes for example:

   Theorem 3.5 ([55]) *Let M and N be closed aspherical topological manifolds of dimensions $\neq 3, 4$. If M is a smooth manifold with a nonpositively curved Riemannian metric and if $\pi_1(M) \xrightarrow{\cong} \pi_1(N)$ is an isomorphism, then this isomorphism is induced by a homeomorphism between M and N.*

4. the work of Farrell and Jones [54] on pseudoisotopies of manifolds of nonpositive curvature. This gives substantial information about the homotopy types of the diffeomorphism groups of these manifolds.

5. examples, constructed by Farrell and Jones ([53], [56], [57]), of manifolds of nonpositive curvature which are homeomorphic but not diffeomorphic.

6. the theorem of Grove and Shiohama [76] that a complete connected Riemannian manifold with dimension ≤ 6, with sectional curvature $\geq \delta > 0$ and with diameter $> \pi/2\sqrt{\delta}$, is diffeomorphic to a standard sphere.

7. work of Grove-Peterson-Wu [75] (see also the work of Ferry [58]) showing that for any integer n, any real number k and positive numbers D and v, the class of closed Riemannian n-manifolds M with sectional

[8] Recall that locally symmetric spaces of non-compact type are included here, by the Cartan-Hadamard Theorem.

curvature $K_M \geq k$, diameter $d_M \leq D$ and volume $V_M \geq v$ contains at most finitely many homeomorphism types when $n \neq 3$, and only finitely many diffeomorphism types if, in addition, $n \neq 4$. (There is a similar result for manifolds with injectivity radius $i_M \geq i_0 > 0$ and volume $V_M \leq v$.)

8. work of Ferry and Weinberger [62] growing out of work on the Novikov Conjecture. This includes the very interesting result that if M^n is an irreducible compact locally symmetric space of noncompact type (with $n > 4$), then the natural forgetful map $\mathrm{Diff}(M) \to \mathrm{Homeo}(M)$ has a continuous splitting.

9. the "packing radius" sphere theorem of Grove and Wilhelm [77], stating that for $n \geq 3$, a closed Riemannian n-manifold M with sectional curvature ≥ 1 and $(n-1)$-packing radius $> \frac{\pi}{4}$ is diffeomorphic to S^n.[9]

10. an improvement of the classical sphere theorem due to Weiss [142], showing that if M^n is a complete simply connected manifold with sectional curvature K satisfying $\frac{1}{4} < K \leq 1$, then not only is M a homotopy sphere, but M has "Morse perfection n," which rules out some of the exotic sphere possibilities for M. See also [78] for further developments. (In the other direction, Wraith [144] has constructed metrics of positive *Ricci* curvature on all homotopy spheres that bound parallelizable manifolds. A few exotic spheres are known to admit metrics of nonnegative sectional curvature (see [71], [119], and work in progress by Grove and Ziller), but the sectional curvatures of the metrics constructed to date are not strictly positive, let alone $\frac{1}{4}$-pinched.)

11. work of Brooks, Perry, and Petersen [31] showing that given a sequence of isospectral manifolds of dimension n for which either all the sectional curvatures are negative or there exists a uniform lower bound on the sectional curvatures, then the sequence contains only finitely many homeomorphism types, and if $n \neq 4$, at most finitely many diffeomorphism types.

12. recent theorems of Belegradek [29] showing that in many cases, given a group π, an integer n larger than the homological dimension of π, and real numbers $a < b < 0$, there are only finitely many diffeomorphism types of complete Riemannian n-manifolds with curvature $a \leq K \leq b$ and fundamental group π. The manifolds involved here are noncompact, and usually have infinite volume.

[9] The $(n-1)$-packing radius is defined to be half the maximum, over all configurations of $(n-1)$ points in M, of the minimum distance between points.

What most of these references have in common is that a geometric assumption, usually based on curvature bounds, is used to deduce some consequences that, while sometimes rather technical and not always directly interesting in themselves, can be plugged into the "surgery machine" to deduce the desired rigidity theorem.

3.9.2 Surgery and positive scalar or Ricci curvature

Surgery enters into differential geometry in another somewhat different way as well: through "surgery theorems" that say that under appropriate hypotheses, a certain geometrical structure on one manifold may be transported via a surgery to some other manifold. In this subsection we will discuss application of this principle to positive scalar or Ricci curvature, in section 3.9.4 we will discuss conformal geometry, and in section 3.9.5 we will discuss application to the study of symplectic or contact structures.

So far the most remarkable and useful surgery theorem is the theorem of [73] and [124] regarding positive scalar curvature. (See also [122] for an exposition and for a correction to one point in the Gromov-Lawson proof.) This says that if M_1^n is a compact manifold (not necessarily connected) with a Riemannian metric of positive scalar curvature, and if M_2^n can be obtained from M_1 by surgery on a sphere of codimension ≥ 3, then M_2 can also be given a metric of positive scalar curvature. This result is so powerful that, when combined with known index obstructions to positive scalar curvature based on the Dirac operator, it has made complete classification of the manifolds admitting positive scalar curvature metrics feasible in many cases. See [122] for a detailed exposition.

In the case of positive Ricci curvature, a surgery theorem as general as this could not be true, for surgery on $S^0 \hookrightarrow S^n$ results in a manifold with infinite fundamental group, which cannot have a metric of positive Ricci curvature by Myers' Theorem. Nevertheless, there is no known reason why surgery on a sphere of dimension ≥ 1 and codimension ≥ 3 in a manifold of positive Ricci curvature cannot result in a manifold of positive Ricci curvature, and in fact there is some positive evidence for this in [126] and [145]. But Stolz in [130], based upon both heuristics of Dirac operators on loop spaces and upon calculations with homogeneous spaces and complete intersections, has conjectured that the Witten genus vanishes for spin manifolds with positive Ricci curvature and with vanishing $\frac{p_1}{2}$.[10] If this is the case, then Stolz has shown [130] that there are simply connected closed manifolds with positive scalar curvature metrics but without metrics of positive Ricci curvature, and thus a surgery theorem this general for positive Ricci curvature cannot hold. So perhaps it should be necessary to

[10]For spin manifolds M, the first Pontrjagin class p_1 is always divisible by 2, and there is an integral characteristic class $\frac{p_1}{2} \in H^4(M; \mathbb{Z})$ which when multiplied by 2 gives p_1.

restrict to surgeries of some greater codimension.

3.9.3 Surgery and the Yamabe invariant

The problem of prescribing scalar curvature on a manifold also has a quantitative formulation in terms of the so-called *Yamabe invariant.* If M^n is a closed manifold and we fix a Riemannian metric g on M, then by the solution of the Yamabe problem, it is always possible to make a (pointwise) conformal change in the metric, i.e., to multiply g by a positive real-valued function, so as to obtain a metric with constant scalar curvature and total volume 1. The minimum possible value of the scalar curvature of such a metric is an invariant of the conformal class of the original metric, known as the *Yamabe constant* of the conformal class. The *Yamabe invariant* $Y(M)$ of M is then defined as the supremum, taken over all conformal classes of metrics on M, of the various Yamabe constants. It is bounded above by a universal constant depending only on n, namely $n(n-1)(\operatorname{vol} S^n(1))^{2/n}$ (the scalar curvature of a round n-sphere of unit volume), and is closely related to the question of determining what real-valued functions can be scalar curvatures of Riemannian metrics on M with volume 1 [92]. Note that $Y(M) > 0$ if and only if M admits a metric of positive scalar curvature. It is known that "most" closed 4-manifolds have negative Yamabe invariant [95]. In a counterpart to the surgery theorem of [73] and [124], it is shown in [115] that if M' can be obtained from M by surgeries in codimension ≥ 3 and if $Y(M) \leq 0$, then $Y(M') \geq Y(M)$. This fact has been applied in [113] to obtain exact calculations of $Y(M)$ for some 4-manifolds M, and in [114] to show that $Y(M) \geq 0$ for every simply connected closed n-manifold M^n with $n \geq 5$.

3.9.4 Surgery and conformal geometry

A *conformal structure* on manifold M^n is an equivalence class of Riemannian structures, in which two metrics are identified if angles (but not necessarily distances) are preserved. For oriented 2-manifolds, this is the same thing as a complex analytic structure. A conformal structure is called *conformally flat* if each point in M has a neighborhood conformally equivalent to Euclidean n-space $\mathbb{R}^n$. (This is true for the standard round metric on S^n, for example.) An immersion $M^n \looparrowright \mathbb{R}^{n+k}$ is called *conformally flat* if the standard flat metric on $\mathbb{R}^{n+k}$ pulls back to a conformally flat structure on M. One of the important problems in conformal geometry is the classification of conformally flat manifolds and conformally flat immersions into Euclidean space. For immersions of hypersurfaces, i.e., immersions in codimension $k = 1$, a complete classification has been given (begun in [93], completed in [41]) using the idea of conformal surgery. The final result is:

Theorem 3.6 ([41]) *A compact connected manifold M^n has a conformal immersion into $\mathbb{R}^{n+1}$ if and only if M can be obtained from S^n by adding finitely many 1-handles, i.e., by doing surgery on a finite set of copies of S^0 in S^n. In particular, any such M has free fundamental group.*

3.9.5 Surgery and symplectic and contact structures

A *symplectic structure* on an even-dimensional manifold M^{2n} is given by a closed 2-form ω such that $\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega$ (n factors) is everywhere non-zero, i.e., is a volume form. A *contact structure* on an odd-dimensional manifold M^{2n+1} is a maximally non-integrable subbundle ξ of TM of codimension 1, and thus is locally given by $\xi = \ker \alpha$, where α is a 1-form such that $\alpha \wedge (d\alpha)^n$ is a volume form. Symplectic and contact structures arise naturally in classical mechanics, and there is a close link between them.

The problem of determining what manifolds admit symplectic or contact structures is not so easy, though there are some obvious necessary conditions. If M^{2n} is a closed connected manifold which admits a symplectic form ω, then if $[\omega] \in H^2(M;\mathbb{R})$ denotes its de Rham class, $[\omega]^n$ must generate $H^{2n}(M;\mathbb{R}) \cong \mathbb{R}$. In particular, M is oriented, and ω gives a reduction of the structure group of TM from $GL(2n,\mathbb{R})$ to $Sp(2n,\mathbb{R})$, which has maximal compact subgroup $U(n)$; thus ω defines an isomorphism class of almost complex structures J on M. The most familiar examples of symplectic manifolds are Kähler, in other words, admit a Riemannian metric g and an integrable (and parallel) almost complex structure J for which $\omega(X,Y) = g(JX,Y)$ for all vector fields X and Y. However, it is known that there are plenty of symplectic manifolds without Kähler structures [135]. A promising line of attack in constructing symplectic structures is therefore to start with the standard examples and try construct new ones using fiber bundles, "blow-ups," and surgery methods. (See [102] for a detailed exposition.) In particular, "symplectic surgery" has been studied in [70] and [132]. With it Gompf has proved [70] that every finitely presented group is the fundamental group of a compact symplectic 4-manifold, even though there are constraints on the fundamental groups of Kähler manifolds. It is not always possible to put a symplectic structure on the connected sum of two symplectic manifolds, since in dimension 4, Taubes [133] has shown using Seiberg-Witten theory that a closed symplectic manifold cannot split as a connected sum of two manifolds each with $b_1^+ > 0$. Gompf's "symplectic connected sum" construction is therefore somewhat different: if M_1^{2n}, M_2^{2n}, and N^{2n-2} are symplectic and one has symplectic embeddings $N \hookrightarrow M_1$, $N \hookrightarrow M_2$ whose normal bundle Euler classes are negatives of one another, then Gompf's $M_1 \#_N M_2$ is obtained by joining the complements of tubular neighborhoods of N in M_1 and in M_2 along their common boundary (a sphere bundle over N).

Surgery has also played an important role in a number of other problems connected with symplectic geometry: the theory of Lagrangian embeddings[11] (see, e.g., [99], [49], and [116]) and Eliashberg's topological classification [47] of Stein manifolds.[12] A Stein manifold of complex dimension n is known to admit a proper Morse function with all critical points of index $\leq n$ (so that, roughly speaking, M is the thickening of an n-dimensional CW-complex), and Eliashberg showed that for $n > 2$, a $2n$-dimensional almost complex manifold admits a Stein structure exactly when it satisfies this condition.

A contact structure on an odd-dimensional manifold appears at first sight to be a very "flabby" object. If we consider only contact structures ξ for which TM/ξ is orientable (this is only a slight loss of generality, and turns out to be automatic if M is orientable and n even), then every contact structure ξ is defined by a global 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$ everywhere, and α is determined by ξ up to multiplication by an everywhere non-zero real function. Note also that as $d\alpha$ defines a symplectic structure on ξ, α defines an *almost contact structure* on M, that is, an isomorphism class of reductions of the structure group of TM from $GL(2n+1, \mathbb{R})$ to $1 \times U(n)$. So a natural question is whether an odd-dimensional manifold always admits a contact structure within every homotopy class of almost contact structures. When $n = 1$, i.e., $\dim M = 3$, the answer is known to be "yes," though Eliashberg showed that there are basically two distinct types of contact structure, "tight" and "overtwisted." Furthermore, if M is a closed oriented 3-manifold, then every class in $H^2(M; \mathbb{Z})$ is the Euler class of an overtwisted contact structure, but only finitely many homology classes in $H^2(M; \mathbb{Z})$ can be realized as the Euler class of a tight contact structure. (For surveys, see [69] and [48].) In higher dimensions, it is not known if every manifold with an almost contact structure admits a contact structure, though the experts seem to doubt this. And it is known that S^{2n+1} has at least two non-isomorphic contact structures in the homotopy class of the standard almost contact structure ([48], Theorem 3.1).

Nevertheless, in many cases one can construct contact structures in a given homotopy class of almost contact structures through a process of "contact surgery." The key tools for doing this may be found in [141] and in [47]. These references basically prove that if (M_1^{2n+1}, ξ_1) is a contact manifold and M_2^{2n+1} can be obtained from M_1 by surgery on $S^k \subset M_1^{2n+1}$,

[11] If (M^{2n}, ω) is a symplectic manifold, an embedding $f\colon N^n \hookrightarrow M^{2n}$ is called *Lagrangian* if $f^*\omega = 0$. Aside from the obvious bundle-theoretic consequence, that ω induces an isomorphism between the cotangent bundle of N and the normal bundle for the embedding, this turns out to put considerable constraints on isotopy class of the embedding.

[12] A complex manifold M is called a *Stein manifold* if $H^j(M, \mathcal{S}) = 0$ for all $j > 0$ for any coherent analytic sheaf $\mathcal{S}$ on M (though it is enough to assume this for $j = 1$), or equivalently, if M has a proper holomorphic embedding into some $\mathbb{C}^k$, that is, M is an affine subvariety of $\mathbb{C}^k$. An open subset of $\mathbb{C}^n$ is a Stein manifold if and only if it is a domain of holomorphy.

then M_2 also admits a contact structure ξ_2 (in the corresponding homotopy class of almost contact structures), provided that S^k is tangent to the contact structure ξ_1, and has trivial "conformal symplectic normal" (CSN) bundle. Since ξ_1 is maximally non-integrable, the first condition (S^k tangent to ξ_1) forces TS^k to be isotropic in ξ_1 for the symplectic form $d\alpha$ on ξ_1, α a 1-form defining ξ_1. In other words, if $(TS^k)^\perp$ denotes the orthogonal complement of TS^k in ξ_1, which has rank $2n-k$, then $TS^k \subseteq (TS^k)^\perp$, so $k \leq n$. The CSN bundle is then $(TS^k)^\perp/TS^k$, and a trivialization of this bundle determines a homotopy class of almost contact structures on M_2. Applications of this theorem may be found in [27], [65], [66], [67], and [68]. Some of the results are that:

1. Every finitely presented group is the fundamental group of a closed contact manifold of dimension $2n+1$, for any $n > 1$ [27].

2. Every simply connected spin^c 5-manifold admits a contact structure in every homotopy class of almost contact structures [65]. (The spin^c condition is necessary for existence of an almost contact structure.)

3. Every closed spin 5-manifold with finite fundamental group of odd order not divisible by 9 and with periodic cohomology admits a contact structure [68].

3.10 Manifold-like spaces

While the original applications of surgery theory were to the classification and study of manifolds, in recent years surgery has also been applied quite successfully to spaces which are not manifolds but which share some of the features of manifolds. We list just a few examples:

1. **Poincaré spaces:** Poincaré spaces have already appeared in this survey; they are spaces with the homotopy-theoretic features of manifolds. Thus for example it makes sense to talk about the bordism theory Ω_*^P defined like classical oriented bordism Ω_*, but using oriented Poincaré complexes in place of oriented smooth manifolds. Since Poincaré complexes do not satisfy transversality, this theory does *not* agree with the homology theory defined by the associated Thom spectrum MSG (whose homotopy groups are all finite), but the two are related by an exact sequence where the relative groups are the Wall surgery groups. The proof uses surgery on Poincaré spaces, and may be found in [97], or in slightly greater generality, in [9]. Another interesting issue is the extent to which Poincaré spaces can be built up by pasting together manifolds with boundary, using homotopy equivalences (instead of diffeomorphisms or homeomorphisms) between boundary components. It turns out that all Poincaré spaces

can be pieced together this way (at least if one avoids the usual problems with dimensions 3 and 4), and that the minimal number of manifold pieces required is an interesting invariant. See [84] and [9] for more details, as well as [91] for a survey of several other issues about Poincaré spaces.

2. **Stratified spaces:** *Stratified spaces* are locally compact spaces X which are not themselves manifolds but which have a filtration $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k = X$ by closed subspaces such that each $X_j \setminus X_{j-1}$ is a manifold and the strata fit together in a suitable way. There are many different categories of such spaces, depending on the exact patching conditions assumed. But two important sets of examples motivate most of theory: algebraic varieties over $\mathbb{R}$ or $\mathbb{C}$, and quotients of manifolds by actions of compact Lie groups. Surgery theory has been very effective in classifying and studying such spaces. There is no room to go into details here, but see [26] and [83] for surveys.

3. **ENR homology manifolds:** Still another way to weaken the definition of a manifold is to consider *homology n-manifolds*, spaces X with the property that for every $x \in X$, $H_j(X, X \setminus \{x\}; \mathbb{Z}) = \begin{cases} 0, & j \neq n, \\ \mathbb{Z}, & j = n. \end{cases}$ In order for such a space to look more like a topological manifold, it is natural to assume also that it is an ENR (Euclidean neighborhood retract). So a natural question is: is every ENR homology n-manifold M homeomorphic to a topological n-manifold? It has been known for a long time that the answer to this question is "no" (the simplest counterexample is the suspension of the Poincaré homology 3-sphere), so to make the question interesting, let's throw in the additional assumption that M has the "disjoint disks property." Then M has (at least) a very weak kind of transversality, and is thus quite close to looking like a manifold. Does this make it a manifold? This question has a long history, and the surprising answer of "no," due to Bryant, Ferry, Mio, and Weinberger [35], is discussed in this collection in [108].

3.11 Non-compact manifolds

Almost all the applications of surgery theory which we have discussed so far are for compact manifolds, but surgery can also be used to study non-compact manifolds as well. Here we just mention a few cases:

1. Siebenmann's characterization [127] of when a non-compact manifold X^n (without boundary) is the interior of some compact manifold W^n with boundary. Obvious necessary conditions are that X have finite

homotopy type and have finitely many ends. Furthermore, the fundamental group "at infinity" in each end must be finitely presented, and the Wall finiteness obstruction (see [60]) of the end must vanish. Siebenmann's Theorem ([127] or [26], §§1.5–1.6) says that these obvious necessary conditions are sufficient if $n \geq 6$.

2. Siebenmann's characterization of when a non-compact manifold W with boundary is an open collar of its boundary, or in other words, when $W \cong \partial W \times [0, \infty)$. It turns out ([128], Theorem 1.3) that necessary and sufficient conditions when $\dim M \geq 5$ (in any of the three categories TOP, PL, or DIFF) are that $(W, \partial W)$ is $(n-2)$-connected, W has one end, and π_1 is "essentially constant at ∞" with "$\pi_1(\infty)$"$\cong \pi_1(W)$. An alternative statement is that $W \cong \partial W \times [0, \infty)$ if and only if $(W, \partial W)$ is $(n-2)$-connected and W is proper homotopy equivalent to $\partial W \times [0, \infty)$. An elegant application ([128], Theorem 2.7) is a characterization of $\mathbb{R}^n$: if X^n is a noncompact oriented n-manifold, $n \geq 5$, then $X^n \cong \mathbb{R}^n$ (in any of the three categories TOP, PL, or DIFF) if and only if there exists a degree-1 proper map $\mathbb{R}^n \to X^n$.

3. Classification in a proper homotopy type. Surgery theory can be used to classify noncompact manifolds with a given *proper* homotopy type. For example, Siebenmann's Theorem 2.7 in [128] can be restated as saying that a non-compact n-manifold of dimension ≥ 5 is isomorphic to $\mathbb{R}^n$ if and only if it has the proper homotopy type of $\mathbb{R}^n$. Similarly much of the proof of the Farrell Fibration Theorem in section 3.7 above may be interpreted as a classification of manifolds with the proper homotopy type of $N \times \mathbb{R}$, for some compact manifold N.

4. There is a close connection between the classification of compact manifolds with fundamental group $\mathbb{Z}^n$ and the classification of noncompact manifolds with a proper map to $\mathbb{R}^n$, which played a vital role in Novikov's proof of the topological invariance of rational Pontrjagin classes (see [110]).

5. Finally (and probably most importantly), controlled surgery classifies noncompact manifolds in various "bounded" and "controlled" categories. See [111] and [112] for surveys and references.

4 Future directions

So where is surgery theory heading today? A glance at the dates on the papers in the bibliography to this article shows that history has proved wrong those who felt that surgery is a dead subject. At the risk of being another false prophet, I would predict that future development of the subject, at least over the next ten years, will lie mostly in the following areas:

- **Surgery in dimension 4.** Some very basic (and very hard!) questions remain concerning surgery in the topological category in dimension 4. (See [117].) In particular, is the surgery exact sequence valid without any restriction on fundamental groups? We can probably expect more work on this question, and also on the question of whether the smooth s-cobordism theorem is valid for 4-dimensional s-cobordisms (between 3-manifolds).

- **Differential geometry.** One of the areas of application of surgery theory that is developing most rapidly is that of applications to differential geometry. I would expect to see further growth in this area, especially in the areas of application to positive Ricci curvature (section 3.9.2 above) and to symplectic and contact geometry (section 3.9.5 above). In these areas what we basically have now are a lot of tantalizing examples, but very little in the way of definitive results, so there is lots of room for innovative new ideas.

- **Coarse geometry.** Still another area of very rapid current development is the study of "behavior at infinity" of noncompact manifolds. Especially fruitful ideas in this regard have been the "macroscopic" or "asymptotic" notions of Gromov [72] in geometry and geometric group theory and Roe's notion of "coarse geometry" [23]. But the Gromovian approach to geometry has not yet been fully integrated with surgery theory. The author expects a synthesis of these subjects to be a major theme in coming years. Ideas of what we might expect may be found in the work of Attie on classification of manifolds of bounded geometry [28] and in the work of Block and Weinberger [30].

- **Manifold-like spaces.** Last but not least, I think we can expect much more work on surgery theory applied to manifold-like spaces which are not manifolds (section 3.10 above). While outlines of basic surgery theories for stratified and singular spaces are now in place, major applications are only beginning to be developed. When it comes to homology manifolds, the situation is even more mysterious, due to the fact that all current arguments for "constructing" exotic ENR homology manifolds are basically non-constructive. It is also not clear if these spaces are homogeneous (like manifolds) or not. (See [108] for a discussion of some of the key unsolved problems.) So we can expect to see much further investigation of these topics.

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Reflections on C. T. C. Wall's work on cobordism

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1 Introduction

This book is intended to fulfill two main functions—to celebrate C. T. C. Wall's contributions to topology, and to discuss the state of surgery theory today. This article concentrates on Wall's first contribution to topology, still of great significance: the completion of the calculation of the cobordism ring Ω defined by René Thom. This subject, while not directly a part of surgery theory, is still vital to it. Surgery theory classifies manifolds (smooth, PL, or topological) within a given homotopy type, starting from the observation that one can construct new manifolds from old ones by means of surgery. It is obvious that if two manifolds are related by surgery, then they are cobordant, and the fundamental theorem of Morse theory, brought to the fore in this context by Milnor, shows that two (smooth) manifolds are cobordant *if and only if* they are related by a sequence of surgeries. It turns out that, in some sense, refinements of this idea are basic to surgery theory. The notions of normal cobordism of maps, algebraic cobordism of algebraic Poincaré complexes, and so on, play a major role. Also, from a historical point of view, Wall's interest in the surgery classification of manifolds grew out of his earlier work on cobordism. But there are also more subtle reasons for studying oriented bordism in connection with surgery theory. For example, the symmetric signature of Mishchenko-Ranicki may be viewed as a natural transformation from oriented bordism to symmetric L-theory ([Ran1], §7.1; [Ran2]), and Wall's work on the structure of the oriented bordism spectrum may be used to deduce facts about the structure of surgery spectra, such as the fact that they are Eilenberg-MacLane when localized at 2 [TayW].

To explain Wall's contribution we first need to say a few words about Thom's work.

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2 Thom's work on cobordism

Two closed (smooth) n-manifolds M_1^n and M_2^n are called *cobordant* if there is a compact manifold with boundary, say W^{n+1}, whose boundary is (diffeomorphic to) the disjoint union of M_1 and M_2. It is trivial to note that this is an equivalence relation, and that with the addition operation defined by the disjoint union of manifolds, the set $\mathfrak{N}_n$ of cobordism classes of closed n-manifolds is an abelian group. This group has exponent 2, since the boundary of $M^n \times [0, 1]$ is (diffeomorphic to) $M \amalg M$. Furthermore, $\mathfrak{N} = \bigoplus_{n=0}^{\infty} \mathfrak{N}_n$ is a commutative graded algebra over $\mathbb{Z}/2$, with multiplication defined by the Cartesian product of manifolds. Thom [Thom] had succeeded in computing this ring by reducing the calculation to a problem in homotopy theory. This is done using the famous *Thom-Pontrjagin construction*. For N sufficiently large, a closed n-manifold M^n may (by the easy part of the Whitney Embedding Theorem) be embedded smoothly in $\mathbb{R}^{n+N}$, and (provided N is large enough) any two such embeddings are isotopic to one another. Given such an embedding, choose a closed tubular neighborhood T of M in $\mathbb{R}^{n+N}$; this is diffeomorphic to the unit disk bundle $D(E)$ of the normal bundle E for the embedding. By the classification theorem for vector bundles, E is obtained by pulling back the universal rank-N vector bundle $E_N^u = EO(N) \times_{O(N)} \mathbb{R}^N$ over $BO(N)$, via a classifying map $f : M \to BO(N)$ whose homotopy class is independent of all choices made. Let MO_N be the space obtained from the unit disk bundle $D(E_N^u)$ of E_N^u by collapsing the sphere bundle $S(E_N^u)$ to a point; this is called the *Thom space* of the vector bundle E_N^u. Map $S^{n+N} = \mathbb{R}^{n+N} \cup \{\infty\}$ to $T/\partial T$ by collapsing the complement of the interior of T to a point, and then map to MO_N by the obvious map obtained from the lift of f to the bundle map $\tilde{f} : E \to E_N^u$. The result is a map $t_M : S^{n+N} \to MO_N$ whose homotopy class only depends on the cobordism class of M as a smooth manifold, since a cobordism W between M and M' gives rise (by a similar construction with W in place of M) to a homotopy between the maps t_M and $t_{M'}$. Thom showed in his fundamental paper [Thom] that in this way one obtains a natural isomorphism between $\mathfrak{N}_n$ and $\pi_n(\mathbf{MO}) = \lim_{N\to\infty} \pi_{n+N}(MO_N)$. The inverse of the map is obtained by transversality: given $t : S^{n+N} \to MO_N$, we make t transverse to the 0-section $\mathbf{0}$ of E_N^u (this requires a little technical fiddling since MO_N isn't quite a manifold), and we recover M^n up to cobordism as $t^{-1}(\mathbf{0})$. (For more details, an excellent reference is [Stong].)

It's convenient to rework Thom's work in more modern language. There are maps $\Sigma MO_N \to MO_{N+1}$ coming from the fact that the product with $\mathbb{R}$ of the universal bundle E_N^u over $BO(N)$ is pulled back from the universal bundle E_{N+1}^u over $BO(N+1)$. Thus the sequence of spaces $\{MO_N\}$ forms a *spectrum* $\mathbf{MO}$ in the sense of stable homotopy theory (the best reference on these for the beginner is [Ad2], Part III), and Thom's result shows that $\pi_*(\mathbf{MO}) \cong \mathfrak{N}_*$. In fact, $\mathbf{MO}$ is even a ring spectrum, with product $\mathbf{MO} \wedge \mathbf{MO} \to \mathbf{MO}$ coming from the Whitney sum of vector bundles, and $\pi_*(\mathbf{MO}) \cong \mathfrak{N}_*$ is a ring isomorphism. Finally, to restate Thom's work in the modern language of spectra, Thom was able to show that $\mathbf{MO}$ is a generalized Eilenberg-MacLane spectrum, and that $\mathfrak{N}$ is a polynomial ring over $\mathbb{Z}/2$, with one generator x_k in each degree $k \geq 1$ not of the form $2^j - 1$. (Thus the first few generators are in dimensions 2 and 4, corresponding to the manifolds $\mathbb{RP}^2$ and $\mathbb{RP}^4$.) The fact that the spectrum $\mathbf{MO}$ is of Eilenberg-MacLane type means that the mod-2 Hurewicz map $\pi_*(\mathbf{MO}) \to H_*(\mathbf{MO}; \mathbb{Z}/2)$ is an injection. This has the geometric interpretation that all information about (unoriented) cobordism comes from mod-2 homology, and thus that a manifold is determined up to cobordism by its Stiefel-Whitney characteristic numbers. Since these only depend on the homotopy type of the manifold, homotopy equivalent manifolds are necessarily cobordant.

Now Thom had observed in [Thom] that one can do something quite similar with *oriented* manifolds. Two *oriented* n-manifolds M_1^n and M_2^n are called *oriented cobordant* if there is a compact oriented manifold with boundary, say W^{n+1}, whose boundary is (diffeomorphic to) the disjoint union of M_1 and M_2, and if the orientation on W induces the given orientation on M_1 and the reverse of the given orientation on M_2. This is again an equivalence relation, since the standard orientation on $M \times [0, 1]$ induces the given orientation on $M \times \{0\}$ and the reverse of the given orientation on $M \times \{1\}$. So oriented cobordism classes of n-manifolds form an abelian group Ω_n under disjoint union, with the inverse of the class of M given by the class of $\overline{M}$, the same manifold as M, but with opposite orientation. One finds that $\Omega = \bigoplus_{n=0}^{\infty} \Omega_n$ is a graded commutative graded ring, the *oriented cobordism ring*, with multiplication defined by the Cartesian product of (oriented) manifolds. Thom's work (slightly rephrased) showed that the Thom-Pontrjagin construction works as before, to give an isomorphism $\Omega_* \cong \pi_*(\mathbf{MSO})$, where the ring spectrum $\mathbf{MSO}$ is defined the same way as $\mathbf{MO}$, but using the universal *oriented* bundle $E_N^{u,o}$ over $BSO(N)$ in place of the universal unoriented bundle E_N^u over $BO(N)$.

Rationally, Thom had no trouble in computing Ω, for $\Omega_* \otimes_{\mathbb{Z}} \mathbb{Q} \cong \pi_*(\mathbf{MSO}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and rational stable homotopy is the same thing as rational homology. Furthermore, by the *Thom isomorphism* for oriented vector bundles (also developed in [Thom]), $\tilde{H}_{k+N}(MSO_N; \mathbb{Q}) \cong H_k(BSO(N); \mathbb{Q})$. Since $H^*(BSO; \mathbb{Q})$ is the power series algebra in the rational Pontrjagin classes, it easily follows that *rationally*, oriented cobordism is determined by the Pontrjagin characteristic numbers. Thom deduced from this that $\Omega \otimes_{\mathbb{Z}} \mathbb{Q}$ is a polynomial algebra on the classes of the complex projective spaces $\mathbb{CP}^{2j}$, $j \geq 1$. The problem left open by Thom and solved by Wall was the *integral* calculation of Ω.

3 Wall's work on cobordism

The first major step in the integral calculation of Ω was taken by Milnor, who proved:

Theorem 3.1 [Mil] *Ω has no torsion of odd order.*

But the hardest part of the calculation, involving the 2-primary torsion, still remained. Some work on this had been done by Rokhlin, but not all of it was correct. Wall's original method for completing the calculation of Ω was both ingenious and geometric. He introduced a new cobordism theory $\mathfrak{W} = \bigoplus_{n=0}^{\infty} \mathfrak{W}_n$, where $\mathfrak{W}_n$ consists of cobordism classes of pairs (M^n, f), where M^n is a closed (unoriented) n-manifold, $f : M \to S^1$, and the pull-back under f of the canonical generator of $H^1(S^1; \mathbb{Z})$ reduces mod 2 to $w_1(M)$. Two such pairs (M_1^n, f_1) and (M_2^n, f_2) are equivalent if there is a compact manifold with boundary W^{n+1} and a map $f : W \to S^1$ such that $\partial W = M_1 \amalg M_2$, f restricts on M_j to f_j, and the pull-back under f of the canonical generator of $H^1(S^1; \mathbb{Z})$ reduces mod 2 to $w_1(W)$. (Example: suppose $M = S^1$ and $f : M \to S^1$ is a map of degree 2. Then the pair (M, f) defines an element of $\mathfrak{W}$, since the degree of f vanishes mod 2. This element of $\mathfrak{W}$ is trivial, since M bounds a Möbius strip W, and f extends to a map $\tilde{f} : W \to S^1$, inducing an isomorphism on π_1 and pulling back the generator on $H^1(S^1; \mathbb{Z}/2)$ to $w_1(W)$.) Note that if M^n is any closed n-manifold, one can find an $f : M \to S^1$ for which (M^n, f) represents a class in $\mathfrak{W}_n$ if and only if $w_1(M)$ is the reduction of an integral class, or if and only if $\beta w_1 = 0$, where $\beta : H^1(M; \mathbb{Z}/2) \to H^2(M; \mathbb{Z})$ is the Bockstein operator. Since the reduction mod 2 of the Bockstein coincides with the Steenrod operation Sq^1, this condition implies $\mathrm{Sq}^1 w_1 = 0$, or $w_1^2 = 0$. If we drop the w_1 condition, there is a forgetful map $\mathfrak{W}_n \to \mathfrak{N}_n(S^1)$ to the cobordism classes of pairs (M^n, f), where M^n is a closed (unoriented) n-manifold, and $f : M \to S^1$. (The equivalence relation on these pairs is as before, but without the w_1 condition.) One can easily check (cf. [Atiyah]; this is also related to results in [Wall4]) that $X \mapsto \mathfrak{N}_n(X)$ defines a homology theory, which can be computed as $\mathfrak{N}_n(X) \cong \pi_*(X_+ \wedge \mathbf{MO})$. (The subscript $_+$ denotes the addition of a disjoint basepoint, which comes from the fact that we are using an unreduced homology theory here.) Thus $\mathfrak{N}_n(S^1) \cong \mathfrak{N}_n \oplus \mathfrak{N}_{n-1}$, and the class of a pair (M^n, f) in $\mathfrak{N}_n(S^1)$ is determined by characteristic numbers of M involving the Stiefel-Whitney classes and the pull-back $f^*(x)$ under f of the generator x of $H^1(S^1; \mathbb{Z}/2)$. When the pair (M^n, f) comes from $\mathfrak{W}_n$, then $f^*(x) = w_1$, so all these characteristic numbers are thus ordinary Stiefel-Whitney numbers. It is not hard to see from this that the forgetful map $\mathfrak{W}_n \to \mathfrak{N}_n$ is injective, and identifies $\mathfrak{W}_n$ with the classes in $\mathfrak{N}_n$ of manifolds for which all Stiefel-Whitney numbers involving w_1^2 vanish.

Now we can explain Wall's approach [Wall1, Wall2] to the calculation of the cobordism ring Ω. (However we will make use of some simplifications subsequently found in the proofs. Still another variant on the argument can be found in [Ad1].) The idea is to set up an exact triangle

$$\begin{tikzcd} \Omega \arrow[rr, "2"] & & \Omega \arrow[dl, "s"] \\ & \mathfrak{W}, \arrow[ul, "\partial"] & \end{tikzcd} \tag{1}$$

where 2 denotes multiplication by 2 on the abelian group Ω, s is the forgetful map from Ω to $\mathfrak{W}$ (an oriented manifold has $w_1 = 0$, hence defines a class in $\mathfrak{W}$), and ∂ is the boundary map (decreasing degree by 1) obtained by "dualizing w_1." (In other words, given M^n with $w_1(M)$ the reduction of an integral cohomology class, let $N^{n-1} \subset M^n$ represent the mod-2 homology class which is the Poincaré dual (mod 2) to $w_1(M)$. This manifold inherits an orientation from the implicit map $f : M \to S^1$. Furthermore, N as an oriented manifold is well-defined up to cobordism.) Wall's original proof of the exactness of the triangle was somewhat complicated, but he gave a simpler and more geometric proof in [Wall5]. The first part is the exactness of $\Omega \xrightarrow{s} \mathfrak{W} \xrightarrow{\partial} \Omega$. We have $\partial \circ s = 0$, since if M is an oriented manifold, then dualizing w_1 gives the empty submanifold. In the other direction, if M^n gives a class in $\mathfrak{W}$ and $\partial[M^n] = 0$, that means that the Poincaré dual of $w_1(M)$ is null-cobordant, so that all the Stiefel-Whitney numbers of M involving w_1 vanish. From Thom's description of generators of $\mathfrak{N}$, this implies that M is cobordant to something orientable. A slightly harder part of the proof is the exactness of $\Omega \xrightarrow{2} \Omega \xrightarrow{s} \mathfrak{W}$. That the composite of the two arrows is trivial is clear, since $\mathfrak{W} \subset \mathfrak{N}$, which as an abelian group has exponent 2. We need to show that if $[M] \in \Omega_n$ maps to 0 in $\mathfrak{W}_n$, then $[M]$ is divisible by 2. The condition that $[M] \in \Omega_n$ maps to 0 in $\mathfrak{W}_n$ means that M is the boundary of an unoriented manifold W^{n+1}, and that there is a map $f : W \to S^1$ whose restriction to M is null-homotopic and such that $f^*(x) = w_1(W)$. We may assume that f sends M to a point in S^1, say 1. Then the inverse image under f of a regular value $\neq 1$ of f is a compact submanifold A^n which doesn't meet the boundary of W and thus is closed. Furthermore, A is "dual" to $w_1(W)$ and has trivial normal bundle. Removing an open tubular neighborhood of A from W, we obtain a compact oriented manifold V with boundary $\partial V \cong A \amalg A \amalg M$, so $[M]$ is divisible by 2 in Ω_n. Finally, we check exactness of $\mathfrak{W} \xrightarrow{\partial} \Omega \xrightarrow{2} \Omega$. That $2 \circ \partial = 0$ is obvious since $\mathfrak{W}$ is a group of exponent 2. On the other hand, if M^n is closed and oriented and $2[M] = 0$ in Ω_n, choose W^{n+1} oriented with boundary $\partial W \cong M \amalg M$, then glue the two copies of M together to get a closed (non-oriented) manifold V^{n+1} in which M is dual to w_1. One can easily show that in this case $w_1(V)$ comes from a map to S^1, so that V represents an element of $\mathfrak{W}_{n+1}$ with $\partial[V]=[M]$.

From the exact triangle (1), Wall deduced his main result on the cobordism ring:

Theorem 3.2 (Wall [Wall2]) *All torsion in Ω is of order 2. The oriented cobordism class of an oriented closed manifold is determined by its Stiefel-Whitney and Pontrjagin numbers.*

This is not just a simple corollary of the exact triangle (1), and Wall needed some additional ingredients. One of these was a Milnor's result quoted above. Another was a more detailed analysis of $\mathfrak{N}$ and $\mathfrak{W}$, based in part on [Dold]. Wall showed that $\mathfrak{W}$ is a subalgebra of $\mathfrak{N}$ (this is obvious from the description above involving vanishing of Stiefel-Whitney numbers involving w_1^2), and that one can choose the polynomial generators x_k ($k\neq 2^j-1$) for $\mathfrak{N}$ so that x_k is represented by an oriented manifold for k odd and $\mathfrak{W}$ is the subalgebra generated by x_k, k not of the forms 2^j or 2^j-1, and by the $x_{2^j}^2$. (This is done using variants on manifolds constructed by Dold [Dold]. The class $x_{2^j}^2$ is represented by a complex projective space.) Finally, Wall showed that with these choices of generators, the map $s\circ\partial:\mathfrak{W}\to\mathfrak{W}$ is the derivation sending $x_{2j}\mapsto x_{2j-1}$ (j not a power of 2) and killing the classes x_{2j-1} and $x_{2^j}^2$. The main step in the proof of Theorem 3.2 is therefore:

Lemma 3.3 (Wall [Wall2]) *Ω has no torsion of order 2^t, $t>1$.*

Proof. Suppose $c\in\Omega_n$ and $2^tc=0$, $2^{t-1}c\neq 0$, $t>1$. Then $2\cdot 2^{t-1}c=0$, so by (1), $2^{t-1}c=\partial y$, for some $y\in\mathfrak{W}_{n+1}$. Also, since $t>1$, $2^{t-1}c=2\cdot 2^{t-2}c$ and hence, again by (1), $s\left(2^{t-1}c\right)=0$. So $s\circ\partial y=0$. From the description of $s\circ\partial$ above, this means y is a sum of monomials in the x_k's in which no x_{2j} (j not a power of 2) occurs to odd order. But each x_k^2 is represented by an oriented manifold (that's because one has alternative generators for $\mathfrak{N}$ in even degrees given by real projective spaces, whose squares are cobordant to complex projective spaces ([Wall2], Proposition 3)), hence $y\in\operatorname{im}\Omega$ and thus $\partial y=0$ by (1), i.e., $2^{t-1}c=0$. This is a contradiction. □

Note as well that once one has proved this lemma, it follows that the forgetful map $\Omega\to\mathfrak{N}$ is injective on the torsion in Ω, and thus that the torsion classes in Ω are detected by Stiefel-Whitney numbers. Indeed, since all torsion in Ω is of order 2, the torsion injects into $\Omega/2\Omega$, which then injects into $\mathfrak{W}\subset\mathfrak{N}$ by (1). Wall went further than this and computed the multiplicative structure of Ω; this depends on the fact that in the triangle (1), $\mathfrak{W}$ is an algebra, s is a ring homomorphism, and $s\circ\partial$ is a derivation. (One may find details in Chapter IX of [Stong].)

Aside from the above proof, there is another way of approaching Theorem 3.2, which to the modern algebraic topologist might seem more natural. Namely, one can work entirely homotopy-theoretically, and bypass direct geometrical considerations altogether. The problem is one of computing $\pi_*(\mathbf{MSO})$, so one can localize at a prime p and apply the Adams spectral sequence ([Ad2], Part III, and [Swit]) to compute $\pi_*(\mathbf{MSO})$ from $H_*(\mathbf{MSO};\, \mathbb{Z}/p)$, viewed as a comodule over the dual mod-p Steenrod algebra $\mathcal{A}^p_*$. The Hurewicz map $\pi_*(\mathbf{MSO}) \to H_*(\mathbf{MSO};\, \mathbb{Z}/p)$ appears as an edge homomorphism of this spectral sequence. Indeed, except for the fact that he replaced homology by cohomology here, this was the approach used in [Mil] in proving Theorem 3.1. Wall's papers [Wall3] and [Wall6] on the Steenrod algebra may be viewed as being motivated by this approach. In fact one can show that the Steenrod comodule structure of $H_*(\mathbf{MSO};\, \mathbb{Z}/p)$ is relatively simple. For p odd, $\mathcal{A}^p_*$ is the tensor product of an exterior algebra and a polynomial algebra, and the polynomial part is a Hopf subalgebra. It turns out ([Swit], Lemma 20.38) that $H_*(\mathbf{MSO};\, \mathbb{Z}/p)$ is a free comodule on the polynomial part of $\mathcal{A}^p_*$, and thus the Adams spectral sequence is fairly easy to analyze. For $p = 2$, $\mathcal{A}^2_* \cong (\mathbb{Z}/2)[\xi_1, \xi_2, \ldots]$ with ξ_i of degree $2^i - 1$ and with coproduct

$$\Delta(\xi_k) = \sum_{i+j=k} \xi_j^{2^i} \otimes \xi_i,$$

and $H_*(\mathbf{MSO};\, \mathbb{Z}/2)$ turns out to be the direct sum of a free $\mathcal{A}^2_*$-comodule and a free comodule over $\left(\mathcal{A}^*/\mathcal{A}^*\mathrm{Sq}^1\right)^*$; this fact is basically equivalent to one of the main results in [Wall5], for which Wall gave a simpler proof in [Wall6]. Another proof is given in [Peng2], and the algebra structure instead of the comodule structure is computed in [Pap1] and [Peng1]. Once again, the Adams spectral sequence is computable (see [Swit], pages 510–516 for details). One can obtain in this way another proof of Wall's Theorem 3.2, with the drawback that the geometry of (1) is submerged. The compensation is that the Hurewicz map $\pi_*(\mathbf{MSO}) \to H_*(\mathbf{MSO};\, \mathbb{Z}/p)$ can be read off directly, giving a clearer explanation of the second part of Wall's Theorem (the fact that classes in Ω are determined by Pontrjagin and Stiefel-Whitney numbers). A few more variants of Wall's proof have been given, most notably:

1. a very quick and slick proof by Taylor [Tayl] of Theorem 3.2, which uses the Atiyah-Hirzebruch spectral sequence to study the Hurewicz maps for **MSO** with $\mathbb{Z}/2$ and $\mathbb{Z}/4$ coefficients;

2. a proof of Gray [Gray], which uses a purely algebraic description of $\mathfrak{W}$, together with multiplicative structure of the Atiyah-Hirzebruch spectral sequence; and

3. a proof of Papastavridis [Pap2], a variant on the Adams spectral sequence approach.

4 Cobordism theory since Wall's work

Wall's paper [Wall2] on cobordism contains the words:

> In fact, all properties of Ω are as simple as they could possibly be (if my results should strike the reader as complicated, let him try and work out cobordism theory for the spinor group).

The implicit challenge here was to compute Ω^{spin}, the cobordism ring of oriented manifolds equipped with spin structures (or equivalently, lifts of the stable normal bundle $M \to BSO$ to a map $M \to BSpin$, where $Spin(n)$ is the double cover of $SO(n)$). This challenge was eventually met by Anderson, Brown, and Peterson [AnBrP], and indeed the answer is quite complicated. However, there are a few similarities to Wall's work on Ω, which certainly served as an important model. Again, all torsion is of order 2, and again, the $\mathbb{Z}/2$-homology of the relevant Thom spectrum **MSpin** is a direct sum of relatively simple $\mathcal{A}^2_*$-comodules of only a few types. The difference between **MSpin** and **MSO** at the prime 2 turns out to involve, as one would expect, primarily Sq^2. More precisely, $\mathcal{A}(1)_* = (\mathbb{Z}/2)[\xi_1, \xi_2, \ldots]/(\xi_1^4, \xi_2^2, \ldots)$ is dual to the subalgebra of $\mathcal{A}_2^*$ generated by Sq^1 and Sq^2, and $H_*(\mathbf{MSpin}; \mathbb{Z}/2)$ turns out to be a direct sum of $\mathcal{A}^2_*$-comodules of the form $\mathcal{A}_* \square_{\mathcal{A}(1)_*} M$, with M one of three $\mathcal{A}(1)_*$-comodules, just as if $\mathcal{A}(0)_*$ is dual to the subalgebra of $\mathcal{A}_2^*$ generated by Sq^1, then $H_*(\mathbf{MSO}; \mathbb{Z}/2)$ turns out to be a direct sum of $\mathcal{A}^2_*$-comodules of the form $\mathcal{A}_* \square_{\mathcal{A}(0)_*} M$, with M one of two $\mathcal{A}(0)_*$-comodules. In fact, the [AnBrP] proof of this relies quite heavily on the technique developed by Wall in [Wall6]. This technology can be carried quite far (see for example [Stolz]), and has applications which will be discussed elsewhere in these volumes, especially in the chapters [RS] on positive scalar curvature and [Tms] on elliptic cohomology.

While it would take us quite far afield to discuss everything that's happened in cobordism theory since Wall's fundamental papers, much of which can be found in the book [Stong] or the survey [Land1], we should mention what are perhaps the most important trends. They stem from remarkable properties of the spectrum **MU** for complex cobordism Ω^U (the cobordism theory for manifolds with an almost complex structure on the stable normal bundle), which are described in detail in the books [Ad2] and [Rav]. It turns out not only that Ω^U is a polynomial ring over $\mathbb{Z}$ [Mil], but also that this ring carries a natural "formal group" structure coming from the Conner-Floyd Chern classes. Quillen [Quil] proved that the formal group law on Ω^U is "universal," i.e., that it can be mapped to every other (one-dimensional commutative) formal group law. From Quillen's theorem one can deduce (see for instance [Rav], Chapter 4, or [Ad2], Part II, §§15–16) that when localized at a prime p, **MU** splits as a sum of suspensions of a very special spectrum **BP**, which plays a very important role in modern stable homotopy theory. For example, one obtains from it the "Morava K-theory" spectra $\mathbf{K}(n)$, which are periodic of periods $2(p^n-1)$. When $p=2$ and $n=1$, $\mathbf{K}(n)$ is just ordinary complex K-theory with $\mathbb{Z}/2$ coefficients, but the other cases have no such simple geometric interpretations. This fact is related (see [Rav], p. 135) to the Conner-Floyd Theorem [CoF] that complex K-theory may be simply constructed out of complex cobordism as

$$K_*(X) \cong \Omega^U_*(X) \otimes_{\Omega^U_*} K_*(\mathrm{pt}). \tag{2}$$

There is no direct link between all this and Wall's work on cobordism, though it does turn out that **MSO** when localized at an *odd* prime splits as a sum of suspensions of **BP**, even though **MSO** localized at 2 looks quite different and is a direct sum of (shifted) Eilenberg-MacLane spectra for the groups $\mathbb{Z}_{(2)}$ and $\mathbb{Z}/2$. As pointed out in [TayW], this is immediately related to the structure of the L-spectra which are discussed elsewhere in this volume, which are **KO**-module spectra at odd primes and Eilenberg-MacLane at 2. The reason is that $\mathbf{L}^\bullet(\mathbb{Z})$ is an **MSO**-module spectrum, and all other L-spectra are module spectra over $\mathbf{L}^\bullet(\mathbb{Z})$ and hence also over **MSO**.

A related development is the recent interest in "elliptic homology" theories, and their connection with cobordism. To explain something about these one first needs the idea of a "genus," which in its most general formulation is simply a ring homomorphism from some cobordism ring to some standard commutative ring Λ (such as $\mathbb{Q}$). The important classical examples are the mod 2 Euler characteristic $\mathfrak{N} \to \mathbb{Z}/2$, the signature $\Omega \to \mathbb{Z}$ for oriented manifolds, the Todd genus $\Omega^U \to \mathbb{Z}$ for complex manifolds, and the $\hat{A}$-genus $\Omega^{\mathrm{spin}} \to \mathbb{Z}$. Under favorable circumstances, a genus is really the induced map on homotopy groups of some map of ring spectra. For example, the K-orientation of complex manifolds gives a map of ring spectra $\mathbf{MU} \to \mathbf{K}$, which when composed with the Chern character to ordinary homology, gives rise on taking homotopy groups to the Todd genus, and the KO-orientation of spin manifolds gives a map of ring spectra $\mathbf{MSpin} \to \mathbf{KO}$ which, when composed with the Pontrjagin character to ordinary homology, gives rise on taking homotopy groups to the $\hat{A}$-genus.

Now if Λ is a $\mathbb{Q}$-algebra and $\varphi : \Omega \to \Lambda$ is a genus, Hirzebruch associated to φ the formal power series

$$g(x) = \sum_{n=0}^{\infty} \tfrac{1}{2n+1} \varphi(\mathbb{CP}^{2n}) x^{2n+1}.$$

It was discovered by Ochanine [Och1] that the genus φ vanishes on all manifolds of the form $\mathbb{CP}(\xi)$ with ξ an even-dimensional complex vector bundle over a closed oriented manifold if and only if $g(x)$ is an elliptic integral

$$g(x) = \int_0^x \frac{1}{\sqrt{1 - 2\delta z^2 + \varepsilon z^4}}\, dz. \tag{3}$$

In this case φ is called an *elliptic genus*. (For expositions of all this, see the papers in [Land2], especially Landweber's survey on pages 55–68, or the book [HiBeJ], as well as [Tms].) This suggested the search for multiplicative homology theories, related to cobordism spectra, whose coefficient rings are rings of automorphic forms (of which elliptic integrals are a special case). The first result in this direction was the discovery by Landweber, Ravenel, and Stong [LaRS] of a multiplicative homology theory at odd primes, having coefficient ring $M = \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$. They showed the elliptic genus associated to (3) gives a ring homomorphism $\Omega \to M$, and that while $X \mapsto \Omega_*(X) \otimes_\Omega M$ is not a homology theory, it becomes a homology theory after inverting ε, $\delta^2 - \varepsilon$, or the discriminant $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$. (The proof relies on the fact that at odd primes, **MSO** closely resembles **MU**, together with universality of the formal group law on **MU**.) For another proof and related results, see also [Fr], [Bryl], [Bak], and [Tms]. More dramatic are recent elegant results of [KrSt] and [Hov], which construct *integral* homology theories El_* out of spin cobordism, which agree with various versions of the Landweber, Ravenel, and Stong theory at odd primes. For the work of Kreck and Stolz, an understanding of the very delicate structure of $H_*(\mathbf{MSpin}; \mathbb{Z}/2)$ as a comodule over $\mathcal{A}^2_*$ is the key. This structure gives rise to theorems of Conner-Floyd type analogous to (2):

$$\begin{aligned} El_*(X) &\cong \Omega^{\mathrm{spin}}_*(X) \otimes_{\Omega^{\mathrm{spin}}_*} El_*(\mathrm{pt}), \\ KO_*(X) &\cong El_*(X) \otimes_{El_*(\mathrm{pt})} KO_*(\mathrm{pt}) \end{aligned}$$

([HoHo], [Hov]). As we've seen, the roots of our understanding of this may be traced back to Wall's work on cobordism.

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A survey of Wall's finiteness obstruction

Steve Ferry and Andrew Ranicki

Contents

Introduction

Wall's finiteness obstruction is an algebraic K-theory invariant which decides if a finitely dominated space is homotopy equivalent to a finite CW complex. The invariant was originally formulated in the context of surgery on CW complexes, generalizing Swan's application of algebraic K-theory to the study of free actions of finite groups on spheres. In the context of surgery on manifolds, the invariant first arose as the Siebenmann obstruction to closing a tame end of a non-compact manifold. The object of this survey is to describe the Wall finiteness obstruction and some of its many applications to the surgery classification of manifolds. The book of Varadarajan [38] and the survey of Mislin [24] deal with the finiteness obstruction from a more homotopy theoretic point of view.

1. Finite domination

A space is finitely dominated if it is a homotopy retract of a finite complex. More formally:

Definition 1.1. A topological space X is *finitely dominated* if there exists a finite CW complex K with maps $d : K \to X$, $s : X \to K$ and a homotopy

$$d \circ s \simeq \mathrm{id}_X : X \to X \ . \qquad \square$$

Example 1.2. (i) A compact ANR X is finitely dominated (Borsuk [3]). In fact, a finite dimensional ANR X can be embedded in $\mathbb{R}^N$ (N large), and X is a retract of an open neighbourhood $U \subset \mathbb{R}^N$ – there exist a retraction $r : U \to X$ and a compact polyhedron $K \subset U$ such that $X \subset K$, so that the restriction $d = r| : K \to X$ and the inclusion $s : X \to K$ are such that $d \circ s = \mathrm{id}_X : X \to X$.
(ii) A compact topological manifold is a compact ANR, and hence finitely dominated. □

The problem of deciding if a compact ANR is homotopy equivalent to a finite CW complex was first formulated by Borsuk [4]. (The problem was solved affirmatively for manifolds by Kirby and Siebenmann in 1969, and in general by West in 1974 – see section 8 below.) The problem of deciding if a finitely dominated space is homotopy equivalent to a finite CW complex was first formulated by J.H.C.Whitehead. Milnor [23] remarked: "It would be interesting to ask if every space which is dominated by a finite complex actually has the homotopy type of a finite complex. This is true in the simply connected case, but seems like a difficult problem in general."

Here is a useful criterion for recognizing finite domination:

Proposition 1.3. *A CW complex X is finitely dominated if and only if there is a homotopy $h_t : X \to X$ such that $h_0 = id$ and $h_1(X)$ has compact closure.*

Proof. If $d : K \to X$ is a finite domination with right inverse s, let h_t be a homotopy from the identity to $d \circ s$. Since $h_1(X) \subset d(K)$, the closure of $h_1(X)$ is compact in X. Conversely, if the closure of $h_1(X)$ is compact in X, let K be a finite subcomplex of X containing $h_1(X)$. Setting d equal to the inclusion $K \to X$ and s equal to $h_1 : X \to K$ shows that X is finitely dominated. □

It is possible to relate finitely dominated spaces, finitely dominated CW complexes and spaces of the homotopy type of CW complexes, as follows.

Proposition 1.4. (i) *A finitely dominated topological space X is homotopy equivalent to a countable CW complex.*
(ii) *If X is homotopy dominated by a finite k-dimensional CW complex, then X is homotopy equivalent to a countable $(k + 1)$-dimensional CW complex.*

Proof. The key result is the trick of Mather [22], which shows that if $d : K \to X$, $s : X \to K$ are maps such that $d \circ s \simeq \mathrm{id}_X : X \to X$ then X is homotopy equivalent to the mapping telescope of $s \circ d : K \to K$. This requires the calculus of mapping cylinders, which we now recall.
By definition, the mapping cylinder of a map $f : K \to L$ is the identification space

$$M(f) \ = \ (K \times [0,1] \cup L)/((x,1) \sim f(x)) \ .$$

We shall use three general facts about mapping cylinders:

- If $f : K \to L$ and $g : L \to M$ are maps and $k : K \to M$ is homotopic to $g \circ f$, the mapping cylinder $M(k)$ is homotopy equivalent rel $K \cup M$ to the concatenation of the mapping cylinders $M(f)$ and $M(g)$ rel $K \cup M$.
- If $f, g : K \to L$ with $f \sim g$, then the mapping cylinder of f is homotopy equivalent to the mapping cylinder of g rel $K \cup L$.
- Every mapping cylinder is homotopy equivalent to its base rel the base.

The mapping telescope of a map $\alpha : K \to K$ is the countable union

$$\bigcup_{i=0}^{\infty} M(\alpha) \;=\; \bigcup_{i=0}^{\infty} K \times [i, i+1]/\{(x, i) \sim (\alpha(x), i+1)\} \;.$$

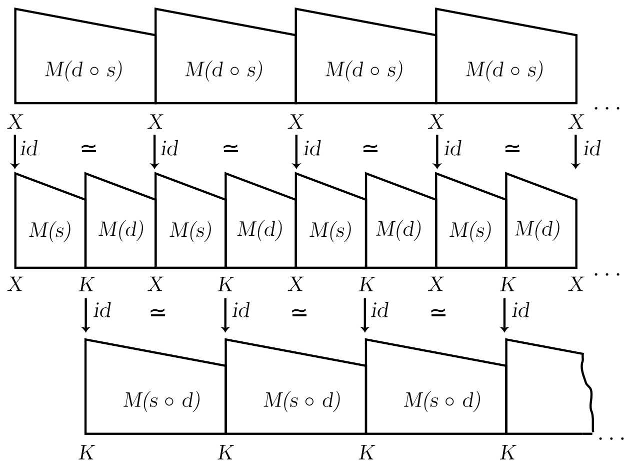
For any maps $d : K \to X$, $s : X \to K$ we have

$$\bigcup_{i=0}^{\infty} M(d \circ s) \;=\; X \times I \cup \bigcup_{i=0}^{\infty} M(s \circ d)$$

with $\bigcup_{i=0}^{\infty} M(s \circ d)$ a deformation retract, so that

$$\bigcup_{i=0}^{\infty} M(d \circ s) \;\simeq\; \bigcup_{i=0}^{\infty} M(s \circ d) \;.$$

To see why this holds, note that $\bigcup_{i=0}^{\infty} M(d \circ s)$ is homotopy equivalent to an infinite concatenation of alternating $M(d)$'s and $M(s)$'s which can also be thought of as an infinite concatenation of $M(s)$'s and $M(d)$'s. Essentially, we're reassociating an infinite product. Here is a picture of this part of the construction.



(i) If $d : K \to X$, $s : X \to K$ are such that $d \circ s \simeq \mathrm{id}_X : X \to X$ there is defined a homotopy idempotent of a finite CW complex

$$\alpha \;=\; s \circ d : K \to K \;,$$

with $\alpha \circ \alpha \simeq \alpha : K \to K$. We have homotopy equivalences

$$X \simeq X \times [0,\infty) \simeq \bigcup_{i=0}^{\infty} M(\mathrm{id}_X) \simeq \bigcup_{i=0}^{\infty} M(d \circ s) \simeq \bigcup_{i=0}^{\infty} M(s \circ d) = \bigcup_{i=0}^{\infty} M(\alpha).$$

The mapping telescope $\bigcup_{i=0}^{\infty} M(\alpha)$ is a countable CW complex.
(ii) As for (i), but with K k-dimensional. □

This proposition is comforting because it shows that the finiteness problem for arbitrary topological spaces reduces to the finiteness problem for CW complexes. One useful consequence of this is that we can use the usual machinery of algebraic topology, including the Hurewicz and Whitehead theorems, to detect homotopy equivalences.

Proposition 1.5. (Mather [22]) *A topological space X is finitely dominated if and only if $X \times S^1$ is homotopy equivalent to a finite CW complex.*

Proof. The mapping torus of a map $\alpha : K \to K$ is defined (as usual) by

$$T(\alpha) \;=\; (K \times [0,1])/\{(x,0) \sim (\alpha(x),1)\} \;.$$

For any maps $d : K \to X$, $s : X \to K$ there is defined a homotopy equivalence

$$T(d \circ s : X \to X) \to T(s \circ d : K \to K) \;;\; (x,t) \mapsto (s(x),t) \;.$$

If $d \circ s \simeq \mathrm{id}_X : X \to X$ and K is a finite CW complex we thus have homotopy equivalences

$$X \times S^1 \simeq T(\mathrm{id}_X) \simeq T(s \circ d)$$

with $T(s \circ d)$ a finite CW complex.
Conversely, if $X \times S^1$ is homotopy equivalent to a finite CW complex K then the maps

$$d : K \simeq X \times S^1 \xrightarrow{proj.} X \;,\; s : X \xrightarrow{incl.} X \times S^1 \simeq K$$

are such that $d \circ s \simeq \mathrm{id}_X$, and X is dominated by K. □

2. The projective class group K_0

Let Λ be a ring (associative, with 1).

Definition 2.1. A Λ-module P is *f. g. projective* if it is a direct summand of a f. g. (= finitely generated) free Λ-module Λ^n, with $P \oplus Q = \Lambda^n$ for some direct complement Q. □

A Λ-module P is f. g. projective if and only if P is isomorphic to $\mathrm{im}(p)$ for some projection $p = p^2 : \Lambda^n \to \Lambda^n$.

Definition 2.2. (i) The *projective class group* $K_0(\Lambda)$ is the Grothendieck group of stable isomorphism classes of f. g. projective Λ-modules.
(ii) The *reduced projective class group* $\widetilde{K}_0(\Lambda)$ is the quotient of $K_0(\Lambda)$ by the subgroup generated by formal differences $[\Lambda^m]-[\Lambda^n]$ of f. g. free modules. □

Thus an element of $\widetilde{K}_0(\Lambda)$ is an equivalence class $[P]$ of f. g. projective Λ-modules, with $[P_1]=[P_2]$ if and only if there are f. g. free Λ-modules F_1 and F_2 so that $P_1\oplus F_1$ is isomorphic to $P_2\oplus F_2$. In particular, $[P]$ is trivial if and only if P is *stably free*, that is, if there is a f. g. free module F so that $P\oplus F$ is free.

Chapter 1 of Rosenberg [35] is a general introduction to the projective class groups $K_0(\Lambda)$, $\widetilde{K}_0(\Lambda)$ and their applications, including the Wall finiteness obstruction.

Example 2.3. There are many groups π for which

$$\widetilde{K}_0(\mathbb{Z}[\pi]) \ = \ 0 \ ,$$

including virtually polycyclic groups, a class which includes free and free abelian groups. □

At present, no example is known of a torsion-free infinite group π with $\widetilde{K}_0(\mathbb{Z}[\pi])\neq 0$. Indeed, Hsiang has conjectured that $\widetilde{K}_0(\mathbb{Z}[\pi])=0$ for any torsion-free group π. (See Farrell and Jones [11], pp. 9–11). On the other hand:

Example 2.4. (i) There are many finite groups π for which

$$\widetilde{K}_0(\mathbb{Z}[\pi]) \ \neq \ 0 \ ,$$

including the cyclic group $\mathbb{Z}_{23}$.
(ii) The reduced projective class group of the quaternion group

$$Q(8) \ = \ \{\pm 1, \pm i, \pm j, \pm k\}$$

is the cyclic group with 2 elements

$$\widetilde{K}_0(\mathbb{Z}[Q(8)]) \ = \ \mathbb{Z}_2 \ ,$$

generated by the f. g. projective $\mathbb{Z}[Q(8)]$-module

$$P \ = \ \mathrm{im}\left(\begin{pmatrix} 1-8N & 21N \\ -3N & 8N \end{pmatrix} \ : \ \mathbb{Z}[Q(8)]\oplus\mathbb{Z}[Q(8)] \to \mathbb{Z}[Q(8)]\oplus\mathbb{Z}[Q(8)]\right)$$

with $N=\sum\limits_{g\in Q(8)} g$. □

We refer to Oliver [25] for a survey of the computations of $\widetilde{K}_0(\mathbb{Z}[\pi])$ for finite groups π.

3. The finiteness obstruction

Here is the statement of Wall's theorem.

Theorem 3.1. ([39],[40]) (i) *A finitely dominated space X has a finiteness obstruction*

$$[X] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$$

such that $[X] = 0$ if and only if X is homotopy equivalent to a finite CW complex.
(ii) *If π is a finitely presented group then every element $\sigma \in \widetilde{K}_0(\mathbb{Z}[\pi])$ is the finiteness obstruction of a finitely dominated CW complex X with $[X] = \sigma$, $\pi_1(X) = \pi$.*
(iii) *A CW complex X is finitely dominated if and only if $\pi_1(X)$ is finitely presented and the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex $C_*(\widetilde{X})$ of the universal cover $\widetilde{X}$ is chain homotopy equivalent to a finite chain complex $\mathcal{P}$ of f. g. projective $\mathbb{Z}[\pi_1(X)]$-modules.*

Outline of proof (i) Here is an extremely condensed sketch of Wall's argument from [39]. If $d : K \to X$ is a finite domination with X a CW complex, we can assume that d is an inclusion by replacing X, if necessary, by the mapping cylinder of d. For each $\ell \geq 2$, we then have a split short exact sequence of abelian groups

$$0 \to \pi_{\ell+1}(X, K) \to \pi_\ell(K) \to \pi_\ell(X) \to 0 \ .$$

Wall gives a special argument to show that d can be taken to induce an isomorphism on π_1 and then shows that $\pi_{\ell+1}(X, K)$ is f. g. as a module over $\mathbb{Z}[\pi_1(X)]$, provided that $\pi_q(X, K) = 0$ for $q \leq \ell$, $\ell \geq 2$. This allows him to attach $\ell+1$-cells to form a complex $\overline{K} \supset K$ and a map $\overline{d} : \overline{K} \to X$ extending d so that $\overline{d}$ induces isomorphisms on homotopy groups through dimension ℓ. Since $\overline{d}$ is a domination with the same right inverse s, this process can be repeated. In the case $\ell \geq \dim(K)$, Wall shows that $\pi_{\ell+1}(X, K)$ is a f. g. *projective* module over $\mathbb{Z}[\pi_1(X)]$. If $\pi_{\ell+1}(X, K)$ is free (or even stably free) we can attach $\ell+1$-cells to kill $\pi_{\ell+1}(X, K)$ without creating new problems in higher dimensions. The result is that $\overline{d}$ is a homotopy equivalence from $\overline{K}$ to X. If this module is not stably free, we are stuck and the finiteness obstruction is defined to be

$$[X] \ = \ (-1)^{\ell+1}[\pi_{\ell+1}(X, K)] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) \ .$$

(ii) Given a finite CW complex K and a nontrivial $\sigma \in \widetilde{K}_0(\mathbb{Z}[\pi_1(K)])$, here is one way to construct a CW complex with finiteness obstruction $\pm\sigma$: let σ be represented by a f. g. projective module P and let $F = P \oplus Q$ be free of rank n. Let A be the matrix of the projection $p : F \to P \to F$ with respect to a standard basis for F. Now let

$$L = K \vee \bigvee_{i=1}^{n} S_i^{\ell}.$$

There is a split short exact sequence

$$0 \longrightarrow \pi_\ell(K) \underset{i_*}{\overset{r_*}{\leftrightarrows}} \pi_\ell(L) \longrightarrow \pi_\ell(L,K) \longrightarrow 0,$$

where $r : L \to K$ is the retraction which sends the S^ℓ's to the basepoint. Since

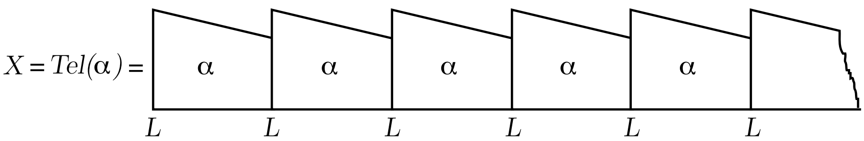
$$\pi_\ell(L,K) \cong \pi_\ell(\tilde{L},\tilde{K}) \cong H_\ell(\tilde{L},\tilde{K}) \cong F,$$

we can define $\alpha : L \to L$ so that $\alpha|K = id$ and so that $\alpha_* : \pi_\ell(L) \to \pi_\ell(L)$ has the matrix

$$\begin{pmatrix} id & 0 \\ 0 & A \end{pmatrix}$$

with respect to the direct sum decomposition $\pi_\ell(L) \cong \pi_\ell(K) \oplus F$. Since $A^2 = A$, it is easy to check that α is *homotopy idempotent*, i.e. that $\alpha \circ \alpha \sim \alpha$ rel K.

Let X be the infinite direct mapping telescope of α pictured below.



Let $d : L \to X$ be the inclusion of L into the top level of the leftmost mapping cylinder of X and define $s' : X \to L$ by setting s' equal to α on each copy of L and using the homotopies $\alpha \circ \alpha \sim \alpha$ to extend over the rest of X. One sees easily that $d \circ s'$ induces the identity on the homotopy groups of X and is therefore a homotopy equivalence. If ϕ is a homotopy inverse for $d \circ s'$, we have $d \circ s \sim id$, where $s = s' \circ \phi$. This means the d is a finite domination with right inverse s. It turns out that $[X] = (-1)^{\ell+1}[P]$.

(iii) If X is dominated by a finite CW complex K then $\pi_1(X)$ is a retract of the finitely presented group $\pi_1(K)$, and is thus also finitely presented. The cellular chain complex $C_*(\widetilde{X})$ is a chain homotopy direct summand of the finite f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex $\mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}[\pi_1(K)]} C_*(\widetilde{K})$, with $\widetilde{K}$ the universal cover of K. It follows from the algebraic theory of Ranicki [29] (or by the original geometric argument of Wall [40]) that

$C_*(\widetilde{X})$ is chain equivalent to a finite f.g. projective $\mathbb{Z}[\pi_1(X)]$-module chain complex $\mathcal{P}$.

Conversely, if $\pi_1(X)$ is finitely presented and $C_*(\widetilde{X})$ is chain equivalent to a finite f.g. projective $\mathbb{Z}[\pi_1(X)]$-module chain complex $\mathcal{P}$ the cellular $\mathbb{Z}[\pi_1(X)][z,z^{-1}]$-module chain complex of the universal cover $\widetilde{X \times S^1} = \widetilde{X} \times \mathbb{R}$ of $X \times S^1$

$$C_*(\widetilde{X \times S^1}) \;=\; C_*(\widetilde{X}) \otimes_{\mathbb{Z}} C_*(\mathbb{R})$$

is chain equivalent to a finite f.g. free $\mathbb{Z}[\pi_1(X)][z,z^{-1}]$-module chain complex, so that $X \times S^1$ is homotopy equivalent to a finite CW complex (by the proof of (i), using $[X \times S^1] = 0$) and X is finitely dominated by 1.5. □

In particular, if π is a finitely presented group such that $\widetilde{K}_0(\mathbb{Z}[\pi]) \neq 0$ then there exists a finitely dominated CW complex X with $\pi_1(X) = \pi$ and such that X is not homotopy equivalent to a finite CW complex. See Ferry [12] for the construction of finitely dominated compact metric spaces (which are not ANR's, still less CW complexes) which are not homotopy equivalent to a finite CW complex.

Wall [40] obtained the finiteness obstruction of a finitely dominated CW complex X from $C_*(\widetilde{X})$, using any finite f.g. projective $\mathbb{Z}[\pi_1(X)]$-module chain complex

$$\mathcal{P} : \cdots \to 0 \to P_n \xrightarrow{\partial} P_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \;.$$

chain equivalent to $C_*(\widetilde{X})$.

Definition 3.2. The *projective class* of X is the projective class of $\mathcal{P}$

$$[X] \;=\; \sum_{i=0}^{\infty} (-1)^i [P_i] \in K_0(\mathbb{Z}[\pi_1(X)]) \;. \qquad \square$$

The projective class is a well-defined chain-homotopy invariant of $C_*(\widetilde{X})$, with components

$$[X] \;=\; (\chi(X), [X]) \in K_0(\mathbb{Z}[\pi_1(X)]) \;=\; K_0(\mathbb{Z}) \oplus \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) \;,$$

where

$$\chi(X) \;=\; \sum_{i=0}^{\infty} (-1)^i \# \text{ of } i\text{-cells} \in K_0(\mathbb{Z}) \;=\; \mathbb{Z}$$

is the Euler characteristic of X, and $[X]$ is the finiteness obstruction.

The *instant finiteness obstruction* (Ranicki [29]) of a finitely dominated CW complex X is a f. g. projective $\mathbb{Z}[\pi_1(X)]$-module P representing the finiteness obstruction

$$[X] \;=\; [P] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$$

which is obtained directly from a finite domination $d : K \to X$, $s : X \to K$, a homotopy $h : d \circ s \simeq \mathrm{id}_X : X \to X$ and the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex $C_*(\widetilde{K})$ of the cover $\widetilde{K} = d^*\widetilde{X}$ of K obtained by pullback from the universal cover $\widetilde{X}$ of X, namely

$$P \;=\; \mathrm{im}(p : \mathbb{Z}[\pi_1(X)]^n \to \mathbb{Z}[\pi_1(X)]^n)$$

with $p = p^2$ the projection

$$p \;=\; \begin{pmatrix} s\circ d & -\partial & 0 & \dots \\ -s\circ h\circ d & 1 - s\circ d & \partial & \dots \\ s\circ h^2\circ d & s\circ h\circ d & s\circ d & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} :$$

$$\mathbb{Z}[\pi_1(X)]^n \;=\; \sum_{i=0}^{\infty} C_*(\widetilde{K})_i \to \sum_{i=0}^{\infty} C_*(\widetilde{K})_i$$

of a f. g. free $\mathbb{Z}[\pi_1(X)]$-module of rank

$$n \;=\; \sum_{i=0}^{\infty} \#\text{ of } i\text{-cells of } K\ .$$

In fact, the finiteness obstruction can be obtained in this way using only the chain homotopy projection $q = s\circ d \simeq q^2 : C_*(\widetilde{K}) \to C_*(\widetilde{K})$ induced by the homotopy idempotent $q = s\circ d \simeq q^2 : K \to K$ (Lück and Ranicki [20]).

The finiteness obstruction has many of the usual properties of the Euler characteristic χ. For instance, if X is the union of finitely dominated complexes X_1 and X_2 along a common finitely dominated subcomplex X_0, then

$$[X] = i_{1*}[X_1] + i_{2*}[X_2] - i_{0*}[X_0].$$

This is the *sum theorem* for finiteness obstructions, which was originally proved in Siebenmann's thesis [36].

The projective class of the product $X \times Y$ of finitely dominated *CW* complexes X, Y is given by

$$[X \times Y] \;=\; [X] \otimes [Y] \in K_0(\mathbb{Z}[\pi_1(X \times Y)])\ ,$$

leading to the *product formula* of Gersten [14] for the finiteness obstruction

$$[X \times Y] \;=\; \chi(X) \otimes [Y] + [X] \otimes \chi(Y) + [X] \otimes [Y] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X \times Y)])\ .$$

In particular, $[X \times S^1] = 0$, giving an algebraic proof of the result (1.5) that $X \times S^1$ is homotopy equivalent to a finite *CW* complex.

A fibration $p : E \to B$ with finitely dominated fibre F induces *transfer maps* in the projective class groups

$$p^! \;:\; K_0(\mathbb{Z}[\pi_1(B)]) \to K_0(\mathbb{Z}[\pi_1(E)])\ ;\ [X] \mapsto [Y]$$

sending the projective class of a finitely dominated CW complex X with a π_1-isomorphism $f : X \to B$ to the projective class of the pullback $Y = f^{\,!}E$, which is a finitely dominated CW complex with a π_1-isomorphism $f^{\,!} : Y \to E$

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F \arrow[d] \\ Y \arrow[r, "f^{\,!}"] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

Lück [18] obtained the following algebraic description of $p^{\,!}$, generalizing the product formula.[1] Let $\widetilde{F}$ be the pullback to F of the universal cover $\widetilde{E}$ of E. The fibration p induces a morphism of rings

$$U \ : \ \mathbb{Z}[\pi_1(B)] \to H_0(\mathrm{Hom}_{\mathbb{Z}[\pi_1(E)]}(C_*(\widetilde{F}), C_*(\widetilde{F})))^{op}$$

sending the homotopy class of a loop $\omega : S^1 \to B$ to the chain homotopy class of the parallel transport chain equivalence $U(\omega) : C_*(\widetilde{F}) \to C_*(\widetilde{F})$. A f. g. projective $\mathbb{Z}[\pi_1(B)]$-module

$$Q \ = \ \mathrm{im}(q : \mathbb{Z}[\pi_1(B)]^n \to \mathbb{Z}[\pi_1(B)]^n) \quad (q = q^2)$$

induces a $\mathbb{Z}[\pi_1(E)]$-module chain complex

$$Q^{\,!} \ = \ \mathcal{C}(U(q) : C_*(\widetilde{F})^n \to C_*(\widetilde{F})^n) \quad (U(q) \simeq U(q)^2)$$

which is algebraically finitely dominated, i.e. chain equivalent to a finite f. g. projective chain complex. The transfer map is given algebraically by

$$p^{\,!}[Q] \ = \ [Q^{\,!}] \in K_0(\mathbb{Z}[\pi_1(E)]) \ .$$

4. The topological space-form problem

Another problem in which a finiteness obstruction arises is the *topological space-form problem.* This is the problem of determining which groups can act freely and properly discontinuously on S^n for some n.

Swan, [37], solved a homotopy version of this problem by proving that a finite group G of order n which has periodic cohomology of period q acts freely on a finite complex of dimension $dq-1$ which is homotopy equivalent to a $(dq-1)$-sphere. Here, d is the greatest common divisor of n and $\phi(n)$, ϕ being Euler's ϕ-function.

One might ask whether such a G can act on S^{q-1}, but this refinement leads to a finiteness obstruction. It follows from Swan's argument that G

[1] See Lück and Ranicki [19] for the algebraic transfer map in the surgery obstruction groups.

acts freely on a countable $q-1$-dimensional complex X homotopy equivalent to S^{q-1} and that X/G is finitely dominated. The finiteness obstruction of X/G need not be zero, however, so not every group with cohomology of period q can act freely on a *finite* complex homotopy equivalent to S^{q-1}. Algebraically, the point is that finite groups with q-periodic cohomology have q-periodic resolutions by f. g. projective modules but need not have q-periodic resolutions by f. g. free modules.

After a great deal of work involving both the finiteness obstruction and surgery theory, see Madsen, Thomas and Wall [21], it turned out that a group G acts freely on S^n for some n if and only if all of its subgroups of order p^2 and $2p$ are cyclic (the condition of Milnor). This is in contrast to the linear case. A group G acts linearly on S^n for some n if and only if all subgroups of order pq, p and q not necessarily distinct primes, are cyclic. See Davis and Milgram [9] for a book-length treatment, and Weinberger [41], p. 110, for a brief discussion.

5. The Siebenmann end obstruction

The most significant application of the finiteness obstruction to the topology of manifolds is via the end obstruction.

An end ϵ of an open n-dimensional manifold W is *tame* if there exists a sequence $W \supset U_1 \supset U_2 \supset \dots$ of finitely dominated neighbourhoods of ϵ with

$$\bigcap_i U_i \ = \ \emptyset \ , \ \pi_1(U_1) \cong \pi_1(U_2) \cong \cdots \cong \pi_1(\epsilon) \ .$$

The end is *collared* if there exists a neighbourhood of the type $M \times [0, \infty)$ for some closed $(n-1)$-dimensional manifold M, i.e. if ϵ is the interior of a compactification $W \cup M$ with boundary component M.

Theorem 5.1. (Siebenmann [36]) *A tame end ϵ of an open n-dimensional manifold W has an end obstruction*

$$[\epsilon] \ = \ \varinjlim_i \, [U_i] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)])$$

such that $[\epsilon] = 0$ if (and for $n \geq 6$ only if) ϵ can be collared.

Novikov's 1965 proof of the topological invariance of the rational Pontrjagin classes made use of the end obstruction in the unobstructed case when π is a free abelian group. The subsequent work of Lashof, Rothenberg, Casson, Sullivan, Kirby and Siebenmann on the Hauptvermutung for high-dimensional manifolds made overt use of the end obstruction ([33]).

See sections 7 and 8 below for brief accounts of the applications of the end obstruction to splitting theorems and triangulation of high-dimensional manifolds.

See Chapman [7], Quinn [27], [28] and Connolly and Vajiac [8] for the controlled end obstruction, and Ranicki [32] for the bounded end obstruction.

See Hughes and Ranicki [15] for a book-length treatment of ends and the end obstruction.

6. Connections with Whitehead torsion

The finiteness obstruction deals with the existence of a finite CW complex K in a homotopy type, while Whitehead torsion deals with the uniqueness of K. There are many deep connections between the finiteness obstruction and Whitehead torsion, which on the purely algebraic level correspond to the connections between the algebraic K-groups K_0, K_1 (or rather $\widetilde{K}_0$, Wh).

The splitting theorem of Bass, Heller and Swan [2]

$$Wh(\pi \times \mathbb{Z}) \;=\; Wh(\pi) \oplus \widetilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi])$$

involves a split injection

$$\widetilde{K}_0(\mathbb{Z}[\pi]) \to Wh(\pi \times \mathbb{Z}) \; ; \; [P] \mapsto \tau(z : P[z, z^{-1}] \to P[z, z^{-1}]) \; . \qquad (*)$$

If X is a finitely dominated space then 1.5 gives a homotopy equivalence $\phi : X \times S^1 \to K$ to a finite CW complex K, uniquely up to simple homotopy equivalence. Ferry [13] identified the finiteness obstruction $[X] \in \widetilde{K}_0(\mathbb{Z}[\pi])$ ($\pi = \pi_1(X)$) with the Whitehead torsion $\tau(f) \in Wh(\pi \times \mathbb{Z})$ of the composite self homotopy equivalence of a finite CW complex

$$f \;:\; K \xrightarrow{\phi^{-1}} X \times S^1 \xrightarrow{\mathrm{id}_X \times -1} X \times S^1 \xrightarrow{\phi} K \; .$$

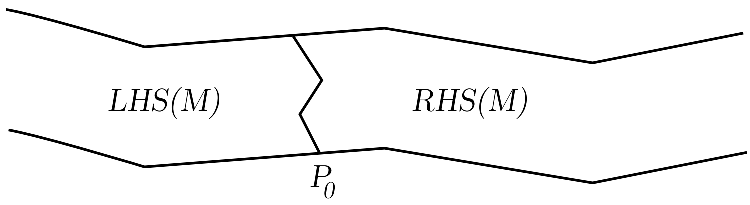
In Ranicki [30],[31] it was shown that $[X] \mapsto \tau(f)$ corresponds to the split injection

$$\widetilde{K}_0(\mathbb{Z}[\pi]) \to Wh(\pi \times \mathbb{Z}) \; ; \; [P] \mapsto \tau(-z : P[z, z^{-1}] \to P[z, z^{-1}])$$

which is different from the original split injection $(*)$ of [2].

7. The splitting obstruction

The finiteness obstruction arises in most classification problems in high-dimensional topology. Loosely speaking, proving that two manifolds are homeomorphic involves decomposing them into homeomorphic pieces. The finiteness obstruction is part of the obstruction to splitting a manifold into pieces. The nonsimply-connected version of Browder's $M \times \mathbb{R}$ Theorem is a case in point. In [5], Browder proved that if M^n, $n \geq 6$, is a PL manifold without boundary, $f : M \to K \times \mathbb{R}^1$ is a (PL) proper homotopy equivalence, and K is a simply-connected finite complex, then M is homeomorphic to $N \times \mathbb{R}^1$ for some closed manifold N homotopy equivalent to K.



When K is connected but not simply-connected, a finiteness obstruction arises. Here is a quick sketch of the argument: It is not difficult to show that M is 2-ended. The proper homotopy equivalence $f : M \to K \times \mathbb{R}^1$ gives us a proper PL map $p : M \to \mathbb{R}$. If $c \in \mathbb{R}$ is not the image of any vertex, then $p^{-1}(c)$ is a bicollared PL submanifold of M which separates the ends. Connected summing components along arcs allows us to assume that $P_0 = p^{-1}(c)$ is connected and a disk-trading argument similar to one in Browder's paper allows us to assume that $\pi_1 P_0 \to \pi_1 M$ is an isomorphism. See Siebenmann [36] for details. An application of the recognition criterion discussed in the third paragraph of this paper shows that the two components of $M - P_0$, which we denote by $RHS(M)$ and $LHS(M)$, respectively, are finitely dominated. By the sum theorem,

$$[RHS(M)] + [LHS(M)] \ = \ 0 \in \widetilde{K}_0(\mathbb{Z}[\pi_1(M)]) \ .$$

It turns out that the vanishing of $[RHS(M)] = -[LHS(M)]$ is necessary and sufficient for M to be homeomorphic to a product $N \times \mathbb{R}$, provided that $\dim(M) \geq 6$. This is one of the main results of [36].

It is possible to realize the finiteness obstruction $\sigma \in \widetilde{K}_0(\mathbb{Z}[\pi_1(K)])$ for an n-dimensional manifold M^n proper homotopy equivalent to $K \times \mathbb{R}$ for some finite K whenever $\sigma + (-1)^{n-1}\sigma^* = 0$ and $n \geq 6$. If we only require that M be properly dominated by some $K \times \mathbb{R}$, then any finiteness obstruction σ can be realized (cf. Pedersen and Ranicki [26]). A similar obstruction arises in the problem of determining whether a map $p : M^n \to S^1$ is homotopic to the projection map of a fiber bundle (Farrell [10]).

The geometric splitting of two-ended open manifolds into right and left sides is closely related to the proof of the algebraic splitting theorem of Bass, Heller and Swan [2] for $Wh(\pi \times \mathbb{Z})$ – see Ranicki [32].

8. The triangulation of manifolds

The finiteness obstruction arises in connection with another of the fundamental problems of topology: *Is every compact topological manifold without boundary homeomorphic to a finite polyhedron?* We will examine this problem in much greater detail.

The triangulation problem was solved affirmatively for two-dimensional manifolds by Rado in 1924 and for three-dimensional manifolds by Moise in 1952. Higher dimensions proved less tractable,[2] a circumstance which encouraged the formulation of weaker questions such as the following *homotopy triangulation problem*: *Does every compact topological manifold have the homotopy type of some finite polyhedron?*

The first solution of this problem came as a corollary to Kirby and Siebenmann's theory of PL triangulations of high-dimensional topological manifolds. By a theorem of Hirsch, every topological manifold M^n has a well-defined stable topological normal disk bundle. The total space of this bundle is a closed neighborhood of M in some high-dimensional euclidean space. In [16], Kirby and Siebenmann proved that a topological n-manifold, $n \geq 6$, has a PL structure if and only if this stable normal bundle reduces from TOP to PL. As an immediate corollary, they deduced that every compact topological manifold has the homotopy type of a finite polyhedron, since each M is homotopy equivalent to the total space of the unit disk bundle of its normal disk bundle and the total space of the normal disk bundle is a PL manifold because its normal bundle is trivial. The argument of Kirby and Siebenmann also shows that each compact topological manifold has a well-defined simple homotopy type. A more refined argument, see p.104 of Kirby and Siebenmann [17], shows that every closed topological manifold of dimension ≥ 6 is a TOP handlebody. From this it follows immediately that every compact topological manifold is homotopy equivalent to a finite CW complex and therefore to a finite polyhedron.

This positive solution to the homotopy-triangulation problem suggests that we should look for large naturally-occurring classes of compact topological spaces which have the homotopy types of finite polyhedra. In 1954, K. Borsuk [4] conjectured that every compact metrizable ANR should have the homotopy type of a finite polyhedron. This became widely known as *Borsuk's Conjecture*.

The Borsuk Conjecture was solved by J. E. West, [42], using results of T. A. Chapman, which, in turn, were based on an infinite-dimensional version of Kirby-Siebenmann's handle-straightening argument. In a nutshell, Chapman proved that every compact manifold modeled on the Hilbert cube ($\equiv \prod_{i=1}^{\infty}[0,1]$) is homotopy equivalent to a finite complex and West showed that every compact ANR[3] is homotopy equivalent to a compact manifold

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[2] In fact, Casson has shown that there are compact four-manifolds without boundary which are not homeomorphic to finite polyhedra (Akbulut and McCarthy [1], p.xvi). The question is still open in dimensions greater than or equal to five.

[3] A compact metrizable space X is an ANR if and only if it embeds as a neighborhood retract in separable Hilbert space. If X has finite covering dimension $\leq n$, separable Hilbert space can be replaced by $\mathbb{R}^{2n+1}$.

modeled on the Hilbert cube. A rather short finite-dimensional proof of the topological invariance of Whitehead torsion, together with the Borsuk Conjecture was given by Chapman in [6]. See Ranicki and Yamasaki [34] for a more recent proof, which makes use of controlled algebraic K-theory.

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An introduction to algebraic surgery

Andrew Ranicki

Introduction

Surgery theory investigates the homotopy types of manifolds, using a combination of algebra and topology. It is the aim of these notes to provide an introduction to the more algebraic aspects of the theory, without losing sight of the geometric motivation.

0.1 Historical background

A closed m-dimensional topological manifold M has Poincaré duality isomorphisms

$$H^{m-*}(M) \;\cong\; H_*(M) \;.$$

In order for a space X to be homotopy equivalent to an m-dimensional manifold it is thus necessary (but not in general sufficient) for X to be an m-dimensional Poincaré duality space, with $H^{m-*}(X) \cong H_*(X)$. The *topological structure set* $\mathbb{S}^{TOP}(X)$ is defined to be the set of equivalence classes of pairs

$$(m\text{-dimensional manifold } M \;,\; \text{homotopy equivalence } h : M \to X)$$

subject to the equivalence relation

$$\begin{array}{ll} (M,h) \sim (M',h') & \text{if there exists a homeomorphism} \\ & f : M \to M' \text{ such that } h'f \simeq h : M \to X. \end{array}$$

The basic problem of surgery theory is to decide if a Poincaré complex X is homotopy equivalent to a manifold (i.e. if $\mathbb{S}^{TOP}(X)$ is non-empty), and if so to compute $\mathbb{S}^{TOP}(X)$ in terms of the algebraic topology of X.

Surgery theory was first developed for differentiable manifolds, and then extended to PL and topological manifolds.

The classic Browder–Novikov–Sullivan–Wall obstruction theory for deciding if a Poincaré complex X is homotopy equivalent to a manifold has two stages :

(i) the primary topological K-theory obstruction $\nu_X \in [X, B(G/TOP)]$ to a TOP reduction $\widetilde{\nu}_X : X \to BTOP$ of the Spivak normal fibration $\nu_X : X \to BG$, which vanishes if and only if there exists a manifold M with a normal map $(f, b) : M \to X$, that is a degree 1 map $f : M \to X$ with a bundle map $b : \nu_M \to \widetilde{\nu}_X$,

(ii) a secondary algebraic L-theory obstruction

$$\sigma_*(f, b) \in L_m(\mathbb{Z}[\pi_1(X)])$$

in the surgery obstruction group of Wall [29], which is defined if the obstruction in (i) vanishes, and which depends on the choice of TOP reduction $\widetilde{\nu}_X$, or equivalently on the bordism class of the normal map $(f, b) : M \to X$. The surgery obstruction is such that $\sigma_*(f, b) = 0$ if (and for $m \geq 5$ only if) (f, b) is normal bordant to a homotopy equivalence.

There exists a TOP reduction $\widetilde{\nu}_X$ of ν_X for which the corresponding normal map $(f, b) : M \to X$ has zero surgery obstruction if (and for $m \geq 5$ only if) the structure set $\mathbb{S}^{TOP}(X)$ is non-empty. A relative version of the theory gives a two-stage obstruction for deciding if a homotopy equivalence $M \to X$ from a manifold M is homotopic to a homeomorphism, which is traditionally formulated as the *surgery exact sequence*

$$\ldots \to L_{m+1}(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}^{TOP}(X) \to [X, G/TOP] \to L_m(\mathbb{Z}[\pi_1(X)]) \ .$$

See the paper by Browder [2] for an account of the original Sullivan-Wall surgery exact sequence in the differentiable category in the case when X has the homotopy type of a differentiable manifold

$$\ldots \to L_{m+1}(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}^{O}(X) \to [X, G/O] \to L_m(\mathbb{Z}[\pi_1(X)]) \ .$$

The algebraic L-groups $L_*(\Lambda)$ of a ring with involution Λ are defined using quadratic forms over Λ and their automorphisms, and are 4-periodic

$$L_m(\Lambda) \ = \ L_{m+4}(\Lambda) \ .$$

The surgery classification of exotic spheres of Kervaire and Milnor [7] included the first computation of the L-groups, namely

$$L_m(\mathbb{Z}) \ = \ \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 \pmod 4 \\ 0 & \text{if } m \equiv 1 \pmod 4 \\ \mathbb{Z}_2 & \text{if } m \equiv 2 \pmod 4 \\ 0 & \text{if } m \equiv 3 \pmod 4 \ . \end{cases}$$

The relationship between topological and PL manifolds was investigated using surgery methods in the 1960's by Novikov, Casson, Sullivan, Kirby and Siebenmann [8] (cf. Ranicki [23]), culminating in a disproof of the manifold Hauptvermutung: there exist homeomorphisms of PL manifolds which are not homotopic to PL homeomorphisms, and in fact there exist topological manifolds without PL structure. The surgery exact sequence for the PL manifold structure set $\mathbb{S}^{PL}(M)$ for a PL manifold M was related to the exact sequence for $\mathbb{S}^{TOP}(M)$ by a commutative braid of exact sequences

$$\begin{tikzcd}
H^3(M;\mathbb{Z}_2) \arrow[dr] \arrow[rr, bend left] & & {[M,G/PL]} \arrow[dr] \arrow[rr, bend left] & & L_m(\mathbb{Z}[\pi_1(M)]) \\
& \mathbb{S}^{PL}(M) \arrow[ur] \arrow[dr] & & {[M,G/TOP]} \arrow[ur] \arrow[dr] & \\
L_{m+1}(\mathbb{Z}[\pi_1(M)]) \arrow[ur] \arrow[rr, bend right] & & \mathbb{S}^{TOP}(M) \arrow[ur] \arrow[rr, bend right] & & H^4(M;\mathbb{Z}_2)
\end{tikzcd}$$

with

$$\pi_*(G/TOP) \ = \ L_*(\mathbb{Z}) \ .$$

Quinn [17] gave a geometric construction of a spectrum of simplicial sets for any group π

$$\mathbb{L}_\bullet(\mathbb{Z}[\pi]) \ = \ \{\mathbb{L}_n(\mathbb{Z}[\pi]) \,|\, \Omega\mathbb{L}_n(\mathbb{Z}[\pi]) \simeq \mathbb{L}_{n+1}(\mathbb{Z}[\pi])\}$$

with homotopy groups

$$\pi_n(\mathbb{L}_\bullet(\mathbb{Z}[\pi])) \ = \ \pi_{n+k}(\mathbb{L}_{-k}(\mathbb{Z}[\pi])) \ = \ L_n(\mathbb{Z}[\pi]) \ ,$$

and

$$\mathbb{L}_0(\mathbb{Z}) \ \simeq \ L_0(\mathbb{Z}) \times G/TOP \ .$$

The construction included an assembly map

$$A \ : \ H_*(X;\mathbb{L}_\bullet(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi_1(X)])$$

and for a manifold X the surgery obstruction function is given by

$$[X,G/TOP] \subset [X,L_0(\mathbb{Z}) \times G/TOP] \ \cong \ H_n(X;\mathbb{L}_\bullet(\mathbb{Z})) \xrightarrow{A} L_m(\mathbb{Z}[\pi_1(X)]) \ .$$

The surgery classifying spectra $\mathbb{L}_\bullet(\Lambda)$ and the assembly map A were constructed algebraically in Ranicki [22] for any ring with involution Λ, using quadratic Poincaré complex n-ads over Λ. The spectrum $\mathbb{L}_\bullet(\mathbb{Z})$ is appropriate for the surgery classification of homology manifold structures (Bryant, Ferry, Mio and Weinberger [3]); for topological manifolds it is necessary to work with the 1-connective spectrum $\mathbb{L}_\bullet = \mathbb{L}_\bullet(\mathbb{Z})\langle 1\rangle$, such that $\mathbb{L}_n$ is n-connected with $\mathbb{L}_0 \simeq G/TOP$. The relative homotopy groups of the spectrum-level assembly map

$$\mathbb{S}_m(X) \ = \ \pi_m(A : X_+ \wedge \mathbb{L}_\bullet \to \mathbb{L}_\bullet(\mathbb{Z}[\pi_1(X)]))$$

fit into the *algebraic surgery exact sequence*

$$\begin{aligned}\dots \to L_{m+1}(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_{m+1}(X) &\\ \to H_m(X;\mathbb{L}_\bullet) &\xrightarrow{A} L_m(\mathbb{Z}[\pi_1(X)]) \to \dots\ .\end{aligned}$$

The algebraic surgery theory of Ranicki [20], [22] provided one-stage obstructions:

(i) An m-dimensional Poincaré duality space X has a *total surgery obstruction* $s(X) \in \mathbb{S}_m(X)$ such that $s(X) = 0$ if (and for $m \geq 5$ only if) X is homotopy equivalent to a manifold.

(ii) A homotopy equivalence of m-dimensional manifolds $h : M' \to M$ has a *total surgery obstruction* $s(h) \in \mathbb{S}_{m+1}(M)$ such that $s(h) = 0$ if (and for $m \geq 5$ only if) h is homotopic to a homeomorphism.

Moreover, if X is an m-dimensional manifold and $m \geq 5$ the geometric surgery exact sequence is isomorphic to the algebraic surgery exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & L_{m+1}(\mathbb{Z}[\pi_1(X)]) \arrow[r] \arrow[d, equal] & \mathbb{S}^{TOP}(X) \arrow[r] \arrow[d, "\cong"] & {[X, G/TOP]} \arrow[r] \arrow[d, "\cong"] & L_m(\mathbb{Z}[\pi_1(X)]) \arrow[d, equal] \\
\cdots \arrow[r] & L_{m+1}(\mathbb{Z}[\pi_1(X)]) \arrow[r] & \mathbb{S}_{m+1}(X) \arrow[r] & H_m(X;\mathbb{L}_\bullet) \arrow[r, "A"] & L_m(\mathbb{Z}[\pi_1(X)])
\end{tikzcd}$$

with

$$\mathbb{S}^{TOP}(X) \to \mathbb{S}_{m+1}(X)\ ;\ (M, h : M \to X) \longmapsto s(h)\ .$$

Given a normal map $(f, b) : M^m \to X$ it is possible to *kill* an element $x \in \pi_r(f)$ by surgery if and only if x can be represented by an embedding $S^{r-1} \times D^{n-r+1} \hookrightarrow M$ with a null-homotopy in X, in which case the *trace* of the surgery is a normal bordism

$$((g, c); (f, b), (f', b'))\ :\ (N; M, M') \to X \times ([0, 1]; \{0\}, \{1\})$$

with

$$\begin{aligned} N^{m+1} &= M \times I \cup D^r \times D^{m-r+1}\ , \\ M'^m &= (M \backslash S^{r-1} \times D^{m-r+1}) \cup D^r \times S^{m-r}\ . \end{aligned}$$

The normal map $(f', b') : M' \to X$ is the *geometric effect* of the surgery on (f, b). Surgery theory investigates the extent to which a normal map can be made bordant to a homotopy equivalence by killing as much of $\pi_*(f)$ as possible. The original definition of the non-simply-connected surgery obstruction $\sigma_*(f, b) \in L_m(\mathbb{Z}[\pi_1(X)])$ (Wall [29]) was obtained after preliminary surgeries below the middle dimension, to kill the relative homotopy groups

$\pi_r(f)$ for $2r \leq m$. It could thus be assumed that $(f,b) : M \to X$ is $[m/2]$-connected, with $\pi_r(f) = 0$ for $2r \leq m$, and $\sigma_*(f,b)$ was defined using the Poincaré duality structure on the middle-dimensional homotopy kernel(s). The surgery obstruction theory is much easier in the even-dimensional case $m = 2n$ when $\pi_r(f)$ can be non-zero at most for $r = m+1$ than in the odd-dimensional case $m = 2n+1$ when $\pi_r(f)$ can be non-zero for $r = m+1$ and $r = m+2$.

Wall [29,§18G] asked for a chain complex formulation of surgery, in which the obstruction groups $L_m(\Lambda)$ would appear as the cobordism groups of chain complexes with m-dimensional quadratic Poincaré duality, by analogy with the cobordism groups of manifolds Ω_*. Mishchenko [15] initiated such a theory of "m-dimensional symmetric Poincaré complexes" (C,ϕ) with C an m-dimensional f. g. free Λ-module chain complex

$$C \ : \ C_m \xrightarrow{d} C_{m-1} \xrightarrow{d} C_{m-2} \to \ldots \to C_1 \xrightarrow{d} C_0$$

and ϕ a quadratic structure inducing m-dimensional Poincaré duality isomorphisms $\phi_0 : H^*(C) \to H_{m-*}(C)$. The cobordism groups $L^m(\Lambda)$ (which are covariant in Λ) are such that for any m-dimensional geometric Poincaré complex X there is defined a symmetric signature invariant

$$\sigma^*(X) \ = \ (C(\widetilde{X}),\phi) \in L^m(\mathbb{Z}[\pi_1(X)]) \ .$$

The corresponding quadratic theory was developed in Ranicki [19]; the m-dimensional quadratic L-groups $L_m(\Lambda)$ for any $m \geq 0$ were obtained as the groups of equivalence classes of "m-dimensional quadratic Poincaré complexes" (C,ψ). The surgery obstruction of an m-dimensional normal map $(f,b) : M^m \to X$ was expressed as a cobordism class

$$\sigma_*(f,b) \ = \ (C,\psi) \in L_m(\mathbb{Z}[\pi_1(X)])$$

with

$$H_*(C) \ = \ K_*(M) \ = \ H_{*+1}(\widetilde{f} : \widetilde{M} \to \widetilde{X}) \ .$$

The symmetrization maps $1+T : L_*(\Lambda) \to L^*(\Lambda)$ are isomorphisms modulo 8-torsion, and the symmetrization of the surgery obstruction is the difference of the symmetric signatures

$$(1+T)\sigma_*(f,b) \ = \ \sigma^*(M) - \sigma^*(X) \in L^m(\mathbb{Z}[\pi_1(X)]) \ .$$

However, the theory of [19] is fairly elaborate. The algebra of [18] and [19] is used in these notes to simplify the original theory of Wall [29] in the odd-dimensional case, without invoking the full theory of [19]. Ranicki [25] is a companion paper to this one, which provides an introduction to the use of algebraic Poincaré complexes in surgery theory.

0.2 What is in these notes

These notes give an elementary account of the construction of the L-groups L_* and the surgery obstruction σ_* for differentiable manifolds. For the more computational aspects of the L-groups see the papers by Hambleton and Taylor [4] and Stark [26].

The even-dimensional L-groups $L_{2n}(\Lambda)$ are the Witt groups of nonsingular $(-1)^n$-quadratic forms over Λ. It is relatively easy to pass from an n-connected $2n$-dimensional normal map $(f,b) : M^{2n} \to X$ to a $(-1)^n$-quadratic form representing $\sigma_*(f,b)$, and to see how the form changes under a surgery on (f,b). This will be done in §§1–5 of these notes.

The odd-dimensional L-groups $L_{2n+1}(\Lambda)$ are the stable automorphism groups of nonsingular $(-1)^n$-quadratic forms over Λ. It is relatively hard to pass from an n-connected $(2n+1)$-dimensional normal map (f,b) : $M^{2n+1} \to X$ to an automorphism of a $(-1)^n$-quadratic form representing $\sigma_*(f,b)$, and even harder to follow through in algebra the effect of a surgery on (f,b). Novikov [16] suggested the reformulation of the odd-dimensional theory in terms of the language of hamiltonian physics, and to replace the automorphisms by ordered pairs of lagrangians (= maximal isotropic subspaces). This reformulation was carried out in Ranicki [18], where such pairs were called 'formations', but it was still hard to follow the algebraic effects of individual surgeries. This became easier after the further reformulation of Ranicki [19] in terms of chain complexes with Poincaré duality – see §§8,9 for a description of how the kernel formation changes under a surgery on (f,b).

The original definition of $L_*(\Lambda)$ in Wall [29] was for the category of based f. g. free Λ-modules and simple isomorphisms, for surgery up to simple homotopy equivalence. Here, f. g. stands for finitely generated and simple means that the Whitehead torsion is trivial, as in the hypothesis of the s-cobordism theorem. These notes will only deal with free L-groups $L_*(\Lambda) = L^h_*(\Lambda)$, the obstruction groups for surgery up to homotopy equivalence.

The algebraic theory of ϵ-quadratic forms (K,λ,μ) over a ring Λ with an involution $\Lambda \to \Lambda; a \mapsto \bar{a}$ is developed in §§1,2, with $\epsilon = \pm 1$ and

$$\lambda \ : \ K \times K \to \Lambda \ ; \ (x,y) \ \longmapsto \lambda(x,y)$$

an ϵ-symmetric pairing on a Λ-module K such that

$$\lambda(x,y) \ = \ \epsilon\overline{\lambda(y,x)} \in \Lambda \ \ (x,y \in K)$$

and

$$\mu \ : \ K \to Q_\epsilon(\Lambda) \ = \ \Lambda/\{a - \epsilon\bar{a} \,|\, a \in \Lambda\}$$

an ϵ-quadratic refinement of λ such that

$$\lambda(x,x) \ = \ \mu(x)+\epsilon\overline{\mu(x)} \in \Lambda \quad (x \in K) \ .$$

For an n-connected $2n$-dimensional normal map $(f,b) : M^{2n} \to X$ geometric (intersection, self-intersection) numbers define a $(-1)^n$-quadratic form $(K_n(M),\lambda,\mu)$ on the kernel stably f. g. free $\mathbb{Z}[\pi_1(X)]$-module

$$K_n(M) \ = \ \ker(\widetilde{f}_* : H_n(\widetilde{M}) \to H_n(\widetilde{X}))$$

with $\widetilde{X}$ the universal cover of X and $\widetilde{M} = f^*\widetilde{X}$ the pullback of $\widetilde{X}$ to M.

The hyperbolic ϵ-quadratic form on a f. g. free Λ-module Λ^k

$$H_\epsilon(\Lambda^k) \ = \ (\Lambda^{2k},\lambda,\mu)$$

is defined by

$$\lambda \ : \ \Lambda^{2k} \times \Lambda^{2k} \to \Lambda \ ;$$

$$((a_1,a_2,\dots,a_{2k}),(b_1,b_2,\dots,b_{2k})) \ \longmapsto \ \sum_{i=1}^{k}(b_{2i-1}\bar{a}_{2i}+\epsilon b_{2i}\bar{a}_{2i-1}) \ ,$$

$$\mu \ : \ \Lambda^{2k} \to Q_\epsilon(\Lambda) \ ; \ (a_1,a_2,\dots,a_{2k}) \longmapsto \ \sum_{i=1}^{k} a_{2i-1}\bar{a}_{2i} \ .$$

The even-dimensional L-group $L_{2n}(\Lambda)$ is defined in §3 to be the abelian group of stable isomorphism classes of nonsingular $(-1)^n$-quadratic forms on (stably) f. g. free Λ-modules, where stabilization is with respect to the hyperbolic forms $H_{(-1)^n}(\Lambda^k)$. A nonsingular $(-1)^n$-quadratic form (K,λ,μ) represents 0 in $L_{2n}(\Lambda)$ if and only if there exists an isomorphism

$$(K,\lambda,\mu)\oplus H_{(-1)^n}(\Lambda^k) \ \cong \ H_{(-1)^n}(\Lambda^{k'})$$

for some integers $k,k' \geq 0$. The surgery obstruction of an n-connected $2n$-dimensional normal map $(f,b) : M^{2n} \to X$ is defined by

$$\sigma_*(f,b) \ = \ (K_n(M),\lambda,\mu) \in L_{2n}(\mathbb{Z}[\pi_1(X)]) \ .$$

The algebraic effect of a geometric surgery on an n-connected $2n$-dimensional normal map (f,b) is given in §5. Assuming that the result of the surgery is still n-connected, the effect on the kernel form of a surgery on $S^{n-1}\times D^{n+1} \hookrightarrow M$ (resp. $S^n \times D^n \hookrightarrow M$) is to add on (resp. split off) a hyperbolic $(-1)^n$-quadratic form $H_{(-1)^n}(\mathbb{Z}[\pi_1(X)])$.

§6 introduces the notion of a "$(2n+1)$-complex" (C,ψ), which is a f. g. free Λ-module chain complex of the type

$$C \ : \ \dots \to 0 \to C_{n+1} \xrightarrow{d} C_n \ \to 0 \to \dots$$

together with a quadratic structure ψ inducing Poincaré duality isomorphisms $(1+T)\psi \ : \ H^{2n+1-*}(C) \to H_*(C)$. (This is just a $(2n+1)$-

dimensional quadratic Poincaré complex (C,ψ) in the sense of [19], with $C_r = 0$ for $r \neq n, n+1$.) An n-connected $(2n+1)$-dimensional normal map $(f,b) : M^{2n+1} \to X$ determines a kernel $(2n+1)$-complex (C,ψ) (or rather a homotopy equivalence class of such complexes) with

$$H_*(C) \;=\; K_*(M) \;=\; \ker(\widetilde{f}_* : H_*(\widetilde{M}) \to H_*(\widetilde{X})) \;.$$

The cobordism of $(2n+1)$-complexes is defined in §7. The odd-dimensional L-group $L_{2n+1}(\Lambda)$ is defined in §8 as the cobordism group of $(2n+1)$-complexes. The surgery obstruction of an n-connected normal map $(f,b) : M^{2n+1} \to X$ is the cobordism class of the kernel complex

$$\sigma_*(f,b) \;=\; (C,\psi) \in L_{2n+1}(\mathbb{Z}[\pi_1(X)]) \;.$$

The odd-dimensional L-group $L_{2n+1}(\Lambda)$ was originally defined in [29] as a potentially non-abelian quotient of the stable unitary group of the matrices of automorphisms of hyperbolic $(-1)^n$-quadratic forms over Λ

$$L_{2n+1}(\Lambda) \;=\; U_{(-1)^n}(\Lambda)/EU_{(-1)^n}(\Lambda)$$

with

$$U_{(-1)^n}(\Lambda) \;=\; \bigcup_{k=1}^{\infty} \mathrm{Aut}_\Lambda H_{(-1)^n}(\Lambda^k)$$

and $EU_{(-1)^n}(\Lambda) \triangleleft U_{(-1)^n}(\Lambda)$ the normal subgroup generated by the elementary matrices of the type

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} \;,\; \begin{pmatrix} 1 & 0 \\ \beta + (-1)^{n+1}\beta^* & 1 \end{pmatrix} \;,\; \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}$$

for any invertible matrix α, and any square matrix β. The group $L_{2n+1}(\Lambda)$ is abelian, since

$$[U_{(-1)^n}(\Lambda), U_{(-1)^n}(\Lambda)] \subseteq EU_{(-1)^n}(\Lambda) \;.$$

The surgery obstruction $\sigma_*(f,b) \in L_{2n+1}(\mathbb{Z}[\pi_1(X)])$ of an n-connected $(2n+1)$-dimensional normal map $(f,b) : M^{2n+1} \to X$ is represented by an automorphism of a hyperbolic $(-1)^n$-quadratic form obtained from a high-dimensional generalization of the Heegaard decompositions of 3-dimensional manifolds as twisted doubles.

§8, §9 and §10 describe three equivalent ways of defining $L_{2n+1}(\Lambda)$, using unitary matrices, formations and chain complexes. In each case it is necessary to make some choices in passing from the geometry to the algebra, and to verify that the equivalence class in the L-group is independent of the choices.

The definition of $L_{2n+1}(\Lambda)$ using complexes given in §8 is a special case of the general theory of chain complexes with Poincaré duality of Ranicki

[19]. The 4-periodicity in the quadratic L-groups

$$L_m(\Lambda) \;=\; L_{m+4}(\Lambda)$$

(given geometrically by taking product with $\mathbb{CP}^2$, as in Chapter 9 of Wall [29]) was proved in [19] using an algebraic analogue of surgery below the middle dimension: it is possible to represent every element of $L_m(\Lambda)$ by a quadratic Poincaré complex (C,ψ) which is "highly-connected", meaning that

$$C_r \;=\; 0 \text{ for } \begin{cases} r \neq n & \text{if } m = 2n \\ r \neq n, n+1 & \text{if } m = 2n+1 \;. \end{cases}$$

In these notes only the highly-connected $(2n+1)$-dimensional quadratic Poincaré complexes are considered, namely the "$(2n+1)$-complexes" of §6.

I am grateful to the referee for suggesting several improvements.

The titles of the sections are:

§1. Duality

§1 considers rings Λ equipped with an "involution" reversing the order of multiplication. An involution allows right Λ-modules to be regarded as left Λ-modules, especially the right Λ-modules which arise as the duals of left Λ-modules. In particular, the group ring $\mathbb{Z}[\pi_1(M)]$ of the fundamental group $\pi_1(M)$ of a manifold M has an involution, which allows the Poincaré duality of the universal cover $\widetilde{M}$ to be regarded as $\mathbb{Z}[\pi_1(M)]$-module isomorphisms.

Let X be a connected space, and let $\widetilde{X}$ be a regular cover of X with group of covering translations π. The action of π on $\widetilde{X}$ by covering translations

$$\pi \times \widetilde{X} \to \widetilde{X} \;;\; (g,x) \longmapsto gx$$

induces a left action of the group ring $\mathbb{Z}[\pi]$ on the homology of $\widetilde{X}$

$$\mathbb{Z}[\pi] \times H_*(\widetilde{X}) \to H_*(\widetilde{X}) \;;\; (\sum_{g\in\pi} n_g g, x) \longmapsto \sum_{g\in\pi} n_g gx$$

so that the homology groups $H_*(\widetilde{X})$ are left $\mathbb{Z}[\pi]$-modules. In dealing with cohomology let

$$H^*(\widetilde{X}) \; = \; H^*_{cpt}(\widetilde{X})$$

be the compactly supported cohomology groups, regarded as left $\mathbb{Z}[\pi]$-modules by

$$\mathbb{Z}[\pi] \times H^*(\widetilde{X}) \to H^*(\widetilde{X}) \; ; \; (\sum_{g\in\pi} n_g g, x) \; \longmapsto \; \sum_{g\in\pi} n_g x g^{-1} \; .$$

(For finite π $H^*(\widetilde{X})$ is just the ordinary cohomology of $\widetilde{X}$.) Cap product with any homology class $[X] \in H_m(X)$ defines $\mathbb{Z}[\pi]$-module morphisms

$$[X] \cap - \; : \; H^*(\widetilde{X}) \to H_{m-*}(\widetilde{X}) \; .$$

Definition 1.1 An *oriented m-dimensional geometric Poincaré complex* (Wall [28]) is a finite CW complex X with a fundamental class $[X] \in H_m(X)$ such that cap product defines $\mathbb{Z}[\pi_1(X)]$-module isomorphisms

$$[X] \cap - \; : \; H^*(\widetilde{X}) \xrightarrow{\cong} H_{m-*}(\widetilde{X})$$

with $\widetilde{X}$ the universal cover of X. □

See 1.14 below for the general definition of a geometric Poincaré complex, including the nonorientable case.

Example 1.2 A compact oriented m-dimensional manifold is an oriented m-dimensional geometric Poincaré complex. □

In order to also deal with nonorientable manifolds and Poincaré complexes it is convenient to have an involution:

Definition 1.3 Let Λ be an associative ring with 1. An *involution* on Λ is a function

$$\Lambda \to \Lambda \; ; \; a \longmapsto \overline{a}$$

satisfying

$$\overline{(a+b)} \; = \; \overline{a} + \overline{b} \; , \; \overline{(ab)} \; = \; \overline{b}.\overline{a} \; , \; \overline{\overline{a}} \; = \; a \; , \; \overline{1} \; = \; 1 \in \Lambda \; . \qquad □$$

Example 1.4 A commutative ring Λ admits the identity involution

$$\Lambda \to \Lambda \; ; \; a \longmapsto \overline{a} \; = \; a \; . \qquad □$$

Definition 1.5 Given a group π and a group morphism

$$w \; : \; \pi \to \mathbb{Z}_2 \; = \; \{\pm 1\}$$

define the *w-twisted involution* on the integral group ring $\Lambda = \mathbb{Z}[\pi]$

$$\Lambda \to \Lambda \; ; \; a \;=\; \sum_{g\in\pi} n_g g \longmapsto \bar{a} \;=\; \sum_{g\in\pi} w(g) n_g g^{-1} \; (n_g \in \mathbb{Z}) \; . \qquad \square$$

In the topological application π is the fundamental group of a space with $w : \pi \to \mathbb{Z}_2$ an orientation character. In the oriented case $w(g) = +1$ for all $g \in \pi$.

EXAMPLE 1.6 Complex conjugation defines an involution on the ring of complex numbers $\Lambda = \mathbb{C}$

$$\mathbb{C} \to \mathbb{C} \; ; \; z \;=\; a + ib \longmapsto \overline{z} = a - ib \; . \qquad \square$$

A "hermitian" form is a symmetric form on a (finite-dimensional) vector space over $\mathbb{C}$ with respect to this involution. The study of forms over rings with involution is sometimes called "hermitian K-theory", although "algebraic L-theory" seems preferable.

The dual of a left Λ-module K is the right Λ-module

$$K^* \;=\; \mathrm{Hom}_\Lambda(K, \Lambda)$$

with

$$K^* \times \Lambda \to K^* \; ; \; (f, a) \longmapsto (x \longmapsto f(x).a) \; .$$

An involution $\Lambda \to \Lambda; a \longmapsto \bar{a}$ determines an isomorphism of categories

$$\{\text{right } \Lambda\text{-modules}\} \xrightarrow{\cong} \{\text{left } \Lambda\text{-modules}\} \; ; \; L \longmapsto L^{op} \; ,$$

with L^{op} the left Λ-module with the same additive group as the right Λ-module L and Λ acting by

$$\Lambda \times L^{op} \to L^{op} \; ; \; (a, x) \longmapsto x\bar{a} \; .$$

From now on we shall work with a ring Λ which is equipped with a particular choice of involution $\Lambda \to \Lambda$. Also, Λ-modules will always be understood to be left Λ-modules.

For any Λ-module K the Λ-module $(K^*)^{op}$ is written as K^*. Here is the definition of K^* all at once:

DEFINITION 1.7 The *dual* of a Λ-module K is the Λ-module

$$K^* \;=\; \mathrm{Hom}_\Lambda(K, \Lambda) \; ,$$

with Λ acting by

$$\Lambda \times K^* \to K^* \; ; \; (a, f) \longmapsto (x \longmapsto f(x).\overline{a})$$

for all $a \in \Lambda$, $f \in K^*$, $x \in K$. $\square$

There is a corresponding notion for morphisms:

DEFINITION 1.8 The *dual* of a Λ-module morphism $f : K \to L$ is the Λ-module morphism

$$f^* \;:\; L^* \to K^* \;;\; g \longmapsto \big(x \longmapsto g(f(x))\big) \;. \qquad \square$$

Thus duality is a contravariant functor

$$* \;:\; \{\Lambda\text{-modules}\} \to \{\Lambda\text{-modules}\} \;;\; K \longmapsto K^* \;.$$

DEFINITION 1.9 For any Λ-module K define the Λ-module morphism

$$e_K \;:\; K \to K^{**} \;;\; x \longmapsto (f \longmapsto \overline{f(x)}) \;. \qquad \square$$

The morphism e_K is natural in the sense that for any Λ-module morphism $f : K \to L$ there is defined a commutative diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "e_K"'] & L \arrow[d, "e_L"] \\ K^{**} \arrow[r, "f^{**}"] & L^{**} \end{tikzcd}$$

DEFINITION 1.10 (i) A Λ-module K is *f. g. projective* if there exists a Λ-module L such that $K \oplus L$ is isomorphic to the f. g. free Λ-module Λ^n, for some $n \geq 0$.
(ii) A Λ-module K is *stably f. g. free* if $K \oplus \Lambda^m$ is isomorphic to Λ^n, for some $m, n \geq 0$. $\square$

In particular, f. g. free Λ-modules are stably f. g. free, and stably f. g. free Λ-modules are f. g. projective.

PROPOSITION 1.11 *The dual of a f. g. projective Λ-module K is a f. g. projective Λ-module K^*, and $e_K : K \to K^{**}$ is an isomorphism. Moreover, if K is stably f. g. free then so is K^*.*
PROOF: For any Λ-modules K, L there are evident identifications

$$(K \oplus L)^* \;=\; K^* \oplus L^* \;,$$
$$e_{K\oplus L} \;=\; e_K \oplus e_L \;:\; K \oplus L \to (K \oplus L)^{**} \;=\; K^{**} \oplus L^{**} \;,$$

so it suffices to consider the special case $K = \Lambda$. The Λ-module isomorphism

$$f \;:\; \Lambda \xrightarrow{\cong} \Lambda^* \;;\; a \longmapsto (b \longmapsto b\overline{a}) \;,$$

can be used to construct an explicit inverse for e_Λ

$$(e_\Lambda)^{-1} \ : \ \Lambda^{**} \to \Lambda \ ; \ g \longmapsto g(f(1)) \ . \qquad \square$$

In dealing with f. g. projective Λ-modules K use the natural isomorphism $e_K : K \cong K^{**}$ to identify $K^{**} = K$. For any morphism $f : K \to L$ of f. g. projective Λ-modules there is a corresponding identification

$$f^{**} \ = \ f \ : \ K^{**} \ = \ K \to L^{**} \ = \ L \ .$$

Remark 1.12 The additive group $\mathrm{Hom}_\Lambda(\Lambda^m,\Lambda^n)$ of the morphisms $\Lambda^m \to \Lambda^n$ between f. g. free Λ-modules Λ^m,Λ^n may be identified with the additive group $M_{m,n}(\Lambda)$ of $m \times n$ matrices $(a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ with entries $a_{ij} \in \Lambda$, using the isomorphism

$$\begin{aligned}&M_{m,n}(\Lambda) \xrightarrow{\cong} \mathrm{Hom}_\Lambda(\Lambda^m,\Lambda^n) \ ;\\ &(a_{ij}) \longmapsto ((x_1,x_2,\dots,x_m) \longmapsto (\sum_{i=1}^m x_i a_{i1}, \sum_{i=1}^m x_i a_{i2}, \dots, \sum_{i=1}^m x_i a_{in})) \ .\end{aligned}$$

The composition of morphisms

$$\begin{aligned}\mathrm{Hom}_\Lambda(\Lambda^m,\Lambda^n) \times \mathrm{Hom}_\Lambda(\Lambda^n,\Lambda^p) &\to \mathrm{Hom}_\Lambda(\Lambda^m,\Lambda^p) \ ;\\ (f,g) &\longmapsto \big(gf : x \longmapsto (gf)(x) = g(f(x))\big)\end{aligned}$$

corresponds to the multiplication of matrices

$$M_{m,n}(\Lambda) \times M_{n,p}(\Lambda) \to M_{m,p}(\Lambda) \ ; \ ((a_{ij}),(b_{jk})) \longmapsto (c_{ik})$$

$$(\, c_{ik} \ = \ \sum_{j=1}^n a_{ij} b_{jk} \ \ (1 \leq i \leq m, \ 1 \leq k \leq p)\,) \ .$$

Use the isomorphism of f. g. free Λ-modules

$$\Lambda^m \xrightarrow{\cong} (\Lambda^m)^* \ ; \ (x_1,x_2,\dots,x_m) \longmapsto ((y_1,y_2,\dots,y_m) \ \longmapsto \sum_{i=1}^m y_i \overline{x}_i)$$

to identify

$$(\Lambda^m)^* \ = \ \Lambda^m \ .$$

The duality isomorphism

$$\begin{aligned}* \ : \ \mathrm{Hom}_\Lambda(\Lambda^m,\Lambda^n) &\xrightarrow{\cong} \mathrm{Hom}_\Lambda((\Lambda^n)^*,(\Lambda^m)^*) \ = \ \mathrm{Hom}_\Lambda(\Lambda^n,\Lambda^m) \ ;\\ f &\longmapsto f^*\end{aligned}$$

can be identified with the isomorphism defined by conjugate transposition of matrices

$$M_{m,n}(\Lambda) \xrightarrow{\cong} M_{n,m}(\Lambda) \ ; \ \alpha \ = \ (a_{ij}) \longmapsto \alpha^* \ = \ (b_{ji}) \ , \ b_{ji} = \overline{a}_{ij} \ . \qquad \square$$

EXAMPLE 1.13 A 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(\Lambda)$$

corresponds to the Λ-module morphism

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \Lambda \oplus \Lambda \to \Lambda \oplus \Lambda \; ; \; (x,y) \longmapsto (xa + yb, xc + yd) \; .$$

The conjugate transpose matrix

$$\begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \in M_{2,2}(\Lambda)$$

corresponds to the dual Λ-module morphism

$$f^* = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} : (\Lambda \oplus \Lambda)^* = \Lambda \oplus \Lambda \to (\Lambda \oplus \Lambda)^* = \Lambda \oplus \Lambda \; ;$$

$$(x,y) \longmapsto (x\overline{a} + y\overline{c}, x\overline{b} + y\overline{d}) \; . \qquad \square$$

The dual of a chain complex of modules over a ring with involution Λ

$$C : \ldots \longrightarrow C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \longrightarrow \ldots$$

is the cochain complex

$$C^* : \ldots \longrightarrow C^{r-1} \xrightarrow{d^*} C^r \xrightarrow{d^*} C^{r+1} \longrightarrow \ldots$$

with

$$C^r = (C_r)^* = \mathrm{Hom}_\Lambda(C_r, \Lambda) \; .$$

DEFINITION 1.14 An *m-dimensional geometric Poincaré complex* (Wall [28]) is a finite CW complex X with an orientation character $w(X) : \pi_1(X) \to \mathbb{Z}_2$ and a $w(X)$-twisted fundamental class $[X] \in H_m(X; \mathbb{Z}^{w(X)})$ such that cap product defines $\mathbb{Z}[\pi_1(X)]$-module isomorphisms

$$[X] \cap - : H^*_{w(X)}(\widetilde{X}) \xrightarrow{\cong} H_{m-*}(\widetilde{X})$$

with $\widetilde{X}$ the universal cover of X. The $w(X)$-twisted cohomology groups are given by

$$H^*_{w(X)}(\widetilde{X}) = H^*(C(\widetilde{X})^*)$$

with $C(\widetilde{X})$ the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex, using the $w(X)$-twisted involution on $\mathbb{Z}[\pi_1(X)]$ (1.5) to define the left $\mathbb{Z}[\pi_1(X)]$-module structure on the dual cochain complex

$$C(\widetilde{X})^* = \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(C(\widetilde{X}), \mathbb{Z}[\pi_1(X)]) \; . \qquad \square$$

The orientation character $w(X) : \pi_1(X) \to \mathbb{Z}_2$ sends a loop $g : S^1 \to X$ to $w(g) = +1$ (resp. $= -1$) if g is orientation-preserving (resp. orientation-reversing).

An oriented Poincaré complex X (1.1) is just a Poincaré complex (1.14) with $w(X) = +1$.

Example 1.15 A compact m-dimensional manifold is an m-dimensional geometric Poincaré complex. □

§2. Quadratic forms

In the first instance suppose that the ground ring Λ is commutative, with the identity involution $\bar{a} = a$ (1.4). A symmetric form (K, λ) over Λ is a Λ-module K together with a bilinear pairing

$$\lambda \ : \ K \times K \to \Lambda \ ; \ (x, y) \longmapsto \lambda(x, y)$$

such that for all $x, y, z \in K$ and $a \in \Lambda$

$$\begin{aligned} \lambda(x, ay) \ &= \ a\lambda(x, y) \ , \\ \lambda(x, y + z) \ &= \ \lambda(x, y) + \lambda(x, z) \ , \\ \lambda(x, y) \ &= \ \lambda(y, x) \in \Lambda \ . \end{aligned}$$

A quadratic form (K, λ, μ) over Λ is a symmetric form (K, λ) together with a function

$$\mu \ : \ K \to Q_{+1}(\Lambda) \ = \ \Lambda \ ; \ x \longmapsto \mu(x)$$

such that for all $x, y \in K$ and $a \in \Lambda$

$$\begin{aligned} \mu(x + y) \ &= \ \mu(x) + \mu(y) + \lambda(x, y) \ , \\ \mu(ax) \ &= \ a^2\mu(x) \in Q_{+1}(\Lambda) \ . \end{aligned}$$

In particular, for every $x \in K$

$$2\mu(x) \ = \ \lambda(x, x) \in Q^{+1}(\Lambda) \ = \ \Lambda \ .$$

If $2 \in \Lambda$ is invertible (e.g. if Λ is a field of characteristic $\neq 2$, such as $\mathbb{R}, \mathbb{C}, \mathbb{Q}$) there is no difference between symmetric and quadratic forms, with μ determined by λ according to $\mu(x) = \lambda(x, x)/2$.

A symplectic form (K, λ) over a commutative ring Λ is a Λ-module K together with a bilinear pairing $\lambda : K \times K \to \Lambda$ such that for all $x, y, z \in K$ and $a \in \Lambda$

$$\begin{aligned} \lambda(x, at) \ &= \ a\lambda(x, y) \ , \\ \lambda(x, y + z) \ &= \ \lambda(x, y) + \lambda(x, z) \ , \\ \lambda(x, y) \ &= \ -\lambda(y, x) \in \Lambda \ . \end{aligned}$$

A (-1)-quadratic form (K,λ,μ) over Λ is a symplectic form (K,λ) together with a function

$$\mu \ : \ K \to Q_{-1}(\Lambda) \ = \ \Lambda/\{2a \,|\, a \in \Lambda\} \ ; \ x \longmapsto \mu(x)$$

such that for all $x,y \in K$ and $a \in \Lambda$

$$\begin{aligned} \mu(x+y) \ &= \ \mu(x) + \mu(y) + \lambda(x,y) \ , \\ \mu(ax) \ &= \ a^2\mu(x) \in Q_{-1}(\Lambda) \ . \end{aligned}$$

In particular, for every $x \in K$

$$2\mu(x) \ = \ \lambda(x,x) \in Q^{-1}(\Lambda) \ = \ \{a \in \Lambda \,|\, 2a = 0\} \ .$$

If $2 \in \Lambda$ is invertible then $Q_{-1}(\Lambda) = 0$ and there is no difference between symplectic and (-1)-quadratic forms, with $\mu = 0$.

In the applications of forms to surgery theory it is necessary to work with quadratic and (-1)-quadratic forms over noncommutative group rings with the involution as in 1.5. §2 develops the general theory of forms over rings with involution, taking account of these differences.

Let X be an m-dimensional geometric Poincaré complex with universal cover $\widetilde{X}$ and fundamental group ring $\Lambda = \mathbb{Z}[\pi_1(X)]$, with the $w(X)$-twisted involution. The Poincaré duality isomorphism

$$\phi \ = \ [X] \cap - \ : \ H^{m-r}_{w(X)}(\widetilde{X}) \xrightarrow{\cong} H_r(\widetilde{X})$$

and the evaluation pairing

$$H^r_{w(X)}(\widetilde{X}) \to H_r(\widetilde{X})^* \ = \ \mathrm{Hom}_\Lambda(H_r(\widetilde{X}),\Lambda) \ ; y \longmapsto (x \longmapsto \langle y,x\rangle)$$

can be combined to define a sesquilinear pairing

$$\lambda \ : \ H_r(\widetilde{X}) \times H_{m-r}(\widetilde{X}) \to \Lambda \ ; \ (x,\phi(y)) \longmapsto \langle y,x\rangle$$

such that

$$\lambda(y,x) \ = \ (-1)^{r(m-r)}\overline{\lambda(x,y)}$$

with $\Lambda \to \Lambda; a \longmapsto \bar{a}$ the involution of 1.5.

If M is an m-dimensional manifold with fundamental group ring $\Lambda = \mathbb{Z}[\pi_1(M)]$ the pairing $\lambda : H_r(\widetilde{M}) \times H_{m-r}(\widetilde{M}) \to \Lambda$ can be interpreted geometrically using the geometric intersection numbers of cycles. For any two immersions $x : S^r \looparrowright \widetilde{M}$, $y : S^{m-r} \looparrowright \widetilde{M}$ in general position

$$\lambda(x,y) \ = \ \sum_{g \in \pi_1(M)} n_g g \in \Lambda$$

with $n_g \in \mathbb{Z}$ the algebraic number of intersections in $\widetilde{M}$ of x and gy. In particular, for $m = 2n$ there is defined a $(-1)^n$-symmetric pairing

$$\lambda \ : \ H_n(\widetilde{M}) \times H_n(\widetilde{M}) \to \Lambda$$

which is relevant to surgery in the middle dimension n. An element $x \in \pi_n(M)$ can be killed by surgery if and only if it is represented by an embedding $S^n \times D^n \hookrightarrow M^{2n}$. The condition that the Hurewicz image $x \in H_n(\widetilde{M})$ be such that $\lambda(x, x) = 0 \in \Lambda$ is necessary but not sufficient to kill $x \in \pi_n(M)$ by surgery. The theory of forms developed in §2 is required for an algebraic formulation of the necessary and sufficient condition for an element in the kernel $K_n(M)$ of an n-connected $2n$-dimensional normal map $(f, b) : M \to X$ to be killed by surgery, assuming $n \geq 3$.

As in §1 let Λ be a ring with involution, not necessarily a group ring.

DEFINITION 2.1 A *sesquilinear pairing* (K, L, λ) on Λ-modules K, L is a function

$$\lambda \ : \ K \times L \to \Lambda \ ; \ (x, y) \longmapsto \lambda(x, y)$$

such that for all $w, x \in K$, $y, z \in L$, $a, b \in \Lambda$

(i) $\lambda(w + x, y + z) \ = \ \lambda(w, y) + \lambda(w, z) + \lambda(x, y) + \lambda(x, z) \in \Lambda$,

(ii) $\lambda(ax, by) \ = \ b\lambda(x, y)\overline{a} \in \Lambda$.

The *dual* (or *transpose*) sesquilinear pairing is

$$T\lambda \ : \ L \times K \to \Lambda \ ; \ (y, x) \longmapsto T\lambda(y, x) \ = \ \overline{\lambda(x, y)} \ . \qquad \square$$

DEFINITION 2.2 Given Λ-modules K, L let $S(K, L)$ be the additive group of sesquilinear pairings $\lambda : K \times L \to \Lambda$. Transposition defines an isomorphism

$$T \ : \ S(K, L) \ \cong \ S(L, K)$$

such that

$$T^2 \ = \ \text{id.} \ : \ S(K, L) \xrightarrow{\cong} S(L, K) \xrightarrow{\cong} S(K, L) \ . \qquad \square$$

PROPOSITION 2.3 *For any Λ-modules K, L there is a natural isomorphism of additive groups*

$$S(K, L) \xrightarrow{\cong} \mathrm{Hom}_\Lambda(K, L^*) \ ;$$
$$(\lambda \ : \ K \times L \to \Lambda) \longmapsto (\lambda \ : \ K \to L^* \, ; \, x \longmapsto (\, y \longmapsto \lambda(x, y)\,))\, .$$

For f. g. projective K, L the transposition isomorphism $T : S(K, L) \cong S(L, K)$ corresponds to the duality isomorphism

$$* : \mathrm{Hom}_\Lambda(K, L^*) \xrightarrow{\cong} \mathrm{Hom}_\Lambda(L, K^*) \, ;$$
$$(\lambda : K \to L^*) \longmapsto (\lambda^* \ : \ L \ \to \ K^* \, ; \, y \longmapsto (x \ \longmapsto \ \overline{\lambda(y, x)}))\, . \qquad \square$$

Use 2.3 to identify

$$S(K,L) \;=\; \mathrm{Hom}_\Lambda(K,L^*)\ ,\ S(K) \;=\; \mathrm{Hom}_\Lambda(K,K^*)\ ,$$
$$Q^\epsilon(K) \;=\; \ker(1-T_\epsilon : \mathrm{Hom}_\Lambda(K,K^*) \to \mathrm{Hom}_\Lambda(K,K^*))\ ,$$
$$Q_\epsilon(K) \;=\; \mathrm{coker}(1-T_\epsilon : \mathrm{Hom}_\Lambda(K,K^*) \to \mathrm{Hom}_\Lambda(K,K^*))$$

for any f. g. projective Λ-modules K,L.

REMARK 2.4 For f. g. free Λ-modules Λ^m, Λ^n it is possible to identify $S(\Lambda^m,\Lambda^n)$ with the additive group $M_{m,n}(\Lambda)$ of $m\times n$ matrices (a_{ij}) with entries $a_{ij}\in\Lambda$, using the isomorphism

$$M_{m,n}(\Lambda) \xrightarrow{\cong} S(\Lambda^m,\Lambda^n)\ ;\ (a_{ij}) \longmapsto \lambda$$

defined by

$$\lambda((x_1,x_2,\dots,x_m),(y_1,y_2,\dots,y_n)) \;=\; \sum_{i=1}^m\sum_{j=1}^n y_j a_{ij}\overline{x}_i\ .$$

The transposition isomorphism $T : S(\Lambda^m,\Lambda^n) \cong S(\Lambda^n,\Lambda^m)$ corresponds to the isomorphism defined by conjugate transposition of matrices

$$T \;:\; M_{m,n}(\Lambda) \xrightarrow{\cong} M_{n,m}(\Lambda)\ ;\ (a_{ij}) \longmapsto (b_{ji})\ ,\ b_{ji} \;=\; \overline{a}_{ij}\ . \qquad \square$$

The group $S(K,L)$ is particularly significant in the case $K=L$:

DEFINITION 2.5 (i) Given a Λ-module K let

$$S(K) \;=\; S(K,K)$$

be the abelian group of sesquilinear pairings $\lambda : K\times K\to\Lambda$.
(ii) The *ϵ-transposition involution* is given for $\epsilon=\pm1$ by

$$T_\epsilon \;:\; S(K) \xrightarrow{\cong} S(K)\ ;\ \lambda \longmapsto T_\epsilon\lambda \;=\; \epsilon(T\lambda)\ ,$$

such that

$$T_\epsilon\lambda(x,y) \;=\; \epsilon\overline{\lambda(y,x)}\in\Lambda\ ,\ (T_\epsilon)^2 \;=\; \mathrm{id} \;:\; S(K)\to S(K)\ . \qquad \square$$

DEFINITION 2.6 The *ϵ-symmetric group* of a Λ-module K is the additive group

$$Q^\epsilon(K) \;=\; \ker(1-T_\epsilon : S(K)\to S(K))\ .$$

The *ϵ-quadratic group* of K is the additive group

$$Q_\epsilon(K) \;=\; \mathrm{coker}(1-T_\epsilon : S(K)\to S(K))\ .$$

The *ϵ-symmetrization morphism* is given by

$$1+T_\epsilon \;:\; Q_\epsilon(K)\to Q^\epsilon(K)\ ;\ \psi \longmapsto \psi+T_\epsilon\psi\ . \qquad \square$$

For $\epsilon = +1$ it is customary to refer to ϵ-symmetric and ϵ-quadratic objects as symmetric and quadratic, as in the commutative case.

For $K = \Lambda$ there is an isomorphism of additive groups with involution

$$\Lambda \xrightarrow{\cong} S(\Lambda) \ ; \ a \longmapsto ((x,y) \longmapsto ya\overline{x})$$

allowing the identifications

$$\begin{aligned} Q^{\epsilon}(\Lambda) &= \{a \in \Lambda \,|\, \epsilon\overline{a} = a\} \ , \\ Q_{\epsilon}(\Lambda) &= \Lambda/\{a - \epsilon\overline{a} \,|\, a \in \Lambda\} \ , \\ 1 + T_{\epsilon} &: Q_{\epsilon}(\Lambda) \to Q^{\epsilon}(\Lambda) \ ; \ a \longmapsto a + \epsilon\overline{a} \ . \end{aligned}$$

Example 2.7 Let $\Lambda = \mathbb{Z}$. The ϵ-symmetric and ϵ-quadratic groups of $K = \mathbb{Z}$ are given by

$$\begin{aligned} Q^{\epsilon}(\mathbb{Z}) &= \begin{cases} \mathbb{Z} & \text{if } \epsilon = +1 \\ 0 & \text{if } \epsilon = -1 \ , \end{cases} \\ Q_{\epsilon}(\mathbb{Z}) &= \begin{cases} \mathbb{Z} & \text{if } \epsilon = +1 \\ \mathbb{Z}/2 & \text{if } \epsilon = -1 \end{cases} \end{aligned}$$

with generators represented by $1 \in \mathbb{Z}$, and with

$$1 + T_{+} = 2 : Q_{+1}(\mathbb{Z}) = \mathbb{Z} \to Q^{+1}(\mathbb{Z}) = \mathbb{Z} \ . \qquad \square$$

Definition 2.8 An *ϵ-symmetric form* (K, λ) over Λ is a Λ-module K together with an element $\lambda \in Q^{\epsilon}(K)$. Thus λ is a sesquilinear pairing

$$\lambda : K \times K \to \Lambda \ ; \ (x,y) \longmapsto \lambda(x,y)$$

such that for all $x, y \in K$

$$\lambda(x,y) = \epsilon\overline{\lambda(y,x)} \in \Lambda \ .$$

The *adjoint* of (K, λ) is the Λ-module morphism

$$K \to K^* \ ; \ x \longmapsto (y \longmapsto \lambda(x,y))$$

which is also denoted by λ. The form is *nonsingular* if $\lambda : K \to K^*$ is an isomorphism. $\square$

Unless specified otherwise, only forms (K, λ) with K a f. g. projective Λ-module will be considered.

Example 2.9 The symmetric form (Λ, λ) defined by

$$\lambda = 1 : \Lambda \to \Lambda^* \ ; \ a \longmapsto (b \longmapsto b\overline{a})$$

is nonsingular. $\square$

DEFINITION 2.10 For any f. g. projective Λ-module L define the nonsingular *hyperbolic ϵ-symmetric form*

$$H^\epsilon(L) \ = \ (L \oplus L^*, \lambda)$$

by

$$\lambda \ = \ \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \ : \ L \oplus L^* \to (L \oplus L^*)^* \ = \ L^* \oplus L \ ;$$

$$(x, f) \longmapsto ((y, g) \longmapsto f(y) + \epsilon \overline{g(x)}) \ . \qquad \square$$

EXAMPLE 2.11 Let X be an m-dimensional geometric Poincaré complex, and let $\widetilde{X}$ be a regular oriented covering of X with group of covering translations π and orientation character $w : \pi \to \mathbb{Z}_2$. An element $g \in \pi$ has $w(g) = +1$ (resp. -1) if and only if the covering translation $g : \widetilde{X} \to \widetilde{X}$ is orientation-preserving (resp. reversing).
(i) Cap product with the fundamental class $[X] \in H_m(X; \mathbb{Z}^w)$ defines the Poincaré duality $\mathbb{Z}[\pi]$-module isomorphisms

$$[X] \cap - \ : \ H^{m-*}_w(\widetilde{X}) \xrightarrow{\cong} H_*(\widetilde{X}) \ .$$

If $m = 2n$ and X is a manifold geometric intersection numbers define a $(-1)^n$-symmetric form $(H_n(\widetilde{X}), \lambda)$ over $\mathbb{Z}[\pi]$ with adjoint the composite

$$\lambda \ : \ H_n(\widetilde{X}) \xrightarrow{([X]\cap -)^{-1}} H^n_w(\widetilde{X}) \xrightarrow{\text{evaluation}} H_n(\widetilde{X})^* \ .$$

(ii) In general $H_n(\widetilde{X})$ is not a f. g. projective $\mathbb{Z}[\pi]$-module. If $H_n(\widetilde{X})$ is f. g. projective then the evaluation map is an isomorphism, and $(H_n(\widetilde{X}), \lambda)$ is a nonsingular form. $\square$

REMARK 2.12 (i) Let M be a $2n$-dimensional manifold, with universal cover $\widetilde{M}$ and intersection pairing $\lambda : H_n(\widetilde{M}) \times H_n(\widetilde{M}) \to \mathbb{Z}[\pi_1(M)]$. An element $x \in \operatorname{im}(\pi_n(M) \to H_n(\widetilde{M}))$ can be killed by surgery if and only if it can be represented by an embedding $x : S^n \times D^n \hookrightarrow M$, in which case the homology class $x \in H_n(\widetilde{M})$ is such that $\lambda(x, x) = 0$. However, the condition $\lambda(x, x) = 0$ given by the symmetric structure alone is not sufficient for the existence of such an embedding – see (ii) below for an explicit example.
(ii) The intersection form over $\mathbb{Z}$ for $M^{2n} = S^n \times S^n$ is the hyperbolic form (2.10)

$$(H_n(S^n \times S^n), \lambda) \ = \ H^{(-1)^n}(\mathbb{Z}) \ .$$

The element $x = (1, 1) \in H_n(S^n \times S^n)$ is such that

$$\lambda(x, x) \ = \ \chi(S^n) \ = \ 1 + (-1)^n \in \mathbb{Z} \ ,$$

so that $\lambda(x, x) = 0$ for odd n. The diagonal embedding $\Delta : S^n \hookrightarrow S^n \times S^n$ has normal bundle $\nu_\Delta = \tau_{S^n} : S^n \to BO(n)$, which is non-trivial for

$n \neq 1, 3, 7$, so that it is not possible to kill $x = \Delta_*[S^n] \in H_n(S^n \times S^n)$ by surgery in these dimensions. □

DEFINITION 2.13 An ϵ-*quadratic form* (K, λ, μ) over Λ is an ϵ-symmetric form (K, λ) together with a function

$$\mu \ : \ K \to Q_\epsilon(\Lambda) \ ; \ x \longmapsto \mu(x)$$

such that for all $x, y \in K$, $a \in \Lambda$

$$\begin{aligned} \text{(i)} \ \ \mu(x+y) - \mu(x) - \mu(y) \ &= \ \lambda(x,y) \in Q_\epsilon(\Lambda) \ , \\ \text{(ii)} \ \ \mu(x) + \epsilon\overline{\mu(x)} \ &= \ \lambda(x,x) \in \operatorname{im}(1 + T_\epsilon : Q_\epsilon(\Lambda) \to Q^\epsilon(\Lambda)) \ , \\ \text{(iii)} \ \ \mu(ax) \ &= \ a\mu(x)\overline{a} \in Q_\epsilon(\Lambda) \ . \end{aligned}$$

□

DEFINITION 2.14 For any f. g. projective Λ-module L define the nonsingular *hyperbolic* ϵ-*quadratic form* over Λ by

$$H_\epsilon(L) \ = \ (L \oplus L^*, \lambda, \mu)$$

with

$$\begin{aligned} \lambda \ &= \ \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \ : \ L \oplus L^* \to (L \oplus L^*)^* \ = \ L^* \oplus L \ ; \\ &\qquad\qquad (x, f) \longmapsto ((y, g) \longmapsto f(y) + \epsilon\overline{g(x)}) \ , \\ \mu \ &: \ L \oplus L^* \to Q_\epsilon(\Lambda) \ ; \ (x, f) \longmapsto f(x) \ . \end{aligned}$$

$(L \oplus L^*, \lambda)$ is the hyperbolic ϵ-symmetric form $H^\epsilon(L)$ of 2.10. □

EXAMPLE 2.15 (Wall [29, Chapter 5]) An n-connected normal map $(f, b) : M^{2n} \to X$ from a $2n$-dimensional manifold with boundary $(M, \partial M)$ to a geometric Poincaré pair $(X, \partial X)$ with $\partial f = f| : \partial M \to \partial X$ a homotopy equivalence determines a $(-1)^n$-quadratic form $(K_n(M), \lambda, \mu)$ over $\Lambda = \mathbb{Z}[\pi_1(X)]$ with the $w(X)$-twisted involution (1.5), with

$$K_n(M) \ = \ \pi_{n+1}(f) \ = \ H_{n+1}(\widetilde{f}) \ = \ \ker(\widetilde{f}_* : H_n(\widetilde{M}) \to H_n(\widetilde{X}))$$

the stably f. g. free kernel Λ-module, and $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ a $\pi_1(X)$-equivariant lift of f to the universal covers. Note that $K_n(M) = 0$ if (and for $n \geq 2$ only if) $f : M \to X$ is a homotopy equivalence, by the theorem of J.H.C. Whitehead.

(i) The pairing $\lambda : K_n(M) \times K_n(M) \to \Lambda$ is defined by geometric intersection numbers, as follows. Every element $x \in K_n(M)$ is represented by an X-nullhomotopic framed immersion $g : S^n \looparrowright M$ with a choice of path in $g(S^n) \subset M$ from the base point $* \in M$ to $g(1) \in M$. Any two elements $x, y \in K_n(M)$ can be represented by such immersions $g, h : S^n \looparrowright M$ with transverse intersections and self-intersections. The intersection of g and h

$$D(g, h) \ = \ \{(a, b) \in S^n \times S^n \,|\, g(a) = h(b) \in M\}$$

is finite. For each intersection point $(a,b)\in D(g,h)$ let

$$\gamma(a,b)\in\pi_1(M)\ =\ \pi_1(X)$$

be the homotopy class of the loop in M obtained by joining the path in $g(S^n)\subset M$ from the base point $*\in M$ to $g(a)$ to the path in $h(S^n)\subset M$ from $h(b)$ back to the base point. Choose an orientation for $\tau_*(M)$ and transport it to an orientation for $\tau_{g(a)}(M)=\tau_{h(b)}(M)$ by the path for g, and let

$$\epsilon(a,b)\ =\ [\tau_a(S^n)\oplus\tau_b(S^n):\tau_{g(a)}(M)]\in\{\pm1\}$$

be $+1$ (resp. -1) if the isomorphism

$$(dg\ dh)\ :\tau_a(S^n)\oplus\tau_b(S^n)\xrightarrow{\cong}\tau_{g(a)}(M)$$

is orientation-preserving (resp. reversing). The *geometric intersection* of $x,y\in K_n(M)$ is given by

$$\lambda(x,y)\ =\sum_{(a,b)\in D(g,h)}I(a,b)\in\Lambda$$

with

$$I(a,b)\ =\ \epsilon(a,b)\gamma(a,b)\in\Lambda\ .$$

It follows from

$$\begin{aligned}
\epsilon(b,a)\ &=\ [\tau_bS^n\oplus\tau_aS^n:\tau_aS^n\oplus\tau_bS^n]\epsilon(a,b)\\
&=\ \det(\begin{pmatrix}0&1\\1&0\end{pmatrix}:\mathbb{R}^n\oplus\mathbb{R}^n\to\mathbb{R}^n\oplus\mathbb{R}^n)\epsilon(a,b)\\
&=\ (-1)^n\epsilon(a,b)\in\{\pm1\}\ ,\\
\gamma(b,a)\ &=\ w(X)(\gamma(a,b))\gamma(a,b)^{-1}\in\pi_1(X)\ ,\\
I(b,a)&=(-1)^n\overline{I(a,b)}\in\Lambda
\end{aligned}$$

that

$$\lambda(y,x)\ =\ (-1)^n\overline{\lambda(x,y)}\in\Lambda$$

(which also holds from purely homological considerations).
(ii) The quadratic function $\mu:K_n(M)\to Q_{(-1)^n}(\Lambda)$ is defined by geometric self-intersection numbers, as follows. Represent $x\in K_n(M)$ by an immersion $g:S^n\looparrowright M$ as in (i), with transverse self-intersections. The double point set of g

$$\begin{aligned}
D_2(g)\ &=\ D(g,g)\backslash\Delta(S^n)\\
&=\ \{(a,b)\in S^n\times S^n\,|\,a\neq b\in S^n, g(a)=g(b)\in M\}
\end{aligned}$$

is finite, with a free $\mathbb{Z}_2$-action $(a,b)\mapsto(b,a)$. For each $(a,b)\in D_2(g)$ let $\gamma(a,b)$ be the loop in M obtained by transporting to the base point the image under g of a path in S^n from a to b. The *geometric self-intersection* of x is defined by

$$\mu(x) \ = \sum_{(a,b)\in D_2(g)/\mathbb{Z}_2} I(a,b) \in Q_{(-1)^n}(\Lambda) \ ,$$

with $I(a,b) = \epsilon(a,b)\gamma(a,b)$ as in (i). Note that $\mu(x)$ is independent of the choice of ordering of (a,b) since $I(b,a) \ = \ (-1)^n\overline{I(a,b)} \in \Lambda$.

(iii) The kernel $(-1)^n$-quadratic form $(K_n(M),\lambda,\mu)$ is such that $\mu(x)=0$ if (and for $n\geq 3$ only if) $x\in K_n(M)$ can be killed by surgery on $S^n\subset M^{2n}$, i.e. represented by an embedding $S^n\times D^n\hookrightarrow M$ with a nullhomotopy in X – the condition $\mu(x)=0$ allows the double points of a representative framed immersion $g: S^n \looparrowright M$ to be matched in pairs, which for $n\geq 3$ can be cancelled by the Whitney trick. The effect of the surgery is a bordant $(n-1)$-connected normal map

$$(f',b') \ : \ M'^{2n} \ = \ \mathrm{cl.}(M\backslash S^n\times D^n)\cup D^{n+1}\times S^{n-1}\to X$$

with kernel Λ-modules

$$K_i(M') \ = \ \begin{cases} \mathrm{coker}(x^*\lambda : K_n(M)\to\Lambda^*) & \text{if } i=n-1 \\ \dfrac{\ker(x^*\lambda : K_n(M)\to\Lambda^*)}{\mathrm{im}(x:\Lambda\to K_n(M))} & \text{if } i=n \\ \ker(x:\Lambda\to K_n(M)) & \text{if } i=n+1 \\ 0 & \text{otherwise .}\end{cases}$$

Thus (f',b') is n-connected if and only if x generates a direct summand $L=\langle x\rangle\subset K_n(M)$, in which case L is a sublagrangian of $(K_n(M),\lambda,\mu)$ in the terminology of §5, with

$$\begin{aligned} L\subseteq L^{\perp} \ &= \ \{y\in K_n(M)\,|\,\lambda(x,y)=0\} \ , \\ (K_n(M'),\lambda',\mu') \ &= \ (L^{\perp}/L,[\lambda],[\mu]) \ , \\ (K_n(M),\lambda,\mu) \ &\cong \ (K_n(M'),\lambda',\mu')\oplus H_{(-1)^n}(\Lambda) \ . \end{aligned}$$

(iv) The effect on (f,b) of a surgery on an X-nullhomotopic embedding $S^{n-1}\times D^{n+1}\hookrightarrow M$ is an n-connected bordant normal map

$$(f'',b'') \ : \ M''^{2n} \ = \ \mathrm{cl.}(M\backslash S^{n-1}\times D^{n+1})\cup D^n\times S^n \ = \ M\#(S^n\times S^n)\to X$$

with kernel Λ-modules

$$K_i(M'') \ = \ \begin{cases} K_n(M)\oplus\Lambda\oplus\Lambda^* & \text{if } i=n \\ 0 & \text{otherwise}\end{cases}$$

and kernel $(-1)^n$-quadratic form

$$(K_n(M''),\lambda'',\mu'') \ = \ (K_n(M),\lambda,\mu)\oplus H_{(-1)^n}(\Lambda) \ .$$

(v) The main result of even-dimensional surgery obstruction theory is that for $n\geq 3$ an n-connected $2n$-dimensional normal map $(f,b): M^{2n}\to X$ is normal bordant to a homotopy equivalence if and only if there exists an isomorphism of $(-1)^n$-quadratic forms over $\Lambda=\mathbb{Z}[\pi_1(X)]$ of the type

$$(K_n(M),\lambda,\mu)\oplus H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} H_{(-1)^n}(\Lambda^{k'})$$

for some $k, k' \geq 0$. □

Example 2.16 There is also a relative version of 2.15. An n-connected $2n$-dimensional normal map of pairs $(f,b) : (M^{2n}, \partial M) \to (X, \partial X)$ has a kernel $(-1)^n$-quadratic form $(K_n(M), \lambda, \mu)$ over $\mathbb{Z}[\pi_1(X)]$ is nonsingular if and only if $\partial f : \partial M \to \partial X$ is a homotopy equivalence (assuming $\pi_1(\partial X) \cong \pi_1(X)$). □

Remark 2.17 (Realization of even-dimensional surgery obstructions, Wall [29, 5.8])
(i) Let X^{2n-1} be a $(2n-1)$-dimensional manifold, and suppose given an embedding $e : S^{n-1} \times D^n \hookrightarrow X$, together with a null-homotopy δe of $e| : S^{n-1} \hookrightarrow X$ and a null-homotopy of the map $S^{n-1} \to O$ comparing the (stable) trivializations of $\nu_{e|} : S^{n-1} \to BO(n)$ given by e and δe. Then there is defined an n-connected $2n$-dimensional normal map

$$(f,b) \ : \ (M; \partial_- M, \partial_+ M) \to X \times ([0,1]; \{0\}, \{1\})$$

with

$$\begin{aligned} \partial_- f \ &= \ \text{id.} \ : \ \partial_- M \ = \ X \to X \ , \\ M^{2n} \ &= \ X \times [0,1] \cup_e D^n \times D^n \ , \\ \partial_+ M \ &= \ \text{cl.}(X \backslash e(S^{n-1} \times D^n)) \cup D^n \times S^{n-1} \ . \end{aligned}$$

The kernel $(-1)^n$-quadratic form (Λ, λ, μ) over Λ (2.16) is the (self-)-intersection of the framed immersion $S^{n-1} \times [0,1] \looparrowright X \times [0,1]$ defined by the track of a regular homotopy $e_0 \simeq e : S^{n-1} \times D^n \to X$ from a trivial unlinked embedding

$$e_0 \ : \ S^{n-1} \times D^n \hookrightarrow S^{2n-1} \ = \ S^{n-1} \times D^n \cup D^n \times S^{n-1} \hookrightarrow X \# S^{2n-1} \ = \ X \ .$$

Moreover, every form (Λ, λ, μ) arises in this way: starting with e_0 construct a regular homotopy $e_0 \simeq e$ to a (self-)linked embedding e such that the track has (self-)intersection (λ, μ).
(ii) Let (K, λ, μ) be a $(-1)^n$-quadratic form over $\mathbb{Z}[\pi]$, with π a finitely presented group and $K = \mathbb{Z}[\pi]^k$ f. g. free. Let $n \geq 3$, so that there exists a $(2n-1)$-dimensional manifold X^{2n-1} with $\pi_1(X) = \pi$. For any such $n \geq 3$, X there exists an n-connected $2n$-dimensional normal map

$$(f,b) \ : \ (M^{2n}; \partial_- M, \partial_+ M) \to X^{2n-1} \times ([0,1]; \{0\}, \{1\})$$

with kernel form (K, λ, μ) and

$$\begin{aligned} \partial_- f \ &= \ \text{id.} \ : \ \partial_- M \ = \ X \to X \ , \ K_n(M) \ = \ K \ , \\ K_{n-1}(\partial_+ M) \ &= \ \text{coker}(\lambda : K \to K^*) \ , \\ K_n(\partial_+ M) \ &= \ \ker(\lambda : K \to K^*) \ . \end{aligned}$$

The map $\partial_+ f : \partial_+ M \to X$ is a homotopy equivalence if and only if the form (K, λ, μ) is nonsingular. Given (K, λ, μ), X the construction of (f,b)

proceeds as in (i). □

EXAMPLE 2.18 For $\pi_1(X) = \{1\}$ the realization of even-dimensional surgery obstructions (2.17) is essentially the same as the Milnor [11], [12] construction of $(n-1)$-connected $2n$-dimensional manifolds by *plumbing* together n-plane bundles over S^n. Let G be a finite connected graph without loops (= edges joining a vertex to itself), with vertices $v_1, v_2, \ldots, v_k$. Suppose given an oriented n-plane bundle over S^n at each vertex

$$\omega_1, \omega_2, \ldots, \omega_k \in \pi_n(BSO(n)) \;=\; \pi_{n-1}(SO(n)) \;,$$

regarded as a weight. Let $(\mathbb{Z}^k, \lambda)$ be the $(-1)^n$-symmetric form over $\mathbb{Z}$ defined by the $(-1)^n$-symmetrized adjacency matrix of G and the Euler numbers $\chi(\omega_i) \in \mathbb{Z}$, with

$$\lambda_{ij} \;=\; \begin{cases} \text{no. of edges in } G \text{ joining } v_i \text{ to } v_j & \text{if } i < j \\ (-1)^n(\text{no. of edges in } G \text{ joining } v_i \text{ to } v_j) & \text{if } i > j \\ \chi(\omega_i) & \text{if } i = j \;, \end{cases}$$

$$\lambda \;:\; \mathbb{Z}^k \times \mathbb{Z}^k \to \mathbb{Z} \;;\; ((x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, y_k)) \mapsto \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_{ij} x_i y_j \;.$$

The graph G and the Euler numbers $\chi(\omega_i)$ determine and are determined by the form $(\mathbb{Z}^k, \lambda)$.

(i) See Browder [1, Chapter V] for a detailed account of the plumbing construction which uses G to glue together the (D^n, S^{n-1})-bundles

$$(D^n, S^{n-1}) \to (E(\omega_i), S(\omega_i)) \to S^n \quad (i = 1, 2, \ldots, k)$$

to obtain a connected $2n$-dimensional manifold with boundary

$$(P, \partial P) \;=\; (P(G, \omega), \partial P(G, \omega))$$

such that P is an identification space

$$P \;=\; (\coprod_{i=1}^{k} E(\omega_i))/\sim$$

with 1-skeleton homotopy equivalent to G, fundamental group

$$\pi_1(P) \;=\; \pi_1(G) \;=\; *_g \mathbb{Z}$$

the free group on $g = 1 - \chi(G)$ generators, homology

$$H_r(P) \;=\; \begin{cases} \mathbb{Z} & \text{if } r = 0 \\ \mathbb{Z}^g & \text{if } r = 1 \\ \mathbb{Z}^k & \text{if } r = n \\ 0 & \text{otherwise} \;, \end{cases}$$

and intersection form $(H_n(P), \lambda)$. Killing $\pi_1(P)$ by surgeries removing g embeddings $S^1 \times D^{2n-1} \subset P$ representing the generators, there is obtained an $(n-1)$-connected $2n$-dimensional manifold with boundary

$$(M, \partial M) \;=\; (M(G,\omega), \partial M(G,\omega))$$

such that

$$H_r(M) \;=\; \begin{cases} \mathbb{Z} & \text{if } r = 0 \\ \mathbb{Z}^k & \text{if } r = n \\ 0 & \text{otherwise ,} \end{cases}$$

$$\lambda \;:\; H_n(M) \times H_n(M) \to \mathbb{Z} \;;$$

$$((x_1, x_2, \dots, x_k), (y_1, y_2, \dots, y_k)) \mapsto \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_{ij} x_i y_j \;,$$

$$\tau_M \;\simeq\; \bigvee_{i=1}^{k} (\omega_i \oplus \epsilon^n) \;:\; M \;\simeq\; \bigvee_{i=1}^{k} S^n \to BSO(2n) \;,$$

$$H_r(\partial M) \;=\; \begin{cases} \mathbb{Z} & \text{if } r = 0, 2n-1 \\ \operatorname{coker}(\lambda : \mathbb{Z}^k \to \mathbb{Z}^k) & \text{if } r = n-1 \\ \ker(\lambda : \mathbb{Z}^k \to \mathbb{Z}^k) & \text{if } r = n \\ 0 & \text{otherwise .} \end{cases}$$

If G is a tree then $g = 0$, $\pi_1(P(G,\omega)) = \{1\}$, and

$$(M(G,\omega), \partial M(G,\omega)) \;=\; (P(G,\omega), \partial P(G,\omega)) \;.$$

(ii) By Wall [27] for $n \geq 3$ an integral $(-1)^n$-symmetric matrix $(\lambda_{ij})_{1 \leq i,j \leq k}$ and elements $\omega_1, \omega_2, \dots, \omega_k \in \pi_n(BSO(n))$ with

$$\lambda_{ii} \;=\; \chi(\omega_i) \in \mathbb{Z} \;\; (i = 1, 2, \dots, k)$$

determine an embedding

$$x \;=\; \bigcup_k x_i \;:\; \bigcup_k S^{n-1} \times D^n \hookrightarrow S^{2n-1}$$

such that:

(a) for $1 \leq i < j \leq k$

$$\text{linking number}(x_i(S^{n-1} \times 0) \cap x_j(S^{n-1} \times 0) \hookrightarrow S^{2n-1}) \;=\; \lambda_{ij} \in \mathbb{Z} \;,$$

(b) for $1 \leq i \leq k$ $x_i : S^{n-1} \times D^n \hookrightarrow S^{2n-1}$ is isotopic to the embedding

$$\begin{aligned} e_{\omega_i} \;:\; S^{n-1} \times D^n \hookrightarrow S^{2n-1} \;&=\; S^{n-1} \times D^n \cup D^n \times S^{n-1} \;; \\ (s,t) &\mapsto (s, \omega_i(s)(t)) \;. \end{aligned}$$

Using x to attach k n-handles to D^{2n} there is obtained an oriented $(n-1)$-connected $2n$-dimensional manifold

$$M(G,\omega) \;=\; D^{2n} \cup_x \bigcup_k n\text{-handles } D^n \times D^n$$

with boundary an oriented $(n-2)$-connected $(2n-1)$-dimensional manifold

$$\partial M(G,\omega) \;=\; \mathrm{cl.}(S^{2n-1}\backslash x(\bigcup_k S^{n-1}\times D^n))\cup\bigcup_k D^n\times S^{n-1}\ .$$

Moreover, every oriented $(n-1)$-connected $2n$-dimensional manifold with non-empty $(n-2)$-connected boundary is of the form $(M(G,\omega),\partial M(G,\omega))$, with (λ_{ij},ω_i) the complete set of diffeomorphism invariants.
(iii) Stably trivialized n-plane bundles over S^n are classified by $Q_{(-1)^n}(\mathbb{Z})$, with an isomorphism

$$Q_{(-1)^n}(\mathbb{Z}) \xrightarrow{\cong} \pi_{n+1}(BSO,BSO(n))\ ;\ 1\mapsto(\delta\tau_{S^n},\tau_{S^n})$$

with

$$\delta\tau_{S^n}\ :\ \tau_{S^n}\oplus\epsilon\ \cong\ \epsilon^{n+1}$$

the stable trivialization given by the standard embedding $S^n\subset S^{n+1}$. The map

$$Q_{(-1)^n}(\mathbb{Z})\;=\;\pi_{n+1}(BSO,BSO(n))\to\pi_n(BSO(n))\ ;\ 1\mapsto\tau_{S^n}$$

is an injection for $n\neq 1,3,7$. With G as above, suppose now that the vertices $v_1,v_2,\dots,v_k$ are weighted by elements

$$\mu_1,\mu_2,\dots,\mu_k\in\pi_{n+1}(BSO,BSO(n))\;=\;Q_{(-1)^n}(\mathbb{Z})\ .$$

Define

$$\begin{aligned}\omega_i\;&=\;[\mu_i]\in\mathrm{im}(\pi_{n+1}(BSO,BSO(n))\to\pi_n(BSO(n)))\\&=\;\ker(\pi_n(BSO(n))\to\pi_n(BSO))\end{aligned}$$

and let $(\mathbb{Z}^k,\lambda,\mu)$ be the $(-1)^n$-quadratic form over $\mathbb{Z}$ with λ as before and

$$\mu\ :\ \mathbb{Z}^k\to Q_{(-1)^n}\mathbb{Z}\ ;\ (x_1,x_2,\dots,x_k)\mapsto\sum_{1\le i<j\le k}\lambda_{ij}x_ix_j+\sum_{i=1}^k\mu_i(x_i)^2\ ,$$

such that

$$\lambda_{ii}\;=\;\chi(\omega_i)\;=\;(1+(-1)^n)\mu_i\in\mathbb{Z}\ .$$

The $(n-1)$-connected $2n$-dimensional manifold

$$M(G,\mu_1,\mu_2,\dots,\mu_n)\;=\;M(G,\omega_1,\omega_2,\dots,\omega_n)$$

is stably parallelizable, with an n-connected normal map

$$(M(G,\mu_1,\mu_2,\dots,\mu_n),\partial M(G,\mu_1,\mu_2,\dots,\mu_n))\to(D^{2n},S^{2n-1})$$

with kernel form $(\mathbb{Z}^k,\lambda,\mu)$.
(iv) For $n\ge 3$ the realization of a $(-1)^n$-quadratic form $(\mathbb{Z}^k,\lambda,\mu)$ over $\mathbb{Z}$ (2.17) is an n-connected $2n$-dimensional normal map

$$\begin{aligned}(f,b)\ &:(M^{2n};S^{2n-1},\partial_+M)\\&=\ (\mathrm{cl.}(M(G,\mu_1,\dots,\mu_k)\backslash D^{2n});S^{2n-1},\partial M(G,\mu_1,\dots,\mu_k))\\&\qquad\to S^{2n-1}\times([0,1];\{0\},\{1\})\ .\end{aligned}$$

with kernel form $(\mathbb{Z}^k, \lambda, \mu)$. If $(\mathbb{Z}^k, \lambda, \mu)$ is nonsingular then

$$\partial_+ f \ : \ \Sigma^{2n-1} \ = \ \partial M(G, \mu_1, \ldots, \mu_k) \to S^{2n-1}$$

is a homotopy equivalence, and Σ^{2n-1} is a homotopy sphere with a potentially exotic differentiable structure (Milnor [10], Kervaire and Milnor [7]) – see 2.20, 3.6 and 3.7 below. □

EXAMPLE 2.19 (i) Consider the special case $k = 1$ of 2.18 (i). Here $G = \{v_1\}$ is the graph with one vertex, and

$$\omega \in \pi_n(BSO(n)) \ = \ \pi_{n-1}(SO(n))$$

classifies an n-plane bundle over S^n. The plumbed $(n-1)$-connected $2n$-dimensional manifold with boundary is the (D^n, S^{n-1})-bundle over S^n

$$(M(G, \omega), \partial M(G, \omega)) \ = \ (E(\omega), S(\omega))$$

with

$$\begin{aligned} E(\omega) \ &= \ S^{n-1} \times D^n \cup_{(x,y) \sim (x, \omega(x)(y))} S^{n-1} \times D^n \\ &= \ D^{2n} \cup_{e_\omega} D^n \times D^n \end{aligned}$$

obtained from D^{2n} by attaching an n-handle along the embedding

$$\begin{aligned} e_\omega \ : \ S^{n-1} \times D^n \hookrightarrow S^{2n-1} \ &= \ S^{n-1} \times D^n \cup D^n \times S^{n-1} \ ; \\ (x, y) &\mapsto (x, \omega(x)(y)) \ . \end{aligned}$$

(ii) Consider the special case $k = 1$ of 2.18 (iii), the realization of a $(-1)^n$-quadratic form $(\mathbb{Z}, \lambda, \mu)$ over $\mathbb{Z}$, with $G = \{v_1\}$ as in (i). An element

$$\begin{aligned} \mu \ = \ (\delta\omega, \omega) \in \pi_{n+1}(BSO, BSO(n)) \ &= \ Q_{(-1)^n}(\mathbb{Z}) \\ &= \ \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 (\bmod 2) \\ \mathbb{Z}_2 & \text{if } n \equiv 1 (\bmod 2) \end{cases} \end{aligned}$$

classifies an n-plane bundle $\omega : S^n \to BSO(n)$ with a stable trivialization

$$\delta\omega \ : \ \omega \oplus \epsilon^\infty \ \cong \ \epsilon^{n+\infty} \ ,$$

and

$$(M(G, \mu), \partial M(G, \mu)) \ = \ (E(\omega), S(\omega)) \ .$$

For $n \neq 1, 3, 7$ $\delta\omega$ is determined by ω. For even n $\mu \in Q_{+1}(\mathbb{Z}) = \mathbb{Z}$ and

$$\omega \ = \ \mu^* \tau_{S^n} \ : \ S^n \xrightarrow{\ \mu \ } S^n \xrightarrow{\ \tau_{S^n} \ } BSO(n)$$

is the unique stably trivial n-plane bundle over S^n with Euler number

$$\chi(\omega) \ = \ 2\mu \in \mathbb{Z} \ .$$

For odd $n \neq 1, 3, 7$ $\mu \in Q_{-1}(\mathbb{Z}) = \mathbb{Z}_2$ and

$$\omega \ = \ \begin{cases} \tau_{S^n} & \text{if } \mu = 1 \\ \epsilon^n & \text{if } \mu = 0 \ . \end{cases}$$

For $n = 1, 3, 7$

$$\omega \; = \; \tau_{S^n} \; = \; \epsilon^n \; : \; S^n \to BSO(n)$$

and $\delta\omega$ is the (stable) trivialization of ω with mod 2 Hopf invariant μ. The plumbed $(n-1)$-connected $2n$-dimensional manifold

$$(M(G,\mu), \partial M(G,\mu)) \; = \; (M(G,\omega), \partial M(G,\omega)) \; = \; (E(\omega), S(\omega))$$

(as in (i)) is stably parallelizable. The trace of the surgery on the normal map

$$(f_-, b_-) \; = \; \text{id.} \; : \; \partial_- M_\omega \; = \; S^{2n-1} \to S^{2n-1}$$

killing $e_\omega : S^{n-1} \times D^n \hookrightarrow S^{2n-1}$ is an n-connected $2n$-dimensional normal map

$$(f_\omega, b_{\delta\omega}) \; : \; (M^{2n}_\omega; \partial_- M_\omega, \partial_+ M_\omega) \to S^{2n-1} \times ([0,1]; \{0\}, \{1\})$$

with

$$\begin{aligned} M_\omega \; &= \; \text{cl.}(M(G,\mu) \backslash D^{2n}) \; = \; \text{cl.}(E(\omega) \backslash D^{2n}) \\ &= \; S^{2n-1} \times [0,1] \cup_{e_\omega} D^n \times D^n \; , \\ \partial_+ M_\omega \; &= \; \text{cl.}(S^{2n-1} \backslash e_\omega(S^{n-1} \times D^n)) \cup D^n \times S^{n-1} \\ &= \; D^n \times S^{n-1} \cup_\omega D^n \times S^{n-1} \; = \; S(\omega) \; , \\ K_n(M_\omega) \; &= \; \mathbb{Z} \end{aligned}$$

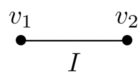
and kernel form $(\mathbb{Z}, \lambda, \mu)$. If $\mu = 0 \in Q_{(-1)^n}(\mathbb{Z})$ then

$$\omega \; = \; \epsilon^n \; : \; S^n \to BSO(n) \quad , \quad \partial_+ M_\omega \; = \; S(\epsilon^n) \; = \; S^{n-1} \times S^n \; ,$$

If $\mu = 1 \in Q_{(-1)^n}(\mathbb{Z})$ then

$$\omega \; = \; \tau_{S^n} \; : \; S^n \to BSO(n) \quad , \quad \partial_+ M_\omega \; = \; S(\tau_{S^n}) \; = \; O(n+1)/O(n-1) \; .$$

(iii) Consider the special case $k = 2$ of 2.18 (i), with $G = I$ the graph with 1 edge and 2 vertices



For any weights $\omega_1, \omega_2 \in \pi_n(BSO(n))$ there is obtained an $(n-1)$-connected $2n$-dimensional manifold

$$M(I, \omega_1, \omega_2) \; = \; D^{2n} \cup_{e_{\omega_1} \cup e_{\omega_2}} (D^n \times D^n \cup D^n \times D^n)$$

by plumbing as in Milnor [11], [12], with intersection form the $(-1)^n$-symmetric form $(\mathbb{Z} \oplus \mathbb{Z}, \lambda)$ over $\mathbb{Z}$ defined by

$$\begin{aligned} \lambda \; &: \mathbb{Z} \oplus \mathbb{Z} \times \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \; ; \\ &((x_1, x_2), (y_1, y_2)) \mapsto \chi(\omega_1) x_1 y_1 + \chi(\omega_2) x_2 y_2 + x_1 y_2 + (-1)^n x_2 y_1 \; . \end{aligned}$$

(iv) Consider the special case $k = 2$ of 2.18 (iii), with $G = I$ as in (iii). For $\mu_1, \mu_2 \in Q_{(-1)^n}(\mathbb{Z})$ and

$$\omega_i \; = \; [\mu_i] \in \text{im}(Q_{(-1)^n}(\mathbb{Z}) \to \pi_n(BSO(n)))$$

the $(-1)^n$-quadratic form $(\mathbb{Z}\oplus\mathbb{Z},\lambda,\mu)$ over $\mathbb{Z}$ defined by

$$\mu \ : \ \mathbb{Z}\oplus\mathbb{Z}\to Q_{(-1)^n}(\mathbb{Z}) \ ; \ (x_1,x_2)\mapsto \mu_1(x_1)^2+\mu_2(x_2)^2+x_1x_2$$

is the kernel form of an n-connected $2n$-dimensional normal map

$$(f,b) \ : \ M(I,\mu_1,\mu_2) \ = \ M(I,\omega_1,\omega_2)\to D^{2n} \ .$$

If $\mu_1=\mu_2=0$ then $(\mathbb{Z}\oplus\mathbb{Z},\lambda,\mu)=H_{(-1)^n}(\mathbb{Z})$ is hyperbolic $(-1)^n$-quadratic form over $\mathbb{Z}$, with

$$\begin{aligned}\lambda \ &: \ \mathbb{Z}\oplus\mathbb{Z}\times\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z} \ ; \ ((x_1,x_2),(y_1,y_2))\mapsto x_1y_2+(-1)^nx_2y_1 \ , \\ \mu \ &: \ \mathbb{Z}\oplus\mathbb{Z}\to Q_{(-1)^n}(\mathbb{Z}) \ = \ \mathbb{Z}/\{1+(-1)^{n-1}\} \ ; \ (x_1,x_2)\mapsto x_1x_2 \ ,\end{aligned}$$

and the plumbed manifold is a punctured torus

$$(M(I,0,0)^{2n},\partial M(I,0,0)) \ = \ (\mathrm{cl.}(S^n\times S^n\backslash D^{2n}),S^{2n-1}) \ .$$

The hyperbolic form is the kernel of the n-connected $2n$-dimensional normal map

$$\begin{aligned}(f,b) \ : \ (M;\partial_-M,\partial_+M) \ &= \ (\mathrm{cl.}(M(I,0,0)\backslash D^{2n});S^{2n-1},S^{2n-1}) \\ &\to S^{2n-1}\times([0,1];\{0\},\{1\})\end{aligned}$$

defined by the trace of surgeries on the linked spheres

$$S^{n-1}\cup S^{n-1}\hookrightarrow S^{2n-1} \ = \ S^{n-1}\times D^n\cup D^n\times S^{n-1}$$

with no self-linking. These are the attaching maps for the cores of the n-handles in the decomposition

$$M(I,0,0) \ = \ D^{2n}\cup D^n\times D^n\cup D^n\times D^n \ ,$$

using the standard framings of $S^{n-1}\subset S^{2n-1}$. If n is odd, say $n=2k+1$, and $\mu_0=\mu_1=1\in Q_{-1}(\mathbb{Z})$ the form in 2.18 (i) is just the Arf (-1)-quadratic form over $\mathbb{Z}$ $(\mathbb{Z}\oplus\mathbb{Z},\lambda,\mu')$ with

$$\mu' \ : \ \mathbb{Z}\oplus\mathbb{Z}\to Q_{(-1)}(\mathbb{Z}) \ = \ \mathbb{Z}_2 \ ; \ (x,y)\longmapsto x^2+xy+y^2 \ .$$

The plumbed manifold

$$M(I,1,1)^{4k+2} \ = \ D^{4k+2}\cup D^{2k+1}\times D^{2k+1}\cup D^{2k+1}\times D^{2k+1}$$

has the same attaching maps for the cores of the $(2k+1)$-handles as $M(I,0,0)$, but now using the framings of $S^{2k}\subset S^{4k+1}$ classified by

$$\tau_{S^{2k+1}}\in\pi_{2k+1}(BSO(2k+1)) \ = \ \pi_{2k}(SO(2k+1))$$

(which is zero if and only if $k=0,1,3$). The Arf form is the kernel of the $(2k+1)$-connected $(4k+2)$-dimensional normal map

$$\begin{aligned}(f',b') \ : \ (M';\partial_-M',\partial_+M') \ &= \ (\mathrm{cl.}(M(I,1,1)\backslash D^{4k+2});S^{4k+1},\Sigma^{4k+1}) \\ &\to S^{4k+1}\times([0,1];\{0\},\{1\})\end{aligned}$$

defined by the trace of surgeries on the linked spheres

$$S^{2k}\cup S^{2k}\hookrightarrow S^{4k+1} \ = \ S^{2k}\times D^{2k+1}\cup D^{2k+1}\times S^{2k} \ ,$$

with self-linking given by the non-standard framing. (See 3.7 below for a brief account of the exotic sphere Σ^{4k+1}). □

EXAMPLE 2.20 The sphere bundles $S(\omega)$ of certain oriented 4-plane bundles ω over S^4 (the special case $n=4$ of 2.19 (i)) give explicit exotic 7-spheres. An oriented 4-plane bundle $\omega : S^4 \to BSO(4)$ is determined by the Euler number and first Pontrjagin class

$$\chi(\omega), p_1(\omega) \in H^4(S^4) \ = \ \mathbb{Z} \ ,$$

which must be such that

$$2\chi(\omega) \equiv p_1(\omega) (\bmod 4) \ ,$$

with an isomorphism

$$\pi_4(BSO(4)) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z} \ ; \ \omega \longmapsto ((2\chi(\omega) + p_1(\omega))/4, (2\chi(\omega) - p_1(\omega))/4) \ .$$

If $\chi(\omega) = 1$ then $S(\omega)$ is a homotopy 7-sphere, and

$$p_1(\omega) \ = \ 2\ell \in H^4(S^4) \ = \ \mathbb{Z}$$

for some odd integer ℓ. The 7-dimensional differentiable manifold $\Sigma^7_\ell = S(\omega)$ is homeomorphic to S^7 (by Smale's generalized Poincaré conjecture, or by a direct Morse-theoretic argument). If Σ^7_ℓ is diffeomorphic to S^7 then

$$M^8 \ = \ E(\omega) \cup_{\Sigma^7_\ell} D^8$$

is a closed 8-dimensional differentiable manifold with

$$p_1(M) \ = \ p_1(\omega) \ = \ 2\ell \in H^4(M) \ = \ \mathbb{Z} \quad , \quad \sigma(M) \ = \ 1 \in \mathbb{Z} \ .$$

By the Hirzebruch signature theorem

$$\begin{aligned} \sigma(M) \ &= \ \langle \mathcal{L}(M), [M] \rangle \\ &= \ (7p_2(M) - p_1(M)^2)/45 \\ &= \ (7p_2(M) - 4\ell^2)/45 \ = \ 1 \in \mathbb{Z} \ . \end{aligned}$$

If $\ell \not\equiv \pm 1 (\bmod 7)$ then

$$p_2(M) \ = \ (45 + 4\ell^2)/7 \notin H^8(M) \ = \ \mathbb{Z}$$

so that there is no such diffeomorphism, and Σ^7_ℓ is an exotic 7-sphere (Milnor [10], Milnor and Stasheff [14, p.247]). □

DEFINITION 2.21 An *isomorphism* of ϵ-symmetric forms

$$f \ : \ (K, \lambda) \xrightarrow{\cong} (K', \lambda')$$

is a Λ-module isomorphism $f : K \cong K'$ such that

$$\lambda'(f(x), f(y)) \ = \ \lambda(x, y) \in \Lambda \ .$$

An *isomorphism* of ϵ-quadratic forms $f : (K, \lambda, \mu) \cong (K', \lambda', \mu')$ is an isomorphism of the underlying ϵ-symmetric forms $f : (K, \lambda) \cong (K', \lambda')$ such that

$$\mu'(f(x)) \;=\; \mu(x) \in Q_\epsilon(\Lambda) \;. \qquad \square$$

PROPOSITION 2.22 *If there exists a central element $s \in \Lambda$ such that*

$$s + \overline{s} = 1 \in \Lambda$$

there is an identification of categories

$$\{\epsilon\text{-}quadratic\ forms\ over\ \Lambda\} \;=\; \{\epsilon\text{-}symmetric\ forms\ over\ \Lambda\} \;.$$

PROOF: The ϵ-symmetrization map $1+T_\epsilon : Q_\epsilon(K) \to Q^\epsilon(K)$ is an isomorphism for any Λ-module K, with inverse

$$Q^\epsilon(K) \to Q_\epsilon(K) \;;\; \lambda \longmapsto ((x,y) \longmapsto s\lambda(x,y)) \;.$$

For any ϵ-quadratic form (K,λ,μ) the ϵ-quadratic function μ is determined by the ϵ-symmetric pairing λ, with

$$\mu(x) \;=\; s\lambda(x,x) \in Q_\epsilon(\Lambda) \;. \qquad \square$$

EXAMPLE 2.23 If $2 \in \Lambda$ is invertible then 2.22 applies with $s = 1/2 \in \Lambda$. $\square$

For any ϵ-symmetric form (K,λ) and $x \in K$

$$\lambda(x,x) \in Q^\epsilon(\Lambda) \;.$$

DEFINITION 2.24 An ϵ-symmetric form (K,λ) is *even* if for all $x \in K$

$$\lambda(x,x) \in \mathrm{im}(1+T_\epsilon : Q_\epsilon(\Lambda) \to Q^\epsilon(\Lambda)) \;. \qquad \square$$

PROPOSITION 2.25 *Let $\epsilon = 1$ or -1. If the ϵ-symmetrization map*

$$1 + T_\epsilon \;:\; Q_\epsilon(\Lambda) \to Q^\epsilon(\Lambda)$$

is an injection there is an identification of categories

$$\{\epsilon\text{-}quadratic\ forms\ over\ \Lambda\} \;=\; \{even\ \epsilon\text{-}symmetric\ forms\ over\ \Lambda\} \;.$$

PROOF: Given an even ϵ-symmetric form (K,λ) over Λ there is a unique function $\mu : K \to Q_\epsilon(\Lambda)$ such that for all $x \in K$

$$(1+T_\epsilon)(\mu(x)) \;=\; \lambda(x,x) \in Q^\epsilon(\Lambda) \;,$$

which then automatically satisfies the conditions of 2.13 for (K,λ,μ) to be an ϵ-quadratic form. $\square$

EXAMPLE 2.26 The symmetrization map

$$1 + T \;=\; 2 \;:\; Q_{+1}(\mathbb{Z}) \;=\; \mathbb{Z} \to Q^{+1}(\mathbb{Z}) \;=\; \mathbb{Z}$$

is an injection, so that quadratic forms over $\mathbb{Z}$ coincide with the even symmetric forms. $\square$

EXAMPLE 2.27 The (-1)-symmetrization map

$$1+T_- \ :\ Q_{-1}(\mathbb{Z}) \ = \ \mathbb{Z}_2 \to Q^{-1}(\mathbb{Z}) \ = \ 0$$

is not an injection, so that (-1)-quadratic forms over $\mathbb{Z}$ have a richer structure than even (-1)-symmetric forms. The hyperbolic (-1)-symmetric form $(K,\lambda)=H^{-1}(\mathbb{Z})$ over $\mathbb{Z}$

$$K \ = \ \mathbb{Z}\oplus\mathbb{Z} \ \ , \ \ \lambda \ :\ K\times K\to\mathbb{Z}\ ;\ ((a,b),(c,d))\longmapsto ad-bc$$

admits two distinct (-1)-quadratic refinements (K,λ,μ), (K,λ,μ'), with

$$\begin{aligned}\mu \ &:\ K\to Q_{-1}(\mathbb{Z}) \ = \ \mathbb{Z}/2\ ;\ (x,y)\longmapsto xy\ ,\\ \mu' \ &:\ K\to Q_{-1}(\mathbb{Z}) \ = \ \mathbb{Z}/2\ ;\ (x,y)\longmapsto x^2+xy+y^2\ .\end{aligned}$$

See §3 below for the definition of the Arf invariant, which distinguishes the hyperbolic (-1)-quadratic form $(K,\lambda,\mu)=H_{-1}(\mathbb{Z})$ from the Arf form (K,λ,μ') (which already appeared in 2.19 (iv)). □

§3. The even-dimensional L-groups

The even-dimensional surgery obstruction groups $L_{2n}(\Lambda)$ will now be defined, using the following preliminary result.

LEMMA 3.1 *For any nonsingular ϵ-quadratic form (K,λ,μ) there is defined an isomorphism*

$$(K,\lambda,\mu)\oplus(K,-\lambda,-\mu) \ \cong \ H_\epsilon(K)\ ,$$

with $H_\epsilon(K)$ the hyperbolic ϵ-quadratic form (2.14).

PROOF: Let L be a f. g. projective Λ-module such that $K\oplus L$ is f. g. free, with basis elements $\{x_1,x_2,\dots,x_k\}$ say. Let

$$\lambda_{ij} \ = \ (\lambda\oplus 0)(x_j,x_i)\in\Lambda \ \ (1\le i<j\le k)$$

and choose representatives $\mu_i\in\Lambda$ of $\mu(x_i)\in Q_\epsilon(\Lambda)$ $(1\le i\le k)$. Define the Λ-module morphism

$$\begin{aligned}&\psi_{K\oplus L} \ :\ K\oplus L\to(K\oplus L)^*\ ;\\ &\qquad\sum_{i=1}^k a_ix_i\longmapsto(\sum_{j=1}^k b_jx_j\longmapsto\sum_{i=1}^k b_i\mu_i\overline{a}_i+\sum_{1\le i<j\le k}b_j\lambda_{ij}\overline{a}_i)\ .\end{aligned}$$

The Λ-module morphism defined by

$$\psi\ :\ K\xrightarrow{\text{inclusion}}K\oplus L\xrightarrow{\psi_{K\oplus L}}(K\oplus L)^* \ = \ K^*\oplus L^*\xrightarrow{\text{projection}}K^*$$

is such that

$$\begin{aligned}\lambda \ &= \ \psi+\epsilon\psi^* \ :\ K\to K^*,\\ \mu(x) \ &= \ \psi(x,x)\in Q_\epsilon(\Lambda)\ (x\in K)\ .\end{aligned}$$

As (K,λ,μ) is nonsingular $\psi+\epsilon\psi^* : K \to K^*$ is an isomorphism. The Λ-module morphism defined by

$$\widetilde{\psi} = (\psi+\epsilon\psi^*)^{-1}\psi(\psi+\epsilon\psi^*)^{-1} : K^* \to K$$

is such that

$$(\psi+\epsilon\psi^*)^{-1} = \widetilde{\psi}+\epsilon\widetilde{\psi}^* : K^* \to K .$$

Define an isomorphism of ϵ-quadratic forms

$$f : H_\epsilon(K) \xrightarrow{\cong} (K\oplus K, \lambda\oplus -\lambda, \mu\oplus -\mu)$$

by

$$f = \begin{pmatrix} 1 & -\epsilon\widetilde{\psi}^* \\ 1 & \widetilde{\psi} \end{pmatrix} : K\oplus K^* \to K\oplus K . \qquad \square$$

DEFINITION 3.2 The $2n$*-dimensional L-group* $L_{2n}(\Lambda)$ is the group of equivalence classes of nonsingular $(-1)^n$-quadratic forms (K,λ,μ) on stably f. g. free Λ-modules, subject to the equivalence relation

$(K,\lambda,\mu) \sim (K',\lambda',\mu')$

if there exists an isomorphism of $(-1)^n$-quadratic forms

$$(K,\lambda,\mu)\oplus H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} (K',\lambda',\mu')\oplus H_{(-1)^n}(\Lambda^{k'})$$

for some f. g. free Λ-modules $\Lambda^k, \Lambda^{k'}$.

Addition and inverses in $L_{2n}(\Lambda)$ are given by

$$\begin{aligned}(K_1,\lambda_1,\mu_1)+(K_2,\lambda_2,\mu_2) &= (K_1\oplus K_2, \lambda_1\oplus\lambda_2, \mu_1\oplus\mu_2) , \\ -(K,\lambda,\mu) &= (K,-\lambda,-\mu) \in L_{2n}(\Lambda) . \end{aligned} \qquad \square$$

The groups $L_{2n}(\Lambda)$ only depend on the residue $n(\bmod 2)$, so that only two L-groups have actually been defined, $L_0(\Lambda)$ and $L_2(\Lambda)$. Note that 3.2 uses Lemma 3.1 to justify $(K,\lambda,\mu)\oplus(K,-\lambda,-\mu)\sim 0$.

REMARK 3.3 The surgery obstruction of an n-connected $2n$-dimensional normal map $(f,b) : M^{2n} \to X$ is an element

$$\sigma_*(f,b) = (K_n(M),\lambda,\mu) \in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

such that $\sigma_*(f,b)=0$ if (and for $n\geq 3$ only if) (f,b) is normal bordant to a homotopy equivalence. $\square$

EXAMPLE 3.4 Let $M = M^2_g$ be the orientable 2-manifold (= surface) of genus g, with degree 1 map $f : M \to S^2$. A choice of framing of the stable normal bundle of an embedding $M \hookrightarrow \mathbb{R}^3$ determines a 1-connected

2-dimensional normal map $(f,b) : M \to S^2$. For a standard choice of framing (i.e. one which extends to a 3-manifold N with $\partial N = M$) the kernel form and the surgery obstruction are given by

$$\sigma_*(f,b) \;=\; H_{-1}(\mathbb{Z}^g) \;=\; 0 \;\in L_2(\mathbb{Z})$$

and (f,b) is normal bordant to a homotopy equivalence, i.e. M is framed null-cobordant. □

EXAMPLE 3.5 The even-dimensional L-groups of the ring $\Lambda = \mathbb{R}$ of real numbers with the identity involution are given by

$$L_{2n}(\mathbb{R}) \;=\; \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd .} \end{cases}$$

Since $1/2 \in \mathbb{R}$ there is no difference between symmetric and quadratic forms over $\mathbb{R}$. □

The *signature* (alias index) of a nonsingular symmetric form (K,λ) over $\mathbb{R}$ is defined by

$$\begin{aligned} \sigma(K,\lambda) \;=\; & \text{no. of positive eigenvalues of } \lambda \\ & \qquad - \text{no. of negative eigenvalues of } \lambda \in \mathbb{Z} \;. \end{aligned}$$

Here, the symmetric form $\lambda \in Q^{+1}(K)$ is identified with the symmetric $k \times k$ matrix $(\lambda(x_i,x_j)_{1\leq i,j\leq k}) \in M_{k,k}(\mathbb{R})$ determined by any choice of basis $x_1,x_2,\dots,x_k$ for K. By Sylvester's law of inertia the rank k and the signature $\sigma(K,\lambda)$ define a complete set of invariants for the isomorphism classification of nonsingular symmetric forms (K,λ) over $\mathbb{R}$, meaning that two forms are isomorphic if and only if they have the same rank and signature. A nonsingular quadratic form (K,ψ) over $\mathbb{R}$ is isomorphic to a hyperbolic form if and only if it has signature 0. Two such forms (K,λ), (K',λ') are related by an isomorphism

$$(K,\lambda) \oplus H_+(\mathbb{R}^m) \xrightarrow{\cong} (K',\lambda') \oplus H_+(\mathbb{R}^{m'})$$

if and only if they have the same signature

$$\sigma(K,\lambda) \;=\; \sigma(K',\lambda') \in \mathbb{Z} \;.$$

Moreover, every integer is the signature of a form, since $1 \in \mathbb{Z}$ is the signature of the nonsingular symmetric form $(\mathbb{R},1)$ with

$$1 \;:\; \mathbb{R} \to \mathbb{R}^* \;;\; x \longmapsto (y \longmapsto xy)$$

and for any nonsingular symmetric forms $(K,\lambda),(K',\lambda')$ over $\mathbb{R}$

$$\begin{aligned} \sigma((K,\lambda)\oplus(K',\lambda')) \;&=\; \sigma(K,\lambda)+\sigma(K',\lambda') \;, \\ \sigma(K,-\lambda) \;&=\; -\sigma(K,\lambda) \in \mathbb{Z} \;. \end{aligned}$$

The isomorphism of 3.5 in the case $n \equiv 0 (\bmod 2)$ is defined by

$$L_0(\mathbb{R}) \xrightarrow{\cong} \mathbb{Z} \; ; \; (K, \lambda) \longmapsto \sigma(K, \lambda) \; .$$

$L_2(\mathbb{R}) = 0$ because every nonsingular (-1)-symmetric (alias symplectic) form over $\mathbb{R}$ admits is isomorphic to a hyperbolic form.

It is not possible to obtain a complete isomorphism classification of nonsingular symmetric and quadratic forms over $\mathbb{Z}$ – see Chapter II of Milnor and Husemoller [13] for the state of the art in 1973. Fortunately, it is much easier to decide if two forms become isomorphic after adding hyperbolics then whether they are actually isomorphic. Define the signature of a nonsingular symmetric form (K, λ) over $\mathbb{Z}$ to be the signature of the induced nonsingular symmetric form over $\mathbb{R}$

$$\sigma(K, \lambda) \;=\; \sigma(\mathbb{R} \otimes K, 1 \otimes \lambda) \in \mathbb{Z} \; .$$

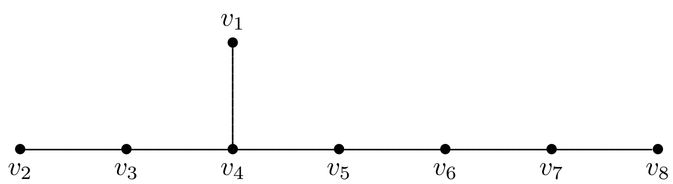
It is a non-trivial theorem that two nonsingular even symmetric forms (K, λ), (K', λ') are related by an isomorphism

$$(K, \lambda) \oplus H_+(\mathbb{Z}^m) \xrightarrow{\cong} (K', \lambda') \oplus H_+(\mathbb{Z}^{m'})$$

if and only if they have the same signature

$$\sigma(K, \lambda) \;=\; \sigma(K', \lambda') \in \mathbb{Z} \; .$$

Moreover, not every integer arises as the signature of an even symmetric form, only those divisible by 8. The Dynkin diagram of the exceptional Lie group E_8 is a tree



Weighing each vertex by $1 \in Q_{+1}(\mathbb{Z}) = \mathbb{Z}$ gives (by the method recalled in 2.18) a nonsingular quadratic form $(\mathbb{Z}^8, \lambda_{E_8}, \mu_{E_8})$ with signature

$$\sigma(\mathbb{Z}^8, \lambda_{E_8}) \;=\; 8 \in \mathbb{Z} \; ,$$

where

$$\lambda_{E_8} \;=\; \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \;:\; \mathbb{Z}^8 \to (\mathbb{Z}^8)^*$$

and μ_{E_8} is determined by λ_{E_8}.

EXAMPLE 3.6 (i) The signature divided by 8 defines an isomorphism

$$\sigma \;:\; L_{4k}(\mathbb{Z}) \to \mathbb{Z} \;;\; (K, \lambda, \mu) \mapsto \sigma(K, \lambda)/8$$

so that $(\mathbb{Z}^8, \lambda_{E_8}, \mu_{E_8}) \in L_{4k}(\mathbb{Z})$ represents a generator.
(ii) See Kervaire and Milnor [7] and Levine [9] for the surgery classification of high-dimensional exotic spheres, including the expression of the h-cobordism group Θ^n of n-dimensional exotic spheres for $n \geq 5$ as

$$\Theta^n \;=\; \pi_n(TOP/O) \;=\; \pi_n(PL/O)$$

and the exact sequence

$$\ldots \to \pi_{n+1}(G/O) \to L_{n+1}(\mathbb{Z}) \to \Theta^n \to \pi_n(G/O) \to \ldots \;.$$

(iii) In the original case $n = 7$ (Milnor [10]) there is defined an isomorphism

$$\Theta^7 \xrightarrow{\cong} \mathbb{Z}_{28} \;;\; \Sigma^7 \mapsto \sigma(W)/8$$

for any framed 8-dimensional manifold W with $\partial W = \Sigma^7$. The realization (2.17) of $(\mathbb{Z}^8, \lambda_{E_8}, \mu_{E_8})$ as the kernel form of a 4-connected 8-dimensional normal bordism

$$\begin{aligned}(f, b) \;:\; (M^8, S^7, \partial_+ M) \;&=\; (\mathrm{cl.}(M(E_8, 1, \ldots, 1)\backslash D^8); S^7, \partial M(E_8, 1, \ldots, 1)) \\ &\qquad\qquad \to S^7 \times ([0,1]; \{0\}, \{1\})\end{aligned}$$

gives the exotic sphere

$$\Sigma^7 \;=\; \partial_+ M \;=\; \partial M(E_8, 1, \ldots, 1)$$

generating Θ^7: the framed 8-dimensional manifold $W = M(E_8, 1, \ldots, 1)$ obtained by the E_8-plumbing of 8 copies of τ_{S^4} (2.18) has $\sigma(W) = 8$. The 7-dimensional homotopy sphere Σ^7_ℓ defined for any odd integer ℓ in 2.20 is the boundary of a framed 8-dimensional manifold W_ℓ with $\sigma(W_\ell) = 8(\ell^2 - 1)$.
$\square$

For any nonsingular (-1)-quadratic form (K, λ, μ) over $\mathbb{Z}$ there exists a symplectic basis $x_1, \ldots, x_{2m}$ for K, such that

$$\lambda(x_i, x_j) \;=\; \begin{cases} 1 & \text{if } i - j = m \\ -1 & \text{if } j - i = m \\ 0 & \text{otherwise .} \end{cases}$$

The *Arf invariant* of (K, λ, μ) is defined using any such basis to be

$$c(K, \lambda, \mu) \;=\; \sum_{i=1}^{m} \mu(x_i)\mu(x_{i+m}) \in \mathbb{Z}_2 \ .$$

Example 3.7 (i) The Arf invariant defines an isomorphism

$$c \ : \ L_{4k+2}(\mathbb{Z}) \to \mathbb{Z}_2 \ ; \ (K, \lambda, \mu) \mapsto c(K, \lambda, \mu)$$

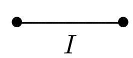
The nonsingular (-1)-quadratic form $(\mathbb{Z} \oplus \mathbb{Z}, \lambda, \mu)$ over $\mathbb{Z}$ defined by

$$\begin{aligned} \lambda((x, y), (x', y')) \;&=\; x'y - xy' \in \mathbb{Z} \ , \\ \mu(x, y) \;&=\; x^2 + xy + y^2 \in Q_{-1}(\mathbb{Z}) \;=\; \mathbb{Z}_2 \end{aligned}$$

has Arf invariant $c(\mathbb{Z} \oplus \mathbb{Z}, \lambda, \mu) = 1$, and so generates $L_{4k+2}(\mathbb{Z})$.
(ii) The realization (2.19 (iv)) of the Arf form $(\mathbb{Z} \oplus \mathbb{Z}, \lambda, \mu)$ as the kernel form of a 5-connected 10-dimensional normal bordism

$$\begin{aligned} (f, b) \ : (M^{10}, S^9, \partial_+ M) \;&=\; (\mathrm{cl.}(M(I,1,1)\backslash D^{10}); S^9, \partial M(I,1,1)) \\ &\qquad \to S^9 \times ([0,1]; \{0\}, \{1\}) \end{aligned}$$

is obtained by plumbing together 2 copies of τ_{S^5} (2.18) where I is the tree with 1 edge and 2 vertices



I

and $\Sigma^9 = \partial_+ M = \partial M(I,1,1)$ is the exotic 9-sphere generating $\Theta^9 = \mathbb{Z}_2$. Coning off the boundary components gives the closed 10-dimensional PL manifold $cS^9 \cup M \cup c\Sigma^9$ without differentiable structure of Kervaire [5]. □

§4. Split forms

A "split form" on a Λ-module K is an element

$$\psi \in S(K) \;=\; \mathrm{Hom}_\Lambda(K, K^*) \ ,$$

which can be regarded as a sesquilinear pairing

$$\psi \ : \ K \times K \to \Lambda \ ; \ (x, y) \mapsto \psi(x, y) \ .$$

Split forms are more convenient to deal with than ϵ-quadratic forms in describing the algebraic effects of even-dimensional surgery (in §5 below), and are closer to the geometric applications such as knot theory.

The main result of §4 is that the ϵ-quadratic structures (λ, μ) on a f.g. projective Λ-module K correspond to the elements of the ϵ-quadratic group of 2.6

$$Q_\epsilon(K) \;=\; \mathrm{coker}(1 - T_\epsilon : S(K) \to S(K)) \ .$$

The pair of functions (λ, μ) used to define an ϵ-quadratic form (K, λ, μ) can thus be replaced by an equivalence class of Λ-module morphisms $\psi : K \to K^*$ such that

$$\begin{aligned} \lambda(x,y) \;&=\; \psi(x,y) + \epsilon\overline{\psi(y,x)} \in \Lambda \ , \\ \mu(x) \;&=\; \psi(x,x) \in Q_\epsilon(\Lambda) \end{aligned}$$

i.e. by an equivalence class of split forms.

DEFINITION 4.1 (i) A *split form* (K, ψ) over Λ is a f. g. projective Λ-module K together with an element $\psi \in S(K)$.
(ii) A *morphism* (resp. *isomorphism*) of split forms over Λ

$$f \;:\; (K, \psi) \to (K', \psi')$$

is a Λ-module morphism (resp. isomorphism) $f : K \to K'$ such that

$$f^*\psi' f \;=\; \psi \;:\; K \to K^* \ .$$

(iii) An *ϵ-quadratic morphism* (resp. *isomorphism*) of split forms over Λ

$$(f, \chi) \;:\; (K, \psi) \to (K', \psi')$$

is a Λ-module morphism (resp. isomorphism) $f : K \to K'$ together with an element $\chi \in Q_{-\epsilon}(K)$ such that

$$f^*\psi' f - \psi \;=\; \chi - \epsilon\chi^* \;:\; K \to K^* \ .$$

(iv) A split form (K, ψ) is *ϵ-nonsingular* if $\psi + \epsilon\psi^* : K \to K^*$ is a Λ-module isomorphism. □

PROPOSITION 4.2 (i) *A split form (K, ψ) determines an ϵ-quadratic form (K, λ, μ) by*

$$\begin{aligned} &\lambda \;=\; \psi + \epsilon\psi^* \;:\; K \to K^* \ ; \ x \longmapsto (y \longmapsto \psi(x,y) + \epsilon\overline{\psi(y,x)}) \ , \\ &\mu \;:\; K \to Q_\epsilon(\Lambda) \ ; \ x \longmapsto \psi(x,x) \ . \end{aligned}$$

(ii) *Every ϵ-quadratic form (K, λ, μ) is determined by a split form (K, ψ), which is unique up to*

$$\psi \;\sim\; \psi' \ \textit{if} \ \psi' - \psi \;=\; \chi - \epsilon\chi^* \ \textit{for some} \ \chi : K \to K^* \ .$$

(iii) *The isomorphism classes of (nonsingular) ϵ-quadratic forms (K, λ, μ) over Λ are in one-one correspondence with the ϵ-quadratic isomorphism classes of (ϵ-nonsingular) split forms (K, ψ) over Λ.*
PROOF: (i) By construction.
(ii) There is no loss of generality in taking K to be f.g. free, $K = \Lambda^k$. An ϵ-quadratic form $(\Lambda^k, \lambda, \mu)$ over Λ is determined by a $k \times k$-matrix

$\lambda = \{\lambda_{ij} \in \Lambda \,|\, 1 \leq i,j \leq k\}$ such that

$$\overline{\lambda_{ij}} \ = \ \epsilon\lambda_{ji} \in \Lambda$$

and a collection of elements $\mu = \{\mu_i \in Q_\epsilon(\Lambda) \,|\, 1 \leq i \leq k\}$ such that

$$\mu_i + \epsilon\overline{\mu_i} \ = \ \lambda_{ii} \in Q^\epsilon(\Lambda) \ .$$

Choosing any representatives $\mu_i \in \Lambda$ of $\mu_i \in Q_\epsilon(\Lambda^k)$ there is defined a split form (Λ^k, ψ) with $\psi = \{\psi_{ij} \in \Lambda \,|\, 1 \leq i,j \leq k\}$ the $k \times k$ matrix defined by

$$\psi_{ij} \ = \ \begin{cases} \lambda_{ij} & \text{if } i < j \\ \mu_i & \text{if } i = j \\ 0 & \text{otherwise .} \end{cases}$$

(iii) An ϵ-quadratic (iso)morphism $(f,\chi) : (K,\psi) \to (K',\psi')$ of split forms determines an (iso)morphism $f : (K,\lambda,\mu) \to (K',\lambda',\mu')$ of ϵ-quadratic forms. Conversely, an ϵ-quadratic form (K,λ,μ) determines an ϵ-quadratic isomorphism class of split forms (K,ψ) as in 3.1, and every (iso)morphism of ϵ-quadratic forms lifts to an ϵ-quadratic (iso)morphism of split forms. □

Thus $Q_\epsilon(K)$ is both the group of isomorphism classes of ϵ-quadratic forms and the group of ϵ-quadratic isomorphism classes of split forms on a f. g. projective Λ-module K.

The following algebraic result will be used in 4.6 below to obtain a homological split form ψ on the kernel $\mathbb{Z}[\pi_1(X)]$-module $K_n(M)$ of an n-connected $2n$-dimensional normal map $(f,b) : M \to X$ with some extra structure, which determines the kernel $(-1)^n$-quadratic form $(K_n(M),\lambda,\mu)$ as in 4.2 (i).

LEMMA 4.3 *Let (K,λ,μ) be an ϵ-quadratic form over Λ.*
(i) *If $s : K \to K$ is an endomorphism such that*

$$\begin{pmatrix} s \\ 1-s \end{pmatrix} \ : \ (K,0,0) \to (K,\lambda,\mu) \oplus (K,-\lambda,-\mu)$$

defines a morphism of ϵ-quadratic forms then $(K,\lambda s)$ is a split form which determines the ϵ-quadratic form (K,λ,μ).
(ii) *If (K,λ,μ) is nonsingular and (K,ψ) is a split form which determines (K,λ,μ) then*

$$s \ = \ \lambda^{-1}\psi \ : \ K \to K$$

is an endomorphism such that

$$\begin{pmatrix} s \\ 1-s \end{pmatrix} \ : \ (K,0,0) \to (K,\lambda,\mu) \oplus (K,-\lambda,-\mu)$$

defines a morphism of ϵ-quadratic forms.
PROOF: (i) By 4.2 there exist a split form (K,ψ) which determines

(K,λ,μ) and an ϵ-quadratic morphism of split forms

$$(\begin{pmatrix} s \\ 1-s \end{pmatrix}, \chi) \ : \ (K,0) \to (K,\psi) \oplus (K,-\psi) \ .$$

It follows from

$$\lambda \ = \ \psi + \epsilon\psi^* \ : \ K \to K^* \ ,$$
$$\begin{pmatrix} s \\ 1-s \end{pmatrix}^* \begin{pmatrix} \psi & 0 \\ 0 & -\psi \end{pmatrix} \begin{pmatrix} s \\ 1-s \end{pmatrix} \ = \ \chi - \epsilon\chi^* \ : \ K \to K^*$$

that

$$\lambda s - \psi \ = \ \chi' - \epsilon\chi'^* \ : \ K \to K^*$$

with

$$\chi' \ = \ \chi - s^*\psi \ : \ K \to K^* \ .$$

(ii) From the definitions. □

In the terminology of §5 the morphism of 4.3 (ii)

$$\begin{pmatrix} s \\ 1-s \end{pmatrix} \ : \ (K,0,0) \to (K,\lambda,\mu) \oplus (K,-\lambda,-\mu)$$

is the inclusion of a lagrangian

$$\begin{aligned} L \ &= \ \mathrm{im}(\begin{pmatrix} s \\ 1-s \end{pmatrix} : K \to K \oplus K) \\ &= \ \ker((\,(-1)^{n-1}\psi^* \quad \psi\,) : K \oplus K \to K^*) \ . \end{aligned}$$

Example 4.4 A $(2n-1)$-*knot* is an embedding of a homotopy $(2n-1)$-sphere in a standard $(2n+1)$-sphere

$$\ell \ : \ \Sigma^{2n-1} \hookrightarrow S^{2n+1} \ .$$

For $n=1$ this is just a classical knot $\ell : \Sigma^1 = S^1 \hookrightarrow S^3$; for $n \geq 3$ Σ^{2n-1} is homeomorphic to S^{2n-1}, by the generalized Poincaré conjecture, but may have an exotic differentiable structure. Split forms (K,ψ) first appeared as the Seifert forms over $\mathbb{Z}$ of $(2n-1)$-knots, originally for $n=1$. See Ranicki [21, 7.8], [24] for a surgery treatment of high-dimensional knot theory. In particular, a Seifert form is an integral refinement of an even-dimensional surgery kernel form, as follows.

(i) A $(2n-1)$-knot $\ell : \Sigma^{2n-1} \hookrightarrow S^{2n+1}$ is *simple* if

$$\pi_r(S^{2n+1}\backslash\ell(\Sigma^{2n-1})) \ = \ \pi_r(S^1) \ \ (1 \leq r \leq n-1) \ .$$

(Every 1-knot is simple). A simple $(2n-1)$-knot ℓ has a simple Seifert surface, that is an $(n-1)$-connected framed codimension 1 submanifold $M^{2n} \subset S^{2n+1}$ with boundary

$$\partial M \ = \ \ell(\Sigma^{2n-1}) \subset S^{2n+1} \ .$$

The kernel of the n-connected normal map

$$(f,b) \;=\; \text{inclusion} \;:\; (M,\partial M) \to (X,\partial X) \;=\; (D^{2n+2},\ell(\Sigma^{2n-1}))$$

is a nonsingular $(-1)^n$-quadratic form $(H_n(M),\lambda,\mu)$ over $\mathbb{Z}$. The *Seifert form* of ℓ with respect to M is the refinement of $(H_n(M),\lambda,\mu)$ to a $(-1)^n$-nonsingular split form $(H_n(M),\psi)$ over $\mathbb{Z}$ which is defined using Alexander duality and the universal coefficient theorem

$$\psi \;=\; i_* \;:\; H_n(M) \to H_n(S^{2n+1}\backslash M) \;\cong\; H^n(M) \;\cong\; H_n(M)^*$$

with $i : M \to S^{2n+1}\backslash M$ the map pushing M off itself along a normal direction in S^{2n+1}. If $i' : M \to S^{2n+1}\backslash M$ pushes M off itself in the opposite direction

$$i'_* \;=\; (-1)^{n+1}\psi^* \;:\; H_n(M) \to H_n(S^{2n+1}\backslash M) \;\cong\; H^n(M) \;\cong\; H_n(M)^*$$

with

$$\begin{aligned} i_* - i'_* \;&=\; \psi + (-1)^n\psi^* \;=\; \lambda \\ &=\; ([M]\cap -)^{-1} \;:\; H_n(M) \to H^n(M) \;\cong\; H_n(M)^* \end{aligned}$$

the Poincaré duality isomorphism. If $x_1,x_2,\dots,x_k \in H_n(M)$ is a basis then $(\psi(x_j,x_{j'}))$ is a *Seifert matrix* for the $(2n-1)$-knot ℓ. For any embeddings $x,y : S^n \hookrightarrow M$

$$\begin{aligned} \psi(x,y) \;&=\; \text{linking number}(ix(S^n)\cup y(S^n) \subset S^{2n+1}) \\ &=\; \text{degree}(y^*ix : S^n \to S^n) \in \mathbb{Z} \end{aligned}$$

with

$$y^*ix \;:\; S^n \xrightarrow{\;x\;} M \xrightarrow{\;i\;} S^{2n+1}\backslash M \xrightarrow{\;y^*\;} S^{2n+1}\backslash y(S^n) \simeq S^n \;.$$

For $n \geq 3$ every element $x \in H_n(M)$ is represented by an embedding $e : S^n \hookrightarrow M$, using the Whitney embedding theorem, and $\pi_1(M) = \{1\}$. Moreover, for any embedding $x : S^n \hookrightarrow M$ the framed embedding $M \hookrightarrow S^{2n+1}$ determines a stable trivialization of the normal bundle $\nu_x : S^n \to BSO(n)$

$$\delta\nu_x \;:\; \nu_x \oplus \epsilon \;\cong\; \epsilon^{n+1}$$

such that

$$\psi(x,x) \;=\; (\delta\nu_x,\nu_x) \in \pi_{n+1}(BSO(n+1),BSO(n)) \;=\; \mathbb{Z} \;.$$

Every element $x \in H_n(M)$ is represented by an embedding

$$e_1 \times e_2 \;:\; S^n \hookrightarrow M \times \mathbb{R}$$

with $e_1 : S^n \looparrowright M$ a framed immersion such that the composite

$$S^n \xrightarrow{\;e_1\times e_2\;} M \times \mathbb{R} \hookrightarrow S^{2n+1}$$

is isotopic to the standard framed embedding $S^n \hookrightarrow S^{2n+1}$. Then

$$\psi(x,x) \ = \sum_{(a,b)\in D_2(e_1),e_2(a)<e_2(b)} I(a,b) \in \mathbb{Z}$$

is an integral lift of the geometric self-intersection (2.15 (ii))

$$\mu(x) \ = \sum_{(a,b)\in D_2(e_1)/\mathbb{Z}_2} I(a,b) \in Q_{(-1)^n}(\mathbb{Z}) \ ,$$

with

$$D_2(e_1) \ = \ \{(a,b)\in S^n\times S^n \,|\, a\neq b\in S^n, e_1(a)=e_1(b)\in M\}$$

the double point set. For even n $\psi(x,x) = \mu(x) \in Q_{+1}(\mathbb{Z}) = \mathbb{Z}$, while for odd n $\psi(x,x) \in \mathbb{Z}$ is a lift of $\mu(x) \in Q_{-1}(\mathbb{Z}) = \mathbb{Z}_2$. The Seifert form $(H_n(M),\psi)$ is such that $\psi(x,x) = 0$ if (and for $n \geq 3$ only if) $x \in H_n(M)$ can be killed by an ambient surgery on $M^{2n} \subset S^{2n+1}$, i.e. represented by a framed embedding of pairs

$$x \ : \ (D^{n+1}\times D^n, S^n\times D^n) \hookrightarrow (S^{2n+1}\times[0,1], M\times\{0\})$$

so that the effect of the surgery on M is another Seifert surface for the $(2n-1)$-knot ℓ

$$M' \ = \ \mathrm{cl.}(M\times x(S^n\times D^n))\cup D^{n+1}\times S^{n-1}\subset S^{2n+1} \ .$$

If $x \in H_n(M)$ generates a direct summand $L = \langle x\rangle \subset H_n(M)$ then M' is also $(n-1)$-connected, with Seifert form

$$(H_n(M'),\psi') \ = \ (L^\perp/L,[\psi]) \ ,$$

where

$$L^\perp \ = \ \{y\in H_n(M)\,|\,(\psi+(-1)^n\psi^*)(x)(y)=0 \text{ for } x\in L\}\subseteq H_n(M) \ .$$

(ii) Every $(-1)^n$-nonsingular split form (K,ψ) over $\mathbb{Z}$ is realized as the Seifert form of a simple $(2n-1)$-knot $\ell : \Sigma^{2n-1} \hookrightarrow S^{2n+1}$ (Kervaire [6]). From the algebraic surgery point of view the realization proceeds as follows. By 2.17 the nonsingular $(-1)^n$-quadratic form (K,λ,μ) determined by (K,ψ) (4.2 (i)) is the kernel form of an n-connected $2n$-dimensional normal map

$$(f,b) \ : \ (M^{2n},\Sigma^{2n-1}) \to (D^{2n},S^{2n-1})$$

with $f| : \Sigma^{2n-1} \to S^{2n-1}$ a homotopy equivalence. The double of (f,b) defines an n-connected $2n$-dimensional normal map

$$(g,c) \ = \ (f,b)\cup-(f,b) \ : \ N^{2n} \ = \ M\cup_{\Sigma^{2n-1}}-M \to D^{2n}\cup_{S^{2n-1}}-D^{2n} \ = \ S^{2n}$$

with kernel form $(K\oplus K,\lambda\oplus-\lambda,\mu\oplus-\mu)$. The direct summand

$$L \ = \ \ker(\begin{pmatrix}(-1)^{n-1}\psi^* & \psi\end{pmatrix} : K\oplus K\to K^*)\subset K\oplus K$$

is such that for any $(x,y)\in L$

$$\mu(x)-\mu(y) \ = \ (1+(-1)^{n-1})\psi(x,x) \ = \ 0\in Q_{(-1)^n}(\mathbb{Z}) \ .$$

Let $k = \text{rank}_{\mathbb{Z}}(K)$. The trace of the k surgeries on (g,c) killing a basis $(x_j, y_j) \in K \oplus K$ $(j = 1, 2, \dots, k)$ for L is an n-connected $(2n+1)$-dimensional normal map

$$(W^{2n+1}; N, S^{2n}) \to S^{2n} \times ([0,1]; \{0\}, \{1\})$$

such that

$$\ell \;:\; \Sigma^{2n-1} \hookrightarrow (\Sigma^{2n-1} \times D^2) \cup (W \cup D^{2n+1}) \cup (M \times [0,1]) \;\cong\; S^{2n+1}$$

is a simple $(2n-1)$-knot with Seifert surface M and Seifert form (K, ψ). Note that M itself is entirely determined by the $(-1)^n$-quadratic form (K, λ, μ), with $\text{cl.}(M \backslash D^{2n})$ the trace of k surgeries on S^{2n-1} removing

$$\bigcup_k S^{n-1} \times D^n \hookrightarrow S^{2n-1}$$

with (self-)linking numbers (λ, μ). The embedding $M \hookrightarrow S^{2n+1}$ is determined by the choice of split structure ψ for (λ, μ).
(iii) In particular, (ii) gives a knot version of the plumbing construction (2.18): let G be a finite graph with vertices $v_1, v_2, \dots, v_k$, weighted by $\mu_1, \mu_2, \dots, \mu_k \in Q_{(-1)^n}(\mathbb{Z})$, so that there are defined a $(-1)^n$-quadratic form $(\mathbb{Z}^k, \lambda, \mu)$ and a plumbed stably parallelizable $(n-1)$-connected $2n$-dimensional manifold with boundary

$$M^{2n} \;=\; M(G, \mu_1, \mu_2, \dots, \mu_k) \;,$$

killing $H_1(G)$ by surgery if G is not a tree. A choice of split form ψ for (λ, μ) determines a compression of a framed embedding $M \hookrightarrow S^{2n+j}$ (j large) to a framed embedding $M \hookrightarrow S^{2n+1}$, so that $\partial M \hookrightarrow S^{2n+1}$ is a codimension 2 framed embedding. The form $(\mathbb{Z}^k, \lambda, \mu)$ is nonsingular if and only if $\Sigma^{2n-1} = \partial M$ is a homotopy $(2n-1)$-sphere, in which case $\Sigma^{2n-1} \hookrightarrow S^{2n+1}$ is a simple $(2n-1)$-knot with simple Seifert surface M.
(iv) Given a simple $(2n-1)$-knot $\ell : \Sigma^{2n-1} \hookrightarrow S^{2n+1}$ and a simple Seifert surface $M^{2n} \hookrightarrow S^{2n+1}$ there is defined an n-connected $2n$-dimensional normal map

$$(f,b) \;=\; \text{inclusion} \;:\; (M, \partial M) \to (X, \partial X) \;=\; (D^{2n+2}, \ell(\Sigma^{2n-1}))$$

as in (i). The knot complement is a $(2n+1)$-dimensional manifold with boundary

$$(W, \partial W) \;=\; (\text{cl.}(S^{2n+1} \backslash (\ell(\Sigma^{2n-1}) \times D^2)), \ell(\Sigma^{2n-1}) \times S^1)$$

with a $\mathbb{Z}$-homology equivalence $p : (W, \partial W) \to S^1$ such that

$$\begin{aligned} p| \;&=\; \text{projection} \;:\; \partial W \;=\; \Sigma^{2n-1} \times S^1 \to S^1 \;, \\ p^{-1}(\text{pt.}) \;&=\; M \subset W \;. \end{aligned}$$

Cutting W along $M \subset W$ there is obtained a cobordism $(N; M, M')$ with M' a copy of M, and N a deformation retract of $S^{2n+1} \backslash M$, such that (f, b)

extends to an n-connected normal map

$$(g,c) \;:\; (N;M,M') \to X \times ([0,1];\{0\},\{1\})$$

with $(g,c)| = (f',b') : M' \to X$ a copy of (f,b). The n-connected $(2n+1)$-dimensional normal map

$$\begin{aligned}(h,d) \;&=\; (g,c)/((f,b)=(f',b')) \;:\\ (W,\partial W) \;&=\; (N;M,M')/(M=M') \to (X,\partial X)\times S^1\end{aligned}$$

is a $\mathbb{Z}$-homology equivalence which is the identity on ∂W, and such that

$$(f,b) \;=\; (h,d)| \;:\; (M,\partial M) \;=\; h^{-1}((X,\partial X)\times\{\text{pt.}\}) \to (X,\partial X)\;. \quad \square$$

EXAMPLE 4.5 (i) Split forms over group rings arise in the following geometric situation, generalizing 4.4 (iv).

Let X be a $2n$-dimensional Poincaré complex, and let $(h,d) : W \to X \times S^1$ be an n-connected $(2n+1)$-dimensional normal map which is a $\mathbb{Z}[\pi_1(X)]$-homology equivalence. Cut (h,d) along $X \times \{\text{pt.}\} \subset X \times S^1$ to obtain an n-connected $2n$-dimensional normal map

$$(f,b) \;=\; (h,d)| \;:\; M \;=\; h^{-1}(\{\text{pt.}\}) \to X$$

and an n-connected normal bordism

$$(g,c) \;:\; (N;M,M') \to X \times ([0,1];\{0\},\{1\})$$

with N a deformation retract of $W\backslash M$, such that $(g,c)| = (f,b) : M \to X$, and such that $(g,c)| = (f',b') : M' \to X$ is a copy of (f,b). The inclusions $i : M \hookrightarrow N$, $i' : M' \hookrightarrow N$ induce $\mathbb{Z}[\pi_1(X)]$-module morphisms

$$i_* \;:\; K_n(M) \to K_n(N) \quad,\quad i'_* \;:\; K_n(M') \;=\; K_n(M) \to K_n(N)$$

which fit into an exact sequence

$$K_{n+1}(W) = 0 \longrightarrow K_n(M) \xrightarrow{i_* - i'_*} K_n(N) \longrightarrow K_n(W) = 0\;,$$

so that $i_* - i'_* : K_n(M) \to K_n(N)$ is an isomorphism. Let $(K_n(M),\lambda,\mu)$ be the kernel $(-1)^n$-quadratic form of (f,b). The endomorphism

$$s \;=\; (i_* - i'_*)^{-1} i_* \;:\; K_n(M) \to K_n(M)$$

is such that

$$\begin{pmatrix} s \\ 1-s \end{pmatrix} \;:\; (K_n(M),0,0) \to (K_n(M),\lambda,\mu) \oplus (K_n(M),-\lambda,-\mu)$$

defines a morphism of $(-1)^n$-quadratic forms, so that by 4.3 the split form $(K_n(M),\psi)$ with

$$\psi \;=\; \lambda s \;:\; K_n(M) \to K_n(M)^*$$

determines $(K_n(M),\lambda,\mu)$. Every element $x \in K_n(M)$ can be represented by a framed immersion $x : S^n \looparrowright M$ with a null-homotopy $fx \simeq * : S^n \to$

X. Use the null-homotopy and the normal $\mathbb{Z}[\pi_1(X)]$-homology equivalence $(h,d): W \to X\times S^1$ to extend x to a framed immersion $\delta x : D^{n+1} \looparrowright W$. If $x_1, x_2, \dots, x_k \in K_n(M)$ is a basis for the kernel f.g. free $\mathbb{Z}[\pi_1(X)]$-module then

$$s(x_j) \;=\; \sum_{j'=1}^{k} s_{jj'}x_{j'} \in K_n(M)$$

with

$$\begin{aligned} s_{jj'} \;&=\; \text{linking number}(ix_j(S^n)\cup x_{j'}(S^n)\subset W) \\ &=\; \text{intersection number}(ix_j(S^n)\cap \delta x_{j'}(D^{n+1})\subset W)\in \mathbb{Z}[\pi_1(X)]\ . \end{aligned}$$

The split form $(K_n(M),\psi)$ is thus a (non-simply connected) Seifert form.
(ii) Suppose given an n-connected $2n$-dimensional normal map (f,b) : $(M,\partial M)\to(X,\partial X)$, with kernel $(-1)^n$-quadratic form $(K_n(M),\lambda,\mu)$ over $\mathbb{Z}[\pi_1(X)]$. A choice of split form ψ for (λ,μ) can be realized by an $(n+1)$-connected $(2n+2)$-dimensional normal map

$$(g,c)\ :\ (L,\partial L)\to (X\times D^2, X\times S^1\cup \partial X\times D^2)$$

which is a $\mathbb{Z}[\pi_1(X)]$-homology equivalence with

$$\begin{aligned} (f,b) \;&=\; (g,c)|\ :\ (M,\partial M) \;=\; g^{-1}((X,\partial X)\times\{0\})\to (X,\partial X)\ , \\ H_{*+1}(\widetilde{L},\widetilde{M}) \;&=\; K_*(M)\ (=\ 0 \text{ for } *\neq n) \end{aligned}$$

as follows. The inclusion $\partial M\hookrightarrow \partial L$ is a codimension 2 embedding with Seifert surface $M\hookrightarrow \partial L$ and Seifert form $(K_n(M),\psi)$ as in the relative version of (i), with

$$(h,d) \;=\; (g,c)|\ :\ W \;=\; \partial L\to X\times S^1\cup \partial X\times D^2\ .$$

The choice of split form ψ for (λ,μ) determines a sequence of surgeries on the n-connected $(2n+1)$-dimensional normal map

$$(f,b)\times 1_{[0,1]}\ :\ M\times([0,1];\{0\},\{1\})\to X\times([0,1];\{0\},\{1\})$$

killing the (stably) f. g. free $\mathbb{Z}[\pi_1(X)]$-module

$$K_n(M\times[0,1]) \;=\; K_n(M)\ ,$$

obtaining a rel ∂ normal bordant map

$$(f_N,b_N)\ :\ (N;M,M')\to X\times([0,1];\{0\},\{1\})$$

with $K_i(N)=0$ for $i\neq n$. The $\mathbb{Z}[\pi_1(X)]$-module morphisms induced by the inclusions $i: M\hookrightarrow N$, $i' : M'\hookrightarrow N$

$$i_*\ :\ K_n(M)\to K_n(N)\quad ,\quad i'_*\ :\ K_n(M) \;=\; K_n(M')\to K_n(N)$$

are such that $i_*-i'_*: K_n(M)\to K_n(N)$ is a $\mathbb{Z}[\pi_1(X)]$-module isomorphism, with

$$\psi\ :\ K_n(M)\xrightarrow{(i_*-i'_*)^{-1}i_*} K_n(M)\xrightarrow{\text{adjoint}(\lambda)} K_n(M)^*\ .$$

Thus it is possible to identify

$$i_* \;=\; \psi \;:\; K_n(M) \to K_n(N) \;\cong\; K_n(M)^* \;,$$

$$i'_* \;=\; (-1)^{n+1}\psi^* \;:\; K_n(M) \;=\; K_n(M') \to K_n(N) \;\cong\; K_n(M)^*$$

with

$$i_* - i'_* \;=\; \psi + (-1)^n\psi^* \;=\; \text{adjoint}(\lambda) \;:\; K_n(M) \xrightarrow{\cong} K_n(M)^* \;.$$

The $(2n+1)$-dimensional manifold with boundary defined by

$$(V, \partial V) \;=\; (N/(M = M'), \partial M \times S^1)$$

is equipped with a normal map

$$(f_V, b_V) \;:\; (V, \partial V) \to (X \times S^1, \partial X \times S^1)$$

which is an n-connected $\mathbb{Z}[\pi_1(X)]$-homology equivalence, with $K_j(V) = 0$ for $j \neq n+1$ and

$$K_{n+1}(V) \;=\; \text{coker}(z\psi + (-1)^n\psi^* : K_n(M)[z, z^{-1}] \to K_n(M)^*[z, z^{-1}])$$

identifying

$$\mathbb{Z}[\pi_1(X \times S^1)] \;=\; \mathbb{Z}[\pi_1(X)][z, z^{-1}] \;\; (\overline{z} = z^{-1}) \;.$$

The trace of the surgeries on $(f, b) \times 1_{[0,1]}$ gives an extension of (f_V, b_V) to an $(n+1)$-connected $(2n+2)$-dimensional normal bordism

$$(f_U, b_U) \;:\; (U; V, M \times S^1) \to X \times S^1 \times ([0,1]; \{0\}, \{1\})$$

with $K_i(U) = 0$ for $i \neq n+1$ and (singular) kernel $(-1)^{n+1}$-quadratic form over $\mathbb{Z}[\pi_1(X)][z, z^{-1}]$

$$\begin{aligned}&(K_{n+1}(U), \lambda_U, \mu_U)\\ &\;=\; (K_n(M)[z, z^{-1}], (1-z)\psi + (-1)^{n+1}(1-z^{-1})\psi^*, (1-z)\psi) \;.\end{aligned}$$

The $(2n+2)$-dimensional manifold with boundary defined by

$$(W, \partial W) \;=\; (M \times D^2 \cup U, \partial M \times D^2 \cup V)$$

is such that (f, b) extends to an $(n+1)$-connected normal map

$$(g, c) \;=\; (f, b) \times 1_{D^2} \cup (f_U, b_U) \;:\; (W, \partial W) \to (X \times D^2, \partial X \times D^2 \cup X \times S^1)$$

which is a $\mathbb{Z}[\pi_1(X)]$-homology equivalence, with $H_{n+1}(\widetilde{W}, \widetilde{M}) = K_n(M)$. See Example 27.9 of Ranicki [24] for further details (noting that the split form ψ here corresponds to the asymmetric form λ there).

(iii) Given a simple knot $\ell : \Sigma^{2n-1} \hookrightarrow S^{2n+1}$ and a simple Seifert surface $M^{2n} \subset S^{2n+1}$ there is defined an n-connected normal map

$$(f, b) \;=\; \text{inclusion} \;:\; (M^{2n}, \partial M) \to (X, \partial X) \;=\; (D^{2n+2}, \ell(\Sigma^{2n-1}))$$

with a Seifert form ψ on $K_n(M) = H_n(M)$, as in 4.4. For $n \geq 2$ the surgery construction of (i) applied to (f, b), ψ recovers the knot

$$\ell \;:\; \Sigma^{2n-1} \;=\; \partial M \hookrightarrow \partial W \;=\; S^{2n+1}$$

with $M^{2n} \subset W = D^{2n+2}$ the Seifert surface pushed into the interior of D^{2n+2}. The knot complement

$$(V^{2n+1}, \partial V) = (\mathrm{cl.}\big(S^{2n+1}\backslash(\ell(\Sigma^{2n-1}) \times D^2)\big), \ell(\Sigma^{2n-1}) \times S^1)$$

is such that there is defined an n-connected $(2n+1)$-dimensional normal map

$$(f_V, b_V) : (V, \partial V) \to (X, \partial X) \times S^1$$

which is a homology equivalence, with

$$(f_V, b_V)| = (f, b) : (M, \partial M) = (f_V)^{-1}((X, \partial X) \times \{*\}) \to (X, \partial X) .$$

Cutting (f_V, b_V) along (f, b) results in a normal map as in (i)

$$(f_N, b_N) : (N^{2n+1}; M^{2n}, M'^{2n}) \to X \times ([0,1]; \{0\}, \{1\}) . \qquad \square$$

§5. Surgery on forms

§5 develops algebraic surgery on forms. The effect of a geometric surgery on an n-connected $2n$-dimensional normal map is an algebraic surgery on the kernel $(-1)^n$-quadratic form. Moreover, geometric surgery is possible if and only if algebraic surgery is possible.

Given an ϵ-quadratic form (K, λ, μ) over Λ it is possible to kill an element $x \in K$ by algebraic surgery if and only if $\mu(x) = 0 \in Q_\epsilon(\Lambda)$ and x generates a direct summand $\langle x \rangle = \Lambda x \subset K$. The effect of the surgery is the ϵ-quadratic form (K', λ', μ') defined on the subquotient $K' = \langle x \rangle^\perp / \langle x \rangle$ of K, with $\langle x \rangle^\perp = \{y \in K \,|\, \lambda(x, y) = 0 \in \Lambda\}$.

Definition 5.1 (i) Given an ϵ-symmetric form (K, λ) and a submodule $L \subseteq K$ define the *orthogonal* submodule

$$\begin{aligned} L^\perp &= \{x \in K \,|\, \lambda(x, y) = 0 \in \Lambda \text{ for all } y \in L\} \\ &= \ker(i^*\lambda : K \to L^*) \end{aligned}$$

with $i : L \to K$ the inclusion. If (K, λ) is nonsingular and L is a direct summand of K then so is $L^\perp$.
(ii) A *sublagrangian* of a nonsingular ϵ-quadratic form (K, λ, μ) over Λ is a direct summand $L \subseteq K$ such that

$$\mu(L) = \{0\} \subseteq Q_\epsilon(\Lambda) ,$$

and

$$\lambda(L)(L) = \{0\} \ , \ L \subseteq L^\perp .$$

(iii) A *lagrangian* of (K, λ, μ) is a sublagrangian L such that $L^\perp = L$. $\square$

The main result of §5 is that the inclusion of a sublagrangian is a morphism of ϵ-quadratic forms

$$i \ : \ (L,0,0) \to (K,\lambda,\mu)$$

which extends to an isomorphism

$$f \ : \ H_\epsilon(L) \oplus (L^\perp/L,[\lambda],[\mu]) \xrightarrow{\cong} (K,\lambda,\mu)$$

with $H_\epsilon(L)$ the hyperbolic ϵ-quadratic form (2.14).

Example 5.2 Let $(f,b) : M^{2n} \to X$ be an n-connected $2n$-dimensional normal map with kernel $(-1)^n$-quadratic form $(K_n(M),\lambda,\mu)$ over $\mathbb{Z}[\pi_1(X)]$, and $n \geq 3$. An element $x \in K_n(M)$ generates a sublagrangian $L = \langle x \rangle \subset K_n(M)$ if and only if it can be killed by surgery on $S^n \times D^n \hookrightarrow M$ with trace an n-connected normal bordism

$$((g,c);(f,b),(f',b')) \ : \ (W^{2n+1};M^{2n},M'^{2n}) \to X \times ([0,1];\{0\},\{1\})$$

such that $K_{n+1}(W,M') = 0$. The kernel form of the effect of such a surgery

$$(f',b') \ : \ M' \ = \ \mathrm{cl.}(M \backslash S^n \times D^n) \cup D^{n+1} \times S^{n-1} \to X$$

is given by

$$(K_n(M'),\lambda',\mu') \ = \ (L^\perp/L,[\lambda],[\mu]) \ .$$

There exists an n-connected normal bordism (g,c) of (f,b) to a homotopy equivalence (f',b') with $K_{n+1}(W,M') = 0$ if and only if $(K_n(M),\lambda,\mu)$ admits a lagrangian. □

Remark 5.3 There are other terminologies. In the classical theory of quadratic forms over fields a lagrangian is a "maximal isotropic subspace". Wall called hyperbolic forms "kernels" and the lagrangians "subkernels". Novikov called hyperbolic forms "hamiltonian", and introduced the name "lagrangian", because of the analogy with the hamiltonian formulation of physics. □

Example 5.4 An n-connected $(2n+1)$-dimensional normal bordism

$$((g,c);(f,b),(f',b')) \ : \ (W^{2n+1};M^{2n},M'^{2n}) \to X \times ([0,1];\{0\},\{1\})$$

with $K_{n+1}(W,M') = 0$ determines a sublagrangian

$$L \ = \ \mathrm{im}(K_{n+1}(W,M) \to K_n(M)) \subset K_n(M)$$

of the kernel $(-1)^n$-quadratic form (K,λ,μ) of (f,b), with $K = K_n(M)$. The sublagrangian L is a lagrangian if and only if (f',b') is a homotopy equivalence. W has a handle decomposition on M of the type

$$W \ = \ M \times I \cup \bigcup_k (n+1)\text{-handles } D^{n+1} \times D^n \ ,$$

and $L \cong K_{n+1}(W,M) \cong \mathbb{Z}[\pi_1(X)]^k$ is a f. g. free $\mathbb{Z}[\pi_1(X)]$-module with rank the number k of $(n+1)$-handles. The exact sequences of stably f. g. free $\mathbb{Z}[\pi_1(X)]$-modules

$$0 \to K_{n+1}(W,M) \to K_n(M) \to K_n(W) \to 0 \ ,$$
$$0 \to K_n(M') \to K_n(W) \to K_n(W,M') \to 0$$

are isomorphic to

$$0 \to \ L \xrightarrow{\ i\ } K \to K/L \to 0 \ ,$$
$$0 \to L^{\perp}/L \to K/L \xrightarrow{[i^*\lambda]} L^* \to 0 \ . \qquad \square$$

Definition 5.5 (i) A *sublagrangian* of an ϵ-nonsingular split form (K,ψ) is an ϵ-quadratic morphism of split forms

$$(i,\theta) \ : \ (L,0) \to (K,\psi)$$

with $i : L \to K$ a split injection.
(ii) A *lagrangian* of (K,ψ) is a sublagrangian such that the sequence

$$0 \to L \xrightarrow{\ i\ } K \xrightarrow{i^*(\psi+\epsilon\psi^*)} L^* \to 0$$

is exact. $\square$

An ϵ-nonsingular split form (K,ψ) admits a (sub)lagrangian if and only if the associated ϵ-quadratic form (K,λ,μ) admits a (sub)lagrangian. (Sub)lagrangians in split ϵ-quadratic forms are thus (sub)lagrangians in ϵ-quadratic forms with the $(-\epsilon)$-quadratic structure θ, which (following Novikov) is sometimes called the "hessian" form.

Definition 5.6 The ϵ-nonsingular *hyperbolic* split form $H_\epsilon(L)$ is given for any f. g. projective Λ-module L by

$$H_\epsilon(L) \ = \ (L\oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ : \ L\oplus L^* \to (L\oplus L^*)^* \ = \ L^*\oplus L) \ ,$$

with lagrangian $(i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0) : (L,0) \to H_\epsilon(L)$. $\square$

Theorem 5.7 *An ϵ-nonsingular split form (K,ψ) admits a lagrangian if and only if it is ϵ-quadratic isomorphic isomorphic to the hyperbolic form $H_\epsilon(L)$. Moreover, the inclusion $(i,\theta) : (L,0) \to (K,\psi)$ of a lagrangian extends to an ϵ-quadratic isomorphism of split forms $(f,\chi) : H_\epsilon(L) \cong (K,\psi)$.*
Proof: An isomorphism of forms sends lagrangians to lagrangians, so any form isomorphic to a hyperbolic has at least one lagrangian. Conversely suppose that (K,ψ) has a lagrangian $(i,\theta) : (L,0) \to (K,\psi)$. An

extension of (i,θ) to an ϵ-quadratic isomorphism $(f,\chi) : H_\epsilon(L) \cong (K,\psi)$ determines a lagrangian $f(L^*) \subset K$ complementary to L. Construct an isomorphism f by choosing a complementary lagrangian to L in (K,ψ). Let $i \in \mathrm{Hom}_\Lambda(L,K)$ be the inclusion, and choose a splitting $j' \in \mathrm{Hom}_\Lambda(L^*,K)$ of $i^*(\psi+\epsilon\psi^*) \in \mathrm{Hom}_\Lambda(K,L^*)$, so that

$$i^*(\psi+\epsilon\psi^*)j' \ = \ 1 \in \mathrm{Hom}_\Lambda(L^*,L^*) \ .$$

In general, $j' : L^* \to K$ is not the inclusion of a lagrangian, with $j'^*\psi j' \neq 0 \in Q_\epsilon(L^*)$. Given any $k \in \mathrm{Hom}_\Lambda(L^*,L)$ there is defined another splitting

$$j \ = \ j' + ik \ : \ L^* \to K$$

such that

$$\begin{aligned} j^*\psi j \ &= \ j'^*\psi j' + k^*i^*\psi ik + k^*i^*\psi j' + j'^*\psi ik \\ &= \ j'^*\psi j' + k \in Q_\epsilon(L^*) \ . \end{aligned}$$

Choosing a representative $\psi \in \mathrm{Hom}_\Lambda(K,K^*)$ of $\psi \in Q_\epsilon(K)$ and setting

$$k \ = \ -j'^*\psi j' \ : \ L^* \to L^*$$

there is obtained a splitting $j : L^* \to K$ which is the inclusion of a lagrangian

$$(j,\nu) \ : \ (L^*,0) \to (K,\psi) \ .$$

The isomorphism of ϵ-quadratic forms

$$(i \ j) \ = \ \begin{pmatrix} \theta & 0 \\ j^*\psi i & \psi \end{pmatrix} \ : \ H_\epsilon(L) \xrightarrow{\cong} (K,\psi)$$

is an ϵ-quadratic isomorphism of split forms. □

Remark 5.8 Theorem 5.7 is a generalization of Witt's theorem on the extension to isomorphism of an isometry of quadratic forms over fields. The procedure for modifying the choice of complement to be a lagrangian is a generalization of the Gram-Schmidt method of constructing orthonormal bases in an inner product space. Ignoring the split structure 5.7 shows that a nonsingular ϵ-quadratic form admits a lagrangian (in the sense of 5.1 (iii)) if and only if it is isomorphic to a hyperbolic form. □

Corollary 5.9 *For any ϵ-nonsingular split form (K,ψ) the diagonal inclusion*

$$\Delta \ : \ K \to K \oplus K \ ; \ x \longmapsto (x,x)$$

extends to an ϵ-quadratic isomorphism of split forms

$$H_\epsilon(K) \xrightarrow{\cong} (K,\psi) \oplus (K,-\psi) \ .$$

Proof: Apply 5.7 to the inclusion of the lagrangian

$$(\Delta,0) \ : \ (K,0) \to (K \oplus K, \psi \oplus -\psi) \ .$$

(This result has already been used in 3.1). □

PROPOSITION 5.10 *The inclusion $(i,\theta):(L,0)\to(K,\psi)$ of a sublagrangian in an ϵ-nonsingular split form (K,ψ) extends to an isomorphism of forms*

$$(f,\chi)\ :\ H_\epsilon(L)\oplus(L^\perp/L,[\psi])\xrightarrow{\cong}(K,\psi)\ .$$

PROOF: For any direct complement L_1 to $L^\perp$ in K there is defined a Λ-module isomorphism

$$e\ :\ L_1\xrightarrow{\cong}L^*\ ;\ x\longmapsto(y\longmapsto(1+T_\epsilon)\psi(x,y))\ .$$

Define a Λ-module morphism

$$j\ :\ L^*\xrightarrow{e^{-1}}L_1\xrightarrow{\text{inclusion}}K\ .$$

The ϵ-nonsingular split form defined by

$$(H,\phi)\ =\ (L\oplus L^*,\begin{pmatrix}0&1\\0&j^*\psi j\end{pmatrix})$$

has lagrangian L, so that it is isomorphic to the hyperbolic form $H_\epsilon(L)$ by 5.7. Also, there is defined an ϵ-quadratic morphism of split forms

$$(g\ =\ (i\ j)\ ,\begin{pmatrix}\theta&i^*\psi j\\0&0\end{pmatrix})\ :\ (H,\phi)\to(K,\psi)$$

with $g:H\to K$ an injection split by

$$h\ =\ ((1+T_\epsilon)\phi)^{-1}g^*(1+T_\epsilon)\psi\ :\ K\to H\ .$$

The direct summand of K defined by

$$\begin{aligned}H^\perp\ &=\ \{x\in K\,|\,(1+T_\epsilon)\psi(x,gy)\ =\ 0\text{ for all }y\in H\}\\&=\ \ker(g^*(1+T_\epsilon)\psi:K\to H^*)\ =\ \ker(h:K\to H)\end{aligned}$$

is such that

$$K\ =\ g(H)\oplus H^\perp\ .$$

It follows from the factorization

$$i^*(1+T_\epsilon)\psi\ :\ K\xrightarrow{h}H\ =\ L\oplus L^*\xrightarrow{\text{projection}}L^*$$

that

$$L^\perp\ =\ \ker(i^*(1+T_\epsilon)\psi:K\to L^*)\ =\ L\oplus H^\perp\ .$$

The restriction of $\psi\in S(K)$ to $H^\perp$ defines an ϵ-nonsingular split form $(H^\perp,\phi^\perp)$. The injection g and the inclusion $g^\perp:H^\perp\to K$ are the components of a Λ-module isomorphism

$$f\ =\ (g\ g^\perp)\ :\ H\oplus H^\perp\to K$$

which defines an ϵ-quadratic isomorphism of split forms

$$(f,\chi)\ :\ (H,\phi)\oplus(H^\perp,\phi^\perp)\xrightarrow{\cong}(K,\psi)$$

with

$$(H^{\perp},\phi^{\perp}) \;\cong\; (L^{\perp}/L,[\psi])\ . \qquad \square$$

Example 5.11 An n-connected $(2n+1)$-dimensional normal bordism

$$((g,c);(f,b),(f',b'))\ :\ (W^{2n+1};M^{2n},M'^{2n}) \to X\times([0,1];\{0\},\{1\})$$

is such that W has a handle decomposition on M of the type

$$W \;=\; M\times I\cup\bigcup_k n\text{-handles } D^n\times D^{n+1}\cup\bigcup_{k'}(n+1)\text{-handles } D^{n+1}\times D^n\ .$$

Let

$$(W;M,M') \;=\; (W';M,M'')\cup_{M''}(W'';M'',M')$$

with

$$\begin{aligned} W' &= M\times[0,1]\cup\bigcup_k n\text{-handles } D^n\times D^{n+1}\ ,\\ M'' &= \text{cl.}(\partial W'\backslash M)\ ,\\ W'' &= M''\times[0,1]\cup\bigcup_{k'}(n+1)\text{-handles } D^{n+1}\times D^n\ . \end{aligned}$$

The restriction of (g,c) to M'' is an n-connected $2n$-dimensional normal map

$$(f'',b'')\ :\ M'' \;\cong\; M\#(\#_k S^n\times S^n) \;\cong\; M'\#(\#_{k'}S^n\times S^n)\to X$$

with kernel $(-1)^n$-quadratic form

$$\begin{aligned} (K_n(M''),\lambda'',\mu'') &\cong (K_n(M),\lambda,\mu)\oplus H_{(-1)^n}(\mathbb{Z}[\pi_1(X)]^k)\\ &\cong (K_n(M'),\lambda',\mu')\oplus H_{(-1)^n}(\mathbb{Z}[\pi_1(X)]^{k'})\ . \end{aligned}$$

Thus $(K_n(M''),\lambda'',\mu'')$ has sublagrangians

$$\begin{aligned} L &= \text{im}(K_{n+1}(W',M'')\to K_n(M'')) \;\cong\; \mathbb{Z}[\pi_1(X)]^k\ ,\\ L' &= \text{im}(K_{n+1}(W'',M'')\to K_n(M'')) \;\cong\; \mathbb{Z}[\pi_1(X)]^{k'} \end{aligned}$$

such that

$$\begin{aligned} (L^{\perp}/L,[\lambda''],[\mu'']) &\cong (K_n(M),\lambda,\mu)\ ,\\ (L'^{\perp}/L',[\lambda'']',[\mu'']') &\cong (K_n(M'),\lambda',\mu')\ . \end{aligned}$$

Note that L is a lagrangian if and only if $(f,b):M\to X$ is a homotopy equivalence. Similarly for L' and $(f',b'):M'\to X$. $\square$

§6. Short odd complexes

A "$(2n+1)$-complex" is the algebraic structure best suited to describing the surgery obstruction of an n-connected $(2n+1)$-dimensional normal map. In essence it is a 1-dimensional chain complex with $(-1)^n$-quadratic Poincaré duality.

As before, let Λ be a ring with involution.

DEFINITION 6.1 A $(2n+1)$-*complex over* Λ (C,ψ) is a f. g. free Λ-module chain complex of the type

$$C\ :\ \dots\to 0\to\ C_{n+1}\ \xrightarrow{d}\ C_n\to 0\to\dots$$

together with two Λ-module morphisms

$$\psi_0\ :\ C^n\ =\ (C_n)^*\to C_{n+1}\ ,\ \psi_1\ :\ C^n\to C_n$$

such that

$$d\psi_0+\psi_1+(-1)^{n+1}\psi_1^*\ =\ 0\ :\ C^n\to C_n\ ,$$

and such that the chain map

$$(1+T)\psi_0\ :\ C^{2n+1-*}\to C$$

defined by

$$\begin{aligned} d_{C^{2n+1-*}}\ &=\ (-1)^{n+1}d^*\ :\\ &\quad(C^{2n+1-*})_{n+1}\ =\ C^n\to(C^{2n+1-*})_n\ =\ C^{n+1}\ ,\\ (1+T)\psi_0\ &=\ \begin{cases}\psi_0\ :\ (C^{2n+1-*})_{n+1}\ =\ C^n\to C_{n+1}\\ \psi_0^*\ :\ (C^{2n+1-*})_n\ =\ C^{n+1}\to C_n\ ,\end{cases}\\ (C^{2n+1-*})_r\ &=\ C^{2n+1-r}\ =\ 0\text{ for }r\neq n,n+1\end{aligned}$$

is a chain equivalence

$$\begin{tikzcd} C^{2n+1-*}:\dots \arrow[r] \arrow[d, "(1+T)\psi_0"'] & 0 \arrow[r] \arrow[d] & C^n \arrow[r, "(-1)^{n+1}d^*"] \arrow[d, "\psi_0"'] & C^{n+1} \arrow[r] \arrow[d, "\psi_0^*"'] & 0 \arrow[r] \arrow[d] & \cdots \\ C\ :\dots \arrow[r] & 0 \arrow[r] & C_{n+1} \arrow[r, "d"] & C_n \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

□

REMARK 6.2 A $(2n+1)$-complex is essentially the inclusion of a lagrangian in a hyperbolic split $(-1)^n$-quadratic form

$$(\begin{pmatrix}\psi_0\\ d^*\end{pmatrix},-\psi_1)\ :\ (C^n,0)\to H_{(-1)^n}(C_{n+1})\ .$$

The chain map $(1+T)\psi_0:C^{2n+1-*}\to C$ is a chain equivalence if and only if the algebraic mapping cone

$$0 \to C^n \xrightarrow{\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix}} C_{n+1} \oplus C^{n+1} \xrightarrow{\begin{pmatrix} d & (-1)^n \psi_0^* \end{pmatrix}} C_n \to 0$$

is contractible, which is just the lagrangian condition. The triple

$$(\text{ form ; lagrangian , lagrangian }) \ = \ (H_{(-1)^n}(C_{n+1}); C_{n+1}, \mathrm{im}\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix})$$

is an example of a "$(-1)^n$-quadratic formation". Formations will be studied in greater detail in §9 below. □

EXAMPLE 6.3 Define a *presentation* of an n-connected $(2n+1)$-dimensional normal map $(f, b) : M^{2n+1} \to X$ to be a normal bordism

$$((g, c); (f, b), (f', b')) \ : \ (W^{2n+2}; M^{2n+1}, M'^{2n+1}) \to X \times ([0, 1]; \{0\}, \{1\})$$

such that $W \to X \times [0, 1]$ is n-connected, with

$$K_r(W) \ = \ 0 \text{ for } r \neq n+1 \ .$$

Then $K_{n+1}(W)$ a f. g. free $\mathbb{Z}[\pi_1(X)]$-module and W has a handle decomposition on M of the type

$$W \ = \ M \times I \cup \bigcup_k (n+1)\text{-handles } D^{n+1} \times D^{n+1} \ ,$$

and $K_{n+1}(W, M) \cong \mathbb{Z}[\pi_1(X)]^k$ is a f. g. free $\mathbb{Z}[\pi_1(X)]$-module with rank the number k of $(n+1)$-handles. Thus $(W; M, M')$ is the trace of surgeries on k disjoint embeddings $S^n \times D^{n+1} \hookrightarrow M^{2n+1}$ with null-homotopy in X representing a set of $\mathbb{Z}[\pi_1(X)]$-module generators of $K_n(M)$. For every n-connected $(2n+1)$-dimensional normal map $(f, b) : M^{2n+1} \to X$ the kernel $\mathbb{Z}[\pi_1(X)]$-module $K_n(M)$ is f. g., so that there exists a presentation $(g, c) : (W; M, M') \to X \times ([0, 1]; \{0\}, \{1\})$. Poincaré duality and the universal coefficient theorem give natural identifications of f. g. free $\mathbb{Z}[\pi_1(X)]$-modules

$$K_{n+1}(W) \ = \ K^{n+1}(W, \partial W) \ = \ K_{n+1}(W, \partial W)^* \ \ (\partial W = M \cup M') \ ,$$

$$K_{n+1}(W, M) \ = \ K^{n+1}(W, M') \ = \ K_{n+1}(W, M')^* \ .$$

The presentation determines a $(2n+1)$-complex (C, ψ) such that

$$H_*(C) \ = \ K_*(M) \ ,$$

with

$$d \ = \ (\text{inclusion})_* \ : \ C_{n+1} \ = \ K_{n+1}(W, M') \ \to C_n \ = \ K_{n+1}(W, \partial W) \ ,$$

$$\psi_0 \ = \ (\text{inclusion})_* \ : \ C^n \ = \ K_{n+1}(W) \to \ C_{n+1} \ = \ K_{n+1}(W, M') \ .$$

The hessian $(C^n, -\psi_1 \in Q_{(-1)^{n+1}}(C^n))$ is the geometric self-intersection $(-1)^{n+1}$-quadratic form on the kernel $C^n = K_{n+1}(W)$ of the normal map

$W^{2n+2} \to X \times [0,1]$, such that

$$\begin{aligned} -(\psi_1 + (-1)^{n+1}\psi_1^*) \ &= \ d\psi_0 \ = \ \text{inclusion}_* \ : \\ C^n \ = \ K_{n+1}(W) \to C_n \ &= \ K_{n+1}(W, \partial W) \ = \ K_{n+1}(W)^* \ . \end{aligned}$$

The chain equivalence $(1+T)\psi_0 : C^{2n+1-*} \to C$ induces the Poincaré duality isomorphisms

$$[M] \cap - \ : \ H^{2n+1-*}(C) \ = \ K^{2n+1-*}(M) \xrightarrow{\cong} H_*(C) \ = \ K_*(M) \ . \qquad \square$$

REMARK 6.4 The $(2n+1)$-complex (C, ψ) of 6.3 can also be obtained by working inside M, assuming that X has a single $(2n+1)$-cell

$$X \ = \ X_0 \cup D^{2n+1}$$

(as is possible by the Poincaré disc theorem of Wall [28]) so that there is defined a degree 1 map

$$\text{collapse} \ : \ X \to X/X_0 \ = \ S^{2n+1} \ .$$

Let $U \subset M^{2n+1}$ be the disjoint union of the k embeddings $S^n \times D^{n+1} \hookrightarrow M$ with null-homotopies in X, so that (f,b) has a Heegaard splitting as a union of normal maps

$$\begin{aligned} &(f,b) \ = \ (e,a) \cup (f_0, b_0) \ : \\ &M \ = \ (U, \partial U) \cup (M_0, \partial M_0) \to X \ = \ (D^{2n+1}, S^{2n}) \cup (X_0, \partial X_0) \end{aligned}$$

with the inclusion (6.2) of the lagrangian

$$\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix} \ : \ C^n \to C_{n+1} \oplus C^{n+1}$$

in the hyperbolic $(-1)^n$-quadratic form $H_{(-1)^n}(C_{n+1})$ given by

$$\text{inclusion}_* \ : \ K_{n+1}(M_0, \partial U) \to K_n(\partial U) \ = \ K_{n+1}(U, \partial U) \oplus K_n(U) \ .$$

Wall obtained the surgery obstruction of (f,b) using an extension (cf. 5.7) of this inclusion to an automorphism

$$\alpha \ : \ H_{(-1)^n}(C_{n+1}) \xrightarrow{\cong} H_{(-1)^n}(C_{n+1}) \ ,$$

which will be discussed further in §10 below. The presentation of (f,b) used to obtain (C,ψ) in 6.3 is the trace of the k surgeries on $U \subset M$

$$\begin{aligned} &(g,c) \ = \ (e_1, a_1) \cup (f_0, b_0) \times \text{id} \ : \\ &(W; M, M') \ = \ (V; U, U') \cup M_0 \times ([0,1]; \{0\}, \{1\}) \to X \times ([0,1]; \{0\}, \{1\}) \end{aligned}$$

with

$$(V; U, U') \ = \ \bigcup_k (D^{n+1} \times D^{n+1}; S^n \times D^{n+1}, D^{n+1} \times S^n) \ . \qquad \square$$

Example 6.5 There is also a relative version of 6.3. A presentation of an n-connected normal map $(f,b) : M^{2n} \to X$ from a $(2n+1)$-dimensional manifold with boundary $(M, \partial M)$ to a geometric Poincaré pair $(X, \partial X)$ with $\partial f = f| : \partial M \to \partial X$ a homotopy equivalence is a normal map of triads

$$\begin{aligned}&(W^{2n+2}; M^{2n+1}, M'^{2n+1}; \partial M \times [0,1])\\ &\qquad \to (X \times [0,1]; X \times \{0\}, X \times \{1\}; \partial X \times [0,1])\end{aligned}$$

such that $W \to X \times [0,1]$ is n-connected. Again, the presentation determines a $(2n+1)$-complex (C, ψ) over $\mathbb{Z}[\pi_1(X)]$ with

$$C_n \;=\; K_{n+1}(W, \partial W)\ ,\ C_{n+1} \;=\; K_{n+1}(W, M')\ ,\ H_*(C) \;=\; K_*(M)\ .\ \square$$

Remark 6.6 (Realization of odd-dimensional surgery obstructions, Wall [29, 6.5]) The theorem of [29] realizing automorphisms of hyperbolic forms as odd-dimensional surgery obstructions has the following interpretation in terms of complexes. Let (C, ψ) be a $(2n+1)$-complex over $\mathbb{Z}[\pi]$, with π a finitely presented group. Let $n \geq 2$, so that there exists a $2n$-dimensional manifold X^{2n} with $\pi_1(X) = \pi$. For any such $n \geq 2$, X there exists an n-connected $(2n+1)$-dimensional normal map

$$(f,b)\ :\ (M^{2n+1}; \partial_- M, \partial_+ M) \to X^{2n} \times ([0,1]; \{0\}, \{1\})$$

with $\partial_- M = X \to X$ the identity and $\partial_+ M \to X$ a homotopy equivalence, and with a presentation with respect to which (f,b) has kernel $(2n+1)$-complex (C, ψ). Such a normal map is constructed from the identity $X \to X$ in two stages. First, choose a basis $\{b_1, b_2, \dots, b_k\}$ for C_{n+1}, and perform surgeries on k disjoint trivial embeddings $S^{n-1} \times D^{n+1} \hookrightarrow X^{2n}$ with trace

$$\begin{aligned}(U; X, \partial_+ U) \;&=\; (X \times [0,1] \cup \bigcup_k D^n \times D^{n+1}; X \times \{0\}, X \#\#_k S^n \times S^n)\\ &\qquad \to X \times ([0,1/2]; \{0\}, \{1/2\})\ .\end{aligned}$$

The n-connected $2n$-dimensional normal map $\partial_+ U \to X \times \{1/2\}$ has kernel $(-1)^n$-quadratic form

$$(K_n(\partial_+ U), \lambda, \mu) \;=\; H_{(-1)^n}(\mathbb{Z}[\pi]^k) \;=\; H_{(-1)^n}(C_{n+1})\ .$$

Second, choose a basis $\{c_1, c_2, \dots, c_k\}$ for C^n and realize the inclusion of the lagrangian in $H_{(-1)^n}(C_{n+1})$ by surgeries on k disjoint embeddings $S^n \times D^n \hookrightarrow \partial_+ U$ with trace

$$(M_0; \partial_+ U, \partial_+ M) \to X \times ([0,1]; \{0\}, \{1\})$$

such that

$$\begin{aligned}\begin{pmatrix}\psi_0\\ d^*\end{pmatrix} \;&=\; \partial\ :\ C^n \;=\; K_{n+1}(M_0, \partial_+ U)\\ &\to C_{n+1} \oplus C^{n+1} \;=\; K_{n+1}(U, \partial_+ U) \oplus K_n(U) \;=\; K_n(\partial_+ U)\ .\end{aligned}$$

The required $(2n+1)$-dimensional normal map realizing (C,ψ) is the union

$$(M;\partial_-M,\partial_+M) \;=\; (U;X,\partial_+U)\cup(M_0;\partial_+U,\partial_+M) \to X\times([0,1];\{0\},\{1\})\ .$$

The corresponding presentation is the trace of surgeries on k disjoint embeddings $S^n\times D^{n+1}\hookrightarrow U\subset M^{2n+1}$. This is the terminology (and result) of Wall [29, Chapter 6]. □

The choice of presentation (6.3) for an n-connected $(2n+1)$-dimensional normal map $(f,b):M^{2n+1}\to X$ does not change the "homotopy type" of the associated $(2n+1)$-complex (C,ψ), in the following sense.

Definition 6.7 (i) A *map* of $(2n+1)$-complexes over Λ

$$f\;:\;(C,\psi)\to(C',\psi')$$

is a chain map $f:C\to C'$ such that there exist Λ-module morphisms

$$\chi_0\;:\;C'^{n+1}\to C'_{n+1}\quad,\quad \chi_1\;:\;C'^n\to C'_n$$

with

$$\begin{aligned} f\psi_0f^*-\psi'_0 \;&=\; (\chi_0+(-1)^{n+1}\chi_0^*)d'^*\;:\;C'^n\to C'_{n+1}\ ,\\ f\psi_1f^*-\psi'_1 \;&=\; -d'\chi_0d'^*+\chi_1+(-1)^n\chi_1^*\;:\;C'^n\to C'_n\ .\end{aligned}$$

(ii) A *homotopy equivalence* of $(2n+1)$-complexes is a map with $f:C\to C'$ a chain equivalence.
(iii) An *isomorphism* of $(2n+1)$-complexes is a map with $f:C\to C'$ an isomorphism of chain complexes. □

Proposition 6.8 *Homotopy equivalence is an equivalence relation on* $(2n+1)$*-complexes.*

Proof: For $m\geq 0$ let $E(m)$ be the contractible f. g. free Λ-module chain complex defined by

$$\begin{aligned} d_{E(m)} \;&=\; 1\;:\;E(m)_{n+1}\;=\;\Lambda^m\to E(m)_n\;=\;\Lambda^m\ ,\\ E(m)_r \;&=\; 0 \text{ for } r\neq n,n+1\ .\end{aligned}$$

A map $f:(C,\psi)\to(C',\psi')$ is a homotopy equivalence if and only if for some $m,m'\geq 0$ there exists an isomorphism

$$f'\;:\;(C,\psi)\oplus(E(m),0)\xrightarrow{\cong}(C',\psi')\oplus(E(m'),0)$$

such that the underlying chain map f' is chain homotopic to

$$f\oplus 0\;:\;C\oplus E(m)\to C'\oplus E(m')\ .$$

Isomorphism is an equivalence relation on $(2n+1)$-complexes, and hence so is homotopy equivalence. □

Example 6.9 The $(2n+1)$-complexes (C,ψ), (C',ψ') associated by 6.3 to any two presentations

$$(W; M, \widehat{M}) \to X \times ([0,1]; \{0\}, \{1\}) \ , \ (W'; M, \widehat{M}') \to X \times ([0,1]; \{0\}, \{1\})$$

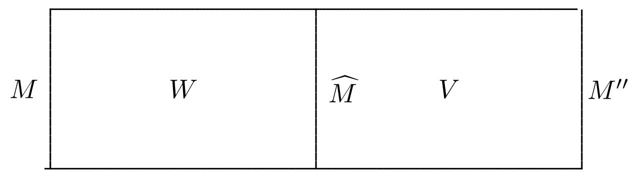
of an n-connected normal map $M^{2n+1} \to X$ are homotopy equivalent. Without loss of generality it may be assumed that W and W' are the traces of surgeries on disjoint embeddings

$$g^i \ : \ S^n \times D^{n+1} \hookrightarrow M \ , \ g'^j \ : \ S^n \times D^{n+1} \hookrightarrow M \ ,$$

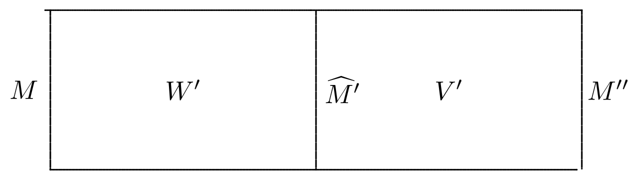
corresponding to two sets of $\mathbb{Z}[\pi_1(X)]$-module generators of $K_n(M)$. Define a presentation of $M \to X$

$$(W''; M, M'') \ = \ (W; M, \widehat{M}) \cup (V; \widehat{M}, M'') \ = \ (W'; M, \widehat{M}') \cup (V'; \widehat{M}', M'')$$

with $(V; \widehat{M}, M'')$ the presentation of $\widehat{M} \to X$ defined by the trace of the surgeries on the copies $\widehat{g}'^j : S^n \times D^{n+1} \hookrightarrow \widehat{M}$ of $g'^j : S^n \times D^{n+1} \hookrightarrow M$, and $(V'; \widehat{M}', M'')$ the presentation of $\widehat{M}' \to X$ defined by the trace of the surgeries on the copies $\widehat{g}^i : S^n \times D^{n+1} \hookrightarrow \widehat{M}'$ of $g^i : S^n \times D^{n+1} \hookrightarrow M$.



$$W'' = W \cup_{\widehat{M}} V = W' \cup_{\widehat{M}'} V'$$



The projections $C'' \to C$, $C'' \to C'$ define homotopy equivalences of $(2n+1)$-complexes

$$(C'', \psi'') \to (C, \psi) \quad , \quad (C'', \psi'') \to (C', \psi') \ . \qquad \square$$

DEFINITION 6.10 A $(2n+1)$-complex (C, ψ) over Λ is *contractible* if it is homotopy equivalent to the zero complex $(0,0)$, or equivalently if $d : C_{n+1} \to C_n$ is a Λ-module isomorphism. $\square$

EXAMPLE 6.11 A $(2n+1)$-complex (C, ψ) associated to an n-connected $(2n+1)$-dimensional normal map $(f, b) : M^{2n+1} \to X$ is contractible if (and for $n \geq 2$ only if) f is a homotopy equivalence, by the theorem of J.H.C. Whitehead. The $(2n+1)$-complexes (C, ψ) associated to the various presentations of a homotopy equivalence $(f, b) : M^{2n+1} \to X$ are contractible, by 6.9. The zero complex $(0,0)$ is associated to the presentation

$$(f,b)\times \mathrm{id.}\ :\ M\times([0,1];\{0\},\{1\})\to X\times([0,1];\{0\},\{1\})\ . \qquad \square$$

§7. Complex cobordism

The cobordism of $(2n+1)$-complexes is the equivalence relation which corresponds to the normal bordism of n-connected $(2n+1)$-dimensional normal maps. The $(2n+1)$-dimensional surgery obstruction group $L_{2n+1}(\Lambda)$ will be defined in §8 below to be the cobordism group of $(2n+1)$-complexes over Λ.

DEFINITION 7.1 A *cobordism* of $(2n+1)$-complexes (C,ψ), (C',ψ')

$$((j\ j'):C\oplus C'\to D,(\delta\psi,\psi\oplus-\psi'))$$

is a f. g. free Λ-module chain complex of the type

$$D\ :\ \ldots\to 0\to D_{n+1}\to 0\to\ldots$$

together with Λ-module morphisms

$$\begin{aligned} j\ :\ C_{n+1}\to D_{n+1}\quad &,\quad j'\ :\ C'_{n+1}\to D_{n+1}\ ,\\ \delta\psi_0\ :\ D^{n+1}\ &=\ (D_{n+1})^*\to D_{n+1}\end{aligned}$$

such that the *duality* Λ-module chain map

$$(1+T)(\delta\psi_0,\psi_0\oplus-\psi'_0)\ :\ \mathcal{C}(j')^{2n+2-*}\to\mathcal{C}(j)$$

defined by

$$\begin{aligned}(1+T)(\delta\psi_0,\psi_0\oplus-\psi'_0)\ &=\ \begin{pmatrix}\delta\psi_0+(-1)^{n+1}\delta\psi_0^* & j'\psi'_0\\ \psi_0^*j^* & 0\end{pmatrix}\\ :\ \mathcal{C}(j')^{2n+1}\ =D^{n+1}\oplus C'^n&\to\mathcal{C}(j)_{n+1}\ =\ D_{n+1}\oplus C_n\end{aligned}$$

is a chain equivalence, with $\mathcal{C}(j)$, $\mathcal{C}(j')$ the algebraic mapping cones of the chain maps $j:C\to D$, $j':C'\to D$. $\square$

The duality chain map $\mathcal{C}(j')^{2n+2-*}\to\mathcal{C}(j)$ is given by

$$\begin{tikzcd}[column sep=small]
\mathcal{C}(j')^{2n+2-*}:\ldots \arrow[r] \arrow[d, "(1+T)\delta\psi_0"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & D^{n+1}\oplus C'^n \arrow[r] \arrow[d] & C'^{n+1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\mathcal{C}(j):\ldots \arrow[r] & 0 \arrow[r] & C_{n+1} \arrow[r] & D_{n+1}\oplus C_n \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

The condition for it to be a chain equivalence is just that the Λ-module morphism

$$\begin{pmatrix} d & 0 & \psi_0^* j^* \\ 0 & d'^* & j'^* \\ (-1)^{n+1} j & j'\psi_0' & \delta\psi_0 + (-1)^{n+1}\delta\psi_0^* \end{pmatrix}$$
$$: \ C_{n+1} \oplus C'^n \oplus D^{n+1} \to C_n \oplus C'^{n+1} \oplus D_{n+1}$$

be an isomorphism.

EXAMPLE 7.2 Suppose given two n-connected $(2n+1)$-dimensional normal maps $M^{2n+1} \to X$, $M'^{2n+1} \to X$ with presentations (6.3)

$$(W^{2n+2}; M^{2n+1}, \widehat{M}^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\}) \ ,$$
$$(W'^{2n+2}; M'^{2n+1}, \widehat{M}'^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\})$$

and corresponding $(2n+1)$-complexes (C, ψ), (C', ψ'). An n-connected normal bordism

$$(V^{2n+2}; M^{2n+1}, M'^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\})$$

determines a cobordism $((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$ (again, up to some choices) from (C, ψ) to (C', ψ'). Define an n-connected normal bordism

$$\begin{aligned}(V'; \widehat{M}, \widehat{M}') \ &= \ (W; \widehat{M}, M) \cup (V; M, M') \cup (W'; M', \widehat{M}') \\ &\qquad \to X \times ([0,1]; \{0\}, \{1\}) \ .\end{aligned}$$

The exact sequence of stably f. g. free $\mathbb{Z}[\pi_1(X)]$-modules

$$\begin{aligned}0 \to K_{n+1}(V) \ &\to K_{n+1}(V', \partial V') \\ &\to K_{n+1}(W, \partial W) \oplus K_{n+1}(W', \partial W') \to 0\end{aligned}$$

splits. Choosing any splitting $K_{n+1}(V', \partial V') \to K_{n+1}(V)$ define j, j' by

$$\begin{aligned}(j\ j') \ : \ C_{n+1} \oplus C'_{n+1} \ &= \ K_{n+1}(W, \widehat{M}) \oplus K_{n+1}(W', \widehat{M}') \\ &\xrightarrow{\text{incl}_* \oplus \text{incl}_*} K_{n+1}(V', \partial V') \ \to K_{n+1}(V) \ = \ D_{n+1} \ .\end{aligned}$$

Geometric intersection numbers provide a $(-1)^{n+1}$-quadratic form $(D^{n+1}, \delta\psi_0)$ over $\mathbb{Z}[\pi_1(X)]$ such that the duality chain map $\mathcal{C}(j')^{2n+2-*} \to \mathcal{C}(j)$ is a chain equivalence inducing the Poincaré duality isomorphisms

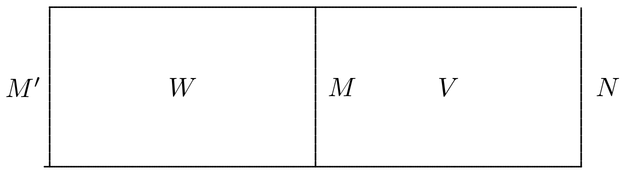
$$[V] \cap - \ : \ H^{2n+2-*}(j') \ = \ K^{2n+2-*}(V, M') \xrightarrow{\cong} H_*(j) \ = \ K_*(V, M) \ . \ \square$$

DEFINITION 7.3 A *null-cobordism* of a $(2n+1)$-complex (C, ψ) is a cobordism $(j : C \to D, (\delta\psi, \psi))$ to $(0, 0)$. □

EXAMPLE 7.4 Let $(W^{2n+2}; M'^{2n+1}, M^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\})$ be a presentation of an n-connected $(2n+1)$-dimensional normal map $M \to X$, with $(2n+1)$-complex (C,ψ). For $n \geq 2$ there is a one-one correspondence between n-connected normal bordisms of $M \to X$

$$(V^{2n+2}; M^{2n+1}, N^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\})$$

to homotopy equivalences $N \to X$ and null-cobordisms $(j : C \to D, (\delta\psi, \psi))$. (Every normal bordism of n-connected $(2n+1)$-dimensional normal maps can be made n-connected by surgery below the middle dimension on the interior.)



$V' = W \cup_M V$

Any such $(V; M, N) \to X \times ([0,1]; \{0\}, \{1\})$ determines by 7.2 a null-cobordism $(j : C \to D, (\delta\psi, \psi))$ of (C, ψ). $\square$

Cobordisms of $(2n+1)$-complexes arise in the following way:

Construction 7.5 An isomorphism of hyperbolic split $(-1)^n$-quadratic forms over Λ

$$\left(\begin{pmatrix} \gamma & \widetilde{\gamma} \\ \mu & \widetilde{\mu} \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ \widetilde{\gamma}^*\mu & \widetilde{\theta} \end{pmatrix}\right) : H_{(-1)^n}(G) \xrightarrow{\cong} H_{(-1)^n}(F)$$

with F, G f. g. free determines a cobordism of $(2n+1)$-complexes

$$((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$$

by

$$\begin{aligned}
d &= \mu^* : C_{n+1} = F \to C_n = G^* ,\\
\psi_0 &= \gamma : C^n = G \to C_{n+1} = F ,\\
\psi_1 &= -\theta : C^n = G \to C_n = G^* ,\\
j &= \widetilde{\mu}^* : C_{n+1} = F \to D_{n+1} = G ,\\
d' &= \gamma^* : C'_{n+1} = F^* \to C_n = G^* ,\\
\psi'_0 &= \mu : C'^n = G \to C'_{n+1} = F^* ,\\
\psi'_1 &= -\theta : C'^n = G \to C'_n = G^* ,\\
j' &= \widetilde{\gamma}^* : C'_{n+1} = F^* \to D_{n+1} = G ,\\
\delta\psi_0 &= 0 : D^{n+1} = G^* \to D_{n+1} = G .
\end{aligned}$$

$\square$

It can be shown that every cobordism of $(2n+1)$-complexes is homotopy equivalent to one constructed as in 7.5.

Example 7.6 An n-connected $(2n+2)$-dimensional normal bordism

$$((g,c);(f,b),(f',b'))\ :\ (W^{2n+2};M^{2n+1},M'^{2n+1})\to X\times([0,1];\{0\},\{1\})$$

with $g:W\to X\times[0,1]$ n-connected can be regarded both as a presentation of (f,b) and as a presentation of (f',b'). The cobordism of $(2n+1)$-complexes $((j\ j'):C\oplus C'\to D,(\delta\psi,\psi\oplus-\psi'))$ obtained in 7.2 with $W=V=W'$, $\widehat{M}=M'$, $\widehat{M'}=M'$ is the construction of 7.5 for an extension of the inclusion of the lagrangian (6.2)

$$(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)\ =\ (\begin{pmatrix}\psi_0\\ d^*\end{pmatrix},-\psi_1)\ :\ (C^n,0)\to H_{(-1)^n}(C_{n+1})$$

to an isomorphism of hyperbolic split $(-1)^n$-quadratic forms

$$\left(\begin{pmatrix}\gamma & \widetilde{\gamma}\\ \mu & \widetilde{\mu}\end{pmatrix},\begin{pmatrix}\theta & 0\\ \widetilde{\gamma}^*\mu & \widetilde{\theta}\end{pmatrix}\right)\ :\ H_{(-1)^n}(C^n)\xrightarrow{\cong}H_{(-1)^n}(C_{n+1})\ ,$$

with

$$\begin{aligned} j\ &=\ \widetilde{\mu}^*\ :\ C_{n+1}\ =\ K_{n+1}(W,M')\ \to D_{n+1}=K_{n+1}(W)\ ,\\ j'\ &=\ \widetilde{\gamma}^*\ :\ C'_{n+1}\ =\ K_{n+1}(W,M)\to D_{n+1}=K_{n+1}(W)\ . \end{aligned}$$

□

Remark 7.7 Fix a $(2n+1)$-dimensional geometric Poincaré complex X with reducible Spivak normal fibration, and choose a stable vector bundle $\nu_X:X\to BO$ in the Spivak normal class, e.g. a manifold with the stable normal bundle. Consider the set of n-connected normal maps $(f:M^{2n+1}\to X,b:\nu_M\to\nu_X)$. The relation defined on this set by

$$(M\to X)\sim(M'\to X)\ \text{if there exists an }(n+1)\text{-connected normal bordism }(W;M,M')\to X\times([0,1];\{0\},\{1\})$$

is an equivalence relation. Symmetry and transitivity are verified in the same way as for any geometric cobordism relation. For reflexivity form the cartesian product of an n-connected normal map $M^{2n+1}\to X$ with $([0,1];\{0\},\{1\})$, as usual. The product is an n-connected normal bordism

$$M\times([0,1];\{0\},\{1\})\to X\times([0,1];\{0\},\{1\})$$

which can be made $(n+1)$-connected by surgery killing the n-dimensional kernel $K_n(M\times[0,1])=K_n(M)$. The following verification that the cobordism of $(2n+1)$-complexes is an equivalence relation uses algebraic surgery in exactly the same way. □

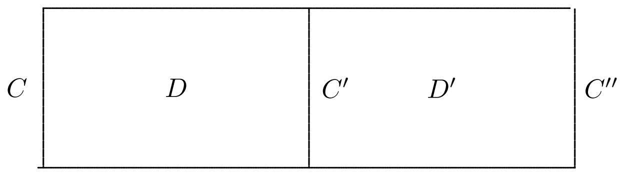
Proposition 7.8 *Cobordism is an equivalence relation on $(2n+1)$-complexes (C,ψ) over Λ, such that $(C,\psi)\oplus(C,-\psi)$ is null-cobordant. Homotopy equivalent complexes are cobordant.*

PROOF: Symmetry is easy: if $((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$ is a cobordism from (C, ψ) to (C', ψ') then

$$((j'\ j) : C' \oplus C \to D', (-\delta\psi, \psi' \oplus -\psi))$$

is a cobordism from (C', ψ') to (C, ψ). For transitivity, suppose given adjoining cobordisms of $(2n+1)$-complexes

$$\begin{aligned} &((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))\ , \\ &((\widetilde{j}'\ j'') : C' \oplus C'' \to D', (\delta\psi', \psi' \oplus -\psi''))\ . \end{aligned}$$



$D'' = D \cup_{C'} D'$

Define the *union* cobordism between (C, ψ) and (C'', ψ'')

$$((\widetilde{j}\ \widetilde{j}'') : C \oplus C'' \to D'', (\delta\psi'', \psi \oplus -\psi''))$$

by

$$\begin{aligned} D''_{n+1} &= \operatorname{coker}(i = \begin{pmatrix} j' \\ d' \\ \widetilde{j}' \end{pmatrix} : C'_{n+1} \to D_{n+1} \oplus C'_n \oplus D'_{n+1})\ , \\ \widetilde{j} &= [j \oplus 0 \oplus 0]\ :\ C_{n+1} \to D''_{n+1}, \\ \widetilde{j}'' &= [0 \oplus 0 \oplus j'']\ :\ C''_{n+1} \to D''_{n+1}\ , \\ \delta\psi''_0 &= \begin{bmatrix} \delta\psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta\psi'_0 \end{bmatrix}\ :\ D''^{n+1} \to D''_{n+1}\ . \end{aligned}$$

The Λ-module morphism $i : C'_{n+1} \to D_{n+1} \oplus C'_n \oplus D'_{n+1}$ is a split injection since the dual Λ-module morphism i^* is a surjection, as follows from the Mayer-Vietoris exact sequence

$$H^{n+2}(D, C') \oplus H^{n+2}(D', C')\ =\ 0 \oplus 0 \to H^{n+2}(D'') \to H^{n+2}(C')\ =\ 0\ .$$

Given any $(2n+1)$-complex (C, ψ) let (C', ψ') be the $(2n+1)$-complex defined by

$$\begin{aligned} d' &= (-1)^n \psi_0^*\ :\ C'_{n+1}\ =\ C^{n+1} \to C'_n\ =\ C_n\ , \\ \psi'_0 &= d^*\ :\ C'^n\ =\ C^n \to C'_{n+1}\ =\ C^{n+1}\ , \\ \psi'_1 &= -\psi_1\ :\ C'^n\ =\ C^n \to C'_n\ =\ C_n\ . \end{aligned}$$

Apply 5.7 to extend the inclusion of the lagrangian in $H_{(-1)^n}(C_{n+1})$

$$\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix} : C^n \to C_{n+1} \oplus C^{n+1}$$

to an isomorphism of $(-1)^n$-quadratic forms

$$\begin{pmatrix} \psi_0 & \widetilde{\psi}_0 \\ d^* & \widetilde{d}^* \end{pmatrix} : H_{(-1)^n}(C^n) \xrightarrow{\cong} H_{(-1)^n}(C_{n+1})$$

with $\widetilde{\psi}_0 \in \mathrm{Hom}_\Lambda(C_n, C_{n+1})$, $\widetilde{d} \in \mathrm{Hom}_\Lambda(C_{n+1}, C^n)$. Now apply 7.5 to construct from any such extension a cobordism

$$((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$$

with

$$\begin{aligned}
j &= \widetilde{d} : C_{n+1} \to D_{n+1} = C^n , \\
j' &= \widetilde{\psi}_0^* : C'_{n+1} = C^{n+1} \to D_{n+1} = C^n , \\
d' &= \psi_0^* : C'_{n+1} = C^{n+1} \to C'_n = C_n , \\
\psi'_0 &= d^* : C'^n = C^n \to C'^{n+1} = C^{n+1} , \\
\delta\psi_0 &= 0 : D^{n+1} = C_n \to D_{n+1} = C^n .
\end{aligned}$$

(This is the algebraic analogue of the construction of a presentation (6.3)

$$(W^{2n+2}; M^{2n+1}, M'^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\})$$

of an n-connected $(2n+1)$-dimensional normal map $M^{2n+1} \to X$ by surgery on a finite set of $\mathbb{Z}[\pi_1(X)]$-module generators of $K_n(M)$). The union of the cobordisms

$$\begin{aligned}
&((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi')) , \\
&((j'\ j) : C' \oplus C \to D, (-\delta\psi, \psi' \oplus -\psi))
\end{aligned}$$

is a cobordism

$$((\widetilde{j}\ \widetilde{j}') : C \oplus C \to D', (\delta\psi', \psi \oplus -\psi))$$

with a Λ-module isomorphism

$$\begin{bmatrix} 1 & 0 & -1 \\ \psi_0 & (-1)^n\widetilde{\psi}_0 & 0 \end{bmatrix} :$$

$$D'_{n+1} = \mathrm{coker}(\begin{pmatrix} \widetilde{\psi}_0^* \\ \psi_0^* \\ \widetilde{\psi}_0^* \end{pmatrix} : C^{n+1} \to C^n \oplus C_n \oplus C^n) \xrightarrow{\cong} C^n \oplus C_{n+1} .$$

This verifies that cobordism is reflexive, and also that $(C, \psi) \oplus (C, -\psi)$ is null-cobordant.

Suppose given a homotopy equivalence of $(2n+1)$-complexes

$$f : (C, \psi) \to (C', \psi') ,$$

with $\chi_0 : C'^{n+1} \to C'_{n+1}$ as in 6.6. By reflexivity there exists a cobordism $((j''\ j') : C' \oplus C' \to D, (\delta\psi', \psi' \oplus -\psi'))$ from (C', ψ') to itself. Define a cobordism $((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$ from (C, ψ) to (C', ψ') by

$$\begin{aligned} j &= j''f \ :\ C_{n+1} \xrightarrow{f} C'_{n+1} \xrightarrow{j''} D_{n+1} \ , \\ \delta\psi_0 &= \delta\psi'_0 + j''\chi_0 j''^* \ :\ D^{n+1} \to D_{n+1} \ . \end{aligned}$$ □

DEFINITION 7.9 (i) A *weak map* of $(2n+1)$-complexes over Λ

$$f \ :\ (C, \psi) \to (C', \psi')$$

is a chain map $f : C \to C'$ such that there exist Λ-module morphisms

$$\chi_0 \ :\ C'^{n+1} \to C'_{n+1} \ \ ,\ \ \chi_1 \ :\ C'^n \to C'^n$$

with

$$f\psi_0 f^* - \psi'_0 \ =\ (\chi_0 + (-1)^{n+1}\chi_0^*)d'^* \ :\ C'^n \to C'_{n+1} \ .$$

(ii) A *weak equivalence* of $(2n+1)$-complexes is a weak map with $f : C \to C'$ a chain equivalence.
(iii) A *weak isomorphism* of $(2n+1)$-complexes is a weak map with $f : C \to C'$ an isomorphism of chain complexes. □

PROPOSITION 7.10 *Weakly equivalent $(2n+1)$-complexes are cobordant.*
PROOF: The proof in 7.8 that homotopy equivalent $(2n+1)$-complexes are cobordant works just as well for weakly equivalent ones. □

Given a $(2n+1)$-complex (C, ψ) let

$$(\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix}, -\psi_1) \ :\ (C^n, 0) \to H_{(-1)^n}(C_{n+1})$$

be the inclusion of a lagrangian in a hyperbolic split $(-1)^n$-quadratic form given by 6.2. The result of 7.10 is that the cobordism class of (C, ψ) is independent of the hessian $(-1)^{n+1}$-quadratic form $(C^n, -\psi_1)$.

§8. The odd-dimensional L-groups

The odd-dimensional surgery obstruction groups $L_{2n+1}(\Lambda)$ of a ring with involution Λ will now be defined to be the cobordism groups of $(2n+1)$-complexes over Λ.

DEFINITION 8.1 Let $L_{2n+1}(\Lambda)$ be the abelian group of cobordism classes of $(2n+1)$-complexes over Λ, with addition and inverses by

$$\begin{aligned} (C, \psi) + (C', \psi') &= (C \oplus C', \psi \oplus \psi') \ , \\ -(C, \psi) &= (C, -\psi) \in L_{2n+1}(\Lambda) \ . \end{aligned}$$ □

The groups $L_{2n+1}(\Lambda)$ only depend on the residue $n(\bmod 2)$, so that only two L-groups have actually been defined, $L_1(\Lambda)$ and $L_3(\Lambda)$. Note that 8.1 uses 7.8 to justify $(C,\psi)\oplus(C,-\psi)=0\in L_{2n+1}(\Lambda)$.

EXAMPLE 8.2 The odd-dimensional L-groups of $\Lambda=\mathbb{Z}$ are trivial

$$L_{2n+1}(\mathbb{Z}) \;=\; 0\;. \qquad \square$$

8.2 was implicit in the work of Kervaire and Milnor [7] on the surgery classification of even-dimensional exotic spheres.

EXAMPLE 8.3 The surgery obstruction of an n-connected $(2n+1)$-dimensional normal map $(f,b): M^{2n+1}\to X$ is the cobordism class

$$\sigma_*(f,b) \;=\; (C,\psi)\in L_{2n+1}(\mathbb{Z}[\pi_1(X)])$$

of the $(2n{+}1)$-complex (C,ψ) associated in 6.3 to any choice of presentation

$$(W;M,M')\to X\times([0,1];\{0\},\{1\})\;.$$

The surgery obstruction vanishes $\sigma_*(f,b)=0$ if (and for $n\geq 2$ only if) (f,b) is normal bordant to a homotopy equivalence. $\square$

DEFINITION 8.4 A *surgery* $(j:C\to D,(\delta\psi,\psi))$ on a $(2n+1)$-complex (C,ψ) is a Λ-module chain map $j:C\to D$ with $D_r=0$ for $r\neq n+1$ and D_{n+1} a f. g. free Λ-module, together with a Λ-module morphism

$$\delta\psi_0\;:\;D^{n+1}\;=\;(D_{n+1})^*\to D_{n+1}\;,$$

such that the Λ-module morphism

$$(\,d\quad \psi_0^*j^*\,)\;:\;C_{n+1}\oplus D^{n+1}\to C_n$$

is onto. The *effect* of the surgery is the $(2n+1)$-complex (C',ψ') defined by

$$\begin{aligned}
d' \;&=\; \begin{pmatrix} d & \psi_0^*j^* \\ (-1)^{n+1}j & \delta\psi_0+(-1)^{n+1}\delta\psi_0^* \end{pmatrix} \\
&\qquad :\; C'_{n+1} \;=\; C_{n+1}\oplus D^{n+1}\to C'_n \;=\; C_n\oplus D_{n+1}\;,\\
\psi'_0 \;&=\; \begin{pmatrix} \psi_0 & 0 \\ 0 & 1 \end{pmatrix} \;:\; C'^n \;=\; C^n\oplus D^{n+1}\to C'_{n+1} \;=\; C_{n+1}\oplus D^{n+1}\;,\\
\psi'_1 \;&=\; \begin{pmatrix} \psi_1 & -\psi_0^*j^* \\ 0 & -\delta\psi_0 \end{pmatrix} \;:\; C'^n = C^n\oplus D^{n+1}\to C'_n \;=\; C_n\oplus D_{n+1}\;.
\end{aligned}$$

The *trace* of the surgery is the cobordism of $(2n+1)$-complexes $((j'\ j''):C\oplus C'\to D',(0,\psi\oplus-\psi'))$, with

$$\begin{aligned}
j'' \;=\; (j'\ k)\; &:C'_{n+1} \;=\; C_{n+1}\oplus D^{n+1}\\
&\to D'_{n+1} \;=\; \ker((\,d\quad \psi_0j^*\,):C_{n+1}\oplus D^{n+1}\to C_n)
\end{aligned}$$

a splitting of the split injection $(d\ \psi_0^* j^*) : C_{n+1} \oplus D^{n+1} \to C_n$. □

EXAMPLE 8.5 Let

$$((e,a);(f,b),(f',b'))\ :\ (V^{2n+2};M^{2n+1},M'^{2n+1}) \to X \times ([0,1];\{0\},\{1\})$$

be the trace of a sequence of k surgeries on an n-connected $(2n+1)$-dimensional normal map $(f,b) : M \to X$ killing elements $x_1, x_2, \dots, x_k \in K_n(M)$, with e n-connected and f' n-connected. V has a handle decomposition on M of the type

$$V\ =\ M \times I \cup \bigcup_k (n+1)\text{-handles } D^{n+1} \times D^{n+1}\ ,$$

and also a handle decomposition on M' of the same type

$$V\ =\ M' \times I \cup \bigcup_k (n+1)\text{-handles } D^{n+1} \times D^{n+1}\ .$$

A presentation of (f,b)

$$((g,c);(\widehat{f},\widehat{b}),(f,b))\ :\ (W^{2n+2};\widehat{M}^{2n+1},M^{2n+1}) \to X \times ([0,1];\{0\},\{1\})$$

with $(2n+1)$-complex (C,ψ) determines a presentation of (f',b')

$$\begin{aligned}((g',c');(\widehat{f},\widehat{b}),(f',b'))\ &=\ ((g,c);(\widehat{f},\widehat{b}),(f,b)) \cup ((e,a);(f,b),(f',b'))\ :\\ (W';\widehat{M},M')\ &=\ (W;\widehat{M},M) \cup (V;M,M') \to X \times ([0,1];\{0\},\{1\})\end{aligned}$$

such that the $(2n+1)$-complex (C',ψ') is the effect of a surgery $(j : C \to D,(\delta\psi,\psi))$ on (C,ψ) with

$$\begin{aligned}D_{n+1}\ &=\ K_{n+1}(V,M')\ =\ \mathbb{Z}[\pi_1(X)]^k\ ,\\ C'_{n+1}\ &=\ K_{n+1}(W',\widehat{M})\ =\ K_{n+1}(W,\widehat{M}) \oplus K_{n+1}(V,M)\ =\ C_{n+1} \oplus D^{n+1}\ ,\\ C'_n\ &=\ K_{n+1}(W',\partial W')\ =\ K_{n+1}(W,\partial W) \oplus K_{n+1}(V,M')\ =\ C_n \oplus D_{n+1}\ .\end{aligned}$$

Also, the geometric trace determines the algebraic trace, with

$$D'_{n+1}\ =\ K_{n+1}(V)\ .$$

□

It can be shown that $(2n+1)$-complexes (C,ψ), (C',ψ') are cobordant if and only if (C',ψ') is homotopy equivalent to the effect of a surgery on (C,ψ). This result will only be needed for $(C',\psi') = (0,0)$, so it will only be proved in this special case:

PROPOSITION 8.6 *A $(2n+1)$-complex (C,ψ) represents 0 in $L_{2n+1}(\Lambda)$ if and only if there exists a surgery $(j : C \to D,(\delta\psi,\psi))$ with contractible effect.*

PROOF : The effect of a surgery is contractible if and only if it is a null-cobordism. □

Given an n-connected $(2n+1)$-dimensional normal map $(f,b) : M^{2n+1} \longrightarrow X$ it is possible to kill every element $x \in K_n(M)$ by an embedding $S^n \times D^{n+1} \hookrightarrow M$ to obtain a bordant normal map

$$(f',b') \ : \ M'^{2n+1} \ = \ \mathrm{cl.}(M \backslash S^n \times D^{n+1}) \cup D^{n+1} \times S^n \to X \ .$$

There are many ways of carrying out the surgery, which are quantified by the surgeries on the kernel $(2n+1)$-complex (C,ψ). In general, $K_n(M')$ need not be smaller than $K_n(M)$.

Example 8.7 The kernel $(2n+1)$-complex (C,ψ) over $\mathbb{Z}$ of the identity normal map

$$(f,b) \ = \ \mathrm{id.} \ : \ M^{2n+1} \ = \ S^{2n+1} \to S^{2n+1}$$

is $(0,0)$. For any element

$$\mu \in \pi_{n+1}(SO, SO(n+1)) \ = \ Q_{(-1)^{n+1}}(\mathbb{Z})$$

let $\omega = \partial\mu \in \pi_n(SO(n+1))$, and define a null-homotopic embedding of S^n in M

$$e_\omega \ : \ S^n \times D^{n+1} \hookrightarrow M \ ; \ (x,y) \ \longmapsto (x, \omega(x)(y)) / \parallel (x,\omega(x)(y)) \parallel \ .$$

Use μ to kill $0 \in K_n(M)$ by surgery on (f,b), with effect a normal bordant n-connected $(2n+1)$-dimensional normal map

$$(f_\mu, b_\mu) \ : \ M^{2n+1}_\mu \ = \ \mathrm{cl.}(M \backslash e_\omega(S^n \times D^{n+1})) \cup D^{n+1} \times S^n \to S^{2n+1}$$

exactly as in 2.18, with the kernel complex (C',ψ') given by

$$d' \ = \ (1 + T_{(-1)^{n+1}})(\mu) \ : \ C'_{n+1} \ = \ \mathbb{Z} \to C'_n \ = \ \mathbb{Z} \ .$$

In particular, for $\mu = 0, 1$ this gives the $(2n+1)$-dimensional manifolds

$$\begin{aligned} M' \ &= \ M_0 \ = \ S^n \times S^{n+1} \ , \\ M'' \ &= \ M_1 \ = \ S(\tau_{S^{n+1}}), \ \text{the tangent } S^n\text{-bundle of } S^{n+1} \\ &= \ O(n+2)/O(n) \\ &= \ V_{n+2,2}, \ \text{the Stiefel manifold of orthonormal 2-frames in } \mathbb{R}^{n+2} \\ &(= \ SO(3) \ = \ \mathbb{RP}^3 \text{ for } n = 1 \ ) \ , \end{aligned}$$

corresponding to the algebraic surgeries on $(0,0)$

$$(0 : 0 \to D, (\delta\psi', 0)) \ , \ (0 : 0 \to D, (\delta\psi'', 0))$$

with

$$D_{n+1} \ = \ \mathbb{Z} \ , \ \delta\psi'_0 \ = \ 0 \ , \ \delta\psi''_0 \ = \ 1 \ . \qquad \square$$

§9. Formations

As before, let Λ be a ring with involution, and let $\epsilon = \pm 1$.

Definition 9.1 An *ϵ-quadratic formation over* Λ $(Q, \phi; F, G)$ is a nonsingular ϵ-quadratic form (Q, ϕ) together with an ordered pair of lagrangians F,G. □

Formations with $\epsilon = (-1)^n$ are essentially the $(2n+1)$-complexes of §6 expressed in the language of forms and lagrangians of §4. In the general theory it is possible to consider formations $(Q, \phi; F, G)$ with Q, F, G f. g. projective, but in view of the more immediate topological applications only the f. g. free case is considered here. Strictly speaking, 9.1 defines a "nonsingular formation". In the general theory a formation $(Q, \phi; F, G)$ is a nonsingular form (Q, ϕ) together with a lagrangian F and a sublagrangian G. The automorphisms of hyperbolic forms in the original treatment due to Wall [29] of odd-dimensional surgery theory were replaced by formations by Novikov [16] and Ranicki [18].

In dealing with formations assume that the ground ring Λ is such that the rank of f. g. free Λ-modules is well-defined (e.g. $\Lambda = \mathbb{Z}[\pi]$). The rank of a f. g. free Λ-module K is such that

$$\mathrm{rank}_\Lambda(K) \; = \; k \in \mathbb{Z}^+$$

if and only if K is isomorphic to Λ^k. Also, since $\Lambda^k \cong (\Lambda^k)^*$

$$\mathrm{rank}_\Lambda(K) \; = \; \mathrm{rank}_\Lambda(K^*) \in \mathbb{Z}^+ \; .$$

Definition 9.2 An *isomorphism* of ϵ-quadratic formations over Λ

$$f \; : \; (Q, \phi; F, G) \xrightarrow{\cong} (Q', \phi'; F', G')$$

is an isomorphism of forms $f : (Q, \phi) \cong (Q', \phi')$ such that

$$f(F) \; = \; F' \quad , \quad f(G) \; = \; G' \; . \qquad \square$$

Proposition 9.3 (i) *Every ϵ-quadratic formation $(Q, \phi; F, G)$ is isomorphic to one of the type $(H_\epsilon(F); F, G)$.*
(ii) *Every ϵ-quadratic formation $(Q, \phi; F, G)$ is isomorphic to one of the type $(H_\epsilon(F); F, \alpha(F))$ for some automorphism $\alpha : H_\epsilon(F) \cong H_\epsilon(F)$.*
Proof : (i) By Theorem 5.7 the inclusion of the lagrangian $F \to Q$ extends to an isomorphism of forms $f : H_\epsilon(F) \cong (Q, \phi)$, defining an isomorphism of formations

$$f \; : \; (H_\epsilon(F); F, f^{-1}(G)) \xrightarrow{\cong} (Q, \phi; F, G) \; .$$

(ii) As in (i) extend the inclusions of the lagrangians to isomorphisms of forms

$$f \;:\; H_\epsilon(F) \xrightarrow{\cong} (Q,\phi)\ ,\ g \;:\; H_\epsilon(G) \xrightarrow{\cong} (Q,\phi)\ .$$

Then

$$\mathrm{rank}_\Lambda(F) \;=\; \mathrm{rank}_\Lambda(Q)/2 \;=\; \mathrm{rank}_\Lambda(G) \in \mathbb{Z}^+\ ,$$

so that F is isomorphic to G. Choosing a Λ-module isomorphism $\beta : G \cong F$ there is defined an automorphism of $H_\epsilon(F)$

$$\alpha \;:\; H_\epsilon(F) \xrightarrow{\;f\;} (Q,\phi) \xrightarrow{\;g^{-1}\;} H_\epsilon(G) \xrightarrow{\begin{pmatrix} \beta & 0 \\ 0 & \beta^{*-1} \end{pmatrix}} H_\epsilon(F)$$

such that there is defined an isomorphism of formations

$$f \;:\; (H_\epsilon(F);F,\alpha(F)) \xrightarrow{\cong} (Q,\phi;F,G)\ . \qquad \square$$

PROPOSITION 9.4 *The weak isomorphism classes of* $(2n+1)$*-complexes* (C,ψ) *over* Λ *are in natural one-one correspondence with the isomorphism classes of* $(-1)^n$*-quadratic formations* $(Q,\phi;F,G)$ *over* Λ*, with*

$$H_n(C) \;=\; Q/(F+G)\quad,\quad H_{n+1}(C) \;=\; F\cap G\ .$$

Moreover, if the complex (C,ψ) *corresponds to the formation* $(Q,\phi;F,G)$ *then* $(C,-\psi)$ *corresponds to* $(Q,-\phi;F,G)$*.*

PROOF: Given a $(2n+1)$-complex (C,ψ) define a $(-1)^n$-quadratic formation

$$(Q,\phi;F,G) \;=\; (H_{(-1)^n}(C_{n+1});C_{n+1},\mathrm{im}(\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix} : C^n \to C_{n+1}\oplus C^{n+1}))\ .$$

The formation associated in this way to the $(2n+1)$-complex $(C,-\psi)$ is isomorphic to $(Q,-\phi;F,G)$, by the isomorphism

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} : (Q,-\phi;F,G) \xrightarrow{\cong}$$
$$(H_{(-1)^n}(C_{n+1});C_{n+1},\mathrm{im}(\begin{pmatrix} -\psi_0 \\ d^* \end{pmatrix} : C^n \to C_{n+1}\oplus C^{n+1}))\ .$$

Conversely, suppose given an $(-1)^n$-quadratic formation $(Q,\phi;F,G)$. By 9.3 (i) this can be replaced by an isomorphic formation with $(Q,\phi) = H_{(-1)^n}(F)$. Let $\gamma \in \mathrm{Hom}_\Lambda(G,F)$, $\mu \in \mathrm{Hom}_\Lambda(G,F^*)$ be the components of the inclusion

$$i \;=\; \begin{pmatrix} \gamma \\ \mu \end{pmatrix} \;:\; G \to Q \;=\; F\oplus F^*\ .$$

Choose any $\theta \in \mathrm{Hom}_\Lambda(G,G^*)$ such that

$$\gamma^*\mu \;=\; \theta + (-1)^{n+1}\theta^* \in \mathrm{Hom}_\Lambda(G,G^*)\ .$$

Define a $(2n+1)$-complex (C,ψ) by

$$\begin{aligned} d &= \mu^* : C_{n+1} = F \to C_n = G^* , \\ \psi_0 &= \gamma : C^n = G \to C_{n+1} = F , \\ \psi_1 &= (-1)^n\theta : C^n = G \to C_n = G^* . \end{aligned}$$

The exact sequence

$$0 \to G \xrightarrow{i} Q \xrightarrow{i^*(\phi+(-1)^n\phi^*)} G^* \to 0$$

is the algebraic mapping cone

$$0 \to G \xrightarrow{\begin{pmatrix}\gamma \\ \mu\end{pmatrix}} F \oplus F^* \xrightarrow{(\mu^* \quad (-1)^n\gamma^*)} G^* \to 0$$

of the chain equivalence $(1+T)\psi_0 : C^{2n+1-*} \to C$. □

EXAMPLE 9.5 An n-connected $(2n+1)$-dimensional normal map $M^{2n+1} \to X$ together with a choice of presentation $(W;M,M') \to X \times ([0,1];\{0\},\{1\})$ determine by 9.3 a $(2n+1)$-complex (C,ψ), and hence by 9.4 a $(-1)^n$-quadratic formation $(Q,\phi;F,G)$ over $\mathbb{Z}[\pi_1(X)]$ such that

$$\begin{aligned} Q/(F+G) &= H_n(C) = K_n(M) , \\ F \cap G &= H_{n+1}(C) = K_{n+1}(M) . \end{aligned}$$

□

The following equivalence relation on formations corresponds to the weak equivalence (7.9) of $(2n+1)$-complexes.

DEFINITION 9.6 (i) An ϵ-quadratic formation $(Q,\phi;F,G)$ is *trivial* if it is isomorphic to $(H_\epsilon(L);L,L^*)$ for some f. g. free Λ-module L.
(ii) A *stable isomorphism* of ϵ-quadratic formations

$$[f] : (Q,\phi;F,G) \xrightarrow{\cong} (Q',\phi';F',G')$$

is an isomorphism of ϵ-quadratic formations of the type

$$f : (Q,\phi;F,G) \oplus (\text{trivial}) \xrightarrow{\cong} (Q',\phi';F',G') \oplus (\text{trivial}') .$$

□

EXAMPLE 9.7 The $(-1)^n$-quadratic formations associated in 9.5 to all the presentations of an n-connected $(2n+1)$-dimensional normal map $M^{2n+1} \to X$ define a stable isomorphism class. □

PROPOSITION 9.8 *The weak equivalence classes of $(2n+1)$-complexes over Λ are in natural one-one correspondence with the stable isomorphism classes of $(-1)^n$-quadratic formations over Λ.*

PROOF: The $(2n+1)$-complex (C,ψ) associated (up to weak equivalence) to a $(-1)^n$-quadratic formation $(Q,\phi;F,G)$ in 9.4 is contractible if and only if the formation is trivial. □

The following formations correspond to the null-cobordant complexes.

DEFINITION 9.9 The *boundary* of a $(-\epsilon)$-quadratic form (K,λ,μ) is the ϵ-quadratic formation

$$\partial(K,\lambda,\mu) \;=\; (H_\epsilon(K);K,\Gamma_{(K,\lambda)})$$

with $\Gamma_{(K,\lambda)}$ the *graph* lagrangian

$$\Gamma_{(K,\lambda)} \;=\; \{(x,\lambda(x))\in K\oplus K^* \,|\, x\in K\}\ . \qquad □$$

Note that the form (K,λ,μ) may be singular, that is the Λ-module morphism $\lambda : K \to K^*$ need not be an isomorphism. The graphs $\Gamma_{(K,\lambda)}$ of $(-\epsilon)$-quadratic forms (K,λ,μ) are precisely the lagrangians of $H_\epsilon(K)$ which are direct complements of K^*.

PROPOSITION 9.10 *A $(-1)^n$-quadratic formation $(Q,\phi;F,G)$ is stably isomorphic to a boundary $\partial(K,\lambda,\mu)$ if and only if the corresponding $(2n+1)$-complex (C,ψ) is null-cobordant.*

PROOF: Given a $(-1)^{n+1}$-quadratic form (K,λ,μ) choose a split form $\theta : K\to K^*$ (4.2) and let (C,ψ) be the $(2n+1)$-complex associated by 9.4 to the boundary formation $\partial(K,\lambda,\mu)$, so that

$$\begin{aligned}
d \;=\; \lambda \;=\; \theta+(-1)^{n+1}\theta^* \;&:\; C_{n+1} \;=\; K \to C_n \;=\; K^*\ ,\\
\psi_0 \;=\; 1 \;&:\; C^n \;=\; K\to C_{n+1} \;=\; K\ ,\\
\psi_1 \;=\; -\theta \;&:\; C^n \;=\; K\to C_n \;=\; K^*\ .
\end{aligned}$$

Then (C,ψ) is null-cobordant, with a null-cobordism $(j:C\to D,(\delta\psi,\psi))$ defined by

$$\begin{aligned}
j \;=\; 1 \;&:\; C_{n+1} \;=\; K\to D_{n+1} \;=\; K\ ,\\
\delta\psi_0 \;=\; 0 \;&:\; D^{n+1} \;=\; K^*\to D_{n+1} \;=\; K\ .
\end{aligned}$$

Conversely, suppose given a $(2n+1)$-complex (C,ψ) with a null-cobordism

$(j : C \to D, (\delta\psi, \psi))$ as in 8.1. The $(2n+1)$-complex (E, θ) defined by

$$d = \begin{pmatrix} \psi_1 + (-1)^{n+1}\psi_1^* & d & \psi_0^* j^* \\ (-1)^{n+1}d^* & 0 & -j^* \\ (-1)^{n+1}j\psi_0^* & (-1)^n j & \delta\psi_0 + (-1)^{n+1}\delta\psi_0^* \end{pmatrix} :$$

$$E_{n+1} = C^n \oplus C_{n+1} \oplus D^{n+1} \to E_n = C_n \oplus C^{n+1} \oplus D_{n+1} ,$$

$$\theta_0 = 1 : E^n = C^n \oplus C_{n+1} \oplus D^{n+1} \to E_{n+1} = C^n \oplus C_{n+1} \oplus D^{n+1} ,$$

$$\theta_1 = \begin{pmatrix} -\psi_1 & -d & -\psi_0^* j^* \\ 0 & 0 & j^* \\ 0 & 0 & -\delta\psi_0 \end{pmatrix} :$$

$$E^n = C^n \oplus C_{n+1} \oplus D^{n+1} \to E_n = C_n \oplus C^{n+1} \oplus D_{n+1}$$

corresponds to the boundary $(-1)^n$-quadratic formation $\partial(E^n, \lambda_1, \mu_1)$ of the $(-1)^{n+1}$-quadratic form (E^n, λ_1, μ_1) determined by the split form θ_1, and there is defined a homotopy equivalence $f : (E, \theta) \to (C, \psi)$ with

$$f_n = (1 \;\; \psi_0^* \;\; 0) : E_n = C_n \oplus C^{n+1} \oplus D_{n+1} \to C_n ,$$

$$f_{n+1} = (0 \;\; 1 \;\; 0) : E_{n+1} = C^n \oplus C_{n+1} \oplus D^{n+1} \to C_{n+1} . \qquad \square$$

PROPOSITION 9.11 *The cobordism group $L_{2n+1}(\Lambda)$ of $(2n+1)$-complexes is naturally isomorphic to the abelian group of equivalence classes of $(-1)^n$-quadratic formations over Λ, subject to the equivalence relation*

$(Q, \phi; F, G) \sim (Q', \phi'; F', G')$ *if there exists a stable isomorphism*

$$[f] : (Q, \phi; F, G) \oplus (Q', -\phi'; F', G') \xrightarrow{\cong} \partial(K, \lambda, \mu)$$

for some $(-1)^{n+1}$-quadratic form (K, λ, μ) over Λ ,

with addition and inverses by

$$(Q, \phi; F, G) + (Q', \phi'; F', G') = (Q \oplus Q', \phi \oplus \phi'; F \oplus F', G \oplus G') ,$$

$$-(Q, \phi; F, G) = (Q, -\phi; F, G) \in L_{2n+1}(\Lambda) .$$

PROOF: This is just the translation of the definition (8.1) of $L_{2n+1}(\Lambda)$ into the language of $(-1)^n$-quadratic formations, using 9.4, 9.8 and 9.10. $\square$

Use 9.11 as an identification of $L_{2n+1}(\Lambda)$ with the group of equivalence classes of $(-1)^n$-quadratic formations over Λ.

COROLLARY 9.12 *A $(-1)^n$-quadratic formation $(Q, \phi; F, G)$ over Λ is such that $(Q, \phi; F, G) = 0 \in L_{2n+1}(\Lambda)$ if and only if it is stably isomorphic to the boundary $\partial(K, \lambda, \mu)$ of a $(-1)^{n+1}$-quadratic form (K, λ, μ) on a f. g. free Λ-module K.*

PROOF: Immediate from 9.10. $\square$

Next, it is necessary to establish the relation

$$(Q,\phi;F,G)\oplus(Q,\phi;G,H) \;=\; (Q,\phi;F,H)\in L_{2n+1}(\Lambda)\ .$$

This is the key step in the identification in §10 below of $L_{2n+1}(\Lambda)$ with a stable unitary group.

LEMMA 9.13 (i) *An ϵ-quadratic formation $(Q,\phi;F,G)$ is trivial if and only if the lagrangians F and G are direct complements in Q.*
(ii) *An ϵ-quadratic formation $(Q,\phi;F,G)$ is isomorphic to a boundary if and only if (Q,ϕ) has a lagrangian H which is a direct complement of both the lagrangians F,G.*
PROOF: (i) If F and G are direct complements in Q express any representative $\phi\in \mathrm{Hom}_\Lambda(Q,Q^*)$ of $\phi\in Q_\epsilon(Q)$ as

$$\phi \;=\; \begin{pmatrix}\lambda-\epsilon\lambda^* & \gamma\\ \delta & \mu-\epsilon\mu^*\end{pmatrix} \;:\; Q \;=\; F\oplus G\to Q^* \;=\; F^*\oplus G^*\ .$$

Then $\gamma+\epsilon\delta^*\in \mathrm{Hom}_\Lambda(G,F^*)$ is an Λ-module isomorphism, and there is defined an isomorphism of ϵ-quadratic formations

$$\begin{pmatrix}1 & 0\\ 0 & (\gamma+\epsilon\delta^*)^{-1}\end{pmatrix} \;:\; (H_\epsilon(F);F,F^*) \xrightarrow{\cong} (Q,\phi;F,G)$$

so that $(Q,\phi;F,G)$ is trivial. The converse is obvious.
(ii) For the boundary $\partial(K,\lambda,\mu)$ of a $(-\epsilon)$-quadratic form (K,λ,μ) the lagrangian K^* of $H_\epsilon(K)$ is a direct complement of both the lagrangians $K,\Gamma_{(K,\lambda)}$. Conversely, suppose that $(Q,\phi;F,G)$ is such that there exists a lagrangian H in (Q,ϕ) which is a direct complement to both F and G. By the proof of (i) there exists an isomorphism of formations

$$f \;:\; (H_\epsilon(F);F,F^*) \xrightarrow{\cong} (Q,\phi;F,H)$$

which is the identity on F. Now $f^{-1}(G)$ is a lagrangian of $H_\epsilon(F)$ which is a direct complement of F^*, so that it is the graph $\Gamma_{(F,\lambda)}$ of a $(-\epsilon)$-quadratic form (F,λ,μ), and f defines an isomorphism of ϵ-quadratic formations

$$f \;:\; \partial(F,\lambda,\mu) \;=\; (H_\epsilon(F);F,\Gamma_{(F,\lambda)}) \xrightarrow{\cong} (Q,\phi;F,G)\ . \qquad \square$$

PROPOSITION 9.14 *For any lagrangians F,G,H in a nonsingular $(-1)^n$-quadratic form (Q,ϕ) over Λ*

$$(Q,\phi;F,G)\oplus(Q,\phi;G,H) \;=\; (Q,\phi;F,H)\in L_{2n+1}(\Lambda)\ .$$

PROOF: Choose lagrangians F^*,G^*,H^* in (Q,ϕ) complementary to F,G,H respectively. The $(-1)^n$-quadratic formations $(Q_i,\phi_i;F_i,G_i)$ $(1\le i\le 4)$

defined by

$$
\begin{aligned}
(Q_1,\phi_1;F_1,G_1) &= (Q,-\phi;G^*,G^*)\ ,\\
(Q_2,\phi_2;F_2,G_2) &= (Q\oplus Q,\phi\oplus -\phi;F\oplus F^*,H\oplus G^*)\\
&\qquad\qquad \oplus (Q\oplus Q,-\phi\oplus\phi;\Delta_Q,H^*\oplus G)\ ,\\
(Q_3,\phi_3;F_3,G_3) &= (Q\oplus Q,\phi\oplus -\phi,F\oplus F^*,G\oplus G^*)\ ,\\
(Q_4,\phi_4;F_4,G_4) &= (Q\oplus Q,\phi\oplus -\phi;G\oplus G^*,H\oplus G^*)\\
&\qquad\qquad \oplus (Q\oplus Q,-\phi\oplus\phi;\Delta_Q,H^*\oplus G)
\end{aligned}
$$

are such that

$$
\begin{aligned}
(Q,\phi;F,G)\ &\oplus (Q,\phi;G,H)\oplus(Q_1,\phi_1;F_1,G_1)\oplus(Q_2,\phi_2;F_2,G_2)\\
&= (Q,\phi;F,H)\oplus(Q_3,\phi_3;F_3,G_3)\oplus(Q_4,\phi_4;F_4,G_4)\ .
\end{aligned}
$$

Each of $(Q_i,\phi_i;F_i,G_i)$ $(1\leq i\leq 4)$ is isomorphic to a boundary, since there exists a lagrangian H_i in (Q_i,ϕ_i) complementary to both F_i and G_i, so that 9.13 (ii) applies and $(Q_i,\phi_i;F_i,G_i)$ represents 0 in $L_{2n+1}(\Lambda)$. Explicitly, take

$$
\begin{aligned}
H_1 &= G\subset Q_1\ =\ Q\ ,\\
H_2 &= \Delta_{Q\oplus Q}\subset Q_2\ =\ (Q\oplus Q)\oplus(Q\oplus Q)\ ,\\
H_3 &= \Delta_Q\subset Q_3\ =\ Q\oplus Q\ ,\\
H_4 &= \Delta_{Q\oplus Q}\subset Q_4\ =\ (Q\oplus Q)\oplus(Q\oplus Q)\ .
\end{aligned}
$$

□

REMARK 9.15 It is also possible to express $L_{2n+1}(\Lambda)$ as the abelian group of equivalence classes of $(-1)^n$-quadratic formations over Λ subject to the equivalence relation generated by

(i) $(Q,\phi;F,G)\sim(Q',\phi';F',G')$ if $(Q,\phi;F,G)$ is stably isomorphic to $(Q',\phi';F',G')$,

(ii) $(Q,\phi;F,G)\oplus(Q,\phi;G,H)\sim(Q,\phi;F,H)$, with addition and inverses by

$$
\begin{aligned}
(Q,\phi;F,G)+(Q',\phi';F',G') &= (Q\oplus Q',\phi\oplus\phi';F\oplus F',G\oplus G')\ ,\\
-(Q,\phi;F,G) &= (Q,\phi;G,F)\in L_{2n+1}(\Lambda)\ .
\end{aligned}
$$

This is immediate from 9.13 and the observation that for any $(-1)^{n+1}$-quadratic form (K,λ,μ) on a f. g. free Λ-module K the lagrangian K^* in $H_{(-1)^n}(K)$ is a complement to both K and the graph $\Gamma_{(K,\lambda)}$, so that

$$
\begin{aligned}
\partial(K,\lambda,\mu)\ &\sim\ (H_{(-1)^n}(K);K,\Gamma_{(K,\lambda)})\oplus(H_{(-1)^n}(K);\Gamma_{(K,\lambda)},K^*)\\
&\sim\ (H_{(-1)^n}(K);K,K^*)\ \sim\ 0\ .
\end{aligned}
$$

□

§10. Automorphisms

The $(2n+1)$-dimensional L-group $L_{2n+1}(\Lambda)$ of a ring with involution Λ is identified with a quotient of the stable automorphism group of hyperbolic $(-1)^n$-quadratic forms over Λ, as in the original definition of Wall [29].

Given a Λ-module K let $\mathrm{Aut}_\Lambda(K)$ be the group of automorphisms $K \to K$, with the composition as group law.

EXAMPLE 10.1 The automorphism group of the f. g. free Λ-module Λ^k is the general linear group $GL_k(\Lambda)$ of invertible $k \times k$ matrices in Λ

$$\mathrm{Aut}_\Lambda(\Lambda^k) \;=\; GL_k(\Lambda)$$

with the multiplication of matrices as group law (cf. Remark 1.12). The general linear group is not abelian for $k \geq 2$, since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \qquad \square$$

DEFINITION 10.2 For any ϵ-quadratic form (K,λ,μ) let $\mathrm{Aut}_\Lambda(K,\lambda,\mu)$ be the subgroup of $\mathrm{Aut}_\Lambda(K)$ consisting of the automorphisms $f:(K,\lambda,\mu) \to (K,\lambda,\mu)$. $\square$

DEFINITION 10.3 The (ϵ,k)-*unitary group* of Λ is defined for $\epsilon = \pm 1$, $k \geq 0$ to be the automorphism group

$$U_{\epsilon,k}(\Lambda) \;=\; \mathrm{Aut}_\Lambda(H_\epsilon(\Lambda^k))$$

of the ϵ-quadratic hyperbolic form $H_\epsilon(\Lambda^k)$. $\square$

PROPOSITION 10.4 $U_{\epsilon,k}(\Lambda)$ is the group of invertible $2k \times 2k$ matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_{2k}(\Lambda)$ such that

$$\alpha^*\delta + \epsilon\gamma^*\beta \;=\; 1 \in M_{k,k}(\Lambda)\ ,\ \alpha^*\gamma \;=\; \beta^*\delta \;=\; 0 \in Q_\epsilon(\Lambda^k)\ .$$

PROOF: This is just the decoding of the condition

$$\begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Q_\epsilon(\Lambda^k \oplus (\Lambda^k)^*)$$

for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ to define an automorphism of the hyperbolic (split) ϵ-quadratic form

$$H_\epsilon(\Lambda^k) \;=\; (\Lambda^k \oplus (\Lambda^k)^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})\ . \qquad \square$$

Use 10.4 to express the automorphisms of $H_\epsilon(\Lambda^k)$ as matrices.

EXAMPLE 10.5 $U_{\epsilon,1}(\Lambda)$ is the subgroup of $GL_2(\Lambda)$ consisting of the 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$d\bar{a}+\epsilon b\bar{c} \;=\; 1\in\Lambda\ ,\ c\bar{a} \;=\; d\bar{b} \;=\; 0\in Q_\epsilon(\Lambda)\ . \qquad \square$$

DEFINITION 10.6 The *elementary* (ϵ,k)*-quadratic unitary group* of Λ is the normal subgroup

$$EU_{\epsilon,k}(\Lambda)\subseteq U_{\epsilon,k}(\Lambda)$$

of the full (ϵ,k)-quadratic unitary group generated by the elements of the following two types:

(i) $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}$ for any automorphism $\alpha\in GL_k(\Lambda)$,

(ii) $\begin{pmatrix} 1 & 0 \\ \theta-\epsilon\theta^* & 1 \end{pmatrix}$ for any split $(-\epsilon)$-quadratic form (Λ^k,θ). $\square$

LEMMA 10.7 *For any* $(-\epsilon)$*-quadratic form* $(\Lambda^k,\theta\in Q_{-\epsilon}(\Lambda^k))$

$$\begin{pmatrix} 1 & \theta-\epsilon\theta^* \\ 0 & 1 \end{pmatrix}\in EU_{\epsilon,k}(\Lambda)\ .$$

PROOF: This is immediate from the identity

$$\begin{pmatrix} 1 & \theta-\epsilon\theta^* \\ 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ \theta-\epsilon\theta^* & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad \square$$

Use the identifications

$$\Lambda^{k+1} \;=\; \Lambda^k\oplus\Lambda\ ,\ H_\epsilon(\Lambda^{k+1}) \;=\; H_\epsilon(\Lambda^k)\oplus H_\epsilon(\Lambda)$$

to define injections of groups

$$U_{\epsilon,k}(\Lambda)\to U_{\epsilon,k+1}(\Lambda)\ ;\ f\ \longmapsto f\oplus 1,$$

such that $EU_{\epsilon,k}(\Lambda)$ is sent into $EU_{\epsilon,k+1}(\Lambda)$.

DEFINITION 10.8 (i) The *stable* ϵ*-quadratic unitary group* of Λ is the union

$$U_\epsilon(\Lambda) \;=\; \bigcup_{k=1}^{\infty} U_{\epsilon,k}(\Lambda)\ .$$

(ii) The *elementary stable* ϵ*-quadratic unitary group* of Λ is the union

$$EU_\epsilon(\Lambda) \;=\; \bigcup_{k=1}^{\infty} EU_{\epsilon,k}(\Lambda)\ ,$$

a normal subgroup of $U_\epsilon(\Lambda)$.
(iii) The ϵ*-quadratic* M*-group* of Λ is the quotient

$$M_\epsilon(\Lambda) \;=\; U_\epsilon(\Lambda)/\{EU_\epsilon(\Lambda),\sigma_\epsilon\}$$

with $\sigma_\epsilon = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \in U_{\epsilon,1}(\Lambda) \subseteq U_\epsilon(\Lambda)$. □

The automorphism group $M_\epsilon(\Lambda)$ is the original definition due to Wall [29, Chap. 6] of the odd-dimensional L-group $L_{2n+1}(\Lambda)$, with $\epsilon = (-1)^n$. The original verification that $M_\epsilon(\Lambda)$ is abelian used a somewhat complicated matrix identity ([29, p.66]), corresponding to the formation identity 9.14. Formations will now be used to identify $M_{(-1)^n}(\Lambda)$ with the a priori abelian L-group $L_{2n+1}(\Lambda)$ defined in §8.

Given an automorphism of a hyperbolic $(-1)^n$-quadratic form

$$\alpha \;=\; \begin{pmatrix} \gamma & \widetilde{\gamma} \\ \mu & \widetilde{\mu} \end{pmatrix} \;:\; H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} H_{(-1)^n}(\Lambda^k)$$

define a $(2n+1)$-complex (C,ψ) by

$$\begin{aligned} d \;&=\; \mu^* \;:\; C_{n+1} \;=\; \Lambda^k \to C_n \;=\; \Lambda^k \;, \\ \psi_0 \;&=\; \gamma \;:\; C^n \;=\; \Lambda^k \to C_{n+1} \;=\; \Lambda^k \;, \end{aligned}$$

corresponding to the $(-1)^n$-quadratic formation

$$\Phi_k(\alpha) \;=\; (H_{(-1)^n}(\Lambda^k); \Lambda^k, \mathrm{im}(\begin{pmatrix} \gamma \\ \mu \end{pmatrix} : \Lambda^k \to \Lambda^k \oplus (\Lambda^k)^*)) \;.$$

LEMMA 10.9 *The formations $\Phi_k(\alpha_1)$, $\Phi_k(\alpha_2)$ associated to two automorphisms*

$$\alpha_i \;=\; \begin{pmatrix} \gamma_i & \widetilde{\gamma}_i \\ \mu_i & \widetilde{\mu}_i \end{pmatrix} \;:\; H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} H_{(-1)^n}(\Lambda^k) \quad (i = 1,2)$$

are isomorphic if and only if there exist $\beta_i \in GL_k(\Lambda)$, $\theta_i \in S(\Lambda^k)$ such that

$$\begin{aligned} &\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1^{*-1} \end{pmatrix} \begin{pmatrix} 1 & \theta_1 + (-1)^{n+1}\theta_1^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 & \widetilde{\gamma}_1 \\ \mu_1 & \widetilde{\mu}_1 \end{pmatrix} \\ &\qquad = \begin{pmatrix} \gamma_2 & \widetilde{\gamma}_2 \\ \mu_2 & \widetilde{\mu}_2 \end{pmatrix} \begin{pmatrix} \beta_2 & 0 \\ 0 & \beta_2^{*-1} \end{pmatrix} \begin{pmatrix} 1 & \theta_2 + (-1)^{n+1}\theta_2^* \\ 0 & 1 \end{pmatrix} \\ &\qquad\qquad : H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} H_{(-1)^n}(\Lambda^k) \;. \end{aligned}$$

PROOF: An automorphism α of the hyperbolic $(-1)^n$-quadratic form $H_{(-1)^n}(\Lambda^k)$ preserves the lagrangian $\Lambda^k \subset \Lambda^k \oplus (\Lambda^k)^*$ if and only if there exist $\beta \in GL_k(\Lambda)$, $\theta \in S(\Lambda^k)$ such that

$$\alpha \;=\; \begin{pmatrix} \beta & 0 \\ 0 & \beta^{*-1} \end{pmatrix} \begin{pmatrix} 1 & \theta + (-1)^{n+1}\theta^* \\ 0 & 1 \end{pmatrix} \;:\; H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} H_{(-1)^n}(\Lambda^k) \;.$$

□

PROPOSITION 10.10 *The function*

$$\Phi \ : \ M_{(-1)^n}(\Lambda) \to L_{2n+1}(\Lambda) \ ; \ \alpha \ \longmapsto \Phi_k(\alpha) \quad (\alpha \in U_{(-1)^n,k}(\Lambda))$$

is an isomorphism of groups.

PROOF: The function

$$\Phi_k \ : \ U_{(-1)^n,k}(\Lambda) \to L_{2n+1}(\Lambda) \ ; \ \alpha \longmapsto \Phi_k(\alpha)$$

is a group morphism by 9.14. Each of the generators (10.6) of the elementary subgroup $EU_{(-1)^n,k}(\Lambda)$ is sent to 0 with

(i) $\Phi_k\begin{pmatrix} \beta & 0 \\ 0 & \beta^{*-1}\end{pmatrix} = (H_{(-1)^n}(\Lambda^k);\Lambda^k,\Lambda^k) = \partial(\Lambda^k,0,0) = 0 \in L_{2n+1}(\Lambda)$,

(ii) $\Phi_k\begin{pmatrix} 1 & 0 \\ \theta+(-1)^{n+1}\theta^* & 1\end{pmatrix} = \partial(\Lambda^k,\theta+(-1)^{n+1}\theta^*,\theta) = 0 \in L_{2n+1}(\Lambda)$.

Also, abbreviating $\sigma_{(-1)^n}$ to σ

$$\Phi_1(\sigma) \ = \ (H_{(-1)^n}(\Lambda);\Lambda,\Lambda^*) \ = \ 0 \ ,$$

$$\Phi_{k+1}(\alpha\oplus 1) \ = \ \Phi_k(\alpha)\oplus(H_{(-1)^n}(\Lambda);\Lambda,\Lambda) \ = \ \Phi_k(\alpha) \in L_{2n+1}(\Lambda) \ .$$

Thus the morphisms Φ_k $(k \geq 0)$ fit together to define a group morphism

$$\Phi \ : \ M_{(-1)^n}(\Lambda) \to L_{2n+1}(\Lambda) \ ; \ \alpha \longmapsto \Phi_k(\alpha) \text{ if } \alpha \in U_{(-1)^n,k}(\Lambda)$$

such that

$$\Phi(\alpha_1\alpha_2) \ = \ \Phi(\alpha_1\oplus\alpha_2) \ = \ \Phi(\alpha_1)\oplus\Phi(\alpha_2) \in L_{2n+1}(\Lambda) \ .$$

Φ is onto by 9.3 (ii). It remains to prove that Φ is one-one.

For any $\alpha_i \in U_{(-1)^n,k_i}(\Lambda)$ $(i=1,2)$

$$\alpha_1\oplus\alpha_2 \ = \ \alpha_2\oplus\alpha_1 \in M_{(-1)^n}(\Lambda) \ ,$$

since

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2\end{pmatrix} \ = \ \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}^{-1}\begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$$

$$: \ H_{(-1)^n}(\Lambda^{k_1+k_2}) \to H_{(-1)^n}(\Lambda^{k_1+k_2}) \ .$$

Now $\sigma = 1 \in M_{(-1)^n}(\Lambda)$ (by construction), so that for any $\alpha \in U_{(-1)^n,k}(\Lambda)$

$$\alpha\oplus\sigma \ = \ \sigma\oplus\alpha \ = \ (\sigma\oplus 1)(1\oplus\alpha) \ = \ \alpha \in M_{(-1)^n}(\Lambda) \ .$$

It follows that for every $m \geq 1$

$$\sigma\oplus\sigma\oplus\ldots\oplus\sigma \ = \ 1 \in M_{(-1)^n}(\Lambda) \ (m\text{-fold sum}) \ .$$

If $\alpha \in U_{(-1)^n,k}(\Lambda)$ is such that $\Phi(\alpha) = 0 \in L_{2n+1}(\Lambda)$ then by 9.12 the $(-1)^n$-quadratic formation $\Phi_k(\alpha)$ is stably isomorphic to the boundary $\partial(\Lambda^{k'},\lambda,\mu)$ of a $(-1)^{n+1}$-quadratic form $(\Lambda^{k'},\lambda,\mu)$. Choosing a split form

$\theta \in S(\Lambda^{k'})$ for (λ, μ) this can be expressed as

$$\partial(\Lambda^{k'}, \lambda, \mu) \;=\; \Phi_{k'} \begin{pmatrix} 1 & 0 \\ \theta + (-1)^{n+1}\theta^* & 1 \end{pmatrix} .$$

Thus for a sufficiently large $k'' \geq 0$ there exist by 10.9 $\beta_i \in GL_{k''}(\Lambda)$, $\theta_i \in S(\Lambda^{k''})$ $(i = 1, 2)$ such that

$$\begin{aligned} &\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1^{*-1} \end{pmatrix} \begin{pmatrix} 1 & \theta_1 + (-1)^{n+1}\theta_1^* \\ 0 & 1 \end{pmatrix} (\alpha \oplus \sigma \oplus \ldots \oplus \sigma) \\ &= \left(\begin{pmatrix} 1 & 0 \\ \theta + (-1)^{n+1}\theta^* & 1 \end{pmatrix} \oplus \sigma \oplus \ldots \oplus \sigma \right) \begin{pmatrix} \beta_2 & 0 \\ 0 & \beta_2^{*-1} \end{pmatrix} \\ &\qquad \begin{pmatrix} 1 & \theta_2 + (-1)^{n+1}\theta_2^* \\ 0 & 1 \end{pmatrix} \;:\; H_{(-1)^n}(\Lambda^{k''}) \to H_{(-1)^n}(\Lambda^{k''}) \end{aligned}$$

so that by another application of 10.7

$$\alpha \;=\; \begin{pmatrix} 1 & 0 \\ \theta + (-1)^{n+1}\theta^* & 1 \end{pmatrix} \;=\; 1 \in M_{(-1)^n}(\Lambda) ,$$

verifying that Φ is one-one. □

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Automorphisms of manifolds

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0. Introduction

This survey is about homotopy types of spaces of automorphisms of topological and smooth manifolds. Most of the results available are *relative*, i.e., they compare different types of automorphisms.

In chapter 1, which motivates the later chapters, we introduce our favorite types of manifold automorphisms and make a comparison by (mostly elementary) geometric methods. Chapters 2, 3, and 4 describe algebraic models (involving L–theory and/or algebraic K–theory) for certain spaces of "structures" associated with a manifold M, that is, spaces of other manifolds sharing some geometric features with M. The algebraic models rely heavily on

- Wall's work in surgery theory, e.g. [Wa1] ,
- Waldhausen's work in h–cobordism theory alias concordance theory, which includes a parametrized version of Wall's theory of the finiteness obstruction, [Wa2] .

The structure spaces are of interest for the following reason. Suppose that two different notions of automorphism of M are being compared. Let $X_1(M)$ and $X_2(M)$ be the corresponding automorphism spaces; suppose that $X_1(M) \subset X_2(M)$. As a rule, the space of cosets $X_2(M)/X_1(M)$ is a union of connected components of a suitable structure space.

Chapter 5 contains the beginnings of a more radical approach in which one tries to calculate the classifying space $BX_1(M)$ in terms of $BX_2(M)$, rather than trying to calculate $X_2(M)/X_1(M)$. Chapter 6 contains some examples and calculations.

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Not included in this survey is the disjunction theory of automorphism spaces and embedding spaces begun by Morlet, see [BLR], and continued in [Go4], [Go1], [Go2], [We2], [We3], [GoWe], [GoKl]. It calls for a survey of its own.

1. Stabilization and descent

1.1. Notation, terminology

1.1.1. Terminology. *Space* with a capital S means *simplicial set.* We will occasionally see simplicial Spaces (=bisimplicial sets). A simplicial Space $k \mapsto Z_k$ determines a Space $(\amalg_k\ \Delta^k \times Z_k)/\sim$, where Δ^k is the k–simplex viewed as a Space (= simplicial set) and $\sim$ refers to the relations $(f_*x, y) \sim (x, f^*y)$. The Space $(\amalg_k\ \Delta^k \times Z_k)/\sim$ is isomorphic to the Space $k \mapsto Z_k(k)$, the *diagonal* of $k \mapsto Z_k$. See [Qui], for example.

A *euclidean k–bundle* is a fiber bundle with fibers homeomorphic to $\mathbb{R}^k$. Trivial euclidean k–bundles are often denoted ε^k.

The homotopy fiber of a map $B \to C$ of Spaces, where C is based, will be denoted hofiber$[\,B \to C\,]$. A *homotopy fiber sequence* is a diagram of spaces $A \longrightarrow B \longrightarrow C$ where C is based, together with a nullhomotopy of the composition $A \to C$ which makes the resulting map from A to hofiber$[\,B \to C\,]$ a (weak) homotopy equivalence.

The term *cartesian square* is synonymous with *homotopy pullback square.* More generally, an *n–cartesian square* ($n \leq \infty$) is a commutative diagram of Spaces and maps

$$\begin{array}{ccc} A & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & D \end{array}$$

such that the resulting map from A to the homotopy pullback of the diagram $B \longrightarrow D \longleftarrow C$ is n–connected.

A *commutative diagram* of Spaces is a functor F from some small category $\mathcal{D}$ to Spaces. We say that $\mathcal{D}$ is the shape of the diagram. When we represent such a diagram graphically, we usually only show the maps $F(g_i)$ for a set $\{g_i\}$ of morphisms generating $\mathcal{D}$. For example, the commutative square just above is a functor from a category with four objects and five non–identity morphisms to Spaces. The notion of a *homotopy commutative diagram* of shape $\mathcal{D}$ has been made precise by [Vogt]. It is a continuous functor from a certain topological category $\mathcal{WD}$ (determined by $\mathcal{D}$) to Spaces. In more detail, $\mathcal{WD}$ is a small topological category with discrete object set, and comes with a continuous functor $\mathcal{WD} \to \mathcal{D}$ which restricts to a homeomorphism (=bijection) of object sets, and to a homotopy equivalence of morphism spaces. Graphically, we represent homotopy commutative diagrams of shape $\mathcal{D}$ like commutative diagrams of shape $\mathcal{D}$.

1.1.2. Notation. For a topological manifold M, we denote by $\mathrm{TOP}(M)$ the Space of homeomorphisms $f : M \to M$ which agree with the identity on ∂M. (A k–simplex in $\mathrm{TOP}(M)$ is a homeomorphism $f : M \times \Delta^k \to M \times \Delta^k$ over Δ^k which agrees with the identity on $\partial M \times \Delta^k$.) References: [BLR], [Bu1].

We use the abbreviations $\mathrm{TOP}(n) = \mathrm{TOP}(\mathbb{R}^n)$ and $\mathrm{TOP} = \bigcup_n \mathrm{TOP}(n)$, where we include $\mathrm{TOP}(n)$ in $\mathrm{TOP}(n+1)$ by $f \mapsto f \times \mathrm{id}_{\mathbb{R}}$.

Let $\partial_+ M$ be a codimension zero submanifold of ∂M, closed as a subspace of ∂M. Let $\partial_- M$ be the closure of $\partial M \smallsetminus \partial_+ M$. Let $\mathrm{TOP}(M, \partial_+ M)$ be the Space of homeomorphisms $M \to M$ which agree with the identity on $\partial_- M$. (This is $\mathrm{TOP}(M)$ if $\partial_+ M = \emptyset$.) Special case: the Space of concordances alias pseudo–isotopies of a compact manifold N, which is $\mathrm{C}(N) := \mathrm{TOP}(N \times I, N \times 1)$ where $I = [0, 1]$. References: [Ce], [HaWa], [Ha], [DIg], [Ig]. Let $\mathrm{G}(M, \partial_+ M)$ be the Space of homotopy equivalences of triads, $(M; \partial_+ M, \partial_- M) \to (M; \partial_+ M, \partial_- M)$, which are the identity on $\partial_- M$. We abbreviate $\mathrm{G}(M, \emptyset)$ to $\mathrm{G}(M)$. *Warning:* If $\partial M = \emptyset$, then $\mathrm{G}(M)$ is the Space of homotopy equivalences $M \to M$, but in general it is not.

1.1.3. Definitions. An *h–structure* on a closed manifold M^n is a pair (N, f) where N^n is another closed manifold and $f : N \to M$ is a homotopy equivalence. If f is a *simple* homotopy equivalence, (N, f) is an *s–structure*. An *isomorphism* from an h–structure (N_1, f_1) to another h–structure (N_2, f_2) on M is a homeomorphism $N_1 \to N_2$ over M.

We see that the h–structures on M form a groupoid. Better, they form a *simplicial groupoid*: Objects in degree k are pairs (N, f) where N^n is another closed manifold and $f : N \times \Delta^k \to M \times \Delta^k$ is a homotopy equivalence over Δ^k. Morphisms in degree k are homeomorphisms over $M \times \Delta^k$.

Let $\mathcal{S}(M)$ be the diagonal nerve (= diagonal of degreewise nerve) of this simplicial groupoid; also let $\mathcal{S}^s(M)$ be the diagonal nerve of the simplicial subgroupoid of s–structures. Think of $\mathcal{S}(M)$ and $\mathcal{S}^s(M)$ as the *Spaces* of h–structures on M and s–structures on M, respectively. (They are actually simplicial *classes*, not simplicial sets, as it stands. The reader can either accept this, or avoid it by working in a Grothendieck "universe".) The forgetful functor $(N, f) \mapsto N$ induces a map from $\mathcal{S}(M)$ to the diagonal nerve of the simplicial groupoid of all closed n–manifolds and homeomorphisms between such. This map is a Kan fibration. Its fiber over the point corresponding to M is $\mathrm{G}(M)$. Hence there is a homotopy fiber sequence

$$\mathrm{TOP}(M) \to \mathrm{G}(M) \to \mathcal{S}(M)\,.$$

More generally, given compact M and $\partial_+ M \subset \partial M$ as above, there is an h–structure Space $\mathcal{S}(M, \partial_+ M)$ and an s–structure Space $\mathcal{S}^s(M, \partial_+ M)$ and a homotopy fiber sequence

$$\mathrm{TOP}(M, \partial_+ M) \to \mathrm{G}(M, \partial_+ M) \to \mathcal{S}(M, \partial_+ M)\,.$$

We omit the details. Important special cases: the *Space of h–cobordisms* and the *Space of s–cobordisms* on a compact manifold,

$$\mathcal{H}(N) := \mathcal{S}(N \times I, N \times 1)$$
$$\mathcal{H}^s(N) := \mathcal{S}^s(N \times I, N \times 1)\,.$$

Since $\mathrm{G}(N \times I, N \times 1) \simeq *$, we have $\Omega\mathcal{H}(N) \simeq \mathrm{C}(N)$. There is a stabilization map $\mathcal{H}(N) \to \mathcal{H}(N \times I)$, *upper stabilization* to be precise [Wah2], [HaWa]. Let

$$\mathcal{H}^\infty(N) := \operatorname*{hocolim}_k \mathcal{H}(N \times I^k)\,, \qquad \mathrm{C}^\infty(N) := \operatorname*{hocolim}_k \mathrm{C}(N \times I^k)\,.$$

1.1.4. More definitions. The *block automorphism Space* $\widetilde{\mathrm{TOP}}(M)$ has as its k–simplices the homeomorphisms $g : M \times \Delta^k \to M \times \Delta^k$ which satisfy $g(M \times s) = M \times s$ for each face $s \subset \Delta^k$, and restrict to the identity on $\partial M \times \Delta^k$. References: [ABK], [Bu1] [Br1]. There is also a block s-structure Space

$$\widetilde{\mathcal{S}}^s(M),$$

defined as the diagonal nerve of a simplicial groupoid. The objects of the simplicial groupoid in degree k are of the form (N, f) where N^n is closed and f is a *simple* homotopy equivalence $N \times \Delta^k \to M \times \Delta^k$ such that $f(N \times t) \subset M \times t$ for each face t of Δ^k, and f restricts to a homeomorphism $\partial N \times \Delta^k \to \partial M \times \Delta^k$. The morphisms in degree k are homeomorphisms respecting the reference maps to $M\times\Delta^k$. References: [Qun1], [Wa1, §17.A], [Ni], [Rou1]. There is a homotopy fiber sequence

$$\widetilde{\mathrm{TOP}}(M) \to \widetilde{\mathrm{G}}^s(M) \to \widetilde{\mathbb{S}}^s(M),$$

where $\widetilde{\mathrm{G}}^s(M)$ is defined like $\widetilde{\mathrm{TOP}}(M)$, but with simple homotopy equivalences instead of homeomorphisms.— These definitions have relative versions (details omitted); for example, there is a homotopy fiber sequence

$$\widetilde{\mathrm{TOP}}(M,\partial_+M) \to \widetilde{\mathrm{G}}^s(M,\partial_+M) \to \widetilde{\mathbb{S}}^s(M,\partial_+M)\,.$$

Let $\mathrm{G}^s(M,\partial_+M) \subset \mathrm{G}(M,\partial_+M)$ consist of the components containing those f which are simple homotopy automorphisms and induce simple homotopy automorphisms of ∂_+M. The inclusion

$$\mathrm{G}^s(M,\partial_+M) \to \widetilde{\mathrm{G}}^s(M,\partial_+M)$$

is a homotopy equivalence (because it induces an isomorphism on homotopy groups; both Spaces are fibrant).

1.2. Open stabilization versus closed stabilization

Let M^n be compact, $M_0 = M \smallsetminus \partial M$. *Open stabilization* refers to the map

$$\mathrm{TOP}(M,\partial M) \to \bigcup_k \mathrm{TOP}(M_0\times\mathbb{R}^k)$$

given by $f\mapsto f|M_0$. We include $\mathrm{TOP}(M_0\times\mathbb{R}^k)$ in $\mathrm{TOP}(M_0\times\mathbb{R}^{k+1})$ by $g\mapsto g\times\mathrm{id}_{\mathbb{R}}$. *Closed stabilization* refers to the inclusion

$$\mathrm{TOP}(M,\partial M) \to \bigcup_k \mathrm{TOP}(M\times I^k, \partial(M\times I^k))\,.$$

Open stabilization factors through closed stabilization, by means of the restriction maps $\mathrm{TOP}(M\times I^k,\partial(M\times I^k)) \to \mathrm{TOP}(M_0\times I_0^k)$ and an identification $I_0\cong\mathbb{R}$. Here in §1.2 we describe the homotopy type of $\bigcup_k \mathrm{TOP}(M_0\times\mathbb{R}^k)$, and descend from there to $\bigcup_k \mathrm{TOP}(M\times I^k,\partial(M\times I^k))$. For a more algebraic version of this, see §5.2.

Let $\hat\tau : M_0 \to B\mathrm{TOP}(n)$ classify the tangent bundle [Mi1], [Kis], [Maz2], [KiSi,IV.1]. We map $\mathrm{G}(M_0)$ to the mapping Space $\mathrm{map}(M_0, B\mathrm{TOP})$ by $f\mapsto\hat\tau f$ and we map $\bigcup_k \mathrm{TOP}(M_0\times\mathbb{R}^k)$ to $\mathrm{G}(M_0)$ by $f\mapsto pfi$, where $p: M_0\times\mathbb{R}^k\to M_0$ and $i: M_0\cong M_0\times\mathbf{0}\to M_0\times\mathbb{R}^k$ are projection and inclusion, respectively.

1.2.1. Theorem [CaGo]**.** *The resulting diagram*

$$\bigcup_k \mathrm{TOP}(M_0 \times \mathbb{R}^k) \longrightarrow \mathrm{G}(M_0) \longrightarrow \mathrm{map}(M_0, B\mathrm{TOP})$$

is a homotopy fiber sequence.

The proof uses immersion theory [Gau], general position, and the half–open s–cobordism theorem [Sta]. See also [Maz1].

Choose a collar for M, that is, an embedding $c : \partial M \times I \to M$ extending the map $(x, 0) \to x$ on $\partial M \times 0$. Reference: [Brn], [KiSi, I App. A]. Any homeomorphism $f : M_0 \to M_0$ determines an h–cobordism W_f on ∂M: the region of M enclosed by ∂M and $fc(\partial M \times 1)$. The bundle on the geometric realization of $\mathrm{TOP}(M_0)$ with fiber W_f over the vertex f is a bundle of h–cobordisms, classified by a map v from $\mathrm{TOP}(M_0)$ to $\mathcal{H}(\partial M)$.

If $f : M_0 \to M_0$ is the restriction of some homeomorphism $g : M \to M$, then $W_f \cong gc(\partial M \times I)$ is trivialized. Conversely, a trivialization of W_f can be used to construct a homeomorphism $g : M \to M$ with an isotopy from $g|M_0$ to f. Therefore: the diagram

$$\mathrm{TOP}(M, \partial M) \xrightarrow{\mathrm{res}} \mathrm{TOP}(M_0) \xrightarrow{v} \mathcal{H}(\partial M) \tag{1.2.2}$$

is a homotopy fiber sequence. See [Cm] for details. The special case where $M = \mathbb{D}^n$ is due to [KuLa].

This observation can be stabilized. Let $u : \mathcal{H}(\partial M) \to \mathcal{H}(\partial(M \times I))$ be the composition of stabilization $\mathcal{H}(\partial M) \to \mathcal{H}(\partial M \times I)$ with the map induced by the inclusion of $\partial M \times I$ in $\partial(M \times I)$. Then

$$\begin{array}{ccc} \mathrm{TOP}(M_0) & \xrightarrow{\quad v \quad} & \mathcal{H}(\partial M) \\ \big\downarrow & & \big\downarrow u \\ \mathrm{TOP}(M_0 \times I_0) & \xrightarrow{\qquad} & \mathcal{H}(\partial(M \times I)) \end{array}$$

is homotopy commutative. The homotopy colimit of the $\mathcal{H}(\partial(M \times I^k))$ under the u–maps becomes $\simeq \mathcal{H}^\infty(M)$. Therefore:

1.2.3. Theorem. *There exists a homotopy fiber sequence*

$$\bigcup_k \mathrm{TOP}(M \times I^k, \partial(M \times I^k)) \xrightarrow{\mathrm{res}} \bigcup_k \mathrm{TOP}(M_0 \times \mathbb{R}^k) \longrightarrow \mathcal{H}^\infty(M)\,.$$

Let $Q = I^\infty$ be the Hilbert cube. The product $M \times Q$ is a *Hilbert cube manifold* [Cha] without boundary. Let $\mathrm{TOP}(M \times Q)$ be the Space of homeomorphisms $M \times Q \to M \times Q$ and let $\mathrm{G}(M \times Q)$ be the Space of homotopy equivalences $M \times Q \to M \times Q$. Chapman and Ferry have shown [Bu2] that an evident map from $\bigcup_k \mathrm{TOP}(M \times I^k, \partial(M \times I^k))$ to the homotopy fiber of the composition

$$\mathrm{TOP}(M \times Q) \xrightarrow{\subset} \mathrm{G}(M \times Q) \;\simeq\; G(M_0) \to \mathrm{map}(M_0, B\mathrm{TOP})$$

(last arrow as in 1.2.1) is a homotopy equivalence. Therefore

$$\bigcup_k \mathrm{TOP}(M \times I^k, \partial(M \times I^k)) \longrightarrow \mathrm{TOP}(M \times Q) \longrightarrow \mathrm{map}(M_0, B\mathrm{TOP})$$

is a homotopy fiber sequence. Comparison with 1.2.1 gives the next result.

1.2.4. Theorem. *The following homotopy commutative diagram is cartesian:*

$$\begin{array}{ccc} \bigcup_k \mathrm{TOP}((M \times I^k, \partial(M \times I^k)) & \xrightarrow{\mathrm{res}} & \bigcup_k \mathrm{TOP}(M_0 \times \mathbb{R}^k) \\ \big\downarrow & & \big\downarrow \\ \mathrm{TOP}(M \times Q) & \longrightarrow & \mathrm{G}(M_0)\,. \end{array}$$

This suggests that the map $\bigcup_k \mathrm{TOP}(M_0 \times \mathbb{R}^k) \to \mathcal{H}^\infty(M)$ in 1.2.3 factors through $\mathrm{G}(M_0)$. We will obtain such a factorization in 1.5.3.

Remark. Looking at horizontal homotopy fibers in 1.2.4, and using 1.2.3, and the homotopy equivalence $\mathrm{G}(M_0) \simeq \mathrm{G}(M \times Q)$, one finds that the homotopy fiber of the inclusion $\mathrm{TOP}(M \times Q) \to \mathrm{G}(M \times Q)$ is $\mathrm{C}^\infty(M)$. This can also be deduced from [Cha2], [Cha3].

1.3. Bounded stabilization versus no stabilization

Let M^n be compact. A homeomorphism $f : M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ is *bounded* if $\{p_2 f(z) - p_2(z) \mid z \in M \times \mathbb{R}^k\}$ is a bounded subset of $\mathbb{R}^k$, where $p_2 : M \times \mathbb{R}^k \to \mathbb{R}^k$ is the projection. Let $\mathrm{TOP}^b(M \times \mathbb{R}^k)$ be the Space of bounded homeomorphisms $M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ which agree with the identity on $\partial M \times \mathbb{R}^k$. Note $\mathrm{TOP}(M) = \mathrm{TOP}^b(M \times \mathbb{R}^0)$. *Bounded stabilization* refers to the inclusion

$$\mathrm{TOP}(M) \to \bigcup_k \mathrm{TOP}^b(M \times \mathbb{R}^k)\,.$$

Surgery theory describes the homotopy type of $\bigcup_k \mathrm{TOP}^b(M \times \mathbb{R}^k)$, modulo the mysteries of $\mathrm{G}(M)$. See §2.4; here in §1.3 we analyze the difference between $\bigcup_k \mathrm{TOP}^b(M \times \mathbb{R}^k)$ and $\mathrm{TOP}(M)$.

The Space $\mathrm{TOP}^b(M \times \mathbb{R}^{k+1})/\,\mathrm{TOP}^b(M \times \mathbb{R}^k)$ for $k \geq 0$ and fixed M is the k–th Space in a spectrum $\mathbf{H}(M)$, by analogy with the sphere spectrum, which is made out of the spaces $\mathrm{O}(\mathbb{R}^{k+1})/\,\mathrm{O}(\mathbb{R}^k)$. Compare [BuLa1]. Anderson and Hsiang, who introduced bounded homeomorphisms in [AH1], [AH2] showed that $\Omega^{\infty+1}(\mathbf{H}(M)) \simeq \mathrm{C}^\infty(M)$. In more detail: they introduced *bounded concordance spaces*

$$\mathrm{C}^b(M \times \mathbb{R}^k) = \mathrm{TOP}^b(M \times I \times \mathbb{R}^k, M \times 1 \times \mathbb{R}^k)$$

and proved the following. See also [WW1,§1+App.5], [Ha, App.II].

1.3.1. Theorem [AH1]**,** [AH2]**.** *Assume $n > 4$. Then*
 i) $\Omega(\mathrm{TOP}^b(M \times \mathbb{R}^{k+1})/\,\mathrm{TOP}^b(M \times \mathbb{R}^k)) \simeq \mathrm{C}^b(M \times \mathbb{R}^k)$*;*
 ii) $\Omega\,\mathrm{C}^b(M \times \mathbb{R}^k) \simeq \mathrm{C}^b(M \times I \times \mathbb{R}^{k-1})$*.*

Part ii) of 1.3.1 shows that the spaces $\mathrm{C}^b(M \times \mathbb{R}^k)$ for $k \geq 0$ form a spectrum, with structure maps

$$\mathrm{C}^b(M \times \mathbb{R}^{k-1}) \xrightarrow{\text{stab.}} \mathrm{C}^b(M \times I \times \mathbb{R}^{k-1}) \;\simeq\; \Omega\,\mathrm{C}^b(M \times \mathbb{R}^k);$$

then part i) of 1.3.1 with some extra work [WW1,§1] identifies the new spectrum with $\Omega\mathbf{H}(M)$. It is also shown in [AH1] that the homotopy groups $\pi_j\mathbf{H}(M)$ for $j \leq 0$ are lower K–groups [Ba]:

1.3.2. Theorem. *Let $j \leq 0$ be an integer. Then*

$$\pi_j\,\mathrm{C}^b(M \times \mathbb{R}^k) = \begin{cases} K_{j-k+2}(\mathbb{Z}\pi_1(M)) & (j < k-2) \\ \widetilde{K}_0(\mathbb{Z}\pi_1(M)) & (j = k-2) \\ \textit{Whitehead gp. of } \pi_1(M) & (j = k-1)\,. \end{cases}$$

Remark. Madsen and Rothenberg [MaRo1] , [MaRo2] have proved equivariant analogs of 1.3.1 and 1.3.2, and Chapman [Cha4], Hughes [Hu] have proved a Hilbert cube analog. Carter [Ca1], [Ca2], [Ca3] has shown that $K_r(\mathbb{Z}\pi)$ vanishes if π is finite and $r < -1$.

Remark. It is shown in [WW1,§5] that $\Omega^\infty \mathbf{H}(M) \simeq \mathcal{H}^\infty(M)$; this improves slightly on $\Omega^{\infty+1}\mathbf{H}(M) \simeq \mathrm{C}^\infty(M)$.

Theorems 1.3.1 and 1.3.2 are about descent from $\mathrm{TOP}^b(M \times \mathbb{R}^{k+1})$ to $\mathrm{TOP}^b(M \times \mathbb{R}^k)$. For instant descent from $\mathrm{TOP}^b(M \times \mathbb{R}^{k+1})$ to $\mathrm{TOP}(M)$, there is the *hyperplane test* [WW1,§3], [We4]. Think of $\mathbb{R}P^k$ as the Grassmannian of codimension one linear subspaces $W \subset \mathbb{R}^{k+1}$. Let Γ_k be the Space of sections of the bundle $E(k) \to \mathbb{R}P^k$ with fibers

$$E(k)_W := \mathrm{TOP}^b(M \times \mathbb{R}^{k+1}) / \mathrm{TOP}^b(M \times W)$$

(see the remark just below). Note that $E(k) \to \mathbb{R}P^k$ has a trivial section picking the coset [id] in each fiber; so Γ_k is a based Space.

Remark. The "bundle" $E(k) \to \mathbb{R}P^{k+1}$ is really a *twisted cartesian product* [Cu] with base Space equal to the singular simplicial set of $\mathbb{R}P^k$, and with fibers $E(k)_W$ over a vertex W as stated.

We define a map $\Phi_k : \mathrm{TOP}^b(M \times \mathbb{R}^{k+1}) / \mathrm{TOP}(M) \to \Gamma_k$ by taking the coset $f \cdot \mathrm{TOP}(M)$ to the section $W \mapsto f \cdot \mathrm{TOP}^b(M \times W)$. For $k > 0$, it is easy to produce an embedding v_k making the square

$$\begin{array}{ccc} \mathrm{TOP}^b(M \times \mathbb{R}^k) / \mathrm{TOP}(M) & \xrightarrow{\Phi_{k-1}} & \Gamma_{k-1} \\ \cap \big\downarrow & & v_k \big\downarrow \\ \mathrm{TOP}^b(M \times \mathbb{R}^{k+1}) / \mathrm{TOP}(M) & \xrightarrow{\Phi_k} & \Gamma_k \end{array}$$

commutative. Let $\Phi : \bigcup_k \mathrm{TOP}^b(M \times \mathbb{R}^k) / \mathrm{TOP}(M) \to \bigcup_k \Gamma_k$ be the union of the Φ_{k-1} for $k \geq 0$. It turns out that Φ is highly connected (1.3.5 below), under mild conditions on M.

1.3.3. Definition. An integer j is in the *topological, resp. smooth, concordance stable range* for M if the upper stabilization maps from $\mathrm{C}(M \times I^r)$ to $\mathrm{C}(M \times I^{r+1})$, resp. the smooth versions, are j–connected, for all $r \geq 0$.

1.3.4. Theorem [Ig]. *If M is smooth and $n \geq \max\{2j+7, 3j+4\}$, then j is in the smooth and in the topological concordance stable range for M.* (The estimate for the topological concordance stable range is due to Burghelea–Lashof and Goodwillie. Their argument uses smoothness of M, and Igusa's estimate of the smooth concordance stable range. See [Ig, Intro.])

1.3.5. Proposition [WW1]. *If j is in the topological concordance stable range for M, and $n > 4$, then $\Phi : \bigcup_k \mathrm{TOP}^b(M \times \mathbb{R}^k)/\,\mathrm{TOP}(M) \to \bigcup_k \Gamma_{k-1}$ is $(j+1)$–connected.*

Outline of proof. For $-1 \leq \ell \leq k$ let $\Gamma_{k,\ell} \subset \Gamma_k$ consist of the sections s for which $s(W) = *$ whenever W contains the standard copy of $\mathbb{R}^{\ell+1}$ in $\mathbb{R}^{k+1}$. Let $\Phi_{k,\ell}$ be the restriction of Φ_k to $\mathrm{TOP}^b(M \times \mathbb{R}^{\ell+1})/\,\mathrm{TOP}(M)$, viewed as a map with codomain $\Gamma_{k,\ell}$. One shows by induction on ℓ that the $\Phi_{k,\ell}$ for fixed ℓ define a highly connected map

$$\mathrm{TOP}^b(M \times \mathbb{R}^{\ell+1})/\,\mathrm{TOP}(M) \longrightarrow \bigcup_{k \geq \ell} \Gamma_{k,\ell}\,. \quad \square$$

1.3.6 Theorem [WW1]. *There exists a homotopy equivalence*

$$\bigcup_k \Gamma_k \simeq \Omega^\infty(\mathbf{H}(M)_{h\mathbb{Z}/2})$$

for some involution on $\mathbf{H}(M)$. Here $\mathbf{H}(M)_{h\mathbb{Z}/2} := (E\mathbb{Z}/2)_+ \wedge_{\mathbb{Z}/2} \mathbf{H}(M)$ is the homotopy orbit spectrum.

Sketch proof. Note $\bigcup_k \Gamma_k = \bigcup_{\ell>0} \bigcup_{k>\ell} \Gamma_{k,\ell}$ and $\bigcup_{k>\ell} \Gamma_{k,\ell}$ is homotopy equivalent to $\Omega^\infty(\mathbb{S}^\ell_+ \wedge_{\mathbb{Z}/2} \mathbf{H}(M))$ by Poincaré duality [WW1, 2.4], using 1.3.1 and 1.3.2. $\square$

1.3.7. Summary. There exist a spectrum $\mathbf{H}(M)$ with involution and a $(j+1)$–connected map

$$\bigcup_k \mathrm{TOP}^b(M \times \mathbb{R}^k)/\,\mathrm{TOP}(M) \longrightarrow \Omega^\infty(\mathbf{H}(M)_{h\mathbb{Z}/2})$$

where j is the largest integer in the topological concordance stable range for M. Further, $\Omega^\infty \mathbf{H}(M) \simeq \mathcal{H}^\infty(M)$ and

$$\pi_r \mathbf{H}(M) := \begin{cases} \mathrm{Wh}_1(\pi_1(M)) & r = 0 \\ \widetilde{K}_0(\mathbb{Z}\pi_1(M)) & r = -1 \\ K_{r+1}(\mathbb{Z}\pi_1(M)) & r < -1\,. \end{cases}$$

(See §3 for information about $\pi_r \mathbf{H}(M)$ when $r > 0$.)

1.4. Block automorphism Spaces

The property $\pi_k \widetilde{\mathrm{TOP}}(M) \cong \pi_0 \widetilde{\mathrm{TOP}}(M \times \Delta^k)$ is a consequence of the definitions and has made the block automorphism Spaces popular. See also §2. In homotopy theory terms, the block automorphism Space of M is more closely related to $\bigcup_k \mathrm{TOP}^b(M \times \mathbb{R}^k)$ than to $\mathrm{TOP}(M)$. To explain this we need the bounded block automorphism Spaces

$$\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^k)$$

(definition left to the reader). The following *Rothenberg* type sequence is obtained by inspection, using 1.3.2. For notation, see 1.3.7. Compare [Sha], [Ra1, §1.10].

1.4.2. Proposition [AnPe], [WW1]. *For $k \geq 0$ there exists a long exact sequence*

$$\begin{aligned} \cdots \to \pi_r \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^k) \to \pi_r \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^{k+1}) \to H_{r+k}(\mathbb{Z}/2; \pi_{-k}\mathbf{H}(M)) \\ \to \pi_{r-1} \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^k) \to \pi_{r-1} \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^{k+1}) \to \cdots \end{aligned}$$

The inclusion $\bigcup_k \mathrm{TOP}^b(M \times \mathbb{R}^k) \to \bigcup_k \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^k)$ is a homotopy equivalence [WW1, 1.14]. Together with 1.4.2, this shows for example that

$$\widetilde{\mathrm{TOP}}(M) \simeq \bigcup_k \mathrm{TOP}^b(M \times \mathbb{R}^k)$$

if M is simply connected, because then $\pi_k \mathbf{H}(M) = 0$ for $k \leq 0$.

Another way to relate $\widetilde{\mathrm{TOP}}(M)$ and $\bigcup_k \mathrm{TOP}^b(M \times \mathbb{R}^k)$ is to use a filtered version of Postnikov's method for making highly connected covers. Let X be a fibrant Space with a filtration by fibrant subSpaces

$$X(0) \quad \subset \quad X(1) \quad \subset \quad X(2) \quad \subset \ldots$$

so that X is the union of the $X(k)$. Call an i–simplex in X *positive* if its characteristic map $\Delta^i \to X$ is filtration–preserving, i.e., takes the k–skeleton of Δ^i to $X(k)$. The positive simplices form a subSpace ${}^{\mathrm{pos}}X$ of X, and we let ${}^{\mathrm{pos}}X(k) := {}^{\mathrm{pos}}X \cap X(k)$. Then ${}^{\mathrm{pos}}X(k)$ is fibrant. If $X(0)$ is based, then

$$\pi_i({}^{\mathrm{pos}}X(k), {}^{\mathrm{pos}}X(k-1)) \xrightarrow{\cong} \pi_i(X(k), X(k-1))$$

for $i \geq k$, and $\pi_i({}^{\mathrm{pos}}X(k), {}^{\mathrm{pos}}X(k-1)) = *$ for $i < k$. For example: if $X(k) = *$ for $k \leq m$ and $X(k) = X$ for $k > m$, then ${}^{\mathrm{pos}}X$ is the m–connected Postnikov cover of X. And if $X(k)$ is $\mathrm{TOP}^b(M \times \mathbb{R}^k)/\mathrm{TOP}(M)$, then

$${}^{\mathrm{pos}}X \simeq \widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)\,.$$

See [WW1, 4.10] for a more precise statement, and a proof.

1.4.3. Corollary [Ha]**.** *There exists a spectral sequence with E^1–term given by $E^1_{pq} = \pi_{q-1}(\mathrm{C}(M \times I^p))$, converging to the homotopy groups of*

$$\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)\,.$$

Hatcher also described E^2_{pq} for $p + n \gg q$. What he found is explained by the next theorem, which uses naturality of the pos–construction and the results of §1.3.

1.4.4. Theorem [WW1, Thm. C]**.** *There exists a homotopy commutative cartesian square of the form*

$$\begin{array}{ccc}
\begin{array}{c}\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \\ \simeq \\ {}^{\mathrm{pos}}\Big(\bigcup_k \mathrm{TOP}^b(M\times\mathbb{R}^k)/\mathrm{TOP}(M)\Big)\end{array} & \longrightarrow & \Omega^\infty(\mathbf{H}^s(M)_{h\mathbb{Z}/2}) \\
\big\downarrow \cap & & \big\downarrow \\
\bigcup_k \mathrm{TOP}(M\times\mathbb{R}^k)/\mathrm{TOP}(M) & \xrightarrow{\Phi} & \Omega^\infty(\mathbf{H}(M)_{h\mathbb{Z}/2})
\end{array}$$

where $\mathbf{H}^s(M)$ *is the 0–connected cover of* $\mathbf{H}(M)$*, the right–hand vertical arrow is induced by the canonical map* $\mathbf{H}^s(M) \to \mathbf{H}(M)$*, and* Φ *is the map from 1.3.7.*

By 1.3.7, lower and hence upper horizontal arrow in 1.4.4 are j–connected for j in the topological concordance stable range of M.

1.5. h–STRUCTURES AND h–COBORDISMS

Our main goal in this section is to construct a *Whitehead torsion* map $w : \mathcal{S}(M) \longrightarrow \mathcal{H}^\infty(M)$, and a simple version, $\mathcal{S}^s(M) \longrightarrow (\mathcal{H}^\infty)^s(M) = \Omega^\infty \mathbf{H}^s(M)$, which makes the following diagram homotopy commutative (see 1.1.1):

$$\begin{array}{ccc} \widetilde{\mathrm{TOP}}(M)/\,\mathrm{TOP}(M) & \xrightarrow{\ \delta\ } & \mathcal{S}^s(M) \\ \Big\downarrow{\scriptstyle 1.4.4} & & \Big\downarrow{\scriptstyle w} \\ \Omega^\infty(\mathbf{H}^s(M)_{h\mathbb{Z}/2}) & \xrightarrow{\text{transfer}} & \Omega^\infty \mathbf{H}^s(M)\,. \end{array} \tag{1.5.1}$$

The map δ comes from $\widetilde{\mathrm{TOP}}(M)/\,\mathrm{TOP}(M) \to \mathcal{S}^s(M) \to \widetilde{\mathcal{S}}^s(M)$, a homotopy fiber sequence. In our description of w, we assume for simplicity that M^n is closed.

Let $Z \subset \mathcal{S}(M)$ be a finitely generated subSpace, that is to say, $|Z|$ is compact. Let $p : E(1) \to |Z|$ be the tautological bundle whose fiber over some vertex (N, f), for example, is N. (Here N^n is closed and $f : N \to M$ is a homotopy equivalence.) Let $E(2) = M \times |Z|$. We have a canonical fiber homotopy equivalence $\lambda : E(1) \to E(2)$ over $|Z|$.

Let τ_1 and τ_2 be the vertical tangent bundles of $E(1)$ and $E(2)$, respectively. Choose $k \gg 0$, and a k–disk bundle ξ on $E(1)$ with associated euclidean bundle $\xi^\natural$, and an isomorphism ι of euclidean bundles $\tau_1 \oplus \xi^\natural \cong \lambda^*(\tau_2 \oplus \varepsilon^k)$. Let $E(1)^\xi$ be the total space of the disk bundle ξ; this fibers over $|Z|$. Immersion theory [Gau] says that λ and ι together determine up to contractible choice a fiberwise codimension zero immersion, over $|Z|$, from $E(1)^\xi$ to $E(2) \times \mathbb{R}^k$. We can arrange that the image of this fiberwise immersion is contained in $E(2) \times \mathbb{B}^k$ where $\mathbb{B}^k \subset \mathbb{R}^k$ is the open unit ball. Also, by choosing k sufficiently large and using general position arguments, we can arrange that the fiberwise immersion is a fiberwise embedding. In this situation, the closure of

$$E(2) \times \mathbb{D}^k \smallsetminus \mathrm{im}(E(1)^\xi)$$

is the total space of a fibered family of h–cobordisms over $|Z|$, with fixed base $M \times \mathbb{S}^{k-1}$. This family is classified by a map $Z \to \mathcal{H}(M \times \mathbb{S}^{k-1})$. Letting $k \to \infty$ we have

$$w_Z : Z \longrightarrow \operatorname*{hocolim}_k \mathcal{H}(M \times \mathbb{S}^{k-1}) \simeq \mathcal{H}^\infty(M),$$

a map well defined up to contractible choice. Finally view Z as a variable, use the above ideas to make a map w from the homotopy colimit of the various Z to $\mathcal{H}^\infty(M)$, and note that $\operatorname{hocolim} Z \simeq \mathcal{S}(M)$. This completes the construction.

It is evident that w takes $\mathcal{S}^s(M)$, the Space of s–structures on M, to the Space of s–cobordisms, $\Omega^\infty \mathbf{H}^s(M)$. Homotopy commutativity of (1.5.1) is less evident, but we omit the proof.

1.5.2. Corollary [BuLa2]**,** [BuFi1]. *In the topological concordance stable range for M, and localized at odd primes, there is a product decomposition*

$$\mathcal{S}^s(M) \simeq \widetilde{\mathcal{S}}^s(M) \times \widetilde{\mathrm{TOP}}(M)/\,\mathrm{TOP}(M)\,.$$

Proof. The left–hand vertical map in (1.5.1) is a homotopy equivalence in the concordance stable range, and the lower horizontal map is a split monomorphism in the homotopy category, at odd primes. Consequently the homotopy fiber of

$$\mathcal{S}^s(M) \hookrightarrow \widetilde{\mathcal{S}}^s(M)$$

is a retract up to homotopy of $\mathcal{S}^s(M)$ (localized at odd primes, in the concordance stable range). $\square$

1.5.3. Remark. Suppose that M^n is compact with boundary. Let $Z \subset \mathcal{S}(M)$ be finitely generated. A modification of the construction above gives w_Z from Z to $\operatorname{hocolim}_k \mathcal{H}(M \times \mathbb{S}^{k-1})$ and then $w : \mathcal{S}(M) \to \mathcal{H}^\infty(M)$.

Now let $\mathcal{T}_n(M)$ be the Space of pairs (N, f) where N^n is a compact manifold and $f : N \to M$ is *any* homotopy equivalence, not subject to boundary conditions. Let $Z \subset \mathcal{T}_n(M)$ be finitely generated. Another modification of the construction described above gives w_Z from Z to $\operatorname{hocolim}_k \mathcal{H}(\partial(M \times \mathbb{S}^{k-1}))$, and then $w : \mathcal{T}_n(M) \to \mathcal{H}^\infty(M)$ because again

$$\operatorname*{hocolim}_k \mathcal{H}(\partial(M \times \mathbb{S}^{k-1})) \simeq \mathcal{H}^\infty(M)\,.$$

It follows that $w : \mathcal{S}(M) \to \mathcal{H}^\infty(M)$ extends to $w : \mathcal{T}_n(M) \to \mathcal{H}^\infty(M)$. Let $\mathrm{G}^\natural(M)$ be the Space of *all* homotopy equivalences $M \to M$, not subject to boundary conditions. Then $\mathrm{G}(M_0) \simeq \mathrm{G}^\natural(M) \hookrightarrow \mathcal{T}_n(M)$, and we can think of w as a map from $\mathrm{G}(M_0)$ to $\mathcal{H}^\infty(M)$. This is the map promised directly after 1.2.4.

1.6. Diffeomorphisms

Suppose that M^n is a closed topological manifold, $n > 4$, with tangent (micro)bundle τ. Morlet [Mo1], [Mo2], [BuLa3], [KiSi] proves that the forgetful map from a suitably defined Space of smooth structures on M, denoted $V(M)$, to a suitably defined Space $V(\tau)$ of vector bundle structures on τ is a weak homotopy equivalence. Earlier Hirsch and Mazur [HiMa] had proved that the map in question induces a bijection on π_0. The Space $V(\tau)$ is homotopy equivalent to the homotopy fiber of the inclusion $\mathrm{map}(M, B\mathrm{O}(n)) \to \mathrm{map}(M, B\mathrm{TOP}(n))$ over $\hat{\tau}$.

The Space of smooth structures on M can be defined as the disjoint union of Spaces $\mathrm{TOP}(N)/\,\mathrm{DIFF}(N)$, where N runs through a set of representatives of diffeomorphism classes of smooth manifolds homeomorphic to M. Therefore Morlet's theorem gives, in the case where M is smooth, a homotopy fiber sequence

$$\mathrm{DIFF}(M) \to \mathrm{TOP}(M) \xrightarrow{a} V(M)\,. \tag{1.6.1}$$

The map a is obtained from an action of $\mathrm{TOP}(M)$ on $V(M)$ by evaluating at the base point of $V(M)$. The homotopy fiber sequence (1.6.1) remains meaningful when M is smooth compact with boundary; in this case allow only vector bundle structures on τ extending the standard structure over ∂M.

Traditionally, 1.6.1 has been an excuse for neglecting $\mathrm{DIFF}(M)$ in favor of $\mathrm{TOP}(M)$. But concordance theory has changed that. See §3. It is therefore best to develop a theory of smooth automorphisms parallel to the theory of topological automorphisms where possible. For example, 1.2.1–3, 1.3.1–2, 1.3.5–8, 1.4.2–4 and 1.5.1–3 have smooth analogs; but 1.2.4 does not.
Notation: A subscript d will often be used to indicate smoothness, as in $\mathcal{H}_d(M)$ for a Space of smooth h–cobordisms (when M is smooth).

2. L–theory and structure Spaces

The major theorems in this chapter are to be found in §2.3 and §2.5. Sections 2.1, 2.2 and 2.4 introduce concepts needed to state those theorems.

2.1. Assembly

2.1.1. Definitions. Fix a space Y. Let $\mathcal{W}_Y$ be the category of spaces over Y. A morphism in $\mathcal{W}_Y$ is a *weak homotopy equivalence* if the underlying map of spaces is a weak homotopy equivalence. A commutative square in $\mathcal{W}_Y$ is *cartesian* if the underlying square of spaces is cartesian.

A functor $\mathbf{J}$ from $\mathcal{W}_Y$ to CW–spectra [A, III] is *homotopy invariant* if it takes weak homotopy equivalences to weak homotopy equivalences. It is *excisive* if, in addition, it takes cartesian squares to cartesian squares, takes $\emptyset$ to a contractible spectrum, and satisfies a wedge axiom,

$$\vee_i \mathbf{J}(X_i \to Y) \xrightarrow{\simeq} \mathbf{J}(\amalg_i X_i \to Y)\,.$$

2.1.2. Proposition. *For every homotopy invariant functor $\mathbf{J}$ from $\mathcal{W}_Y$ to CW–spectra, there exist an excisive functor $\mathbf{J}^{\%}$ from $\mathcal{W}_Y$ to CW–spectra and a natural transformation $\alpha_{\mathbf{J}} : \mathbf{J}^{\%} \to \mathbf{J}$ such that $\alpha_{\mathbf{J}} : \mathbf{J}^{\%}(X) \to \mathbf{J}(X)$ is a homotopy equivalence whenever X is a point (over Y).*

The natural transformation $\alpha_{\mathbf{J}}$ is essentially characterized by these properties; it is called the *assembly.* It is the best approximation (from the left) of $\mathbf{J}$ by an excisive functor. We write $\mathbf{J}_{\%}(X)$ for the homotopy fiber of $\alpha_{\mathbf{J}} : \mathbf{J}^{\%}(X) \to \mathbf{J}(X)$, for any X in $\mathcal{W}_Y$.

Remark. Assume that X and Y above are homotopy equivalent to CW–spaces. If $Y \simeq *$, then $\mathbf{J}^{\%}(X) \simeq X_+ \wedge \mathbf{J}(*)$ by a chain of natural homotopy equivalences. If $Y \not\simeq *$, build a quasi–fibered spectrum on Y with fiber $\mathbf{J}(y \hookrightarrow Y)$ over $y \in Y$. Pull it back to X using $X \to Y$. Collapse the zero section, a copy of X. The result is $\simeq \mathbf{J}^{\%}(X)$.

The assembly concept is due to Quinn, [Qun1], [Qun2], [Qun3]. See also [Lo]. For a proof of 2.1.2, see [WWa]. For applications to block s–structure Spaces we need the case where $Y = \mathbb{R}P^{\infty}$ and $\mathbf{J} = \mathbf{L}^s_{\bullet}$ is the L–theory functor $X \mapsto \mathbf{L}^s_{\bullet}(X)$ which associates to X the L–theory spectrum of $\mathbb{Z}\pi_1(X)$, say in the description of Ranicki [Ra2 §13]. Note the following technical points.

- $\mathbf{L}^s_\bullet$ is the quadratic L–theory with decoration s which we make 0–connected (by force).
- Because of 2.1.1 we have no use for a base point in X. This makes it harder to say what $\mathbb{Z}\pi_1(X)$ should mean. For details see [WWa, 3.1], where $\mathbb{Z}\pi_1(X)$ or more precisely $\mathbb{Z}^w\pi_1(X)$ is a ringoid with involution depending on the double cover $w : X^\sim \to X$ induced by $X \to \mathbb{R}P^\infty$.

2.2. Tangential invariants

Geometric topology tradition requires that any classification of you–name–it structures on a manifold or Poincaré space [Kl] be accompanied by a classification of analogous structures on the normal bundle or Spivak normal fibration [Spi], [Ra3], [Br2] of the manifold or Poincaré space. We endorse this. However, we find tangent bundle language more convenient than normal bundle language. The constructions here in §2.2 will also be used in §3 and §4.

Terminology. When we speak of a *stable fiber homotopy equivalence* between euclidean bundles β and γ on a space X, we mean a fiber homotopy equivalence over X between the spherical fibrations associated with $\beta \oplus \varepsilon^k$ and $\gamma \oplus \varepsilon^k$, respectively, for some k.

Let M^n be a closed topological manifold with a choice [Kis], [Maz2] of euclidean tangent bundle τ. An *h–structure* on τ is a pair (ξ, ϕ) where ξ^n is a euclidean bundle on M and ϕ is a stable fiber homotopy equivalence from (the spherical fibration associated with) ξ to τ. The h–structures on τ and their isomorphisms form a groupoid. Enlarge the groupoid to a simplicial groupoid by allowing families parametrized by Δ^k , and let

$$\mathcal{S}(\tau) := \text{ diagonal nerve of the simplicial groupoid.}$$

Then $\mathcal{S}(\tau) \simeq \text{hofiber}\,[\,\text{map}(M, B\text{TOP}(n)) \hookrightarrow \text{map}(M, B\text{G})\,]$, where $\hat{\tau}$ serves as base point in $\text{map}(M, B\text{TOP}(n))$ and $\text{map}(M, B\text{G})$. There is a *tangential invariant map*, well defined up to contractible choice,

$$\nabla : \mathcal{S}(M) \longrightarrow \mathcal{S}(\tau)\,.$$

Sketchy description: A homotopy equivalence $f : N \to M$ determines up to contractible choice a stable fiber homotopy equivalence ψ from $f^*\nu(M)$ to $\nu(N)$ because Spivak normal fibrations [Spi], [Ra3], [Br3], [Wa5] are homotopy invariants. It then determines up to contractible choice a stable fiber homotopy equivalence $\psi^{\mathrm{ad}} : \tau(N) \to f^*\tau(M)$. Now choose a euclidean bundle ξ^n on M and a stable fiber homotopy equivalence $\phi : \xi \to \tau(M)$ *together* with an isomorphism $j : f^*\xi \to \tau(N)$ and a homotopy $f^*\phi \simeq \psi^{\mathrm{ad}} \cdot j$. This is a contractible choice. Let ∇ take (N, f) to (ξ, ϕ). (This defines ∇ on the 0–skeleton; using the same ideas, complete the construction of ∇ by induction over skeletons.)

When $\partial M \neq \emptyset$, define $\mathcal{S}(\tau)$ in such a way that it is homotopy equivalent to the homotopy fiber of

$$\mathrm{map}_{\mathrm{rel}}(M, B\mathrm{TOP}(n)) \xrightarrow{\subset} \mathrm{map}_{\mathrm{rel}}(M, B\mathrm{G})$$

where $\mathrm{map}_{\mathrm{rel}}$ indicates maps from M which on ∂M agree with $\hat{\tau}$. Again $\hat{\tau}$ serves as base point everywhere.

When $\partial M \neq \emptyset$ and $\partial_+ M \subset \partial M$ is specified, with tangent bundle τ' of fiber dimension $n-1$, we define $\mathcal{S}(\tau, \tau')$ in such a way that it is homotopy equivalent to the homotopy fiber of

$$\mathrm{map}_{\mathrm{rel}}((M, \partial_+ M)\,,\,(B\mathrm{TOP}(n), B\mathrm{TOP}(n-1)) \xrightarrow{\subset} \mathrm{map}_{\mathrm{rel}}(M, B\mathrm{G})$$

where $\mathrm{map}_{\mathrm{rel}}$ indicates maps from M which on $\partial_- M$ agree with the classifying map for the tangent bundle of $\partial_- M$. The pair $(\hat{\tau}, \hat{\tau}')$ serves as base point. There is a *tangential invariant map* $\nabla : \mathcal{S}(M, \partial_+ M) \to \mathcal{S}(\tau, \tau')$ which fits into a homotopy commutative diagram where the rows are homotopy fiber sequences:

$$\begin{array}{ccccc}
\mathcal{S}(M) & \xrightarrow{\subset} & \mathcal{S}(M, \partial_+ M) & \longrightarrow & \mathcal{S}(\partial_+ M) \\
\big\downarrow \nabla & & \big\downarrow \nabla & & \big\downarrow \nabla \\
\mathcal{S}(\tau) & \xrightarrow{\subset} & \mathcal{S}(\tau, \tau') & \longrightarrow & \mathcal{S}(\tau')\,.
\end{array}$$

2.2.1. Illustration. Suppose that $(M, \partial_+ M) = (N \times I, N \times 1)$ where N^{n-1} is compact. Then $\mathcal{S}(\tau, \tau')$ is homotopy equivalent to the homotopy fiber of

$$\mathrm{map}_{\mathrm{rel}}(N, B\mathrm{TOP}(n-1)) \hookrightarrow \mathrm{map}_{\mathrm{rel}}(N, B\mathrm{TOP}(n))\,.$$

Finally we need a space $\widetilde{\mathcal{S}}(\tau)$ of *stable h–structures* on τ (assuming again that τ is the tangent bundle of a compact M). Define this as the space of pairs (ξ, ϕ) where ξ^p is a euclidean bundle on M, of arbitrary fiber dimension p, and ϕ is a stable fiber homotopy equivalence $\xi \to \tau$, represented by an actual fiber homotopy equivalence between the spherical fibrations associated with $\xi \oplus \varepsilon^{k-p}$ and $\tau \oplus \varepsilon^{k-n}$ for some large k. Then

$$\widetilde{\mathcal{S}}(\tau) \simeq \text{hofiber}[\text{map}_{\text{rel}}(M, B\text{TOP}) \to \text{map}_{\text{rel}}(M, B\text{G})]$$

which is homotopy equivalent to the space of based maps from $M/\partial M$ to G/TOP. Again there is a tangential invariant map— better known, and in this case more easily described, as the *normal invariant map* :

$$\nabla : \widetilde{\mathcal{S}}^s(M) \to \widetilde{\mathcal{S}}(\tau)\,.$$

2.3. Block s–structures and L–theory

2.3.1. Fundamental Theorem of Surgery (Browder, Novikov, Sullivan, Wall, Quinn, Ranicki)**.** *For compact M^n with tangent bundle τ, where $n > 4$, there exists a homotopy commutative square of the form*

$$\begin{array}{ccc}
\widetilde{\mathcal{S}}^s(M) & \xrightarrow{\quad\nabla\quad} & \widetilde{\mathcal{S}}(\tau) \\
\downarrow\simeq & & \downarrow\simeq \\
\Omega^{\infty+n}((\mathbf{L}^s_\bullet)_\%(M)) & \xrightarrow{\text{forget}} & \Omega^{\infty+n}((\mathbf{L}^s_\bullet)^\%(M))\,.
\end{array}$$

References: [Br3], [Br4], [Nov] for the smooth analog, [Rou1], [Su1], [Su2], [Wa1,§10], [ABK] for the PL case, and [KiSi] for the topological case, all without explicit use of assembly ; [Ra4], [Ra2], [Qun1] for formulations with assembly.

Illustration. 2.3.1 gives $\widetilde{\mathcal{S}}^s(\mathbb{S}^n) \simeq \Omega^\infty \mathbf{L}^s_\bullet(*) \simeq \text{G}/\text{TOP}$ for $n > 4$. (Remember that $\mathbf{L}^s_\bullet(*)$ is 0–connected by definition, §2.1.) The honest structure space is

$$\mathcal{S}(\mathbb{S}^n) \simeq \text{G}(\mathbb{S}^n)/\text{TOP}(\mathbb{S}^n)$$

(this uses the Poincaré conjecture). The inclusion $\mathcal{S}(\mathbb{S}^n) \to \widetilde{\mathcal{S}}^s(\mathbb{S}^n)$ becomes the inclusion of $\text{G}(\mathbb{S}^n)/\text{TOP}(\mathbb{S}^n)$ in $\text{hocolim}_k\, \text{G}(\mathbb{S}^k)/\text{TOP}(\mathbb{S}^k) \simeq \text{G}/\text{TOP}$; in particular, it is n–connected.

2.4. $\mathbb{S}^1$–stabilization

$\mathbb{S}^1$–stabilization is a method for making new homotopy functors out of old ones. It was introduced in [Ra5] and applied in [AnPe], [HaMa] in a special case (the one we will need in §2.6 just below). It is motivated by the definition of the negative K–groups in [Ba].

Let $\mathbf{J}$ be a homotopy functor from $\mathcal{W}_Y$ (see 2.1.2) to CW–spectra. Let $\mathbb{S}^1(+)$ and $\mathbb{S}^1(-)$ be upper half and lower half of $\mathbb{S}^1$, respectively. For X in $\mathcal{W}_Y$ let $\sigma\mathbf{J}(X)$ be the homotopy pullback of

$$\frac{\mathbf{J}(X\times\mathbb{S}^1(+))}{\mathbf{J}(X\times *)} \longrightarrow \frac{\mathbf{J}(X\times\mathbb{S}^1)}{\mathbf{J}(X\times *)} \longleftarrow \frac{\mathbf{J}(X\times\mathbb{S}^1(-))}{\mathbf{J}(X\times *)}\,.$$

Note that $\sigma\mathbf{J}(X)\simeq\Omega[\,\mathbf{J}(X\times\mathbb{S}^1)/\mathbf{J}(X\times *)\,]$, and that $\sigma\mathbf{J}$ is a homotopy functor from $\mathcal{W}_Y$ to CW–spectra. There are natural transformations

$$\mathbf{J}(X)\longrightarrow\frac{\mathbf{J}(X\times\mathbb{S}^0)}{\mathbf{J}(X\times *)}\longrightarrow\sigma\mathbf{J}(X)\,,$$

the first induced by the inclusion $x\mapsto(x,-1)$ of X in $X\times\mathbb{S}^0$, and the second induced by the inclusions of $\mathbb{S}^0$ in $\mathbb{S}^1(-)$ and $\mathbb{S}^1(+)$. Let $\psi:\mathbf{J}(X)\to\sigma\mathbf{J}(X)$ be the composition. Finally let $\sigma^\infty\mathbf{J}(X)$ be the homotopy colimit of the $\sigma^k\mathbf{J}(X)$ for $k\geq 0$, using the maps $\psi:\sigma^{k-1}\mathbf{J}(X)\to\sigma(\sigma^{k-1}\mathbf{J}(X))$ to stabilize. We call $\sigma^\infty\mathbf{J}$ the $\mathbb{S}^1$*–stabilization* of $\mathbf{J}$.

2.5. Bounded h–structures and L–theory

Let M^n be compact. A *bounded* h–structure on $M\times\mathbb{R}^k$ is a pair (N,f) where N^{n+k} is a manifold and $f:N\to M\times\mathbb{R}^k$ is a *bounded* homotopy equivalence restricting to a homeomorphism $\partial N\to\partial M\times\mathbb{R}^k$. (That is, there exist $c>0$ and $g:M\times\mathbb{R}^k\to N$ and homotopies $h:fg\simeq\mathrm{id}$, $j:gf\simeq\mathrm{id}$ such that the sets $\{p_2h_t(x)\mid t\in I\}$, $\{p_2fj_t(y)\mid t\in I\}$ have diameter $<c$ for all $x\in M\times\mathbb{R}^k$ and $y\in N$; moreover h_t, j_t agree with the identity maps on $\partial M\times\mathbb{R}^k$ and ∂N, respectively.) References: [AnPe], [FePe].

A Space $\mathcal{S}^b(M\times\mathbb{R}^k)$ of such bounded h–structures can be constructed in the usual way, as the diagonal nerve of a simplicial groupoid. There is a homotopy fiber sequence

$$\mathrm{TOP}^b(M\times\mathbb{R}^k)\to\mathrm{G}^b(M\times\mathbb{R}^k)\to\mathcal{S}^b(M\times\mathbb{R}^k)$$

where $G^b(M\times\mathbb{R}^k)$ is the Space of bounded homotopy automorphisms of $M\times\mathbb{R}^k$, relative to $\partial M\times\mathbb{R}^k$.

Again we need a *tangential invariant map* $\nabla:\mathcal{S}^b(M\times\mathbb{R}^k)\to\mathcal{S}(\tau\times\varepsilon^k)$ where $\tau=\tau(M)$ and $\tau\times\varepsilon^k$ is the tangent bundle of $M\times\mathbb{R}^k$. Its definition resembles that of the tangential invariant maps in §2.2. Additional subtlety: one needs to know that *open* Poincaré spaces and open Poincaré pairs have Spivak normal fibrations which are invariants of proper homotopy type. See [Tay], [Mau], [FePe], [PeRa]. Note

$$\mathcal{S}(\tau\times\varepsilon^k)\ \simeq\ \text{hofiber}\,[\text{map}_{\text{rel}}(M,B\text{TOP}(n+k))\to\text{map}_{\text{rel}}(M,B\text{G})\,]$$

where map_{rel} indicates maps which on ∂M agree with $\hat\tau$.

Let $\mathbf{L}_\bullet^{\langle-\infty\rangle}(X)$ be the 0–connected cover of $(\sigma^\infty\mathbf{L}^s_\bullet)(X)$, in the notation of §2.4. Here X is a space over $\mathbb{R}P^\infty$.

2.5.1. Theorem. *For compact M^n with tangent bundle τ, where $n>4$, there exists a homotopy commutative square of the form*

$$\begin{array}{ccc}
\bigcup_k \mathcal{S}^b(M\times\mathbb{R}^k) & \xrightarrow{\ \nabla\ } & \bigcup_k \mathcal{S}(\tau\times\varepsilon^k)\\
\big\downarrow\simeq & & \big\downarrow\simeq\\
\Omega^{\infty+n}((\mathbf{L}_\bullet^{\langle-\infty\rangle})_{\%}(M)) & \xrightarrow{\text{forget}} & \Omega^{\infty+n}((\mathbf{L}_\bullet^{\langle-\infty\rangle})^{\%}(M))\,.
\end{array}$$

Remark. $(\mathbf{L}_\bullet^{\langle-\infty\rangle})^{\%}\simeq(\mathbf{L}^s_\bullet)^{\%}$.

3. Algebraic K-theory and structure Spaces

3.1. Algebraic K-theory of Spaces

Waldhausen's homotopy functor $\mathbf{A}$ from spaces to CW–spectra is a composition $\mathbf{K}\cdot\mathcal{R}$, where $\mathcal{R}$ is a functor from spaces to *categories with cofibrations and weak equivalences* alias *Waldhausen categories*, and $\mathbf{K}$ is a functor from Waldhausen categories to CW–spectra.

For a space X, let $\mathcal{R}(X)$ be the Waldhausen category of homotopy finite retractive spaces over X. The objects of $\mathcal{R}(X)$ are spaces Z equipped with maps

$$Z \underset{i}{\overset{r}{\rightleftarrows}} X \qquad\qquad (ri=\text{id}_X)$$

subject to a finiteness condition. Namely, Z must be homotopy equivalent, relative to X, to a relative CW–space built from X by attaching a finite number of cells. A morphism in $\mathcal{R}(X)$ is a map relative to and over X. We call it a weak equivalence if it is a homotopy equivalence relative to X, and a cofibration if it has the homotopy extension property relative to X. See [Wah3, ch.2] for more information.

In [Wah3], [Wah1], Waldhausen associates with any Waldhausen category $\mathcal{C}$ a connective spectrum $\mathbf{K}(\mathcal{C})$, generalizing Quillen's construction [Qui] of the K–theory spectrum of an exact category. For us it is important that $\mathbf{K}(\mathcal{C})$ comes with a map reminiscent of "group completion",

$$|w\mathcal{C}| \hookrightarrow \Omega^\infty \mathbf{K}(\mathcal{C})$$

where $w\mathcal{C}$ is the category of weak equivalences in $\mathcal{C}$ and $|w\mathcal{C}|$ is its classifying space (geometric realization of the nerve).

3.2. Algebraic K–theory of spaces, and h–cobordisms

Let M^n be compact, $n \geq 5$, with fundamental group(oid) π. The s–cobordism theorem due to Smale [Sm], [Mi2] in the simply connected smooth case and Barden–Mazur–Stallings [Ke] in the nonsimply connected smooth case states that $\pi_0\mathcal{H}_d(M)$ is isomorphic to the Whitehead group of π, that is, $K_1(\mathbb{Z}\pi)/\{\pm\pi^{\mathrm{ab}}\}$. See [RoSa] for the PL version and [KiSi, Essay III] for the TOP version. Cerf [Ce] showed that $\pi_1\mathcal{H}_d(M)$ is trivial when M is smooth, simply connected and $n \geq 5$, and Rourke [Rou2] established the analogous statement in the PL category. In the early 70's Hatcher and Wagoner [HaWa], working with a smooth but possibly nonsimply connected M, constructed a surjective homomorphism from $\pi_1\mathcal{H}_d(M)$ to a certain quotient of $K_2(\mathbb{Z}\pi)$, and they were able to describe the kernel of that homomorphism in terms of π and $\pi_2(M)$. See also [DIg]. These results follow from Waldhausen's theorem 3.2.1, 3.2.2 below, which describes the homotopy types of $\mathcal{H}(M)$ and $\mathcal{H}_d(M)$ in a stable range, in algebraic K–theory terms. The size of the stable range is estimated by Igusa's stability theorem, 1.3.4.

Remark. Note that the block s–cobordism Space $\widetilde{\mathcal{S}}^s(M \times I, M \times 1)$ is not a very useful approximation to $\mathcal{H}^s(M)$, because it is contractible (either by a relative version of 2.3.1 which we did not state, or by a direct geometric argument). Hence surgery theory as in §2 does not elucidate the homotopy type of $\mathcal{H}^s(M)$.

For compact M^n write $\mathcal{H}(\tau(M)) := \mathcal{S}(\tau(M \times I), \tau(M \times 1))$ so that there is a tangential invariant map $\nabla : \mathcal{H}(M) \to \mathcal{H}(\tau(M))$. See 2.2.1. There is a stabilization map from $\mathcal{H}(\tau(M))$ to $\mathcal{H}(\tau(M \times I))$, analogous to the stabilization map from $\mathcal{H}(M)$ to $\mathcal{H}(M \times I)$. We let $\tau = \tau(M)$ and $\mathcal{H}^\infty(\tau) = \operatorname{hocolim}_k \mathcal{H}(\tau(M \times I^k))$ and obtain, since ∇ commutes with stabilization,

$$\nabla : \mathcal{H}^\infty(M) \to \mathcal{H}^\infty(\tau)\,.$$

The following result is essentially contained in [Wah2].

3.2.1. Theorem (Waldhausen)**.** *There exists a homotopy commutative square*

$$\begin{array}{ccc}
\mathcal{H}^\infty(M) & \xrightarrow{\nabla} & \mathcal{H}^\infty(\tau) \\
\big\downarrow \simeq & & \big\downarrow \simeq \\
\Omega^\infty(\mathbf{A}_\%(M)) & \xrightarrow{\text{forget}} & \Omega^\infty(\mathbf{A}^\%(M))\,.
\end{array}$$

3.2.2. Remark. Suppose that M is smooth. Then $\mathcal{H}^\infty(M)$ and $\mathcal{H}^\infty(\tau)$ have smooth analogues $\mathcal{H}^\infty_d(M)$ and $\mathcal{H}^\infty_d(\tau)$, and by smoothing theory there is a homotopy commutative cartesian square

$$\begin{array}{ccc}
\mathcal{H}^\infty_d(M) & \xrightarrow{\nabla} & \mathcal{H}^\infty_d(\tau) \\
\big\downarrow & & \big\downarrow \\
\mathcal{H}^\infty(M) & \xrightarrow{\nabla} & \mathcal{H}^\infty(\tau)
\end{array}$$

with forgetful vertical arrows. One shows by direct geometric arguments that $\mathcal{H}^\infty_d(\tau) \simeq \Omega^\infty\Sigma^\infty(M_+)$ and that $\nabla : \mathcal{H}^\infty_d(M) \to \mathcal{H}^\infty_d(\tau)$ is nullhomotopic. In this way 3.2.1 implies

$$\text{(3.2.3)} \qquad \mathcal{H}^\infty_d(M) \times \Omega^{\infty+1}\Sigma^\infty(M_+) \quad \simeq \quad \Omega^{\infty+1}(\mathbf{A}(M))$$

which is better known than 3.2.1. Conversely, 3.2.1 can be deduced from (3.2.3) with functor calculus arguments, if we add the information that (3.2.3) comes from a spectrum level splitting, $\mathbf{H}^h_d(M) \vee \Omega\Sigma^\infty(M_+) \simeq \Omega\mathbf{A}(M)$ or equivalently $\mathbf{Whd}(M) \vee \Sigma^\infty(M_+) \simeq \mathbf{A}(M)$, where $\mathbf{Whd}(M)$ is the delooping of $\mathbf{H}^h_d(M)$ and $\mathbf{H}^h_d(M)$ is the (-1)–connected cover of

$\mathbf{H}_d(M)$. We prefer formulation 3.2.1 because of its amazing similarity with 2.3.1 and 2.5.1.

References: 3.2.3 is stated in [Wah2]. It is reduced in [Wah4] to the spectrum level analog of the left–hand column in (3.2.1),

$$\mathbf{H}^h(M) \simeq \mathbf{A}_\%(M)\ , \tag{3.2.4}$$

where $\mathbf{H}^h(M)$ denotes the (-1)–connected cover of $\mathbf{H}(M)$.) For the proof of (3.2.4), see [Wah3, §3] and the preprints [WaVo1] and [WaVo2]. The papers [Stb] and [Cha5] contain results closely related to [WaVo1] and [WaVo2], respectively. A very rough but helpful guide to this vast circle of ideas is [Wah5]. See also [DWWc].

4. Mixing L–theory and algebraic K–theory of spaces

Introduction. We now have a large amount of indirect knowledge about the s–structure Space $\mathcal{S}^s(M)$ for a compact M. Namely, from the definitions there is a homotopy fiber sequence

$$\widetilde{\mathrm{TOP}}(M)/\,\mathrm{TOP}(M) \longrightarrow \mathcal{S}^s(M) \longrightarrow \widetilde{\mathcal{S}}^s(M).$$

In 2.3.1 we have an expression for $\widetilde{\mathcal{S}}^s(M)$ in terms of L–theory. In the concordance stable range, we also have the expression 1.3.7 for

$$\widetilde{\mathrm{TOP}}(M)/\,\mathrm{TOP}(M)$$

in terms of stabilized concordance theory. But 3.2.1 expresses stabilized concordance theory through the algebraic K–theory of spaces. Therefore, in the concordance stable range, $\mathcal{S}^s(M)$ must be a concoction of L–theory and algebraic K–theory of spaces. It remains to find out what concoction exactly. This problem was previously addressed by Hsiang–Sharpe (roughly speaking, using only the Postnikov 2–coskeleton of the algebraic K–theory of spaces), by Burghelea–Fiedorowicz (rationally), by Burghelea–Lashof (at odd primes), by Fiedorowicz–Schwänzl–Vogt (at odd primes); see references [HsiSha], [BuFi1], [BuFi2], [FiSVo1], [FiSVo2], [FiSVo3], [BuLa2]. In addition, the literature contains many results about $\mathcal{S}^s(M)$ or $\mathcal{S}(M)$, or the differentiable analogs, for specific M; see §6 for a selection and further references.

Our analysis, Thm. 4.2.1 and remark 4.2.3 below, is based on the following idea. Using 3.2.1, Poincaré duality notions, and a more algebraic description of w in (1.5.1), we find that w lifts to an *equivariant Whitehead torsion* map

$$w^\sharp : \mathcal{S}^s(M) \longrightarrow \Omega^\infty(\mathbf{H}^s(M)^{h\mathbb{Z}/2}) \simeq \Omega^\infty((\mathbf{A}^s_{\%}(M)^{h\mathbb{Z}/2}))$$

where $(\text{—})^{h\mathbb{Z}/2}$ indicates homotopy fixed points for a certain action of $\mathbb{Z}/2$. This refinement of w fits into a homotopy commutative diagram whose top portion refines (1.5.1),

$$\begin{array}{ccc}
\widetilde{\mathrm{TOP}}(M)/\,\mathrm{TOP}(M) & \longrightarrow & \Omega^\infty((\mathbf{A}^s_{\%}(M)_{h\mathbb{Z}/2})) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{norm}} \\
\mathcal{S}^s(M) & \xrightarrow{w^\sharp} & \Omega^\infty((\mathbf{A}^s_{\%}(M)^{h\mathbb{Z}/2})) \\
\Big\downarrow & & \Big\downarrow \\
\widetilde{\mathcal{S}}^s(M) & \longrightarrow & \Omega^\infty((\mathbf{A}^s_{\%}(M)^{th\mathbb{Z}/2}))\,.
\end{array}$$

The right–hand column is a homotopy fiber sequence of infinite loop spaces which we will say more about below. The left–hand column is also a homotopy fiber sequence. The upper horizontal arrow is highly connected. Hence the lower square of the diagram is *approximately* cartesian. Since we have an algebraic description for the lower square with $\mathcal{S}^s(M)$ deleted, we obtain an approximate algebraic description of $\mathcal{S}^s(M)$.

Curiously, the map $w^\sharp$ does not have an easy analog in the smooth category. See however §4.3.

This work is still in progress. Currently available: [WW1], [WW2], [WWa], [WWx], [WWd], [WWp], [We1]. The papers [DWW], [DWWc] are closely related and use identical technology.

4.1. *LA*–theory

We describe a functor $\mathbf{LA}^h_\bullet$ from $\mathcal{W}_{BG} \times \mathbb{N}$ to CW–spectra. Here $\mathcal{W}_{BG}$ is the category of spaces over BG (alternatively, spaces equipped with a stable spherical fibration) and $\mathbb{N}$ is regarded as a category with exactly one morphism $m \to n$ if $m \le n$, and no morphism $m \to n$ if $m > n$. For fixed n, the functor $\mathbf{LA}^h_\bullet(\text{—}, n)$ is a homotopy functor. It is a composition $\mathbf{F}_n \cdot \mathcal{R}^\infty$, where $\mathcal{R}^\infty$ is a functor from $\mathcal{W}_{BG}$ to the category of Waldhausen categories with Spanier–Whitehead (SW) product [WWd], and $\mathbf{F}_n$ is a functor from certain Waldhausen categories with SW product to CW–spectra.

Let X be a space over BG. In the Waldhausen category $\mathcal{R}(X)$ of 3.1, we have notions of mapping cylinder, mapping cone and suspension Σ_X. Let $\mathcal{R}^\infty(X)$ be the colimit of the direct system of categories

$$\mathcal{R}(X) \xrightarrow{\Sigma_X} \mathcal{R}(X) \xrightarrow{\Sigma_X} \mathcal{R}(X) \xrightarrow{\Sigma_X} \dots .$$

Again, $\mathcal{R}^\infty(X)$ is a Waldhausen category. We will need additional structure on $\mathcal{R}^\infty(X)$ in the shape of an *SW product* which depends on the reference map from X to BG. The SW product is a functor $(Z_1, Z_2) \mapsto Z_1 \odot Z_2$ from $\mathcal{R}^\infty(X) \times \mathcal{R}^\infty(X)$ to based spaces. Its main properties are:

- *symmetry*, that is, $Z_1 \odot Z_2 \cong Z_2 \odot Z_1$ by a natural involutory homeomorphism;
- *bilinearity*, that is, for fixed Z_2 the functor $Z_1 \mapsto Z_1 \odot Z_2$ takes the zero object to a contractible space and takes pushout squares where the horizontal arrows are cofibrations to cartesian squares;
- *w–invariance*, that is, a weak equivalence $Z_1 \to Z_1'$ induces a weak homotopy equivalence $Z_1 \odot Z_2 \to Z_1' \odot Z_2$ for any Z_2.

Modulo technicalities, the definition of $Z_1 \odot Z_2$ for Z_1, Z_2 in $\mathcal{R}(X) \subset \mathcal{R}^\infty(X)$ is as follows. Let γ be the spherical fibration on X pulled back from BG; we can assume that it comes with a distinguished section and has fibers $\simeq \mathbb{S}^k$ for some k. Convert the retraction maps $Z_1 \to X$ and $Z_2 \to X$ into fibrations, with total spaces $Z_1^\sim$ and $Z_2^\sim$; form the fiberwise smash product (over X) of $Z_1^\sim$, $Z_2^\sim$ and the total space of γ; collapse the zero section to a point; finally apply $\Omega^{\infty+k}\Sigma^\infty$.

Imitating [SpaW], [Spa,§8 ex.F] we say that $\eta \in Z' \odot Z$ is a *duality* if it has certain nondegeneracy properties. For every Z in $\mathcal{R}(X)$ there exists a Z' and $\eta \in Z' \odot Z$ which is a duality; the pair (Z', η) is determined up to contractible choice by Z, and we can say that Z' is the *dual* of Z. Modulo technicalities, an involution on the K–theory spectrum $\mathbf{K}(\mathcal{R}^\infty(X))$ results, induced by $Z \mapsto Z'$. The inclusion $\mathcal{R}(X) \to \mathcal{R}^\infty(X)$ induces a homotopy equivalence of the K–theory spectra, so that we are talking about an involution on $\mathbf{A}(X)$. For all details, we refer to [WWd]. The involution on $\mathbf{A}(X)$ was first constructed in [Vo]. See also [KVWW2].

More generally, suppose that $\mathcal{B}$ is any Waldhausen category with SW product, satisfying the axioms of [WWd,§2] which assure existence and essential uniqueness of SW duals. Then the K–theory spectrum $\mathbf{K}(\mathcal{B})$ has a preferred involution. In this setup it is also possible to define spectra $\mathbf{L}_\bullet(\mathcal{B})$ (quadratic L–theory), $\mathbf{L}^\bullet(\mathcal{B})$ (symmetric L–theory), a forgetful map from quadratic to symmetric L–theory, and a map

$$\Xi : \mathbf{L}^\bullet(\mathcal{B}) \longrightarrow \mathbf{K}(\mathcal{B})^{th\mathbb{Z}/2}$$

where $\mathbf{K}(\mathcal{B})^{th\mathbb{Z}/2}$ is the mapping cone of the *norm map* [AdCD], [GrMa], [WW2, 2.4],

$$\mathbf{K}(\mathcal{B})_{h\mathbb{Z}/2} \longrightarrow \mathbf{K}(\mathcal{B})^{h\mathbb{Z}/2}\,,$$

from the homotopy orbit spectrum to the homotopy fixed point spectrum of the involution on $\mathbf{K}(\mathcal{B})$. The norm map refines the transfer map from $\mathbf{K}(\mathcal{B})_{h\mathbb{Z}/2}$ to $\mathbf{K}(\mathcal{B})$.

Our constructions of $\mathbf{L}_\bullet(\mathcal{B})$ and $\mathbf{L}^\bullet(\mathcal{B})$ are bordism–theoretic and follow [Ra2] very closely, except that with a view to the applications here we need 0–connected versions. In particular, if $\mathcal{B} = \mathcal{R}^\infty(X)$, then $\mathbf{L}_\bullet(\mathcal{B})$ is homotopy equivalent to the 0–connected cover of the quadratic L–theory spectrum (decoration h) of the ring(oid) with involution $\mathbb{Z}\pi_1(X)$. Regarding Ξ, we offer the following explanations. Let $\mathcal{B}^*[i]$ be the category of covariant functors from the poset of nonempty faces of Δ^i to $\mathcal{B}$. Then $\mathcal{B}^*[i]$ inherits from $\mathcal{B}$ the structure of a Waldhausen category with SW product, with weak equivalences and cofibrations defined coordinatewise. The axioms of [WWd,§2] are still satisfied. Modulo technicalities, there is a duality involution on $|\,w\mathcal{B}^*[i]\,|$ for each $i \geq 0$, and an inclusion of simplicial spaces:

$$\begin{array}{ccc} i & \mapsto & |\,w\mathcal{B}^*[i]\,|^{h\mathbb{Z}/2} \\ & & \big\downarrow \\ i & \mapsto & K(\mathcal{B}^*[i])^{h\mathbb{Z}/2}\,. \end{array}$$

The geometric realizations of these simplicial spaces turn out to be Ω^∞ of $\mathbf{L}^\bullet(\mathcal{B})$ and $\mathbf{K}(\mathcal{B})^{th\mathbb{Z}/2}$ respectively. (Recognition is easy in the first case, harder in the second case.) The inclusion map of geometric realizations is Ω^∞ of Ξ , by definition.

We come to the description of $\mathbf{F}_n(\mathcal{B})$, promised at the beginning of this section. Let $\mathbb{S}^n_! = \mathbb{R}^n \cup \infty$ with the involution $z \mapsto -z$ for $z \in \mathbb{R}^n$. This has fixed point set $\{0,\infty\} \cong \mathbb{S}^0$. Let $\mathbf{K}(\mathcal{B},n) := \mathbb{S}^n_! \wedge \mathbf{K}(\mathcal{B})$ with the diagonal involution. The inclusion of $\mathbf{K}(\mathcal{B}) \cong \mathbf{K}(\mathcal{B}, 0)$ in $\mathbf{K}(\mathcal{B}, n)$ induces a homotopy equivalence of Tate spectra,

$$\mathbf{K}(\mathcal{B})^{th\mathbb{Z}/2} \xrightarrow{t^n} \mathbf{K}(\mathcal{B}, n)^{th\mathbb{Z}/2}$$

(proof by induction on n). Write ? for the forgetful map from quadratic L–theory to symmetric L–theory. Let $\mathbf{F}_n(\mathcal{B})$ be the homotopy pullback of

$$\begin{array}{ccc} & & \mathbf{K}(\mathcal{B}, n)^{h\mathbb{Z}/2} \\ & & \big\downarrow \cap \\ \mathbf{L}_\bullet(\mathcal{B}) & \xrightarrow{t^n \cdot \Xi \cdot ?} & \mathbf{K}(\mathcal{B}, n)^{th\mathbb{Z}/2} . \end{array}$$

4.1.1. Summary. $\mathbf{LA}^h_\bullet(X, n)$ is a spectrum defined for any space X over BG and any $n \geq 0$. It is the homotopy pullback of a diagram

$$\begin{array}{ccc} & & \mathbf{A}(X, n)^{h\mathbb{Z}/2} \\ & & \big\downarrow \cap \\ \mathbf{L}^h_\bullet(X) & \xrightarrow{t^n \cdot \Xi \cdot ?} & \mathbf{A}(X, n)^{th\mathbb{Z}/2} \end{array}$$

in which $\mathbf{L}^h_\bullet(X)$ denotes the 0–connected L–theory spectrum of $\mathbb{Z}\pi_1(X)$ with decoration h , and $\mathbf{A}(X, n) = \mathbb{S}^n_! \wedge \mathbf{A}(X)$. Hence there are homotopy fiber sequences

$$\begin{aligned} \mathbf{A}(X, n)_{h\mathbb{Z}/2} &\to \mathbf{LA}^h_\bullet(X, n) \to \mathbf{L}^h_\bullet(X) \ , \\ \mathbf{LA}^h_\bullet(X, n-1) &\to \mathbf{LA}^h_\bullet(X, n) \xrightarrow{v} \mathbf{A}(X, n) \, . \end{aligned} \tag{4.1.2}$$

4.2. LA–THEORY AND h–STRUCTURE SPACES

In the following theorem, we mean by $(\mathbf{LA}^h_\bullet)^\%(—, n)$ and $(\mathbf{LA}^h_\bullet)_\%(—, n)$ domain and homotopy fiber, respectively, of the assembly transformation for the homotopy functor $\mathbf{LA}^h_\bullet(—, n)$ on $\mathcal{W}_{BG}$. The manifold M becomes an object in $\mathcal{W}_{BG}$ by means of the classifying map for $\nu(M)$.

4.2.1. Theorem. *For compact M^n there exists a homotopy commutative square with highly connected (see* 4.2.2*) vertical arrows*

$$\begin{array}{ccc} \mathcal{S}(M) & \xrightarrow{\ \nabla\ } & \mathcal{S}(\tau) \\ \big\downarrow & & \big\downarrow \\ \Omega^{\infty+n}(\mathbf{LA}^h_\bullet)_{\%}(M,n) & \xrightarrow{\text{forget}} & \Omega^{\infty+n}(\mathbf{LA}^h_\bullet)^{\%}(M,n)\,. \end{array}$$

4.2.2. Details. The right-hand vertical arrow in 4.2.1 is $(j+2)$–connected if j is in the smooth concordance stable range for $\mathbb{D}^n$ and $j \le n-2$. The left–hand vertical arrow in 4.2.1 induces a bijection on π_0. Each component of $\mathcal{S}(M)$ determines a homeomorphism class of manifolds N homotopy equivalent to M. If j is in the topological concordance stable range for N, then the left–hand vertical arrow in 4.2.1 restricted to that component (and the corresponding component of the codomain) is $(j+1)$–connected.

4.2.3. Remark. There is an s–decorated version of 4.2.1, in which the Space of s–structures $\mathcal{S}^s(M)$ replaces $\mathcal{S}(M)$ and $\mathbf{LA}^s_\bullet$ replaces $\mathbf{LA}^h_\bullet$. To define $\mathbf{LA}^s_\bullet$ use L–theory and algebraic K–theory of spaces with an s–decoration in §4.1. The homotopy groups of $\mathbf{LA}^s_\bullet(X,n)$ differ from those of $\mathbf{LA}^h_\bullet(X,n)$ only in dimensions $\le n$.

A $\langle -\infty\rangle$–decorated version of 4.2.1 exists, but does not give anything new since the homotopy groups of

$$\mathbf{LA}^{\langle -\infty\rangle}_\bullet(—,n)$$

differ from those of $\mathbf{LA}^h_\bullet(—,n)$ only in dimensions $< n$. Nevertheless, it is good to have this in mind when making the comparison with 2.5.1 (next remark).

4.2.4. Remark. Theorems 4.2.1 and 2.5.1 are compatible: the commutative squares in 4.2.1 and 2.5.1 are opposite faces of a commutative cube. Of the 12 arrows in the cube, the four not mentioned in 4.2.1 or 2.5.1 are inclusion maps (top) and forgetful maps (bottom). Also, the s–version of 4.2.1 is compatible with 2.3.1 in the same sense.

4.2.5. Remark. The left–hand column of the diagram in 4.2.1 matches the left–hand column of the diagram in 3.2.1. In detail: there exists a commutative diagram with highly connected vertical arrows

$$\begin{array}{ccc} \mathcal{H}^\infty(M) & \xleftarrow{\;w\;} & \mathcal{S}(M) \\ \big\downarrow & & \big\downarrow \\ \Omega^\infty(\mathbf{A}_\%(M)) & \xleftarrow{\;v_\%\;} & \Omega^{\infty+n}(\mathbf{LA}^h_\bullet)_\%(M,n) \end{array}$$

where w is the Whitehead torsion map of 1.5 and $v_\%$ is induced by v of (4.1.2).

4.3. Special features of the smooth case

In this section we assume that M^n is smooth. We use the notation of 4.2.1 and 2.5.1. By 4.2.4, there is a commutative diagram

$$(4.3.1)\qquad \begin{array}{ccc} \mathcal{S}(\tau) & \xrightarrow{\;\subset\;} & \bigcup_k \mathcal{S}(\tau\oplus\varepsilon^k) \\ \big\downarrow & & \big\downarrow \\ \Omega^{\infty+n}(\mathbf{LA}^h_\bullet)^\%(M,n) & \longrightarrow & \Omega^{\infty+n}(\mathbf{L}^h_\bullet)^\%(M,n)\,. \end{array}$$

We can write the resulting map of horizontal homotopy fibers in the form

$$(4.3.2)\qquad \begin{array}{c} u\mathcal{S}(\tau) \\ \big\downarrow \\ \Omega^{\infty+n}(\mathbf{A}^\%(M,n)_{h\mathbb{Z}/2}) \end{array}$$

where the prefix u indicates *unstable* structures. It is highly connected ; details as in 4.2.2. Our goal here is to describe a smooth analog of (4.3.2).

4.3.3. Proposition. *For a space X over $B\mathrm{O}$, the map $\Sigma^\infty(X_+)\to\mathbf{A}(X)$ of* 3.2.2 *has a canonical refinement to a $\mathbb{Z}/2$–map.*

Remarks, Notation. As in §4.1, we work in "naive" stable $\mathbb{Z}/2$–homotopy theory. That is, whenever we see a $\mathbb{Z}/2$–map $\mathbf{Y}'\to\mathbf{Y}$ which is an ordinary homotopy equivalence, we are allowed to replace $\mathbf{Y}$ by $\mathbf{Y}'$.

The involution on $\mathbf{A}(X)$ needed here is determined by $X \to B\mathrm{O} \hookrightarrow B\mathrm{G}$ as in §4.1. The involution on $\Sigma^\infty(X_+)$ that we have in mind is as follows. For simplicity, assume that X is a compact CW–space ; then the reference map $X \to B\mathrm{O}$ factors through $B\mathrm{O}(k)$ for some k. Let γ^k be the vector bundle on X pulled back from $B\mathrm{O}(k)$, and let η^ℓ be a complementary vector bundle on X , so that $\gamma \oplus \eta$ is trivialized. Now, to see the involution, replace $\Sigma^\infty(X_+)$ by the homotopy equivalent

$$\Omega^\ell_!\Omega^k\Sigma^\infty\Sigma^\gamma\Sigma^\eta_!(X_+)$$

where $\Sigma^\eta(X_+)$ and $\Sigma^\gamma\Sigma^\eta(X_+)$ are the Thom spaces of η and $\gamma\oplus\eta$, respectively. Subscripts ! indicate that $\mathbb{Z}/2$ acts on loop or suspension coordinates by scalar multiplication with -1. Compare §4.1. — We abbreviate

$$\Sigma^\infty(X_+\ ,n) := \mathbb{S}^n_! \wedge \Sigma^\infty(X_+)\,.$$

While the map $\Sigma^\infty(X_+) \to \mathbf{A}(X)$ of 3.2.2 can be described in algebraic K–theory terms, including the algebraic K–theory of finite sets over X , our proof of 4.3.3 is not entirely K–theoretic. It uses 3.2.1 to interpret the involution on $A(X)$ geometrically.

4.3.4. Theorem. *There is a commutative diagram with highly connected vertical arrows (and lower row resulting from* 4.3.3*)*

$$\begin{tikzcd}
u\mathcal{S}(\tau) \arrow[d] & u\mathcal{S}_d(\tau) \arrow[l] \arrow[d] \\
\Omega^{\infty+n}(\mathbf{A}^{\%}(M,n)_{h\mathbb{Z}/2}) & \Omega^{\infty+n}((\Sigma^\infty(X_+\ ,n))_{h\mathbb{Z}/2})\,. \arrow[l]
\end{tikzcd}$$

Remarks. The right–hand column of this diagram is $(n-2)$–connected. It is essentially an old construction due to Toda and James; see [Jm] for references.

Something should be said about compatibility between 4.3.4 and 4.2.5, but we will leave it unsaid.

4.3.5. Remark. In calculations involving 4.3.4 the concept of *stabilization* is often useful. Stabilization is a way to make new homotopy functors on $\mathcal{W}_Y$ (spaces over Y) from old ones. Idea: Given a homotopy functor $\mathbf{J}$ from $\mathcal{W}_Y$ to spectra, and X in $\mathcal{W}_Y$, let $s\mathbf{J}(X)$ be the homotopy colimit of the $\Omega^n(\mathbf{J}(X\times\mathbb{S}^n)/\mathbf{J}(X))$ for $n\ge 0$. There is a natural transformation $\mathbf{J}(X)\to s\mathbf{J}(X)$ induced by $x\mapsto(x,-1)\in X\times\mathbb{S}^0$ for $x\in X$. The formalities are much as in §2.4, even though the result is quite different. The main examples for us are these:

- Take $\mathbf{J}(X)=\mathbf{A}(X)$ for X in $\mathcal{W}_{BO}$. Then $s\mathbf{A}(X)\simeq\Sigma^\infty(\Lambda X_+)$ where ΛX is the free loop space. (See [Go1] for details.) Hence $(s\mathbf{A})^{\%}(X)\simeq\Sigma^\infty(X_+)$.
- Take $\mathbf{J}(X)=\mathbf{L}^h_\bullet(X)$ for X in $\mathcal{W}_{BO}$. Then $s\mathbf{L}^h_\bullet(X)$ is contractible for all X by the π–π–theorem. Hence $(s\mathbf{L}^h_\bullet)^{\%}(X)$ is also contractible.

One can use these facts to split the lower row in 4.3.4, up to homotopy. See 6.5 for another application.

5. Geometric structures on fibrations

5.1. Block bundle structures

Here we address the following question. Given a fibration p on a Space B whose fibers are Poincaré duality spaces of formal dimension n, can we find a block bundle p_0 on B with closed manifold fibers, fiber homotopy equivalent to p ? For earlier work on this problem, see [Qun4], [Qun1]. We combine this with ideas from [Ra4] and [Ra2].

The *block s–structure Space* $\widetilde{\mathbb{S}}^s(X)$ of a simple Poincaré duality space X of formal dimension n (alias *finite* Poincaré space, [Wa1, §2]) is defined literally as in the case of a closed manifold. Any simple homotopy equivalence $M^n\to X$, where M^n is a closed manifold, induces a homotopy equivalence

$$\widetilde{\mathbb{S}}^s(M)\to\widetilde{\mathbb{S}}^s(X)$$

and $\widetilde{\mathbb{S}}^s(M)$ was described in L–theoretic terms in 2.3.1. But such a homotopy equivalence $M\to X$ might not exist, and even if it does, we might want to see an L–theoretic description of the block s–structure Space of X which does not use a choice of homotopy equivalence $M\to X$.

Ranicki [Ra4], [Ra2, §17] associates to a simple Poincaré duality space X of formal dimension n its *total surgery obstruction*, a point

$$\partial\sigma^*(X)\in\Omega^{\infty+n-1}((\mathbf{L}^s_\bullet)_{\%}(X))\,.$$

The element has certain naturality properties. For example, a homotopy equivalence $g : X \to Y$ determines a path in $\Omega^{\infty+n-1}((\mathbf{L}^s_\bullet)_\%(X))$ from $g_*\partial\sigma^*(X)$ to $\partial\sigma^*(Y)$; more later.

5.1.1. Theorem. *$\widetilde{\mathbb{S}}^s(X)$ is (naturally) homotopy equivalent to the space of paths from $\partial\sigma^*(X)$ to the base point in $\Omega^{\infty+n-1}((\mathbf{L}^s_\bullet)_\%(X))$.*

Note that any choice of (base) point in the space of paths in 5.1.1 leads to an identification of it with $\Omega^{\infty+n}((\mathbf{L}^s_\bullet)_\%(X))$, up to homotopy equivalence. In this way, we recover the result

$$\widetilde{\mathbb{S}}^s(M) \simeq \Omega^{\infty+n}((\mathbf{L}^s_\bullet)_\%(M))\,.$$

The advantages of 5.1.1 become clearer when it is applied to *families*, that is, fibrations $p : E(p) \to B$ whose fibers are Poincaré spaces of formal dimension n. (One must pay attention to simple homotopy types, so we assume that B is connected and p is classified by a map $B \to B\mathrm{G}^s(X)$ for some simple Poincaré duality space X as above.) Given such a fibration $p : E(p) \to B$, we obtain an associated fibration $q : E(q) \to B$ with fiber

$$\Omega^{\infty+n-1}((\mathbf{L}^s_\bullet)_\%(p^{-1}(b)))$$

over $b \in B$, and a section $\partial\sigma^*(p)$ of q selecting the total surgery obstruction $\partial\sigma^*(p^{-1}(b))$ in $q^{-1}(b)$, for $b \in B$. The fibers of q are infinite loop spaces, so we also have a zero section. (Technical point: For these constructions it is convenient to assume that B is a simplicial complex, and to apply a suitable $(n+2)$–ad version of 5.1.1 to $E(p)|\sigma$ for each n–simplex σ in B.) We say that p *admits a block bundle structure* if the classifying map $B \to B\mathrm{G}^s(X)$ lifts to a map

$$B \to B\widetilde{\mathrm{TOP}}(M)$$

for some closed manifold M equipped with a simple homotopy equivalence to X.

5.1.2. Corollary. *The fibration $p : E(p) \to B$ with Poincaré duality space fibers admits a block bundle structure if and only if $\partial\sigma^*(p)$ is vertically nullhomotopic.*

In the case $B = B\mathrm{G}^s(X)$ we can add the following:

5.1.3. Corollary. *Let $p : E(p) \to BG^s(X)$ be the canonical fibration with fibers $\simeq X$. There is a cartesian square*

$$\begin{tikzcd}
\coprod_M B\widetilde{\mathrm{TOP}}(M) \arrow[r] \arrow[d] & BG^s(X) \arrow[d, "\text{total surgery obstruction section}"] \\
BG^s(X) \arrow[r, "\text{zero section}"] & E(q)
\end{tikzcd}$$

where M runs through a maximal set of pairwise non–homeomorphic closed n–manifolds in the simple homotopy type of X.

A few words on how $\partial\sigma^*(X)$ is constructed: Ranicki creates a homotopy functor $\mathbf{VL}^\bullet_s$ on spaces over $\mathbb{R}P^\infty$, and a natural transformation $\mathbf{L}^s_\bullet \to \mathbf{VL}^\bullet_s$ with the property that

$$\begin{tikzcd}
(\mathbf{L}^s_\bullet)^{\%}(X) \arrow[r] \arrow[d, "\text{assembly}"] & (\mathbf{VL}^\bullet_s)^{\%}(X) \arrow[d, "\text{assembly}"] \\
\mathbf{L}^s_\bullet(X) \arrow[r] & \mathbf{VL}^\bullet_s(X)
\end{tikzcd}$$

is cartesian for any X over $\mathbb{R}P^\infty$. (The functor $\mathbf{VL}^\bullet_s$ we have in mind is defined in [Ra, §15], and we should really call it $\mathbf{VL}^\bullet_s\langle 1/2\rangle$ to conform with Ranicki's notation.) Any finite Poincaré duality space X of formal dimension n determines an element $\sigma^*(X) \in \Omega^{\infty+n}(\mathbf{VL}^\bullet_s(X))$, the *visible symmetric signature* of X. The image of $\sigma^*(X)$ under the boundary map

$$\Omega^{\infty+n}(\mathbf{VL}^\bullet_s(X)) \longrightarrow \Omega^{\infty+n-1}((\mathbf{VL}^\bullet_s)_{\%}(X)) \quad \simeq \quad \Omega^{\infty+n-1}((\mathbf{L}^s_\bullet)_{\%}(X))$$

is the total surgery obstruction $\partial\sigma^*(X)$. Therefore: X is simple homotopy equivalent to a closed n–manifold if and only if the component of $\sigma^*(X)$ is in the image of the assembly homomorphism,

$$\pi_n(\mathbf{VL}^\bullet_s)^{\%}(X) \to \pi_n\mathbf{VL}^\bullet_s(X)\,.$$

The functor $\mathbf{VL}^\bullet_s$ has a *ring structure*, that is, for X_1 and X_2 over $\mathbb{R}P^\infty$ there is a multiplication

$$\mu : \mathbf{VL}^\bullet_s(X_1) \wedge \mathbf{VL}^\bullet_s(X_2) \longrightarrow \mathbf{VL}^\bullet_s(X_1 \times X_2),$$

with a unit in $\mathbf{VL}^\bullet_s(*)$. The visible symmetric signature is multiplicative:

$$\mu(\sigma^*(X_1), \sigma^*(X_2)) = \sigma^*(X_1 \times X_2),$$

up to a canonical path, for Poincaré duality spaces X_1 and X_2. This property makes σ^* useful (more useful than $\partial\sigma^*$ alone) in dealing with products, say, in giving a description along the lines of 5.1.1 of the product map

$$\widetilde{\mathcal{S}}^s(X_1) \times \widetilde{\mathcal{S}}^s(X_2) \longrightarrow \widetilde{\mathcal{S}}^s(X_1 \times X_2)\,.$$

5.2. Fiber bundle structures

Here the guiding question is: Given a fibration p over some space B, with fibers homotopy equivalent to finitely dominated CW–spaces, does there exist a bundle p_0 on B with compact manifolds as fibers, fiber homotopy equivalent to p ? We do not assume that the fibers of p satisfy Poincaré duality. We do not ask that the fibers of p_0 be closed and we do not care what dimension they have. See [DWW], [DWWc] for all details.

Let Z be a compact CW–space, equipped with a euclidean bundle ξ. Let $\mathcal{T}_n(Z,\xi)$ be the Space of pairs (M,f,j) where M^n is a compact manifold with boundary, $f : M \to Z$ is a homotopy equivalence, and j is a stable isomorphism $f^*\xi \to \tau(M)$. Let $\mathcal{T}(Z,\xi)$ be the colimit of the $\mathcal{T}_n(Z,\xi)$ under stabilization (product with I).

Any choice of vertex (M,f,j) in $\mathcal{T}(Z,\xi)$ leads to a homotopy equivalence from $\mathcal{T}(M,\tau)$ to $\mathcal{T}(Z,\xi)$. There is a homotopy fiber sequence

$$\bigcup_k \mathrm{TOP}(M \times I^k, \partial(M \times I^k)) \longrightarrow \bigcup_k \mathrm{G}_{\tan}(M_0 \times \mathbb{R}^k) \longrightarrow \mathcal{T}(M,\tau)$$

where $M_0 = M \smallsetminus \partial M$, and $\mathrm{G}_{\tan}(\dots)$ refers to homotopy automorphisms f of $M_0 \times \mathbb{R}^k$ covered by isomorphisms $\tau(M_0 \times \mathbb{R}^k) \to f^*(\tau(M_0 \times \mathbb{R}^k))$. By 1.2.1 we may write $\bigcup_k \mathrm{TOP}(M_0 \times \mathbb{R}^k)$ instead of $\bigcup_k \mathrm{G}_{\tan}(M_0 \times \mathbb{R}^k)$. Now 1.2.3 implies

$$\Omega\mathcal{T}(M,\tau) \simeq \Omega\mathcal{H}^\infty(M)$$

and suggests $\mathcal{T}(M,\tau) \simeq \mathcal{H}^\infty(M)$. This is easily confirmed with the methods of §1.2. Summarizing these observations: any choice of vertex (M,f,j) in $\mathcal{T}(Z,\xi)$ leads to a homotopy equivalence $\mathcal{T}(Z,\xi) \to \mathcal{H}^\infty(M)$. Furthermore, $\mathcal{H}^\infty(M) \simeq \Omega^\infty \mathbf{A}_\%(M) \simeq \Omega^\infty \mathbf{A}_\%(Z)$ by 3.2.1.

Again, we might want to see a description of $\mathcal{T}(Z,\xi)$ in terms of $\mathbf{A}(Z)$ which does not depend on a choice of base point in $\mathcal{T}(Z,\xi)$. To get such a description we proceed very much as in §5.1, by associating to Z a *characteristic*

$$\chi(Z) \in \Omega^\infty \mathbf{A}(Z),$$

analogous to Ranicki's $\sigma^*(X) \in \Omega^{\infty+n}\mathbf{VL}^\bullet_s(X)$ for a simple Poincaré duality space X of formal dimension n. The element $\chi(Z)$ is the image of the object/vertex

$$\mathbb{S}^0 \times Z \underset{i}{\overset{r}{\rightleftarrows}} Z \qquad (r(x,z) = z\ ,\ i(z) = (1,z))$$

in $\mathcal{R}(Z)$ under the inclusion $|w\mathcal{R}(Z)| \hookrightarrow \Omega^\infty\mathbf{A}(Z)$ mentioned in §3.1. If Z is connected, then the component of $\chi(Z)$ in $\pi_0\Omega^\infty\mathbf{A}(Z) \cong \mathbb{Z}$ is the Euler characteristic of Z.

5.2.1. Theorem. $\mathcal{T}(Z,\xi)$ *is (naturally) homotopy equivalent to the homotopy fiber of the assembly map* $\Omega^\infty(\mathbf{A}^\%(Z)) \longrightarrow \Omega^\infty\mathbf{A}(Z)$ *over the point* $\chi(Z)$.

Note that the A–theoretic expression for $\mathcal{T}(Z,\xi)$ does not depend on ξ. Again, *naturality* in 5.2.1 is a license to apply the statement to families. Let $p : E \to B$ be a fibration where the fibers E_b are homotopy equivalent to compact CW–spaces. Let $\Omega^\infty\mathbf{A}(p)$ and $\Omega^\infty\mathbf{A}^\%(p)$ be the associated fibrations on B with fibers $\Omega^\infty\mathbf{A}(E_b)$ and $\Omega^\infty\mathbf{A}^\%(E_b)$, respectively, over a point $b \in B$. The rule $b \mapsto \chi(E_b)$ defines a section of $\Omega^\infty\mathbf{A}^\%(p)$ which we call $\chi(p)$. See [DWW] for explanations regarding the *continuity* of this construction. Assembly gives a map over B from the total space of $\Omega^\infty\mathbf{A}^\%(p)$ to that of $\Omega^\infty\mathbf{A}(p)$.

5.2.2. Corollary. *The fibration* $p : E \to B$ *is fiber homotopy equivalent to a bundle with compact manifolds as fibers if and only if the section* $\chi(p)$ *of* $\Omega^\infty\mathbf{A}(p)$ *lifts (after a vertical homotopy) to a section of* $\Omega^\infty\mathbf{A}^\%(p)$.

We leave it to the reader to state an analog of 5.1.3, and turn instead to the smooth case. Suppose that ξ is a *vector bundle* over Z. There is then a smooth variant $\mathcal{T}_d(Z,\xi)$ of $\mathcal{T}(Z,\xi)$. Any choice of base vertex (M,f,ξ) in $\mathcal{T}_d(Z,\xi)$ leads to a homotopy equivalence

$$\mathcal{T}(Z,\xi) \simeq \mathcal{H}^\infty_d(M)\,.$$

Remember now (3.2.2) that $\mathcal{H}_d^\infty(M)$ can be A–theoretically described as the homotopy fiber of some map $\eta : \Omega^\infty\Sigma^\infty(M_+) \to \Omega^\infty\mathbf{A}(M)$. The map η is, in homotopy invariant terms, Ω^∞ of the composition

$$M_+ \wedge \mathbf{S}^0 \xrightarrow{\mathrm{id}\wedge u} M_+ \wedge \mathbf{A}(*) \quad \simeq \quad \mathbf{A}^{\%}(M) \longrightarrow \mathbf{A}(M)$$

where $u : \mathbf{S}^0 \to \mathbf{A}(*)$ is the unit of the ring spectrum $\mathbf{A}(*)$.

5.2.3. Theorem. *$\mathcal{T}_d(Z,\xi)$ is (naturally) homotopy equivalent to the homotopy fiber of $\eta : \Omega^\infty\Sigma^\infty(Z_+) \longrightarrow \Omega^\infty\mathbf{A}(Z)$ over the point $\chi(Z)$.*

Returning to the notation and hypotheses of 5.2.2, we are compelled to introduce yet another fibration $\Omega^\infty\Sigma^\infty_+ p$ on B, with fiber $\Omega^\infty\Sigma^\infty(E_b)_+$ over $b \in B$.

5.2.4. Corollary. *The fibration $p : E(p) \to B$ is fiber homotopy equivalent to a bundle with smooth compact manifolds as fibers if and only if the section $\chi(p)$ of $\Omega^\infty\mathbf{A}(p)$ lifts (after a vertical homotopy) to a section of $\Omega^\infty\Sigma^\infty_+ p$.*

Remarks. In 5.2.4, *bundle with smooth compact manifolds as fibers* means, say in the case where B is connected, a fiber bundle with fibers $\cong M$ where M is smooth compact, and structure group $\mathrm{DIFF}(M, \partial M)$.

Corollary 5.2.4 is closely related to something we shall discuss in §6.7: the Riemann–Roch theorem of [BiLo], see also [DWW].

5.2.5. Corollary. *Let $p : E \to B$ be a fibration with fibers homotopy equivalent to compact CW–spaces. If Y is any compact connected CW–space of Euler characteristic 0, then the composition $pq : Y \times E \to B$ (where $q : Y \times E \to E$ is the projection) is fiber homotopy equivalent to a bundle with smooth compact manifolds as fibers.*

Proof. A suitable product formula implies that $\chi(pq)$ is vertically homotopic to $\chi(Y) \times \chi(p)$. We saw earlier that the component of $\chi(Y)$ in $\pi_0\mathbf{A}(Y) \cong \mathbb{Z}$ is the Euler characteristic of Y. $\square$

Statements 5.2.1–5 can easily be generalized to the case where Z is a *finitely dominated* CW–space. But it is then necessary to use a variant $\mathbf{A}^p(Z)$ of $\mathbf{A}(Z)$ with a larger π_0 isomorphic to $K_0(\mathbb{Z}\pi_1(X))$. Then $\chi(Z)$

in $\Omega^\infty \mathbf{A}^p(Z)$ is defined, and 5.2.1 and 5.2.3 remain correct as stated. In this more general formulation, 5.2.1 includes Wall's theory [Wa2] of the finiteness obstruction.

Casson and Gottlieb [CaGo] established 5.2.5 in the case $Y = (\mathbb{S}^1)^n$, with a large n depending on $p : E \to B$.

6. Examples and Calculations

6.1. Smooth automorphisms of disks. The smooth version of (1.2.2) gives a homotopy fiber sequence

$$\mathrm{DIFF}(\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow \mathrm{DIFF}(\mathbb{R}^n) \to \mathcal{H}_d(\mathbb{S}^{n-1})$$

where $\mathcal{H}_d$ is a Space of differentiable h–cobordisms. The composition of group homomorphisms $\mathrm{O}(n) \hookrightarrow \mathrm{DIFF}(\mathbb{D}^n, \mathbb{S}^{n-1}) \to \mathrm{DIFF}(\mathbb{R}^n)$ is a homotopy equivalence, so that

$$\mathrm{DIFF}(\mathbb{D}^n, \mathbb{S}^{n-1}) \;\simeq\; \mathrm{O}(n) \times \Omega\mathcal{H}_d(\mathbb{S}^{n-1})\,.$$

By 1.3.4 and 3.2.2, there is a map from $\Omega\mathcal{H}_d(\mathbb{S}^{n-1})$ to $\Omega^{\infty+2}\mathbf{Whd}(\mathbb{S}^{n-1})$ which is an isomorphism on π_j for $j < \phi(n-1)$, where $\phi(n)$ is the minimum of $(n-4)/3$ and $(n-7)/2$. Further, the map $\mathbf{Whd}(\mathbb{S}^{n-1}) \to \mathbf{Whd}(*)$ induced by $\mathbb{S}^{n-1} \to *$ is approximately $2n$–connected [Wah2] and the rational homotopy groups of $\mathbf{Whd}(*)$ in dimensions > 1 are those of $\mathbf{K}(\mathbb{Q})$, which are known [Bo]. Therefore:

$$\pi_j\, \mathrm{DIFF}(\mathbb{D}^n, \mathbb{S}^{n-1}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & \text{if } 0 < j < \phi(n-1) \text{ and } 4 \mid j+1 \\ 0 & \text{if } 0 < j < \phi(n-1) \text{ and } 4 \nmid j+1. \end{cases}$$

For a calculation of the rational homotopy groups of $\mathrm{DIFF}(\mathbb{D}^n)$ in the concordance stable range, following Farrell and Hsiang [FaHs], we note $\mathrm{DIFF}(\mathbb{D}^n) \simeq \Omega\mathcal{S}_d(\mathbb{D}^n)$ and use the smooth version of 1.5.2, which gives (at odd primes and in the concordance stable range)

$$\mathcal{S}_d(\mathbb{D}^n) \simeq \widetilde{\mathcal{S}}_d(\mathbb{D}^n) \times \widetilde{\mathrm{DIFF}}(\mathbb{D}^n)/\,\mathrm{DIFF}(\mathbb{D}^n) \simeq \widetilde{\mathcal{S}}_d(\mathbb{D}^n) \times \Omega^\infty(\mathbf{H}_d(\mathbb{D}^n)_{h\mathbb{Z}/2})\,.$$

Here $\widetilde{\mathcal{S}}_d(\mathbb{D}^n) \simeq \Omega^n(\mathrm{TOP}\,/\,\mathrm{O})$, which is rationally trivial, and $\mathbf{H}_d(\mathbb{D}^n)$ is homotopy equivalent to $\Omega\mathbf{Whd}(*)$, so 3.2.2 and [Bo] give

$$\pi_j \mathbf{H}_d(\mathbb{D}^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } 4 | j \\ 0 & \text{otherwise} \end{cases}$$

provided $0 < j \le \phi(n)$. The canonical involution on $\mathbf{H}_d(\mathbb{D}^n)$ acts trivially on these rationalized homotopy groups if n is odd, and nontrivially if n is even. Therefore, if $0 \le j < \phi(n)$, then [FaHs]

$$\pi_j \,\mathrm{DIFF}(\mathbb{D}^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & n \text{ odd and } 4 \mid j+1 \\ 0 & \text{otherwise.} \end{cases}$$

Beware that Farrell and Hsiang write $\mathrm{DIFF}(\mathbb{D}^n, \partial)$ for our $\mathrm{DIFF}(\mathbb{D}^n)$.

6.2. Smooth automorphisms of spherical space forms. Let M^n be smooth closed orientable, with universal cover $\simeq \mathbb{S}^n$, where $n \ge 5$. Hsiang and Jahren [HsiJ] calculate $\pi_* \,\mathrm{DIFF}(M) \otimes \mathbb{Q}$ in the smooth concordance stable range, assuming that n is odd. They begin with the observation that $\pi_j \,\mathrm{G}^s(M)$ is finite for all j. Therefore

$$\pi_j \,\mathrm{DIFF}(M) \otimes \mathbb{Q} \cong \pi_{j+1} \mathcal{S}^s_d(M) \otimes \mathbb{Q}$$

for $j > 0$. By the smooth versions of 1.5.2 and 1.4.4 we have a splitting

$$\mathcal{S}^s_d(M) \simeq \widetilde{\mathcal{S}}^s_d(M) \times \Omega^\infty(\mathbf{H}^s_d(M)_{h\mathbb{Z}/2})$$

at odd primes, in the concordance stable range. Therefore, rationally,

$$\pi_j \,\mathrm{DIFF}(M) \cong \pi_{j+1} \mathcal{S}^s_d(M) \cong \pi_{j+n+2} \widetilde{\mathbf{L}}^h_\bullet(M) \oplus \pi_{j+1} \mathbf{L}^h_\bullet(*) \oplus \pi^-_{j+1} \mathbf{H}^s_d(M)$$

for $0 < j < \phi(n)$, where $\widetilde{\mathbf{L}}^h_\bullet$ is the reduced L–theory and $\pi^-_{j+1}\mathbf{H}^s_d(M)$ is the quotient of $\pi_{j+1}\mathbf{H}^s_d(M)$ by the fixed subgroup of the $\mathbb{Z}/2$–action. This is the Hsiang–Jahren result. Rationally, the multisignature homomorphisms on $\pi_* \mathbf{L}^h_\bullet(M)$ are isomorphisms [Wa3]. Rationally, $\pi_*(\mathbf{H}^s_d(M)) \cong \pi_{*+1}\mathbf{K}(\mathbb{Q}\pi_1(M))$ for $0 < * < n-1$. The calculation of $\pi_* \mathbf{K}(\mathbb{Q}\pi) \otimes \mathbb{Q}$ for a finite group π can often be accomplished with [Bo], certainly in the case where $\pi_1(M)$ is commutative.

6.3. Automorphisms of negatively curved manifolds. Let M^n be smooth, closed, connected, with a Riemannian metric of sectional curvature < 0. In the course of their proof of the Borel conjecture for such M , Farrell and Jones [FaJo1], [FaJo5] show that $\alpha : (\mathbf{L}_\bullet)^\%(M) \to \mathbf{L}_\bullet(M)$ is a homotopy equivalence (with a 4–periodic definition of $\mathbf{L}_\bullet(M)$, decoration s or h). With our 0–connected definition of $\mathbf{L}_\bullet(M)$, it is still true that

$$\begin{gathered}\Omega^{\infty+n}((\mathbf{L}_\bullet)_\%(M)) \simeq * \\ \Omega^{\infty+n}((\mathbf{LA}^h_\bullet)_\%(M,n)) \simeq \Omega^{\infty+n}(\mathbf{A}_\%(M,n)_{h\mathbb{Z}/2})\,.\end{gathered}$$

For a simple closed geodesic T in M, let $T^\sharp$ be the "desingularization" of T, so that $T^\sharp \cong \mathbb{S}^1$. Farrell and Jones also show that the map

$$\bigvee_T \mathbf{A}_\%(T^\sharp) \to \mathbf{A}_\%(M)$$

induced by $T^\sharp \to T \hookrightarrow M$ for all simple closed geodesics T in M is a homotopy equivalence [FaJo3]; see also [FaJo4], [FaJo2] for extensions. Now there is a *fundamental theorem* in the algebraic K–theory of spaces: The assembly from $\mathbb{S}^1 \wedge \mathbf{A}(*) \simeq \mathbf{A}^\%(\mathbb{S}^1)$ to $\mathbf{A}(\mathbb{S}^1)$ is a split monomorphism in the homotopy category [KVWW1], [KVWW2] and its mapping cone splits up to homotopy into two copies of a spectrum $\mathbf{Nil}_\mathbf{A}(*)$. Under any of the involutions constructed by the method of §4.1, these two copies are interchanged. Therefore $\Omega^{\infty+n}((\mathbf{LA}_\bullet)_\%(M,n)) \simeq \Omega^{\infty+1}(\bigvee_T \mathbf{Nil}_\mathbf{A}(*))$ and we get from 4.2.1 a map

$$\mathcal{S}(M) \to \Omega^{\infty+1}(\bigvee_T \mathbf{Nil}_\mathbf{A}(*))$$

which is approximately $(n/3)$–connected (see 4.2.2). It is known that $\mathbf{Nil}_\mathbf{A}(*)$ is rationally trivial and 1–connected, but $\pi_2\mathbf{Nil}_\mathbf{A}(*) \neq 0$. See [HaWa],[Wah1]. From [Dun], [BHM], one has a homological algebra description of $\mathbf{Nil}_\mathbf{A}(*)$, as explained in [Ma1, 4.5] and [Ma2, §5]. But the homological algebra is over the ring spectrum $\mathbf{S}^0$ and it is not considered easy.— From the fiber sequence $\mathrm{TOP}(M) \to \mathrm{G}(M) \to \mathcal{S}(M)$ we get $\pi_j\,\mathrm{TOP}(M) \cong \bigoplus_T \pi_{j+2}\mathbf{Nil}_\mathbf{A}(*)$ if $1 < j < \phi(n)$, and an exact sequence

$$\begin{aligned}\bigoplus_T \pi_3\mathbf{Nil}_\mathbf{A}(*) \rightarrowtail \pi_1\,\mathrm{TOP}(M) \to \mathrm{center}(\pi_1(M)) \\ \longrightarrow \bigoplus_T \pi_2\mathbf{Nil}_\mathbf{A}(*) \to \pi_0\,\mathrm{TOP}(M) \twoheadrightarrow \mathrm{Out}(\pi_1(M)).\end{aligned}$$

6.4. The h–structure Space of $\mathbb{S}^n$, for $n \geq 5$. By 4.2.1 and 4.2.2 there is a commutative square

$$\begin{array}{ccc} \mathcal{S}(\mathbb{S}^n) & \longrightarrow & \Omega^{\infty+n}((\mathbf{LA}_\bullet)_\%(\mathbb{S}^n, n)) \\ \big\downarrow \cap & & \big\downarrow \\ \widetilde{\mathcal{S}}(\mathbb{S}^n) & \xrightarrow{\simeq} & \Omega^{\infty+n}((\mathbf{L}^s_\bullet)_\%(\mathbb{S}^n)), \end{array}$$

where the top horizontal arrow is highly connected. Here $\Omega^n(\mathbf{L}^s_\bullet)_\%(\mathbb{S}^n)$ simplifies to $\mathbf{L}_\bullet(*)$, and $\Omega^n(\mathbf{LA}_\bullet)_\%(\mathbb{S}^n, n)$ has an analogous simplifying map (not a homotopy equivalence, but highly connected) to $\mathbf{LA}_\bullet(*, n)$. Summarizing: there is a homotopy commutative square

$$\begin{array}{ccc} \mathcal{S}(\mathbb{S}^n) & \longrightarrow & \Omega^\infty \mathbf{LA}_\bullet(*, n) \\ \big\downarrow \cap & & \big\downarrow \text{forget} \\ \widetilde{\mathcal{S}}(\mathbb{S}^n) & \xrightarrow{\simeq} & \Omega^\infty \mathbf{L}_\bullet(*) \end{array}$$

and a homotopy fiber sequence $\mathbf{LA}_\bullet(*, n) \to \mathbf{L}_\bullet(*) \xrightarrow{\beta} \mathbb{S}^1 \wedge (\mathbf{A}(*, n))_{h\mathbb{Z}/2}$ from (4.1.2). Calculations [WWp] using connective K–theory $\mathbf{bo}$ as a substitute for $\mathbf{A}(*)$, via $\mathbf{A}(*) \to \mathbf{K}(\mathbb{Z}) \to \mathbf{bo}$, show that β detects all elements in $\pi_{n+q}\mathbf{L}(*)$ whose signature is not divisible by $2a_q$ if 4 divides $n + q$, and by $4a_q$ if 4 divides both $n + q$ and q. Here $a_q = 1, 2, 4, 4, 8, 8, 8, 8$ for $q = 1, 2, \ldots, 8$ and $a_{q+8} = 16a_q$; the numbers a_q are important in the theory of Clifford modules [ABS]. Consequently, if 4 divides $n + q$, then the image of the inclusion–induced homomorphism

$$\pi_{n+q}\mathcal{S}(\mathbb{S}^n) \longrightarrow \pi_{n+q}\widetilde{\mathcal{S}}(\mathbb{S}^n) \cong 8\mathbb{Z}$$

is contained in $2a_q\mathbb{Z}$ if 4 does not divide q, and in $4a_q\mathbb{Z}$ if 4 does divide q. Note the similarity of this statement with [At, 3.3], [LM, IV.2.7], and [Tho, Thm.14].

6.5. Obstructions to unblocking smooth block automorphisms. One of the main points of §4.2 and the introduction to §4 is a homotopy commutative diagram, for compact M^n with $n \geq 5$,

$$\begin{array}{ccc} \Omega^{\infty+n+1}(\mathbf{L}^s_\bullet)_\%(M) & \longrightarrow & \Omega^{\infty+n}(\mathbf{A}^s_\%(M, n)_{h\mathbb{Z}/2}) \\ \big\downarrow \simeq & & \big\uparrow \\ \Omega\widetilde{\mathcal{S}}^s(M) & \longrightarrow & \widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \end{array} \tag{6.5.1}$$

in which the upper row is the connecting map from the first of the two homotopy fiber sequences in (4.1.2),

$$\Omega\mathbf{L}^s_\bullet(M) \to \mathbf{A}^s(M,n)_{h\mathbb{Z}/2}\ , \tag{6.5.2}$$

with $\%$ and decoration s and $\Omega^{\infty+n}$ inflicted. Modulo the identification $\mathbf{A}^s_\%(M) \simeq \mathbf{H}^s(M)$ (the s–decorated version of 3.2.2), the right–hand column of (6.5.1) is the highly connected map which we found at the end of §1.4 using purely geometric methods. The lower row of (6.5.1) is the connecting map from the homotopy fiber sequence

$$\widetilde{\mathrm{TOP}}(M)/\,\mathrm{TOP}(M) \longrightarrow \mathbb{S}^s(M) \longrightarrow \widetilde{\mathbb{S}}^s(M)\,.$$

One of the main points of §4.3 is that much of (6.5.1) has a smooth analog, in the shape of a homotopy commutative diagram

$$\begin{array}{ccc} \Omega^{\infty+n+2}\mathbf{L}^s_\bullet(M) & \longrightarrow & \Omega^{\infty+n+1}(\mathbf{Whd}^s(M,n)_{h\mathbb{Z}/2}) \\ \Big\downarrow & & \Big\uparrow \\ \Omega\widetilde{\mathbb{S}}^s_d(M) & \longrightarrow & \widetilde{\mathrm{DIFF}}(M)/\,\mathrm{DIFF}(M) \end{array} \tag{6.5.3}$$

defined for smooth compact M. Here $\mathbf{Whd}^s(M)$ is the mapping cone of the map $\Sigma^\infty(M_+) \to \mathbf{A}^s(M)$ discussed in 4.3.3 (except for a decoration s which we add here), and $\mathbf{Whd}^s(M,n) := \mathbb{S}^n_! \wedge \mathbf{Whd}^s(M)$. The upper row in (6.5.3) is $\Omega^{\infty+n+1}$ of (6.5.2) composed with the projection $\mathbf{A}^s(M,n)_{h\mathbb{Z}/2} \longrightarrow \mathbf{Whd}^s(M,n)_{h\mathbb{Z}/2}$. Again, modulo an identification of $\mathbf{Whd}^s(M)$ with $\mathbf{H}^s_d(M)$, coming from (3.2.3), the right–hand column of (6.5.3) is a purely geometric construction going back to (the smooth version of) §1.4. It is highly connected. The left–hand column of (6.5.3) is not a homotopy equivalence, which makes the analogy a little imperfect. We arrive at (6.5.3) by first making full use of 4.2.1 and 4.2.4 to produce a *framed* version of (6.5.1), with upper left–hand vertex $\Omega^{\infty+n+2}\mathbf{L}^s_\bullet(M)$, upper right–hand vertex $\Omega^{\infty+n+1}(\mathbf{A}(M,n)_{h\mathbb{Z}/2})$, and lower left–hand vertex equal to Ω of the homotopy fiber of

$$\nabla : \widetilde{\mathbb{S}}^s \to \widetilde{\mathbb{S}}(\tau)\,.$$

Now assume that the classifying map $M \to B\mathrm{O}$ for the stable tangent bundle factors up to homotopy through an aspherical space. (For example, this is the case if M is stably framed.) Stabilization arguments as in 4.3.5 show that then $\mathbf{Whd}^s(M,n)_{h\mathbb{Z}/2}$ splits off $\mathbf{A}^s(M,n)_{h\mathbb{Z}/2}$, up to homotopy. Also, the connecting map (6.5.2) factors through the summand $\mathbf{Whd}^s(M,n)_{h\mathbb{Z}/2}$, up to homotopy. It follows that elements in $L^s_k(\mathbb{Z}\pi)$ detected under (6.5.2) are also detected under

$$L^s_k(\mathbb{Z}\pi) \longrightarrow \pi_{k-n-2}(\widetilde{\mathrm{DIFF}}(M)/\,\mathrm{DIFF}(M))\ , \tag{6.5.4}$$

the homomorphism coming from (6.5.3). To exhibit such elements in $L^s_k(\mathbb{Z}\pi)$, we suppose in addition that M is orientable and $\pi = \pi_1(M)$ is finite. Then we have the multisignature homomorphisms

$$L^s_k(\mathbb{Z}\pi) \to L^p_k(\mathbb{R}\pi) \cong \bigoplus_V L^p_k(E_V) \tag{6.5.5}$$

where the direct sum is over a maximal set of pairwise non–isomorphic irreducible real representations V of π, and E_V is the endomorphism ring of V, isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ equipped with a standard conjugation involution [Le], [Wa3]. It is known that $L^p_k(E_V) \cong \mathbb{Z}$ if $4|k$, and also $L^p_k(E_V) \cong \mathbb{Z}$ if $2|k$ and $E_V \cong \mathbb{C}$; otherwise $L^p_k(E_V) = 0$. A calculation similar to that mentioned in 6.4 shows that an element in $L^s_k(\mathbb{Z}\pi) = \pi_k(\mathbf{L}^s_\bullet(M))$ will be detected by the homomorphism associated with (6.5.2) if, for some irreducible real representation V of π, the V–component of its multisignature is not divisible by

$$\begin{array}{ll} 2a_{k-n} & \text{(assuming } 4|k \text{ and } E_V \cong \mathbb{R}) \\ 2a^c_{k-n} & \text{(assuming } 2|k \text{ and } E_V \cong \mathbb{C}) \\ a_{k-n+4}/4 & \text{(assuming } 4|k \text{ and } E_V \cong \mathbb{H}) \end{array}$$

where $a^c_q = 1$ if $q = 1,2$ and $a^c_{q+2} = 2a^c_q$ for $q > 2$. If $4|n$ and V is the trivial 1–dimensional representation, we can can do a little better: the element will also be detected if the V–component of its multisignature is not divisible by

$$4a_{k-n} \qquad \text{(assuming } 4|k)\,.$$

Now the multisignature homomorphisms (6.5.5) are of course periodic in k with period 4, and are rational isomorphisms [Wa1]. Therefore many elements in $L^s_*(\mathbb{Z}\pi)$ are indeed detected by (6.5.2), and a fortiori by (6.5.4).

Remark. This calculation can be viewed as a cousin of Rochlin's theorem. To make this clearer we switch from block diffeomorphisms to bounded diffeomorphisms, i.e. we look at

$$L^{\langle -\infty\rangle}_k(\mathbb{Z}\pi) \quad \longrightarrow \quad \pi_{k-n-2}(\mathrm{DIFF}^b(M\times\mathbb{R}^\infty)/\,\mathrm{DIFF}(M)) \tag{6.5.6}$$

instead of (6.5.4). Compare 2.5.1 and 4.2.3. The same calculations as before show that an element x in the domain of (6.5.6) maps nontrivially if, for some irreducible real representation V of π, the V–component of the multisignature of x is not divisible by certain powers of 2, depending on V, k and n, exactly as above. Note in passing that

$$L_k^p(E_V) \cong L_k^{\langle -\infty \rangle}(E_V)$$

so that we can indeed speak of multisignatures as before. Specializing to $M = *$ and $4|k$, with $k > 0$, we see that elements x whose signature is not divisible by $4a_k$ are detected by (6.5.6). But in the case $M = *$ we also have $\mathrm{DIFF}^b(M \times \mathbb{R}^\infty) \simeq \Omega(\mathrm{TOP/O})$ and we may identify (6.5.6) with the boundary map in the long exact sequence of homotopy groups associated with the homotopy fiber sequence

$$\mathrm{TOP/O} \longrightarrow \mathrm{G/O} \longrightarrow \mathrm{G/TOP}\ .$$

Therefore, if $4|k$ and $k > 0$, the image of $\pi_k(\mathrm{G/O})$ in $\pi_k(\mathrm{G/TOP}) = 8\mathbb{Z}$ is contained in $4a_k \cdot \mathbb{Z}$. For $k = 4$, this statement is (one form of) Rochlin's theorem. For $k > 4$, it is also well known as the 2–primary aspect of the Kervaire–Milnor work on homotopy spheres [KeM], [Lev].

6.6. Gromoll filtration. The Gromoll filtration of $x \in \pi_0\widetilde{\mathrm{DIFF}}(\mathbb{D}^{i-1})$ is the largest number $j = j(x)$ such that x lifts from

$$\pi_0\widetilde{\mathrm{DIFF}}(\mathbb{D}^{i-1}) \cong \pi_{j-1}\widetilde{\mathrm{DIFF}}(\mathbb{D}^{i-j})$$

to $\pi_{j-1}\,\mathrm{DIFF}(\mathbb{D}^{i-j})$. This is the original definition of [Grom]; see also [Hit].

To obtain upper bounds on $j(x)$ in special cases, we use 6.5, with $k = i{+}1$ and $n = i - j$ and $M = \mathbb{D}^n$. Therefore: if 4 divides $i+1$ and x has Gromoll filtration $\geq j$, and is the image of $\bar{x} \in L_{i+1}(\mathbb{Z})$, then the signature of $\bar{x}$ is divisible by $2a_{j+1}$ (by $4a_{j+1}$ in the case where 4 divides $j + 1$).

6.7. Riemann–Roch for smooth fiber bundles. Let $p : E \to B$ be a fiber bundle with fibers $\cong M$ and structure group $\mathrm{DIFF}(M, \partial M)$. Let R be a ring, and let V be a bundle (with discrete structure group) of f.g. left proj. R–modules on E. This determines $[V] : E \to \Omega^\infty\mathbf{K}(R)$. Let V_i be the bundle on B with fiber $H_i(p^{-1}(b); V)$ over b. We assume that the fibers of V_i are projective; then each V_i determines $[V_i] : B \to K(R)$. Then the following Riemann–Roch formula holds:

$$\text{(6.7.1)} \qquad \mathrm{tr}^*[V] = \sum (-1)^i [V_i] \qquad \in [B, \Omega^\infty \mathbf{K}(R)]$$

where $\mathrm{tr} : \Sigma^\infty B_+ \to \Sigma^\infty E_+$ is the Becker–Gottlieb–Dold transfer [BeGo], [Do], [DoP], a stable map determined by p . Both sides of (6.7.1) have meaning when $p : E \to B$ is a fibration whose fibers are homotopy equivalent to compact CW–spaces. However, (6.7.1) does not hold in this generality. It can fail for a fiber bundle with compact (and even closed) *topological* manifolds as fibers.

Formula (6.7.1) is a distant corollary of 5.2.4. Namely, both (6.7.1) and 5.2.4 are ways of saying that certain generalized Euler characteristics of a smooth compact M lift canonically to $\Omega^\infty \Sigma^\infty (M_+)$. For the proof of (6.7.1), see [DWW]. Earlier, Bismut and Lott [BiLo] had proved by analytic methods that (6.7.1) holds in the case $R = \mathbb{C}$ after certain characteristic classes are applied to both sides of the equation.

6.8. Obstructions to finding block bundle structures. Suppose that $p : E \to B$ is a fibration with connected base whose fibers are oriented Poincaré duality spaces of formal dimension $2k$. Let $E^\sim \to E$ be a normal covering with translation group π. With these data we can associate a map

$$\text{(6.8.1)} \qquad B \longrightarrow \Omega^\infty \mathbf{bhm}_\pi(k)$$

where $\mathbf{bhm}_\pi(k)$ is the (topological, connective) K–theory of f.g. projective $\mathbb{R}\pi$–modules with nondegenerate $(-1)^k$–hermitian form [Wa1], [Wa4]. The map (6.8.1) stably classifies the hermitian bundle on B with fiber $H^k(E^\sim_x; \mathbb{R})$ over $x \in B$. There is a *hyperbolic map* [Wa4] from $\mathbf{bo}_\pi$, the (topological, connective) K–theory of f.g. projective $\mathbb{R}\pi$–modules, to $\mathbf{bhm}_\pi(k)$. Let $\mathbf{bhm}_\pi(k)/\mathbf{bo}_\pi$ be its mapping cone. The map

$$\text{(6.8.2)} \qquad B \longrightarrow \Omega^\infty (\mathbf{bhm}_\pi(k)/\mathbf{bo}_\pi)$$

obtained by composing (6.8.1) with Ω^∞ of $\mathbf{bhm}_\pi(k) \to \mathbf{bhm}_\pi(k)/\mathbf{bo}_\pi$ is the *family multisignature* of p. Now suppose that p admits a block bundle structure; see §5.1. Then (6.8.2) factors *rationally* as

$$B \longrightarrow \Omega^\infty (\mathbf{bhm}_e(k)/\mathbf{bo}_e) \xrightarrow{\otimes \mathbb{R}\pi} \Omega^\infty (\mathbf{bhm}_\pi(k)/\mathbf{bo}_\pi)$$

where e is the trivial group. The case $B = *$ appears in [Wa1, §13B]. The general statement can be proved by expressing the rationalized family multisignature in terms of the family visible symmetric signature of §5.1.

6.9. Relative calculations of h–structure Spaces. Using §1.6 and 4.2.1 one obtains, in the case where M is smooth, a diagram

$$\mathcal{S}_d(M^n) \xrightarrow{\nabla} \mathcal{S}_d(\tau) \xrightarrow{\lambda} \Omega^{\infty+n}\mathbf{LA}^h_\bullet(M,n) \tag{6.9.1}$$

which is a homotopy fiber sequence in the concordance stable range; more precisely, the composite map in (6.9.1) is trivial and the resulting map from $\mathcal{S}_d(M)$ to the homotopy fiber of λ is approximately $(n-1)/3$–connected. This formulation has some relative variants which are attractive because relative LA–theory is often easier to describe than absolute LA–theory, and the estimates available for the relative concordance stable range are often better than those for the absolute concordance stable range.

Illustration: Let M be smooth, compact, connected, with connected boundary, and suppose the inclusion $\partial M \to M$ induces an isomorphism of fundamental groups. Let $M_0 = M \smallsetminus \partial M$. Define $\mathcal{S}_d(M_0)$ as the Space of pairs (N, f) where $f : N \to M_0$ is a proper homotopy equivalence of smooth manifolds without boundary. Let $\mathcal{S}_d(\tau_0)$ be the corresponding Space of n–dimensional vector bundles on M_0 equipped with a stable fiber homotopy equivalence to $\tau_0 := \tau|M_0$. Using some controlled L–theory and algebraic K–theory of spaces, one obtains the following variation on (6.9.1): a diagram

$$\mathcal{S}_d(M_0) \xrightarrow{\nabla} \mathcal{S}_d(\tau_0) \longrightarrow \Omega^{\infty+n}(\mathbf{LA}^h_\bullet(M,n)/\mathbf{LA}^h_\bullet(\partial M,n)) \tag{6.9.2}$$

which is a homotopy fiber sequence in the $\leq (n/3 - c_1)$ range, where c_1 is a constant independent of n. Our hypothesis on fundamental groups implies that $\mathbf{LA}^h_\bullet(M,n)/\mathbf{LA}^h_\bullet(\partial M,n) \simeq (\mathbb{S}^n_! \wedge (\mathbf{A}(M)/\mathbf{A}(\partial M)))_{h\mathbb{Z}/2}$ and [BHM], [Go3] imply

$$\mathbf{A}(M)/\mathbf{A}(\partial M) \simeq \mathbf{TC}(M)/\mathbf{TC}(\partial M)$$

where $\mathbf{TC}$ is the *topological cyclic homology* viewed as a spectrum–valued functor. See also [Ma1], [Ma2], [HeMa]. So (6.9.2) simplifies to

$$\mathcal{S}_d(M_0) \xrightarrow{\nabla} \mathcal{S}_d(\tau_0) \longrightarrow \Omega^{\infty+n}((\mathbb{S}^n_! \wedge (\mathbf{TC}(M)/\mathbf{TC}(\partial M)))_{h\mathbb{Z}/2})\,.$$

This is still a homotopy fiber sequence in the $\leq (n/3 - c_1)$ range. Unpublished results of T.Goodwillie and G.Meng indicate that it is a fibration sequence in the $\leq n - c_2$ range for some constant c_2, provided $\partial M \to M$ is 2–connected.

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Spaces of smooth embeddings, disjunction and surgery

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Abstract. We describe progress in the theory of smooth embeddings over more than 50 years, starting with Whitney's embedding theorem, continuing with the generalized Whitney tricks of Haefliger and Dax, early disjunction results for embeddings due to Hatcher and Quinn, the surgery methods for constructing embeddings due to Browder and Levine, respectively, moving on to a systematic theory of multiple disjunction which builds on all the foregoing, and concluding with a functor calculus approach which reformulates the main theorem on multiple disjunction as a convergence theorem. Convergence takes place when the codimension is at least 3, giving a decomposition of the space of embeddings under scrutiny into 'homogeneous layers' which admit an attractive combinatorial description. The divergent cases are not devoid of interest, since they suggest a view of low–dimensional topology as a 'divergent' analogue of high–dimensional topology.

Section headings

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0. Preliminaries

0.1. Overview

This survey traces the development, over more than 50 years, of a theory of smooth embeddings resting today on two pillars: the methods of disjunction and surgery. More precisely, the theory is about homotopical and homological properties of spaces of smooth embeddings $\mathrm{emb}(M^m, N^n)$. It is more satisfactory when $n - m \geq 3$, but has something to offer in the other cases, too.

Chapter 1 is about embeddings in the metastable range, $m < 2n/3$ approximately, and the idea of producing an embedding $M \to N$ by starting with an immersion and removing self–intersections. This goes back to Whitney [Wh2], of course, and was pursued further by Haefliger [Hae1], [Hae2], Dax [Da], and Hatcher–Quinn [HaQ]. In the process, two important new insights emerged. The first of these [Hae2] is that embeddings in the metastable range are determined up to isotopy by their local behavior. However, this is only true with an unusual definition of *local* where the loci are small tubular neighborhoods of subsets of M of cardinality 1 or 2. The second insight [HaQ] is that, in the metastable range, practically any method for disjunction (here: removing mutual intersections of two embedded manifolds in a third by subjecting the embedded manifolds to isotopies) can serve as a method for removing self–intersections of one manifold in another.

Chapter 2, about surgery methods for constructing embeddings of M in N, gives about equal weight to the Browder approach [Br2], which is to start with an embedding $M \to N'$ and a degree one normal map $N' \to N$, and the slightly older Levine approach [Lev], which is to start with a degree one normal map $M' \to M$ and an embedding $M' \to N$. The Browder approach leads eventually to the Browder–Casson–Sullivan–Wall theorem which, assuming $n - m \geq 3$ and $n \geq 5$, essentially expresses the block embedding space $\mathrm{emb}^{\sim}(M, N)$, a rough approximation to $\mathrm{emb}(M, N)$, in terms of the space of Poincaré duality (block) embeddings from M to N. The Levine approach does not give such a neat reduction, but in contrast to the Browder approach it does lead to some ideas on how to construct embeddings of one Poincaré duality space in another. These ideas inspired work by Williams [Wi], Richter [Ric], and more recently by Klein [Kl1], [Kl2], [Kl3], which is summarized in the later parts of chapter 2.

Chapter 3 is a systematic account of multiple disjunction alias higher excision (here: an obstruction theory for making a finite number of submanifolds $M_i \subset N$ pairwise disjoint by subjecting them to isotopies in N). The most difficult ingredient is [Go1], a multiple disjunction theorem for smooth concordance embeddings (concordances alias pseudo–isotopies from a fixed smooth embedding $f_0\colon M \to N$ to a variable one, $f_1\colon M \to N$). Another important ingredient is a multiple disjunction theorem for (spaces of) Poincaré embeddings [GoKl], which uses [Go6] and some of the results described at the end of chapter 2. Via the Browder–Casson–Sullivan–Wall theorem, this leads to a disjunction theorem for block embedding spaces, which combines well with the aforementioned multiple disjunction theorem for concordance embeddings, resulting in a multiple disjunction theorem for honest embeddings. See [Go7].

In chapter 4, we take up and develop further Haefliger's localization ideas described in chapter 1. Specifically, we construct a sequence of approximations $T_k\,\mathrm{emb}(M, N)$ to $\mathrm{emb}(M, N)$. A point in $T_k\,\mathrm{emb}(M, N)$ is a coherent family of embeddings $V \to N$, where V runs through the tubular neighborhoods of subsets of M of cardinality $\leq k$; in particular, $T_2\,\mathrm{emb}(M, N)$ is Haefliger's approximation to $\mathrm{emb}(M, N)$, and $T_1\,\mathrm{emb}(M, N)$ is homotopy equivalent to the space of smooth immersions from $M \to N$, if $m < n$. Just as Hatcher–Quinn disjunction can be used to prove that the Haefliger approximation is a good one, so the higher disjunction results of chapter 3 are used to show that the approximations $T_k\,\mathrm{emb}(M, N)$ converge to $\mathrm{emb}(M, N)$ as $k \to \infty$, provided $n - m \geq 3$. Actually, in the cases when $2m < n-2$, only a very easy result from chapter 3 is used. In all cases, the relative homotopy of the forgetful maps $T_k\,\mathrm{emb}(M, N) \to T_{k-1}\,\mathrm{emb}(M, N)$ is fairly manageable.

Chapter 5 applies the same localization ideas to the (generalized) homology of $\mathrm{emb}(M, N)$. What we get turns out to be a generalization of the generalization due to Rector [Re] and Bousfield [Bou] of the Eilenberg–Moore spectral sequence [EM]. The convergence issue is more complex in this case, but we have a satisfactory result for the cases where $n > 2m+1$. For $m = 1$ and $n \geq 3$ we make a connection with the Vassiliev theory of knot invariants [Va1], [Va2], [Va3], [BaN], [BaNSt], [Ko].

0.2. Notation, Terminology

Sets. Given a set X and $x \in X$, we often write x for the subset $\{x\}$. In particular, if $x_1, x_2, ...$ are elements of X and $f\colon X \to Y$ is any map, we may write $f|x_1$ and $f|x_1 \cup x_2$ etc. for the restrictions of f to $\{x_1\}$, $\{x_1, x_2\}$ etc. .

Spaces. All spaces in sight are understood to be compactly generated weak Hausdorff. (A space X is compactly generated weak Hausdorff if and only if the canonical map $\mathrm{colim}_{K\subset X} K \to X$, with K ranging over the compact Hausdorff subspaces of X, is a homeomorphism). Products and mapping spaces are formed in the category of such spaces in the usual way, and are related by adjunction. Pointed spaces (alias based spaces) are understood to have nondegenerate basepoints.

As is customary, we write QX for $\Omega^\infty\Sigma^\infty X$ where X is a based space, and $Q(X_+)$ or $Q_+(X)$ for $\Omega^\infty\Sigma^\infty(X_+)$ where X is unbased. Occasionally we will need a twisted version of $Q_+(X)$, as follows. Suppose that X is finite dimensional, and equipped with two real vector bundles ζ and ξ. Choose a vector bundle monomorphism $\xi \to \varepsilon^i$ where ε^i is a trivial vector bundle on X. Let

$$Q_+(X; \zeta - \xi)$$

be $\Omega^i Q$ of the Thom space of the vector bundle $\zeta \oplus \varepsilon^i/\xi$ on X. This is essentially independent of the choice of vector bundle monomorphism $\xi \to \varepsilon$ made. We will also use this notation when X is infinite dimensional, and the bundle ξ is in some obvious way pulled back from a finite dimensional space.

More generally, with X, ζ, ξ as before and $A \subset X$ a closed subset for which the inclusion is a cofibration, we let

$$Q(X/A; \zeta - \xi)$$

be $\Omega^i Q$ of a certain quotient of Thom spaces (Thom space of $\zeta \oplus \varepsilon^i/\xi$ on X, modulo Thom space of the restriction of $\zeta \oplus \varepsilon^i/\xi$ to A).

Cubical diagrams. Let S be a finite set. An S–cube of spaces is a covariant functor $R \mapsto X(R)$ from the poset of subsets of S to spaces. It is k*–cartesian* if the canonical map (whose homotopy fibers are the *total homotopy fibers* of X)

$$X(\emptyset) \longrightarrow \operatorname*{holim}_{R \neq \emptyset} X(R)$$

is k–connected. (Here the homotopy inverse limit can be described explicitly as the space of natural transformations from $R \mapsto \Delta(R)$ to $R \mapsto X(R)$, where $\Delta(R)$ is the simplex of dimension $|R|-1$ spanned by R, assuming $R \neq \emptyset$.) The cube is k*–cocartesian* if the canonical map (whose homotopy cofiber is the *total homotopy cofiber* of X)

$$\operatorname*{hocolim}_{R \neq S} X(R) \longrightarrow X(S)$$

is k–connected. (Here the homotopy colimit can be described explicitly as the quotient of $\coprod_{R \neq S} \Delta(S \smallsetminus R) \times X(R)$ by relations $(i_* a, b) \simeq (a, i_* b)$ where $i\colon R_1 \to R_2$ is an inclusion of proper subsets of S.) In both cases, $k = \infty$ is allowed. If X is a functor from the poset of subsets of S to pointed spaces, then the canonical map $X(\emptyset) \to \operatorname{holim}_{R \neq \emptyset} X(R)$ is pointed; its homotopy fiber over the base point will be called *the* total homotopy fiber of X.

The poset of subsets of S is isomorphic to its own opposite, so we use similar language for contravariant functors from it to spaces.

An S–cube is *strongly* ∞*–cocartesian* if all its 2–dimensional subcubes are ∞–cocartesian, and *strongly* ∞*–cartesian* if all its 2–dimensional subcubes are ∞–cartesian. For $|S| \geq 2$, strongly ∞–cocartesian/cartesian implies ∞–cocartesian/cartesian.

A contravariant S–cube X of spaces in which $S = \{1, \ldots, n-1\}$ is called an n*–ad* if the maps from $X(R)$ to $X(\emptyset)$ are inclusions, for any $R \subset S$, and $X(R) = \bigcap_{i \in R} X(i) \subset X_\emptyset$. The n–ad is *special* if $X(S) = \emptyset$. The n–ad is a *manifold* n–ad if each $X(R)$ is a manifold with boundary $\bigcup_{i \notin R} X(R \cup i)$. In the smooth setting, each $X(R)$ is required to be a smooth manifold with appropriate corners in the boundary.

Homotopy (co–)limits. For homotopy limits and homotopy colimits in general, see [BK]. We like the point of view of [Dr] and [DwK2], which is as follows, in outline. A functor E from a small category $\mathcal{C}$ to spaces is a *CW–functor* if it is a monotone union of subfunctors E_{-1}, E_0, E_1, E_2, $\ldots$, where E_i has been obtained from E_{i-1} by attachment of so–called i–cells. These are functors of the form $c \mapsto \mathbb{D}^i \times \operatorname{mor}_{\mathcal{C}}(c, d)$, for some d in $\mathcal{C}$. Every functor F from $\mathcal{C}$ to spaces has a CW–approximation $F^{\sim} \to F$ (in which $F^{\sim}$ is a CW–functor, and $F^{\sim} \to F$ specializes to weak homotopy equivalences $F^{\sim}(c) \to F(c)$ for each c in $\mathcal{C}$). Put

$$\text{hocolim } F := \text{colim } F^{\sim}\,, \qquad \text{holim } F := \text{nat}((*_{\mathcal{C}})^{\sim}, F)$$

where nat denotes the space of natural transformations and $*_{\mathcal{C}}$ is the constant functor $c \mapsto *$ on $\mathcal{C}$. For colimits, see [MaL]. For more on homotopy limits and homotopy colimits, see also [DwK1].

MANIFOLDS. All manifolds in this survey are assumed to have a countable base for their topology. Manifolds are without boundary unless otherwise stated; a manifold *with boundary* may of course have empty boundary.

We write $\text{emb}(M, N)$ for the space of smooth embeddings from M to N, and $\text{imm}(M, N)$ for the space of smooth immersions, both defined as geometric realizations of certain simplicial sets. Unless otherwise stated, M and N are assumed to be smooth without boundary.

Let G be a finite group. A map $f\colon K \to L$ of manifolds with G–action is *equivariant* if it is a G–map, and *isovariant* if, in addition, $f^{-1}(L^H) = K^H$ for every subgroup $H \leq G$. If K, L are smooth and f is a smooth map, it is natural to combine isovariance as above with "infinitesimal" isovariance: call f *strongly isovariant* if it is isovariant and, for each $H \leq G$ and $x \in K^H$, the differential $T_x f$ of f at x is an isovariant linear map from $T_x K$ to $T_{f(x)} L$.

POINCARÉ SPACES. Poincaré space is short for simple Poincaré duality space, alias simple Poincaré complex [Wa2, 2nd ed., §2]; Poincaré pair is short for simple Poincaré duality pair. The fundamental class $[X]$ of a Poincaré space X of formal dimension n lives in $H_n(X; \mathbb{Z}^t)$, where $\mathbb{Z}^t$ denotes a local coefficient system on X with fibers isomorphic to $\mathbb{Z}$. Together, $[X]$ and $\mathbb{Z}^t$ are determined by X, up to a unique isomorphism between local coefficient systems on X.

What is more, there exist a fibration ν^k on X with fibers $\simeq \mathbb{S}^{k-1}$, and a 'degree one' map ρ from $\mathbb{S}^{n+k}$ to the Thom space (mapping cone) of ν ; together, ν and ρ are unique up to contractible choice if k is allowed to tend to ∞. See [Br3], [Ra]. The fibration ν is known as the Spivak normal fibration of X. The image of $[\rho]$ under Hurewicz homomorphism and Thom isomorphism is a fundamental class in $H_n(X; \mathbb{Z}^t)$ where $\mathbb{Z}^t$ is the twisted integer coefficient system associated with ν. Something analogous is true for Poincaré pairs.

1. Double point obstructions

1.1. The Whitney embedding theorem

1.1.1. Theorem [Wh2]**.** *For $m > 0$, every smooth m–manifold M can be embedded in $\mathbb{R}^{2m}$.*

Whitney's proof of 1.1.1 relies on the fact [Wh1] that M^m can be immersed in $\mathbb{R}^{2m}$. He also knew [Wh1] that any immersion $M^m \to \mathbb{R}^{2m}$ can be approximated by one with transverse self–intersections. The other main ideas are these:

- (i) Without loss of generality, M^m is connected. Suppose that M is also closed. Then any immersion $f\colon M^m \to \mathbb{R}^{2m}$ has an algebraic self–intersection number I_f (to be defined below) which is an integer if m is even and orientable, otherwise an integer modulo 2.
- (ii) (*Whitney trick*) In the situation of (i), the immersion f is regularly homotopic to an immersion with exactly $|I_f|$ transverse self–intersections (and no other self–intersections), provided $m > 2$. Here $|I_f|$ should be read as 0 or 1 if $I_f \in \mathbb{Z}/2$.
- (iii) For every $m > 0$, there exists an immersion $g\colon \mathbb{S}^m \longrightarrow \mathbb{R}^{2m}$ having algebraic self–intersection number $I_g = 1$.

Assuming (i), (ii), (iii), the proof of 1.1.1 for $m > 2$ is completed as follows. We start by choosing some immersion $f_0\colon M^m \to \mathbb{R}^{2m}$. In the closed connected case, we use (iii) to modify it, so that an immersion $f\colon M \to \mathbb{R}^{2m}$ with $I_f = 0$ results. Then (ii) can be applied. In the case where M is open and connected, and all self–intersections are transverse, it is easy to "indent" M appropriately, i.e. to find an embedding $e\colon M \to M$ isotopic to the identity such that $f := f_0 e$ is an embedding. See [Wh2, §8] for details.

1.1.2. Definitions. Whitney gives two definitions of I_f. For the first, assume that $f\colon M \to \mathbb{R}^{2m}$ is an immersion with transverse self–intersections only. Count the self–intersections (with appropriate sign ± 1 if m is even and M is orientable, otherwise modulo 2). The result is I_f.

For the second definition, let $f\colon M \to \mathbb{R}^{2m}$ be any immersion. Define

$$\beta\colon M \times M \smallsetminus \Delta_M \longrightarrow \mathbb{R}^{2m}$$

by $\beta(x,y) := f(x) - f(y)$. Then $\beta^{-1}(0)$ is compact and β is $\mathbb{Z}/2$–equivariant, where the generator of $\mathbb{Z}/2$ acts on the domain (freely) and codomain (not freely) by $(x,y) \mapsto (y,x)$ and by $z \mapsto -z$, respectively.

Hence β has a well defined degree I_f in $\mathbb{Z}$ or $\mathbb{Z}/2$. It can be found by deforming β in a small neighborhood of $\beta^{-1}(0)$ so that β becomes transverse to 0, and counting $\mathbb{Z}/2$–orbits in the inverse image of 0 (with appropriate signs when m is even and M is orientable).

We assume that statements (ii) and (iii) above are well known through [Mi1]. We will see plenty of generalizations quite soon.

Remark. Whitney's I_f has precursors in [van]. See also [Sha].

1.2. Scanning

The theorem of Haefliger that we are about to present dates back to the early sixties. The immersion classification theorem was available [Sm1], [Hi1]; see also [Hae3]. It states that if M^m and N^n are smooth, $m < n$, or $m = n$ and M open, then an evident map from $\mathrm{imm}(M,N)$ to the space of pairs (f,g), with $f\colon M \to N$ continuous and $g\colon \tau_M \to f^*\tau_N$ fiberwise monomorphic (and linear), is a (weak) homotopy equivalence. In addition, transversality concepts had conquered differential topology. In particular, it was known that a "generic" smooth immersion $M^m \to N^n$ would have transverse self–intersections only, of multiplicity $\leq n/(n-m)$. It was therefore natural for Haefliger to impose the condition $n/(n-m) < 3$, equivalently $m < 2n/3$ (*metastable range*), which ensures that all self–intersection points in a generic immersion $M \to N$ are double points, and to view an embedding $M \to N$ as an immersion without double points.

Notation. In 1.2.1 below we write $\mathrm{map}(\dots)$, $\mathrm{map}^G(\dots)$, $\mathrm{ivmap}^G(\dots)$ for spaces of smooth maps, equivariant smooth maps, strongly isovariant smooth maps, respectively, all to be defined as geometric realizations of simplicial sets.

1.2.1. Theorem [Hae2]**.** *If $m+1 < 2n/3$, then the following square is 1–cartesian :*

$$\begin{array}{ccc}
\mathrm{emb}(M,N) & \xrightarrow{\subset} & \mathrm{map}(M,N) \\
\big\downarrow & & \big\downarrow{\scriptstyle f\mapsto f\times f} \\
\mathrm{ivmap}^{\mathbb{Z}/2}(M\times M, N\times N) & \xrightarrow{\subset} & \mathrm{map}^{\mathbb{Z}/2}(M\times M, N\times N)\,.
\end{array}$$

Remark. Haefliger's original statement is slightly different: in his definitions of the mapping spaces involved, other than $\mathrm{emb}(M,N)$, he does not ask for *smooth* maps. A vector bundle theoretic argument [HaeH, 4.3.a] shows that the two versions are equivalent.

It will turn out that the square in 1.2.1 is $(2n-3-3m)$–cartesian, an improvement which is essentially due to Dax [Da]. We will sketch the proof in section 1.3, and again in section 1.4, following Dax more closely.

1.2.2. Example [Hae2]. Let $N = \mathbb{R}^n$. Then $\mathrm{map}(M,N)$ is contractible and so is $\mathrm{map}^{\mathbb{Z}/2}(M \times M, N \times N) \cong \mathrm{map}(M \times M, N)$. Therefore 1.2.1 implies that

$$\mathrm{emb}(M, \mathbb{R}^n) \longrightarrow \mathrm{ivmap}^{\mathbb{Z}/2}(M \times M, \mathbb{R}^n \times \mathbb{R}^n)$$

given by $f \mapsto f \times f$ is 1–connected, if $m+1 < 2n/3$. Now an isovariant map g from $M \times M$ to $\mathbb{R}^n \times \mathbb{R}^n$ determines an equivariant map vgj from $M \times M \smallsetminus \Delta_M$ to $\mathbb{S}^{n-1}$, where $j\colon M \times M \smallsetminus \Delta_M \to M \times M$ is the inclusion and v is the map $(x,y) \mapsto (x-y)/|x-y|$ from $\mathbb{R}^n \times \mathbb{R}^n$ minus diagonal to $\mathbb{S}^{n-1}$. It follows easily from [HaeH, 4.3.a] that $g \mapsto vgj$ is 1–connected if $m+1 < 2n/3$. Hence isotopy classes of smooth embeddings of M^m in $\mathbb{R}^n$, for $m+1 < 2n/3$, are in bijection with homotopy classes of equivariant maps from $M \times M \smallsetminus \Delta_M$ to $\mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is equipped with the antipodal action of $\mathbb{Z}/2$.

We now briefly justify our use of the word *scanning* in the title of this subsection. The upper horizontal map in the diagram in 1.2.1 captures, for each $f \in \mathrm{emb}(M,N)$, the restricted embeddings $f|S$ where S runs through the one–element subsets of M. The left–hand vertical map captures, for each $f \in \mathrm{emb}(M,N)$, the restricted embeddings $f|S$ where S runs through the 2–element subsets of M (the two elements are allowed to 'collide'); it also captures the tangent bundle monomorphism induced by f. The remaining two arrows capture coherence.

1.3. Disjunction

Disjunction theory, as we understand it here, is about the elimination of intersections of two or more manifolds, each embedded in a common ambient manifold, by means of isotopies of the embedded manifolds. Families of such elimination problems are also considered. An important theme is that disjunction homotopies can often be improved to disjunction isotopies, as in the following theorem.

1.3.1. Theorem. *Let L^ℓ, M^m, N^n be smooth, L and M closed, L contained in N as a smooth submanifold. The following square of inclusion* *maps is* $(2n-3-2m-\ell)$*–cartesian:*

$$
\begin{array}{ccc}
\mathrm{emb}(M, N \smallsetminus L) & \longrightarrow & \mathrm{emb}(M,N) \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{map}(M, N \smallsetminus L) & \longrightarrow & \mathrm{map}(M,N).
\end{array}
$$

Idea of proof. Let $\{h_t\colon M \times \Delta^k \to N \mid 0 \le t \le 1\}$ be a smooth homotopy such that $h_0|M \times y$ is an embedding and $h_1|M \times y$ has image in $N \smallsetminus L$, for all $y \in \Delta^k$. Suppose also that $h = \{h_t\}$ is a constant homotopy on $M \times \partial\Delta^k$. Let $Z \subset M \times \Delta^k \times [0,1]$ consist of all points (x,y,t) such that $h_s|M \times y$ is singular at $x \in M$ for some $s \le t$. If $h = \{h_t\}$ is 'generic' and k is not too large, for example $k \le (2n-3-2m-\ell)-1$, then Z will have empty intersection with $h^{-1}(L)$. Then it is easy to find a smooth function $\psi\colon M \times \Delta^k \to [0,1]$ such that Z lies above the graph of ψ, and $h^{-1}(L)$ lies below it. Using this, one deforms h to the homotopy $h^!$ given by $h^!_t(x,y) = h_{\psi(x)t}(x,y)$. Now $h^!$ is adjoint to a homotopy of maps $\Delta^k \to \mathrm{emb}(M,N)$ and $h^!_1$ is adjoint to a map $\Delta^k \to \mathrm{emb}(M, N \smallsetminus L)$. This shows that the square in 1.3.1 is k–cartesian, with $k = (2n-3-2m-\ell)-1$. A little extra work improves the estimate to $2n-3-2m-\ell$. $\square$

Earlier results in the direction of 1.3.1 can be found in [Sta], [Wa1], [Lau1], [Ti], [Lau2] and [Lau3]. The method of proof is a simple example of *sunny collapsing*, an idea which appears to originate in Zeeman's PL unknotting work [Ze]; see also [Hu1].

1.3.2. Corollary. *Let* L^ℓ, M^m, N^n *be smooth,* L *and* M *closed,* L *and* M *contained in* N *as smooth submanifolds. The homotopy fiber of*

$$\mathrm{emb}(L \amalg M, N) \to \mathrm{emb}(L,N) \times \mathrm{emb}(M,N)$$

has a $\min\{2n-2m-\ell-3, 2n-2\ell-m-3\}$*–connected scanning map to the section space* $\Gamma(u)$ *, where* u *is a fibration over* $M \times L$ *with fiber over* (x,y) *equal to the homotopy fiber of*

$$\mathrm{emb}(x \amalg y, N) \to \mathrm{emb}(x,N) \times \mathrm{emb}(y,N).$$

Proof. We use a Fubini type argument. First, scan along M. The homotopy fiber of $\mathrm{emb}(L \amalg M, N) \to \mathrm{emb}(L,N) \times \mathrm{emb}(M,N)$ is homotopy equivalent to the homotopy fiber of $\mathrm{emb}(M, N \smallsetminus L) \hookrightarrow \mathrm{emb}(M,N)$. The homotopy fiber of $\mathrm{emb}(L \amalg x, N) \to \mathrm{emb}(L, N) \times \mathrm{emb}(x, N)$ is homotopy equivalent to the homotopy fiber of $\mathrm{emb}(x, N \smallsetminus L) \hookrightarrow \mathrm{emb}(x, N)$, for every $x \in M$. So by 1.3.1, scanning along M gives a $(2n-2m-\ell-3)$–connected map from the homotopy fiber of $\mathrm{emb}(L \amalg M, N) \to \mathrm{emb}(L, N) \times \mathrm{emb}(M, N)$ to $\Gamma(v)$, where v is a fibration on M whose fiber over $x \in M$ is the homotopy fiber of $\mathrm{emb}(L \amalg x, N) \to \mathrm{emb}(L, N) \times \mathrm{emb}(x, N)$.

We get from $\Gamma(v)$ to $\Gamma(u)$ by scanning along L. Note that for each x in M, the homotopy fiber of $\mathrm{emb}(L \amalg x, N) \to \mathrm{emb}(L, N) \times \mathrm{emb}(x, N)$ is homotopy equivalent to the homotopy fiber of the inclusion of $\mathrm{emb}(L, N \smallsetminus x)$ in $\mathrm{emb}(L, N)$. Therefore another application of 1.3.1 shows that our second scanning map is $((2n - 2\ell - 0 - 3) - m)$–connected. Hence the composite scanning map is $\min\{2n - 2m - \ell - 3, 2n - 2\ell - m - 3\}$–connected. $\square$

Terminology. Eventually we will need a relative version of 1.3.2. In the most general relative version, N is a manifold with boundary, and L, M are compact triads. For L this means that ∂L is the union of smooth codimension zero submanifolds $\partial_0 L$ and $\partial_1 L$ with

$$\partial\partial_0 L = \partial\partial_1 L = \partial_0 L \cap \partial_1 L\,.$$

L is viewed as a manifold with corners (corner set $\partial_0 L \cap \partial_1 L$). We assume that L is contained in N in such a way that $\partial_0 L = L \cap \partial N$ and the inclusion $\partial_1 L \hookrightarrow N$ is transverse to ∂N. We make analogous assumptions for M and the inclusion $M \to N$. In addition, we assume that $\partial_0 M$ and $\partial_0 L$ are disjoint, and allow only embeddings $M \to N$ and $L \to N$ which agree with the inclusions on $\partial_0 M$ and $\partial_0 L$ respectively. The appropriate section space $\Gamma(u)$ consists of sections of a certain fibration on $M \times L$ as before, but the sections are prescribed on $(\partial_0 M \times L) \cup (M \times \partial_0 L)$.

1.3.3. Corollary. *The square in* 1.2.1 *is* $(2n - 3 - 3m)$ *—connected.*

Proof, in outline. Let $\mathrm{emb}_h(M, N)$ be the Haefliger approximation to $\mathrm{emb}(M, N)$. That is, $\mathrm{emb}_h(M, N)$ is the homotopy pullback of the lower left hand, upper right hand and lower right hand terms in 1.2.1. We have to show that Haefliger's map

$$\mathrm{emb}(M, N) \to \mathrm{emb}_h(M, N)$$

is $(2n - 3 - 3m)$–connected. It suffices to establish this in the case where $M = \bar{M} \smallsetminus \partial\bar{M}$ for a compact smooth manifold $\bar{M}$ with boundary. We can suppose that $\bar{M}$ comes with a handle decomposition. More specifically, suppose the handles are all of index $\leq r$, and the number of handles of index r is a_r. We proceed by induction on r, and for fixed r we proceed by induction on a_r.

Choose a handle $H \cong \mathbb{D}^r \times \mathbb{D}^{m-r}$ in M of maximal index r. If $r=0$ there is not much to prove, so we assume $r>0$. We can then choose two disjoint index r subhandles H_1 and H_2 of H. (In the coordinates for H, these would correspond to $C_1 \times \mathbb{D}^{m-r}$ and $C_2 \times \mathbb{D}^{m-r}$ where C_1 and C_2 are small disjoint disks in $\mathbb{D}^r$.)

Let $M_i = M \smallsetminus H_i$ for $i=1,2$, and $M_T = \cap_{i\in T} M_i$ for $T \subset \{1,2\}$. For $T \neq \emptyset$, the closure of M_T in $\bar{M}$ has a handle decomposition with fewer than a_r handles of index r, and no handles of index $>r$. By induction, $\mathrm{emb}(M_T, N) \to \mathrm{emb}_h(M_T, N)$ is $(2n-3-3m)$–connected for $T \neq \emptyset$.

The spaces $\mathrm{emb}(M_T, N)$ and the restriction maps between them form a commutative square, denoted $\mathrm{emb}(M_\bullet, N)$. We have another commutative square $\mathrm{emb}_h(M_\bullet, N)$ and a Haefliger map

$$\mathrm{emb}(M_\bullet, N) \longrightarrow \mathrm{emb}_h(M_\bullet, N)\,.$$

Looking at the induced map from any of the total homotopy fibers of $\mathrm{emb}(M_\bullet, N)$ to the corresponding homotopy fiber of $\mathrm{emb}_h(M_\bullet, N)$, one finds that it is an instance of scanning essentially as in 1.3.2 (see the details just below). By 1.3.4, it is $(2n-3m-3)$–connected. Combined with the inductive assumption, that the map $\mathrm{emb}(M_T, N) \longrightarrow \mathrm{emb}_h(M_T, N)$ is $(2n-3m-3)$–connected for $T \neq \emptyset$, this shows that Haefliger's map $\mathrm{emb}(M_T, N) \longrightarrow \mathrm{emb}_h(M_T, N)$ is also $(2n-3m-3)$–connected when T is empty. $\square$

Details. To understand the total homotopy fibers of $\mathrm{emb}(M_\bullet, N)$ in the above proof, replace $\mathrm{emb}(M_\bullet, N)$ by $\mathrm{emb}(\bar{M}_\bullet, N)$ where $\bar{M}_T$ is the closure of M_T in $\bar{M}$. (Our notation $\mathrm{emb}(\bar{M}_T, N)$ is legalized by the remark just before 1.3.3, provided we decree $\partial_0 \bar{M}_T = \emptyset$.) By the isotopy extension theorem, all maps in $\mathrm{emb}(\bar{M}_\bullet, N)$ are fibrations. Hence we can obtain all total homotopy fibers as homotopy fibers of subsquares of the form $\mathrm{emb}(\bar{M}_\bullet, N; g)$, where $g\colon \bar{M}_{\{1,2\}} \to N$ is an embedding and $\mathrm{emb}(\bar{M}_T, N; g)$ denotes the space of embeddings $\bar{M}_T \to N$ extending g. Modulo natural homotopy equivalences, these subsquares can be rewritten in the form $\mathrm{emb}(H_\bullet, N_g)$ where $H_T = \cup_{i\in T} H_i$ for $T \subset \{1,2\}$ and $N_g \subset N$ is the closure of the complement of a thickening of $\mathrm{im}(g)$ in N. Boundary conditions are understood: $\partial_0 H_i = H_i \cap \bar{M}_i$. With that, we are in the situation of 1.3.2 (relative version) and obtain a scanning map to a section space $\Gamma(u_g)$, where u_g is a fibration on $H_1 \times H_2$.

We can also recast the relevant total homotopy fibers of the square $\mathrm{emb}_h(\bar{M}_\bullet, N)$ as total homotopy fibers of subsquares $\mathrm{emb}_h(\bar{M}_\bullet, N; g)$ with g as before. Here $\mathrm{emb}_h(\bar{M}_T, N; g)$ is the fiber of $\mathrm{emb}_h(\bar{M}_T, N)$ over the element of $\mathrm{emb}_h(\bar{M}_{\{1,2\}}, N)$ determined by the embedding g. There is a scanning map (which is a homotopy equivalence) from the total homotopy fiber of $\mathrm{emb}_h(\bar{M}_\bullet, N; g)$ to a section space $\Gamma(v_g)$. The sections are subject to boundary conditions as usual. Again v_g is a fibration on $H_1 \times H_2$, containing u_g. The fiber of v_g over $(x, y) \in H_1 \times H_2$ is

$$\begin{aligned} &\mathrm{hofiber}\,[\,\mathrm{emb}_h(x \amalg y, N) \longrightarrow \mathrm{emb}_h(x, N) \times \mathrm{emb}_h(y, N)\,] \\ \simeq\; &\mathrm{hofiber}\,[\,\mathrm{emb}(x \amalg y, N) \longrightarrow \mathrm{emb}(x, N) \times \mathrm{emb}(y, N)\,]\,. \end{aligned}$$

The inclusion $u_g \to v_g$ is not a fiber homotopy equivalence in general (because N_g is not the same as N), but it is $(2n - m - 3)$–connected on fibers. Hence the induced map $\Gamma(u_g) \to \Gamma(v_g)$ is $(2n - 3m - 3)$–connected. $\square$

1.4. The stable point of view

Although Haefliger's scanning idea was a new departure, his proof of 1.2.1 used "conservative" double point elimination methods as in 1.1. About ten years later, Dax [Da] and Hatcher–Quinn [HaQ] developed the double point elimination methods into a full–blown theory, of which we want to give an idea. (See [Sa] and [LLZ] and for the analogous double point elimination approach to block embedding spaces $\mathrm{emb}^{\sim}(M, N)$, defined in 2.2 below.)

Suppose that $f\colon M \to N$ is any smooth immersion which is generic (the tangent spaces of M at self–intersection points in N are in general position). Suppose that M is closed. Let $E^{\curlyvee}(f, f)$ be the space of triples (x, y, ω) where $(x, y) \in M \times M \smallsetminus \Delta_M$ and $\omega\colon [-1, +1] \to N$ is a path from $f(x)$ to $f(y)$ in N. Think of it as a space over $M \times M \smallsetminus \Delta_M$. There is an involution on $E^{\curlyvee}(f, f)$ given by $(x, y, \omega) \mapsto (y, x, \omega^{-1})$, where ω^{-1} is ω in reverse. The projection to $M \times M \smallsetminus \Delta_M$ is equivariant. Let

$$\langle f \pitchfork f \rangle \subset E^{\curlyvee}(f, f)_{\mathbb{Z}/2}$$

consist of all (orbits of) triples (x, y, ω) in $E^{\curlyvee}(f, f)$ with constant path ω. Then $\langle f \pitchfork f \rangle$ is a smooth manifold which maps to the self–intersection set of $f(M)$ in N and should be viewed as a resolution of it. If $m < 2n/3$, then the resolving map is a diffeomorphism.

Next we discuss normal data. There are maps from $E^{\curlyvee}(f, f)$ to N and $M \times M \smallsetminus \Delta_M$ given by $(x, y, \omega) \mapsto \omega(0)$ and $(x, y, \omega) \mapsto (x, y)$, respectively, which we can use to pull back the tangent bundles τ_N and

$\tau_{M\times M}$. The maps are equivariant (trivial involution on N), so we have canonical choices κ_1 and κ_2 of involutions on the pullback bundles covering the standard involution on $E^{\curlyvee}(f,f)$. However, we use $-\operatorname{id}\cdot\kappa_1$ and κ_2 to view τ_N and $\tau_{M\times M}$, and then $\tau_N-\tau_{M\times M}$, as (virtual) vector bundles on $E^{\curlyvee}(f,f)_{\mathbb{Z}/2}$. Then we can say that the (absolute) normal bundle of $\langle f \pitchfork f\rangle$ is identified with the virtual vector bundle which is the pullback of $\tau_N-\tau_{M\times M}$ under $\langle f \pitchfork f\rangle \hookrightarrow E^{\curlyvee}(f,f)_{\mathbb{Z}/2}$. Therefore $\langle f \pitchfork f\rangle$ can be viewed as a "bordism element" or, by the Thom–Pontryagin construction, as a point in the infinite loop space

$$Q_+(E^{\curlyvee}(f,f)_{\mathbb{Z}/2}\,;\,\tau_N-\tau_{M\times M}).$$

Next, fix some γ in the homotopy fiber of $\operatorname{emb}(M,N)\to\operatorname{imm}(M,N)$ over f. We assume that γ is smooth and generic when viewed as a map from $M\times[0,1]$ to $N\times[0,1]$ over $[0,1]$; this implies that the self–intersections are transverse. Let $\langle\gamma \pitchfork \gamma\rangle\subset E^{\curlyvee}(f,f)\times[0,1]$ consist of all quadruples (x,y,ω,t) where $\gamma_t(x)=\gamma_t(y)\in N$ and ω is the path

$$s\mapsto\begin{cases}\gamma_{t-s(1-t)}(x) & -1\le s\le 0\\ \gamma_{t+s(1-t)}(y) & 0\le s\le 1.\end{cases}$$

A discussion like the one above shows that $\langle\gamma \pitchfork \gamma\rangle$ determines a path from $*$ to $\langle f \pitchfork f\rangle$ in $Q_+(E^{\curlyvee}(f,f)_{\mathbb{Z}/2}\,;\,\tau_N-\tau_{M\times M})$, via the Thom–Pontryagin construction. The procedure generalizes easily to generic families, more precisely, generic maps from some simplex Δ^k to $\phi(f)$, and in this way gives a map

$$\begin{array}{c}\operatorname{hofiber}_f\,[\,\operatorname{emb}(M,N)\to\operatorname{imm}(M,N)\\ \Big\downarrow\\ \text{paths from } * \text{ to } \langle f \pitchfork f\rangle \text{ in } Q_+(E^{\curlyvee}(f,f)_{\mathbb{Z}/2}\,;\,\tau_N-\tau_{M\times M})\,.\end{array}\tag{1.4.1}$$

1.4.2. Theorem [Da, VII.2.1]**;** see also [HaQ]**.** *This map is* $(2n-3-3m)$*–connected.*

Dax' proof of 1.4.2 is based on a "higher" Whitney trick, a purely geometric statement about the realizability of abstract nullbordisms of a self–intersection manifold (or family of such) by regular homotopies of the immersed manifold (or family of such). The higher Whitney trick is very beautifully distilled in [HaQ].

There is another proof of 1.4.2 by reduction to 1.3.3, as we now explain. The map in 1.4.2 is a composition

$$\begin{array}{c} \mathrm{hofiber}_f\,[\,\mathrm{emb}(M,N) \to \mathrm{imm}(M,N)\,] \\ \Big\downarrow{\scriptstyle \text{scanning}} \\ \Gamma_c^{\mathbb{Z}/2}(p_f) \\ \Big\downarrow \\ \text{paths from } * \text{ to } \langle f \pitchfork f\rangle \text{ in } Q_+(E^{\curlyvee}(f,f)_{\mathbb{Z}/2}\,;\,\tau_N - \tau_{M\times M})\,. \end{array} \tag{1.4.3}$$

Here p_f is the fibration on $M \times M \smallsetminus \Delta_M$ whose fiber over (x,y) is the homotopy fiber of $\mathrm{emb}(x \cup y, N) \to \mathrm{imm}(x \cup y, N)$ over the point $f|x \cup y$. We say that a section s of p_f has compact support if, for every (x,y) in $M \times M$ sufficiently close to but not in Δ_M , the value $s(x,y)$ belongs to the homotopy fiber of the identity map $\mathrm{emb}(x \cup y, N) \to \mathrm{emb}(x \cup y, N)$ over the point $f|x \cup y$. (Note: $f|x \cup y$ is indeed an embedding for (x,y) close to the diagonal.) Restriction of embeddings and immersions from M to $x \cup y$ for $(x,y) \in M \times M \smallsetminus \Delta_M$ gives the first arrow in (1.4.3). The symbol Γ is for sections as usual; the subscript c is for compact support, and the superscript $\mathbb{Z}/2$ indicates that we obtain equivariant sections.

The second arrow in (1.4.3) is a stabilization map combined with Poincaré duality, compare [Go6, ch.7], which results from the following observation.

1.4.4. Observation. *The fiberwise unreduced suspension of p_f is fiberwise homotopy equivalent to the fiberwise (over $M \times M \smallsetminus \Delta_M$) Thom space of the vector bundle τ_N on $E^{\curlyvee}(f,f)$.*

Sketch proof. Fix some $x, y \in M$ with $x \neq y$. The fiber V of p_f over (x,y) is the homotopy fiber of the inclusion $\mathrm{emb}(x \cup y, N) \to \mathrm{map}(x \cup y, N)$ over the point $f|x \cup y$. Let W be the homotopy fiber of

$$\mathrm{id}\colon \mathrm{map}(x \cup y, N) \to \mathrm{map}(x \cup y, N)$$

over the point $f|x \cup y$. Then $V \subset W$. Since W is contractible, the mapping cone of $V \hookrightarrow W$ can be identified with the unreduced suspension of V. But W is also a smooth Banach manifold, and $W \smallsetminus V$ is a codimension n smooth submanifold of W, homeomorphic to the space of paths in N from $f(x)$ to $f(y)$. The normal bundle of $W \smallsetminus V$ in W corresponds to the pullback of τ_N under the midpoint evaluation map. The mapping cone of the inclusion $V \to W$ is homotopy equivalent to the Thom space of the normal bundle of $W \smallsetminus V$ in W. $\square$

Now our deduction of 1.4.2 from 1.3.3. goes like this: the second arrow in (1.4.3) is $(2n-3-2m)$–connected by Freudenthal, while the first is $(2n-3-3m)$–connected by 1.3.3. $\square$

Dax has another result along the lines of 1.4.2, giving a homotopy theoretic analysis in the metastable range of the homotopy fiber of the inclusion $\mathrm{emb}(M,N) \to \mathrm{map}(M,N)$ over some $f \in \mathrm{map}(M,N)$. We can also recover this from 1.3.3. Note that our definition of p_f in (1.4.3) works for any continuous $f\colon M \to N$. In this generality it does not make sense to speak of sections of p_f with compact support, but we can speak of *tempered* sections of p_f ; a section s is tempered if, for (x,y) in $M \times M$ close to but not in the diagonal, the value $s(x,y)$ viewed as a path in $N^{\{x,y\}}$ stays close to the diagonal. Stabilization combined with Poincaré duality gets us from the space of tempered equivariant sections of p_f to

$$Q\left(\frac{E^{\curlyvee}(f,f)_{\mathbb{Z}/2}}{\mathbb{S}(TM)_{\mathbb{Z}/2}}\ ;\ \tau_N - \tau_{M\times M}\right)$$

where $\mathbb{S}(TM)$ is the total space of the unit sphere bundle associated with TM. (Regard it as a $\mathbb{Z}/2$–invariant subspace of $M \times M \smallsetminus \Delta_M$, namely, the boundary of a nice symmetric closed tubular neighborhood of Δ_M in $M \times M$. The inclusion of $\mathbb{S}(TM)$ in $M \times M \smallsetminus \Delta_M$ lifts canonically to an equivariant map from $\mathbb{S}(TM)$ to $E^{\curlyvee}(f,f)$.) Therefore the composition of scanning, fiberwise stabilization and Poincaré duality is a map

$$\begin{array}{c} \mathrm{hofiber}_f\,[\,\mathrm{emb}(M,N) \to \mathrm{map}(M,N)\,] \\ \big\downarrow \\ \text{paths from } * \text{ to } \langle f \cap f\rangle \text{ in } Q\left(\dfrac{E^{\curlyvee}(f,f)_{\mathbb{Z}/2}}{\mathbb{S}(TM)_{\mathbb{Z}/2}}\ ;\ \tau_N - \tau_{M\times M}\right) \end{array} \tag{1.4.5}$$

Here the definition of $\langle f \cap f\rangle$ is a by–product of the stabilization process. To understand where it comes from, note that the fiberwise suspension of p_f (as in 1.4.4) has two distinguished sections, denoted $+1$ and -1. Stabilization and Poincaré duality take $+1$ to the base point by construction, but -1 becomes $\langle f \cap f\rangle$ by definition. — Arguing as we did in the proof of 1.4.2 from 1.3.3, we get:

1.4.6. Theorem [Da,VII.2.1]. *The map* (1.4.5) *is* $(2n{-}3{-}3m)$*–connected.*

Suppose that f in 1.4.6 is k–connected. Then the inclusion of $\mathbb{S}(TM)$ in $E^{\curlyvee}(f,f)$ is $\min\{k-1, m-2\}$–connected, by inspection. (It can be written as a composition $\mathbb{S}(TM) \to E^{\curlyvee}(\mathrm{id}_M, \mathrm{id}_M) \longrightarrow E^{\curlyvee}(f,f)$ in which the second arrow is clearly $(k-1)$-connected. The first arrow can be looked at as a map over M, and the fiber of $E^{\curlyvee}(\mathrm{id}_M, \mathrm{id}_M)$ over $x \in M$ is, up to homotopy equivalence, the homotopy fiber of $M \smallsetminus x \hookrightarrow M$.) This gives a corollary, essentially due to Haefliger again [Hae1]:

1.4.7. Corollary. *Let* $f\colon M \to N$ *be a* k*–connected map. Then the homotopy fiber of the inclusion* $\mathrm{emb}(M,N) \hookrightarrow \mathrm{map}(M,N)$ *over* f *is* $\min\{k-1+n-2m, n-m-2, 2n-3m-4\}$*–connected. In particular, it is nonempty when* $m+1 < 2n/3$ *and* $k > 2m-n$.

2. Surgery methods

We will be concerned with two methods which use surgery to construct smooth embeddings. The older one, initiated by Levine [Lev], aims to construct a smooth embedding $M \to N$ by making hypotheses of a homotopy theoretic nature which, via transversality, translate into a diagram

$$M \xleftarrow{g} M' \xrightarrow{e} N$$

where e is a smooth embedding and g is a degree one normal map, normal cobordant to the identity $M \to M$. The normal cobordism amounts to a finite sequence of elementary surgeries transforming $M' \cong e(M')$ into something diffeomorphic to M, and one tries to perform these surgeries as *embedded* surgeries, inside N. The other method, invented by Browder [Br1], [Br2], aims to construct a smooth embedding $M \to N$ by making hypotheses of a homotopy theoretic nature which, via transversality, translate into a diagram

$$M \xrightarrow{e} N' \xrightarrow{f} N$$

where e is a smooth embedding and f is a degree one normal map, normal cobordant to the identity. The normal cobordism amounts to a finite sequence of elementary surgeries transforming N' into something diffeomorphic to N, and one tries to perform these surgeries away from $e(M)$.

Reversing the historical order once again, we will begin with Browder's method, which reduces the problem of constructing embeddings $M \to N$ to a homotopy theoretic one. Then we will turn to Levine's method, to find that it has a lot to tell us about the homotopy theoretic problem created by Browder's method.

2.1. Smoothing Poincaré embeddings

Let $(M, \partial M)$ and $(N, \partial N)$ be Poincaré pairs, both of formal dimension n. By a (codimension zero) Poincaré embedding of $(M, \partial M)$ in $(N, \partial N)$ we mean a simple homotopy equivalence of Poincaré pairs

$$(M \amalg_{\partial M} C,\, \partial_1 C) \xrightarrow{f} (N, \partial N)$$

where $(C, \partial C)$ is a special Poincaré triad of formal dimension n (that is, a Poincaré pair with $\partial C = \partial_0 C \amalg \partial_1 C$) and $\partial_0 C$ is identified with ∂M. We call C the *formal complement* determined by the Poincaré embedding. For example, if M^n and N^n are smooth compact manifolds, then a smooth embedding g: $M \to N$ avoiding ∂N gives rise to a codimension zero Poincaré embedding whose formal complement is the closure of $N \smallsetminus g(M)$ in N.

Slightly more generally, we will say that a Poincaré embedding f as above is *induced* by a smooth embedding g: $M \to N$ if $f|M = g$, and $f|C$ restricts to a simple homotopy equivalence (of special triads) from C to the closure of $N \smallsetminus g(M)$ in N.

2.1.1. Theorem. *Let M^n and N^n be smooth compact, $n \geq 5$. Let f: $M \amalg_{\partial M} C \to N$ be a codimension zero Poincaré embedding (in shorthand notation). Let ι: $\nu_M \to f^*\nu_N|M$ be a stable vector bundle isomorphism refining the canonical stable fiber homotopy equivalence determined by the codimension zero Poincaré embedding (see explanations below). Assume that f induces an isomorphism $\pi_1 C \to \pi_1 N$. Then, up to a homotopy, the pair (f, ι) is induced by a smooth embedding g: $M \to N$ avoiding ∂N.*

Explanations. By the characterization of Spivak normal fibrations, the codimension zero embedding determines a stable fiber homotopy equivalence from ν_M (viewed as a spherical fibration) to $f^*\nu_n|M$ (ditto). The stable vector bundle isomorphism ι also determines such a stable fiber homotopy equivalence; we want the two to be fiberwise homotopic.

There is a mild generalization of 2.1.1 which involves the concept of a Poincaré embedding of arbitrary (formal) codimension. Assume this time that $(M, \partial M)$ and $(N, \partial N)$ are (simple) Poincaré pairs, of formal dimensions m and n, where $n - m =: q \geq 0$. A Poincaré embedding of $(M, \partial M)$ in $(N, \partial N)$ consists of

- a fibration $E \to M$ with fibers homotopy equivalent to $\mathbb{S}^{q-1}$ (the *unstable normal fibration* of the Poincaré embedding)
- a codimension zero Poincaré embedding of $(zE, \partial zE)$, where zE is the mapping cylinder of $E \to M$ and ∂zE is the union of E and the portion of zE projecting to ∂M.

This concept is due to [Br2], at least in the case where M and N are smooth manifolds, $\partial M = \emptyset = \partial N$.

2.1.2. Corollary. *Let M^m and N^n be smooth compact, where $n \geq 5$, and $q := n - m$. Let a Poincaré embedding f of $(M, \partial M)$ in $(N, \partial N)$ be given, with formal complement $(C, \partial C)$; let a reduction of the structure 'group' $G(q)$ of its unstable normal fibration to $O(q)$ be given, refining the canonical reduction for the stable normal fibration. Suppose that the induced homomorphism $\pi_1 C \to \pi_1 N$ is an isomorphism. Then there exists a smooth embedding $M \to N \smallsetminus \partial N$ inducing (up to a homotopy) the given Poincaré embedding and the unstable refinement of the canonical reduction for the stable normal fibration.*

Explanations. Let f_M be the restriction of f to M. The unstable refinement of the canonical reduction etc. is a point in the homotopy fiber of an evident map

$$BO(q)^M \quad \longrightarrow \quad \operatorname{holim}\,[BG(q)^M \to BG^M \leftarrow BO^M\,]$$

over the point determined by the unstable normal fibration on M, the stable normal (vector) bundle $\nu_M - {f_M}^*\nu_N$ on M, and the stable spherical fibration determined by the stable normal vector bundle.

Browder came close to 2.1.1 in [Br1] and proved in [Br2] the special case of 2.1.2 where M and N are simply connected, $\partial M = \emptyset = \partial N$, and $n - m \geq 3$, which makes the hypothesis on fundamental groups superfluous. One understands that Casson and Sullivan in unpublished but possibly mimeographed work and lectures simplified Browder's proof and obtained the appropriate uniqueness statement (see 2.2). Also, Casson pointed out [Ca] that Browder's hypothesis $n - m \geq 3$ could be replaced by the hypothesis on fundamental groups in 2.1.2. Wall [Wa2, ch.11] proved 2.1.2 in the nonsimply connected case. Therefore 2.1.2 and variations, see 2.2, are known as the Browder–Casson–Sullivan–Wall theorem. For an indication of the proof, see also 2.2.

2.2. Smoothing block families of Poincaré embeddings

Assume that M and N are smooth closed, for simplicity. The smooth embedding $M \to N$ whose existence is asserted in 2.1.2 is not determined up to isotopy, in general. But a relative version of 2.1.2, see [Wa2, 11.3 rel], implies that it is determined up to a concordance of embeddings (smooth embedding $M \times [0,1] \to N \times [0,1]$ taking $M \times i$ to $N \times i$ for $i = 0, 1$). In this way, 2.1.2 and the relative version give a homotopy theoretic expression for $\pi_0 \operatorname{emb}(M,N)$ modulo the concordance relation.

The block embedding space $\operatorname{emb}^{\sim}(M,N)$ is a crude approximation from the right to $\operatorname{emb}(M,N)$. It is the geometric realization of an incomplete simplicial set (alias simplicial set without degeneracy operators, alias Δ–set) whose k–simplices are the smooth embeddings of special manifold $(k+2)$–ads

$$\Delta^k \times M \longrightarrow \Delta^k \times N\,.$$

It is fibrant (has the Kan extension property), so that $\pi_k \operatorname{emb}^{\sim}(M,N)$, with respect to a base vertex $f\colon M \to N$, can be identified with the set of concordance classes of embeddings $\Delta^k \times M \to \Delta^k \times N$ which agree with $\mathrm{id} \times f$ on $\partial\Delta^k \times M$. Therefore 2.1.2 and the relative version give a homotopy theoretic expression for all $\pi_k \operatorname{emb}^{\sim}(M,N)$, $k \geq 0$. This suggests that 2.1.2 plus relative version admits a space level reformulation, involving $\operatorname{emb}^{\sim}(M,N)$ and a Poincaré embedding analogue. We denote that analogue by $\operatorname{emb}_{\mathrm{PD}}{}^{\sim}(M,N)$; it is defined whenever M and N are Poincaré spaces. (There is also an 'unblocked' version, $\operatorname{emb}_{\mathrm{PD}}(M,N)$; but the inclusion of $\operatorname{emb}_{\mathrm{PD}}(M,N)$ in $\operatorname{emb}_{\mathrm{PD}}{}^{\sim}(M,N)$ is a homotopy equivalence.)

We will also need notation and terminology for the complicated normal bundle and normal fibration data. Given Poincaré spaces M and N, of formal dimensions m and n, where $n-m =: q \geq 0$, a *Poincaré immersion* from M to N is a triple (f,ξ,ι) where $f\colon M \to N$ is a map, ξ^q is a spherical fibration on M (fibers $\simeq \mathbb{S}^{q-1}$), and ι is a stable fiber homotopy equivalence of the Spivak normal fibration ν_M with the Whitney sum alias fiberwise join $\xi \oplus f^*\nu_N$. We can make a space $\operatorname{imm}_{\mathrm{PD}}(M,N)$ out of such triples; we can also use the $(k+2)$–ad analogue of the notion of Poincaré immersion to define a block immersion space $\operatorname{imm}_{\mathrm{PD}}{}^{\sim}(M,N)$. It is easy to see that the inclusion of $\operatorname{imm}_{\mathrm{PD}}(M,N)$ in $\operatorname{imm}_{\mathrm{PD}}{}^{\sim}(M,N)$ is a homotopy equivalence.

Remarks. Suppose that M^m and N^n are smooth and closed, $n > m$. The immersion classification theorem, applied craftily to spaces of (smooth) block immersions, implies that the block immersion space $\operatorname{imm}^{\sim}(M,N)$ maps by a homotopy equivalence to the space of triples (f,ξ,ι) where

- $f\colon M \to N$ is a map
- ξ is an $(n-m)$–dimensional vector bundle on M
- $\iota\colon \nu_M \cong \xi \oplus f^*\nu_N$ is a stable vector bundle isomorphism.

This motivates our definition of $\operatorname{imm}_{\mathrm{PD}}{}^{\sim}(M,N)$ for Poincaré spaces M and N, which is taken from [Kln3]. Beware: in the smooth setting, the inclusion $\mathrm{imm}(M,N) \to \mathrm{imm}^{\sim}(M,N)$ is not a homotopy equivalence in general. (Try $M = *$ and $N = \mathbb{S}^n$.)

We will sometimes also speak of Poincaré immersions from a Poincaré pair $(M, \partial M)$ to a Poincaré pair $(N, \partial N)$, of formal dimensions m and n, respectively, $m \le n$. The definition is much the same as before.

2.2.1. Theorem (Browder–Casson–Sullivan–Wall). *For closed smooth M^m and N^n with $n \ge 5$ and $n - m \ge 3$, the following commutative square is ∞–cartesian:*

$$\begin{array}{ccc} \mathrm{emb}^{\sim}(M,N) & \xrightarrow{\subset} & \mathrm{imm}^{\sim}(M,N) \\ \downarrow & & \downarrow \\ \mathrm{emb_{PD}}^{\sim}(M,N) & \xrightarrow{\subset} & \mathrm{imm_{PD}}^{\sim}(M,N)\,. \end{array}$$

Remarks. The vertical arrows are essentially forgetful, but to make the one on the left hand side explicit, we ought to redefine $\mathrm{emb}^{\sim}(M,N)$ using smooth embeddings with specified riemannian tubular neighborhoods. The right hand vertical arrow is $(2n-3-3m)$–connected; therefore so is the left hand one. See [Wa2, Cor. 11.3.2].

If $n < 5$ we can still say that the square becomes ∞–cartesian when Ω^{5-n} is applied — this requires a choice of base vertex in $\mathrm{emb}^{\sim}(M,N)$. However, some condition like $n - m \ge 3$ is essential.

Theorem 2.2.1 has PL and TOP versions. In the PL and TOP settings, the content of the theorem is quite simply that $\mathrm{emb}^{\sim}(M,N)$ maps by a homotopy equivalence to $\mathrm{emb_{PD}}^{\sim}(M,N)$. Namely, in the PL and TOP settings, the right hand vertical arrow in the diagram in 2.2.1 is a homotopy equivalence; here again $n - m \ge 3$ is essential. See [Wa2, Cor. 11.3.1].

Example. We calculate $\mathrm{emb}^{\sim}(*, \mathbb{R}^n)$. Observe first that our definition of $\mathrm{emb}^{\sim}(M,N)$ makes sense for arbitrary smooth M and N without boundaries. We will use the stronger version of 2.2.1 where M and N are allowed to be compact with boundary; only smooth and Poincaré embeddings avoiding ∂N are considered, and we denote the corresponding block embedding spaces by $\mathrm{emb}^{\sim}(M,N)$ and $\mathrm{emb}^{\sim}_{\mathrm{PD}}(M,N)$ for brevity. Our choice is $M = *$ and $N = \mathbb{D}^n$ and we find

$$\begin{aligned} \mathrm{imm}^{\sim}(*, \mathbb{D}^n) &\simeq \mathrm{O}/\mathrm{O}(n)\,, \\ \mathrm{imm_{PD}}^{\sim}(*, \mathbb{D}^n) &\simeq \mathrm{G}/\mathrm{G}(n)\,, \\ \mathrm{emb_{PD}}^{\sim}(*, \mathbb{D}^n) &\simeq *\,. \end{aligned}$$

Therefore $\mathrm{emb}^{\sim}(*, \mathbb{R}^n) \cong \mathrm{emb}^{\sim}(*, \mathbb{D}^n)$ is homotopy equivalent to the homotopy fiber of the inclusion $\mathrm{O}/\mathrm{O}(n) \longrightarrow \mathrm{G}/\mathrm{G}(n)$.

2.2.2. Theorem. *Let M^n and N^n be smooth compact, $n \geq 5$. Assume that M has a handle decomposition with handles of index $\leq n-3$ only. Then the following commutative square is ∞–cartesian:*

$$\begin{array}{ccc} \mathrm{emb}^{\sim}(M,N) & \xrightarrow{\subset} & \mathrm{imm}^{\sim}(M,N) \\ \big\downarrow & & \big\downarrow \\ \mathrm{emb}_{\mathrm{PD}}{}^{\sim}(M,N) & \xrightarrow{\subset} & \mathrm{imm}_{\mathrm{PD}}{}^{\sim}(M,N)\,. \end{array}$$

Remarks. This is the 'family' version of 2.1.1. In particular, $\mathrm{emb}^{\sim}(M,N)$ is short for the space of smooth block embeddings of M in $N \smallsetminus \partial N$, and so on. Precise definitions of the spaces in the diagram are left to the reader. It is easy to deduce 2.2.1 from 2.2.2. We could drop the condition $n \geq 5$ at the price of choosing a base vertex in $\mathrm{emb}^{\sim}(M,N)$ and applying Ω^{5-n}.

We turn to the proof of 2.2.2, assuming $\partial N = \emptyset$ for simplicity. Actually we will just deduce 2.2.2 from the Sullivan–Wall(–Quinn–Ranicki) homotopy fiber sequence. To explain what this is, we fix a (simple) Poincaré space W of formal dimension n. An *s–structure* on W is a pair (M,f), where M is smooth closed and f is a simple homotopy equivalence $f\colon M \to W$. The s–structures on W form a groupoid where an isomorphism from (M_1, f_1) to (M_2, f_2) is a diffeomorphism $g\colon M_1 \to M_2$ satisfying $f_2 g = f_1$. We enlarge this to an incomplete simplicial groupoid $\mathrm{str}_\bullet(W)$ in which $\mathrm{str}_k(W)$ is the groupoid of s–structures, in the special $(k+2)$–ad sense, on $\Delta^k \times W$. The *block structure space* $\mathcal{S}^{\sim}(W)$ of W can be defined as the geometric realization of the incomplete simplicial set whose simplices in degree k are rules ρ which associate

- to each face z of Δ^k, an object $\rho(z)$ in $\mathrm{str}_{|z|}(W)$;
- to each face z of Δ^k and face operator δ from degree $|z|$ to a smaller degree, an isomorphism $\delta\rho(z) \to \rho\delta(z)$ in $\mathrm{str}_{|z|}(W)$. (These isomorphisms are subject to an evident associativity condition.)

There is a second definition of $\mathcal{S}^{\sim}(W)$, homotopy equivalent to the first, according to which $\mathcal{S}^{\sim}(W)$ is the geometric realization of the incomplete simplicial space $k \mapsto |\mathrm{str}_k(W)|$. However, the first definition matches our earlier definition of block embedding spaces better. — The Sullivan–Wall homotopy fiber sequence is then

$$\mathcal{S}^{\sim}(W) \longrightarrow R^{\mathrm{O}}_{\mathrm{G}}(\nu_W) \longrightarrow \Omega^{n+\infty} \boldsymbol{L}^s_\bullet(\mathbb{Z}\pi_1 W).$$

Here $R^{\mathrm{O}}_{\mathrm{G}}(\nu_W)$ is the homotopy fiber of $B\mathrm{O}^W \to B\mathrm{G}^W$ over the point determined by ν_W (think of it as the space of 'reductions' of the structure 'group' of ν_W, from G to O) and $\mathbf{L}^s_\bullet(\mathbb{Z}\pi_1 W)$ is the L–theory spectrum associated with the group ring $\mathbb{Z}\pi_1 W$. We have shortened $\Omega^n\Omega^\infty$ to $\Omega^{n+\infty}$. We need a slightly more complicated version where W is a Poincaré triad, $\partial W = \partial_0 W \cup \partial_1 W$, and the s–structures are fixed (prescribed) on $\partial_0 W$. Consequently the structure 'group' reductions are also fixed (prescribed) over $\partial_0 W$, and the relevant L–theory spectrum is the one associated with the homomorphism of group rings $\mathbb{Z}\pi_1\partial_1 W \to \mathbb{Z}\pi_1 W$ induced by inclusion.

With M and N as in 2.2.2, fix a Poincaré embedding of M in N, say $f\colon M \amalg_{\partial M} C \to N$. Let W be the mapping cylinder of f. Then W is a Poincaré triad with $\partial W \cong (M \amalg_{\partial M} C) \amalg N$ and $\partial_0 W = M \amalg N$, $\partial_1 W = C$. There is a preferred s–structure on $\partial_0 W$, given by the identity; indeed $\partial_0 W$ is a smooth compact manifold. Browder's crucial, highly original and yet trivial observation at this point, slightly reformulated, is that $\mathcal{S}^\sim(W)$, with the definition where structures are prescribed on $\partial_0 W$, is homotopy equivalent to the homotopy fiber (over the point f) of the left hand vertical arrow in the diagram of 2.2.2. It is easy to check that the corresponding homotopy fiber of the right hand vertical arrow is homotopy equivalent to $R^{\mathrm{O}}_{\mathrm{G}}(\nu_W)$ (with the definition where the reductions are fixed over $\partial_0 W$), and that, with these identifications, the map of homotopy fibers becomes the map

$$\mathcal{S}^\sim(W) \longrightarrow R^{\mathrm{O}}_{\mathrm{G}}(\nu_W)$$

from the Sullivan–Wall homotopy fiber sequence. It is a homotopy equivalence because, by the π–π–theorem, the L–theory term in the Sullivan–Wall homotopy fiber sequence is contractible. $\square$

Remark. In this proof $R^{\mathrm{O}}_{\mathrm{G}}(\nu_W)$ can be interpreted, via transversality, as a space of 'degree one normal maps' to W which restrict to identity maps over $\partial_0 W$. Such a normal map to W is exactly the same thing as a smooth embedding $M \to N'$, plus a degree one normal map $N' \to N$, plus a normal cobordism from $N' \to N$ to the identity $N \to N$.

2.3. Embedded Surgery

Let M^m be smooth closed. Following Levine [Lev] and Rigdon–Williams [RiW], we will discuss the construction of embeddings $M \to \mathbb{R}^n$ from the following data:

- a degree one normal map $g\colon M' \to M$ and a normal (co)bordism h from g to id: $M \to M$;
- a smooth embedding $e\colon M' \to \mathbb{R}^n$.

On the set of such triples (g, h, e) there is an evident bordism relation. Surgery methods can be used (some details below) to show that, when $2n-3-3m \geq 0$, every bordism class has a representative (g, h, e) in which h is a product cobordism, so that g is homotopic to a diffeomorphism. A straightforward Thom–Pontryagin construction leads to a homotopy theoretic description of the bordism set. Combining these two ideas, one obtains embeddings $M \to \mathbb{R}^n$ from homotopy theoretic data if $2n-3-3m \geq 0$.

The homotopy theoretic description. Let $\nu = \nu_M$ be the stable normal bundle of M. Let $V^{n-m}(\nu)$ be the tautological $(n-m)$–dimensional vector bundle on the homotopy pullback of

$$M \xrightarrow{\nu} B\mathrm{O} \leftarrow B\mathrm{O}(n-m)\,.$$

There is a forgetful map from the base of $V^{n-m}(\nu)$ to M, and a stable map of vector bundles $V^{n-m}(\nu) \to \nu$ covering it. This leads to another forgetful or stabilization map

$$\Omega^n T(V^{n-m}(\nu)) \to \Omega^{m+\infty}\boldsymbol{T}(\nu) \tag{2.3.1}$$

where the T denotes a Thom space and the (boldface) $\boldsymbol{T}$ denotes a Thom spectrum. In $\Omega^{m+\infty}\boldsymbol{T}(\nu)$ we have a distinguished point ρ, the Spanier–Whitehead or Poincaré dual of $1\colon M \to Q\mathbb{S}^0$. See 0.2, on the subject of Poincaré spaces. By a Thom–Pontryagin construction, the set of triples (g, h, e) as above, modulo bordism, can be identified with π_0 of the homotopy fiber of (2.3.1) over the point ρ.

Digressing a little now, we note that a smooth embedding $M \to \mathbb{R}^n$ determines a triple (g, h, e) as above where $h\colon M \times [0,1] \to M$ is the projection and e from $M \times 1 \cong M$ to $\mathbb{R}^n$ is that embedding. The bordism class of the triple (g, h, e) may be called the (smooth, unstable, etc.) *normal invariant* of the embedding $M \to \mathbb{R}^n$.

The surgery methods. Assume that $m \geq 5$. Let (g, h, e) be one of those triples. To be more specific, we write the normal cobordism in the form $h\colon M'' \to M$, where $\dim(M'') = m+1$. Surgery below the middle dimension on M'' creates a bordism from the triple (g, h, e) to another such triple, (g_1, h_1, e_1), in which $h_1\colon M_1'' \to M$ is $[m/2+1/2]$–connected. Then the inclusion of M in M_1'' is $[m/2-1/2]$–connected. It follows that M_1'' admits a handle decomposition, relative to a collar on $M_1' := \partial M_1'' \smallsetminus M$, with handles of index $\leq m - [m/2-1/2]$ only.

Now choose a handle of lowest index i, giving a framed embedding $u\colon (\mathbb{D}^i, \mathbb{S}^{i-1}) \to (M_1'', M_1')$. We try to create a bordism from (g_1, h_1, e_1) to another triple, (g_2, h_2, e_2), by doing a half–surgery, alias handle subtraction, on $u(\mathbb{D}^i)$. Of course, the resulting surgery on $u(\mathbb{S}^{i-1}) \subset M_1'$ has to be an embedded surgery — embedded in $\mathbb{R}^n \times [0,1]$, to be precise. The 'embedded surgery lemma' in [RiW] shows that the required (partly embedded) half–surgery can be carried out if $n - m > i$. Since $i \le m - [m/2 - 1/2]$, it is enough to require $2n - 3 - 3m \ge 0$. In that case we can also repeat the procedure until all handles in the handle decomposition of M_1'' relative to M_1' have been subtracted. So (g_2, h_2, e_2) is bordant to a triple (g_r, h_r, e_r) in which h_r is a product cobordism. $\square$

Hence we have the existence part of the following (the proof of uniqueness uses the same ideas in a relative setting):

2.3.2. Proposition [RiW]**.** *Assume $m \ge 5$. If $2n - 3 - 3m \ge 0$, then every element in π_0 of the homotopy fiber of* (2.3.1) *over ρ is the (unstable) normal invariant of a smooth embedding $M \to \mathbb{R}^n$. If $2n - 3 - 3m > 0$, such an embedding is unique up to concordance.*

Although 2.3.2 owes a lot to the ideas in [Lev], it has a sharper focus and leads on to a number of new ideas. In particular, 2.3.2 generalizes easily to block families: M can be replaced by $M \times \Delta^k$ and $\mathbb{R}^n$ by $\mathbb{R}^n \times \Delta^k$ in the sketch proof. We must require $2(n+k) - 3 - 3(m+k) \ge 0$, in other words $k \le 2n - 3 - 3m$, and pay some attention to the faces $M \times d_i\Delta^k$. This shows that our unstable normal invariant for embeddings of M in $\mathbb{R}^n$ is really a map

$$\begin{array}{c} \mathrm{emb}^{\sim}(M, \mathbb{R}^n) \\ \big\downarrow \\ \mathrm{hofiber}_\rho\,[\,\Omega^n T(V^{n-m}(\nu)) \to \Omega^{m+\infty}\boldsymbol{T}(\nu)\,] \end{array} \tag{2.3.3}$$

and gives us an estimate for the connectivity:

2.3.4. Theorem. *The map* (2.3.3) *is* $(2n - 3 - 3m)$*–connected (*$m \ge 5$*).*

Let $f\colon M \to \mathbb{R}^n$ be an immersion with normal bundle ν_f ; so ν_f is a vector bundle of dimension $n - m$ on M.

2.3.5. Corollary. *Suppose that $m \ge 5$. There is a $(2n - 3 - 3m)$–connected map*

$$\begin{array}{c} \mathrm{hofiber}_f\,[\,\mathrm{emb}^{\sim}(M, \mathbb{R}^n) \hookrightarrow \mathrm{imm}^{\sim}(M, \mathbb{R}^n)\,] \\ \big\downarrow \\ \mathrm{hofiber}_\rho\,[\,\Omega^n T(\nu_f) \hookrightarrow \Omega^{m+\infty}\boldsymbol{T}(\nu)\,]\,. \end{array}$$

Proof. The map is a variation on (2.3.3); use the fact that an embedding $M \to \mathbb{R}^n$ equipped with a regular homotopy to f has a normal bundle which is canonically identified with ν_f. To show that the map in question is $(2n-3-3m)$–connected, view it as the left column of a commutative square whose right column is (2.3.3). Now we need to show that the square is $(2n-3-3m)$–cartesian. With the abbreviations $\mathrm{emb}^\sim = \mathrm{emb}^\sim(M, \mathbb{R}^n)$ and $\mathrm{imm}^\sim = \mathrm{imm}(M, \mathbb{R}^n)$, this reduces to the assertion that

$$\begin{tikzcd} \mathrm{hofiber}_f[\mathrm{emb}^\sim \to \mathrm{imm}^\sim] \arrow[r, "\text{forget}"] \arrow[d] & \mathrm{emb}^\sim \arrow[d] \\ \Omega^n T(\nu_f) \arrow[r] & \Omega^n T(V^{n-m}(\nu)) \end{tikzcd}$$

is $(2n-3-3m)$–cartesian. Actually this is $(2n-2-3m)$–cartesian. (Use the immersion classification theorem and Blakers–Massey to understand the homotopy fibers of the upper and lower rows, respectively. Then compare.) $\square$

2.4. Poincaré embeddings into disks

Williams was apparently the first to realize [Wi1] that the proper context for Levine's ideas in [Lev] was not manifold geometry, but Poincaré space geometry. To illustrate this point, we will translate 2.3.5 into Poincaré space language, relying on 2.2.1 for the translation. For a Poincaré pair $(M, \partial M)$ of formal dimension n, let $\Omega^n_\vdash(M/\partial M) \subset \Omega^n(M/\partial M)$ consist of the elements which carry a fundamental class for $(M, \partial M)$. This is a union of connected components of $\Omega^n(M/\partial M)$.

2.4.1. Reformulation of 2.3.5. *Let $(M, \partial M)$ be the Poincaré pair of formal dimension n determined by a spherical fibration $\xi^{n-\ell}$ on a smooth closed L^ℓ. That is, ∂M is the total space of ξ, and M is the mapping cylinder of the projection $\partial M \to L$. If $\ell \geq 5$, then the map*

$$\mathrm{emb}^\sim_{\mathrm{PD}}(M, \mathbb{D}^n) \longrightarrow \Omega^n_\vdash(M/\partial M)$$

associating to a Poincaré embedding its collapse map is $(2n-3-3\ell)$–connected.

Explanation. Make a space E_{PD} whose elements are pairs (μ, σ) where $\mu^{n-\ell}$ is a spherical fibration on L, and $\sigma\colon \mathbb{S}^n \to T(\mu)$ carries a fundamental class. The real content of 2.4.1 is that the map

$$\mathrm{emb}^\sim_{\mathrm{PD}}(L, \mathbb{D}^n) \to E_{\mathrm{PD}}$$

which to a Poincaré embedding associates its normal bundle and collapse map is $(2n-3-3\ell)$–connected. We now deduce this from 2.3.5:

By the characterization of the Spivak normal fibration of L, the spherical fibration μ in any $(\mu,\sigma)\in E_{\mathrm{PD}}$ is (canonically) stably fiber homotopy equivalent to ν_L. So there is a forgetful map $E_{\mathrm{PD}}\to \mathrm{imm}^{\sim}_{\mathrm{PD}}(L,\mathbb{D}^n)$. Let E be the homotopy pullback of

$$E_{\mathrm{PD}}\to \mathrm{imm}^{\sim}_{\mathrm{PD}}(L,\mathbb{D}^n)\leftarrow \mathrm{imm}^{\sim}(L,\mathbb{R}^n)\,.$$

We get a commutative square

$$\begin{array}{ccc}
\mathrm{emb}^{\sim}(L,\mathbb{R}^n) & \longrightarrow & E \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{emb}^{\sim}_{\mathrm{PD}}(L,\mathbb{D}^n) & \longrightarrow & E_{\mathrm{PD}}
\end{array}$$

which is ∞–cartesian by 2.2.1. By the remark after 2.2.1, the right–hand vertical arrow is $(2n-3-3\ell)$–connected So it is enough to show that the upper horizontal arrow in the square is $(2n-3-3\ell)$–connected. But that is exactly the content of 2.3.5. □

Williams saw that the peculiar hypotheses on the Poincaré pair $(M/\partial M)$ in 2.4.1 could be replaced by a single much simpler one. (For simplicity we restrict attention to $\pi_0\,\mathrm{emb}^{\sim}_{\mathrm{PD}}(M,\mathbb{D}^n)$. We write $\pi_n^{\vdash}(M/\partial M)$ for the subset of $\pi_n(M/\partial M)$ consisting of the elements which carry a fundamental class.)

2.4.2. Theorem [Wi1]**.** *Let $(M,\partial M)$ be a Poincaré pair of formal dimension $n\geq 6$, where M is homotopy equivalent to a CW–space of dimension m. Assume that $\pi_1\partial M\to\pi_1 M$ is an isomorphism. Then the map*

$$\pi_0\,\mathrm{emb}^{\sim}_{\mathrm{PD}}(M,\mathbb{D}^n)\longrightarrow \pi_n^{\vdash}(M/\partial M)$$

associating to a Poincaré embedding the class of its collapse map is surjective for $2n-3-3m\geq 0$, and bijective for $2n-3-3m>0$.

Williams' proof of 2.4.2 uses Hodgson's thickening theorem, 2.4.4 below. This is a distant corollary of Hudson's embedding theorem:

2.4.3. Theorem [Hu1, 8.2.1], [Hu2, 1.1]**.** *If N^n is a compact smooth manifold and P is a codimension zero compact smooth submanifold of ∂N such that $P\hookrightarrow N$ is j–connected, then any element of $\pi_r(N,P)$ may be represented by a smooth embedding $(\mathbb{D}^r,\mathbb{S}^{r-1})\to(N,P)$ provided that $r\leq n-3$ and $2r\leq n+j-1$.*

2.4.4. Theorem [Ho, 2.3]**.** *Let N^n be a compact smooth manifold, $n \geq 6$, and P a codimension zero smooth submanifold of ∂N. Let K be a CW–space rel P of (relative) dimension $\leq k$, and let $f\colon K \to N$ be any map rel P. If f is $(2k-n+1)$–connected rel P, then f is homotopic rel P to a composition*

$$K \xrightarrow{\simeq} K' \hookrightarrow N$$

where K' is a smooth compact triad contained in N with $\partial_0 K' = P = K' \cap \partial N$, and $K \to K'$ is a simple homotopy equivalence rel P.

Remark. It is an exercise, but a non–trivial one, to deduce the special case of 2.4.4 where K has just one cell rel P from 2.4.3.

Outline of proof of 2.4.2. Two key concepts in Williams' proof are those of compression and decompression. The *decompression* of a codimension zero Poincaré embedding of M in N is an obvious codimension zero Poincaré embedding of $M \times J$ in $N \times I$ where $I = [0,1]$ and $J = [1/3, 2/3]$. Here M, N are short for Poincaré pairs of formal dimension n, and $M \times J$, $N \times I$ are short for certain Poincaré pairs of formal dimension $n+1$. Conversely, to *compress* a Poincaré embedding of $M \times J$ in $N \times I$ means to find a concordance from it to the decompression of some Poincaré embedding of M in N.

Browder points out in [Br2] that a map $\eta\colon \mathbb{S}^n \to M/\partial M$ which carries a fundamental class for the Poincaré pair $(M, \partial M)$ determines a Poincaré embedding of $M \times J$ in $\mathbb{D}^n \times I$. Its formal complement C is the mapping cylinder of

$$\partial(M \times J) \amalg \partial(\mathbb{D}^n \times I) \quad \xrightarrow{q \amalg \eta} \quad M/\partial M$$

where q is the quotient map collapsing $M \times 1/3$ and $\partial M \times J$ to a point. The boundary ∂C is $\partial(M \times J) \amalg \partial(\mathbb{D}^n \times I)$.

This leaves the task of compressing $M \times J \to \mathbb{D}^n \times I$, the Poincaré embedding determined by some η from $\mathbb{S}^n$ to $M/\partial M$ as above, to a Poincaré embedding $M \to \mathbb{D}^n$. Hirsch [Hi2] gives a necessary and often sufficient condition for the existence of such a compression: that the inclusion of $M \times 1/3 \subset \partial(M \times J)$ in the formal complement C of the Poincaré embedding $M \times J \to \mathbb{D}^n \times I$ be nullhomotopic. This is clearly satisfied here — there is a preferred choice of nullhomotopy alias *link trivialization.* Williams shows in fact that the map just described, from $\pi_n^{\vdash}(M/\partial M)$ to concordance classes of Poincaré embeddings $M \times J \to \mathbb{D}^n \times I$ with a link trivialization, is a bijection. (This is not difficult.) He then proceeds to show that the link trivialization determines a compression. His argument has two parts:

(i) Without loss of generality, M and C are compact smooth manifolds. Namely, the existence of $\eta\colon \mathbb{S}^n \to M/\partial M$ implies that the Spivak normal fibration of M is trivial; hence there exists a degree one normal map $(M, \partial M) \to (M, \partial M)$ where M is smooth compact, and by the $\pi - \pi$ theorem (here we use $n \geq 6$ and the condition on fundamental groups) this is normal bordant to a homotopy equivalence. A similar argument works for C ; in this case the manifold structure is already prescribed on $\partial_0 C$ since we want $\partial_0 C \cong \partial M$.

(ii) The nullhomotopy for $M \times 1/3 \hookrightarrow C$ means that the inclusion of $(M \times 1/3) \amalg (\mathbb{D}^n \times 0)$ in ∂C extends to a map $e\colon X \to C$, where X is any CW–space rel $(M \times 1/3) \amalg (\mathbb{D}^n \times 0)$ which is contractible. The metastable range condition in 2.4.2 now makes it possible to use Hodgson's thickening theorem, 2.4.4. The conclusion is that X can be taken to be a compact $n+1$–manifold with $\pi_1 \partial X \cong \pi_1 X$, and $(M \times 1/3) \amalg (\mathbb{D}^n \times 0)$ contained in ∂X ; moreover, e can be taken to be an embedding. Then X is an $(n+1)$–disk and the inclusion of $M \times 1/3$ in the closure of $\partial X \smallsetminus (\mathbb{D}^n \times 0)$ is the compressed embedding we have been looking for. $\square$

Remark. The idea to use Hodgson's thickening theorem 2.4.4 for compression purposes comes from [Lt] and Hirsch [Hi2]. Actually Hirsch had to work with Hudson's embedding theorem, 2.4.3.

Williams noted in [Wi2] that his own proof of 2.4.2 "... consists of converting $(M, \partial M)$ to a manifold and then using smooth embedding theory " and went on to propose an alternative and truly homotopy theoretic proof, along the following lines. Given $\eta\colon \mathbb{S}^m \to M/\partial M$ carrying a fundamental class, Browder's observation gives as before a Poincaré embedding of $M \times J$ in $\mathbb{D}^n \times I$ with a preferred link trivialization, and formal complement homotopy equivalent to $M/\partial M$. If this compresses to a Poincaré embedding of M in $\mathbb{D}^n$ with formal complement A, then there is a square, commutative up to preferred homotopy

$$\begin{tikzcd}
\partial(M \times J) \arrow[r] \arrow[d, "\text{quotient map}"'] & M/\partial M \arrow[d, "\simeq"] \\
\Sigma_u \partial M \arrow[r, "\Sigma_u \iota"] & \Sigma_u A.
\end{tikzcd}$$

Here Σ_u denotes unreduced suspension, and the rows are, respectively and essentially, inclusion of boundary of $M \times J$ in complement of uncompressed embedding, and Σ_u of inclusion of boundary of M in complement of compressed embedding. The left–hand column is (isomorphic to) the projection from $\partial(M \times J)$ to the quotient of $\partial(M \times J)$ by $M \times \partial J$. — Conversely, if the square exists, then the compression exists. Using elementary homotopy theoretic arguments, Williams managed to show that, under the hypotheses of 2.4.2, there is indeed a homotopy commutative square

$$\begin{array}{ccc} \partial(M \times J) & \longrightarrow & M/\partial M \\ {\scriptstyle \text{quotient map}} \big\downarrow & & \big\downarrow {\scriptstyle \simeq} \\ \Sigma_u \partial M & \longrightarrow & \Sigma_u A. \end{array}$$

But he did not show with homotopy theoretic methods that the lower horizontal arrow desuspends. This was done much later by Richter [Ric], who combined desuspension techniques of Hilton and Boardman–Steer [BS], Berstein–Hilton [BH], and Ganea [Ga1], [Ga2], [Ga3].

2.5. Poincaré embeddings: The fiberwise point of view

We turn to the subject of codimension zero Poincaré embeddings with arbitrary codomain. To remain as close as possible to the conceptual framework of 2.4, we use the language and notation of fiberwise homotopy theory (over the codomain, which is fixed). The idea of using fiberwise homotopy theory in the context of Poincaré duality and Poincaré embeddings is due to J. Klein and S. Weinberger, independently.

Notation, terminology. For a space Z, let $\mathcal{R}(Z)$ be the category of retractive spaces over Z. An object of $\mathcal{R}(Z)$ is a space C equipped with maps $r\colon C \to Z$ and $s\colon Z \to C$ such that $rs = \mathrm{id}_Z$. We assume that s is a cofibration. The morphisms from C_1 to C_2 are maps $f\colon C_1 \to C_2$ satisfying $fs_1 = s_2$ and $r_2 f = r_1$ where r_i and s_i are the structure maps for C_i. We call such a morphism a weak equivalence if the underlying map $C_1 \to C_2$ of spaces, without structure maps from and to Z, is a homotopy equivalence. (We will make sure that all spaces in sight are homotopy equivalent to CW–spaces.) If r_2 is a fibration, we define $[C_1, C_2]$ as the set of homotopy classes (vertical and rel Z) of morphisms $C_1 \to C_2$ in $\mathcal{R}(N)$. In general, we choose a weak equivalence $C_2 \to C_2^{\&}$ of retractive spaces over Z, where the structure map $C_2^{\&} \to Z$ is a fibration, and let

$$[C_1, C_2] := [C_1, C_2^{\&}]\,.$$

More notation. For a space X over Z and (well–behaved) subspace A of X, let $X /\!\!/ A$ be the pushout of $X \leftarrow A \to Z$, viewed as an object of $\mathcal{R}(Z)$ with obvious structure maps.

Let $(M, \partial M)$ and $(N, \partial N)$ be Poincaré pairs of the same formal dimension n. Let $g\colon M \to N$ be any map (not necessarily respecting boundaries). If

$$f\colon (M \amalg_{\partial M} C\,,\, \partial_1 C) \longrightarrow (N, \partial N)$$

is any Poincaré embedding of M in N, then we can regard the domain of f as a space over N. If f is equipped with the additional structure of a homotopy from $f|M$ to g, then the identification map from $(M \amalg_{\partial M} C) /\!\!/ \partial_1 C$ to $(M \amalg_{\partial M} C) /\!\!/ C$ can be written, modulo canonical weak equivalences, as a map

$$\eta\colon N/\!\!/\partial N \longrightarrow M/\!\!/\partial M$$

where M is viewed as a space over N by means of g. We call η the collapse map determined by f (and the homotopy from $f|M$ to g). It carries a fundamental class for $(M, \partial M)$. That is, the induced map from $N/\partial N$ to $M/\partial M$ takes any fundamental class for $(N, \partial N)$ to one for $(M, \partial M)$. Let

$$[N/\!\!/\partial N, M/\!\!/\partial M]^{\vdash} \subset [N/\!\!/\partial N, M/\!\!/\partial M]$$

consist of the homotopy classes of retractive maps $N/\!\!/\partial N \to (M/\!\!/\partial M)^{\&}$ which are fundamental–class carrying.

2.5.1. Theorem [Kln3]**.** *Let $(M, \partial M)$ and $(N, \partial N)$ be Poincaré pairs of formal dimension n. Suppose that M has the homotopy type of a CW–space of dimension m, and $\partial M \to M$ induces an isomorphism of fundamental groups. Let $g\colon M \to N$ be any map. Then the map*

$$\pi_0\,\mathrm{hofiber}_g\,(\mathrm{emb}^{\sim}_{\mathrm{PD}}(M, N) \to \mathrm{map}(M, N)) \quad\longrightarrow\quad [N/\!\!/\partial N, M/\!\!/\partial M]^{\vdash}$$

which associates to a Poincaré embedding f (with a homotopy from $f|M$ to g) its collapse map is surjective for $2n - 4 - 3m \geq 0$, and bijective for $2n - 4 - 3m > 0$.

Outline of proof, following [Kln3]. The proof is very similar to that of 2.4.2. Make M into a space over N using g. Every $[\eta]$ in $[N/\!\!/\partial N, M/\!\!/\partial M]$ can be represented by a morphism

$$\eta\colon N/\!\!/\partial N \longrightarrow (M/\!\!/\partial M)^{\&}$$

in $\mathcal{R}(N)$. If η carries a fundamental class, then it determines a Poincaré embedding of $M \times J$ in $N \times I$ whose formal complement C is the mapping cylinder of

$$\partial(M \times J) \amalg \partial(N \times I) \quad\xrightarrow{\;q \amalg \eta\;} (M/\!\!/\partial M)^{\&}.$$

The Poincaré embedding has a preferred link trivialization, a vertical null-homotopy (over N) of the composite map $M \times 1/3 \hookrightarrow \partial C \hookrightarrow C$. One shows that the link trivialization determines a compression. At this point, conversion of M and C to manifolds is not an option. So what one needs is an analogue of Hodgson's thickening theorem, 2.4.4, with Poincaré pairs instead of manifolds with boundary. Klein supplies this in [Kln1], [Kln2]. It is (currently) slightly less sharp than Hodgson's, which accounts for the loss of one dimension ($2n-4-3m$ in 2.5.1 where 2.4.2 has $2n-3-3m$). $\square$

Remarks. This proof is much closer to Williams' original proof of 2.4.2 than to the alternative homotopy theoretic proof of 2.4.2 planned by Williams and carried out by Richter.

Klein's proof of the Poincaré analogue of Hodgson's thickening theorem is homotopy theoretic, and it is the homotopy theory of retractive spaces over some fixed base space which is used.

3. Higher excision, multiple disjunction

Remark. The canonical problem in (approximate) higher excision theory of practically any kind is this. Given a finite set S and a strongly ∞–cocartesian S–cube X of spaces (perhaps subject to some conditions of a geometric nature), and a functor F, covariant or contravariant, from such spaces to spaces, find a large k such that the S–cube FX is k–cartesian. This was apparently first considered for $F = \mathrm{id}$ by Barratt and J.H.C. Whitehead [BaW], following the work of Blakers and Massey [BlM1], [BlM2] in the case $|S| = 2$. The result of [BaW] was later improved on by Ellis and Steiner [ES]; see also [Go3]. For us, X will often be a cube of manifolds, and F will often be something like 'space of embeddings to or from a fixed manifold'.

Notation, conventions. In this chapter, N denotes a compact smooth manifold with boundary, or a Poincaré pair of formal dimension n. The symbols M and L_i are reserved for (smooth compact or Poincaré) triads; here i runs through the elements of a finite set S. We assume that embeddings $\partial_0 M \to N$ and $\partial_0 L_i \to N$ are specified, with 'disjoint' images (in the Poincaré case this means that a Poincaré embedding of the disjoint union of M and the L_i in N is specified).

For $R \subset S$ write $L_R := \coprod_{i \in R} L_i$. In the smooth case, we allow only embeddings from M to N or from L_R to N which agree with the specified ones on $\partial_0 M$ or $\partial_0 L_R$, and for which $\partial_0 M$ or $\partial_0 L_R$ is the transverse preimage of ∂N. Analogous conditions are imposed in the Poincaré case ;

also, *maps* from M to N or from L_R to N are prescribed on $\partial_0 M$ and $\partial_0 L_R$. Spaces of such embeddings and maps will be denoted $\mathrm{emb}(M,N)$, $\mathrm{emb}(L_R\, ,N)$, $\mathrm{map}(M,N)$ and so on, with embellishments as appropriate, e.g., a tilde for block embedding spaces. If we wish to make the subscript R in L_R into a variable, we may write $L_\bullet$. For example, $\mathrm{emb}(L_\bullet, N)$ is short for the (contravariant) S–cube given by $R \mapsto \mathrm{emb}(L_R, N)$.

Sometimes, but not always, we assume $M \subset N$ and/or $L_i \subset N$, in which case the inclusions $M \to N$ and/or $L_i \to N$ are subject to the above conditions.

In the case where the L_i are smooth, let ℓ_i be the smallest number such that L_i can be obtained from a closed collar on $\partial_0 L_i$ by successively attaching handles of index $\leq \ell_i$. In the case where the L_i are Poincaré triads, let ℓ_i be the smallest number such that L_i is homotopy equivalent rel $\partial_0 L$ to a CW–space rel $\partial_0 L$ with cells of dimension $\leq \ell_i$ only. Let m be the corresponding number for M. (These numbers are called 'relative handle dimension' in the smooth case, and 'relative homotopy dimension' in the Poincaré case.) Let $\ell_i' := n - \ell_i - 2$.

3.1. Easy multiple disjunction for embeddings

Here we assume that M, N and L_i for $i \in S$ are smooth, and $L_S \subset N$.

3.1.1. Proposition. *The diagram* $\mathrm{emb}(M, N \smallsetminus L_\bullet)$ *is* $(1 + \Sigma_i(\lambda_i - 2))$–*cartesian, where* λ_i *is the maximum of* $(n - m - \ell_i)$ *and* 0.

Proof. Abbreviate $E_R = \mathrm{emb}(M, N \smallsetminus L_R)$ for $R \subset S$. By an easy multiple induction over the number(s) of handles needed to build M from $\partial_0 M$, and L_i from $\partial_0 L_i$, we can reduce to the case where these numbers are all equal to 1. We may then replace the handles by their cores; so now M and the L_i are disks of dimension m and ℓ_i , respectively, and $\partial_1 M = \emptyset$, $\partial_1 L_i = \emptyset$.

General position arguments show that the complement of $E_{R\cup i}$ in E_R, for $i \in S \smallsetminus R$, has codimension $\geq \lambda_i$ in E_R, and the complement of $\cup_{i \notin R} E_{R \cup i}$ has codimension $\geq \Sigma_{i\notin R}\lambda_i$ in E_R. Therefore each pair

$$(E_R, \cup_{i\notin R} E_{R\cup i})$$

is (k_{S-R})–connected where $k_T = -1 + \Sigma_{i \in T}\lambda_i$ for $T \subset S$. According to [Go3, 2.5] the cubical diagram is then k–cartesian where k is the minimum of $1 - |S| + \Sigma_\alpha k_{S(\alpha)}$ over all partitions $\{S(\alpha)\}$ of S. The minimum is achieved when S is partitioned into singletons. $\square$

In the corollary which follows, we have N and L_i for $i \in S$ as usual; there is no M and there is no preferred embedding of L_S in N.

3.1.2. Corollary. *The diagram* $\mathrm{emb}(L_\bullet, N)$ *is* $(3-n+\Sigma_i(n-2\ell_i-2))$*–cartesian.*

Proof. Choose $j \in S$ for which ℓ_j is minimal. Let $T := S \smallsetminus j$. It is enough to show that for every choice of base point e in $\mathrm{emb}(L_T, N)$, the cube given by

$$R \mapsto \mathrm{hofiber}_e\,[\,\mathrm{emb}(L_{R\cup j}, N) \xrightarrow{\text{res.}} \mathrm{emb}(L_R, N)\,]$$

for $R \subset T$ is $(3-n+\Sigma_{i\in S}(n-2\ell_i-2))$–cartesian. Here homotopy fibers over e may be replaced by fibers over e, so that we have to show that $R \mapsto \mathrm{emb}(L_j, N \smallsetminus e(L_R))$ is $(3-n+\Sigma_{i\in S}(n-2\ell_i-2))$–cartesian. But this follows directly from 3.1.1, with T instead of S and L_j instead of M. $\square$

Remark/Preview. Although 3.1.2 is not sharp, it is an excellent tool in the study of spaces of smooth embeddings $\mathrm{emb}(M, N)$ when $2m < n-2$. To handle all cases $m < n-2$, we need a stronger multiple disjunction theorem for embeddings, 3.5.1 below. This is much harder to prove. We will proceed in historical order, going through multiple disjunction and higher excision theorems for spaces of concordance embeddings, Poincaré embeddings, and block embeddings, before we get to (serious) multiple disjunction and higher excision for spaces of embeddings.

3.2. Multiple disjunction for concordance embeddings

Here we assume that M, N and L_i for $i \in S$ are smooth, $M \subset N$ and $L_i \subset N$, pairwise disjoint in N. A *concordance embedding* of M in N is a concordance of embeddings from the inclusion to some other embedding, i.e. an embedding $M \times [0,1] \to N \times [0,1]$ which

- restricts to the inclusion on $M \times 0$ and $\partial_0 M \times [0,1]$
- takes $M \times 1$ to $N \times 1$
- is transverse to the boundary of $N \times [0,1]$, the inverse image of the boundary being $M \times 0 \cup M \times 1 \cup \partial_0 M \times [0,1]$.

The space of such concordance embeddings is $\mathrm{cemb}(M, N)$. It is not essential here that N be compact. Actually in 3.2.1 and 3.2.2 below we also use concordance embedding spaces $\mathrm{cemb}(M, N \smallsetminus A)$ where A is a closed subset of N, disjoint from M.

The following theorem is a slight reformulation of the main result of [Go1]; see [Go7] for instructions.

3.2.1. Theorem. *If $m \leq n-3$ and $\ell_i \leq n-3$ for $i \in S$, then the contravariant S–cube* $\mathrm{cemb}(M, N \smallsetminus L_\bullet)$ *is $(n-m-2+\Sigma_i \ell_i')$–cartesian.*

We state the cases $|S|=0$ and $|S|=1$ explicitly:

3.2.2. Corollary. *If $m \leq n-3$, then* $\mathrm{cemb}(M,N)$ *is $(n-m-3)$–connected.*

3.2.3. Corollary. *If $m \leq n-3$ and $\ell \leq n-3$, then the inclusion map* $\mathrm{cemb}(M, N \smallsetminus L) \to \mathrm{cemb}(M,N)$ *is $(2n-m-\ell-4)$–connected.*

Corollary 3.2.2 improves on a result due to Hudson [Hu1, Thm. 9.2]. Corollary 3.2.3 is essentially the celebrated Morlet disjunction lemma (Morlet had $2n-m-\ell-4$ only for simply connected N, otherwise $2n-m-\ell-5$). There is no published proof of Morlet's lemma by Morlet, although there were course notes [Mo] at one time. The earliest published proof appears to be the one in [BLR]. For the PL version there is a proof by Millett [Milt1], [Milt2, Thm. 4.2] which uses 'sunny collapsing' (the technique which also Hatcher and Quinn used to prove their disjunction theorem 1.3.1, and which Goodwillie used to prove 3.2.1).

Note that 3.2.3 is not an obvious consequence of a relative version of the Hatcher–Quinn disjunction theorem 1.3.1. There is such a version, but the connectivity estimate we get from it is not good enough. Morlet's lemma is deeper than the Hatcher–Quinn theorem, although it is older. (Conversely, the Hatcher–Quinn theorem is a much better introduction to the subject of disjunction than Morlet's lemma.)

In applications later on, the special case of 3.2.1 where M and the L_i have the same dimension as N is most important. In that case we allow ourselves to mean by $N \smallsetminus L_R$, $N \smallsetminus M$ etc. the *closure* of the complement of L_R, M etc. in N. There is a fibration sequence

$$C(N \smallsetminus M \smallsetminus L_R) \longrightarrow C(N \smallsetminus L_R) \longrightarrow \mathrm{cemb}(M, N \smallsetminus L_R)$$

where C is for spaces of smooth concordances. (A concordance of P is a diffeomorphism $P \times [0,1] \to P \times [0,1]$ restricting to the identity on $\partial P \times [0,1]$ and on $P \times 0$.) From 3.2.2 we also know that $\mathrm{cemb}(M, N \smallsetminus L_R)$ is connected if $m \leq n-3$, in which case we get another homotopy fiber sequence

$$\mathrm{cemb}(M, N \smallsetminus L_R) \longrightarrow BC(N \smallsetminus M \smallsetminus L_R) \longrightarrow BC(N \smallsetminus L_R)\,.$$

Therefore 3.2.1 implies that the diagram $BC(N \smallsetminus M \smallsetminus L_\bullet) \to BC(N \smallsetminus L_\bullet)$ is $(n-m-2+\Sigma_i \ell_i')$–cartesian. Renaming M as one of the L_i, and enlarging S accordingly, we have:

3.2.4. Corollary. *If* $\ell_i \le n-3$ *for all* $i \in S$, *then* $BC(N \smallsetminus L_\bullet)$ *is* $\Sigma_i \ell_i'$*–cartesian.*

3.3. Multiple disjunction for Poincaré embeddings

Here we assume that N is a Poincaré pair and that M and the L_i for $i \in S$ are Poincaré triads, all of the same formal dimension n. A Poincaré embedding e of L_S in N is fixed. For $R \subset S$ we denote by $N \smallsetminus L_R$ the formal complement of $e|L_R$, viewed as a Poincaré pair.

3.3.1. Theorem. *If* $m \le n-3$ *and* $\ell_i \le n-3$ *for* $i \in S$, *then the diagram* $\mathrm{emb}_{\mathrm{PD}}(M, N \smallsetminus L_\bullet) \longrightarrow \mathrm{map}(M, N \smallsetminus L_\bullet)$ *is* $(n-2m-2+\Sigma_i \ell_i')$*–cartesian.*

Remarks. The special case $|S| = 1$ is the (codimension zero) Poincaré version of 1.3.1; notice a loss of 1 in the connectivity estimate. In the general form, 3.3.1 is an important ingredient in the proof of 3.5.3 below, a 'multiple' version of 1.3.1, again for smooth embeddings; somewhat miraculously the loss of 1 can be repaired in the deduction.

There is a version of 3.3.1 where M and the L_i are allowed to have arbitrary formal dimensions $\le n$, and where the relative homotopy dimensions m and ℓ_i are replaced by the formal dimensions of M and the L_i. This is an easy consequence of 3.3.1 as it stands.

The full proof of 3.3.1 is still in preparation [GoKl], but a slightly weaker result is proved in [Go6]. Let $H(N \smallsetminus L_R)$ be the space of homotopy automorphisms of $N \smallsetminus L_R$ relative to the boundary. Select a base vertex in $\mathrm{emb}_{\mathrm{PD}}(M, N \smallsetminus L_S)$ if possible. Let

$$\mathcal{X}_R := \mathrm{hofiber}\,[\,H(N \smallsetminus L_R) \xrightarrow{|M} \mathrm{emb}_{\mathrm{PD}}(M, N \smallsetminus L_R)\,]\,,$$
$$\mathcal{Y}_R := \mathrm{hofiber}\,[\,H(N \smallsetminus L_R) \xrightarrow{|M} \mathrm{map}(M, N \smallsetminus L_R)\,]\,.$$

The forgetful arrows $\mathcal{X}_R \to \mathcal{Y}_R$ lead to a diagram $\mathcal{X}_\bullet \to \mathcal{Y}_\bullet$. It is shown in [Go6] that this is $(n-2m-3+\Sigma_i \ell_i')$–cartesian. The looped version of 3.3.1 follows since the diagram $H(N \smallsetminus L_\bullet) \longrightarrow H(N \smallsetminus L_\bullet)$ given by the identity maps $H(N \smallsetminus L_R) \to H(N \smallsetminus L_R)$ is ∞–cartesian.

3.3.2. Corollary. *If* $m \le n-3$ *and* $\ell_i \le n-3$ *for* $i \in S$, *then the diagram* $\mathrm{emb}_{\mathrm{PD}}(M, N \smallsetminus L_\bullet)$ *is* $(1-m+\Sigma_i \ell_i')$*–cartesian.*

Sketch proof modulo 3.3.1. The diagram $\mathrm{map}(M, N \smallsetminus L_\bullet)$ is $(1-m+\Sigma_i \ell_i')$–cartesian. $\square$

Corollary 3.3.2 has a more symmetrical reformulation as a 'higher excision theorem', obtained by renaming M as one of the L_i. (The hypotheses here are a little different: there is no M anymore, since it has been renamed, and no preferred Poincaré embeddings of the L_i in N are specified; but as usual, the $\partial_0 L_i$ are embedded in ∂N.)

3.3.3. Corollary. *If $\ell_i \le n-3$ for $i \in S$, then the diagram* $\mathrm{emb}_{\mathrm{PD}}(L_\bullet, N)$ *is $(3-n+\Sigma_i \ell'_i)$–cartesian.*

Proof modulo 3.3.2. The case $S=\emptyset$ is trivial. Assume $S \neq \emptyset$. Pick $j \in S$. Let $T = S \smallsetminus j$. By [Go3, 1.18] it suffices to show that for every choice of base point e in $\mathrm{emb}_{\mathrm{PD}}(L_T, N)$, the T–cube

$$\mathrm{hofiber}\,[\,\mathrm{emb}_{\mathrm{PD}}(L_{\bullet\cup j}, N) \to \mathrm{emb}_{\mathrm{PD}}(L_\bullet, N)\,]$$

(where $\bullet$ stands for a variable subset of T) is $(3-n+\Sigma_{i\in S}\ell'_i)$–cartesian, in other words $(1-\ell_j+\Sigma_{i\in T}\ell'_i)$–cartesian. But this follows from 3.3.2 (with T in place of S and L_j in place of M), since the homotopy fiber of $\mathrm{emb}_{\mathrm{PD}}(L_{R\cup j}, N) \to \mathrm{emb}_{\mathrm{PD}}(L_R, N)$ over $e|L_R$ is homotopy equivalent to $\mathrm{emb}_{\mathrm{PD}}(L_j, N \smallsetminus L_R)$. $\square$

3.4. Higher excision for block embeddings

Here we assume that N and the L_i for $i \in S$ are smooth, all of the same formal dimension n. There is no M.

3.4.1. Theorem. *If $n \ge 5$ and $\ell_i \le n-3$ for $i \in S$, then the diagram* $\mathrm{emb}^{\sim}(L_\bullet, N)$ *is $(3-n+\Sigma_i\ell'_i)$–cartesian.*

Proof. By 3.3.2 and [Go3, 1.18], it is enough to show that for every choice of base point in $\mathrm{emb}_{\mathrm{PD}}{}^{\sim}(L_S, N)$, the diagram

$$\mathrm{hofiber}\,[\,\mathrm{emb}^{\sim}(L_\bullet, N) \to \mathrm{emb}_{\mathrm{PD}}{}^{\sim}(L_\bullet, N)\,]$$

is $(3-n+\Sigma_i\ell'_i)$–cartesian. But this is ∞–cartesian since, by a mild generalization of 2.2.2, we can identify it with

$$\mathrm{hofiber}\,[\,\mathrm{imm}^{\sim}(L_\bullet, N) \to \mathrm{imm}_{\mathrm{PD}}{}^{\sim}(L_\bullet, N)\,]. \quad \square$$

3.5. Higher excision for embeddings

3.5.1. Theorem. *Under the hypotheses of* 3.4.1*, the diagram* $\mathrm{emb}(L_\bullet, N)$ *is* $(3-n+\Sigma_i\ell'_i)$*–cartesian.*

There is an equivalent 'multiple disjunction' version:

3.5.2. Theorem. *If* $n \geq 5$ *and* $n-m \geq 3$*,* $n-\ell_i \geq 3$ *for all* i*, then the cube* $\mathrm{emb}(M, N \smallsetminus L_\bullet)$ *is* $(1-m+\Sigma_i\ell'_i)$*–cartesian.*

Outline of proof of 3.5.1. Choose a base vertex e in $\mathrm{emb}^\sim(L_S, N)$. For $R \subset S$ let $\mathcal{X}_R$ be the homotopy fiber (over $e|L_R$) of the inclusion of $\mathrm{emb}(L_R, N)$ in $\mathrm{emb}^\sim(L_R, N)$. By 3.4.1, it suffices to show that $\mathcal{X}_\bullet$ is $(3-n+\Sigma_i\ell'_i)$–cartesian. There are homotopy fiber sequences

$$\frac{\mathrm{diff}^\sim(N \smallsetminus L_R)}{\mathrm{diff}(N \smallsetminus L_R)} \longrightarrow \frac{\mathrm{diff}^\sim(N)}{\mathrm{diff}(N)} \longrightarrow \mathcal{X}_R$$

where $N \smallsetminus L_R$ is short for the closure of the complement of $e(L_R)$ in N, and all diffeomorphisms in sight restrict to the identity on the appropriate boundary. Therefore (and because $\mathcal{X}_S$ is connected, by 3.2.2) it is enough to show that $\mathcal{Y}(N \smallsetminus L_\bullet)$ is $(2-n+\Sigma_i\ell'_i)$–cartesian, where

$$\mathcal{Y}(P) := \frac{\mathrm{diff}^\sim(P)}{\mathrm{diff}(P)}$$

for a compact smooth P. In fact we will show (twice) that $\mathcal{Y}(N \smallsetminus L_\bullet)$ is $\Sigma_i\ell'_i$–cartesian.

First argument. One of the main results of [WW1], motivated by a spectral sequence due to Hatcher [Hat], says that $\mathcal{Y}(P)$ is, up to homotopy equivalence, the homotopy colimit of a diagram

$$* = F_0\mathcal{Y}(P) \to F_1\mathcal{Y}(P) \to F_2\mathcal{Y}(P) \to \dots$$

where each arrow fits into a homotopy fiber sequence

$$F_j\mathcal{Y}(P) \hookrightarrow F_{j+1}\mathcal{Y}(P) \to B^{j+1}C(P \times [0,1]^j)\,.$$

(Here B^{j+1} denotes $(j+1)$–fold j–connected deloopings.) All of this depends naturally on P, with respect to codimension zero embeddings. Hence it is enough to show that $B^{j+1}C((N \smallsetminus L_\bullet)\times[0,1]^j)$ is $\Sigma\ell'_i$–cartesian, and more than enough to show that it is $(j+\Sigma\ell'_i)$-cartesian. But this follows easily from 3.2.4 (use induction on $|S|$).

Second argument. Again we think of $\mathcal{Y}$ as a functor on compact smooth manifolds and codimension zero embeddings. There is a natural homotopy fiber sequence

$$\mathcal{Y}(P\times[0,1]) \longrightarrow \frac{C^{\sim}(P)}{C(P)} \longrightarrow \mathcal{Y}(P)$$

where $C^{\sim}(P)$, the 'block' version of $C(P)$, is contractible, so that the term in the middle is $BC(P)$. Moreover $\mathcal{Y}(P)$ is connected for any P. These properties are strong enough to imply that the higher excision estimates for the functor BC are also valid for the functor $\mathcal{Y}$. See [Go7] for the details, which are quite elementary. $\square$

3.5.1. Theorem [bis]. *The hypothesis* $n \geq 5$ *in* 3.5.1 *and* 3.5.2 *is unnecessary.*

Idea of proof. If $n=3$ then necessarily $m=0$, so that 3.5.2 for $n=3$ follows from 3.1.1. Now assume $n=4$. The looped versions of 3.4.1 and 3.4.4 are then still valid, with the same proof, and for any compatible choice of base points. The looped version of 3.5.2 with $n=4$ follows, as before, for any choice of base point in $\mathrm{emb}(M, N \smallsetminus L_S)$. Moreover 3.1.1 shows that the diagram in 3.5.2 (but with $n=4$) is 1–cartesian. This is enough. $\square$

The higher excision theorem 3.5.1 leads to a multiple disjunction theorem for embeddings and maps, in the style of 1.3.1 and 3.3.1. To state it we return to the setup with N, M and L_i for $i \in S$, all of the same dimension n; an embedding $L_S \to N$ is specified.

3.5.3 Theorem. *If* $m \leq n-3$ *and* $\ell_i \leq n-3$ *for all* $i \in S$*, then the diagram* $\mathrm{emb}(M, N \smallsetminus L_\bullet) \to \mathrm{map}(M, N \smallsetminus L_\bullet)$ *is* $(n-2m-1+\Sigma_i \ell'_i)$*–cartesian.*

The case $|S|=1$ of 3.5.3 is the codimension zero case of 1.3.1. Again there exists a version of 3.5.3 where the codimensions of M and the L_i in N are arbitrary. This follows easily from 3.5.3 as it stands.

Idea of proof of 3.5.3. One reduces to the case where M can be obtained from a closed collar on $\partial_0 M$ by attaching a single handle. That case is dealt with by induction on the handle index. (The case where the handle index is zero is trivial.) The induction step uses 3.5.2 and a device called handle splitting. See [Go7, §4,§6] for all details, also [BLR, pf. of 2.3] for handle splitting. — We will indicate another proof (modulo 3.5.1 or 3.5.2) in §4.

4. Calculus methods: Homotopy aspect

In this chapter we approach the 'calculation' of a space of smooth embeddings $\mathrm{emb}(M, N)$ by viewing it as a special value of the cofunctor

$$V \mapsto \mathrm{emb}(V, N)$$

on the poset $\mathcal{O}(M)$ of open subsets V of M. The multiple disjunction and higher excision theorems of chapter 3 imply that if $m \leq n-3$, then this cofunctor on $\mathcal{O}(M)$ admits a unique decomposition (Taylor tower) into so–called homogeneous cofunctors, one of each degree $k > 0$. The homogeneous cofunctors are easy to understand and classify. So we end up with something like a functorial calculation of the homotopy type of $\mathrm{emb}(V, N)$, up to extension problems. There is no doubt that the extension problems are serious.

4.1. Taxonomy of cofunctors on $\mathcal{O}(M)$

Let U, V be smooth m–manifolds without boundary. A smooth embedding $e_1\colon U \to V$ is an *isotopy equivalence* if there exists a smooth embedding $e_2\colon V \to U$ such that e_1e_2 and e_2e_1 are smoothly isotopic to id_V and id_U , respectively.

4.1.1. Definition. We fix M and write $\mathcal{O} := \mathcal{O}(M)$. A cofunctor F from $\mathcal{O}$ to spaces is *good* if

(i) it takes isotopy equivalences to weak homotopy equivalences (that is, if an inclusion $U \to V$ of open subsets of M is an isotopy equivalence, then the induced map $F(V) \to F(U)$ is a weak homotopy equivalence);

(ii) it takes monotone unions to homotopy inverse limits (that is, if V_i for $i \geq 0$ are open sets in M with $V_i \subset V_{i+1}$, then the canonical map from $F(\bigcup_i V_i)$ to $\mathrm{holim}_i\, F(V_i)$ is a weak homotopy equivalence).

Remark. Call $V \in \mathcal{O}$ *tame* if V is the interior of a compact smooth (codimension zero) submanifold of M. Property (ii) ensures that a good cofunctor F on $\mathcal{O}$ is essentially determined by its behavior on tame open subsets of M. In particular, suppose that F is a cofunctor from $\mathcal{O}(M)$ to spaces having property (i). Then the functor defined by

$$F^\sharp(V) := \operatorname*{holim}_{\text{tame } U \subset V} F(U)$$

for $V \in \mathcal{O}$ is a good cofunctor on $\mathcal{O}$. We call $F^\sharp$ the *taming* of F. Note that $F^\sharp(V) \simeq F(V)$ if V is a tame open subset of M.

4.1.2. Examples. It is not hard to show that the cofunctors given by $V \mapsto \mathrm{emb}(V,N)$, $V \mapsto \mathrm{emb}^{\sim}(V,N)$, $V \mapsto \mathrm{imm}(V,N)$, $V \mapsto \mathrm{imm}^{\sim}(V,N)$ (for fixed smooth N without boundary, and variable V in $\mathcal{O}$) are good. See [We1] for details.

For another example, fix $k \geq 1$, and let $F(V)$ be the space of smooth immersions $g\colon V \to N$ with $|g^{-1}(x)| \leq k$ for all $x \in N$. Then the taming $F^{\sharp}$ of F is good.

4.1.3. Definition. Fix $k \geq 0$. A good cofunctor F on $\mathcal{O}$ is *polynomial of degree* $\leq k$ if, for every $V \in \mathcal{O}$ and pairwise disjoint closed subsets $A_1, \ldots, A_{k+1}$ of V, the $(k+1)$–cube $F(V \smallsetminus A_\bullet)$ is ∞–cartesian. (Here $A_R = \bigcup_{i \in R} A_i$ for a subset R of $\{1, ..., k+1\}$.)

4.1.4. Example. Fix a space X and let $F(V) := \mathrm{map}(V^k, X)$ for $V \in \mathcal{O}$, where V^k means $V \times \cdots \times V$ (k factors). Then F is polynomial of degree $\leq k$. *Idea of proof:* Given V and $A_1, \ldots, A_{k+1}$ as in 4.1.3, one notes using a pigeon hole argument that V^k is the union of the $(V \smallsetminus A_R)^k$ for nonempty $R \subset \{1, ..., k+1\}$. This implies easily that the cubical diagram $(V \smallsetminus A_\bullet)^k$ is ∞–cocartesian. Therefore it turns into an ∞–cartesian diagram when $\mathrm{map}(—, X)$ is applied.

4.1.5. Example. Let $\mathcal{O}k \subset \mathcal{O}$ be the full sub–poset consisting of the V which are diffeomorphic to $\mathbb{R}^m \times S$ with S discrete, $|S| \leq k$. For a good cofunctor F on $\mathcal{O}$, let $T_k F$ be the homotopy right Kan extension (along $\mathcal{O}k \hookrightarrow \mathcal{O}$) of $F|\mathcal{O}k$. Explicitly:

$$T_k F(V) := \underset{\substack{W \subset V \\ W \in \mathcal{O}k}}{\mathrm{holim}}\, F(W)\,.$$

Then $T_k F$ is again a good cofunctor. The 'operator' T_k on good cofunctors comes with an obvious forgetful transformation $\eta_k\colon F(V) \to T_k F(V)$, natural not only in V but also in F. The pair consisting of T_k and η_k has the following properties:

(i) $T_k F$ is polynomial of degree $\leq k$, for any good F.
(i) $\eta_k\colon F(V) \to T_k F(V)$ is a weak homotopy equivalence for all V if F is already polynomial of degree $\leq k$.
(ii) $T_k(\eta_k)\colon T_k F(V) \to T_k(T_k F)(V)$ is (always) a weak homotopy equivalence.

These properties essentially characterize T_k and η_k. One should think of $\eta_k\colon F \to T_k F$ as the best approximation (from the right) of F by a polynomial cofunctor of degree $\leq k$. (We also call it the k–th Taylor approximation of F.) In fact, any natural transformation $v\colon F \to G$ where G is polynomial of degree $\leq k$ can be enlarged to a commutative square of natural transformations

$$\begin{array}{ccc} F & \xrightarrow{v} & G \\ \Big\downarrow{\scriptstyle \eta_k} & & \Big\downarrow{\scriptstyle \eta_k} \\ T_kF & \xrightarrow{T_k v} & T_kG \end{array}$$

where the right–hand column is a natural weak homotopy equivalence by property (i) of T_k and η_k. Thus $v\colon F \to G$ factors through $\eta_k\colon F \to T_kF$, up to formal inversion of a natural weak homotopy equivalence. Property (ii) can be used to show that the factorization is essentially unique (a suitable category of such factorizations has a contractible nerve). See [We1] for all details.

4.1.6. Examples. Suppose that $F(V) = \mathrm{emb}(V, N)$ where N^n is fixed smooth manifold without boundary, and $n > m = \dim(M)$. We will make T_kF explicit for $k = 1$ and $k = 2$. See also 4.3.

Let $F_1(V) := \mathrm{imm}(V, N)$. The natural inclusion $\iota_1\colon F \to F_1$ has the following properties (the first by the immersion classification theorem, the other by inspection):

- the codomain F_1 of ι_1 is polynomial of degree ≤ 1 ;
- ι_1 specializes to a weak homotopy equivalence $F(V) \to F_1(V)$ whenever V is a tubular neighborhood of a single point.

But these two properties of ι_1 essentially characterize $\eta_1\colon F \to T_1F$; so $T_1F(V) \simeq F_1(V) = \mathrm{imm}(V, N)$, by a chain of natural weak homotopy equivalences.

Using the notation from Haefliger's theorem 1.2.1, let $F_2(V)$ be the homotopy pullback (homotopy inverse limit) of the diagram

$$\begin{array}{ccc} & & \mathrm{map}(V, N) \\ & & \Big\downarrow{\scriptstyle f \mapsto f \times f} \\ \mathrm{ivmap}^{\mathbb{Z}/2}(V \times V, N \times N) & \xrightarrow{\subset} & \mathrm{map}^{\mathbb{Z}/2}(V \times V, N \times N)\,. \end{array}$$

Then there is a forgetful natural transformation $\iota_2\colon F \to F_2$. One checks easily that

- the codomain F_2 of ι_2 is polynomial of degree ≤ 2 ;
- ι_2 specializes to a weak homotopy equivalence $F(V) \to F_2(V)$ whenever V is a tubular neighborhood of a subset S of M of cardinality ≤ 2.

(The homotopy inverse limit of a diagram of good cofunctors on $\mathcal{O}$ which are polynomial of degree $\le k$ is again polynomial of degree $\le k$. Therefore, to show that F_2 is polynomial of degree ≤ 2, it suffices to show that the cofunctors

$$\begin{gathered} V \mapsto \mathrm{map}(V, N) \\ V \mapsto \mathrm{map}^{\mathbb{Z}/2}(V \times V, N \times N) \\ V \mapsto \mathrm{ivmap}^{\mathbb{Z}/2}(V \times V, N \times N) \end{gathered}$$

are polynomial of degree $\le 1, 2, 2$ respectively, and this can be done much as in 4.1.4.) These properties of ι_2 essentially characterize $\eta_2\colon F \to T_2 F$, and it follows that $T_2F(V) \simeq F_2(V)$ by a chain of natural weak homotopy equivalences.

4.1.7. Definition. A good cofunctor F on $\mathcal{O}$ is *homogeneous of degree* k if it is polynomial of degree $\le k$ and if $T_{k-1}F(V)$ is weakly homotopy equivalent to a point, for all $V \in \mathcal{O}$.

4.1.8. Example. Let $\binom{M}{k}$ be the space of unordered configurations of k distinct points in M. Let

$$p\colon E \to \binom{M}{k}$$

be a fibration. Suppose that this is equipped with the structure of a germ σ of partial sections, defined 'near' the fat diagonal (complement of $\binom{M}{k}$ in the space of unordered k–tuples of points in M). For $V \in \mathcal{O}$ let $F(V)$ be the space of partial sections of p which are defined on $\binom{V}{k}$ and agree with σ near the fat diagonal. Then F is a good cofunctor which is homogeneous of degree k. There is a classification theorem for homogeneous cofunctors on $\mathcal{O}$ which says that they can all be obtained in this way (up to a natural weak homotopy equivalence), from a pair (p, σ) as above, unique up to fiber homotopy equivalence respecting section germs. We call p the classifying fibration of the homogeneous cofunctor.

If F is any good cofunctor on $\mathcal{O}$, with a preferred base point in $F(M)$, then L_kF defined by

$$L_kF(V) := \mathrm{hofiber}\,[\,T_kF(V) \xrightarrow{\text{forget}} T_{k-1}F(V)\,]$$

is a homogeneous cofunctor of degree k. Its classifying fibration p on $\binom{M}{k}$ must have a preferred global section σ, corresponding to the base point of $L_kF(M)$. The fibration p and the global section σ can be described roughly as follows. For $S \subset M$ with $|S| = k$, and each $x \in S$, choose a small open ball V_x about x. For $R \subset S$ let $V_R = \cup_{x\in R} V_x$. Then the fiber of p over S is the total homotopy fiber of the contravariant S–cube $R \mapsto F(V_R)$. Note that this is a pointed space.

4.1.9. Example. Let $F(V) = \mathrm{emb}(V, N)$. Fix a base point in $F(M)$, alias embedding $M \to N$. We describe the classifying fibration(s) p_k for L_kF, any $k > 0$, simplifying the general description in 4.1.8 as much as possible. First, p_1 is the forgetful map and fibration

$$E_1 \longrightarrow M$$

where $E_1 = \{(x, z, f) \mid x \in M, z \in N,\ f\colon T_xM \to T_zN \text{ linear injective}\}$. Second, p_k for $k > 1$ is the fibration

$$E_k \longrightarrow \binom{M}{k}$$

whose fiber over $S \in \binom{M}{k}$ is the total homotopy fiber of the cubical diagram of pointed spaces given by $R \mapsto \mathrm{emb}(R, N)$ for $R \subset S$. (These spaces are pointed because $R \subset S \subset M \subset N$.) To see that these are correct descriptions, make a forgetful map, between spaces over $\binom{M}{k}$, from the standard description of p_k (classifying fibration for L_kF) as given in 4.1.8 to the new description under scrutiny; then verify that it is a fiberwise homotopy equivalence.

4.1.10. Definition. Let F be a good cofunctor F on $\mathcal{O}$. We say that F is *ρ–analytic with excess c* (where $\rho, c \in \mathbb{Z}$) if it has the following property. For $V \in \mathcal{O}$ and $k > 0$ and pairwise disjoint closed subsets $A_1, \ldots, A_{k+1}$ of V, where each A_i is a smooth submanifold of V of codimension $q_i < \rho$, diffeomorphic to euclidean space, the cube $F(V \smallsetminus A_\bullet)$ is $(c + \Sigma_i(\rho - q_i))$–cartesian.

Remark. To motivate 4.1.10 just a little, we note that the definition of a polynomial cofunctor, 4.1.3, can be reformulated as follows. A good cofunctor F on $\mathcal{O}$ is polynomial of degree $\le k$ if it has the following property. For $V \in \mathcal{O}$ and pairwise disjoint closed subsets $A_1, \ldots, A_{k+1}$ of V, where each A_i is a smooth submanifold of V, diffeomorphic to a euclidean space, the cube $F(V \smallsetminus A_\bullet)$ is ∞–cartesian.

Indeed, if F has the property, then $F(V \smallsetminus B_\bullet)$ will be ∞–cartesian whenever $V \in \mathcal{O}$ is tame, $B_1, \ldots, B_{k+1}$ are pairwise disjoint closed subsets of V, and the closure $\bar{B}_i$ of B_i in M is a compact codimension zero smooth manifold triad embedded in $\bar{V}$, with $\partial_0 \bar{B}_i = \bar{B}_i \cap \partial \bar{V}$ and $\partial_1 \bar{B}_i$ transverse to $\partial \bar{V}$. The proof is by an easy (multiple) induction over the number of handles required to build each $\bar{B}_i$ from a collar on $\partial_0 \bar{B}_i$. An application of the limit axiom for good cofunctors then shows that $F(V \smallsetminus C_\bullet)$ will be ∞–cartesian whenever $V \in \mathcal{O}$ is tame, and $C_1, \ldots, C_{k+1}$ are pairwise disjoint closed subsets of V.

4.1.11. Digression/Definition. Given a finite set S and an S–cube $\mathcal{X}$ of spaces and $z \in \mathbb{R}$, let us say that $\mathcal{X}$ is z–cartesian if the canonical map

$$\mathcal{X}(\emptyset) \longrightarrow \operatorname*{holim}_{\emptyset \neq R \subset S} \mathcal{X}(R)$$

has connectivity $\geq z$. With this convention, 4.1.10 remains meaningful for arbitrary $\rho, c \in \mathbb{R}$. This will become important in §5.

4.1.12. Definitions. The theory has a variant where M is a manifold with boundary, and F is a cofunctor on $\mathcal{O}(M)$, the poset of all open subsets of M containing ∂M. The kind of functor we have in mind is $V \mapsto \mathrm{emb}(V, N)$ where N is fixed, with boundary, and an embedding $e\colon \partial M \to \partial N$ has been specified. In the definition of $\mathrm{emb}(V, N)$ we allow only embeddings $V \to N$ which agree with e on ∂V, and are transverse to ∂M.

A good cofunctor F from $\mathcal{O}(M)$ to spaces is *polynomial of degree* $\leq k$ if $F(V \smallsetminus A_\bullet)$ is ∞–cartesian for any $V \in \mathcal{O}(M)$ and pairwise disjoint subsets $A_0, \ldots, A_k$ of V, closed in V and disjoint from ∂M. The k–th Taylor approximation $T_k F$ of an arbitrary good cofunctor F on $\mathcal{O}(M)$ is defined by

$$T_k F(V) := \operatorname*{holim}_{\substack{W \in \mathcal{O}k \\ W \subset V}} F(W)$$

where $\mathcal{O}k = \mathcal{O}k(M)$ consists of the $W \in \mathcal{O}(M)$ which are tubular neighborhoods of $\partial M \cup S$ for some subset S of $M \smallsetminus \partial M$, with $|S| \leq k$. A homogeneous functor F of degree k on $\mathcal{O}(M)$ has a *classifying fibration*

$$p\colon E \longrightarrow \binom{M}{k}$$

equipped with a germ σ of sections, defined near fat diagonal *and* on the boundary. Then $F(V)$ is, up to a chain of natural homotopy equivalences, the space of (partial) sections of p defined over $\binom{V}{k}$ and agreeing with σ near the fat diagonal and on the boundary. The classifying fibration p_k for the k–th homogeneous layer, $k \geq 2$, of the cofunctor $V \mapsto \mathrm{emb}(V, N)$, as above, has fiber $p_k^{-1}(S)$ equal to the total homotopy fiber of the cube

$$R \mapsto \mathrm{emb}(R, N) \qquad (R \subset S).$$

4.2. The convergence theorem

The *Taylor tower* of a good cofunctor F on $\mathcal{O}$ is the diagram of good cofunctors and (forgetful) transformations

$$\cdots \xrightarrow{r_{k+1}} T_k F \xrightarrow{r_k} T_{k-1} F \xrightarrow{r_{k-1}} T_{k-2} F \xrightarrow{r_{k-2}} \cdots .$$

It should be regarded as a diagram of cofunctors *under* F, since for each k we have $\eta_k \colon F \to T_k F$ and the relations $r_k \eta_k = \eta_{k-1}$ hold.

4.2.1. Theorem. *Suppose that F is ρ–analytic with excess c, and $V \in \mathcal{O}$ has a proper Morse function with critical points of index $\leq q$ only, where $q < \rho$. Then the connectivity of*

$$\eta_{k-1} \colon F(V) \longrightarrow T_{k-1} F(V)$$

is $\geq c + k(\rho - q)$, for $k > 1$. Therefore $F(V) \xrightarrow{\simeq} \mathrm{holim}_k T_k F(V)$. In words, the Taylor tower of F, evaluated at V, converges to $F(V)$.

See [GoWe, 2.3] for the proof, which is quite easy. Although originally intended for the situation where $\rho, c \in \mathbb{Z}$, it goes through with arbitrary $\rho, c \in \mathbb{R}$. Compare 4.1.11.

4.2.2. Corollary. *If F is ρ–analytic, and $\rho > m = \dim(M)$, then $F(V) \simeq \mathrm{holim}_k T_k F(V)$ for all $V \in \mathcal{O}$.*

4.2.3. Theorem–Example. *Let $F(V) = \mathrm{emb}(V, N)$ for $V \in \mathcal{O}$, where N^n is fixed (smooth, without boundary). Then F is $(n-2)$–analytic with excess $3 - n$.*

Idea of proof. Fix a finite set S. It suffices to check that $F(V \smallsetminus A_\bullet)$ is $(3 - n + \Sigma_i (n - q_i - 2))$–cartesian if

- $V \in \mathcal{O}$ is tame;
- $A_i = D_i \cap V$ for $i \in S$, where $D_i \subset \bar{V}$ is a smoothly embedded disk of codimension $q_i < n - 2$, transverse to the boundary of $\bar{V}$, with $\partial D_i = D_i \cap \partial \bar{V}$, and the D_i are pairwise disjoint.

Next, fix some smooth embedding $e\colon V \smallsetminus A_S \to N$. It is enough to show that the cube

$$\text{hofiber}_e \left[F(V \smallsetminus A_\bullet) \longrightarrow F(V \smallsetminus A_S) \right]$$

is $(3 - n + \Sigma_{i \in S}(n - q_i - 2))$–cartesian. We can assume that e extends to a smooth embedding $\bar{e}\colon \bar{V} \to N$, and further, to a codimension zero embedding $f\colon W \to N$ where $W \to \bar{V}$ is the disk bundle of the normal bundle of $\bar{e}$. Let N' be the closure in N of the complement of $f(W)$. Then

$$\text{hofiber}_e \left[F(V \smallsetminus A_R) \longrightarrow F(V \smallsetminus A_S) \right]$$

is naturally homotopy equivalent to $\text{emb}(D_R, N')$ where $D_R = \cup_{i \in R} D_i$ for $R \subset S$. (Note that preferred embeddings $\partial D_i \to \partial N'$ are given.) Hence it is enough to show that the cube $\text{emb}(D_\bullet, N')$ is $(3 - n + \Sigma_{i \in S}(n - q_i - 2))$–cartesian. But this follows from 3.5.1 (actually, the 'arbitrary codimension' version of 3.5.1). $\square$

4.2.4. Corollary/Summary. *Let* $F(V) = \text{emb}(V, N)$*, and assume that the codimension* $n - m$ *is* ≥ 3*. Suppose for simplicity* $M \subset N$*, so that each* $F(V)$ *is a based space. Then*

$$\eta_{k-1}\colon F(V) \longrightarrow T_{k-1}F(V)$$

is $(3 - n + k(n - m - 2))$*–connected; therefore* $F(V) \xrightarrow{\simeq} \text{holim}_k T_k F(V)$*. We have* $T_1 F(V) \simeq \text{imm}(V, N)$*. For* $k > 1$*, the homotopy fiber* $L_k F(V)$ *of* $T_k F(V) \to T_{k-1} F(V)$ *is homotopy equivalent to the space of sections, vanishing near the fat diagonal, of*

$$p_k\colon E_k \longrightarrow \binom{M}{k}$$

where $p_k^{-1}(S)$ *for* $S \in \binom{M}{k}$ *is the total homotopy fiber of the* S*–cube defined by* $R \mapsto \text{emb}(R, N)$ *for* $R \subset S$*.*

Remark. There is a considerable shortcut to corollary 4.2.4 in the cases where $2m < n - 2$. In those cases we can avoid most of chapter 3, using only the easy higher excision theorem, in the symmetric form 3.1.2, to show that F is $(n - m - 2)$–analytic. Since $m < n - m - 2$, this implies according to 4.2.2 that

$$F(V) \xrightarrow{\simeq} \operatorname*{holim}_k T_k F(V)$$

for all $V \in \mathcal{O}$. The analysis of the layers $L_k F(V)$ goes through as before. We can now use 4.1.9 and again 3.1.2 to show that the fibers of the classifying fibration for L_kF are $(k+1)(n-2)$–connected; hence $L_kF(V)$ is $((k+1)(n-2)-mk)$–connected and $T_kF(V) \to T_{k-1}F(V)$ is $((k+1)(n-2)-mk+1)$–connected, i.e., $(3-n+k(n-m-2))$–connected. It follows that η_k from $F(V) \simeq \mathrm{holim}_k\, T_kF(V)$ to $T_{k-1}F(V)$ is $(3-n+k(n-m-2))$–connected. □

4.2.5. Example. This example is meant to illustrate the 'with boundary' variant of 4.2.4. Suppose that $M=[0,1]$ and that N has a boundary, and $M \subset N$ as a submanifold, ∂M being the transverse intersection of M with ∂N. Let $F(V) := \mathrm{emb}(V,N)$ as in 4.1.12. Then

$$\binom{M}{k} \cong \Delta^k$$

and so the k–th homogeneous layer $L_kF(M)$ becomes the k–th loop space of any of the fibers of the classifying fibration for L_kF. If in addition N is homotopy equivalent to a suspension, $N \simeq \Sigma Y$, then this can be analyzed with the Hilton-Milnor theorem, and one finds

$$L_kF(M) \simeq \prod_w{}' \Omega^k \Sigma^{1+\alpha(w)(n-2)} Y^{(\beta(w))}$$

for $k>1$, where the weak product $\prod'$ is over all basic words w in the letters $z_1,\dots,z_k$ involving all letters except possibly z_1. See [GoWe, §5] for more details and explanations.

4.2.6. Remark. Two different calculus approaches to block embedding spaces $\mathrm{emb}^\sim(M,N)$ come to mind. One of these is to view $\mathrm{emb}^\sim(M,N)$ as a special value of a good cofunctor F on $\mathcal{O}(M)$, and to approximate it by the $(T_rF)(M)$ for $r \geq 0$. The other is to think of $\mathrm{emb}^\sim(M,N)$ as the geometric realization of a simplicial space

$$k \mapsto \mathrm{emb}^\sim_{\dots}(M \times \Delta^k,\, N \times \Delta^k)$$

where the dots indicate certain boundary conditions; then, to view each $\mathrm{emb}^\sim_{\dots}(M \times \Delta^k,\, N \times \Delta^k)$ as a special value of a cofunctor F_k defined on the open subsets of $M \times \Delta^k$; then, to approximate $\mathrm{emb}^\sim(M,N)$ by the geometric realizations of

$$k \mapsto (T_rF_k)(M \times \Delta^k)$$

for $r \geq 0$, where T_rF_k is a suitable Taylor approximation to F_k which we have not defined and will not define here.

Taking $M = *$ shows that these approaches give quite different results. The second appears to be superior. It is still very much under construction, and so we will not waste more words on it, except by saying that it sheds light on the Levine problem (section 2.3). In fact it has been used, in the quadratic alias metastable range, in an unpublished paper by Larmore and Williams [LW] on the Levine problem. Their main result, which they prove without surgery, is the generalization of 2.3.2 to the situation where the domain M is compact, smooth, but not necessarily closed.

4.3. Scanning revisited

Let $F(V) = \mathrm{emb}(V, N)$ as in 4.1.6, 4.1.9, 4.2.3, 4.2.4. Our goal here is to give a description of the Taylor approximation $F \to T_k F$, for $k \geq 2$, which generalizes the Haefligeresque description of $F \to T_2 F$ in 1.2.1 and 4.1.6.

Notation. Think of the standard $(k-1)$–simplex Δ^{k-1} as an incomplete simplicial set whose i–simplices are the monotone injections z from $\{0, ..., i\}$ to $\{1, ..., k\}$. With such an i–simplex z we can associate the set $\{1, ..., z(i)\}$, filtered by subsets $\{1, ..., z(j)\}$ for $0 \leq j \leq i$. Let $G(z)$ be the group of permutations of $\{1, ..., z(i)\}$ which respect the filtration, and let $G_0(z)$ be the full permutation group of $\{1, ..., z(0)\}$, so that $G_0(z)$ is a factor in an obvious product decomposition of $G(z)$. Write $[z := z(0)$ and $z] := z(i)$ where $i = |z|$.

4.3.1. Definition. For $k \geq 2$ and a simplex z of Δ^{k-1}, let $J_{M,N,k}(z) = J_M(z)$ be the space of smooth maps

$$M^{z]} \longrightarrow N^{[z}$$

which are strongly isovariant with respect to $G_0(z)$, and equivariant with respect to $G(z)$. (The actions of $G_0(z)$ on $M^{z]}$ and $N^{[z}$ are by permutation of the coordinates labeled 1 through $[z$. The action of $G(z)$ on $M^{z]}$ is by permutation of the coordinates labeled 1 through $z]$. The action of $G(z)$ on $N^{[z}$ is obtained from the action of $G_0(z)$ on $N^{[z}$ just defined by means of the projection $G(z) \to G_0(z)$.)

Then $J_M(z)$ is a functor of the variable z. (If y is a face of z, then we have homomorphisms $G(z) \to G(y)$ and $G_0(z) \hookrightarrow G_0(y)$, and we also have projections $N^{[y} \to N^{[z}$, $M^{z]} \to M^{y]}$ which are both $G(z)$–equivariant and strongly $G_0(z)$–isovariant.)

4.3.2. Definition. We let $\Theta_k(M,N) = \operatorname{holim}_z J_M(z)$ with J_M as in definition 4.3.1. Explicitly, $\Theta_k(M,N)$ is the space of natural transformations from the functor $z \mapsto \Delta^{|z|}$ to the functor $z \mapsto J_M(z)$.

Motivation. Let $\mathcal{D}_k(M)$ be the topological poset of functions $g: M \to \mathbb{N}$ with finite support, and degree $|g| := \sum_x g(x)$ satisfying $1 \le |g| \le k$. Here $\mathbb{N} = \{0, 1, 2, ...\}$; for $f, g \in \mathcal{D}_k(M)$ we decree $g \le f$ if $g(x) \le f(x)$ for all $x \in M$, and we topologize $\mathcal{D}_k(M)$ by identifying it with the coproduct of the M^i/Σ_i for $1 \le i \le k$.

For $g \in \mathcal{D}_k(M)$ let $p(g)$ be the support, a subset of M of cardinality between 1 and k. The idea is that $\Theta_k(M,N)$ is a modified version of the topological homotopy limit of the functor $g \mapsto \operatorname{emb}(p(g), N)$. The expression *topological homotopy limit* indicates that we pay attention to the topological structure of $\mathcal{D}_k(M)$. The modification happens where we ask for strongly isovariant smooth maps rather than just isovariant continuous maps.

4.3.3. Example. Let $k = 2$. Let's denote the simplices of Δ^1 by $I, 0, 1$ in this case. Let $f = \{f_I, f_0, f_1\}$ be any point in $\Theta_2(M,N)$. Then f_1 is a strongly isovariant Σ_2–map from M^2 to N^2, and f_0 is just a smooth map $M \to N$. Finally f_I is a path (parametrized by $[0,1]$) of smooth maps $M^2 \to N$. Its values at time 1 and 0 respectively are the compositions

$$M^2 \xrightarrow{f_1} N^2 \longrightarrow N$$

$$M^2 \longrightarrow M \xrightarrow{f_0} N.$$

It follows that $\Theta_2(M,N)$ is (homeomorphic to) the Haefliger approximation to $\operatorname{emb}(M,N)$ of 1.2.1 and 4.1.6.

4.3.4. Theorem. $\Theta_k(M,N) \simeq T_k \operatorname{emb}(M,N)$, *for* $k \ge 2$.

Idea of proof. Let $F(V) = \operatorname{emb}(V,N)$ for $V \in \mathcal{O}$. We will show that $T_k F(V)$ is naturally weakly homotopy equivalent to $\Theta_k(V,N)$. There is a natural inclusion $F(V) \to \Theta_k(V,N)$. It suffices to show that

(i) $\Theta_k(V,N)$ is polynomial of degree $\le k$ as a functor of V ;
(ii) the natural inclusion $F(V) \to \Theta_k(V,N)$ is a homotopy equivalence whenever V is in $\mathcal{O}k$.

To establish (i) it is enough to show that each of the functors $V \mapsto J_V(z)$ is polynomial of degree $\leq k$. This is easy. For (ii), suppose that V is a tubular neighborhood of $S \subset M$, where $|S| \leq k$. One checks that

$$\begin{array}{ccc} \mathrm{emb}(V,N) & \longrightarrow & \Theta_k(V,N) \\ \Big\downarrow{\scriptstyle \mathrm{res.}} & & \Big\downarrow{\scriptstyle \mathrm{res.}} \\ \mathrm{emb}(S,N) & \longrightarrow & \Theta_k(S,N) \end{array}$$

is ∞–cartesian. With the motivation above, it is not hard to show that $\mathrm{emb}(S,N) \to \Theta_k(S,N)$ is a homotopy equivalence. See [GoKW] for the details. □

5. Calculus methods: Homology aspect

5.1. One–dimensional domains

One of us (Goodwillie) observed long ago that when $M = I = [0,1]$, compare 4.1.12, the calculus of good cofunctors F on $\mathcal{O}(M)$ amounts to a theory of cosimplicial spaces and their corealizations (corealization = Tot). It can therefore give *homological* information about $F(M) = F(I)$ (which tends to play the role of the corealization) by means of the generalized Eilenberg–Moore spectral sequence [Bou], [Re], [EM], the standard tool for calculating the homology of such corealizations. These ideas are explained here. Following Bott [Bo], we make contact with the theory of knot invariants of finite type initiated by Vassiliev [Va1], [Va2], [Va3], [BiL], [BaN], [BaNSt], [Ko], [Bi] and extensions of it used by Kontsevich [Ko] in his calculation of $H^*(\mathrm{emb}(\mathbb{S}^1,\mathbb{R}^n);\mathbb{Q})$ for $n > 3$.

Let $\mathcal{O} = \mathcal{O}(I)$ and $\mathcal{O}k = \mathcal{O}k(I)$, with the conventions of 4.1.12. We want to establish a correspondence between good cofunctors from $\mathcal{O}$ to spaces, and *augmented* cosimplicial spaces, that is, covariant functors from the category of all finite totally ordered sets (including the empty set) to spaces. Let $\mathcal{O}' \subset \mathcal{O}$ consist of all elements which have only finitely many connected components, so that

$$\mathcal{O}' = \{I\} \cup \bigcup_{k \geq 0} \mathcal{O}k\,.$$

A good cofunctor on $\mathcal{O}$ is determined up to natural weak homotopy equivalence by its restriction to $\mathcal{O}'$. The restriction is still an isotopy invariant cofunctor. Hence it is enough to establish a correspondence between isotopy invariant cofunctors from $\mathcal{O}'$ to spaces, and augmented cosimplicial spaces. (We write augmented cosimplicial spaces in the form $S \mapsto \mathfrak{F}_S$, or in the form $\mathfrak{F}_\emptyset \to \mathfrak{F}_\bullet$. Here the bullet stands for a nonempty finite totally ordered set, so that $\mathfrak{F}_\bullet$ is the underlying un–augmented cosimplicial space.)

5.1.1. Constructions. Let κ be the cofunctor from $\mathcal{O}'$ to totally ordered finite sets given by $V \mapsto \pi_0(I \smallsetminus V)$. Pre–composition with κ gets us from augmented cosimplicial spaces to isotopy invariant space–valued cofunctors on $\mathcal{O}'$. Conversely, an isotopy invariant cofunctor F from $\mathcal{O}'$ to spaces determines an augmented cosimplicial space by homotopy right Kan extension along κ,

$$\mathfrak{F}_S := \operatorname*{holim}_{V \text{ with } S \to \kappa(V)} F(V)$$

for a finite totally ordered S. These two construction are inverses of one another, up to natural weak homotopy equivalence.

5.1.2. Definitions. Let $\mathfrak{F}_\bullet$ be any cosimplicial space. For $0 \le k \le \infty$ let $\mathrm{Tot}^k(\mathfrak{F}_\bullet)$ be the space of natural transformations from $S \mapsto \Delta(S)$ to $S \mapsto \mathfrak{F}_S$, for totally ordered finite S with $1 \le |S| \le k$. Here $\Delta(S)$ denotes the simplex spanned by S. When $k = \infty$, we simply write $\mathrm{Tot}(\mathfrak{F}_\bullet)$, and speak of the corealization. There is a tower of forgetful maps (Serre fibrations)

$$\mathrm{Tot}(\mathfrak{F}_\bullet) \cdots \to \mathrm{Tot}^k(\mathfrak{F}_\bullet) \to \mathrm{Tot}^{k-1}(\mathfrak{F}_\bullet) \to \cdots \to \mathrm{Tot}^0(\mathfrak{F}_\bullet)\,.$$

Let $\mathfrak{C}_\bullet$ be a cosimplicial chain complex. For $0 \le k \le \infty$ let $\mathrm{Tot}^k(\mathfrak{C}_\bullet)$ be the chain complex of natural maps of graded abelian groups from $S \mapsto C_*(\Delta_S)$ to $S \mapsto \mathfrak{C}_S$, for totally ordered finite S with $1 \le |S| < \infty$, where C_* is the singular chain complex functor. (The i–chains in $\mathrm{Tot}^k(\mathfrak{C}_\bullet)$ are the natural maps raising degrees by i, for $i \in \mathbb{Z}$.) When $k = \infty$, we write $\mathrm{Tot}(\mathfrak{C}_\bullet)$. There is a tower of chain complexes and forgetful chain maps

$$\mathrm{Tot}(\mathfrak{C}_\bullet) \cdots \to \mathrm{Tot}^k(\mathfrak{C}_\bullet) \to \mathrm{Tot}^{k-1}(\mathfrak{C}_\bullet) \to \cdots \to \mathrm{Tot}^0(\mathfrak{C}_\bullet)\,.$$

Each of these chain maps is a ‘fibration’ (degreewise split onto). With such a tower of fibrations of chain complexes, one can associate in the usual way an exact couple and/or a spectral sequence converging, under mild conditions on $\mathfrak{C}_\bullet$, to the homology of $\mathrm{Tot}(\mathfrak{C}_\bullet)$.

In particular, suppose that $\mathfrak{C}_\bullet = C_*(\mathfrak{F}_\bullet)$ is the cosimplicial chain complex obtained from a cosimplicial space by applying C_*. Then under suitable conditions on $\mathfrak{F}_\bullet$,

- the spectral sequence converges to $H_* \operatorname{Tot}(C_*(\mathfrak{F}_\bullet))$, and
- the canonical map $H_* \operatorname{Tot}(\mathfrak{F}_\bullet) \to H_* \operatorname{Tot}(C_*(\mathfrak{F}_\bullet))$ is an isomorphism.

In that case we can say simply that the spectral sequence converges to $H_* \operatorname{Tot}(\mathfrak{F}_\bullet)$. It is called a 'generalized Eilenberg–Moore spectral sequence' because, according to Rector [Re], the original Eilenberg–Moore spectral sequence [EM] for the calculation of the homology of a homotopy pullback of spaces is a special case.

5.1.3. Remark. Let $\mathcal{A}$ be an abelian category. The Dold–Kan correspondence [Cu] is an equivalence of categories, often denoted N for 'normalization', from simplicial $\mathcal{A}$–objects to chain complexes in $\mathcal{A}$ graded over the integers ≥ 0. In particular, the Dold–Kan correspondence associates to a cosimplicial chain complex $\mathfrak{C}_\bullet$ a cochain complex $N\mathfrak{C}_\bullet$ of chain complexes

$$N\mathfrak{C}_0 \xrightarrow{d_0} N\mathfrak{C}_1 \xrightarrow{d_0} N\mathfrak{C}_2 \xrightarrow{d_0} \cdots .$$

Here each $N\mathfrak{C}_i$ is a chain complex in its own right, the quotient of $\mathfrak{C}_i$ by the chain subcomplex generated by the images of the face operators $d_j \colon \mathfrak{C}_{i-1} \to \mathfrak{C}_i$ for $0 < j \leq i$. It is also (as a chain complex) a direct summand of $\mathfrak{C}_i$. Now $\operatorname{Tot}^k(\mathfrak{C}_\bullet)$ is isomorphic to the 'total chain complex' [CaE] of the truncated double complex

$$N\mathfrak{C}_0 \longrightarrow N\mathfrak{C}_1 \longrightarrow \cdots \longrightarrow N\mathfrak{C}_k .$$

Although this does not help much in explaining the generalized Eilenberg–Moore spectral sequence above, where $\mathfrak{C}_\bullet = C_*(\mathfrak{F}_\bullet)$, it does lead to the insight that the E^1 and E^2–terms are

$$\begin{aligned} E^1_{-p,q} &\cong N^p(H_q\mathfrak{F}_\bullet), \\ E^2_{-p,q} &\cong H^p(N(H_q\mathfrak{F}_\bullet)) . \end{aligned}$$

Here $H_q\mathfrak{F}_\bullet$ for fixed q is a cosimplicial abelian group, and $N(H_q\mathfrak{F}_\bullet)$ is the associated cochain complex, with p–th cochain group $N^p(H_q\mathfrak{F}_\bullet)$. The spectral sequence lives in the second quadrant. With these grading conventions, the differentials on E^r have bidegree $(-r, r-1)$, and $E^\infty_{-p,q}$ is (in the convergent case) a subquotient of $H_{q-p}(\operatorname{Tot}\mathfrak{F}_\bullet)$.

Now suppose that $\mathfrak{F}_\emptyset \to \mathfrak{F}_\bullet$ is the augmented cosimplicial space associated with a good cofunctor F from $\mathcal{O} = \mathcal{O}(I)$ to spaces. Then it follows easily from the definitions that

$$\begin{aligned} \mathfrak{F}_S &\simeq F(I \smallsetminus S) \qquad\qquad \text{for finite } S \subset I \smallsetminus \partial I, \\ \operatorname{Tot}^k \mathfrak{F}_\bullet &\simeq T_k F(I), \\ \operatorname{Tot} \mathfrak{F}_\bullet &\simeq \operatorname*{holim}_k T_k F(I)\,. \end{aligned}$$

Under these identifications the comparison map $F(I) \to \operatorname{holim}_k T_k F(I)$ corresponds to the augmentation–induced map $\mathfrak{F}_\emptyset \to \operatorname{Tot} \mathfrak{F}_\bullet$. In particular, if F is ρ–analytic with $\rho > 1$, then by the convergence theorem

$$\mathfrak{F}_\emptyset \quad \xrightarrow{\simeq} \quad \operatorname{Tot} \mathfrak{F}_\bullet\,.$$

Therefore, assuming Bousfield's convergence criteria [Bou] are satisfied, the spectral sequence constructed above converges to $H_* F(I)$; more precisely, we can write

$$\{\, E^2_{-p,q} = H^p(N(H_q F(I \smallsetminus \bullet)))\,\} \qquad \Rightarrow \qquad \{\, H_{q-p} F(I)\,\}$$

where $\bullet$ runs through a selection of nonempty finite subsets of $I \smallsetminus \partial I$, one for each (finite, nonzero) cardinality.

5.1.4. Example. For $V \in \mathcal{O}$ let $F(V)$ be the homotopy fiber of the inclusion $\operatorname{emb}(V, \mathbb{R}^{n-1} \times I) \hookrightarrow \operatorname{imm}(V, \mathbb{R}^{n-1} \times I)$, where $n \geq 3$. Boundary conditions as in 4.1.12 are understood. Note that $\operatorname{imm}(V, \mathbb{R}^{n-1} \times I)$ is homotopy equivalent to the space of pointed maps from $V/\partial V$ to $\mathbb{S}^{n-1}$ by immersion theory. — The generalized Eilenberg–Moore spectral sequence has

$$E^2_{-p,q} \cong H^p(N(H_q(\operatorname{emb}(\{1,2,...,\bullet\}, \mathbb{R}^n))))$$

where $\bullet$ runs through the integers ≥ 0. The homology of the 'configuration space' $\operatorname{emb}(\{1,2,\ldots,k\}, \mathbb{R}^n)$ is torsion free, therefore dual to the cohomology of $\operatorname{emb}(\{1,\ldots,k\}, \mathbb{R}^n)$. The cohomology ring $H^*(\operatorname{emb}(\{1,\ldots,k\}, \mathbb{R}^n))$ is the quotient of an exterior algebra on generators α_{st} in degree $n-1$, one such for any two distinct elements $s, t \in \{1,\ldots,k\}$, by relations

$$\begin{gathered} \alpha_{st} = (-1)^n \alpha_{ts}\,, \\ \alpha_{rs}\alpha_{st} + \alpha_{st}\alpha_{tr} + \alpha_{tr}\alpha_{rs} = 0\,. \end{gathered}$$

Here α_{st} is the image of the canonical generator under the map in cohomology induced by

$$\operatorname{emb}(\{1,2,\ldots,k\}, \mathbb{R}^n) \longrightarrow \mathbb{R}^n \smallsetminus 0 \quad ; \quad g \mapsto g(t) - g(s).$$

These assertions can be proved by induction on k, using the fact that the Leray–Serre spectral sequence associated with the forgetful fibration $\mathrm{emb}(\{1,\dots,k\},\mathbb{R}^n) \to \mathrm{emb}(\{1,\dots,k-1\},\mathbb{R}^n)$ collapses at E_2. Our description of $H_*(\mathrm{emb}(\{1,\dots,k\},\mathbb{R}^n))$ is so natural that it is in fact a description of the cosimplicial graded abelian group $H_*(\mathrm{emb}(\{1,\dots,\bullet\},\mathbb{R}^n))$, thereby delivering $E^2_{-p,q} \cong H^p(N(H_q(\mathrm{emb}(\{1,2,...,\bullet\},\mathbb{R}^n))))$, the E^2–term of the Eilenberg–Moore spectral sequence. We omit the details, but mention the following points.

(i) When $n > 3$, Bousfield's convergence condition [Bou, Thm.3.4] is satisfied; we will verify this somewhat indirectly in 5.2 below. Therefore the spectral sequence converges to the homology of

$$F(I) = \mathrm{hofiber}\,[\,\mathrm{emb}(I,\mathbb{R}^{n-1}\times I) \to \mathrm{imm}(I,\mathbb{R}^{n-1}\times I)\,]\,.$$

It seems to be very closely related to a spectral sequence developed by Kontsevich in [Ko], for the calculation of the rational cohomology of $\mathrm{emb}(\mathbb{S}^1,\mathbb{R}^n)$ where $n > 3$. However, Kontsevich can also show that his spectral sequence collapses.

(ii) When $n = 3$, the set $\pi_0 F(I)$ can be identified with the set of framed knots in $\mathbb{R}^3$ which are regularly homotopic as framed immersions to the standard one. So we are doing knot theory. — The pieces of the E^1–term of the spectral sequence in total degree < 0 vanish, by inspection. Hence, for the pieces in total degree 0, there are surjections

$$E^1_{-p,p} \to E^2_{-p,p} \to E^3_{-p,p} \to E^4_{-p,p} \to \cdots\ .$$

For odd p we have $E^1_{-p,p} = 0$. For even p, the term $E^1_{-p,p}$ is isomorphic to the free abelian group generated by the set of partitions of $\{1,\dots,p\}$ into $p/2$ subsets of cardinality 2. The relations introduced in passing to $E^2_{-p,p}$ can be calculated from the above information. They are

$$u\cdot\gamma \sim 0, \qquad v\cdot\gamma \sim 0\,,$$

where γ is a generator corresponding to a partition containing two parts of the form $\{r,s\}$ and $\{s+1,t+1\}$ with $r<s<t$, and u,v are certain elements in the group ring of the symmetric group Σ_p (which acts by pushforward). Namely,

$$u = 1-(s,s{+}1)+(t{+}1,t,\dots,s)-(t,t{+}1)(t{+}1,t,\dots,s)\,,$$
$$v = 1-(s,s{+}1)+(r,r{+}1,\dots,s{+}1)-(r,r{+}1)(r,r{+}1,\dots,s{+}1).$$

The reader familiar with the theory of knot invariants of finite type [Va1],[Va2], [Va3], [BiL], [Ko], [BaN], [Bi] will now recognize $E^2_{-p,p}$ as the degree $p/2$ part of $\mathcal{A}$, the graded algebra of chord diagrams modulo the so–called 4T relation; see particularly [BaN].

As Bott points out in [Bo], this suggests that passage from $H_0F(I)$ to $H_0\,\mathrm{Tot}\,\mathfrak{F}_\bullet$ and subsequent analysis of $H_0\,\mathrm{Tot}\,\mathfrak{F}_\bullet$ by means of the spectral sequence is an alternative approach to the theory of (framed) knot invariants of finite type. However, as Bott also points out, it is far from obvious that the surjections

$$E^2_{-p,p} \longrightarrow E^\infty_{-p,p}$$

are bijections (and consequently we do not have a straightforward construction of framed knot invariants in $\mathcal{A}$ using this approach). If they are, we expect that any proof will use substantial parts of the existing theory of knot invariants of finite type, such as the Kontsevich integrals [Ko], [BaN].

5.2. Higher dimensional domains

One conclusion to be drawn from 5.1 is that the notion of an isotopy invariant cofunctor F from $\bigcup_{k\geq 0}\mathcal{O}k(M)$ to spaces is a legitimate generalization of the notion of cosimplicial space (special case $M = I = [0,1]$). In particular, the construction $F \mapsto \mathrm{holim}\, F$ is the correct generalization of Tot, and $F \mapsto \mathrm{holim}\,(F|\mathcal{O}k(M))$ is the correct generalization of Tot^k. The Eilenberg–Moore–Rector–Bousfield question of whether Tot commutes with 'linearization' functors from spaces to spaces

$$\lambda_{\boldsymbol{J}}\colon X \mapsto \Omega^\infty(X_+ \wedge \boldsymbol{J})$$

(where $\boldsymbol{J}$ denotes a fixed CW–spectrum) turns into the question of whether $\lambda_{\boldsymbol{J}}(\mathrm{holim}\; F) \simeq \mathrm{holim}\; \lambda_{\boldsymbol{J}}F$. But we already have a conditional answer to the generalized question. Namely, if F is defined on all of $\mathcal{O}(M)$, and sufficiently analytic, and if $\lambda_{\boldsymbol{J}}F$ is also sufficiently analytic on $\mathcal{O}(M)$, then we will have

$$\begin{array}{ccccc} F(M) & \xrightarrow{\ \simeq\ } & \operatorname*{holim}_k\, T_kF(M) & \xrightarrow{\ \simeq\ } & \operatorname*{holim}_{V\in\bigcup\mathcal{O}k(M)} F(V)\,, \\ \lambda_{\boldsymbol{J}}F(M) & \xrightarrow{\ \simeq\ } & \operatorname*{holim}_k\, T_k(\lambda_{\boldsymbol{J}}F)(M) & \xrightarrow{\ \simeq\ } & \operatorname*{holim}_{V\in\bigcup\mathcal{O}k(M)} \lambda_{\boldsymbol{J}}F(V)\,. \end{array}$$

We then also have a (twice generalized) Eilenberg–Moore type spectral sequence converging to the homotopy of $\lambda_{\boldsymbol{J}}F(M)$, which is essentially the

$\boldsymbol{J}$–homology of $F(M)$. It is the homotopy spectral associated with the tower

$$\cdots \to T_{k+1}(\lambda_{\boldsymbol{J}}F)(M) \to T_k(\lambda_{\boldsymbol{J}}F)(M) \to T_{k-1}(\lambda_{\boldsymbol{J}}F)(M) \to \cdots$$

where $T_k(\lambda_{\boldsymbol{J}}F)(V)$ is nothing but $\operatorname{holim}(\lambda_{\boldsymbol{J}}F|\mathcal{O}k(M))$. From 4.1.8, we have quite a good understanding of its E^1–term. Of course, we do *not* claim that $T_k(\lambda_{\boldsymbol{J}}F)$ agrees in any sensible sense with $\lambda_{\boldsymbol{J}}(T_kF)$, except as it were for $k=\infty$ by Eilenberg–Moore type magic.

In the following lemma M^m is arbitrary (smooth, possibly with boundary). If there is a nonempty boundary, define $\mathcal{O}(M)$ as in 4.1.12. For the first time we use the generalization 4.1.11 of definition 4.1.10 of an analytic cofunctor.

5.2.1. Lemma. *Let F be a good cofunctor on $\mathcal{O}(M)$ and let $\boldsymbol{J}$ be a (-1)–connected CW–spectrum. Suppose that $T_{r-1}F \simeq *$ for some $r>0$, and F is ρ–analytic with excess $c<0$, where $\rho + c/r > m$. Then the taming of $\lambda_{\boldsymbol{J}}F$ is $(\rho+c/r)$–analytic with excess 0.*

See [We2] for the proof.

5.2.2. Example. Let M be compact, oriented, and $M^m \subset N^n$ as a smooth submanifold, $\partial M = M \cap \partial N$ (transverse intersection). Let

$$F(V) = \operatorname{hofiber}[\operatorname{emb}(V,N) \to \operatorname{imm}(V,N)]$$

with conventions as in 4.1.12. Then F is $(n-2)$–analytic with excess $3-n$, by 4.2.3 and §3 of [GoWe]. Applying 5.2.1 with $r=2$ and $\boldsymbol{J} = \boldsymbol{H}\mathbb{Z}$, and writing λ for $\lambda_{\boldsymbol{H}\mathbb{Z}}$, we find that the taming of λF is $(n/2-1/2)$–analytic with excess 0, provided $n/2-1/2 > m$. In that case the connectivity of the Taylor approximations

$$\lambda F(M) \longrightarrow T_k(\lambda F)(M)$$

tends to infinity as $k \to \infty$. Then the spectral sequence determined by the exact couple $(E^1, D^1, \dots)$ with

$$D^1_{-p,q} := \pi_{q-p}(T_{p-1}(\lambda F)(M)),$$
$$E^1_{-p,q} := \pi_{q-p}[T_p(\lambda F)(M) \longrightarrow T_{p-1}(\lambda F)(M)] = \pi_{q-p}L_p(\lambda F)(M)$$

converges to $\{\pi_{q-p}(\lambda F(M))\} = \{H_{q-p}(F(M))\}$. Its E^1–term simplifies by 4.1.8 and Poincaré duality to

$$E^1_{-p,q} \quad \cong \quad \begin{cases} 0 & (p < 2) \\ H_{pm-1-q}(Y(M,N,p)\,;\,\mathbb{Z}^{\pm}) & (p \geq 2) \end{cases}$$

where $Y(M,N,p)$ is the space over $\binom{M}{p}$ whose fiber over $S \in \binom{M}{p}$ is

$$\operatorname*{hocolim}_{\emptyset \neq R \subset S} \text{hofiber}\,[\,\text{emb}(R,N) \to N^R\,].$$

When m is odd, untwisted integer coefficients $\mathbb{Z}^+$ are understood; when m is even, use $\mathbb{Z}^-$, integer coefficients twisted by means of the composition

$$\pi_1 Y(M,N,p) \to \Sigma_p \to \mathbb{Z}/2 = \text{aut}(\mathbb{Z})\,.$$

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Surgery theoretic methods in group actions

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This paper is intended to give a brief introduction to the applications of the ideas of surgery in transformation group theory; it is not intended to be any kind of survey of the latter theory, whose study requires many additional insights and methods. However, despite this disclaimer, there have been a number of signal achievements of the surgery theoretic viewpoint, notably in the directions of producing examples and, on occasion, giving complete classifications of particular sorts of actions.

We have divided this paper into three sections which deal with three different variants of classical surgery. The first deals with "CW surgery", i.e., surgery in the category of CW complexes. Then we discuss some *ad hoc* applications of classical surgery of closed manifolds in various problems, as opposed to the development of equivariant surgery theories. The latter point of view is discussed in the last section. We shall only briefly describe the theory of "pseudoequivalence" [Pe2] and its successes, which have tended to lie in the construction of actions with unusual properties, but instead concentrate on classification theorems for actions up to equivariant isomorphism.

With deep regret, we will deal exclusively with <u>finite</u> group actions in this paper because of space requirements. We also highly recommend the 15 year old conference survey [Sch3] for its many summaries of the state of the art at that turning point in its development.

This paper is dedicated to C.T.C. Wall, whose powerful and inspirational contributions to this story include the finiteness obstruction, non-simply connected surgery obstruction groups (their definition, application, and calculation), and his *tour de force* on free actions on the sphere: i.e., fake lens spaces and the space form problem.

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1 The CW Category

The analogue of the theory of surgery for CW complexes is the calculus of attaching handles to a given complex to produce a finite CW complex (weak) homotopy equivalent to another specified space. In the most classical context this is the combination of Wall's finiteness obstruction theory with Whitehead's simple homotopy theory. We begin our survey of equivariant surgery theory with a discussion of several of the high-points achieved by just CW theory.

1.1 The space form problem

This problem, to which we will return in the next section, is "Which groups act freely on some homotopy sphere or, in particular, on the standard sphere?" One can see ([Wo]) that there is a free action by (linear) isometries on some standard sphere (and the dimension can be computed) if and only if all subgroups of order pq, where p and q are not necessarily distinct primes, are cyclic.

The first nontrivial result on the purely topological problem is due to P. A. Smith who showed that if G acts freely on a finite dimensional complex homotopy equivalent to a sphere, then every subgroup of order p^2 is cyclic. (Cartan and Eilenberg [CE] refined this to the statement that if G acts freely on a homotopy S^d, then G has periodic cohomology with period $d+1$; we will see the reason momentarily.)

An early high point in "CW surgery" was Swan's proof [Sw] of a converse:

Theorem 1.1 *A group G acts freely on a finite dimensional complex X homotopy equivalent to S^{d-1} if and only if G has d-periodic cohomology (which, according to Cartan and Eilenberg* [CE]*, is equivalent, if we allow d to vary, to all subgroups of order p^2 being cyclic, for p prime). Moreover, X can be taken to be a finite complex if and only if an obstruction $w_d(G) \in \widetilde{K}_0(\mathbb{Z}G)/\mathrm{Sw}$ vanishes.*

Here, Sw denotes the subgroup of $\widetilde{K}_0(\mathbb{Z}G)$ represented by finite modules of order prime to $\#(G)$, with trivial action. This is the image of a natural homomorphism, the Swan homomorphism $\mathrm{Sw}\colon (\mathbb{Z}/\#(G))^* \to \widetilde{K}_0(\mathbb{Z}G)$, which assigns, to an integer n, the Euler characteristic of a finite $\mathbb{Z}G$-projective resolution of $\mathbb{Z}/n\mathbb{Z}$ viewed as a module over $\mathbb{Z}G$ by giving it the trivial action.

Since $\widetilde{K}_0(\mathbb{Z}G)$ is a finite abelian group (another fundamental theorem of Swan) and, as will be apparent from the definition, $w_{kd}(G) = kw_d(G)$, it follows that there always exists a finite complex on which G acts freely, but of undetermined dimension.

[Note for instance that dihedral groups D_{2p} (indeed, all of the metacyclic groups, $M_{pq}, q \equiv 1 \bmod p$) satisfy the conditions of Swan's theorem and hence they act on finite complexes homotopy equivalent to a sphere. We will see in the next sections that the dihedral groups don't, however, act on a sphere, but the odd order metacyclic groups do.]

Wall [Wa2] had shown that $w_d(G)$ always has order 1 or 2 and Milgram [Mi] gave the first examples where $w_d(G) \neq 0$ (see also [Da] for different examples).

Essentially, Swan's method is to show that if G has periodic cohomology then $\mathbb{Z}$ has a periodic resolution over $\mathbb{Z}G$. More precisely, there are finitely generated projective modules P_i such that one has an exact sequence

$$0 \to \mathbb{Z} \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0.$$

This chain complex is chain equivalent to the equivariant chain complex for the G-action on X if there is such an action; and, conversely, from such a chain complex Swan observes one can build a well defined equivariant homotopy type. (If X is finite, one can, of course, use the cellular chain complex of X as a finite free resolution.)

Even in the case of cyclic groups, the "resolution" is not well defined, because the equivariant homotopy type can be varied. The indeterminacy is caught by the "k-invariant" in $(\mathbb{Z}/\#(G))^*$ (which is the degree of any equivariant map between these homotopy spheres; this is a number prime to $\#(G)$, by the Borsuk-Ulam theorem, and it is well defined up to $\#(G)$, by obstruction theory).

The periodicity of $H^*(G)$ follows from splicing together such resolutions to obtain:

$$\cdots \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to P_d \to P_{d-1} \to \cdots P_1 \to P_0 \to \mathbb{Z} \to 0$$

and using such a periodic resolution to compute group cohomology. Such a splicing trick also verifies the formula for $w_{kd}(G)$ mentioned above.

Swan's work has been greatly extended by tom Dieck [tD] who has studied (substantially in joint work with Petrie [tD-P]) the theory of "homotopy representations," which consist of G-CW complexes where the fixed sets of every subgroup are homotopy spheres. There are two kinds of invariants for these: dimension functions (like "d") and "generalized degrees" which are the analogues of the k-invariants.

Even for cyclic groups, there are values of dimension functions on the set of subgroups that are realized by homotopy representations, but which don't arise for geometric representations. The simplest is for $G = \mathbb{Z}_p, d(e) =$

$d(G)$, which at least doesn't arise for effective linear actions. (The degree can be an arbitrary number prime to p.) More significantly, one can have for $G = \mathbb{Z}_{pq}, d(G) = -1$ (no fixed points) $d(\mathbb{Z}_p) = \delta, d(\mathbb{Z}_q) = \delta'$ and $d(\mathbb{Z}_{pq}) = \delta''$ whenever δ, δ', and δ'' have the same parity and $\delta'' \geq \max(\delta, \delta')$; for linear representations $\delta'' \geq \delta + \delta'$.

Indeed, tom Dieck's work shows exactly when all dimension functions for a group are linear, and when this is true in the sense of a Grothendieck group. However, here the manifold realization problems are much less well understood than in the traditional space form problem; there are, in addition, a host of problems that arise when the "gap hypothesis" fails, i.e., for the situation where the dimensions of fixed point sets can be large relative to the dimension of the ambient complex (e.g., when $\delta'' \leq 2\max(\delta, \delta')$ in the $\mathbb{Z}_{pq}$ example). We will return to some of these issues in section 4.

1.2 Semifree actions on the disk

The first nonfree problem that commands study is, no doubt, the problem of $\mathbb{Z}_p$ actions on the disk, or for our present *CW* purposes, contractible *CW* complexes. We will discuss the case for general G. (Obviously, general groups G cannot act semifreely on a genuine disk — e.g., $G = \mathbb{Z}_p \times \mathbb{Z}_p$ cannot[1], but it can, however, act on a contractible complex, as G acts on the cone $c(G)$.)

For this problem one has a definitive solution, due to Jones [Jo1] for the cyclic case (where Sw is trivial), and Assadi [As] in general:

Theorem 1.2 *A finite CW complex F is homotopy equivalent to the fixed point set of a semifree G-action on a contractible finite complex if and only if*

(i) $\widetilde{H}_i(F;\mathbb{Z})$ *is a finite group of order prime to* $\#(G)$*, all i; and,*

(ii) $\sum(-1)^i \mathrm{Sw}(\#(\widetilde{H}_i(F)) = 0$ *in* $\widetilde{K}_0(\mathbb{Z}G)$.

Here Sw is the Swan homomorphism described above. The necessity of condition (i) is a conclusion of Smith theory. Bredon's book [Br] is a valuable textbook reference. Note that in this theorem the finiteness obstruction lies in precisely the subgroup that we threw away in the space form problem. Condition (i) alone suffices to make F the fixed point set on a *G-ANR*. (With no finiteness at all, one should reformulate (i) as being mod $\#(G)$ acyclicity, and then it is necessary and sufficient.)

[1] This follows from the observation that the "p^2 condition" discussed above (in 1.1) is necessary even for free actions on mod p homology spheres, e.g., to the complement of the fixed set in the disk. (See Theorem 1.2 (i).)

The proof of this theorem comes about by trying to attach free cells equivariantly to obtain a finite contractible complex. After one has inductively killed all homology through $\dim(F)$, there remains (under (i)) a projective module, which can be identified up to sign with

$$\sum(-1)^i \mathrm{Sw}(\#(\widetilde{H}_i(F)).$$

As in the Wall finiteness theory, this is stably free if and only if one can attach cells to remove this final bit of homology.

Condition (ii) is sometimes called the "Assadi condition".

(Obtaining the G-ANR is a fairly simple "Eilenberg swindle" and will be discussed in the final section: see Theorem 3.2 and the surrounding discussion.)

1.3 Fixed sets for nonfree G-actions on the disk

If G is a p-group then an inductive Smith theory argument shows that the fixed set of a G-action on a finite dimensional mod p acyclic space (and, in particular, a contractible one) is mod p acyclic. Conversely, as G has a cyclic quotient, Jones's theorem from the previous subsection implies that any mod p acyclic finite complex is the fixed set of a G-action on some finite contractible CW complex.

On the other hand, for non-p-groups the situation is more complicated. For G cyclic, the Brouwer fixed point theorem implies that the fixed set is nonempty. Indeed, its Euler characteristic $e(F)$ must be 1, by a refinement of the Lefschetz fixed point theorem ($e(F) = L(f)$ for f a periodic map). Also, in general, for a p-group P acting on a space Y, $e(Y^P) \equiv e(Y) \bmod p$. Combining these observations one can obtain necessary congruence conditions for fixed point sets. Oliver [O] showed that these are almost enough; occasionally, there are more refined congruences that hold, i.e., the primes that actually occur arise to a higher power than one would expect from just the preceding analysis:

Theorem 1.3 *If G is not a p-group, then there is a number $n(G)$ (which is readily computable from G) such that F is homotopy equivalent to the fixed point set of a simplicial G-action on a contractible finite complex if and only if $e(F) \equiv 1 \bmod n(G)$.*

For instance, $n(G) = 1$ if and only if the empty set can arise as a fixed point set, and then any F can arise. (Indeed, it turns out that by an explicit construction one can show that if $n(G) = 1$, then every polyhedron occurs as the fixed point set of some PL G-action on some disk!)

Quinn [Q1, Q2] observed that for G-ANR's the same theorem holds with a possibly smaller value of $n(G)$, (i.e., there is another number $m(G)$

playing an analogous role.) This is closely related to the equivariant topological Whitehead theory discussed below.

Remark. Many of these theorems can be extended to more general "surgery theoretic" situations, and could then be viewed as analogues of a "Kervaire-Milnor" type surgery theory. Only some of this is in the literature, to our knowledge. See, e.g. [OP] for the simply connected case of this perspective.

2 *Ad Hoc* Applications of Surgery

In this section we will describe several of the many applications of ordinary, as distinguished from distinctly equivariant, surgery theory. The next section deals with genuinely equivariant surgery.

2.1 The space form problem

Certainly one of the great areas of application (and motivations for the development) of non simply connected surgery was the "spherical space form problem", which, as noted above is the problem of classifying manifolds whose universal covers are the sphere. To this problem Wall made (at least) two great contributions:

Theorem 2.1 ([BPW]) *If G is an odd order cyclic group, then free PL (or topological) actions on an odd dimensional sphere (of dimension at least 5) are detected by their Reidemeister torsion and "rho" invariants. "Suspension" induces an isomorphism between G-actions on S^{2n-1} and on S^{2n+1}.*

The Reidemeister torsion is explained in [M3]. The "rho" invariant is the equivariant signature defect and can be defined as follows: If G is a finite group acting freely as an odd dimensional manifold M, some multiple $kM = \partial W$ for some G-manifold W, and

$$\rho(M) = \frac{1}{k}(G\text{- sign}\,(W))$$

modulo the regular representation, which is the indeterminacy in G-sign (W) as one varies W among free G-manifolds with given boundary. (See [BPW], and also [AS] for a version allowing $W's$ with nonfree action.) It is essentially equivalent to the Atiyah-Patodi-Singer invariant [APS] associated to the various flat complex line bundles over the quotient manifold, associated to the different representations of G into $U(1)$.

Another point of view using intersection homology applied to the cone can be found in [CSW]. We note that Browder and Livesay had shown earlier [BL] that this is not at all true for $\mathbb{Z}_2$; indeed, they directly constructed obstructions to desuspension. Lopez de Medrano [LdM] and Wall [Wa3] gave a classification of the free involutions. (The second author, in [We 2], left as an exercise a proof of an incorrect description of the general free cyclic actions; we still leave the correct solution to this problem as an exercise![2])

Wall's other great theorem in this subject is:

Theorem 2.2 ([MTW]) *A finite group G acts freely on some* (*homotopy*) *sphere if and only if all subgroups of order $2p$ or p^2, for p a prime, are cyclic.*

The necessity of the p^2 condition is cohomological, as discussed in the previous section. The $2p$ condition is not necessary in the context of actions on finite CW complexes, but is necessary for the actions on manifolds, according to a theorem of Milnor, [M1], based on the Borsuk-Ulam theorem. Ronnie Lee [L] discovered more algebraic proof of this necessity via his theory of semicharacteristics; this was put on a surgery theoretic footing by Jim Davis in his thesis and published in [Da2].

The problem of figuring out exactly which groups act in which dimension, and in how many ways, has spawned a vast literature, much of which is surveyed in [DM]. In the final section, we will mention some results on nonfree actions that had this work as one of its main inspirations.

2.2 Semifree actions on the disk and sphere

Another historically important line of investigation that, at least initially, used essentially only conventional surgery theory is the study of semifree actions. These actions are ones where there are only two orbit types: the fixed point set (of the whole group, which is identical to the fixed set of any nontrivial element of the group), and the free part (where points are not fixed by any element of the group, i.e., on which the action is free).

Essentially the method most often used is this: study fixed point sets and their neighborhoods, and study the complements, and then study the possible ways of gluing these together. An early paper expounding this point of view is Browder's [B2]. (Another related approach has arisen, which is appropriate only to the topological category: study the manifold structures on the quotient of the free part that are controlled homotopy equivalent to the given complement, where the control is with respect to a map to the open cone of the fixed point set. See [Q1, We2, HuW].)

[2]Such problems are incomparably easier now, given our complete knowledge of L-groups, assembly maps, their homotopical foundations and the connections between L-theory and topological K-theory.

In the smooth category, the essential observations about classification of neighborhoods were first systematically exploited by Conner and Floyd [CF]. For instance, for a cyclic group of odd order the neighborhoods of fixed sets are essentially the same as a set of complex vector bundles, "the eigenbundles of the action of the differential," whose sum must have underlying orthogonal bundle agreeing with the normal bundle of the fixed set in the ambient manifold. Many restrictions on the Chern classes of these eigenbundles can be read off from the Atiyah-Singer G-signature theorem [AS]. For instance, for semifree actions of groups of order p^r, p prime, on the sphere with fixed set of dimension 0, i.e., where there is a pair of fixed points, Atiyah and Bott (and Milnor) [AB] showed that the two representations at the fixed point sets must agree, for $p \neq 2$; and when the dimension of the fixed set is at least four, Ewing [E] showed that the Chern classes of these eigenbundles must all vanish!

In the PL case, one has equivariant block bundle neighborhoods whose classifying spaces must be analyzed. This is always done on the basis of Quinn's thesis [Q3] on blocked surgery; see [BLR] and also [Ro] for a description of Casson's prior contributions to this circle of ideas. See [CW1, Jo2, Re] for how blocked surgery can be applied to the classification of equivariant PL regular neighborhoods.

Essentially, PL stratified surgery takes off from the fact that blocked surgery theory is not so different from ordinary surgery (especially if one "spacifies" the latter, i.e., forms an appropriate semi-simplicial space of manifolds homotopy equivalent to a given one, so that π_0 is the structure set usually studied in surgery theory; for blocked stratified surgery, with some computational methods; see [CW2]). Then both steps in the above outline can sometimes be completed simultaneously.

While a certain part of the discussion applies in great generality, the cases that attracted the most attention involve group actions on the disk or sphere. For reasons of space, we will concentrate on these cases.

A useful theorem for many of these investigations is the following, which itself doesn't depend on any surgery theory beyond the "π-π" vanishing theorem for surgery obstructions on manifolds with boundary:

Theorem 2.3 (**Extension across homology collars** [AsB, We3]) *Suppose given a manifold triad $(W^{n+1}; M^n, N^n)$ with W and M simply connected of dimension at least 5, and let G be a finite group acting freely and $\mathbb{Z}[1/|G|]$-homologically trivially on N, with $H_*(W, N; \mathbb{Z}(1/|G|)) = 0$. Then there is an extension of the action on N to one on W if and only if $\Sigma(-1)^i \mathrm{Sw}(\#(H_j(W, N)) = 0$ in $\widetilde{K}_0(\mathbb{Z}G)$. Moreover, such extensions are well defined up to an element of* $\mathrm{Wh}(G)$.

The method of proof is to produce a Poincaré model for "W/G" and

"M/G" using "Zabrodsky mixing" (mixing together different local homotopy types to produce an interesting global homotopy type), identifying the Wall finiteness obstruction with the quantity appearing in the theorem, checking that a normal invariant rel N/G exists, and doing surgery using the π-π theorem to obtain some extension on a manifold homotopy equivalent to W. Obtaining such an action on W itself, once one already has it on something homotopy equivalent, is another surgery calculation.

Remark. Assadi-Vogel [AsV] have a non simply connected extension of this result that uses an algebraic K-group that mixes finiteness obstructions with Whitehead torsions. Chase [Ch] has a version for homologically nontrivial actions, but which requires more hypotheses.

In the cyclic group case, the homomorphism Sw identically vanishes, and one obtains:

Corollary *A submanifold of the disk of codimension greater than* 2 *is the fixed point set of a smooth* (*orientation preserving*) $\mathbb{Z}_p$ *action if and only if*

1. *it is mod p acyclic, and*
2. *it is of even codimension, and*
3. *if p odd, the normal bundle has an almost complex structure.*

This corollary, when the codimension is very large, was first obtained by Jones, essentially by replacing cells by handles in his constructions described in (1.2) above.

In the PL locally linear case one can show that the analogous corollary remains valid without any condition analogous to (3), i.e., just assuming (1) and (2) (and a Swan condition for more general groups). See [CW1]. Remarkably, Jones [Jo2] developed a similar theory for PL actions without local linearity, although for odd order groups there is another interesting characteristic class obstruction. A stratified surgery theoretic approach to the results of that amazing paper is sketched in the exercises of [We1].

The only non-orientation preserving case is $\mathbb{Z}_2$, and was settled in the PL case by Chase [Ch].

There are a number of additional subtleties for actions on the sphere, quite different in the different categories. For instance, by making perspicacious use of the solution to the Segal conjecture for $\mathbb{Z}_p$, Schultz [Sch1] discovered that there is a p-local residue of the smooth structure of spheres that survives as the only obstruction to being the fixed point set of a smooth $\mathbb{Z}_p$ action on a very high dimensional sphere (besides having a normal bundle with a complex structure with torsion Chern classes). (See also [DW3] for some results on obstructions to being a smooth fixed point for some complicated groups.)

The PL locally linear case is solved in [We4] — there are L-theoretic analogues of the Swan condition that arise as well. (In the case of cyclic groups this obstruction vanishes, just as the Swan homomorphism does; an independent earlier proof for the prime power case appeared in [CW1].)

These L-theoretic "Swan homomorphisms" arose first in the work of Davis [Da3]. They also play an important role in the "homology propagation problem", namely, given a G-action on M and a $\mathbb{Z}/|G|$ homology equivalence $M' \to M$, when can one find a G-action on M' such that the map is homotopic to an equivariant map? (Extension across homology collars can be viewed as a special case of a variant of this problem — which itself has applications to more systematic converses to Smith theory.) See [CW4, DL, DW2] for information about homology propagation for closed manifolds, and the references [AsB, AsV, Jo4, Q4] for related material.

Finally, we should mention the general work of Hambleton and Madsen that classified semifree actions on $\mathbb{R}^{n+k}$ with $\mathbb{R}^n$ as fixed point set, by viewing it as an analogue of the classification of free G-actions on S^{k-1}, but replacing the usual L^h surgery groups by L^p for $n = 0$ (the L-group based on projective modules in place of free ones) and by L^{-n} for $n > 0$, where these L-groups are based on negative K-groups. These groups are related to L^{-n+1} in exactly the same way that L^p relates to L^h (or L^h relates to L^s), namely, via a generalized Rothenberg sequence; see [Sh1, R4].

2.3 Nonlinear similarity and the Smith problem

The problem of nonlinear similarity is that of deciding when two linear representations of a finite group are topologically conjugate. The PL version of this problem was solved by de Rham, who showed that PL equivariant representations are linearly equivalent by using Whitehead torsion ideas (see [Rt] for a modern treatment). R. Schultz [Sch2] and D. Sullivan proved that topological and linear similarity coincide in the topological category for odd p-groups, but the first examples of nonlinear similarities were constructed in [CS1]; these counterexamples were for all cyclic groups of orders a multiple of 4 and greater than 4. (Further examples for these groups were later constructed in [CSSW].[3])

All of these results are obtained using tools of classical surgery. The Schultz-Sullivan result can be simply seen, for instance, by realizing that a nonlinear similarity between two cyclic p-groups (by character theory, the critical case) immediately implies, by transversality, that the lens spaces

[3]We should warn the reader, though, that there are some errors in some of the detailed calculations in that paper which are corrected in [HP1,2], that gives a more complete picture, using a wide variety of controlled surgery theoretic ideas that were developed subsequently.

associated to the eigenvalues of largest period are normally cobordant. Moreover, not only are those normally cobordant, but also the result of stabilizing them by adding on free representations remain normally cobordant (because such stabilized representations are, *a fortiori,* topologically conjugate). Using this, a quick application of Wall's rho-invariant criterion for normal cobordism implies that these eigenvalues are the same. Then one can "downward induct" to the remaining eigenvalues of lower period.

The [CS1] examples of nonlinearly similar representations essentially involve computing when non-trivial interval bundles over lens spaces are h-cobordant, which can be viewed as a transfer of the surgery obstruction of a normal cobordism[4] between lens spaces.[5] When this vanishes one modifies the construction to obtain that the unit spheres of certain representations are equivariantly h-cobordant. An "infinite process" $1 - 1 + 1 - 1 + \ldots$ argument, also known as the "Eilenberg swindle" (as in [St]), then would produce the equivariant homeomorphism.

By working carefully, similar methods produced some smooth h-cobordisms, which gave counterexamples to Smith's conjecture that a smooth cyclic group action on the sphere with two fixed points must have the same representations at the two fixed points; see [CS5]. These counterexamples included the case $G = \mathbb{Z}_{2r}, r \geq 4$. (Recall from the above cited result of [AB] this cannot occur for $G = \mathbb{Z}_{p^r}$, p odd.) Much further information can be found in [Sh2, PR, Sch3].

Remark. We should at least mention at this point the deep result of [HP] and [MR] that for odd order groups, nonlinear similarity is equivalent to linear similarity — but the techniques necessary for this result must wait till the next section.

2.4 Actions with one fixed point

At yet a further extreme in the theory of group actions are ones that are not at all modeled on linear ones. A beautiful example of one such is due to E. Stein [Stn], and it is probably the most complicated action, in terms of orbit structure, that has been produced by direct application of usual manifold surgery. It is an action of the dodecahedral group on the sphere with a single fixed point.

One could predict such an action from an observation of Floyd and Richardson [FR] that the dodecahedral group acts by isometries on the Poincaré dodecahedral homology 3-sphere M^3 with just a single fixed point; so one should just make a higher dimensional version of this and do an

[4]The paper also describes useful criteria for normal cobordism between linear lens spaces.

[5]This is done using the "generalized Browder-Livesay theory" of [CS4].

equivariant "plus" construction to get rid of fundamental group. However, the details are quite complicated, and we shall say nothing further about this problem here. Suffice it to say that all subsequent work on the construction of "one-fixed-point actions" depended rather on systematic exploitation of equivariant surgery, which we turn to in the next section.

3 Equivariant Surgery

Suppose one has an equivariant normal map (whatever that is) and tries to "surger" it equivariantly to be a (perhaps equivariant) homotopy equivalence. Depending on the problem, one will inductively assume some level of success at the fixed point sets of smaller groups. At this point, though, one runs into trouble doing the surgery. Homotopy theory will tell you that there usually are spheres in your manifold that need to be killed, but they will often be embedded in a fashion that intersects some fixed point set. To do surgery equivariantly, one has to surger all of the translates under the group action of these spheres, and if these intersect each other, one is stuck. (If one can succeed, then one is led into the algebra of equivariant intersection forms, and of absorbing the perturbations of lower strata into the L-theory.)

There are essentially two approaches to dealing with this difficult problem. The first, emphasized by Petrie and his collaborators, is to assume a gap hypothesis; see [DP, DR, Pe1, PR] for this approach and a number of its applications. The point here is that the spheres one uses for surgery are always of at most half the dimension of the ambient manifold. Thus, if the dimension of the fixed set is strictly less than half that of the manifold, general position will enable the spheres to be moved so as to not intersect the fixed set, and aside from the middle dimensional case itself, not their own translates as well. The remaining middle dimensional difficulties are to be absorbed into the surgery obstruction group.

With this in hand, what happens under gap hypothesis might be summarized as follows: One defines a notion of normal invariants so that surgery is possible. The idea just described works to the extent of leading to a π-π vanishing theorem for relative surgery. (Rothenberg and Weinberger had shown that the π-π theorem fails for equivariant surgery when the gap hypothesis fails; see [DS] for a description.) The approach of [Wa] Chapter 9 shows how such vanishing theorems in surgery theories make possible the geometric definition of a surgery obstruction group. In such general settings, it is extremely rare to have a good algebraic definition of this group, but nevertheless the literature contains *ad hoc* calculations in many cases.

The normal invariants also actually pose a serious problem; that is, it's often quite hard to find them. Here the smooth category looks a lot more workable than the others, since G-transversality is in much better shape (see [CoW] for a modern and elegant systematic development); but, in any case, this part of the subject still seems more art than technology.

Nevertheless, this approach has lead to the constructions of many quite interesting and exotic examples including one-fixed-point actions on the sphere for many groups. See, for instance [Pe3], and the recent very satisfying [LaM], beautiful theorems regarding possible dimension functions for smooth actions [DP], and varied examples of actions on the sphere with two fixed points but different representations at the fixed points (see [PR] and the surveys in [Sch3]).

A remarkable, systematic *tour de force* using this methodology can be found in the papers [MR] which extend Wall's classification theorem from free actions of odd order cyclic groups on the sphere to all odd order actions (assuming a gap hypothesis). This approach depends extensively on proving PL (and Top) G-transversality theorems, and this only works for odd order groups. The Browder-Livesay invariants discussed in the previous section obstruct G-transversality, even for G of order 2, and consequently this whole Madsen-Rothenberg edifice is not applicable to this case.

However, there is another approach to the key issue of equivariant surgery; while it has not been nearly as successful in constructing exotic actions with unusual orbit structures, etc., it has led to more complete classification theories, especially in the PL and topological categories. It is to this topic that we turn now.

An idea of Browder and Quinn [BQ] is that we should look at isovariant maps, i.e., ones which map the free parts to the free parts (precisely: an equivariant map is isovariant if and only if $G_m = G_{f(m)}$ for all m). Then the homology kernel one studies is precisely the kernel of the free part (by excision), from which the homotopy classes of interest already have representatives there, as desired.

An unpublished theorem of Browder's (it's at least 10 years old; see also [Do] for a related result; Sandor Straus' unpublished 1973 Ph.D. thesis [Str] proves essentially the same result for $\mathbb{Z}_p$-actions) asserts that any equivariant homotopy equivalence between G-manifolds satisfying a suitable gap hypothesis is equivariantly homotopic to an isovariant homotopy equivalence; and the inclusion of the space of isovariant homotopy equivalences in the space of equivariant equivalences is as highly connected as the excess that the dimensions satisfy beyond the gap hypothesis. One proof can be obtained by comparing the results of the two theories.

In their paper, Browder and Quinn [BQ] employed a very simple and calculable kind of normal invariants (indeed, the set of normal invariants is isomorphic to the familiar $[M/G, F/CAT]$), but at a great cost. They required a stratified transversality condition for all maps in their category. This is tantamount to having lots of bundle and framing data; objects then tend to have many self maps not homotopic to isomorphisms in this category, so many actions are counted more than once. Moreover, many equivariantly homotopy equivalent manifolds have no such transverse G-maps between them, thus severely restricting the kind of actions produced. (See [HuW] for a discussion of [BQ].) Finally, in the topological category, the types of rigid structures their theory demands simply do not exist. (This nonexistence is manifest in the [CS] example of nonlinear similarity discussed above and in the equivariantly nonfinite G-ANR's of Quinn [Q1, Q2] discussed below. A good local structure of the [BQ] sort would, in particular, enable one to remove open regular neighborhoods of lower strata, and thus yield a reasonable simple homotopy theory, such as exists in the PL category, for the complements.)

Section 4 of [HuW] in these proceedings gives a sketch of the Browder-Quinn theory. What is important for our purposes is not the details, but the basic idea; it is that one does surgery on a stratum, then using transversality extends the solution of the surgery problem (assuming 0 obstruction, of course) to a neighborhood of the stratum, and then deals with the new ordinary surgery obstruction that one is confronted with on the next stratum up.

Thus, the surgery obstruction groups in this theory are "built up" out of the surgery obstruction groups of all of the "pure strata", e.g. the sets of points with a given isotropy group (up to conjugation)[6]. We recommend that the reader consult [HuW] for a few examples which show how the strata can interact in forming the Browder-Quinn L-group. (For instance, in some cases of interest, despite the fact that all the strata have nontrivial L-groups, the global L-group is trivial.)[7] [8]

For the rest of this paper we will focus our attention on the topological category; almost everything we say has a simpler analogue in the PL category — and works out very differently in the smooth category.[9]

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[6]And for a general stratified space, one means the set of points that lie in a given stratum, but not in any lower one.

[7]Easy exercise: Let $G = \mathbb{Z}_2$ and let M be a manifold with no action. Compute S^{iso} $(M \times S^n)$, where the group action is constructed using the involution on S^n with fixed point set S^{n-1}, $n > 2$. Thought question: What do you make of the fact that isovariant homotopy equivalences can automatically be made transverse in this (unusual) case?

[8]Deeper examples of how the strata can interact to cancel or almost cancel their L-theoretic contributions arise in the beautiful work of Davis, Hsiang, and Morgan [DH, DHM] on the concordance classification or certain smooth $U(n)$ and $O(n)$ actions (the so-called "multiaxial actions" on the sphere).

[9]The past fifteen years have seen a shift in emphasis from the smooth category to the topological in transformation group theory. The smooth category had seemed much more comprehensible than the topological, because of the slice theorem, equivariant tubular neighborhood theorem, and clear inductive techniques, as well as deep tools like

The first topic that must be clarified is what we mean by the topological category. Of course arbitrary continuous actions can be quite wild: the fixed set need not have any manifold (or even homology manifold) points. It can be far from nice in any homological sense — every open subset can have infinitely generated rational homology!

Two choices have been commonly chosen. The first, introduced in [B2] and deeply investigated in [MR], is the locally linear category. One assumes that each orbit (G/G_x) has an invariant open neighborhood which is equivariantly homeomorphic to a neighborhood of some orbit within a linear G-representation. Smooth group actions all have this structure, and this condition forces all fixed point sets of all subgroups to be nice locally flat submanifolds of one another. This condition leads, for instance, to fairly simple isotopy extension theorems.

The other choice is to assume just that all fixed point sets are locally flat submanifolds of one another. This is a wider category, called the tame category, but it fortunately turns out — quite non-trivially — that this category also has all the isotopy extension theorems one would want [Q2, Hu], see also [HTWW] and [HuW], and remains suitable for classification theorems, as we will sketch. Quinn's isotopy extension theorem implies homogeneity of the strata; so if one has, for instance, a semifree action with connected locally flat fixed point set, then it is locally linear if and only if it is locally linear at a single fixed point.

Theorem 3.1 ([Q2]) *If G acts tamely on a topological manifold, then one can extend G-isotopies. In particular, any two points p and q in the same component of the submanifold consisting of points with G_p as their isotropy have equivariantly homeomorphic neighborhoods.*

Assadi's obstruction is of course just the Wall finiteness obstruction for the quotient of the complement of the fixed set, which must vanish in the PL category because of regular neighborhood theory. The above theorem shows that one cannot find such neighborhoods in the topological setting. (The first examples of this phenomenon were discovered by Quinn [Q1, Q2].) The idea of the proof is to remove a point from a stratum, work non-compactly, then one point compactify, and use the magic of there being no isolated topological singularities.

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the Atiyah-Singer G-signature formula. At this date, it seems that for finite groups the topological category is the better understood one: for instance, one still does not know exactly which submanifolds of S^n are the fixed point sets of smooth $\mathbb{Z}_p$ actions (even if we avoid the awkward codimension two case) (see [Sch1] for beautiful partial results which already display some fascinating phenomena) or the classification of actions with given fixed point set. In some sense, the key difference between the smooth and topological categories is that, in the latter, local and global issues are essentially the same. In the smooth category, global issues are largely surgery theoretic, while the more rigid local ones involve via linearization, and the unstable homotopy theory of classical groups.

By this point, it does not really matter much which of these two topological categories the reader keeps in mind. When we develop precise classification theorems, it will turn out that the second category is somewhat more convenient; but in principle all one has to do at the end of any discussion is to examine what is occurring at several points in order to check whether one has succeeded in working in the locally linear category.

The first interesting point is that locally linear actions need not have equivariant handle decompositions. The first example of this was Siebenmann's locally triangulable nontriangulable sphere [Si1], whose key feature was nontriangulability for algebraic K-theoretic reasons, rather than because of the Kirby-Siebenmann invariant familiar from the unequivariant theory of topological manifolds. Other examples appear in [Q1, Q2, DR, We1]. Needless to say, once existence is called into question, then there shouldn't be uniqueness, and this had been forcefully shown by the examples [CS1] of nonlinear similarity mentioned above, i.e., of nonequivalent orthogonal representations that cannot be PL equivalent because of torsion considerations (de Rham's theorem), but which are nevertheless topologically conjugate.

Non-uniqueness of handle structure can also be seen quite easily using Milnor's [M2] counterexamples to the Hauptvermutung. The nonexistence is also quite easy using Siebenmann's proper h-cobordism theorem [Si2], if one grants the homogeneity of tame actions.

Example. Suppose that one starts with any smooth $\mathbb{Z}_p$ action on a manifold, where the fixed set is nonempty and has codimension at least 4. Then, for $p > 3$, $\mathrm{Wh}(\mathbb{Z}_p) \neq 0$, and so one can erect an equivariant h-cobordism with nontrivial torsion (working in the quotient of the complement). However, this smoothly nontrivial h-cobordism is topologically trivial, by an Eilenberg swindle (similar to the YG construction in [St]).

Example. Suppose that the fixed set has positive dimension and is disconnected. Instead of realizing an element of the usual Whitehead group of the closed complement, remove two points from the fixed set from different components. Attach to $M \times I$ the realization of an element of the proper Whitehead group of the new closed complement (which is $\widetilde{K}_0(\mathbb{Z}_p)$), and then end-point compactify. One still has a locally linear action by the isotopy homogeneity theorem. However, one can easily check that this space does not have a closed regular neighborhood of its singular set.

A striking application of this overall technique geometrically can be found in [We1]:

Theorem 3.2 *A submanifold Y^n of S^{n+r}, for $r > 2$ and $n + r > 4$, is the fixed set of a semifree locally linear orientation preserving G-action on the sphere if and only if G acts freely and linearly on S^{r-1} and Y is a mod $|G|$ homology sphere.*

One should compare this with the conditions that arose in Assadi's theorem mentioned in (1.2). In that theorem there are restrictions on the orders of the homology modules of Y, depending on G (and they are indeed nontrivial). Here the topological category turns out to be easier to analyze than the PL locally linear category, because in the former there can be no isolated singularities by Quinn's theorem. (See [We4] for the analogous theorem in the PL locally linear setting.)

In short, the proof goes like this: One produces an action on $\mathbb{R}^{n+r}$ with fixed point set $\widetilde{Y}$ ($= Y$ minus a point) using local bundle information [CW1] to get the action on a neighborhood and extension across homology collars [AsB, We3] to extend outwards. Then, as above, one one-point compactifies and is guaranteed local linearity.

Examples like these very naturally suggested bringing to bear the technology of controlled topology [ChF, AH, Q1, Y, FP, ACFP, We2]. Steinberger and West [Stnb] and Quinn [Q2], in the more general setting of stratified spaces (with a missing realization result provided in [HTWW, Hu]) gave an s-cobordism theorem.[10]

Theorem 3.3 *If G acts tamely on a manifold M, with no fixed point set of codimension one or two in another, then*

$$\mathrm{Wh}_G^{\mathrm{top}}(M) \cong \bigoplus \mathrm{Wh}_{NH/H}^{\mathrm{top}}(M^H, \mathrm{rel\ sing}).$$

These latter groups can be computed via an exact sequence:

$$\begin{aligned} H_0(M/G; \mathbf{Wh}(G_m)) \to \mathrm{Wh}(G) \to \mathrm{Wh}_G^{\mathrm{top}}(M, \mathrm{rel\ sing}) \\ \to H_0(M/G; \mathbf{K}_0(G_m)) \to K_0(G). \end{aligned}$$

These "*rel sing*" theories can be interpreted as being the theory of h-cobordisms that are trivial on the lower strata. Because they are not assumed to be trivialized in a neighborhood of these strata (as would be automatic in the PL and smooth categories), we have the nontrivial assembly maps ($H_0(M/G; \mathbf{Wh}(G_m)) \to \mathrm{Wh}(G)$ and $H_0(M/G; \mathbf{K}_0(G_m)) \to K_0(G)$) which control the "leaking" of obstructions through lower strata. The map $\mathrm{Wh}_G^{\mathrm{top}}(M, \textit{rel sing}) \to H_0(M/G; \mathbf{K}_0(G_m))$ measures the obstruction to putting a mapping cylinder around singular set of the h-cobordism (assuming we had one on the bottom of the h-cobordism to begin with). And the map $H_0(M/G; \mathbf{Wh}(G_m)) \to \mathrm{Wh}(G)$ measures the effect of changing mapping cylinder structures. (We recommend the reader rethink the above two examples in light of this theorem in order to appreciate it.)

[10] In the locally linear setting, one has to be careful, because the realization of h-cobordisms when the fixed set consists of isolated points might leave the category.

Corollary *If M is G-simply connected, i.e., if M^H is simply connected for all $H \subset G$, and M satisfies the "no low codimension" gap condition, then* $\mathrm{Wh}_G^{\mathrm{top}}(M) = 0$.

This holds for the sphere of a representation whenever the fixed point set is two- or more dimensional and there are no one- or two- dimensional eigenspaces. (These can be dealt with, of course, but with a bit more calculation.) Thus, one can see that once one has two trivial summands, the question of the existence of a nonlinear similarity is unaffected by stabilization with a third or more.

Another key feature that distinguishes the smooth category from the PL and topological locally linear categories, not to mention the tame category, is the issue (mentioned above) of equivariant transversality. In the smooth category, there is a stable equivariant transversality theorem (see e.g. [Pe2, PR, CoW]). However, for $G = \mathbb{Z}_2$ the Browder-Livesay examples of nondesuspendable involutions on the sphere give rise to counterexamples to transversality, even for maps into $\mathbb{R}$ with its nontrivial involution. (See [MR]; a simple nonlocally linear example arises from the natural map from the open cone of a non-desuspendable free involution on the sphere mapping to $\mathbb{R}$. The transverse inverse image of 0 would have around its fixed point a desuspension of the action.) For odd order groups, one would have similar counterexamples in the tame category, because Wall's desuspension theorem only covers desuspension with respect to the linear $\mathbb{Z}_p$ actions. However, as noted above, the remarkable theorem of Madsen and Rothenberg shows that for locally linear odd order group actions, transversality holds. This enabled them to extend Wall's theorem to nonfree actions, give a classifying space for topological locally linear equivariant normal invariants (assuming a "gap hypothesis") and an associated surgery theory. Furthermore, they gave an extension of Sullivan's $KO_*[1/2]$ orientation of topological manifolds to a $KO_*^G[1/2]$ orientation for locally linear G-manifolds when G is of odd order. (It was this invariant which they used to prove the result on odd order nonlinear similarity mentioned above.)

Soon after, Rothenberg and Weinberger gave an analytic approach to the $KO_*^G[1/2]$ class, which works for all G (see [RsW1, RtW]) and in the tame category. A general and direct controlled topological approach for group actions on topological pseudomanifolds was given by the present authors and Shaneson in [CSW], using intersection chain sheaves. This class fails to be an orientation in general[11]. For smooth G-manifolds, this

[11] A similar thing occurs in the theory of the usual Sullivan class. It is definable as an element of the more refined homology theory $H_*(\_\_;\mathbf{L}^\bullet(e))$, where it is an orientation for topological manifolds [R3]. Actually it can be defined even for ANR homology manifolds, but in that generality the class need not be an orientation. In fact it is an orientation away from 2 if and only if the homology manifold is resolvable (i.e., a cell-like image of a topological manifold). For more of a discussion of the theory of homology manifolds

class is the equivariant symbol of the signature operator. This takes us a certain way towards explaining the following theorem:

Theorem 3.4 (i) (Combination of [MR, RtW, CSW, RsW1, We2]) *Suppose G is a finite group acting tamely on a manifold M. Then one can define a KO^G-homology fundamental class*

$$\triangle(M) \in KO_*^G(M) \otimes \mathbb{Z}[1/2]$$

which is a topological invariant. Its image under the natural map $M \to pt$ sends $\triangle(M)$ to G-signature(M).
(ii) ([CWY] using [We2])
Suppose that there are no low codimensional fixed point sets; then

$$S_G^{\mathrm{iso}}(M) \otimes \mathbb{Z}[1/2] \;\cong\; \bigoplus S_{NH/H}^{\mathrm{iso}}(M^H, \mathrm{rel\ sing}) \otimes \mathbb{Z}[1/2].$$

In fact, $S_G^{\mathrm{iso}}(M, \mathrm{rel\ sing})$ is a covariantly equivariantly functorial abelian group, the elements of which for M G-simply connected are determined away from 2 by $\triangle$, and all values of $\triangle$ are realized subject to the augmentation condition.

Remark. S_G^{iso} is the structure set of a G-manifold defined using isovariant homotopy equivalences. Recall that by Browder's theorem this is the same as the equivariant structure set when a strong gap hypothesis applies.

Remark. The splitting of the structure set here into pieces defined by the fixed point sets means (ignoring the issue of the prime 2) that any equivariant structure on a fixed point set (of any subgroup) can be extended to one on the whole G-manifold. In the case where the subgroup involved is G itself, this is the phenomenon studied earlier by "replacement theorems" [CW2], wherein any manifold homotopy equivalent to the fixed point set is actually the fixed set of an equivariantly homotopy equivalent group action. A typical statement of that theory is the following:

Theorem 3.5 *Suppose G is an odd order abelian group acting locally linearly on a G-manifold M (with no codimension-two fixed point sets), smoothly in a neighborhood of a 1-skeleton. If F' is a manifold homotopy equivalent to the fixed point set F, then there is an equivariantly homotopy equivalent G-action with fixed point set F'.*

Unfortunately that statement is false for even order groups, but it's true away from the prime 2! (The theorem in [CW2] is, in fact, stronger in that one also sees that the new G-action is on the original manifold M.)

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and its analogies to group actions, see [We5].

In any case, it's not even entirely obvious that such a homotopy equivalent manifold even embeds in the manifold under discussion. Indeed, in the smooth category, characteristic class theory easily shows that this false. Implicit in this theorem is the homotopy invariance of PL or topological embedding in codimension larger than 2 (Browder, Casson, Haefliger, Sullivan and Wall), which, of course, involves systematically changing the putative normal bundles to our submanifolds. In other words, this type of phenomenon is exactly the type of thing not studiable by the original theory of [BQ]. Nevertheless, as we will see, calculations of sheaves of Browder-Quinn surgery groups, constructed as noted above for use in a much more constrained setting, play an important role in the topological category.

We can only give a short sketch of a proof here. The first part is to simply recognize that isovariant structures on M are the same thing as "stratified structures on M/G", where one works in the category of homotopically stratified spaces [Q2, We2, Hu, HuW]. [We2] extended [BQ] to (i) apply to these spaces, and (ii) to maps which are stratified but not necessarily transverse. (Surveys of this are [We5, HuW]; an early precursor to this general theory is [CW3].)

The problem then is to compute what the theory actually says. The surgery obstruction groups away from 2 can be analyzed by a trick (see [CSW, LM, DS]). To any G-manifold (G finite) one can define a symmetric signature (see [R1]) $\delta^*_G(M) \in L^*(\mathbb{Q}\Gamma)$, where Γ is the "orbifold fundamental group" (= fundamental group of the Borel construction) which fits into an exact sequence $1 \to \pi_1(M) \to \Gamma \to G \to 1$. This can be done in the same way as symmetric signatures are defined in general, simply observing that the $\mathbb{Q}\Gamma$-chain complex of a G-manifold, that is, the $\mathbb{Q}$-chain complex of the Γ-manifold that is the universal cover of M, is made up out of projective chain complexes, since all isotropy is finite. Now we use the fact that, according to Ranicki [R], changing coefficients from $\mathbb{Z}\Gamma$ to $\mathbb{Q}\Gamma$ only affects L-theory at the prime 2 (as does allowing projective rather than free modules in the definition of L-groups).

If one assembles all of the equivariant signatures of all the strata together, one has an *a priori* method of detecting all of the surgery obstructions that will be inductively arising in the Browder-Quinn process, i.e., many secondary, tertiary, etc., obstructions are, in fact, primary. This kind of argument yields the collapse of some spectral sequence and this gives a splitting away from 2 of the L^{BQ} into ordinary L-groups, just like the splitting asserted on structure sets.

Remark. Results of [LM] can be used to see that for many odd order group actions that there is an integral splitting, as well.

It turns out that this method is sufficiently canonical to apply not just to individual L-groups, but to "cosheaves of L-spectra" as well. As a result, as noted by the present authors with Min Yan, here the normal invariants [We2], which turn out to be a cosheaf homology theory, also break up into pieces defined on the various strata. This is enough to get the splitting of S_G^{iso} into pieces corresponding to the various strata.

Now, to compute these pieces, one shows that the normal invariants (of a stratum *rel sing*) essentially boils down to equivariant K-groups. This is an extension of Sullivan's "characteristic variety" theorem for the structure of G/PL at odd primes and can be proven in a couple of ways, by now. For example, [CSW] explained how to produce the equivariant signature class via the calculation such a cosheaf homology group. On the other hand, [RsW1, We2] uses just the PL version of the equivariant Teleman signature operator to give a calculation of the cosheaf homology (which still is an improvement on the approach of [RtW] in that it does not require the construction of any Lipschitz structures, nor of the harder Lipschitz signature operator.)

Remark. The remaining statement regarding equivariant functoriality can be found in [CWY], where we, together with Min Yan, analyze these *rel sing* structure sets, as well as the issue of integral splitting. (It does not split in general; there is a spectral sequence whose differentials are related to assembly maps for finite groups.) That work also deals with the calculation of normal invariants at the prime 2, and their relation to Bredon homology. Earlier work on that problem can be found in [Na].

Remark. The non simply connected case can be dealt with in a similar fashion; essentially one just has to use non simply connected L-groups on occasion to replace the $KO^G(*)$.

Needless to say, this theorem has implications for nonlinear similarity, in terms of giving necessary conditions, such as the fact that for odd order groups nonlinear and linear similarity coincide ([HP] and [MR]) or the more general topological invariance of generalized Atiyah-Bott numbers of [CSSWW]; it also quickly leads to the "topological rationality principle" for representations of general finite groups of [CS2]. The reader should also consult [HaP1,2] for a great deal of information regarding nonlinear similarity for cyclic groups.

We would like to close this survey with describing some relatively recent ideas on equivariant structure sets when the gap hypothesis does not hold.

One key to this is the work of [Y1] and [WY] (which extends earlier work of [DS] on equivariant L-groups) which shows that there are isovariant periodicity theorems for structure sets, as suggested by the form of the normal invariants away from 2 and equivariant Bott periodicity. Unfortunately, one does not have a general statement, so we will leave precise statements to the references. However, a payoff from this is the following: one can, as a consequence, often show that existence and classification problems do not actually depend on all of the stratified data, but only depend on the equivariant homotopy data.

The model for this is the following. An easy application of the total surgery obstruction theory of [R3], modified in [BFMW], is that a Poincaré complex X is homotopy equivalent to an ANR homology manifold if and only if $X \times \mathbb{CP}^2$ is. Ordinarily this is not too useful because it is hard to verify directly that $X \times \mathbb{CP}^2$ is a manifold. However, in the situation of group actions, say for odd order groups, the substitute is a product of $\mathbb{CP}^2$'s with permutation action, so that after crossing some number of times, the gap hypothesis becomes valid.

Now, using Browder's theorem one sees that the difference between equivariance and isovariance goes away by this process. It follows that if an equivariantly equivalent Poincaré space can be geometrically constructed, then stably and isovariantly one can realize geometrically the Poincaré space, and thus, by Yan's work, one can also realize the original Poincaré complex.

This then leads (under good conditions) to a decomposition $S^{\text{equi}} \cong S^{\text{iso}} \times ?$ where ? is a purely homotopy theoretic object. It consists of the isovariant homotopy types of Poincaré objects within the given equivariant homotopy type. While this does not look particularly computable, it turns out to be closely related to the set of PL (or topologically locally flat) embeddings of the fixed point set homotopic to the given embedding.

This strangely reduces the geometrical problem to homotopy theory, and then relates the homotopy theory back to a different geometrical problem, instead of dealing with it homotopy theoretically. This process leads to, for instance, a result conjectured by the first author:

Theorem 3.6 ([We7]) *A semifree action on a simply connected manifold with simply connected fixed set of codimension greater than 2 is determined within its equivariant homotopy type, up to finite ambiguity, by the equivariant signature class and the underlying topological types of the fixed point set and its embedding in the ambient manifold.*

As a sample application regarding the equivariant version of the Borel rigidity conjecture, i.e., the conjecture that aspherical manifolds are determined up to homeomorphism by their fundamental groups, we have:

Theorem 3.7 ([We7], see [Shi] for more refined information): *The cyclic group $\mathbb{Z}_p$, p a prime, $p > 3$, acts affinely (via a permutation representation on the coordinates) on T^{2p-3} in such a way that $S^{\text{equi}}(T^{2p-3})$ is infinite. Moreover, all of the elements are smoothable and they remain distinct when crossing with any torus with trivial action, but they all do become equivalent on taking a suitable finite sheeted cover.*

One can actually put a group structure on many of these structure sets and give actual calculations. (This last is due to *positive* results on the equivariant Borel conjecture, see [CK]; more results can be obtained by combining [FJ] with [We2].)[12]

All but the smoothability comes from the type of analysis discussed above: one constructs embeddings of the torus $T^{p-1} \hookrightarrow T^{2p-3}$ that are isotopic to their images under the linear $\mathbb{Z}_p$ action and then, using the connections between embeddings and Poincaré embeddings and isovariant Poincaré complexes, one gets a homotopy model that can be geometrically realized.

The smoothability is explained in [FRW].

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[12] [CK] gives a direct geometric proof that, in the presence of the gap hypothesis, the equivariant rigidity conjecture holds for odd order affine action on the torus. [We2] gives an analysis of the rigidity problem (again, assuming isovariance or the gap hypothesis) in terms of assembly maps, which are quite similar to the ones arising in Farrell and Jones' "isomorphism conjecture" for the computation of L-groups. They announced that the isomorphism conjecture is true for $L^{-\infty}$ of uniform lattices in real connected line groups and for the algebraic K-groups of these groups. Combining the ingredients yields a calculation of the isovariant structure set in many cases, although the answer is not always 0! See [CK2] for an example (which they attribute to the second author of this paper) of a nonrigid affine involution on the torus, whose construction is based on the UNil group of [Ca] and another example of a nonrigid affine torus whose construction is based on a nonvanishing Nil group. Unfortunately, we cannot explain any more about this fascinating problem here.

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Surgery and stratified spaces

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0. Introduction

The past couple of decades has seen significant progress in the theory of stratified spaces through the application of controlled methods as well as through the applications of intersection homology. In this paper we will give a cursory introduction to this material, hopefully whetting your appetite to peruse more thorough accounts.

In more detail, the contents of this paper are as follows: the first section deals with some examples of stratified spaces and describes some of the different categories that have been considered by various authors. For the purposes of this paper, we will work in either the PL category or a very natural topological category introduced by Quinn [Q4]. The next section discusses intersection homology and how it provides one with a rich collection of self dual sheaves. These can be manipulated by ideas long familiar to surgery theorists who have exploited Poincaré duality from the start. We will give a few applications of the tight connection between an important class of stratified spaces (Witt spaces), self dual sheaves, and K-theory; one last application will appear in the final section of the paper (where we deal with the classification of "supernormal" spaces with only even codimensional strata).

Section three begins an independent direction, more purely geometric. We describe the local structure of topological stratified spaces in some detail, in particular explaining the teardrop neighborhood theorem ([HTWW], [H2]) and giving applications to isotopy theorems and the like. The last three sections describe the theory of surgery on stratified spaces, building on our understanding of teardrop neighborhoods, and some applications to classification problems (other applications can also be found in the survey [CW4]).

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1. Definitions and Examples of Stratified Spaces

A *stratification* $\Sigma = \{X_i\}$ of a space X is a locally finite decomposition of X into pairwise disjoint, locally closed subsets of X such that each $X_i \in \Sigma$ is a topological manifold. We always assume that X is a locally compact, separable metric space and that Σ satisfies the *Frontier Condition*: $\operatorname{cl} X_i \cap X_j \neq \emptyset$ if and only if $X_j \subseteq \operatorname{cl} X_i$. The index set is then partially ordered by $j \leq i$ if and only if $X_j \subseteq \operatorname{cl} X_i$. The set $X_i \in \Sigma$ is called a *stratum* and $X^i = \operatorname{cl} X_i = \cup\{X_j \mid j \leq i\}$ is a *skeleton* (or *closed stratum* in the terminology of [W4]).

Partitioning non-manifold spaces into manifold pieces is a very old idea — one has only to consider polyhedra in which the strata are differences between successive skeleta. However, it was not until relatively recently that attention was paid to how the strata should fit together, or to the geometry of the neighborhoods of strata. In 1962 Thom [T1] discussed stratifications in which the strata have neighborhoods which fibre over that stratum and which have "tapis" maps (the precursor to the tubular maps in Mather's formulation in 1.2 below). It was also in this paper that Thom conjectured that the topologically stable maps between two smooth manifolds are dense in the space of all smooth maps. In fact, it was Thom's program for attacking that conjecture which led him to a study of stratifications [T2]. The connection between stratifications and topological stability (and, more generally, the theory of singularities of smooth maps) is outside the scope of this paper, but the connections have continued to develop (for a recent account, see the book of du Plessis and Wall [dPW].)

Here we review the major formulations of the conditions on neighborhoods of strata. These are due to Whitney, Mather, Browder and Quinn, Siebenmann, and Quinn. The approaches of Whitney, Mather, Browder and Quinn are closely related to Thom's original ideas. These types of stratifications are referred to as *geometric stratifications*. The approaches of Siebenmann and Quinn are attempts at finding an appropriate topological setting.

1.1 Whitney stratifications. In two fundamental papers [Wh1],[Wh2], Whitney clarified some of Thom's ideas on stratifications and introduced his Conditions A and B. To motivate these conditions consider a real algebraic set $V \subseteq \mathbb{R}^n$, the common locus of finitely many real polynomials. The singular set ΣV of all points where V fails to be a smooth manifold is also an algebraic set. There is then a finite filtration $V = V^m \supseteq V^{m-1} \supseteq \cdots \supseteq V^0 \supseteq V^{-1} = \emptyset$ with $V^{i-1} = \Sigma V^i$. One obtains a stratification of V by considering the strata $V_i = V^i \setminus V^{i-1}$. However, with this naive construction the strata need not have geometrically well-behaved neighborhoods; that is, the local topological type need not be locally constant along strata. For example, consider the famous Whitney umbrella which is the locus of $x^2 = zy^2$, an algebraic set in $\mathbb{R}^3$. The singular set ΣV is the z-axis and is a smooth manifold, so one obtains just two strata, $V \setminus \Sigma V$ and ΣV. However, there is a drastic change in the neighborhood of ΣV in V as one passes through the origin: for negative z small neighborhoods meet only ΣV whereas this is not the case for positive z.

If X, Y are smooth submanifolds of a smooth manifold M, then X is *Whitney regular over* Y if whenever $x_i \in X$, $y_i \in Y$ are sequence of points converging to some $y \in Y$, the lines $l_i = \overline{x_i y_i}$ converge to a line l, and the tangent spaces $T_{x_i}X$ converge to a space τ, then

(A) $T_yY \subseteq \tau$ and
(B) $l \subseteq \tau$.

A stratification $\Sigma = \{X_i\}$ of X is a *Whitney stratification* if whenever $j \leq i$, X_i is Whitney regular over X_j.

In the Whitney umbrella V, $V \setminus \Sigma V$ is not Whitney regular over ΣV at the origin. However, the stratification can be modified to give a Whitney stratification and a similar construction works for a class of spaces more general than algebraic sets: a subset $V \subseteq \mathbb{R}^n$ is a *semi-algebraic set* if it is a finite union of sets which are the common solutions of finitely many polynomial equations and inequalities. Examples include real algebraic sets and polyhedra. In fact, the class of semi-algebraic sets is the smallest class of euclidean subsets containing the real algebraic sets and which is closed under images of linear projections. If V is semi-algebraic, then there is a finite filtration as in the case of an algebraic set discussed above obtained by considering iterated singular sets. This filtration can be modified by removing from the strata the closure of the set of points where the Whitney conditions fail to hold. In this way, semi-algebraic sets are given Whitney stratifications (see [GWdPL]).

In fact, Whitney [Wh2] showed that any real or complex analytic set admits a Whitney stratification. These are sets defined analogously to algebraic sets with analytic functions used instead of polynomials. Łojasiewicz [Lo] then showed that semi-analytic sets (the analytic analogue of semi-algebraic sets) have Whitney stratifications. An even more general class of spaces, namely the *subanalytic sets*, were shown by Hardt [Hr] to admit Whitney stratifications. For a modern and thorough discussion of stratifications for semi-algebraic and subanalytic sets see Shiota [Shi].

1.2 Mather stratifications: tube systems. Mather clarified many of the ideas of Thom and Whitney and gave complete proofs of the isotopy lemmas of Thom. He worked with a definition of stratifications closer to Thom's original ideas than to Whitney's, but then proved that spaces with Whitney stratifications are stratified in his sense.

Definition. For $0 \le k \le +\infty$, a *Mather C^k-stratification* of X is a triple $(X, \Sigma, \mathbf{T})$ such the following hold:

(1) $\Sigma = \{X_i\}$ is a stratification of X such that each stratum $X_i \in \Sigma$ is a C^k-manifold.

(2) $\mathbf{T} = \{T_i, \pi_i, \rho_i\}$ is called a *tube system* and T_i is an open neighborhood of X_i in X, called the *tubular neighborhood* of X_i, $\pi_i : T_i \to X_i$ is a retraction, called the *local retraction* of T_i, and $\rho_i : T_i \to [0,\infty)$ is a map such that $\rho_i^{-1}(0) = X_i$.

(3) For each $X_i, X_j \in \Sigma$, if $T_{ij} = T_i \cap X_j$ and the restrictions of π_i and ρ_i to T_{ij} are denoted π_{ij} and ρ_{ij}, respectively, then the map

$$(\pi_{ij}, \rho_{ij}) : T_{ij} \to X_i \times (0,\infty)$$

is a C^k-submersion.

(4) If $X_i, X_j, X_k \in \Sigma$, then the following compatibility conditions hold for $x \in T_{jk} \cap T_{ik} \cap \pi_{jk}^{-1}(T_{ij})$:

$$\pi_{ij} \circ \pi_{jk}(x) = \pi_{ik}(x),$$

$$\rho_{ij} \circ \pi_{jk}(x) = \rho_{ik}(x).$$

When $k = 0$ above, a C^0-submersion, or topological submersion, means every point in the domain has a neighborhood in which the map is topologically equivalent to a projection (see [S2]).

Mather [Ma1], [Ma2] proved that Whitney stratified spaces have Mather C^∞-stratifications.

The Thom isotopy lemmas mentioned above are closely related to the geometric structure of neighborhoods of strata. For example, the first isotopy lemma says that if $f : X \to Y$ is a proper map between Whitney stratified spaces with the property that for every stratum X_i of X there exists a stratum Y_j of Y such that $f| : X_i \to Y_j$ is a smooth submersion, then f is a fibre bundle projection (topologically — not smoothly!) and has local trivializations which preserve the strata. Mather applied this to the *tubular maps*

$$\pi_i \times \rho_i : T_i \to X_i \times [0,\infty)$$

defined on the tubular neighborhoods of the strata of a Whitney stratified space X in order to show that every stratum X_i has a neighborhood N such that the pair (N, X_i) is homeomorphic to $(\mathrm{cyl}(f), X_i)$ where $\mathrm{cyl}(f)$ is the mapping cylinder of some fibre bundle projection $f : A \to X_i$. The existence of these mapping cylinder neighborhoods was abstracted by Browder and Quinn as is seen next.

1.3 Browder-Quinn stratifications: mapping cylinder neighborhoods. In order to classify stratified spaces Browder and Quinn [BQ] isolated the mapping cylinder structure as formulated by Mather. The mapping cylinder was then part of the data that was to be classified. More will be said about this in §4 below. Here we recall their definition.

Let $\Sigma = \{X_i\}$ be a stratification of a space X such that each stratum X_i is a C^k-manifold. The *singular set* ΣX_i is $\operatorname{cl} X_i \setminus X_i = \cup\{X_j \mid j < i\}$. (This is in general bigger than the singular set as defined in 1.1.)

Definition. Σ is a C^k *geometric stratification* of X if for every i there is a closed neighborhood N_i of ΣX_i in $X^i = \operatorname{cl} X_i$ and a map $\nu_i : \partial N_i \to \Sigma X_i$ such that

(1) ∂N_i is a codimension 0 submanifold of X_i,
(2) N_i is the mapping cylinder of ν_i (with ∂N_i and ΣN_i corresponding to the top and bottom of the cylinder),
(3) if $j < i$ and $W_j = X_j \setminus \operatorname{int} N_j$, then $\nu_i| : \nu_i^{-1}(W_j) \to W_j$ is a C^k-submersion.

The complement of $\operatorname{int} N_i$ in X^i is called a *closed pure stratum* and is denote $\overline{X}^i$.

Note this definition incorporates a topological theory by taking $k = 0$. Browder and Quinn also pointed out that by relaxing the condition on the maps v_i other theories can be considered. For example, one can insist that the strata be PL manifolds and the v_i be PL block bundles with manifold fibers.

1.4 Siebenmann stratifications: local cones. In the late 1960s Cernavskii [Ce] developed intricate geometric techniques for deforming homeomorphisms of topological manifolds. In particular, he proved that the group of self homeomorphisms of a compact manifold is locally contractible by showing that two sufficiently close homeomorphisms are canonically isotopic. The result was reproved by Edwards and Kirby [EK] by use of Kirby's torus trick. Siebenmann [S2] developed the technique further in order to establish the local contractibility of homeomorphism groups for certain nonmanifolds, especially, compact polyhedra.

Siebenmann's technique applied most naturally to stratified spaces and a secondary aim of [S2] was to introduce a class of stratified spaces that he thought might "come to be the topological analogues of polyhedra in the piecewise-linear realm or of Thom's stratified sets in the differentiable realm." These are the *locally conelike TOP stratified sets* whose defining property is that strata are topological manifolds and for each point x in the n–stratum there is a compact locally conelike TOP stratified set L (with fewer strata — the definition is inductive) and a stratum-preserving homeomorphism of $\mathbb{R}^n \times \mathring{c}L$ onto an open neighborhood of x where $\mathring{c}L$ denotes the open cone on L and the homeomorphism takes $0 \times \{\text{vertex}\}$ to x. Simple examples include polyhedra and the topological (C^0) versions

It is important to realize that Siebenmann didn't just take the topological version of Mather's stratified space, but he did have Mather's C^0-tubular maps *locally* at each point. The reason he was able to work in this generality was that the techniques for proving local contractibility of homeomorphism groups were purely local.

As an example, consider a pair (M, N) of topological manifolds with N a locally flat submanifold of M. With the two strata, N and $M \setminus N$, the local flatness verifies that this is a locally conelike stratification. However, Rourke and Sanderson [RS] showed that N need not have a neighborhood given by the mapping cylinder of a fibre bundle projection. Thus, Siebenmann's class is definitely larger than the topological version of the Thom-Whitney-Mather class.

On the other hand, Edwards [E] did establish that locally flat submanifolds of high dimensional topological manifolds do, in general, have mapping cylinder neighborhoods. However, the maps to the submanifold giving the mapping cylinder need not be a fibre bundle projection. It turns out that the map is a *manifold approximate fibration*, a type of map which figures prominently in the discussion of the geometry of homotopically stratified spaces below.

Later, Quinn [Q2,II] and Steinberger and West [StW] gave examples of locally conelike TOP stratified sets in which the strata do not have mapping cylinder neighborhoods of any kind. In fact, their examples are orbit spaces of finite groups acting locally linearly on topological manifolds. Such orbit spaces are an important source of examples of locally conelike stratified sets and many of advances in the theory of stratified spaces were made with applications to locally linear actions in mind. These examples were preceded by an example mentioned by Siebenmann [S3] of a locally triangulable non-triangulable space.

Milnor's counterexamples to the Hauptvermutung [M1] give non-homeomorphic polyhedra whose open cones are homeomorphic. As Siebenmann observed, these show that the links in locally conelike stratified sets are not unique up to homeomorphism. Siebenmann does prove that the links are stably homeomorphic after crossing with a euclidean space of the dimension of the stratum plus 1. The non-uniqueness of links points to the fact that Siebenmann's stratified spaces are too rigid to really be the topological analogue of polyhedra and smoothly stratified sets, whereas the stable uniqueness foreshadows the uniqueness up to controlled homeomorphism of fibre germs of manifold approximate fibrations [HTW1].

The main applications obtained by Siebenmann, namely local contractibility of homeomorphism groups, isotopy extension theorems, and the fact that many proper submersions are bundle projections, can all be generalized to the setting of homotopically stratified sets discussed below.

Siebenmann himself experimented with a less rigid class of stratified spaces, called *locally weakly conelike*. In order to include in this class stratified spaces with isolated singularities which arise as the one-point compactifications of manifolds with nonvanishing Siebenmann obstruction [S], he no longer required the existence of links. However, neighborhoods around strata of dimension n were still required to split off a factor of $\mathbb{R}^n$ locally. In other words, in a locally conelike set X a point in the n-dimensional stratum X_n has a neighborhood U in X with $U \setminus X_n$ homeomorphic to $L \times \mathbb{R}^{n+1}$ with L the compact link. In a weakly conelike set, $U \setminus X_n$ is homeomorphic to $C \times \mathbb{R}^n$ with C a noncompact space with a certain tameness property at infinity. While this generalization was a move in the right direction, the role of the weak link C prevented further developments and it was left to Quinn to make a bolder generalization.

1.5 Quinn stratifications: homotopy mapping cylinders. Quinn [Q5] introduced a class of spaces which we will call *manifold homotopically stratified sets*. His objective was to "give a setting for the study of purely topological stratified phenomena, particularly group actions on manifolds." As has been pointed out above, the previously defined topologically stratified spaces were inadequate. On one hand, the geometrically stratified spaces (that is, the topological version of Thom's spaces as formulated by Mather or Browder and Quinn) require too strong of a condition on neighborhoods of strata (namely, mapping cylinder neighborhoods) ruling out important examples (like locally flat submanifolds and orbit spaces of locally linear group actions). On the other hand, the locally conelike stratified sets of Siebenmann require a very strong local condition which need not propagate to the entire neighborhood of the strata. Without an understanding of the geometry of neighborhoods of strata, topological stratified versions of surgery, transversality, and h-cobordism theorems were missing.

Quinn formulated his definition to be equivalent to saying that for $j < i$, $X_i \cup X_j$ is homotopy equivalent near X_j to the mapping cylinder of some fibration over X_j. This has two pleasant properties. First, besides the geometric condition that the strata be manifolds, the definition is giving in homotopy theoretic terms. Second, the condition concerns neighborhoods of strata rather than closed strata, so that, in particular, there are no complicated compatibility conditions where more than two strata meet. The links are now defined only up to homotopy.

Even without a geometric condition on neighborhoods of strata, Quinn was able to derive geometric results. These will be discussed in §3 below along with a theorem of Hughes, Taylor, Weinberger and Williams which says that neighborhoods of strata do carry a weak geometric structure. One thing that Quinn did not do was to develop a surgery theory for these manifold homotopically stratified sets. That piece of the picture was completed by Weinberger [W4] (see §5 below).

1.6 Group actions. Suppose that G is a finite group acting on a topological manifold M. One attempts to study the action by studying the orbit space M/G and the map $M \to M/G$. For example, if G acts freely, then M/G is a manifold and $M \to M/G$ is a covering projection. Moreover, the surgery theoretic set of equivariant manifold structures on M is in 1-1 correspondence with the set of manifold structures on M/G via the pull-back construction.

When the action is not free, M/G must be viewed as a space with singularities and $M \to M/G$ as a collection of covering projections. The prototypical example occurs when M is a closed Riemann surface and G is a finite cyclic group acting analytically. Then $M \to M/G$ is a *branched covering*.

More generally, if M is a smooth manifold and G acts differentiably, then M has a Whitney stratification with the strata $M_{(H)}$ indexed by conjugacy classes of subgroups of G and consisting of all points with isotropy subgroup conjugate to H. This induces a Whitney stratification of M/G. The standard reference is Lellmann [Le], but Dovermann and Schultz [DS] provide a more accessible proof. In the more general setting of a compact Lie group G, Davis [Dv1] showed how to view $M \to M/G$ as a collection of fibre bundle projections based on the fact that each $M_{(H)} \to M_{(H)}/G$ is a smooth fibre bundle projection with fibre G/H.

Now if the action of the finite group G on the topological manifold M is locally linear (also called locally smooth), then the examples of Quinn and Steinberger and West show (as mentioned above) that M/G need not have a geometric stratification, but it is a locally conelike TOP stratified set, and so Siebenmann's results can be applied. Lashof and Rothenberg [LR] used stratification theory of the orbit space to classify equivariant smoothings of locally smooth G-manifolds. Hsiang and Pardon [HsP] and Madsen and Rothenberg [MR] used stratifications for the classification of linear representations up to homeomorphism (see also [CSSW], [CSSWW], [HP]). Stratifications also played an important role in the work of Steinberger and West [StW] on equivariant s-cobordism theorems and equivariant finiteness obstructions.

The stratification theory of the orbit space actually corresponds with the isovariant, rather than the equivariant, theory of the manifold.

Locally linear actions on topological manifolds have the property that fixed sets are locally flat submanifolds. It is natural to consider all such actions. These are essentially the actions whose orbit space is a manifold homotopically stratified set. After being introduced by Quinn [Q5], Yan [Y] applied Weinberger's stratified surgery (see §5 below) to study equivariant periodicity. More recently, Beshears [Bs] made precise the properties of the map $M \to M/G$ and proved that the isovariant structures on M are in 1-1 correspondence with the stratum preserving structures on M/G.

1.7 Mapping cylinders. Mapping cylinders provide examples of spaces with singularities. The mapping cylinder $\mathrm{cyl}(p)$ of a map $p : M \to N$ between manifolds has a natural stratification with three strata: the top M, the bottom N and the space in between $M \times (0,1)$. The properties of the stratification depend on the map p. With this stratification $\mathrm{cyl}(p)$ is geometrically stratified if and only if $p \times \mathrm{id}_{\mathbb{R}}$ can be approximated arbitrarily closely by fibre fibre bundle projections. On the other hand, $\mathrm{cyl}(p)$ is a manifold homotopically stratified set if and only if p is a manifold approximate fibration. The cylinder is nonsingular; i.e., a manifold with N a locally flat submanifold if and only if p is a manifold approximate fibration with spherical homotopy fibre. (Here and elsewhere in this section, we ignore problems with low dimensional strata.)

More generally, one can consider the mapping cylinder of a map $p : X \to Y$ between stratified spaces which take each stratum of X into a stratum of Y. The natural collection of strata of $\mathrm{cyl}(p)$ contains the strata of X and Y. Cappell and Shaneson [ChS4] observed that even if one considers maps between smoothly stratified spaces which are smooth submersions over each stratum of X, then $\mathrm{cyl}(p)$ need not be smoothly stratified (they refer to an example of Thom [T1]). However, Cappell and Shaneson [CS5] proved that such cylinders are manifold homotopically stratified sets, showing that the stratifications of Quinn arise naturally in the theory of smoothly stratified spaces.

Even more generally, the mapping cylinder $\mathrm{cyl}(p)$ of a stratum preserving map between manifold homotopically stratified sets is itself a manifold homotopically stratified set (with the natural stratification) if and only if p is a manifold stratified approximate fibration [H2].

2. Intersection Homology and Surgery Theory

In the mid 70's Cheeger and Goresky-MacPherson, independently and by entirely different methods, discovered that there is a much larger class of spaces than manifolds that can be assigned a sequence of "homology groups" that satisfy Poincaré duality. Given the central role that Poincaré duality plays in surgery theory, it was inevitable that this would lead to a new environment for the applications of surgery.

2.1. Let X be a stratified space where $X^i \setminus X^{i-1}$ is an i-dimensional F-homology manifold, for a field F. We shall assume that the codimension one stratum is of codimension at least two and that $X \setminus X^{n-1}$ (the nonsingular part) is given an F-orientation; for simplicity we will also mainly be concerned with the case of $F = \mathbb{Q}$. It pays to think PL, as we shall, but see [Q3] for an extension to homotopically stratified sets. A *perversity* p is a nondecreasing function from the natural numbers to the nonnegative integers, with $p(1) = p(2) = 0$, and for each i, $p(i+1) \leq p(i) + 1$. The *zero perversity* is the identically 0 function and the *total perversity* t has $t(i) = i - 2$ for $i \geq 2$. Two perversities, p and q are *dual* if $p + q = t$.

We say that X is *normal* if the link of any simplex of codimension larger than 1 is connected. This terminology is borrowed from algebraic geometry. It is not hard to "normalize" "abnormal" spaces by an analogue of the construction of the orientation two-fold cover of a manifold.

A *chain* is just what it always was in singular homology: we say it is *p-transverse*, or *p-allowable*, if for every simplex in the chain $\Delta \cap X^{n-i}$ has dimension at most i larger than what would be predicted by transversality and the same is true for the simplices in its boundary that have nonzero coefficient.

Note: It is not always the case that the chain complex of p-transverse chains with coefficients in $\mathbb{R}$ is just the complex for $\mathbb{Z}$ tensored with $\mathbb{R}$, as it would be in ordinary homology, because a non-allowable chain can become allowable after tensoring when some simplex in the boundary gets a 0-coefficient.

2.2. $IH_p(X)$ is the homology obtained by considering p-allowable chains. It is classical for normal spaces that IH^t is just ordinary homology; a bit more amusing is the theorem of McCrory that IH^0 is cohomology in the dual dimension. The forgetful map is capping with the fundamental class.

Note that IH is **not** set up to be a functor. It turns out to be functorial with respect to *normally nonsingular* or (*homotopy*) *transverse* maps. (We'll discuss these in a great deal more details in §§4,5.) Thus, it **is** functorial with respect to (PL) homeomorphisms and inclusions of open subsets and collared boundaries.

Note also, that one can give "cellular" versions of IH, which means that one can define perverse finiteness obstructions and Reidemeister and Whitehead torsions in suitable circumstances. (See [Cu, Dr].) Here one would usually want to build in refinements to integer coefficients that we will not discuss till 2.10 below.

2.3. The main theorems of [GM1] are that (1) IH is stratification independent (indeed it is a topological invariant, even a stratified homotopy invariant) and (2) for dual perversities the groups in dual dimensions are dual. The latter boils down to Poincaré duality in case X is a manifold, however it is much more general.

2.4. What is important in many applications is that one can often get a self duality. Unfortunately, there is no self dual perversity function (what should $p(3)$ equal?). However, we have two *middle perversities* $0, 0, 1, 1, 2, 2, \dots$ and $0, 0, 0, 1, 1, 2 \dots$; note that these differ only on the condition of intersections with odd codimensional strata. Consequently, for spaces with only even codimensional strata, the middle intersection homology groups are self dual.

2.5. It turns out that the middle perversity groups have many other amazing properties. Cheeger independently discovered the "De Rham" version of these. He gave a polyhedral X as above a piecewise flat metric (i.e. flat on the simplices, and conelike) and observed that the L^2 cohomology of the incomplete manifold obtained by removing the singular set was very nice. Under a condition that easily holds when one has even codimensional strata, the $*$ operator takes L^2 forms to themselves, and one formally obtains Poincaré duality. A consequence of this is that the **Künneth formula holds**.

In addition, Goresky and MacPherson [GM3] proved that Morse theory takes a very nice form for stratified spaces when you use intersection homology. This leads to a proof of the Lefschetz hyperplane section theorem. (A sheaf theoretic proof appears in [GM2].) [BBD] proved hard Lefschetz for the middle perversity intersection homology of a singular variety using the methods of characteristic p algebraic geometry. This requires the sheaf theoretic reformulation to be discussed below. Finally Saito [Sa] gave an analytic proof of this and a Hodge decomposition for these groups.

2.6. Let us return to pure topology by way of example. Consider a manifold with boundary $W, \partial W$, and the singular space obtained by attaching a cone to ∂W. Normality would correspond to the assumption that ∂W is connected.

What are the intersection homology groups in this case? Fix p. We would ordinarily not expect any chain of dimension less than n to go through the cone point. Once $i + p(i)$ is at least n, we begin allowing all chains to now go through the cone point, so one gets above that dimension all of the reduced homology. Below that dimension, we are insisting that our chains miss the cone point, so one gets $H_*(W)$. There is just one critical dimension where the chain can go through and the boundary cannot: here one gets the image of the ordinary homology in the reduced homology.

Using these calculations, one can reduce the Goresky-MacPherson duality theorem to Poincaré-Lefschetz duality for the manifold with boundary.

If the dimension of W is even, one gets in the middle dimension (for the middle perversity) the usual intersection pairing on $(W, \partial W)$ modulo its torsion elements.

Note though that if W is odd dimensional the failure of self duality is caused by the middle dimensional homology of ∂W. If its homology vanished, we'd still get Poincaré duality.

2.7. Of course, one immediately realizes that one can now define signatures for spaces with even codimensional singularities (that lie in the Witt group $W(F)$ of the ground field.) We'll, for now, only pay attention to $F = \mathbb{R}$ and ordinary signature.

Thom and Milnor's work on PL L-classes and Sullivan's work on $KO[\frac{1}{2}]$ orientations for PL manifolds all just depend on a cobordism invariant notion of signature that is multiplicative with respect to products with closed smooth manifolds. Thus, as observed in [GM I] it is possible to define such invariants lying in ordinary homology and $KO[\frac{1}{2}]$ of any space with even codimensional strata.

2.8. It is very natural to sheafify. Nothing prevents us from considering the intersection homology of open subsets and one sees that for each open set one has duality between locally finite homology and cohomology. It turns out that the usual algebraic apparatus of surgery theory mainly requires self dual sheaves rather than manifolds. So we can define symmetric signatures that take the fundamental group into account, which are just the assemblies (in the sense of assembly maps) of the classes in 2.7.

2.9. The original motivations to sheafify were rather different. Firstly, using sheaf theory there are simple Eilenberg-Steenrod type axioms that can be used to characterize IH; these are useful for calculational purposes and for things like identifying the Cheeger description with the geometric definition of Goresky and MacPherson.

Secondly, using various constructions in the derived category of sheaves, push forwards and proper push forwards and truncations of various sorts, it is possible to give a direct abstract definition of IH without using chains. This definition is appropriate to characteristic p algebraic varieties.

Finally, there is a very simple sheaf theoretic statement, Verdier duality, that can be used to express locally the self duality of the intersection homology of all open subsets of a given X. It says that IC^m is a self-dual sheaf for spaces with even codimensional singularities. We will see below that this is quite a powerful statement.

2.10. We can ask for which spaces is IC self dual? We know that all spaces with even codimensional strata have this property, but they are not all of them, for we saw in 2.6 that if we have an isolated point of odd codimension one still gets Poincaré duality in middle perversity IH if (and only if) the middle dimensional homology of the link – which is a manifold – vanishes. One can generalize this observation to see that if the link of each simplex of odd codimension in X has vanishing middle IH^m, then IC is self dual on X. (Indeed this condition is necessary and sufficient.) Such X's were christened by Siegel [Si], *Witt spaces*. Actually they were introduced somewhat earlier by Cheeger as being the set of spaces for which the $*$ operator on L^2 forms on the nonsingular part behaves properly.

The main point of Siegel's thesis, though, was to compute the bordism of Witt spaces. Obviously the odd dimensional bordism groups vanish, because the cone on an odd dimensional Witt space is a Witt nullcobordism. For even dimensional Witt spaces this only works if there is no middle dimensional IH^m. By a surgery process on middle dimensional cycles, he shows that you can cobord a Witt space to one of those if and only if the quadratic form in middle $IH^m(\ ;\mathbb{Q})$ is hyperbolic – so there is no obstruction in 2 mod 4, but there's an obstruction in $W(\mathbb{Q})$ in 0 mod 4. Moreover, aside from dimension 0, where all that can arise is $\mathbb{Z} \subseteq W(\mathbb{Q})$ given by signature, in all other multiples of 4 all the other (exponent 4 torsion, computed in [MH]) elements can be explicitly constructed, essentially by plumbing. The isomorphism of the bordism with $W(\mathbb{Q})$ is what gives these spaces their name.

However, making use of the natural transformations discussed above, we actually see that Witt spaces form a nice cycle theory for the (connective) spectrum $L(\mathbb{Q})$ if we ignore dimension 0. Siegel phrases it by inverting 2:

Theorem. *Witt spaces form a cycle theory for connective $KO \otimes \mathbb{Z}[1/2]$, i.e.*

$$\Omega^{\mathrm{Witt}}(X) \otimes \mathbb{Z}[1/2] \to KO(X) \otimes \mathbb{Z}[1/2]$$

is an isomorphism.

Pardon, [P] building on earlier work of Goresky and Siegel, [GS], showed that the spaces with **integrally self dual** IC form a class of spaces (which does not include all spaces with even codimensional strata: one needs an extra condition on the torsion of the one off the middle dimension IH group) whose cobordism groups agree with $L^*(\mathbb{Z})$ and then give a cycle theoretic description for the connective version of this spectrum.

Other interesting bordism calculations for classes of singular spaces can be found in [GP].

2.11 (Some remarks of Siegel). The fact that one has a bordism invariant signature for Witt spaces contains useful facts about signature for manifolds. For instance, using the identification of signature for manifolds with boundary with the intersection signature of the associated singular space with an isolated singular point, it is easy to write down a Witt cobordism (the pinch cobordism) which proves Novikov's additivity theorem [AS].

Also, the mapping cylinder of a fiber bundle is not always a Witt cobordism: there is a condition on the middle homology of the fiber. One could have thought that one can still define signature for singular spaces where the links have signature zero (obviously one can't introduce a link type with nonzero signature and have a cobordism invariant signature). However, Atiyah's example of nonmultiplicativity of signature gives a counterexample to this [A]. It is thus quite interesting that having no middle homology **is** enough for doing this.

2.12. Siegel's theorem has had several interesting applications. The first is a purely topological disproof of the integral version of the Hodge conjecture (already disproven by analytic methods in [AH]) on the realization of all (p, p) homology classes of a nonsingular variety by algebraic cycles. If one were looking for nonsingular cycles, then one can use failure of Steenrod representability, or better, Steenrod representability by unitary bordism!, but here we allow singular cycles. Thanks to Hironaka, we could apply resolution of singularities to make the argument work anyway. However, even without resolution one sees that these homology classes have a refinement to K-homology, which is a nontrivial homotopical condition (as in [AH] which develops explicit counterexamples along these lines).

Another application stems from the fact that the bordism theory is homology at the prime 2. Since one can define a signature operator for Witt spaces which is bordism invariant [PRW], one can view the signature operator from the point of view of Witt bordism and thus obtain a refinement at the prime 2 of the K-homology class of the signature operator to ordinary homology [RW]. This, then implies that the K-homology class of the signature operator is a homotopy invariant for, say, $\mathbb{R}P^n$.

Yet other applications concern "eta type invariants". The basic idea for these applications is that if one knows the Novikov conjecture for a group π, then by Siegel's theorem $\Omega^{\text{Witt}}(B\pi) \to L(\mathbb{Q}\pi)$ rationally injects. This means that one can get geometric coboundaries from homotopical hypotheses. Thus, for instance, homotopy equivalent manifolds should be rationally Witt cobordant (preserving their fundamental group).

In particular, then, the invariant of Atiyah-Patodi-Singer [APS] associated to an an odd dimensional manifold with a unitary representation of its fundamental group can only differ, for homotopy equivalent manifolds, that a twisted signature of the cobounding Witt space, e.g. a rational number. In [W3], known results regarding the Novikov conjecture and the deformation results of [FL] are used to prove this unconditionally.

A similar application is made in [W6] to define "higher rho invariants" for various classes of manifolds. For instance, say that a manifold is antisimple if its chain complex is chain equivalent to one with 0 in its middle dimension (this can be detected homologically). Then its symmetric signature vanishes and, therefore, assuming the Novikov conjecture, it is Witt nullcobordant. By gluing together the Witt nullcobordism and the algebraic nullcobordism one obtains a closed object one dimension higher, whose symmetric signature (modulo suitable indeterminacies) is an interesting invariant of such manifolds. It can be used to show that the homeomorphism problem is undecidable even for manifolds which are given with homotopy equivalences to each other [NW].

2.13 (Dedicated to the Cheshire cat). Associated to any Witt space one has a self dual sheaf, namely IC^m. Actually, the cobordism group of self dual sheaves over a space X (assuming the self duality is symmetric) can be identified with $H_*(X; L^*(\mathbb{Q}))$, (see [CSW] for a sketch, and [Ht] for a more general statement including some more general rings[1]).

This statement has some immediate implications: Since IC^m is topologically invariant, all of the characteristic classes introduced for singular spaces in this way are topologically invariant. (This is basically the topological invariance of rational Pontrjagin classes extended to Witt spaces.)

We thus have, away from 2, three rather different descriptions of K-homology: Witt space bordism, homotopy classes of abstract elliptic operators (see [BDF, K]), and bordism of self dual sheaves (and, not so different from this one: controlled surgery obstruction groups).

A number of applications to equivariant and stratified surgery come from these identifications (and generalizations of them). We will return to some of these in §6.

2.14. A very nice application of cobordism of the self dual sheaves associated to IH and its various pushforwards is given in [CS2]. The goal is to extend the usual multiplicativity of signature in fiber bundles (with no monodromy) to stratified maps. We will not give a precise definition of a stratified map, but it is the intuitive notion, e.g. a fiber bundle has just one stratum.

Theorem. *Let $f : X \to Y$ be a stratified map between spaces with even codimensional strata, and suppose that all the strata of f are of even codimension and the pure strata are simply connected. We then have*

$$f_*(L(X)) = \sum \operatorname{sign}(c(\operatorname{star}_f(V)))L(V)\,,$$

[1] In general there are algebraic K-theoretic difficulties with identifying the Witt group self dual sheaves, at 2, with a homology theory. However, as Hutt himself was aware, one can certainly include many more rings than included in that paper.

where V runs over the strata of f (which is a substratification of Y). Here $c(\cdot)$ stands for a compactification – in this case it means the following. If $V = Y$ it is just the generic fiber. If V is a proper stratum, then one can consider $f^{-1}(\mathrm{cone}(L))$, where L is the link of a generic top simplex of V, and then one-point compactify it (i.e, cone off its boundary).

One can deal with nonsimply connected open strata by putting in a correction term for the monodromy action of $\pi_1(\mathrm{int}(V))$ on $IH(c(\mathrm{star}_f(V)))$.

The proof of the theorem in [CS2] is very pretty; it makes use of the machinery on perverse sheaves found in [BBD] but in explicit cases essentially produces an explicit cobordism between $f_*IC(X)$ and an explicit sum of other intersection sheaves: one for each stratum of the map.

Remarks.

(1) To get a feeling for the theorem it is worth considering a few special cases. Firstly, the case of a fiber bundle just reduces to [CHS]. As a second special case, if one considers a pinch map from a union of two manifolds along their common boundary, the formula boils down to Novikov additivity, and the cobordism implicit in the proof is the pinch cobordism of 2.11. As a final example, one can consider the case of a circle action on a manifold. Aside from some fundamental group points, there is a similar cobordism between M and some projective space bundles over the fixed set components of the circle action, and the formula of the theorem generalizes by considering projection to the quotient – with some slight modifications for 0 mod 4 components of fixed set, which lead to non-Witt singularities – (or specializes to) the formula in [W2] that identifies the higher signature of a manifold and that of its fixed point set of any circle action with nonempty fixed point set. The cobordism (discussed in both [W2] and [CS2]) is then the bubble quotient cobordism of [CW3].

(2) In the case of an **algebraic map**, one could directly apply [BBD] which gives a beautiful and deep decomposition theorem for f_*IC (X) and the general machinery on self dual sheaves to prove the existence of a formula like the one in the theorem. However, it is not so clear what the coefficients are.

(3) Still in the algebraic case, it is important to realize that there are many additional characteristic classes that can be defined for singular varieties beyond just the L-classes, for instance, MacPherson Chern classes and Todd classes. In [CS3], there are announced generalizations of the basic formula where the meaning of c is different: one must use projective completion to get a variety, and then the formula must be rewritten to account for the extra topology (think

about the case of intersection Euler characteristic classes, which can be dealt with by the proof of the usual Hurwitz formula for Euler characteristic of branched cover, together with the sheaf version of intersection Künneth). To prove such formulae one uses deformation to the normal cone (see [Fu2]) to replace the cobordism theory.

2.15. It is worth mentioning but beyond the purview of this survey to describe in any detail the applications of 2.14 given in [CS3, CS4, Sh2] to lattice point problems. The connection goes via the theory of toric varieties for which there are several excellent surveys [Od, Da, Fu1], which gives an assignment of a (perhaps singular) variety to every convex polygon with lattice point vertices on which a complex torus acts. (See also [Gu] for a discussion of the purely symplectic aspects of this situation.) Problems like counting numbers of lattice points inside such a polytope (= computation of the Erhart polynomial) and Euler – Maclaurin summation formulae can be reduced to calculations of the Todd class, which are studied in tandem with L-classes using the projection formulae. These, in turn, have substantial number theoretic implications.

3. The geometry of homotopically stratified spaces

One of the strengths of Quinn's formulation of manifold homotopically stratified spaces is that the defining conditions are homotopic theoretical (except, obviously, the geometric condition that the strata be manifolds). This, of course, makes it easier to verify the conditions, but harder to establish geometric facts about manifold homotopically stratified spaces. Nevertheless, Quinn was able to prove two important geometric results: homogeneity and stratified h-cobordism theorems.

Quinn's homogeneity result says that if x, y are two points in the same component of a stratum (with adjacent strata of dimension at least 5) of a manifold homotopically stratified space X, then there is a self-homeomorphism (in fact, isotopy) of X carrying x to y. Quinn obtains this as a consequence of an stratified isotopy extension theorem (an isotopy on a closed union of strata can be extended to a stratum preserving isotopy on the whole space). In turn, Quinn proves the isotopy extension theorem by using the full force of his work on "Ends of maps" (see [Q2,IV]).

As an example of the usefulness of the homogeneity result, consider a finite group acting on a manifold M. Even though the action need not be locally linear, the quotient M/G is often a manifold homotopically stratified space. Thus, the homogeneity result can be used to verify local linearity by establishing local linearity at a single point of each stratum component. Quinn first used this technique to construct locally linear actions whose fixed point set does not have an equivariant mapping cylinder neighborhood

[Q2,II]. Weinberger [W1] and Buchdahl, Kwasik and Schultz [BKS] have also used this result to verify that certain actions were locally linear.

It turns out that there is an alternative way to prove Quinn's homogeneity theorem which is based on engulfing (the classical way that homotopy information is converted into homeomorphism information in manifolds). In fact, this alternative method uses a description of neighborhoods of strata together with MAF (manifold approximate fibration) technology, and is useful for many other geometric results.

We have seen in §1 that in certain formulations of conditions on a stratification $\Sigma = \{X_i\}$ of a space X one considers tubular maps

$$\tau_i : U_i \to X_i \times [0, +\infty)$$

where U_i is a neighborhood of X_i and τ_i restricts to the identity $U_i \setminus X_i \to X_i \times (0, +\infty)$. For Whitney stratifications, the tubular maps are submersions on each stratum and fibre bundles over $X \times (0, +\infty)$. Since strata of manifold homotopically stratified spaces need not have mapping cylinder neighborhoods, such a result is too much to hope for in general. However, there is the following result which was proved by Hughes, Taylor, Weinberger and Williams [HTWW] in the case of two strata and by Hughes [H3] in general.

Theorem. *For manifold homotopically stratified spaces in which all strata have dimension greater than or equal to* 4*, tubular maps exist which are manifold stratified approximate fibrations.*

The neighborhoods of the strata which are the domains of these MSAF (manifold stratified approximate fibration) tubular maps are called *teardrop neighborhoods*. They are effective substitutes for mapping cylinder neighborhoods, and the result should be thought of as a *tubular neighborhood theorem* for stratified spaces. The point is that even though Quinn's definition does not postulate neighborhoods with any kind of reasonable tubular maps, one is able to derive their existence. The situation is optimal: minimal conditions in the definition with much stronger conditions as a consequence. This makes the surgery theory conceptually easier than for geometrically stratified sets for which the geometric neighborhood structure must be part of the data.

As mentioned above, these teardrop neighborhoods can be used to give a different proof of Quinn's isotopy extension theorem. Manifold approximate fibrations have the approximate isotopy covering property [H1]. This property holds in the stratified setting and is used inductively to extend isotopies from strata to their teardrop neighborhoods. In fact, parametric isotopy extension is now possible whereas Quinn's methods only work for a single isotopy.

Similarly, other results in geometric topology can be extended to manifold homotopically stratified sets by using MAF technology. For example, the homeomorphism group of a manifold homotopically stratified set is locally contractible, and a stratified version of the Chapman and Ferry [ChF] α-approximation theorem holds. In short, the case can be made that manifold homotopically stratified sets are the correct topological version of polyhedra and Thom's stratified sets.

4. Browder-Quinn theory

In [BQ], Browder and Quinn introduced an interesting, elegant, and useful general classification theory for strongly stratified spaces. The setting is a category where one has a fixed choice of strong stratification as part of the data one is interested in.

4.1. Let X be a strongly stratified space (e.g. a geometrically stratified space as in §1.3) with *closed pure strata* $\overline{X}^i$ (see §1.3). An h-cobordism with boundary X is a stratified space Z with boundary $X \cup X'$ where the inclusions of X and X' are stratified homotopy equivalences, and the neighborhood data for the strata of Z are the pullbacks with respect to the retractions of the data for X (and of X'). This condition is automatic in the PL and Diff categories when one is dealing with something like the quotient of a group action stratified by orbit types.

Theorem. *The h-cobordisms with boundary X (ignoring low dimensional strata) are in a 1–1 correspondence with a group* $\mathrm{Wh}^{BQ}(X)$. *There is an isomorphism* $\mathrm{Wh}^{BQ}(X) \cong \bigoplus \mathrm{Wh}(\overline{X}^i)$.

The map $\mathrm{Wh}^{BQ}(X) \to \bigoplus \mathrm{Wh}(\overline{X}^i)$ is given by sending

$$(Z, X) \to (\tau(\overline{Z}^i, \overline{X}^i)).$$

One proves the isomorphism (and theorem) inductively, using the classical h-cobordism theorem to begin the induction, and using the strong stratifications to pull up product structures to deal with one more stratum.

4.2. The surgery theory of Browder and Quinn deals with the problem of turning a degree one normal map into a stratified homotopy equivalence which is **transverse**, i.e. one for which the strong stratification data in the domain is the pullback of the data from the range.

This transversality is, in practice, the fly in the ointment. When one is interested in classifying embeddings or group actions usually the bundle data is something one is interested in understanding rather than *a priori* assuming. Still, in some problems (e.g. those mentioned in 6.3) one can sometimes prove that transversality is automatic. Also, of course, if one uses the machinery to construct examples, it is certainly fine if one produces examples that have extra restrictions on the bundle data.

4.3. Either by induction or by mimicking the usual identification of normal invariants, one can prove that $NI^{BQ}(X) \cong [X; F/Cat]$.

4.4. Define $S^{BQ}(X)$ to be the strongly stratified spaces with a transverse stratified simple homotopy equivalence to X up to Cat-strongly stratified simple isomorphism (note this implicitly is keeping track of "framings"). Then, one has groups $L^{BQ}(X)$ and a long exact surgery sequence:

$$\cdots \to L^{BQ}(X \times I \text{ rel } \partial) \to S^{BQ}(X) \to [X; F/Cat] \to L^{BQ}(X).$$

4.5. Note that unlike the Whitehead theory $L^{BQ}(X)$ does not decompose into a sum of L-groups of the closed strata. Indeed, for a manifold with boundary S^{BQ} is just the usual structure set (existence and uniqueness of collars gives the strong stratification structures) and the L-group is the usual L-group of a manifold with boundary, i.e. is a relative L-group, not a sum of absolute groups.

However, there is a connection between the L-groups of the pure strata and $L^{BQ}(X)$. This exact sequence generalizes the exact sequence of a pair in usual L-theory, and expresses the fact that as a space $L^{BQ}(X)$ is the fiber of the composition

$$L(X_0) \to L(\partial\, \text{Neighborhood}(X_0)) \to L(\text{cl}(X \setminus X_0))$$

where the first map is a transfer and the second is an inclusion.

4.6. The proof of this theorem is by the method of chapter 9 of [Wa]: one need only verify the π-π theorem. This is done by induction.

5. Homotopically stratified theory

If one does not want to insist on the transversality condition required in the Browder-Quinn theory, or if one is only dealing with homotopically stratified spaces, it is necessary to proceed somewhat differently. For more complete explanations, see [W4], [W5]. We will only discuss the topological version. The PL version is simpler but slightly more complicated.

5.1 The h-cobordism theorem. That new phenomena would arise in any systematic study of Whitehead torsion on nonmanifolds was clear from the start. Milnor's original counterexamples to the Hauptvermutung for polyhedra were based on torsion considerations [M1]. Siebenmann gave examples of locally triangulable nontriangulable spaces – not at all due to

Kirby-Siebenmann considerations, but rather K_0. More pieces came forward in the work on Anderson-Hsiang [AnH1, AnH2] and then in [Q2], which showed that under appropriate K-theoretic hypotheses one can triangulate, and therefore apply the straightforward PL torsion theory.

Real impetus came from the theory of group actions. Cappell and Shaneson [CS1] gave the striking examples of equivariantly homeomorphic representation spaces, which laid down the gauntlet to the topological community at large to deal with the issue of equivariant classification. h-cobordism theorems suitable for the equivariant category were produced by Steinberger (building on joint work with West) [St] and by Quinn [Q4] in the generality of homotopically stratified spaces (although the theorem in that paper does not include realization of all torsions in an h-cobordism[2]).

The ultimate theorem asserts, as usual, that (ignoring low dimensional issues) h-cobordisms on a stratified space X are classified by an abelian group $\mathrm{Wh}^{\mathrm{top}}(X)$.

Theorem. $\mathrm{Wh}^{\mathrm{top}}(X) \cong \bigoplus \mathrm{Wh}^{\mathrm{top}}(X^i, X^{i-1})$, *and we have an exact sequence*

$$\cdots \to H_0(X^{i-1}; \mathbf{Wh}(\pi_1(\text{holink}))) \to \mathrm{Wh}(\pi_1(X^i \setminus X^{i-1})) \to$$
$$\mathrm{Wh}^{\mathrm{top}}(X^i, X^{i-1}) \to H_0(X^{i-1}; \mathbf{K}_0(\pi_1(\text{holink}))) \to K_0(\pi_1(X^i \setminus X^{i-1}))\,.$$

Boldface terms are spectra. This decomposition of $\mathrm{Wh}^{\mathrm{top}}$ into a direct sum does not respect the involution obtained by turning h-cobordisms upside down, which is a pleasant descendant of the analogous fact in the Browder-Quinn theory. It does not have an analogue in L-theory.

5.2 Stable classification. Ranicki (following a sketch using geometric Poincaré complexes in place of algebraic ones, by Quinn) has elegantly reformulated the usual Browder-Novikov-Sullivan-Wall surgery exact sequence in the topological manifold setting as the assertion that there is a fibration:

$$\mathbf{S}(M) \to \mathbf{H}_n(M; \mathbf{L}(e)) \to \mathbf{L}_n(\pi_1(M))$$

where $\mathbf{X}$ means a space (or better a spectrum) whose homotopy groups are those of the group valued functor ordinarily denoted by X. $S(M)$ is the *structure set* of M, which essentially[3] consists of the manifolds simple homotopy equivalent to M up to homeomorphism. The map $\mathbf{H}_n(M; \mathbf{L}(e)) \to$

[2] As pointed out in [HTWW], the teardrop neighborhood theorem can be used to complete the proof of realization.

[3] In actuality, for our purposes it is best to use the finite dimensional ANR homology manifolds, and the equivalence relation is s-cobordism. See Mio's paper [Mi] in Volume 1 for a discussion of the difference this makes. (It is at most a single $\mathbb{Z}$ if M is connected.)

$\mathbf{L}_n(\pi_1(M))$ is called the *assembly map* and can be defined purely algebraically. Geometrically it has several interpretations: most notably, as the map from normal invariants to surgery obstructions in the topological category, or as a forgetful map from some variant of controlled surgery to uncontrolled surgery.

Since the assembly map has a purely algebraic definition, one can ask whether it computes anything interesting if X is not a manifold? and alternatively, if X is just a stratified space, what is the analogue of this sequence?

Cappell and the second author gave an answer to the first question in [CW2] where it is shown (under some what more restrictive hypotheses) if X is a manifold with singularities, i.e. X contains a subset Σ whose complement is a manifold, and suppose further that Σ is 1-LCC embedded[4] in X, then $S^{\mathrm{alg}}(X) \cong S(X \text{ rel } \Sigma)$ where $S^{\mathrm{alg}}(X)$ denotes the fiber of the algebraically defined assembly map $H_*(X; L(e)) \to L(X)$ and $S(X \text{ rel } \Sigma)$ means

$$\{\varphi : X' \to X \mid \varphi \text{ is a stratified simple}^5 \text{ homotopy equivalence} \\ \text{with } \varphi|\Sigma(X') \text{ already a homeomorphism}\}.$$

The answer to the second question is a bit more complicated, and actually requires two fibrations in general. The first is a stable generalization of the surgery exact sequence:

$$S^{-\infty}(X \text{ rel } Y) \to \mathbf{H}_0(X; \mathbf{L}^{BQ-\infty}(\text{—}, \text{rel } \; Y)) \to \mathbf{L}^{BQ-\infty}(X \text{ rel } Y).$$

Here the superscripts $-\infty$ denote that we are using a stable version of structure theory: we will soon explain that it only differs from the usual thing at the prime 2, and the phenomenon is governed by algebraic K-theory. The coefficients of the homology is with respect to a cosheaf of L-spectra: to each open set U of X one assigns the spectrum $\mathbf{L}(U \text{ rel } U \cap Y \text{with compact support})$. The BQ superscripts are a slight generalization of the theory discussed in §4.

To complete the theory one needs a destabilization sequence. This is given by the following:

$$S(X) \to S^{-\infty}(X) \to \hat{H}(\mathbb{Z}_2; \mathrm{Wh}^{\mathrm{top}}(X)^{\leq 1})$$

Here $S(X)$ is the geometric structure set, and $S^{-\infty}(X)$ is the stabilized version, which differ by a Tate cohomology term. An analogous sequence

[4] This means that maps of 2-complexes into X can be deformed slightly to miss Σ.
[5] The material of 5.1 can be used to make sense of this.

developed for a quite similar purpose appears in [WW]. Indeed in [HTWW] the theory outlined in this subsection is deduced from the [WW] results using blocked surgery [Q1, BLR, CW2] and [HTW1,2] (the classification of manifold approximate fibrations) and the teardrop neighborhood theorem. On the other hand, there are different approaches to all this using controlled end and/or surgery theorems that are sketched in [W4], [W5].

6. Some applications of the stratified surgery exact sequence

In practice the application of the theory of the previous section requires additional input for the calculations to be either possible or comprehensible. See [CW4] for the application to topological group actions. The last 100 pages of [W4] also give more applications than we can hope to discuss here.

6.1. Probably the simplest interesting and illustrative example of the classification theorem is to locally flat topological embeddings. The first point is that this problem is susceptible to study by these methods: Every topological locally flat embedding gives a two stratum homotopically stratified space where the holink is a homotopy sphere *and conversely*. This last is essentially Quinn's characterization of local flatness in [Q2,I].

Things are very different in codimensions one and two from codimension three and higher. We will defer to 6.3 the low codimensional discussion and restrict our attention here to the last of these cases.

Lemma. *If (W, M) is a manifold pair with cod$(M) > 2$, then $L^{BQ}(W, M) \cong L(W) \times L(M)$ where the map is the forgetful map.*

The proof is straightforward. Note that the lemma implies the analogous statement holds at the level of cosheaves of spectra ($\cong$ being quasi-isomorphism). It is quite straightforward in this case to work out the Whitehead theory: there are no surprises. Thus, we obtain:

Corollary. *$S(W, M) \cong S(W) \times S(M)$.*

Note that the discussion makes perfect sense even if (W, M) is just a Poincaré pair (see [Wa]), and then the corollary boils down to the statement that isotopy classes of embeddings of one topological manifold in another (in codimension at least 3) are in a 1-1 correspondence with the Poincaré embeddings (see [Wa]).

(Actually, a bit more work enables one to prove the same thing for M an ANR homology manifold.)

6.2. Using the material from §2 we can also analyze, away from 2, $S(X)$ for a very interesting class of spaces that have even codimensional strata.

We continue to let $S^{\mathrm{alg}}(X)$ denote the fiber of the classical assembly map $H_*(X; L(e)) \to L(X)$. It is what the structure set of X would be if X were a manifold.

Theorem([CW2]**).** *If X is a space with even codimensional strata and all holinks of all strata in one another simply connected, then there is an isomorphism* $\otimes[1/2]$

$$S(X) \cong \bigoplus S^{\mathrm{alg}}(V)$$

where the sum is over closed strata.

The proof consists of building an isomorphism $L^{BQ}(X) \cong \bigoplus L(V)[1/2]$ for arbitrary X satisfying the hypotheses. It is obvious enough how to introduce $\mathbb{Q}$ coefficients into L^{BQ}. Ranicki [R] has shown that introducing coefficients does not change L at the odd primes, but with $\mathbb{Q}$-coefficients one can make forgetful maps to $L(V; \mathbb{Q})$ using the intersection chain complexes.

6.3. To give an example where things work out differently, we shall assume that the holinks are aspherical, and that the Borel conjecture holds for the fundamental groups of these holinks. (This example is a special case of the theory of "crigid holinks" in [W4].)

In this case there is nothing good that can be said about the global L^{BQ} term, in general. However, the assumptions are enough to imply that $H_*(X; L^{BQ}) \cong [X; L(e)]$. (See [W4], [BL] for a discussion.) In particular, for locally flat embeddings in codimension 1 and 2, one sees that the fiber of $S(W, M) \to S(W)$ only involves fundamental group data, not, say, the whole homology of the manifold and submanifolds. This, too, reflects phenomena already found in Wall's book [Wa].

Another amusing example is X = simplicial complex, stratified by its triangulation. Then one gets $L^{BQ}(X) \cong [X; L(e)]$.

There are other interesting examples that display similar phenomena that come up from toric varieties. The theory of multiaxial actions (see §2 and [Dv2]) is another situation where the local cosheaves tend to decompose into pieces that can be written in simple terms involving skyscraper $L(e)$-cosheaves. Not all holinks are crigid and consequently different phenomena appear: indeed signatures 0 and 1 alternate in the simply connected holinks, with quite interesting implications. As a simple exercise, one can see that while $S^{6n-1}/U(n)$ is contractible, its structure set[6] $S(S^{6n-1}/U(n)) \cong \mathbb{Z}_2$. Similarly, $S^{12n-1}/Sp(n)$ is contractible, but its structure set[7] is $\mathbb{Z}$.

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[6] Actually, the structure set is $\mathbb{Z} \times \mathbb{Z}_2$ with the extra $\mathbb{Z}$ corresponding to actions on nonresolvable homology manifolds that are homotopy spheres.

[7] Same caveat as above.

Remark. If all holinks are simply connected (as in the case of multiaxial actions of $U(n)$ and $Sp(n)$) one always has a spectral sequence computing $S(X)$ in terms of the $S^{\mathrm{alg}}(X_i)$. For instance, if there are just two strata $X \supset \Sigma$, there is an exact sequence:

$$\cdots \to S^{\mathrm{alg}}(\Sigma \times I) \to S^{\mathrm{alg}}(X) \to S(X) \to S^{\mathrm{alg}}(\Sigma) \to \dots$$

The sequence continues to the left in the most obvious way. On the right it continues via deloops of the algebraic structure spaces. The map $S^{\mathrm{alg}}(\Sigma \times I) \to S^{\mathrm{alg}}(X)$ depends on the symmetric signature of the holink (and on the monodromy of the holink fibration). The case where the simply connected holink is rigid is essentially that of manifolds with boundary. The normal invariant term here is $[X; L(e)]$, but thought of here as $H(X, \partial X; L(e))$. On the other hand, in 6.2 we gave an important case where this spectral sequence degenerates (at least away from the prime 2).

6.4. As our final example, let us work out in detail a case that is somewhat opposite to the one of the previous paragraph: X = the mapping cylinder of even a PL block bundle $V \to N$, with fiber F, where N is a sufficiently good aspherical manifold. (Sufficiently good is a function of the reader's knowledge. Even the circle is a case not devoid of interest.) We are interested in understanding what the general theory tells us about $S(X \text{ rel } V)$.

Firstly, there is the calculation of the Whitehead group. (Or even pseudoisotopy spaces ...). In this case, the sequence boils down to:

$$\begin{aligned} H_0(N; \mathbf{Wh}_1(F)) \to \mathrm{Wh}_1(V) \to \mathrm{Wh}^{\mathrm{top}}&(X \text{ rel } V) \\ &\to H_0(N; \mathbf{K}_0(F)) \to K_0(V)\,. \end{aligned}$$

In a totally ideal world, the assembly maps $H_0(N; \mathbf{Wh}_1(F)) \to \mathrm{Wh}_1(V)$ and $H_0(N; \mathbf{K}_0(F)) \to K_0(V)$ would be isomorphisms, and $\mathrm{Wh}^{\mathrm{top}}(X \text{ rel } V)$ would vanish. However, even in the case of $N = S^1$ where the assembly map (for the product bundle) was completely analyzed by [BHS], this is not true. In that case, there is an extra piece called *Nil* that obstructs this; however, *Nil* is a split summand. Thus, the assembly maps are still injections, and one obtains an isomorphism of $\mathrm{Wh}^{\mathrm{top}}(X \text{ rel } V)$ with a sum of *Nil*s. In general, the pattern discovered by Farrell and Jones [FJ] shows that the cokernel of these assembly maps is at least reasonably conjectured to be a “sum” of *Nil*s.

The splitting of the K-theory assembly map essentially boils down to the assertion that $\mathrm{Wh}^{BQ}(X \text{ rel } V) \to \mathrm{Wh}^{\mathrm{top}}(X \text{ rel } V)$ has a section. There are fairly direct proofs of this fact when N is a nonpositively curved Riemannian manifold in [FW] and in [HTW3]. The first approach notes that putting a PL structure on stratified spaces can be viewed (essentially following [AnH1, AnH2]) as a problem of reducing the tangent microbundle to the group of block bundle maps: but in the presence of curvature assumptions this can be done in the large by the methods of controlled topology.

The approach in [HTW3] depends on the same controlled topology, but its focus is showing that one can associate a MAF structure to any map whose homotopy fiber is finitely dominated. The teardrop neighborhood theorem of course provides the relation between these approaches.

The same analyses can be done for the (stable) structure set $S(X \text{ rel } V)$. In this case one does often have the vanishing of the analogue of *Nil* (although if there's orientation reversal or complicated monodromy in the bundle, this might not be the case). The structure set is here described as the fiber of the assembly map, and thus it often vanishes.

This has an interesting interpretation. Let us suppose that the fiber is K-flat, i.e. that $\mathrm{Wh}(\pi_1(F) \times \mathbb{Z}^k) = 0$ for all k to avoid any potential end obstructions. In this case one also knows that all MAF's are equivalent to block bundle projections.

The vanishing of $S(X \text{ rel } V)$ means that $S(X) \cong S(V)$ by the "obvious" fibration: $S(X \text{ rel } V) \to S(X) \to S(V)$. (We'll discuss the " " marks in a moment.) Now $S(X)$ is basically the same thing as the F-block bundles on N with fiber a manifold homotopy equivalent to F. Thus we have a generalized fibration theorem for manifolds with maps to N. (Indeed, the Farrell fibration theorem [Fa] is all that is necessary to feed into the machinery to get out the calculation of L-groups: that's the content of Shaneson's thesis [Sh1]!)

Without the K-flatness, we see that there are still only *Nil* obstructions to obtaining MAF structures (but genuine K-theory obstructions to getting block structures).

To return to the "obvious" fibration, a little thought shows that it is not at all obvious. What is obvious is that it is a fibration over the components of $S(V)$ in the image of the map $S(X) \to S(V)$. We are asserting, after the arguments given above, that this image is all the components, but prima facie, the argument in whole is circular.

However, that is not the case as a consequence of the complete general theory. The map $S(X) \to S(V)$ is actually an infinite loop map, isomorphic to its own 4^{th} loop space (see [CW1, We5]). Thus, the fact that we knew exactness at the π_i level for $i = 3, 4$ gives us everything we want for any such ad hoc component problem. (This is exactly the same point involved in continuing the exact sequence of 6.3 further to the right.)

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Metrics of positive scalar curvature and connections with surgery

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1 Introduction

This chapter discusses the connection between geometry of Riemannian metrics of positive scalar curvature and surgery theory. While this is quite a deep subject which has attracted quite a bit of recent attention, the most surprising aspect of this whole area remains the original discovery of Gromov-Lawson and of Schoen-Yau from about 20 years ago—namely, that there *is* a connection between positive scalar curvature metrics and surgery. The Surgery Theorem of Gromov-Lawson and Schoen-Yau remains the most important result in this subject. We discuss it and its variants at length in Section 3. Then in Section 4, we discuss the status of the so-called Gromov-Lawson Conjecture, which relates the existence of positive scalar curvature metrics to index theory and KO-homology. This is preparatory to Section 5, which explains the parallels between the classification of positive scalar curvature metrics and the classification of manifolds via Wall's surgery theory. In the final section, Section 6, we discuss a number of open problems.

All manifolds in this paper will be assumed to be smooth (C^∞). For simplicity, we restrict attention to compact manifolds, although there are also plenty of interesting questions about complete metrics of positive scalar curvature on non-compact manifolds. At some points in the discussion, however, it will be necessary to consider manifolds with boundary.

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2 Background and Preliminaries

One of the most important problems in global differential geometry is to study how curvature relates to topology, or to phrase things differently, to study what constraints topology places on curvature. This problem can be asked in several different contexts. When applied to vector bundles with a connection, it gives rise to Chern-Weil theory and the theory of characteristic classes. Here we will instead ask about the scalar curvature of a Riemannian manifold. The scalar curvature is the weakest curvature invariant one can attach (pointwise) to a Riemannian n-manifold. Its value at any point can be described in several different ways:

1. as the trace of the Ricci tensor, evaluated at that point.
2. as twice the sum of the sectional curvatures over all 2-planes $e_i \wedge e_j$, $i < j$, in the tangent space to the point, where $e_1, \ldots, e_n$ is an orthonormal basis.
3. up to a positive constant depending only on n, as the leading coefficient in an expansion telling how volumes of small geodesic balls differ from volumes of corresponding balls in Euclidean space. Positive scalar curvature means balls of radius r for small r have a smaller volume than balls of the same radius in Euclidean space; negative scalar curvature means they have larger volume.

In the special case $n = 2$, the scalar curvature is just twice the Gaussian curvature.

We can now state the basic problems we will consider in this paper:

Problems 2.1

1. If M^n is a closed n-manifold, when can M be given a Riemannian metric for which the scalar curvature function is everywhere strictly positive? (For simplicity, such a metric will henceforth be called a metric of positive scalar curvature.)
2. If M^n is a closed manifold which admits at least one Riemannian metric of positive scalar curvature, what is the topology of the space $\mathfrak{R}^+(M)$ of all such metrics on M? In particular, is this space connected?
3. If M^n is a compact manifold with boundary, when does M admit a Riemannian metric of positive scalar curvature which is a product metric on a collar neighborhood $\partial M \times [0, a]$ of the boundary? When this is the case, what is the topology of the space of all such metrics? Of the space of all such metrics extending a fixed metric in $\mathfrak{R}^+(\partial M)$?

A few comments on these problems are in order. With regard to question (1), the reader might well ask what is special about positivity. Why not ask about metrics of *negative* scalar curvature, or of vanishing scalar curvature, or of non-negative scalar curvature? More generally, we could ask which smooth functions on a manifold M are realized as the scalar curvature function of some metric on M. It is a remarkable result of Kazdan and Warner that the answer to this question *only* depends on which of the following classes the manifold M belongs to:

1. Closed manifolds admitting a Riemannian metric whose scalar curvature function is non-negative and not identically 0.

2. Closed manifolds admitting a Riemannian metric with non-negative scalar curvature, but not in class (1).

3. Closed manifolds not in classes (1) or (2).

All three classes are non-empty if $n \geq 2$. For example, it is easy to see from the Gauss-Bonnet-Dyck Theorem[1] that if $n = 2$, class (1) consists of S^2 and $\mathbb{RP}^2$; class (2) consists of T^2 and the Klein bottle; and class (3) consists of surfaces with negative Euler characteristic.

Theorem 2.2 (Trichotomy Theorem, [KW1], [KW2]) *Let M^n be a closed connected manifold of dimension $n \geq 3$.*

1. *If M belongs to class (1), every smooth function is realized as the scalar curvature function of some Riemannian metric on M.*

2. *If M belongs to class (2), then a function f is the scalar curvature of some metric if and only if either $f(x) < 0$ for some point $x \in M$, or else $f \equiv 0$. If the scalar curvature of some metric g vanishes identically, then g is Ricci flat.* (I.e., *not only does the scalar curvature vanish identically, but so does the Ricci tensor.*)

3. *If M belongs to class (3), then $f \in C^\infty(M)$ is the scalar curvature of some metric if and only if $f(x) < 0$ for some point $x \in M$.*

We note that the Theorem shows that deciding whether a manifold M belong to class (1) is equivalent to solving Problem 2.1.1. Futaki [Fu] has shown that – at least for simply connected manifolds – class (2) consists of very special manifolds admitting metrics with restricted holonomy groups.

As further justification for our concentrating on *positive* scalar curvature in Problem 2.1.2, one has the following fairly recent result:

[1] The point is that for any choice of Riemannian metric, the integral of the scalar curvature with respect to the measure defined by the metric is 4π times the Euler characteristic. Dyck's role in this is explained in the interesting article by D. Gottlieb, "All the way with Gauss-Bonnet and the sociology of mathematics," Amer. Math. Monthly **103** (1996), no. 6, 457–469.

Theorem 2.3 ([Loh]) *The space $\mathfrak{R}^-(M)$ of negative scalar curvature metrics on M is contractible, for any closed manifold M^n of dimension $n \geq 3$.*

Finally, one might ask the reason for the Riemannian product boundary condition in Problem 2.1.3. The first part of the answer comes from the fact that without a boundary condition, *any* manifold with non-empty boundary admits a metric of positive scalar curvature. (In fact, Gromov [Gr], Theorem 4.5.1, even showed it admits a metric of positive *sectional* curvature, a much stronger condition.) The second part of the answer is that there are other interesting boundary conditions one could impose that are relevant to the study of positive scalar curvature, such as positive mean curvature on the boundary (see [GL1], Theorem 5.7, for example), but we have tried to limit attention to the simplest such condition. Often one can reduce to this condition anyway—see [Gaj1], Theorem 5.

3 The Surgery Theorem and its Variants

The connection between positive scalar curvature metrics and surgery begins with:

Theorem 3.1 (Surgery Theorem, [GL2], Theorem A and [SY]) *Let N^n be a closed manifold, not necessarily connected, with a Riemannian metric of positive scalar curvature, and let M^n be obtained from N by a surgery of codimension $q \geq 3$. Then M can be given a metric of positive scalar curvature.*

Proof. We give the argument of Gromov-Lawson, just briefly sketching their initial reduction of the problem (which is explained well in their paper), but going over the crucial "bending argument" in detail. (The reason for this is that it appears there is a mistake in [GL2] on page 428—in the displayed formula on the middle of that page, there is a factor of $\sin^2\theta_0$ missing, and thus the argument at the bottom of page 428 doesn't work as stated.)

Suppose S^p is an embedded sphere in N of codimension $q = n - p \geq 3$, with trivial normal bundle. By using the exponential map on the normal bundle of S^p, we may assume that we have an embedding of $S^p \times D^q(\bar{r})$ into N for some $\bar{r} > 0$ (the radius of a "good tubular neighborhood of S^p") so that the sphere on which we will do surgery is $S^p \times \{0\}$, the radial coordinate r on $D^q(\bar{r})$ measures distances from $S^p \times \{0\}$, and such that curves of the form $\{y\} \times \ell$, where ℓ is a ray in $D^q(\bar{r})$ starting at the origin, are geodesics. However, we know nothing about the restriction of the metric on N to the sphere $S^p \times \{0\}$.

The key idea of the proof is to choose a suitable C^∞ curve γ (with endpoints) in the t-r plane, and to consider

$$T = \{(y,\, x,\, t) \in \big(S^p \times D^q(\bar{r})\big) \times \mathbb{R} : (t,\, r = \|x\|) \in \gamma\}$$

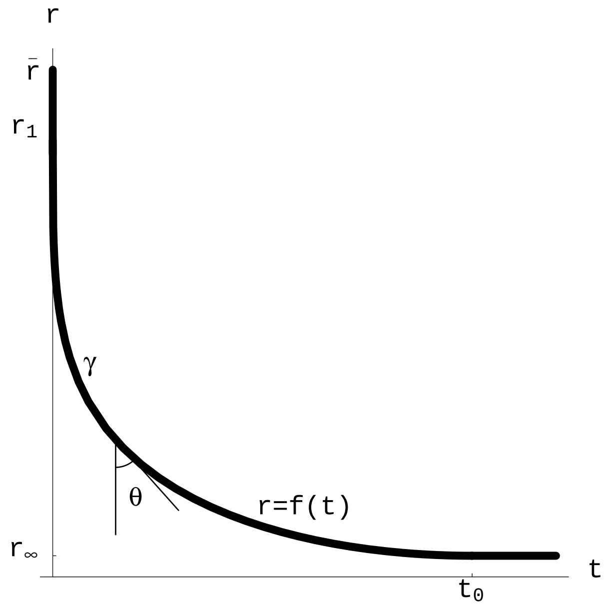
with the induced metric, where $\mathbb{R}$ is given the Euclidean metric and $\big(S^p \times D^q(\bar{r})\big) \times \mathbb{R}$ is given the metric of the Riemannian product $N \times \mathbb{R}$. We choose the curve γ to satisfy the following constraints:

1. γ lies in the region $0 < r \le \bar{r}$ of the t-r plane.

2. γ begins at one end with a vertical line segment $t = 0$, $r_1 \le r \le \bar{r}$. This guarantees that near one of the two components of ∂T, T is isometric to a portion of N.

3. γ ends with a horizontal line segment $r = r_\infty > 0$, with r_∞ very small. This guarantees that near the other component of ∂T, T is isometric to the Riemannian product of a line segment with $S^p \times S^{q-1}(r_\infty)$, where the metric on $S^p \times S^{q-1}(r_\infty)$ (not in general a product metric) is induced by the embedding $S^p \times S^{q-1}(r_\infty) \subset S^p \times D^q(\bar{r}) \subset N$.

4. In the region $r_\infty < r < r_1$, γ is the graph of a function $r = f(t)$ which is decreasing and (weakly) concave upward.



5. γ is chosen so that the scalar curvature of T is everywhere positive. This is the hard part. The Gauss curvature equation says that the sectional curvature of a hypersurface, evaluated on a plane spanned by two principal directions for the second fundamental form, is the corresponding sectional curvature of the ambient manifold, plus the product of the two principal curvatures. So, summing the sectional

curvatures over all the two-planes spanned by pairs of principal directions, one derives for small $r > 0$ the formula:

$$\kappa_T = \kappa_N + O(1)\sin^2\theta + (q-1)(q-2)\frac{\sin^2\theta}{r^2} - (q-1)\frac{k\sin\theta}{r} - O(r)(q-1)k\sin\theta, \tag{3.1}$$

where κ_T and κ_N are the scalar curvatures of T and N, respectively, where k is the curvature of γ (as a curve in the Euclidean plane), and where θ is the angle between γ and a vertical line. (See figure above.)

Assume for the moment that we have constructed γ as required. Since the metric on T is isometric to a portion of N in a collar of one component of ∂T, we can glue T onto $N \smallsetminus \big(S^p \times D^q(\bar{r})\big)$, getting a manifold N' of positive scalar curvature with a single boundary component $S^p \times S^{q-1}(r_\infty)$, and with a metric that is a product metric in a collar neighborhood of the boundary.

Since $q-1 \geq 2$ and r_∞ is very small, there is a homotopy of the metric on $S^p \times S^{q-1}(r_\infty)$ through metrics of positive scalar curvature to a Riemannian product of two standard spheres: $S^p(1)$ and $S^{q-1}(r_\infty)$. Even though $S^p(1)$ has zero curvature if $p \leq 1$, we have large positive scalar curvature since $S^{q-1}(r_\infty)$ has sectional curvature $r_\infty^{-2} \gg 0$. (See [GL2], Lemma 2.) This homotopy can be used to construct a metric of positive scalar curvature on a cylinder $S^p \times S^{q-1}(r_\infty) \times [0, a]$, which in a neighborhood of one boundary component matches the metric on a collar neighborhood of ∂T in T, and which in a neighborhood of the other boundary component is a Riemannian product of standard spheres $S^p(1)$ and $S^{q-1}(r_\infty)$ with an interval. (See Proposition 3.3 below.) We glue this cylinder onto N' to get N'', a manifold of positive scalar curvature with boundary $S^p(1) \times S^{q-1}(r_\infty)$, and with a product metric in a neighborhood of the boundary.

Finally, to finish off the proof, we glue onto N'' a Riemannian product $D^{p+1} \times S^{q-1}(r_\infty)$, where the disk D^{p+1} has not the flat metric but a metric which is a Riemannian product $S^p(1) \times [0, b]$ in a neighborhood of the boundary. (Such metrics on the disk are easy to write down.) The end product of the construction is a metric of positive scalar curvature on M.

We're still left with the most delicate step, which is construction of a curve γ with the properties listed on page 357 above. Obviously, there is no problem satisfying the first four conditions. To satisfy the last condition, we need to choose γ so that $\kappa_T > 0$ in equation (3.1). Since κ_N is bounded below by a positive constant, the constraint will be satisfied provided that

$$(1 + C'r^2)k \leq (q-2)\frac{\sin\theta}{r} + \kappa_0\frac{r}{\sin\theta} - Cr\sin\theta, \tag{3.2}$$

where $\kappa_0 > 0$ is $\frac{1}{q-1}$ times a lower bound for κ_N, and where the constants $C > 0$ and $C' > 0$ come from the $O(1)$ term and the $O(r)$ term in equation (3.1), respectively. (When $\theta = 0$, the right-hand side of inequality (3.2) is to be interpreted as $+\infty$.)

To satisfy this inequality, we begin by choosing

$$0 < \theta_0 < \arcsin\left(\sqrt{\frac{\kappa_0}{C}}\right).$$

Then for $0 \le \theta \le \theta_0$, the second term on the right in inequality (3.2) dominates the last term, and thus we can start at the point $(0, r_1)$ (where θ and k are required to vanish) and find a small "bump function" of compact support for k (as a function of arc length) satisfying (3.2), so that γ bends in a small region around to a line segment with small positive θ. Decreasing θ_0 if necessary, we may assume this "first bend" ends at $\theta = \theta_0$. (So far the details are just as in [GL2], except that we have made the estimates more explicit.)

Next, we choose r_0 with

$$0 < r_0 < \min\left(\sqrt{\frac{1}{4C}}, \sqrt{\frac{1}{2C'}}\right).$$

This insures (since $q - 2 \ge 1$) that for $r \le r_0$,

$$(q-2)\frac{\sin\theta}{r} - Cr\sin\theta \ge \frac{3\sin\theta}{4r}$$

and

$$1 + C'r^2 \le \frac{3}{2},$$

so that k can be as large as $\frac{2}{3} \cdot \frac{3\sin\theta}{4r} = \frac{\sin\theta}{2r}$. When γ crosses the line $r = r_0$, we start the "second bend" by quickly bringing k up to the allowed value of $\frac{\sin\theta}{2r}$ and thereafter following the solution of the differential equation $k = \frac{\sin\theta}{2r}$. If we write $r = f(t)$, then

$$\sin\theta = \frac{1}{\sqrt{1+(f')^2}}, \qquad k = \frac{f''}{(1+(f')^2)^{\frac{3}{2}}}.$$

So our differential equation can be rewritten

$$f'' = \frac{1+(f')^2}{2f}.$$

This equation can be solved explicitly; the solution is

$$f(t) = \frac{1}{C_1} + \frac{C_1}{4}(t - C_2)^2,$$

for constants C_1 and C_2. Suppose we start following the differential equation at $t = t_1 \approx r_1 \arctan\theta_0$. Then we will need to take $f(t_1)$ very close to r_0 and $f'(t_1)$ very close to $-\cot\theta_0$. This can be accomplished by taking C_2 bigger than t_1, $C_1(C_2 - t_1)$ large, and C_1 huge. Then we follow the solution out until t is very close to C_2, at which point $f(t)$ is approximately $\frac{1}{C_1}$, which is very small but positive, and $f'(t)$ is approximately 0, *i.e.*, θ is very close to $\frac{\pi}{2}$. Then we quickly bring k back down to 0 and finish with a horizontal line, thereby satisfying all our requirements. $\square$

There is a slight strengthening of this due to Gajer, which provides information about manifolds with boundary.

Theorem 3.2 (Improved Surgery Theorem, [Gaj1]) *Let N be a closed manifold with a metric of positive scalar curvature ds^2_N, not necessarily connected, and let M be obtained from N by a surgery of codimension ≥ 3. Let W be the trace of this surgery (a cobordism from N to M). Then W can be given a metric of positive scalar curvature ds^2_W which is a product metric $ds^2_N + dt^2$ in a collar neighborhood of N and a product metric $ds^2_M + dt^2$ in a collar neighborhood of M.*

This indeed strengthens Theorem 3.1, since in a neighborhood of M, the scalar curvature of ds^2_W is the same as that of ds^2_M, and thus we have given M a metric of positive scalar curvature.

The study of metrics such as the one in Theorem 3.2, together with the obvious parallels in the theory of automorphisms of manifolds, motivates the following.

Definition. Let ds^2_0 and ds^2_1 be two Riemannian metrics on a compact manifold M, both with positive scalar curvature. (For the moment we take M to be closed, though later we will also consider the case where M has a boundary.) We say these metrics are *isotopic* if they lie in the same path component of the space of positive scalar curvature metrics on M, and *concordant* if there is a positive scalar curvature metric on a cylinder $W = M \times [0, a]$ which restricts to $ds^2_0 + dt^2$ in a collar neighborhood of $M \times \{0\}$ and to $ds^2_1 + dt^2$ in a collar neighborhood of $M \times \{a\}$. We denote by $\widetilde{\pi}_0\mathfrak{R}^+(M)$ the set of concordance classes of positive scalar curvature metrics on M.

There is one important and easy result relating isotopy and concordance of positive scalar curvature metrics.

Proposition 3.3 ([GL2], Lemma 3; [Gaj1], pp. 184–185) *Isotopic metrics of positive scalar curvature are concordant.*

Sketch of Proof. Suppose ds^2_t, $0 \leq t \leq 1$, is an isotopy between positive scalar curvature metrics on M. Consider the metric $ds^2_{t/a} + dt^2$ on $W = M \times [0, a]$. This will have positive scalar curvature for $a \gg 0$, since a calculation shows that the scalar curvature $\kappa(x, t)$ at a point (x, t) will be of the form $\kappa_{t/a}(x) + O(1/a)$, where $\kappa_{t/a}$ is the scalar curvature of M for the metric $ds^2_{t/a}$. (In fact, if one is careful, the $O(1/a)$ can be improved to $O(1/a^2)$, though this doesn't matter to us.) Since M is compact and all the metrics $ds^2_{t/a}$ have positive scalar curvature, we may choose $\kappa_0 > 0$ such that $\kappa_{t/a}(x) \geq \kappa_0 > 0$ for all x and for all t. For a large enough, the error terms will be less than $\kappa_0/2$, so W also has positive scalar curvature. $\square$

It is still not known if the converse holds or not; indeed, there is no known methodology for approaching this question, as there is no known method for distinguishing between isotopy classes of positive scalar curvature metrics which is not based on distinguishing concordance classes. However, dimension 2 is special enough so that for the two closed 2-manifolds which admit positive scalar curvature metrics, S^2 and $\mathbb{RP}^2$, we can give a complete classification up to isotopy, and even say a bit more.

Theorem 3.4 *Any two metrics of positive scalar curvature on S^2 or on $\mathbb{RP}^2$ are isotopic. In fact, the spaces $\mathfrak{R}^+(S^2)$ and $\mathfrak{R}^+(\mathbb{RP}^2)$ are contractible.*

Proof. We begin with a general observation. Let M be any manifold, say for simplicity compact, and let $\operatorname{Diff} M$ be its diffeomorphism group, a topological group in the C^∞ topology. (For M compact, there is only one reasonable topology on $\operatorname{Diff} M$.) When M is oriented, we denote the orientation-preserving subgroup of $\operatorname{Diff} M$ by $\operatorname{Diff}^+ M$. Let $C^\infty(M)$ be the smooth functions on M, viewed as a topological vector space (and, in particular, as a topological group under addition). Then one can form the semidirect product group $C^\infty(M) \rtimes \operatorname{Diff} M$, with $\operatorname{Diff} M$ acting on $C^\infty(M)$ by pre-composition. Note that $C^\infty(M) \rtimes \operatorname{Diff} M$ acts on Riemannian metrics on M on the right by the formula

$$g \cdot (u, \varphi) = \varphi^*(e^u g), \qquad u \in C^\infty(M), \quad \varphi \in \operatorname{Diff} M,$$

and that this action is continuous for the C^∞ topologies. Any two metrics in the same orbit for this action are said to be *conformal* to one another; any two metrics in the same orbit for the action of the subgroup $C^\infty(M)$ are said to be *pointwise conformal* to one another.

Now we need to recall the Uniformization Theorem for Riemann surfaces. When formulated in the language of differential geometry (rather than complex analysis), it says that if M is an oriented connected closed 2-manifold, then $C^\infty(M) \rtimes \operatorname{Diff}^+ M$ acts transitively on the space of Riemannian metrics on M. Let's apply this to S^2. Then we get an identification of the (contractible) space of Riemannian metrics on S^2 with the quotient of $C^\infty(S^2) \rtimes \operatorname{Diff}^+ S^2$ by the subgroup fixing the standard metric g_0 of constant Gaussian curvature 1. This subgroup is identified with

$PSL(2,\mathbb{C})$, the group of Möbius transformations,[2] since a famous result of complex analysis says that all (orientation-preserving) pointwise conformal automorphisms for the standard spherical metric come from holomorphic automorphisms of $S^2 = \mathbb{CP}^1$. Since $PSL(2,\mathbb{C})$ has the homotopy type of its maximal compact subgroup $PSU(2) \cong SO(3)$, and since

$$\left(C^\infty(S^2) \rtimes \mathrm{Diff}^+ S^2\right)/PSL(2,\mathbb{C})$$

must be contractible, it follows that $\mathrm{Diff}^+ S^2$ has a deformation retraction down to its subgroup $SO(3)$, which in turn is the group of orientation-preserving isometries for the standard metric. Also observe that since S^2 is the double cover of $\mathbb{RP}^2$, taking the $\mathbb{Z}/2$-action into account shows that $C^\infty(\mathbb{RP}^2) \rtimes \mathrm{Diff}\,\mathbb{RP}^2$ acts transitively on the Riemannian metrics on $\mathbb{RP}^2$, and that the stabilizer of the standard metric is precisely $SO(3)$, the isometry group. So $\mathrm{Diff}\,\mathbb{RP}^2$ also has a deformation retraction down to $SO(3)$.

Let's come back to metrics of positive scalar curvature. If g_0 and $\bar{g}_0$ denote the standard metrics on S^2 or $\mathbb{RP}^2$ of constant Gaussian curvature 1, then a conformally related metric $g_0 \cdot (u, \varphi)$ (respectively, $\bar{g}_0 \cdot (u, \varphi)$) has positive scalar curvature if and only if $e^u g_0$ (resp., $e^u \bar{g}_0$) does (since positive scalar curvature is preserved under the action of Diff). Since g_0 has scalar curvature $\equiv 2$, the formula computing the change in scalar curvature under a conformal change in the metric (found in [KW1], for example) gives

$$\Delta(u) = 2 - e^u \kappa, \tag{3.3}$$

where Δ is the Laplace-Beltrami operator for the metric g_0 (with the sign convention making this a negative semi-definite operator) and κ is the scalar curvature of the metric $e^u g_0$. We claim that the set

$$\mathcal{S} = \{u \in C^\infty : \kappa \text{ in (3.3) is strictly positive}\}$$

is star-shaped about the origin.

To prove this, suppose u is such that κ in (3.3) is strictly positive. Then if κ_t denotes the scalar curvature of the metric $e^{tu} g_0$, replacing u by tu in (3.3) gives

$$\Delta(tu) = 2 - e^{tu}\kappa_t.$$

Since Δ is linear and $\kappa_0 \equiv 2$, we obtain:

$$2 - e^{tu}\kappa_t = t\Delta(u) = t\left(2 - e^u\kappa\right),$$

[2] Caution: While $PSL(2,\mathbb{C})$ embeds in $\mathrm{Diff}^+ S^2$, the identification of $PSL(2,\mathbb{C})$ with the stabilizer of g_0 is via a "diagonal embedding," since we need to take the "conformal factor" into account.

or

$$e^{tu}\kappa_t = te^u\kappa + 2(1-t).$$

Since, by assumption, κ is everywhere positive and $0 \le t \le 1$, both terms on the right are non-negative. Furthermore, the first term on the right only vanishes when $t = 0$, and the second term only vanishes when $t = 1$. Thus $e^{tu}\kappa_t$ is everywhere positive, and so κ_t is everywhere positive, proving that $\mathcal{S}$ is star-shaped (and thus contractible).

Finally, we see that $\mathfrak{R}^+(S^2)$ is identified with

$$\left(\mathcal{S}(S^2)\cdot \mathrm{Diff}^+(S^2)\right)/PSL(2,\mathbb{C}) \subset \left(C^\infty(S^2)\cdot \mathrm{Diff}^+(S^2)\right)/PSL(2,\mathbb{C}),$$

and similarly $\mathfrak{R}^+(\mathbb{RP}^2)$ is identified with

$$\left(\mathcal{S}(\mathbb{RP}^2)\cdot \mathrm{Diff}(\mathbb{RP}^2)\right)/SO(3) \subset \left(C^\infty(\mathbb{RP}^2)\cdot \mathrm{Diff}(\mathbb{RP}^2)\right)/SO(3).$$

As $\mathrm{Diff}^+(S^2)/PSL(2,\mathbb{C})$, $\mathrm{Diff}(\mathbb{RP}^2)/SO(3)$, $\mathcal{S}(S^2)$, and $\mathcal{S}(\mathbb{RP}^2)$ are all contractible, we see that $\mathfrak{R}^+(S^2)$ and $\mathfrak{R}^+(\mathbb{RP}^2)$ must be contractible. $\square$

4 The Gromov-Lawson Conjecture and its Variants

In the discussion so far, we have not explained (except in the case of dimension 2) why it is that there are closed manifolds which *cannot* admit a positive scalar curvature metric. Most of the known results of this sort, at least for manifolds of large dimension, stem from a fundamental discovery of Lichnerowicz [Li], which is that if $\not{D}$ is the Dirac operator on a spin manifold M (a self-adjoint elliptic first-order differential operator, acting on sections of the spinor bundle), then

$$\not{D}^2 = \nabla^*\nabla + \frac{\kappa}{4}. \tag{4.1}$$

Here ∇ is the covariant derivative on the spinor bundle induced by the Levi-Civita connection, and ∇^* is the adjoint of ∇. Since the operator $\nabla^*\nabla$ is obviously self-adjoint and non-negative, it follows from equation (4.1) that the square of the Dirac operator for a metric of positive scalar curvature is bounded away from 0, and thus that the Dirac operator cannot have any kernel. It follows that any index-like invariant of M which can be computed in terms of harmonic spinors (*i.e.*, the kernel of $\not{D}$) has to vanish. E.g., if M is a spin manifold of dimension n, there is a version of the Dirac operator which commutes with the action of the Clifford algebra $C\ell_n$ (see [LaM], § II.7). In particular, its kernel is a (graded) $C\ell_n$-module, which represents an element $\alpha(M)$ in the real K-theory group $KO_n = KO^{-n}(\mathrm{pt})$ (see [LaM], Def. II.7.4).

Theorem 4.1 (Lichnerowicz [Li]; Hitchin [Hit]) *If M^n is a closed spin manifold for which $\alpha(M) \neq 0$ in KO_n, then M does not admit a metric of positive scalar curvature.*

We recall that $KO_n \cong \mathbb{Z}$ for $n \equiv 0 \mod 4$, that $KO_n \cong \mathbb{Z}/2$ for $n \equiv 1, 2 \mod 8$, and $KO_n = 0$ for all other values of n. Furthermore, for $n \equiv 0 \mod 4$, the invariant $\alpha(M)$ is essentially equal to Hirzebruch's $\widehat{A}$-genus $\widehat{A}(M)$, namely $\alpha(M) = \widehat{A}(M)$ for $n \equiv 0 \mod 8$, and $\alpha(M) = \widehat{A}(M)/2$ for $n \equiv 4 \mod 8$. So this result immediately shows that there are many manifolds, even simply connected ones, which do not lie in class (1) of the Kazdan-Warner trichotomy (see Theorem 2.2). E.g., the Kummer surface K^4, the hyperplane in the complex projective space $\mathbb{CP}^3$ given by the equation $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$, is spin and has $\widehat{A}(K) = 2$, and hence does not admit a metric of positive scalar curvature.

We observe that $\alpha(M)$ depends only on the spin bordism class $[M] \in \Omega_n^{\mathrm{spin}}$. In fact, we can interpret $\alpha(M)$ as the image of $[M]$ under a natural transformation of generalized homology theories as follows. Let $KO_*(X)$ and $ko_*(X)$ denote the periodic and connective real K-homology of a space X, respectively (so $KO_*(X)$ satisfies Bott periodicity, and $ko_* = ko_*(\mathrm{pt})$ is obtained from $KO_* = KO_*(\mathrm{pt})$ by killing the groups in negative degree). Then there are natural transformations

$$\Omega_*^{\mathrm{spin}}(X) \xrightarrow{D} ko_*(X) \xrightarrow{\mathrm{per}} KO_*(X),$$

the first of which sends the bordism class $[M, f]$ to $f_*([M]_{ko})$, where $[M]_{ko} \in ko_*(M)$ denotes the ko-fundamental class of M determined by the spin structure. With this notation, $\alpha(M) = \mathrm{per} \circ D([M])$.

Next, we want to state an important consequence of Theorem 3.1, but first we need a relevant definition.

Definition. Let $B \to BO$ be a fibration. A *B-structure* on a manifold is defined to be a lifting of the (classifying map of the) stable normal bundle to a map into B. Then one has bordism groups Ω_n^B of manifolds with B-structures, defined in the usual way. (For instance, if $B = B\mathrm{Spin}$, mapping as usual to BO, then $\Omega_n^B = \Omega_n^{\mathrm{spin}}$.) We note that given a connected closed manifold M, there is a choice of such a B [3] for which M has a B-structure and the map $M \to B$ is a 2-equivalence. (**Example**: If M is a spin manifold, choose $B = B\pi \times B\mathrm{Spin}$, where $\pi = \pi_1(M)$, and let $B \to BO$ be the projection onto the second factor composed with the map $B\mathrm{Spin} \to BO$ induced by $\mathrm{Spin} \to O$. Map M to the first factor by means of the classifying map for the universal cover, and to the second factor by means of the spin structure.)

The simply connected cases of the following theorem were proved in [GL2]; the general case, with this formulation, is in [RS1].

[3] We will see in Section 5 how to formalize this in a functorial way.

Theorem 4.2 (Bordism Theorem) *Let M^n be a B-manifold with $n = \dim M \geq 5$, and assume that the map $M \to B$ is a 2-equivalence. Then M admits a metric of positive scalar curvature if and only if there is some B-manifold of positive scalar curvature in the same B-bordism class.*

Sketch of Proof. Let N be a B-manifold B-bordant to M. The hypotheses combine (via the method of proof of the s-Cobordism Theorem) to show that M can be obtained from N by surgeries in codimension ≥ 3. Then if N admits a metric of positive scalar curvature, one can apply Theorem 3.1 to conclude that the same is true for M. □

Remark. Note that in the proof of Theorem 4.2, M and N do not quite play symmetrical roles. While M can be obtained from N by surgeries in codimension ≥ 3, the converse may not be the case unless $N \to B$ is also a 2-equivalence. This is useful in applications, since often the "obvious" generators for B-bordism groups do not satisfy the 2-equivalence condition.

Theorem 4.3 (Gromov-Lawson [GL2]) *If M is a simply connected closed manifold of dimension $n \geq 5$, and if $w_2(M) \neq 0$, then M admits a metric of positive scalar curvature.*

Sketch of Proof. If M is simply connected with $w_2(M) \neq 0$, then the appropriate $B \to BO$ to use in Theorem 4.2 is just $BSO \to BO$, and the corresponding bordism theory is oriented bordism. Gromov-Lawson proceed to show that the generators of Ω_* constructed by Wall all admit positive scalar curvature metrics. □

Of course, the restriction $w_2(M) \neq 0$ in Theorem 4.3 is important, because Theorem 4.1 shows that otherwise there can be obstructions to positive scalar curvature. It is also well-known that the maps $D_n : \Omega_n^{\text{spin}} \to ko_n(\text{pt})$ are all surjective, so all potential obstructions are in fact realized. In the simply connected spin case, Gromov and Lawson were not able to get as sharp a result as in the non-spin case, but at least they were able to prove:

Theorem 4.4 *If M is a simply connected closed manifold of dimension $n \geq 5$, and if $w_2(M) = 0$ (so that, once an orientation is fixed, M defines a class $[M] \in \Omega_n^{\text{spin}}$), then a finite connected sum of copies of M admits a metric of positive scalar curvature if and only if $[M]$ maps to $0 \in KO_n(\text{pt}) \otimes_{\mathbb{Z}} \mathbb{Q}$ under α.*

For manifolds with a non-trivial fundamental group, the situation is more complicated, as can already be seen in the 2-dimensional case. (As we have already observed, no closed connected 2-dimensional with an infinite fundamental group admits a positive scalar curvature metric. Nevertheless, oriented surfaces map trivially to $KO_2(\mathrm{pt}) = \mathbb{Z}/2$, at least for the usual (bounding) choice of a spin structure.) It was shown in [GL1] and [SY] that tori never admit positive scalar curvature metrics (in any dimension), and that in general, there are extra obstructions to positive scalar curvature that come from the fundamental group. Extrapolating from Theorem 4.4 and from their results in [GL3], Gromov and Lawson arrived at:

Conjecture 4.5 ("Gromov-Lawson Conjecture" [GL3]) *Suppose M is a connected closed spin manifold of dimension $n \geq 5$ with "reasonable" fundamental group π (in a sense to be discussed below). Let $f : M \to B\pi$ be the classifying map for the universal cover of M, so that (M, f) defines a class $[M, f] \in \Omega_n^{\mathrm{spin}}(B\pi)$. Then M admits a metric of positive scalar curvature if and only if* $\mathrm{per} \circ D([M, f]) = 0$ *in $KO_n(B\pi)$.*

The conjecture in the simply connected case was settled by:

Theorem 4.6 (Stolz [St1]) *If M is a simply connected closed manifold of dimension $n \geq 5$, and if $w_2(M) = 0$ (this means M admits a spin structure, which since M is simply connected is unique once we fix an orientation), then M admits a metric of positive scalar curvature if and only if the Lichnerowicz-Hitchin obstruction $\alpha(M)$ vanishes in $KO_n(\mathrm{pt})$.*

Sketch of Proof. The first step in the proof is to reduce this to a 2-primary problem in homotopy theory. This reduction is primarily due to Miyazaki, who showed [Mi] by explicit construction of enough manifolds of positive scalar curvature that the subgroup of Ω_n^{spin} generated by manifolds of positive scalar curvature is a subgroup of the kernel of α of index a power of 2. The main part of the proof is then based on the observation that the first non-trivial element in the kernel of α is the quaternionic projective space $\mathbb{HP}^2$. A careful transfer argument (relying on the mod 2 Adams spectral sequence) then shows that, after localizing at 2, the kernel of α in general is generated by the total spaces of fiber bundles over spin manifolds with fiber $\mathbb{HP}^2$ and structure group $PSp(3)$, the isometry group of $\mathbb{HP}^2$. It is not hard to show that all such fiber bundles admit positive scalar curvature metrics (since one can rescale the metric so that the positive scalar curvature on the projective space fibers dwarfs any contributions from the base). So the result follows from the simply connected case of Theorem 4.2. □

To explain progress regarding the conjecture in the non-simply connected case, we need one additional ingredient.

Definition. Let π be any discrete group. Then the real group ring $\mathbb{R}\pi$ can be completed in two standard ways to get a C^*-algebra $C^*(\pi)$.[4] (Either one

[4] A C^*-algebra is a Banach algebra with involution which is isometrically $*$-isomorphic to an algebra of operators on a Hilbert space which is closed under the adjoint operation and closed in the operator norm.

lets $\mathbb{R}\pi$ act on $\ell^2(\pi)$ on the left in the usual way, and takes the completion in the operator norm, obtaining what is usually called $C_r^*(\pi)$, or else one lets $\mathbb{R}\pi$ act on the Hilbert space direct sum of the spaces of *all* unitary representations of π (suitably interpreted to avoid set-theoretic problems), and takes the completion in the operator norm, obtaining what is usually called $C^*_{\max}(\pi)$.) The two completions coincide if and only if π is amenable, but for present purposes it will not matter which one we use, so we won't distinguish in the notation.

There is an *assembly map* $A : KO_n(B\pi) \to KO_n(C^*(\pi))$ defined as follows. Form the bundle $\mathcal{V}_{B\pi} = E\pi \times_\pi C^*(\pi)$ over $B\pi$ whose fibers are rank-one free (right) modules over $C^*(\pi)$. As a "$C^*(\pi)$-vector bundle" over $B\pi$, this has a stable class $[\mathcal{V}_{B\pi}]$ in a K-group $KO^0(B\pi; C^*(\pi))$, and A is basically the "slant product" with $[\mathcal{V}_{B\pi}]$. The assembly map A is functorial in π (to the extent that this makes sense). Injectivity of A, often known as the Strong Novikov Conjecture, implies the Novikov Conjecture on homotopy invariance of higher signatures for manifolds with fundamental group π.

The results on one direction of the Gromov-Lawson Conjecture all come from:

Theorem 4.7 ([R2]) *Let M be a closed connected spin manifold of positive scalar curvature, and let $f : M \to B\pi$ be the classifying map for the universal cover of M. Then $A \circ \mathrm{per} \circ D([M, f]) = 0$ in $KO_n(C^*(\pi))$. In particular, if the Strong Novikov Conjecture is true for π* (i.e., *A is injective), then* $\mathrm{per} \circ D([M, f]) = 0$ *in $KO_n(B\pi)$.*

Sketch of Proof. This relies on an index theory, due to Mishchenko and Fomenko, for elliptic operators with coefficients in a $C^*(\pi)$-vector bundle. If M is as in the theorem, then the (Clifford algebra linear) Dirac operator on M, with coefficients in the bundle $\mathcal{V}_{B\pi}$, has an index $\alpha(M, f) \in KO_n(C^*(\pi))$, which one can show by the Kasparov calculus is just $A \circ \mathrm{per} \circ D([M, f])$. Since $\mathcal{V}_{B\pi}$ is by construction a flat bundle, there are no correction terms due to curvature of the bundle, and formula (4.1) applies without change. Hence if M has positive scalar curvature, the square of this Dirac operator is bounded away from 0, and the index vanishes. □

This result seems to be about the best one can do in (in the spin case) in attacking the Gromov-Lawson Conjecture 4.5 via index theory. It indicates that perhaps the "reasonable" groups for purposes of the Conjecture (which Gromov and Lawson did not make precise) should be a subset of the class of those for which the assembly map A is injective.[5] Many torsion-free groups are known to lie in this class, including for example all torsion-free amenable groups, all torsion-free subgroups of $GL(n, \mathbb{Q})$, and all torsion-free hyperbolic groups in the sense of Gromov.

[5] As far as we know at the moment, this class could include all torsion-free groups.

For groups with torsion, even for finite cyclic groups, it is easy to find examples (see [R1]) where Conjecture 4.5 fails. The reason is simply that many classes in $KO_n(B\pi)$ can be represented by manifolds of positive scalar curvature, such as lens spaces. A first attempt at remedying this results in the following modified conjecture (which first appears in [R2], [R3]):

Conjecture 4.8 ("Gromov-Lawson-Rosenberg Conjecture") *Suppose M is a connected closed spin manifold of dimension $n \geq 5$. Let $f : M \to B\pi$ be the classifying map for the universal cover of M, so that (M, f) defines a class $[M, f] \in \Omega_n^{\mathrm{spin}}(B\pi)$. Then M admits a metric of positive scalar curvature if and only if $\alpha(M, f)$, the generalized index of the Dirac operator, vanishes in $KO_n(C^*(\pi))$.*

There are analogues of this conjecture, involving indices of "twisted Dirac operators," for manifolds which are non-spin but which have spin universal covers. Rather than state them now, we will defer these cases to Section 5. However, it is worth pointing out that one way to rephrase Conjecture 4.8 is by saying that "the index of Dirac tells all." If this is the case even in the non-spin case, then it implies:

Conjecture 4.9 *If M is a connected closed manifold of dimension $n \geq 5$, and if the universal cover of M does* not *admit a spin structure, then M admits a metric of positive scalar curvature.*

Conjecture 4.9 is consistent with Theorem 4.3, but unfortunately it is known to fail for manifolds with large fundamental group. A counterexample suggested by [GL3], for which failure of the conjecture can be checked using the "minimal hypersurface technique" of [SY], is $T^6\#(\mathbb{CP}^2 \times S^2)$. This suggests that Conjecture 4.8 should be false as well, though the following counterexample was only discovered recently.

Counterexample 4.10 ([Sch]) *Let M^5 be the closed spin manifold obtained from T^5 by doing spin surgery to cut down the fundamental group to $\mathbb{Z}^4 \times \mathbb{Z}/3$, and let $f : M \to B(\mathbb{Z}^4 \times \mathbb{Z}/3)$ be the classifying map for its universal cover. Then $\alpha(M, f) = 0$ in $KO_n(C^*(\pi))$, but M does not admit a metric of positive scalar curvature.*

What is most amazing about Conjectures 4.8 and 4.9 is not that there are cases where they fail, but that they indeed hold in a great number of cases. This should be viewed as a vindication of the intuition of Gromov and Lawson, since in many cases Conjecture 4.5 is true in its original formulation. Before stating some of these results, we should first explain how it is that one "narrows the gap" between the positive results of the Bordism Theorem, Theorem 4.2, and the results on obstructions in Theorem 4.7. While one could prove some of the results in greater generality, we will state them only in the spin and oriented non-spin cases.

Theorem 4.11 (Stolz, Jung) *Let M^n be a connected closed manifold of dimension $n \geq 5$, and let $f : M \to B\pi$ be the classifying map for its universal cover. If M is spin, then M admits a metric of positive scalar curvature if and only if there is some spin manifold of positive scalar curvature representing the class $D([M, f])$ in $ko_n(B\pi)$. If M is oriented and if the universal cover of M does* not *admit a spin structure, then M admits a metric of positive scalar curvature if and only if there is some oriented manifold of positive scalar curvature representing the class $f_*([M]) \in H_n(B\pi; \mathbb{Z})$.*

Sketch of Proof. This requires a number of techniques. The 2-primary calculation in the spin case is based on a generalization, found in [St2], of the $\mathbb{HP}^2$-bundle method of the proof of Theorem 4.6. The 2-primary calculation in the oriented non-spin case is easier, so we give it here. Localized at 2, the spectrum MSO is known to be Eilenberg-MacLane (see [R4]), so $\Omega_n(B\pi)$, after localizing at 2, splits up as $\bigoplus_j H_{n-j}(B\pi; \Omega_j)$, with the summand $H_{n-j}(B\pi; \Omega_j)$ corresponding to bordism classes of the form

$$N^{n-j} \times P^j \xrightarrow{g} B\pi,$$

with g collapsing P to a point. But by the same calculation as in the proof of Theorem 4.3, each generator of Ω_j with $j > 0$ is represented by a manifold of positive scalar curvature. So by the Bordism Theorem, Theorem 4.2, we are reduced to looking at $H_n(B\pi; \mathbb{Z})$.

The proof at odd primes is based on the theory of homology theories derived from bordism, using "bordism with singularities." □

Using this result, it is easy to check certain cases of Conjectures 4.8 and 4.9. For example, one easily deduces:

Theorem 4.12 *Conjecture 4.9 is true for orientable manifolds with finite cyclic fundamental group.*

Proof. The integral homology of a cyclic group is concentrated in odd degrees n, where (for $n \geq 3$) a generator is represented by a lens space (which has positive scalar curvature). □

Putting together Theorem 4.7 and Theorem 4.11, we obtain the following positive results on Conjecture 4.8:

Theorem 4.13 *Suppose the discrete group π has the following two properties:*

1. *The Strong Novikov Conjecture holds for π,* i.e., *the assembly map $A : KO_*(B\pi) \to KO_*(C^*(\pi))$ is injective.*
2. *The natural map* per $: ko_*(B\pi) \to KO_*(B\pi)$ *is injective.*

Then the Gromov-Lawson Conjecture, Conjecture 4.5, and the Gromov-Lawson-Rosenberg Conjecture, Conjecture 4.8, hold for spin manifolds with fundamental group π.

Proof. Suppose M^n is a spin manifold, with $n \geq 5$, and $f : M \to B\pi$ is the classifying map for its universal cover. If $\mathrm{per} \circ D([M, f]) = 0$ in $KO_n(B\pi)$, then $D([M, f]) = 0$ in $ko_n(B\pi)$ by Condition (2), and so M admits a metric of positive scalar curvature by Theorem 4.11. But if $\mathrm{per} \circ D([M, f]) \neq 0$, condition (1) says that $\alpha(M, f) \neq 0$, and thus M cannot admit a metric of positive scalar curvature, by Theorem 4.7. □

Theorem 4.13 applies to quite a number of torsion-free groups, for example, free groups and free abelian groups. It is not much help in studying finite groups, however. For finite groups, both of the conditions in Theorem 4.13 usually fail. Still, there are so far no counterexamples to the Gromov-Lawson-Rosenberg Conjecture in the case of finite fundamental groups. In fact, the Conjecture is true for the following class of finite groups. Recall that a finite group has periodic cohomology if and only if its Sylow subgroups are all cyclic or generalized quaternion.

Theorem 4.14 ([BGS]) *The Gromov-Lawson-Rosenberg Conjecture, Conjecture 4.8, holds for any spin manifold with finite fundamental group with periodic cohomology.*

One might wonder whether the restriction to dimensions $n \geq 5$ in most of our results is truly necessary. In dimension 2, we already know the full story as far as positive scalar curvature is concerned, and in dimension 3, the Thurston Geometrization Conjecture would basically settle everything. Dimension 4 is different, however. Seiberg-Witten theory gives the following:

Theorem 4.15 (primarily due to Taubes [Tau]; see also [LeB]) *Let M^n be a closed, connected oriented 4-manifold with $b_2^+(M) > 1$. If M admits a symplectic structure (in particular, if M admits the structure of a Kähler manifold of complex dimension 2) then M does not admit a positive scalar curvature metric (even one not well-behaved with respect to the symplectic structure).*

This dramatic result implies that the Gromov-Lawson-Rosenberg Conjecture fails badly in dimension 4, even in the simply connected case.

Counterexample 4.16 *In dimension 4, there exist:*

1. *a simply connected spin manifold M^4 with $\widehat{A}(M) = 0$ but with no positive scalar curvature metric.*

2. *simply connected non-spin manifolds with no positive scalar curvature metric.*

The counterexamples we have listed to Conjectures 4.8 and 4.9, as well as the unusual behavior in dimension 4, suggest that it may be best to divide the Gromov-Lawson-Rosenberg Conjecture into two pieces: an "unstable" part, that may fail in some cases due to low-dimensional difficulties (or other factors), and a "stable" conjecture, which stands a better chance of being true in general. This, as well as the fact that the periodicity in KO-theory has no obvious geometric counterpart as far as positive scalar curvature is concerned, motivates:

Conjecture 4.17 ("Stable Gromov-Lawson-Rosenberg Conjecture") *Let* Bt^8 *be the Bott manifold, a simply connected spin manifold of dimension* 8 *with* $\widehat{A}(\mathrm{Bt}^8) = 1$. (*This manifold is not unique, but any choice will do. What is essential here is that* Bt^8 *geometrically represents Bott periodicity in* KO*-theory.*) *If* M^n *is a spin manifold, and if* $f : M \to B\pi$ *is the classifying map for its universal cover, then* $M \times \mathrm{Bt}^8 \times \cdots \times \mathrm{Bt}^8$ *admits a metric of positive scalar curvature* (*for some sufficiently large number of* Bt^8 *factors*) *if and only if* $\alpha(M, f) = 0$ *in* $KO_n(C^*(\pi))$.

The counterpart of Theorem 4.13 as far as the Stable Conjecture is concerned is simply:

Theorem 4.18 *The Stable Gromov-Lawson-Rosenberg Conjecture, Conjecture 4.17, holds for spin manifolds with fundamental group* π*, provided that the assembly map* $A : KO_*(B\pi) \to KO_*(C^*(\pi))$ *is injective.*

At the other extreme of finite fundamental groups, we have:

Theorem 4.19 ([RS2]) *The Stable Gromov-Lawson-Rosenberg Conjecture, Conjecture 4.17, holds for spin manifolds with finite fundamental group.*

For groups with torsion, the assembly map A is not expected to be injective, so Baum, Connes, and Higson [BCH] suggested replacing it by the so-called Baum-Connes assembly map $KO_*^\pi(\underline{E\pi}) \to KO_*(C^*(\pi))$. Here $\underline{E\pi}$ is the universal *proper* π-space and $KO_*^\pi(\underline{E\pi})$ is its π-equivariant KO-homology. The space $\underline{E\pi}$ coincides with $E\pi$, the universal free π-space, exactly when π is torsion-free, and in this case one recovers the usual assembly map. For a finite group, $\underline{E\pi}$ is a point and the Baum-Connes assembly map is an isomorphism. The following result generalizes Theorems 4.18 and 4.19.

Theorem 4.20 ([St5]) *The Stable Gromov-Lawson-Rosenberg Conjecture, Conjecture 4.17, holds for spin manifolds with fundamental group* π*, provided that the Baum-Connes assembly map* $KO_*^\pi(\underline{E\pi}) \to KO_*(C^*(\pi))$ *is injective.*

The hypothesis of this theorem is known to be satisfied in a great many cases, for example, whenever π can be embedded discretely in a Lie group with finitely many connected components.

5 Parallels with Wall's Surgery Theory

Surgery theory is the main tool in the study of smoothings of Poincaré complexes. As we have seen, it is also the main tool in the study of metrics of positive scalar curvature. In this section we want to discuss similarities and differences between the resulting theories.

A central role in our understanding of smoothings of a Poincaré complex X is played by Wall's surgery obstruction groups $L_i(\pi, w)$; these are abelian groups, which depend on the fundamental group $\pi = \pi_1(X)$, the first Stiefel-Whitney class $w = w_1(X)$, and an integer i. The group relevant for the *existence* of a smoothing of X is $L_n(\pi, w)$, $n = \dim X$, while $L_{n+1}(\pi, w)$ plays a role in the *classification* of smoothings of X.

The analog of the Wall group in the study of positive scalar curvature metrics on a manifold M is an abelian group $R_i(\pi, w, \widehat{\pi})$, which depends on the fundamental group $\pi = \pi_1(M)$ and the first Stiefel-Whitney class $w\colon \pi \to \mathbb{Z}/2$, as well as an extension $\widehat{\pi}$ of π. Geometrically, the extension $\widehat{\pi} \twoheadrightarrow \pi$ is given by applying the fundamental group functor to the fiber bundle $O(M)/\mathbb{Z}/2 \to M$, where $O(M)$ is the frame bundle of M and $\mathbb{Z}/2$ acts on $O(M)$ by mapping an isometry $f\colon \mathbb{R}^n \to T_xM$ to the composition $f \circ r$, where $r\colon \mathbb{R}^n \to \mathbb{R}^n$ is the reflection in the hyperplane perpendicular to $(1, 0, \ldots, 0)$.

Up to isomorphism, the extension $\widehat{\pi} \twoheadrightarrow \pi$ is determined by the second Stiefel-Whitney class $w_2(M)$ as follows. If the universal cover of M is spin, then $w_2(M) = u^*(e)$ for a unique $e \in H^2(B\pi; \mathbb{Z}/2)$ where $u\colon M \to B\pi$ is the classifying map of the universal covering of M; in this case $\widehat{\pi} \to \pi$ is the central $\mathbb{Z}/2$-extension classified by e. Otherwise $\widehat{\pi} \to \pi$ is an isomorphism.

Before defining the groups $R_i(\pi, w, \widehat{\pi})$, we want to state and discuss the following result which shows the central role of these groups for the study of positive scalar curvature metrics.

Theorem 5.1 ([St4]) *Let M be a smooth, connected, compact manifold of dimension $n \geq 5$, possibly with boundary. Let $\pi = \pi_1(M)$ be the fundamental group, $w\colon \pi \to \mathbb{Z}/2$ the first Stiefel-Whitney class, and let $\widehat{\pi} \to \pi$ be the extension described above.*

Existence. *A positive scalar curvature metric h on ∂M extends to a positive scalar curvature metric on M which is a product metric near the boundary if and only if an obstruction $\sigma(M, h) \in R_n(\pi, w, \widehat{\pi})$ vanishes.*

Concordance Classification. *If h extends to a positive scalar curvature metric on M, then the group $R_{n+1}(\pi, w, \widehat{\pi})$ acts freely and transitively on the concordance classes of such metrics.*

The groups $R_i(\gamma)$ for $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$ (corresponding to spin manifolds) were first introduced by Hajduk [Haj]; he also proved the existence statement in that case.

We wish to compare Theorem 5.1 with the corresponding statements concerning smoothings of a Poincaré complex X. We recall that a *smoothing* of X is a (simple) homotopy equivalence $f\colon N \to X$ between a closed manifold N and X; two such pairs (N, f), (N', f') are identified if there is a diffeomorphism $g\colon N \to N'$ such that f is homotopic to $f' \circ g$. A necessary condition for the existence of a smoothing is that the Spivak normal bundle of X is stably fiber homotopy equivalent to the sphere bundle of a vector bundle. In homotopy theoretic terms this condition means that the map $X \to BG$ classifying the Spivak normal bundle factors through the canonical map $BO \to BG$. Since this map fits into a homotopy fibration $BO \to BG \to B(G/O)$, the condition is equivalent to the composition $X \to BG \to B(G/O)$ being homotopic to the constant map.

A fiber homotopy equivalence Φ between the Spivak normal bundle of X and the sphere bundle of a vector bundle determines via the Pontryagin-Thom construction a degree one normal map $f\colon N \to X$ up to bordism. The pair (N, f) is bordant to a smoothing if and only if its "surgery obstruction" $\sigma(N, f) \in L_n(\pi, w)$ vanishes. In particular, if the group $[X, B(G/O)]$ of pointed homotopy classes of maps from X to $B(G/O)$ is trivial, then the vanishing of $\sigma(N, f)$ is sufficient for the existence of a smoothing of X; if in addition the group $[X, G/O]$ is trivial, then the fiber homotopy equivalence Φ is unique up to homotopy. It follows that the bordism class of the degree one normal map $f\colon N \to X$ and hence the surgery obstruction $\sigma(N, f)$ is independent of the choices made in the construction of (N, f). Thus in this case, the vanishing of $\sigma(N, f)$ is also a necessary condition for the existence of a smoothing of X.

Concerning classification, the group $L_{n+1}(\pi, w)$ acts on the set $\mathfrak{S}(X)$ of smoothings of X. The "surgery exact sequence" describes the orbits as well as the isotropy groups of this action. The orbits are the fibers of a map $\mathfrak{S}(X) \to [X, G/O]$, and the isotropy subgroups are the images of homomorphisms $[\Sigma X, G/O] \to L_{n+1}(\pi, w)$. In particular, if the groups $[X, G/O]$ and $[\Sigma X, G/O]$ are trivial, then $L_{n+1}(\pi, w)$ acts freely and transitively on $\mathfrak{S}(X)$.

The upshot of this discussion is that if the groups $[X, B(G/O)]$, $[X, G/O]$, and $[\Sigma X, G/O]$ vanish, then the main result of surgery theory takes precisely the form of Theorem 5.1, with concordance classes of positive scalar curvature metrics replaced by smoothings and $R_i(\pi, w, \widehat{\pi})$ replaced by $L_i(\pi, w)$.

We recall that Wall's L_i-groups have an algebraic description as well as a description as bordism groups. So far, there is only a bordism description of R_i.

Definition 5.2 Let γ be a triple $(\pi, w, \widehat{\pi})$, where $w\colon \pi \to \mathbb{Z}/2$ is a group homomorphism and $\widehat{\pi} \to \pi$ is an extension of π such that $\ker(\widehat{\pi} \to \pi)$ is either $\mathbb{Z}/2$ or the trivial group. Let $\sigma\colon \mathrm{Spin}(n) \to SO(n)$ be the non-trivial double covering of the special orthogonal group $SO(n)$. We note that the conjugation action of $O(n)$ on $SO(n)$ lifts to an action on $\mathrm{Spin}(n)$. Let $\widehat{\pi} \ltimes \mathrm{Spin}(n)$ be the semi direct product, where $\widehat{g} \in \widehat{\pi}$ acts on the normal subgroup $\mathrm{Spin}(n)$ by conjugation by $r^{w(\widehat{g})}$. Here $r \in O(n)$ is the reflection in the hyperplane perpendicular to $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Abusing notation, we also use the notation w for the composition $\widehat{\pi} \to \pi \to \mathbb{Z}/2$. We define $G(\gamma, n)$ to be the quotient of $\widehat{\pi} \ltimes \mathrm{Spin}(n)$ by the central subgroup generated by $(k, -1)$, where $k \in \widehat{\pi}$ is the (possibly trivial) generator of $\ker(\widehat{\pi} \to \pi)$. Sending $[a, b] \in G(\gamma, n)$ to $r^{w(a)}\sigma(b)$ defines a homomorphism $\rho(\gamma, n)\colon G(\gamma, n) \to O(n)$.

A γ-*structure* on an n-dimensional Riemannian manifold M is a principal $G(\gamma, n)$-bundle $P \to M$ together with a $G(\gamma, n)$-equivariant map $\rho\colon P \to O(M)$. Here $O(M)$ is the orthogonal frame bundle of M, a principal bundle for the orthogonal group $O(n)$, and $G(\gamma, n)$ acts on $O(M)$ via the homomorphism $\rho(\gamma, n)$.

Remark 5.3

1. If π is the trivial group, then $G(\gamma, n) = SO(n)$ (resp. $\mathrm{Spin}(n)$) if $\ker(\widehat{\pi} \to \pi)$ is trivial (resp. non-trivial). In this case a γ-structure on M amounts to an orientation (resp. spin structure) on M (cf. [LaM], Def. II.1.3).
2. More generally, if $w = 0$ and $\widehat{\pi} = \pi$ (resp. $\widehat{\pi} = \pi \times \mathbb{Z}/2$), then $G(\gamma, n) = \pi \times SO(n)$ (resp. $G(\gamma, n) = \pi \times \mathrm{Spin}(n)$); in this case, a γ-structure amounts to an orientation (resp. spin structure) on M, together with a principal π-bundle $\widetilde{M} \to M$.
3. A γ-structure determines a principal π-bundle $\widetilde{M} \stackrel{\text{def}}{=} P/G_1 \to M$, where G_1 is the identity component of $G(\gamma, n)$. We note that $G_1 = SO(n)$ if $\ker(\widehat{\pi} \to \pi)$ is trivial, and $G_1 = \mathrm{Spin}(n)$ otherwise. Hence the principal G_1-bundle $P \to \widetilde{M}$ can be identified with the oriented frame bundle of $\widetilde{M}$ or a double cover thereof.

Definition 5.4 Given a triple γ as above, $R_n(\gamma)$ is the bordism group of pairs (N, h), where N is a n-dimensional manifold with γ-structure and h is a positive scalar curvature metric on the boundary ∂N (possibly empty). The obstruction $\sigma(M, h) \in R_n(\gamma(M))$ to extending the positive scalar curvature metric h on ∂M to a positive scalar curvature metric on M is just the bordism class $[M, h]$ (every manifold M has a *canonical* $\gamma(M)$-structure).

Sketch of Proof of Theorem 5.1. Both the existence and the classification statement are fairly direct consequences of the surgery results discussed in

Section 3. Concerning existence, it is easy to see that if h extends to a positive scalar curvature metric on M, then (M, h) represents zero in the bordism group $R_n(\gamma)$, $\gamma = \gamma(M) = (\pi, w, \widehat{\pi})$. (The manifold $M \times [0, 1]$ with some corners suitably rounded represents a zero bordism.) Conversely, a zero bordism for (M, h) provides us with a manifold M' with boundary $\partial M' = \partial M$ over which h extends to a positive scalar curvature metric (which is a product metric near the boundary), and a manifold W of dimension $n + 1$ whose boundary is $\partial W = M \cup_{\partial M} M'$. Moreover, the γ-structure on M extends to a γ-structure on W. Doing some surgery on W if necessary, we may assume that the map $W \to BG(n + 1, \gamma)$ provided by the γ-structure on W is a 3-equivalence (i.e., it induces an isomorphism on homotopy groups π_i for $i < 3$, and a surjection for $i = 3$). The restriction of this map to M is a 2-equivalence (this is a property of the "canonical" $\gamma(M)$-structure of M). It follows that the inclusion $M \subset W$ is a 2-equivalence; this implies that W can be built by attaching handles of dimension ≥ 3 to $M \times [0, 1]$. Reversing the roles of M and M', it follows that W can be constructed from M' by attaching handles of codimension ≥ 3; in particular, M is obtained from M' by a sequence of surgeries in the interior of M' of codimension ≥ 3. Hence the Surgery Theorem 3.1 shows that h extends to a positive scalar curvature metric on M.

We turn to the classification up to concordance. Our first goal is to define the action of $R_{n+1}(\gamma)$ on $\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$. We do so by describing for each $[g] \in \widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$ the map

$$m_{[g]} \colon R_{n+1}(\gamma) \to \widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h) \qquad r \mapsto r \cdot [g].$$

We note that our claim that the action is free and transitive translates into the statement that for each $[g] \in \widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$ the map $m_{[g]}$ is bijective. It seems difficult to describe the map $m_{[g]}$ directly. Instead we construct a map

$$i_{[g]} \colon \widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h) \to R_{n+1}(\gamma),$$

show that it is a bijection, and define $m_{[g]}$ to be the inverse of $i_{[g]}$. To define $i_{[g]}([g'])$, consider the positive scalar curvature metric

$$g \cup (h \times s) \cup g' \quad \text{on} \quad \partial(M \times I) = (M \times \{0\}) \cup (\partial M \times I) \cup (M \times \{1\}),$$

where s is the standard metric on I, and $h \times s$ is the product metric on $\partial M \times I$. We define $i_{[g]}([g'])$ to be the bordism class of $M \times I$ (furnished with its canonical γ-structure) together with the metric $g \cup (h \times s) \cup g'$ on its boundary.

Injectivity of $i_{[g]}$ follows immediately from the existence statement proved above. Surjectivity of $i_{[g]}$ is proved in two steps. First we show that every element of $R_{n+1}(\gamma)$ has a representative of the form (T, q) with $q \in \mathfrak{R}^+(\partial T)$, where T is an $(n + 1)$-thickening of the 2-skeleton of M (i.e.,

$T \subset M \times I$ is a codimension zero submanifold with boundary simply homotopy equivalent to a 2-skeleton of M). To prove this, let (N, k) be a representative of a given element of $R_{n+1}(\gamma)$. After modifying N if necessary by surgeries in the interior, we may assume that the map $N \to BG(\gamma, n+1)$ given by the γ-structure on N is a 3-equivalence. Then using Wall's classification of thickenings in the stable range [Wa], Prop. 5.1, it can be shown that T embeds into the interior of N. Another application of the Improved Surgery Theorem 3.2 then shows that k extends to a positive scalar curvature metric K on $N \setminus \operatorname{int} T$, which implies $[N, k] = [T, K_{|\partial T}]$.

In a second step, the Improved Surgery Theorem 3.2 is used again to argue that the positive scalar curvature metric $g \cup (h \times s) \cup H_{|\partial T}$ which lives on a part of the boundary of $(M \times I) \setminus T$ can always be extended to a positive scalar curvature metric G on $(M \times I) \setminus T$. This shows that $i_{[g]}$ maps $[G_{|M \times \{1\}}] \in \widetilde{\pi}_0 \mathfrak{R}^+(M \operatorname{rel} h)$ to $[T, H_{|\partial T}]$. □

As mentioned above, there is so far no *algebraic* description of the R_n-groups. Worse yet, there is no pair (n, γ), with $n \geq 5$, for which $R_n(\gamma)$ is known. However, in many cases, we can obtain a lower bound for the size of $R_n(\gamma)$ by means of an "index homomorphism"

$$\theta \colon R_n(\gamma) \to KO_n(C_r^*\gamma).$$

Here $C^*\gamma$ is a $\mathbb{Z}/2$-graded C^*-algebra associated to $\gamma = (\pi, w, \widehat{\pi})$.[6] It is defined as an ideal in the group C^*-algebra $C^*\widehat{\pi}$; namely multiplication by the generator k of $\ker(\widehat{\pi} \to \pi)$ is an involution on $C^*\widehat{\pi}$ whose -1-eigenspace is $C^*\gamma$. The $\mathbb{Z}/2$-grading is given by the $\{\pm 1\}$-eigenspaces of the involution $C^*\gamma \to C^*\gamma$ which is the restriction of the involution $C^*\widehat{\pi} \to C^*\widehat{\pi}$ given by $\hat{g} \mapsto (-1)^{\hat{w}(\hat{g})}$ for $\hat{g} \in \widehat{\pi} \subset C^*\widehat{\pi}$, where $\hat{w}$ is the composition of the projection map $\widehat{\pi} \to \pi$ and $w \colon \pi \to \mathbb{Z}/2$. In particular, $C^*\gamma = 0$ if $\widehat{\pi} = \pi$ and $C^*\gamma = C^*\pi$ if $w = 0$ and $\widehat{\pi} = \pi \times \mathbb{Z}/2$.

Remark 5.5 The index homomorphism θ is a generalization of the index $\alpha(N, f) \in KO_n(C^*\pi)$ for n-dimensional closed spin manifolds N equipped with a map $f \colon M \to B\pi$. By remark 5.3, the spin structure and the map f amount to a γ-structure on N, $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$, and hence the closed manifold N represents an element $[N]$ in the bordism group $R_n(\gamma)$. Then

$$\alpha(N, f) = \theta([N]) \in KO_n(C^*\pi) = KO_n(C^*\gamma).$$

In particular, θ generalizes α to non-spin manifolds, and to manifolds with boundary (whose boundary is equipped with a positive scalar curvature metric).

[6] For the meaning of the subscript $_r$, which we henceforth suppress, see the discussion on page 367.

Definition 5.6 To define the index homomorphism θ, it is convenient to describe its range $KO_n(C^*\gamma)$ as equivalence classes of "Kasparov modules" (H, F). Here H is a Hilbert module over the real C^*-algebra $A = C^*\gamma \otimes C\ell_n$ [Bla], §13; i.e., H is a right A-module equipped with a compatible A-valued inner product, which is complete with respect to a norm derived from this inner product. (When $A = \mathbb{R}$ or $\mathbb{C}$, a Hilbert A-module is just a real or complex Hilbert space.) Here F is an A-linear bounded operator on H satisfying certain properties generalizing the main features of elliptic pseudodifferential operators of order 0. (If $A = \mathbb{R}$ or $\mathbb{C}$, these properties imply in particular that F is Fredholm.)

Hence to define θ, we need to describe the pair (H, F) that represents $\theta([N, h])$, where N is manifold with γ-structure and h is a positive scalar curvature metric on ∂N. The Hilbert module H is the space of L^2-sections of a bundle S over the complete manifold without boundary $\widehat{N} = N \cup_{\partial N} \partial N \times [0, \infty)$ obtained by attaching a cylindrical end to N.

The key fact for the construction of S is the existence of a homomorphism from $G(\gamma, n)$ to $O^{ev}(A)$, the group of even orthogonal elements of the C^*-algebra $A = C^*\gamma \otimes C\ell_n$. (An element x of a real C^*-algebra is orthogonal if $x^*x = xx^* = 1$.) This homomorphism is given by

$$\rho\colon G(\gamma, n) = \widehat{\pi} \ltimes_{\mathbb{Z}/2} \operatorname{Spin}(n) \to O^{ev}(A) \qquad [a, b] \mapsto ea \otimes e_1^{w(a)} b.$$

Here $e = (1 - k)/2 \in C^*\widehat{\pi}$ is the unit of the ideal $C^*\gamma \subset C^*\widehat{\pi}$, and $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. We remark that e_1 has order four in $C\ell_n$ (its square is -1); to make the above map well-defined, we decree $w(b) \in \{0, 1\} \subset \mathbb{Z}$ (this gives in fact a homomorphism!).

If $P \to \widehat{N}$ is the principal $G(\gamma, n)$-bundle given by the γ-structure on N extended to $\widehat{N}$, then we define the "spinor" bundle $S_{\widehat{N}}$ by

$$S_{\widehat{N}} \overset{\text{def}}{=} P \times_{G(\gamma,n)} A,$$

where $g \in G(\gamma, n)$ acts on A by left multiplication by $\rho(g)$.

We note that the fibers of $S_{\widehat{N}}$ are right A-modules and are furnished with an A-valued inner product $\langle\ ,\ \rangle$ given by $\langle [p, a], [p, b] \rangle = a^*b \in A$ (we note that two elements in the same fiber of $S_{\widehat{N}}$ can be written in the form $[p, a], [p, b]$ with $p \in P$, $a, b \in A$). Upon integration over $\widehat{N}$, this gives the space $L^2(S_{\widehat{N}})$ of L^2-sections of $S_{\widehat{N}}$ the structure of a Hilbert A-module.

To construct a "Dirac operator" $D_{\widehat{N}}\colon L^2(S_{\widehat{N}}) \to L^2(S_{\widehat{N}})$ it suffices to note that the Levi-Civita connection on $\widehat{N}$ induces a connection on $S_{\widehat{N}}$, and that the γ-structure can be used to make the fiber of $S_{\widehat{N}}$ over a point $x \in \widehat{N}$ a left-module over the Clifford algebra generated by the tangent space T_xM. Then $D_{\widehat{N}}$ is defined by the usual formula (cf. [LaM], Ch. II, formula 5.0).

The operator $D_{\widehat{N}}$ is A-linear, but it is *not* a bounded operator on the Hilbert A-module $H = L^2(S_{\widehat{N}})$ (not even in the classical case $A = \mathbb{C}$). One needs to replace $D_{\widehat{N}}$ by a bounded operator $f(D_{\widehat{N}})$, where f is a suitable real valued function on $\mathbb{R}$, and $f(D_{\widehat{N}})$ is defined by "functional calculus" [Lan]. On a compact manifold the usual choice is $f(x) = x(x^2+1)^{-1/2}$. This doesn't work on the non-compact manifold $\widehat{N}$, since $f(D_{\widehat{N}})^2 - 1$ is not compact, which is one of the requirements for a Kasparov module. However, it is shown in [St4] that if $4c^2$ is a lower bound for the scalar curvature of the metric f on ∂N (and hence a lower bound for the scalar curvature of $\widehat{N}$ outside a compact set), and if $f\colon \mathbb{R} \to \mathbb{R}$ is an odd function with $f(x) = 1$ for $x \geq c$ and $f(x) = -1$ for $x \leq -c$, then $(L^2(S_{\widehat{N}}), f(D_{\widehat{N}}))$ is in fact a Kasparov module. Moreover, its K-theory class $[L^2(S_{\widehat{N}}), f(D_{\widehat{N}})] \in KO(A) = KO_n(C^*\gamma)$ is independent of the choice of f and the Riemannian metric on N extending $h \in \mathfrak{R}^+(\partial N)$.

Bunke's relative index theorem for K-valued indices [Bun], Theorem 1.2, shows furthermore that the K-theory class $[L^2(S_{\widehat{N}}), f(D_{\widehat{N}})]$ depends only on the bordism class of (N, h) in $R_n(\gamma)$; this shows that

$$\theta\colon R_n(\gamma) \to KO_n(C^*\gamma) \qquad [N,h] \mapsto [L^2(S_{\widehat{N}}), f(D_{\widehat{N}})]$$

is a well-defined homomorphism.

We have seen in Section 4 that there are closed spin manifolds with trivial α-invariant, which do not admit a metric of positive scalar curvature. In view of Theorem 5.1 and Remark 5.5 this implies that

$$\theta\colon R_n(\gamma) \to KO_n(C^*\gamma)$$

is not in general injective.

We observe that the target of θ is 8-periodic and that the isomorphism $KO_n(C^*\gamma) \cong KO_{n+8}(C^*\gamma)$ is given by multiplication with the Bott element, the generator of $KO_8(\mathbb{R}) \cong \mathbb{Z}$. Under θ, this corresponds to the map $R_n(\gamma) \to R_{n+8}(\gamma)$ given by Cartesian product with the Bott manifold Bt^8. However, this map is *not* an isomorphism in general; in fact, the above examples represent non-trivial elements of $R_n(\gamma)$, whose product with a suitable power of Bt is trivial.

We note that the groups $R_n(\gamma)$ can be made 8-periodic by "inverting" the Bott manifold; i.e., by defining a "periodic" or "stable" version of the R_n-groups by

$$R_n(\gamma)[\mathrm{Bt}^{-1}] \stackrel{\text{def}}{=} \varinjlim \left(R_n(\gamma) \xrightarrow{\times \mathrm{Bt}} R_{n+8}(\gamma) \xrightarrow{\times \mathrm{Bt}} \dots \right).$$

Then θ factors through a "stable" homomorphism

$$\theta_{st}\colon R_n(\gamma)[\mathrm{Bt}^{-1}] \to KO_n(C^*\gamma).$$

Conjecture 5.7 ([St4]) *The homomorphism θ_{st} is an isomorphism.*

The rest of this section is devoted to discussing the status of this conjecture. First, we look at the case $\ker(\widehat{\pi} \to \pi) = 0$, which corresponds to manifolds whose universal covering is not spin. In this case $C^*\gamma$ and hence also $KO_n(C^*\gamma)$ are trivial. It is a simple observation that also $R_n(\gamma)[\mathrm{Bt}^{-1}]$ vanishes. The argument is the following: Cartesian product gives $R_*(\gamma)$ the structure of a module over the spin bordism ring Ω_*^{spin}; if $\ker(\widehat{\pi} \to \pi)$ is trivial, it is in fact a module over the oriented bordism ring Ω_*^{SO}. In the latter, the Bott manifold is bordant to a linear combination of the quaternionic plane $\mathbb{HP}^2$ and the complex projective space $\mathbb{CP}^4$, which generate $\Omega_8^{SO} \cong \mathbb{Z} \oplus \mathbb{Z}$. Both of these manifolds admit metrics of positive scalar curvature, and hence the product of any element in $R_n(\gamma)$ with Bt^8 is the trivial element in $R_{n+8}(\gamma)$.

Injectivity of θ_{st} is closely related to the Stable Conjecture 4.17. In fact, having the index homomorphism θ at our disposal, we can formulate the following more general conjecture, which agrees with Conjecture 4.17 for spin manifolds.

Conjecture 5.8 *A closed manifold M admits stably a positive scalar curvature metric if and only if $\theta([M])$ vanishes in $KO_n(\gamma(M))$ (here M is equipped with its canonical $\gamma(M)$-structure).*

We note that injectivity of the homomorphism θ_{st} implies Conjecture 5.8, but not vice versa; in fact, Conjecture 5.8 is equivalent to the statement that θ_{st} is injective when restricted to the image of $\Omega_n(\gamma) \to R_n(\gamma)[\mathrm{Bt}^{-1}]$, where $\Omega_n(\gamma)$ is the bordism group of n-dimensional *closed* manifolds with γ-structure. We note that this map factors in the form

$$\Omega_n(\gamma) \to KO_n(\gamma) \overset{\mathrm{def}}{=} (\Omega_n(\gamma)/T_n(\gamma))\,[\mathrm{Bt}^{-1}] \xrightarrow{F} R_n(\gamma)[\mathrm{Bt}^{-1}], \tag{5.1}$$

where $T_n(\gamma) \subset \Omega_n(\gamma)$ consists of the bordism classes represented by total spaces of $\mathbb{HP}^2$-bundles. In the spin case $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$, a (homotopy theoretic) result of Kreck and the second author [KS], Theorem C, implies that $KO_n(\gamma)$ can be identified with the KO-homology of $B\pi$. Composing the forgetful map F and the index map θ we obtain a homomorphism

$$A\colon KO_n(\gamma) \xrightarrow{F} R_n(\gamma)[\mathrm{Bt}^{-1}] \xrightarrow{\theta_{st}} KO_n(C^*\gamma)$$

which agrees with the assembly map in the spin case $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$.

In the authors' opinion, Conjecture 5.8 (assuming as in Theorem 4.20 that a Baum-Connes type map is injective) seems to be within reach; an important ingredient in the proof will be a homotopy theoretic interpretation of $KO_n(\gamma)$ as a 'twisted' KO-homology group of $B\pi$. This is work in progress by Michael Joachim based on his thesis [Joa].

Proving injectivity of θ_{st} seems hard due to an apparent lack of tools; proving injectivity in the simplest case $\gamma = (0, 0, \mathbb{Z}/2)$ is equivalent to giving an affirmative solution to Problem 6.1 discussed in the next section.

Surjectivity of θ_{st} is closely related to the Baum-Connes Conjecture of [BCH]. We recall that for torsion-free groups π this Conjecture claims that the assembly map $A\colon KO_n(B\pi) \to KO_n(C_r^*\pi)$ is an isomorphism. The factorization (5.1) of A shows that surjectivity of A implies that θ_{st} is surjective.

If π is a finite group, then A is in general far from being surjective. Still, Laszlo Feher shows in his thesis [Feh] that θ_{st} is surjective in the "spin case" $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$, provided π is a finite p-group (i.e., a finite group whose order is a power of p for some prime p).

6 Future Directions

In this final section, we mention just a few of the most important open problems concerning positive scalar curvature metrics. These problems appear to be quite hard, but they play such fundamental roles that it seems we will never fully understand the subject of positive scalar curvature until some progress is made on them.

Problem 6.1 *Suppose g is a positive scalar curvature metric on S^n. Then there is an index theoretic obstruction with values in KO_{n+1}, studied in* [Hit]*,* [GL3]*, and in Section 5 above, to extending g to a positive scalar curvature metric on D^{n+1} which is a product metric on a neighborhood of the boundary. Is this the only obstruction? In other words, if the index obstruction vanishes in KO_{n+1}, does g extend to a positive scalar curvature metric on D^{n+1}? If not, is this at least true "stably"* (*after taking a Riemannian product with enough copies of the Bott manifold* Bt^8,[7] *or after taking a Riemannian product with a flat torus of sufficiently high dimension*)*?*

Discussion. This problem is absolutely fundamental, since without its solution, there is no hope for computing the R-groups described in Section 5 above, and thus no hope for a complete concordance classification of positive scalar curvature metrics, even on the very simplest manifolds. At the moment, we know the answer to this question only in the case $n = 2$, where it is easy to see from Theorem 3.4 that every positive scalar curvature metric extends (and the index obstruction always vanishes).

A case which may be exceptional (because of the peculiarities of 4-dimensional smooth topology) is $n = 3$. For this case, Seiberg-Witten

[7]It is worth noting here that it is now known that there is a model for the Bott manifold which admits a Ricci-flat metric [J]. If we use this particular choice, then taking a Riemannian product with Bt^8 does not change the scalar curvature.

theory could conceivably be of use; though it is more likely that Seiberg-Witten theory is only useful in studying the extension problem for more complicated pairs $(M^4, \partial M)$ where $b_2^+(M) > 0$. At the moment, we also do not know anything about the image of the index obstruction in $KO_4 \cong \mathbb{Z}$ when $n = 3$. However, it is proved in [GL3], pp. 130–131, that the obstruction takes all values in $KO_8 \cong \mathbb{Z}$ when $n = 7$.

One possible method of attack in this problem (which could potentially be used in any dimension > 2) is the following. We may as well assume that the scalar curvature of g is a positive constant, say 1. If we extend g any way we like to a metric $\overline{g}$ on D^{n+1} which is a product metric in a neighborhood of $S^n = \partial(D^{n+1})$, then we can try to make a pointwise conformal change in the metric $\overline{g}$, supported away from the boundary, to a metric of positive scalar curvature of the special form $e^f\overline{g}$, f supported on the interior of D. This leads to the study of the "Yamabe equation with Dirichlet boundary conditions." Rewriting the conformal factor e^f as $v^{4/(n-2)}$, we obtain the boundary value problem

$$\begin{aligned} &-\Delta v + \tfrac{n-2}{n-1}\tfrac{\kappa}{4} v = \tfrac{n-2}{n-1}\tfrac{\kappa_1}{4} v^{\frac{n+2}{n-2}} \quad \text{in} \quad \operatorname{int} D^{n+1}, \\ &v > 0 \quad \text{in} \quad \operatorname{int} D^{n+1}, \quad v \equiv 1 \quad \text{near} \quad \partial(D^{n+1}). \end{aligned} \tag{6.1}$$

Here κ is the scalar curvature of the original metric $\overline{g}$, which is 1 on a neighborhood of $\partial(D^{n+1})$ and has unknown behavior in the interior, Δ is the Laplace-Beltrami operator with respect to $\overline{g}$ (with the sign convention for which this operator is non-positive), and κ_1 is the scalar curvature for the new metric (which we want to be everywhere positive).

Note from equation (6.1) that if the "conformal Laplacian," the linear operator

$$L_0 = -\Delta + \frac{n-2}{n-1}\frac{\kappa}{4},$$

has positive spectrum (with Dirichlet boundary conditions, in other words on functions vanishing at the boundary), then it follows that the metric g has an extension with positive scalar curvature. The reasoning, copied in part from [KW1] and [KW2], is as follows. We may assume that the minimum value of κ is $-\kappa_0$, some non-positive number. (Otherwise we're already done.) The eigenfunction φ of L_0 corresponding to the lowest eigenvalue λ cannot change sign, by an application of the maximum principle, so we may assume $\varphi \geq 0$ in $\operatorname{int} D^{n+1}$, and clearly there must be some $\varepsilon > 0$ such that $\varphi > \varepsilon$ on the compact set where $\kappa \leq 0$. Then if $v = 1 + \mu\varphi$, $v > 0$ on D^{n+1}, $v \equiv 1$ on $\partial(D^{n+1})$, and

$$L_0 v = \frac{n-2}{n-1}\frac{\kappa}{4} + \lambda\mu\varphi,$$

which we can arrange to be everywhere positive by taking μ large enough to have $\lambda\mu\varepsilon > \frac{n-2}{n-1}\frac{\kappa_0}{4}$. So v satisfies equation (6.1) except for the condition that v be constant near the boundary. We can achieve this by making a small perturbation in φ near the boundary. (This destroys its being an eigenfunction for L_0, but doesn't change the condition we really need, which is that $L_0(1+\mu\varphi)$ should be everywhere positive.)

A curious feature of equation (6.1), which suggests that the answer to our "stable" question is "yes," is that the operator L_0 bears a remarkable similarity to equation (4.1) for the square of the Dirac operator. (In fact, the lower-order terms $\frac{n-2}{n-1}\frac{\kappa}{4}$ and $\frac{\kappa}{4}$ become the same in the stable limit as $n \to \infty$.) A challenge before us is therefore to figure out how to apply information about the Dirac operator, which acts on spinors, to the study of the scalar equation (6.1). $\square$

Problem 6.2 *Are we missing additional "unstable" obstructions to positive scalar curvature* (*in the closed manifold case, and in dimensions other than* 4) *which do not come from the theory of minimal hypersurfaces?*

Discussion. The existence of counterexamples to Conjectures 4.8 and 4.9, as well as the fact that there are many classes in $H_n(B\pi)$ or $ko_n(B\pi)$ for finite groups π (see Theorem 4.11) which no one has been able to represent by manifolds of positive scalar curvature, suggests that this may be the case. (The minimal hypersurface method of [SY] can only be applied to manifolds which have a covering space with positive first Betti number, clearly a very restrictive condition not applying when the fundamental group is finite.) Conceivably, additional obstructions to positive scalar curvature might come from the study of certain non-linear partial differential equations, for example, from higher-dimensional analogues of Seiberg-Witten theory, that involve coupling of the Dirac operator to something else, or from the study of moduli spaces of solutions to variants of the Yamabe problem. $\square$

Problem 6.3 *Are concordant positive scalar curvature metrics necessarily isotopic?*

Discussion. This question is still wide open. See the comments following Proposition 3.3. In the analogous problem for automorphisms of manifolds, it is known that invariants from algebraic K-theory (especially K_2 and Waldhausen's K-theory of spaces) play a role here. It would be very interesting to see if any similar phenomena occur in the theory of positive scalar curvature metrics. $\square$

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A survey of 4-manifolds through the eyes of surgery

Robion C. Kirby and Laurence R. Taylor

Table of Contents:

§0. Review of Surgery Theory.

Surgery theory is a method for constructing manifolds satisfying a given collection of homotopy conditions. It is usually combined with the s–cobordism theorem which constructs homeomorphisms or diffeomorphisms between two similar looking manifolds. Building on work of Sullivan, Wall applied these two techniques to the problem of computing structure sets. While this is not the only use of surgery theory, it is the aspect on which we will concentrate in this survey. In dimension 4, there are two versions, one in which one builds topological manifolds and homeomorphisms and the second in which one builds smooth manifolds and diffeomorphisms. These two versions are dramatically different. Freedman has shown that the topological case resembles the higher dimensional theory rather closely. Donaldson's work showed that the smooth case differs wildly from what the high dimensional theory would predict. Surgery theory requires calculations in homotopy theory and in low dimensions these calculations become much more manageable. In sections 0 and 1, we review the general theory and describe the general results in dimensions 3 and 4. In sections 2 through 6, we describe precisely what the high dimensional theory predicts. Finally, we describe the current state of affairs for the two versions in sections 7 and 8.

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To begin, let $(X, \partial X)$ be a simple, n–dimensional Poincaré space whose boundary may be empty. In particular, X is homotopy equivalent to a finite CW complex which satisfies Poincaré duality for any coefficients, with a twist in the non–orientable case, and simple means that there is a chain map

$$[X, \partial X]\cap: Hom_{\mathbf{Z}[\pi_1(X)]}(C_*(X), \mathbf{Z}[\pi_1(X)]) \to C_{n-*}(X)$$

which is a simple isomorphism between based chain complexes, [85]. This is the homotopy analogue of a manifold. Let CAT stand for either TOP, the topological category, or DIFF, the differential category. There is also the category of PL–manifolds, but it follows from the work of Cerf, [18], that in dimension 4 PL is equivalent to DIFF, so we will rarely discuss PL here. Fix a CAT–manifold L^{n-1} without boundary and a simple homotopy equivalence $h: L \to \partial X$.

Structure Sets: Define the set $\mathcal{S}^{CAT}(X; \text{rel } h)$ as the set of all simple homotopy equivalences of pairs, $f: (M, \partial M) \to (X, \partial X)$, where $(M, \partial M)$ is a CAT–manifold, and for which there exists a CAT–equivalence $g: L \to \partial M$ such that the composition $L \to \partial M \to \partial X$ is homotopic to h; two such, (M_i, f_i, g_i) $i = 0, 1$, are deemed equal if there exists a CAT–equivalence $F: (M_0, \partial M_0) \to (M_1, \partial M_1)$ so that $f_1 \circ F$ is homotopic, as a map of pairs, to f_0, and $F|_\partial \circ g_0$ is homotopic to g_1. In diagrams,

$$\begin{tikzcd} L \arrow[r, "g"] \arrow[dr, "h"'] & \partial M \arrow[d, "f|_{\partial X}"] \\ & \partial X \end{tikzcd} \qquad\qquad \begin{tikzcd} M_0 \arrow[r, "f_0"] \arrow[d, "F"'] & X \\ M_1 \arrow[ur, "f_1"'] & \end{tikzcd}$$

homotopy commute.

Remark: One can use the homotopy extension theorem to tighten up the definition: one can restrict to manifolds M with $\partial M = L$ and with maps f such that $f|_\partial = h$; $F|_\partial$ can be required to be the identity and the homotopy between $f_1 \circ F$ and f_0 can be required to be constant on L. Finally, base points may be selected in each component of M, X, ∂M and ∂X and all the maps and homotopies may be assumed to preserve the base points. This is a useful remark in identifying various fundamental groups precisely rather than just up to inner automorphism.

The questions now are whether the set $\mathcal{S}^{CAT}(X; \text{rel } h)$ is non–empty (existence) and if non–empty, how many elements does it have (uniqueness). The only 1 and 2 dimensional Poincaré spaces are simple homotopy equivalent to manifolds, [26], [27], and this is conjecturally true in dimension three, [81]. In general, the Borel conjecture asserts that this is true for aspherical Poincaré spaces in all dimensions (see the discussion of Problem 5.29 in [47] and the articles in [28]).

There are bundle–theoretic obstructions to $\mathcal{S}^{CAT}(X; \mathrm{rel}\ h)$ being non–empty. Every Poincaré space has a stable Spivak normal fibration, [75], which is given by a map $\nu_X: X \to BG$. This is the homotopy analogue of the stable normal bundle for a manifold. The space BG can be thought of as the classifying space for stable spherical fibrations, or as the limit of the classifying spaces of $G(m)$, the space of homotopy automorphisms of S^{m-1}. There is a map $BCAT \to BG$ and a necessary condition for $\mathcal{S}^{CAT}(X; \mathrm{rel}\ h)$ to be non–empty is that ν_X lift to $BCAT$. Given a homotopy equivalence between a CAT–manifold and a Poincaré space, X, Sullivan, [77], constructs a *homotopy differential*, a specific lift of ν_X. The lift to $BCAT$ gives a stable CAT bundle η over X and the lift gives a specific fibre homotopy equivalence between the associated sphere bundle to η and the Spivak normal fibration ν_X.

With data as above, the Sullivan homotopy differential gives an explicit lift of $\nu_{\partial X}$ to $BCAT$: a second application of this yoga gives a map

$$N: \mathcal{S}^{CAT}(X; \mathrm{rel}\ h) \to \mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$$

where $\mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$ is the set of homotopy classes of lifts of ν_X to $BCAT$ which restrict to our given lift over $\nu_{\partial X}$.

Boardman and Vogt, [6], prove that the spaces $BCAT$ and BG are infinite loop spaces and that the maps $BCAT \to BG$ are infinite loop maps. It follows that there is a sequence of homotopy fibrations, extending infinitely in both directions,

$$\cdots \to CAT \to G \to G/CAT \to BCAT \to BG \to B(G/CAT) \to \cdots$$

The Spivak normal fibration is a map $\nu_X: X \to BG$, and the Sullivan differential on the boundary gives an explicit null–homotopy of $\nu_X|_{\partial X}$ in $B(G/CAT)$ and so defines a map $b: X/\partial X \to B(G/CAT)$.

The next result follows from standard homotopy theory considerations:

Theorem 1. *$\mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$ is non–empty if and only if $b: X/\partial X \to B(G/CAT)$ is null homotopic. If $\mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$ is non–empty, the abelian group $[\,X/\partial X, G/CAT\,]$ acts simply–transitively on it.*

Remark: If X is already a CAT–manifold, $\mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$ has an obvious choice of base point, namely the normal bundle of X.

Given a point $x \in \mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$ and an element $\eta \in [\,X/\partial X, G/CAT\,]$, let $\eta \bullet x \in \mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$ denote the result of the action.

CAT–transversality allows an interpretation of $\mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$ as a normal bordism theory. We can translate this into a more geometric language where we assume for simplicity that $\partial X = \emptyset$. Choose a simplicial subcomplex of a high dimensional sphere, S^N, which is simple homotopy equivalent to X. Let $W, \partial W$ denote a regular neighborhood. If the map $\partial W \to X$ is made into a fibration then the result is a spherical fibration with fibre S^{N-n-1}, which is the Spivak normal fibration; it corresponds to a classifying map $X \to BG$. Note that by collapsing the complement of W to a point, we get a map from S^N to the Thom space of the Spivak normal fibration. A lift from BG to $BCAT$ provides a fibre homotopy equivalence from the Spivak normal fibration to the CAT bundle over X, and this extends to Thom spaces. Thus a lift from BG to $BCAT$ gives by composition a map from S^N to the Thom space of the CAT bundle; making this map transverse to the 0–section provides a manifold, M^n, and a degree one map, $M \to X$ covered by a CAT bundle map from the stable normal bundle for M to the given bundle over X. Different choices change the data by a normal bordism. Summarizing, $\mathcal{L}^{CAT}(X)$ can be interpreted as bordism classes of degree–one normal maps, that is, degree one maps $f\colon M \to X$ covered by a bundle map from the stable normal bundle of M to some CAT bundle over X.

Given a normal map $M^n \xrightarrow{h} X$, one can try to surger M so that h becomes a simple homotopy equivalence. This allows one to define a surgery obstruction map in general,

$$\theta\colon \mathcal{L}^{CAT}(X; \mathrm{rel}\ h) \to \overset{s}{L}_n(\mathbf{Z}[\pi_1(X)], w_1(X))$$

where $w_1(X)\colon \pi_1(X) \to \pm 1$ is the first Stiefel–Whitney class of the Poincaré space X and $\overset{s}{L}_n$ is the Wall group as defined in [85]. The Wall groups depend only on the group and the first Stiefel–Whitney class and are 4–fold periodic.

In the simply connected case, the only obstruction in dimensions congruent to 0 mod 4 is the difference in the signatures of M and X, so $\overset{s}{L}_0(\mathbf{Z})$ is $\mathbf{Z}$ and the map θ is given by $(\sigma(M) - \sigma(X))/8$. In dimensions congruent to 2 mod 4, do surgery to the middle dimension, put a quadratic enhancement on the kernel in homology and take the Arf invariant to get an invariant in $\overset{s}{L}_2(\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$. The simplest example is the degree one map from T^2 to S^2 with stable normal map given by framing the stable normal bundle to S^2 and taking the "Lie framing" of the stable normal bundle to T^2 defined as follows: identify a normal bundle to T^2 with the product of two stable normal bundles to S^1 and frame each of these with the framing that does not extend over D^2. In odd dimensions, the obstruction is $0 = \overset{s}{L}_1(\mathbf{Z}) = \overset{s}{L}_3(\mathbf{Z})$.

If $\mathcal{S}^{CAT}(X; \mathrm{rel}\ h) \neq \emptyset$, the composite

$$\mathcal{S}^{CAT}(X; \mathrm{rel}\ h) \xrightarrow{N} \mathcal{L}^{CAT}(X; \mathrm{rel}\ h) \xrightarrow{\theta} \overset{s}{L}_n(\mathbf{Z}[\pi_1(X)], w_1(X))$$

sends every element in the structure set to the zero element in the Wall group.

Given $x \in \mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$, let

$$\theta_x \colon [\, X/\partial X, G/CAT\,] \to \overset{s}{L}_n(\mathbf{Z}[\pi_1(X)], w_1(X))$$

be defined by $\theta_x(\eta) = \theta(\eta \bullet x)$. Thus far, there are no dimension restrictions, but one of Wall's fundamental results, [85, Thm 10.3 and 10.8], is

Theorem 2. *If $n \geq 5$ and if $x \in \mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$, the following sequence is exact*

$$(3) \quad \mathcal{S}^{CAT}(X; \mathrm{rel}\ h) \xrightarrow{N_x} [\, X/\partial X, G/CAT\,] \xrightarrow{\theta_x} \overset{s}{L}_n(\mathbf{Z}[\pi_1(X)], w_1(X))$$

in the sense that $\theta_x^{-1}(0)$ equals the image of N_x. If $\mathcal{S}^{CAT}(X; \mathrm{rel}\ h) \neq \emptyset$, there is an action of a Wall group on it:

$$\overset{s}{L}_{n+1}(\mathbf{Z}[\pi_1(X)], w_1(X)) \times \mathcal{S}^{CAT}(X; \mathrm{rel}\ h) \to \mathcal{S}^{CAT}(X; \mathrm{rel}\ h)$$

and N_x is injective on the orbit space. The isotropy subgroups of this action are given by "backing–up" sequence (3), *being careful with base point. Specifically, if $f \colon M \to X$ is in $\mathcal{S}^{CAT}(X; \mathrm{rel}\ h)$, let $f \times 1_{[0,1]}$ be the evident map $M \times [0,1] \to X \times [0,1]$ with $\partial_{f \times 1_{[0,1]}}$ being the evident homeomorphism on the boundary: let $N(f \times 1_{[0,1]}) \in \mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$ be our choice of base point, denoted y below. The isotropy subgroup of $f \colon M \to X$ is the image of θ_y in the version of* (3)

$$\begin{aligned} \mathcal{S}^{CAT}(X \times [0,1]; \mathrm{rel}\ \partial_{f \times 1_{[0,1]}}) &\xrightarrow{N_y} [\, \Sigma(X/\partial X), G/CAT\,] \\ &\xrightarrow{\theta_y} \overset{s}{L}_{n+1}(\mathbf{Z}[\pi_1(X)], w_1(X)) \ . \end{aligned}$$

§1. The Low Dimensional Results.

If $n < 5$, sets $\bar{\mathcal{S}}^{CAT}(X; \mathrm{rel}\ h)$ are defined below so that Theorem 2 remains true if the sets $\bar{\mathcal{S}}^{CAT}$ are used instead of the sets $\mathcal{S}^{CAT}$. By construction there will be a map $\psi_{CAT} \colon \mathcal{S}^{CAT}(X; \mathrm{rel}\ h) \to \bar{\mathcal{S}}^{CAT}(X; \mathrm{rel}\ h)$ and the failure of surgery in low dimensions is the failure of ψ_{CAT} to be a bijection.

It is a fortuitous combination of calculations of Wall groups, the classification of manifolds and the result that 2–dimensional Poincaré spaces have the homotopy type of manifolds, [26], [27], that Theorem 2 holds as stated for $n = 1$ and 2. After this remark, we restrict attention to the three and four dimensional cases.

In dimension 3, for closed manifolds, it is conjectured that $\mathcal{S}^{CAT}(M^3)$ is a point, [47, 3.1Ω]. Computationally, $\bar{\mathcal{S}}^{DIFF}(S^3)$ is two points, S^3 and the Poincaré sphere; however, $\bar{\mathcal{S}}^{TOP}(S^3)$ is still one point, because S^3 and the Poincaré sphere are topologically homology bordant.

In dimension 4, Freedman's work shows ψ_{TOP} is a bijection for "good" fundamental groups; Donaldson's work shows ψ_{DIFF} is not bijective for many 4–manifolds. These points are discussed below in sections 7 and 8.

A mantra of four–dimensional topology is that "things work after adding $S^2 \times S^2$'s": a mantra of three–dimensional topology is that "surgery works up to homology equivalence". The results below lend some precision to these statements.

Let us assume given $(X^3, \partial X)$ with a CAT–homotopy structure $h: L^2 \to \partial X$. Since every 2–dimensional TOP–manifold has a unique smooth structure, it is no loss of generality to assume L is smooth. Define $\bar{\mathcal{S}}^{CAT}(X; \mathrm{rel}\, h)$ as a set of objects modulo an equivalence relation. Each object is a pair consisting of a CAT–manifold, M, and a map, $f: M^3 \to X$, where M^3 is smooth and f induces an isomorphism in homology with coefficients in $\mathbf{Z}[\pi_1(X)]$. Any such map has a Whitehead torsion in $Wh(\mathbf{Z}[\pi_1(X)])$ and we further require that this torsion be 0. Two such objects, M_i, f_i $i = 0$, 1, are deemed equivalent if and only if there exists a normal bordism which will consist of a CAT–manifold W^4 with $\partial W = M_0 \perp\!\!\!\perp M_1$, a map $F: W \to X \times [0, 1]$ extending f_0 and f_1, a CAT–bundle ζ over $X \times [0, 1]$, and a bundle map covering F between the normal bundle for W and ζ. In such a case, there is a well–defined surgery obstruction in $L_4^s(\mathbf{Z}[\pi_1(X)], w_1(X))$ which we further require to be 0. In case TOP–surgery works in dimension 4 for $\pi_1(X)$, this condition is equivalent to the following more geometric statement: if $CAT = TOP$, the normal bordism can be replaced by a topological s–cobordism; if $CAT = DIFF$, the normal bordism can be replaced by a topological s–cobordism with vanishing stable triangulation obstruction.

We now turn to the 4–dimensional case. Let us assume given $(X^4, \partial X)$ with a CAT–homotopy structure $h: L^3 \to \partial X$. Since every 3–dimensional TOP–manifold has a unique smooth structure, it is no loss of generality to assume L is smooth. Following Wall, write X as a 3–dimensional complex, $\overset{\circ}{X} \subset X$, union a single 4–cell. For any integer $r > 0$, one can form the connected sum, $X\#rS^2 \times S^2$ by removing a 4–ball in the interior of the top 4–cell. There are maps $p_X\colon X\#rS^2 \times S^2/\partial X \to X/\partial X$. Define ${}^r\widetilde{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h) = \{f \in \mathcal{S}^{CAT}(X\#rS^2 \times S^2;\mathrm{rel}\ h) \mid N(f) \in \mathrm{Im}\ p_X^*\}$ and for uniformity, let ${}^0\widetilde{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h) = \mathcal{S}^{CAT}(X;\mathrm{rel}\ h)$. There are evident maps ${}^r\widetilde{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h) \to {}^{r+1}\widetilde{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h)$, so define $\bar{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h)$ to be the limit. One can define ${}^r\widetilde{\mathcal{L}}^{CAT}(X;\mathrm{rel}\ h)$ similarly, but the maps ${}^r\widetilde{\mathcal{L}}^{CAT}(X;\mathrm{rel}\ h) \to {}^{r+1}\widetilde{\mathcal{L}}^{CAT}(X;\mathrm{rel}\ h)$ are isomorphisms. We call $\bar{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h)$ the *stable structure set*.

Theorem 4. *If $n = 3$ or 4, and if $x \in \mathcal{L}^{CAT}(X;\mathrm{rel}\ h)$, the following sequence is exact*

$$\bar{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h) \xrightarrow{N_x} [\,X/\partial X, G/CAT\,] \xrightarrow{\theta_x} \overset{s}{L}_n(\mathbf{Z}[\pi_1(X)], w_1(X)) \ .$$

If $\bar{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h) \neq \emptyset$, $\overset{s}{L}_{n+1}(\mathbf{Z}[\pi_1(X)], w_1(X))$ acts on it and N_x is injective on the orbit space. The isotropy subgroups are given as in Theorem 2. Finally, there is a map $\psi_{CAT}\colon \mathcal{S}^{CAT}(X;\mathrm{rel}\ h) \to \bar{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h)$ (and, if $n = 4$, $\psi^r_{CAT}\colon {}^r\widetilde{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h) \to \bar{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h)$).

Addendum. *If $n = 4$ and if $f_i\colon (M_i, L) \to (X, \partial X)$, $i = 0, 1$ are such that $\psi(f_0) = \psi(f_1)$, there exists an s–cobordism, W, from M_0 to M_1 which is a product over L, together with a map of pairs $F\colon (W, \partial W) \to (X \times [0,1], \partial(X \times [0,1]))$ which extends f_0 and f_1 and is $h \times [0,1]$ on $L \times [0,1] \subset \partial W$.*

The calculations above for the smooth and the topological stable structure sets can be compared using the map $G/O \to G/TOP$. A second way to compare them comes from the work of Kirby and Siebenmann, [48], in high dimensions and proceeds as follows. There is a function $k\colon \mathcal{S}^{TOP}(X;\mathrm{rel}\ h) \to [\,X/\partial X, B(TOP/O)\,]$ which sends $f\colon M \to X$ to the smoothing obstruction for M. The group $[\,X/\partial X, TOP/O\,]$ acts on the smooth structure set: an element $\eta \in [\,X/\partial X, TOP/O\,]$ corresponds to a homeomorphism $\hat{\eta}\colon M' \to M$, and let η act on f to yield

$$\eta \bullet f\colon M' \xrightarrow{\hat{\eta}} M \xrightarrow{f} X \ .$$

The evident relation $\bar{\eta} \bullet N(f) = N(\eta \bullet f)$ holds, where $\bar{\eta}$ denotes the composite $X/\partial X \xrightarrow{\eta} TOP/O \to G/O$. In dimension 4, there are similar results on the stable structure sets thanks to the work of Lashof and Shaneson, [56]. In this case $[\,X/\partial X, B(TOP/O)\,] = H^4(X, \partial X; \mathbf{Z}/2\mathbf{Z})$ and $[\,X/\partial X, TOP/O\,] = H^3(X, \partial X; \mathbf{Z}/2\mathbf{Z})$.

Theorem 5. *If $n = 4$, the image of the forgetful map $\bar{\mathcal{S}}^{DIFF}(X; \mathrm{rel}\ h) \to \bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h)$ is $k^{-1}(0)$ ($k: \bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h) \to H^4(X, \partial X; \mathbf{Z}/2\mathbf{Z})$). The group $H^3(X, \partial X; \mathbf{Z}/2\mathbf{Z})$ acts on $\bar{\mathcal{S}}^{DIFF}(X; \mathrm{rel}\ h)$ and the forgetful map induces a bijection between the orbit space and $k^{-1}(0)$.*

Remark: In dimension 4, there is another version of "stably CAT equivalent" that appears sometimes in the literature. One might say M_1 and M_2 were "stably CAT equivalent" if $M_1 \# r S^2 \times S^2$ was CAT equivalent to $M_2 \# r S^2 \times S^2$. We will rarely discuss this concept, but will say M_1 and M_2 are *weakly, stably CAT equivalent* when we do. We say M_1 and M_2 are *stably CAT equivalent* if there is a CAT equivalence $h: M_1 \# r S^2 \times S^2 \to M_2 \# r S^2 \times S^2$ and a homotopy equivalence, $f: M_1 \to M_2$, such that $f \# r 1_{S^2 \times S^2}$ is homotopic to h. As an indication of the difference, consider that the Wall group acts on our stable structure set (non–trivially in some case as we shall see below), whereas the top and bottom of a normal bordism are always weakly, stably CAT equivalent since such a bordism has a handle decomposition with only 2 and 3 handles. It is also easy to give examples of weakly, stably TOP equivalent, simply connected manifolds which are not even homotopy equivalent since there are many distinct definite forms which become isomorphic after adding a single hyperbolic.

Kreck observes that the question of whether two manifolds are weakly, stably CAT equivalent is a bordism question, [50]. More precisely, fix a map $h: M \to K(\pi_1(M), 1)$ inducing an isomorphism on π_1 and use the normal bundle to get a map $h \times \nu: M \to K(\pi_1(M), 1) \times BCAT$. There exists a unique class $\omega_1 \in H^1\big(K(\pi_1(M), 1); \mathbf{Z}/2\mathbf{Z}\big)$ such that $h^*(\omega_1)$ is the first Stiefel–Whitney class of M. Define $E_1(\pi_1(M), \omega_1)$ to be the homotopy fibre of the map $K(\pi_1(M), 1) \times BCAT \xrightarrow{\omega_1 \times 1 + 1 \times w_1} K(\mathbf{Z}/2\mathbf{Z}, 1)$ and note $h \times \nu$ factors through a map $h_1: M \to E_1(\pi_1(M), \omega_1)$. The map h_1 induces an isomorphism on π_1: it induces an epimorphism on π_2 if and only if the universal cover of M is not *Spin*. If the universal cover is *Spin*, there exists a unique class $\omega_2 \in H^2\big(K(\pi_1(M), 1); \mathbf{Z}/2\mathbf{Z}\big)$ such that $h^*(\omega_2)$ is the second Stiefel–Whitney class of M. Define $E_2(\pi_1(M), \omega_1, \omega_2)$ as the homotopy fibre of the map

$$K(\pi_1(M), 1) \times BCAT \xrightarrow{(\omega_1 \times 1 + 1 \times w_1) \times (\omega_2 \times 1 + 1 \times w_2)} K(\mathbf{Z}/2\mathbf{Z}, 1) \times K(\mathbf{Z}/2\mathbf{Z}, 2).$$

Then h factors through a map $h_2: M \to E_2(\pi_1(M), \omega_1, \omega_2)$ which induces an isomorphism on π_1 and an epimorphism on π_2. Over E_i, $i = 1$ or 2, there is a stable bundle coming from the map $E_i \to BCAT$. One can form Thom complexes and take stable homotopy to get bordism groups,

$\Omega_4^{CAT}(\pi_1(M),\omega_1)$ and $\Omega_4^{CAT}(\pi_1(M),\omega_1,\omega_2)$: the pair M and h as above determine an element $[M,h]\in\Omega_4^{CAT}(\pi_1(M),\omega_1,\omega_2)$ or $[M,h]\in\Omega_4^{CAT}(\pi_1(M),\omega_1)$ (depending on whether the universal cover of M is *Spin* or not). For a fixed M, the homotopy classes of maps h correspond bijectively to $\mathrm{Out}\bigl(\pi_1(M)\bigr)$, the outer automorphism group of $\pi_1(M)$. Define two subgroups,

$$\mathrm{Out}\bigl(\pi_1(M),\omega_1,\omega_2\bigr)=\bigl\{h\in\mathrm{Out}\bigl(\pi_1(M)\bigr)\;\big|\;h^*(\omega_1)=\omega_1 \text{ and } h^*(\omega_2)=\omega_2\bigr\}$$

and

$$\mathrm{Out}\bigl(\pi_1(M),\omega_1\bigr)=\bigl\{h\in\mathrm{Out}\bigl(\pi_1(M)\bigr)\;\big|\;h^*(\omega_1)=\omega_1\bigr\}.$$

These subgroups act on the bordism groups and M determines a well–defined element in

$$\Omega_4^{CAT}(\pi_1(M),\omega_1,\omega_2)/\mathrm{Out}\bigl(\pi_1(M),\omega_1,\omega_2\bigr)$$

or

$$\Omega_4^{CAT}(\pi_1(M),\omega_1)/\mathrm{Out}\bigl(\pi_1(M),\omega_1\bigr)$$

depending on whether the universal cover of M is *Spin* or not.

Two manifolds M_1 and M_2 are weakly, stably CAT equivalent if and only if there exists a choice of ω_1 (and ω_2 if the universal covers are *Spin*) such that M_1 and M_2 represent the same element in

$$\Omega_4^{CAT}(\pi_1(M),\omega_1)/\mathrm{Out}\bigl(\pi_1(M),\omega_1\bigr),$$

or, if the universal covers are *Spin*, in

$$\Omega_4^{CAT}(\pi_1(M),\omega_1,\omega_2)/\mathrm{Out}\bigl(\pi_1(M),\omega_1,\omega_2\bigr).$$

The proof is to construct a bordism W^5 between M_1 and M_2 with a map $H\colon W\to E_i$, $i=1$ or 2 as appropriate. Then do surgery to make H as connected as possible and then calculate that this new bordism can be built from 2 and 3 handles.

These bordism groups depend only on the algebraic data, but their calculation can be difficult. One easy case is when M is orientable ($\omega_1=0$) and the universal cover is not *Spin*. Then $\Omega_4^{CAT}(\pi_1(M),\omega_1)$ is just the ordinary oriented CAT bordism group of $K(\pi_1(M),1)$ which is just $H_4\bigl(K(\pi_1(M),1);\mathbf{Z}\bigr)\oplus\mathbf{Z}$ in the smooth case and $H_4\bigl(K(\pi_1(M),1);\mathbf{Z}\bigr)\oplus\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}$ in the topological case: the $\mathbf{Z}$ is given by the signature of

M; the $\mathbf{Z}/2\mathbf{Z}$ is given by the Kirby–Siebenmann invariant; and the element in $H_4\big(K(\pi_1(M),1);\mathbf{Z}\big)$ is just $h_*\big([M\,]\big)$. The action by $\mathrm{Out}\big(\pi_1(M)\big)$ is by the identity on the $\mathbf{Z}$ and the $\mathbf{Z}/2\mathbf{Z}$ and is the usual action on $H_4\big(K(\pi_1(M),1);\mathbf{Z}\big)$.

The proofs of Theorems 4 and 5 are relatively straightforward given Wall's work in high dimensions. In the 3–dimensional case, one simply observes that there are no embedding issues, but because circles now have codimension two, we no longer have complete control over the fundamental group. In the smooth case in dimension 4, Wall, [83], [84], Cappell and Shaneson, [10], and Lawson, [58], prove the necessary results and in the topological case one need only observe that Freedman and Quinn, [32], supply the tools needed to mimic the smooth proofs.

§2. Calculation of Normal Maps.

Given the structure of the surgery exact sequence, we need to be able to compute the space of homotopy classes of maps from complexes into G/TOP and G/O. Standard homotopy theory tells us how to do this in principle.

The first step in this program is to calculate the homotopy groups of these spaces. The surgery sequence helps in this analysis. The L-groups of the trivial group are $\mathbf{Z}$, 0, $\mathbf{Z}/2\mathbf{Z}$, 0.

Using the "exact sequence " (3), the Poincaré conjecture and the L-groups show that $\pi_i(G/TOP) = \mathbf{Z},\ 0,\ \mathbf{Z}/2\mathbf{Z},\ 0,\ i \equiv 0,\ 1,\ 2,\ 3 \pmod 4$. Generators can be constructed as well. In dimensions congruent to 0 mod 4, follow Milnor, [64], and plumb the E_8 form. The boundary is a topological sphere except in dimension 4 where it is the Poincaré homology sphere. Cone the boundary or use Freedman, [30], to complete to a closed manifold, denoted E_8, and construct a normal degree one map to the sphere. In dimensions congruent to 2 mod 4, follow a similar process. Plumb two tangent bundles to S^{2k+1}. The boundary is a homotopy sphere. Cone the boundary to get a PL manifold, M^{4k+2}, and a degree one map $f\colon M \to S^{4k+2}$. This map can be made into a normal map so as to have non–zero surgery obstruction (already done in dimension 2 above as a map $T^2 \to S^2$). See e.g. Browder, [8], §V.

One can do a similar analysis on $\pi_i(G/O)$ except now the Poincaré conjecture fails in high dimension. Still, $\pi_i(G/O) = \pi_i(G/TOP)$ for $i < 8$, although the map $\pi_4(G/O) \to \pi_4(G/TOP)$ is multiplication by 2 (Rochlin's theorem, [71], or [45]). Purists will quibble that the results used above require the calculations they are quoted to justify, but the quoted results are correct and proved ten years before Freedman's work by Sullivan, [77], Kirby and Siebenmann, [48].

The first two stages of a Postnikov decomposition for G/CAT are

$$K(\mathbf{Z},4) \to G/CAT \to K(\mathbf{Z}/2\mathbf{Z},2)\ .$$

Rochlin's theorem shows that normal maps over S^4 have surgery obstruction divisible by 16; on the other hand, there is a normal map $M = CP^2\#8\overline{CP}^2 \to CP^2$, defined as follows. The cohomology class $(3,1,\cdots,1)$ determines a degree one map, $f\colon M \to CP^2$. Note 7 times the Hopf bundle pulls back via f to the normal bundle of M. As Sullivan observes, this means the first k–invariant of G/O is non–zero. This k–invariant lives in $H^5(K(\mathbf{Z}/2\mathbf{Z},2);\mathbf{Z}) = \mathbf{Z}/4\mathbf{Z}$, [7]; G/O is an H–space so its k invariants are primitive [1]. In $H^5(K(\mathbf{Z}/2\mathbf{Z},2);\mathbf{Z})$ only 0 and 2 are primitives, [7]. Hence the first k–invariant for G/O is 2, which as a cohomology operation is δSq^2, the integral Bockstein of the second Steenrod square. Freedman's construction of the E_8 manifold shows that the first k–invariant of G/TOP is trivial. (Again, Kirby and Siebenmann had already shown this result, but the above makes a nice justification for the result.)

The next k invariant for both G/O and G/TOP is trivial, so in particular there are maps

$$\begin{aligned} G/TOP &\to K(\mathbf{Z}/2\mathbf{Z},2) \times \phantom{{}_{\delta Sq^2}} K(\mathbf{Z},4) \\ G/O &\to K(\mathbf{Z}/2\mathbf{Z},2) \times_{\delta Sq^2} K(\mathbf{Z},4) \end{aligned} \tag{6}$$

which are 5–connected.

The first k–invariant of $\Omega G/O$ is the composition $\Omega(\delta)\circ\Omega(Sq^2)$ and $\Omega(Sq^2) = 0$. This remark is useful in computing $[\,\Sigma Y, G/O\,] = [\,Y, \Omega(G/O)\,]$.

Having computed the first k–invariants for these spaces, we want to extract explicit calculations of the groups $[\,Y, G/CAT\,]$ for Y a 4–complex as well as a calculation of the map induced by the map $G/O \to G/TOP$. There is a class $k \in H^4(BTOP;\mathbf{Z}/2\mathbf{Z})$, the stable triangulation obstruction, which restricts to a class, $k \in H^4(G/TOP;\mathbf{Z}/2\mathbf{Z})$. This class certainly vanishes when restricted to G/O and we wish to identify it in $H^4(G/TOP;\mathbf{Z}/2\mathbf{Z})$. Let $f\colon M^4 \to N^4$ be a normal map. By Theorem 1, f corresponds to a map $\hat f\colon N \to G/TOP$ and the composite $N \xrightarrow{\hat f} G/TOP \to BTOP$ determines a bundle ζ over N such that $\nu_N \oplus \zeta$ pulls back via

[1] A primitive in the cohomology of an H–space, $m\colon Y\times Y \to Y$, is a cohomology class y such that $m^*(y) = 1\times y + y\times 1$.

f^* to ν_M. Then $k(\nu_M) = k(\zeta) + k(\nu_N)$ so $\hat{f}^*(k)$ is the difference of the triangulation obstructions for M and N. Now $H^4(G/TOP; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ generated by ι_2^2 and $(\iota_4)_2$. Here $\iota_2 \in H^2(K(\mathbf{Z}/2\mathbf{Z}, 2); \mathbf{Z}/2\mathbf{Z})$ and $\iota_4 \in H^4(K(\mathbf{Z}, 4); \mathbf{Z})$ are generators and $(\iota_4)_2$ denotes the generator of $H^4(K(\mathbf{Z}, 4); \mathbf{Z}/2\mathbf{Z})$. By examining the normal maps, $\widehat{CP}^2 \to CP^2$ (where $\widehat{CP}^2$ is Freedman's Chern manifold, [30]) and $E_8 \to S^4$ one sees

$$k = \iota_2^2 + (\iota_4)_2 \ .$$

One can further see that if $\hat{f}(k) = 0$, then the map $N \to G/TOP$ factors through a map $N \to G/O$.

Let X be a connected 4–dimensional Poincaré space. The maps in (6) induce natural equivalences of abelian groups,

$$[\,X/\partial X, G/TOP\,] = H^2(X, \partial X; \mathbf{Z}/2\mathbf{Z}) \oplus H^4(X, \partial X; \mathbf{Z})$$
$$[\,\Sigma(X/\partial X), G/TOP\,] = H^1(X, \partial X; \mathbf{Z}/2\mathbf{Z}) \oplus H^3(X, \partial X; \mathbf{Z})$$

The calculations for G/O look similar:

$$0 \to H^4(X, \partial X; \mathbf{Z}) \to [\,X/\partial X, G/O\,] \to H^2(X, \partial X; \mathbf{Z}/2\mathbf{Z}) \to 0$$
$$[\,\Sigma(X/\partial X), G/O\,] = H^1(X, \partial X; \mathbf{Z}/2\mathbf{Z}) \oplus H^3(X, \partial X; \mathbf{Z}) \ .$$

In general, the exact sequence for G/O is not split. To describe the result, let $\mathcal{H}^2(X, \partial X)$ denote the kernel of the homomorphism given by the cup square, $H^2(X, \partial X; \mathbf{Z}/2\mathbf{Z}) \to H^4(X, \partial X; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$. Note $\mathcal{H}^2(X, \partial X) = H^2(X, \partial X; \mathbf{Z}/2\mathbf{Z})$ if and only if $v_2(X) = 0$ where v_2 denotes the second Wu class of the tangent bundle.

Lemma 7. *For X a connected 4–dimensional Poincaré space with boundary,*
(8).
$(*)$ $[\,X/\partial X, G/O\,] = H^2(X, \partial X; \mathbf{Z}/2\mathbf{Z}) \oplus H^4(X, \partial X; \mathbf{Z})$ *if* $v_2(X) = 0$

$(**)$ $[\,X/\partial X, G/O\,] = \mathcal{H}^2(X, \partial X) \oplus \begin{cases} \mathbf{Z} & \text{if } w_1(X) = 0 \text{ and } v_2(X) \neq 0 \\ \mathbf{Z}/4\mathbf{Z} & \text{if } w_1(X) \neq 0 \text{ and } v_2(X) \neq 0 \end{cases}$

The splitting in case $(**)$ depends on the choice of an element $x \in H^2(X, \partial X; \mathbf{Z}/2\mathbf{Z})$ of odd square. The map of $[\,X/\partial X, G/O\,]$ into

$$[\,X/\partial X, G/TOP\,] = H^2(X, \partial X; \mathbf{Z}/2\mathbf{Z}) \oplus H^4(X, \partial X; \mathbf{Z})$$

in case $(*)$ is just an isomorphism on H^2 and multiplication by 2 on H^4 and in case $(**)$ it is inclusion on $\mathcal{H}^2$ and sends the generator of the $\mathbf{Z}$ (respectively $\mathbf{Z}/4\mathbf{Z}$) to $(x, 1)$ where 1 denotes a generator of $H^4(X, \partial X; \mathbf{Z}) = \mathbf{Z}$ (respectively $\mathbf{Z}/2\mathbf{Z}$).

Remark: For 3–dimensional Poincaré spaces, the map $G/CAT \to K(\mathbf{Z}/2\mathbf{Z}, 2)$ induces an isomorphism, $[X/\partial X, G/CAT] \to H^2(X, \partial X; \mathbf{Z}/2\mathbf{Z})$.

A proof of Lemma 7 can be constructed along the following lines. A diagram chase shows that $[\,X/\partial X, G/O\,] \to [\,X/\partial X, G/TOP\,]$ is injective whenever X is orientable: the image is the kernel of k. Another diagram chase shows that every element in $\mathcal{H}^2(X,\partial X) \subset H^2(X,\partial X;\mathbf{Z}/2\mathbf{Z})$ lifts to an element of order 2 in $[\,X/\partial X, G/O\,]$ and any lift of an element of odd square to $[\,X/\partial X, G/O\,]$ has infinite order. This is formula 8 in the orientable case.

Assume X is non–orientable. If $\partial X \neq \emptyset$, let $D(X)$ denote the double of X. Since $X \subset D(X) \to X/\partial X$ is a cofibration and since the inclusion $X \subset D(X)$ is split, the case with boundary follows from the closed case. From Thom, [79], there exists a smooth manifold and a map $f\colon M^4 \to X$ which is an isomorphism on $H_4(\,\_\,;\mathbf{Z}/2\mathbf{Z})$. It then follows that f^* is an isomorphism on $H^4(\,\_\,;\mathbf{Z})$ and an injection on $H^2(\,\_\,;\mathbf{Z}/2\mathbf{Z})$. Hence f^* is injective on $[\,\_\,, G/O\,]$ so we may assume X is a smooth manifold. Every 2–dimensional homology class is represented by an embedded submanifold, $F \subset X$, and hence the Poincaré dual is the pull back of a map $X \to T(\eta)$, where η is a 2–plane bundle over F. A diagram chase reduces the proof of Lemma 7 to the calculation for $T(\eta)$. Smashing the part of F outside a disk to a point gives a map $F \to S^2$, and there is a bundle ν over S^2 with a map $T(\eta) \to T(\nu)$. The bundle ν is classified by an integer, its Euler class, and it follows from the oriented result above that

$$[\,T(\nu), G/O\,] = \begin{cases} \mathbf{Z} & \text{if } \chi(\nu) \text{ is odd} \\ \mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z} & \text{if } \chi(\nu) \text{ is even} \end{cases},$$

where the $\mathbf{Z}/2\mathbf{Z}$ in case $\chi(\nu)$ odd maps onto $H^2(T(\nu);\mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$. This implies Lemma 7 in general.

The remaining question concerning normal maps is whether $\mathcal{L}^{CAT}(X;\text{rel } h)$ is empty or not: homotopy theory says that the Spivak normal bundle plus the lift over ∂X defines a map $X/\partial X \to B(G/CAT)$. In the TOP case, $[\,X/\partial X, B(G/TOP)\,] = H^3(X,\partial X;\mathbf{Z}/2\mathbf{Z})$. The class $g_3\colon BG \to B(G/TOP) \to K(\mathbf{Z}/2\mathbf{Z},3)$ was defined by Gitler and Stasheff, [37]. One can show that g_3 evaluates non–trivially on $\pi_3(BG) = \mathbf{Z}/2\mathbf{Z}$. The generator of $\pi_3(BG)$ corresponds to the generator of the stable 2–stem, since $\pi_{k+1}(BG)$ is isomorphic to the stable k–stem for all k. This in turn can be understood via the Pontrjagin–Thom construction as a map from S^4 to S^2 with the inverse image of a point being T^2 with the "Lie group framing".

Hambleton and Milgram, [40], construct a non–orientable Poincaré space with $g_3 \neq 0$. Using the Levitt–Jones–Quinn Poincaré bordism sequence, [44, 4.5 p.90], one can analyze this situation in the oriented case as well. One sees that g_3 always vanishes in the closed, orientable 4–dimensional case, as well as in the 3–dimensional case.

§3. Surgery Theory.

The Quinn–Ranicki theory, [70], of the assembly map can be used to decouple the surgery theory from the specifics of the Poincaré space X. More precisely, this section defines groups which depend only on the fundamental group, the orientation, the fundamental groups of the boundary and the image of the fundamental class of X in the homology of the fundamental group rel the fundamental group(s) of the boundary. One of these groups will be a quotient of L_5 and will act freely on the structure set so that the quotient injects into the set of normal maps. Another acts freely on the smooth structure set so that the orbit space injects into the topological structure set. Yet another gives a piece of the set of normal maps. The results of Quinn and Ranicki are one of the major developments in general surgery theory and provide the following description of the surgery obstruction map.

A Poincaré space with a lift of its Spivak normal fibration to $BTOP$ acquires a fundamental class in a twisted, n–dimensional extraordinary homology theory, $\mathbf{L}^0$. The theory $\mathbf{L}^0$ is a ring theory and there is a theory, $\mathbf{L}\langle 1\rangle$, so that $[\,X/\partial X, G/TOP\,]$ is the 0–th cohomology group for $\mathbf{L}\langle 1\rangle$–theory and $\cap D$ is just the usual Poincaré duality isomorphism given by cap product with the fundamental class, $\cap[X]\colon [\,X/\partial X, G/TOP\,] \to \mathbf{L}\langle 1\rangle_n^{w_1(X)}(X)$. The map classifying the universal cover, $u\colon X \to B\pi$ induces a map $u_*\colon \mathbf{L}\langle 1\rangle_n^{w_1(X)}(X) \to \mathbf{L}\langle 1\rangle_n^{w_1}(B\pi)$. There is a map A, the assembly map,

$$A_{\pi_1,w_1}\colon \mathbf{L}\langle 1\rangle_n^{w_1}(B\pi_1) \to \overset{s}{L}_n(\mathbf{Z}[\pi_1], w_1)\ .$$

The composite $\alpha = A_{\pi_1(X),w_1(X)} \circ u_* \circ (\cap[X])$,

$$\begin{aligned}[\,X/\partial X, G/TOP\,] \xrightarrow{\cap[X]} \mathbf{L}\langle 1\rangle_n^{w_1(X)}(X) &\xrightarrow{u_*} \mathbf{L}\langle 1\rangle_n^{w_1}(B\pi)\\ &\xrightarrow{A} \overset{s}{L}_n(\mathbf{Z}[\pi_1(X)], w_1(X))\end{aligned}$$

is related to surgery via the following formula: let $x \in \mathcal{L}^{CAT}(X; \mathrm{rel}\ h)$ be a chosen basepoint; then for any $\eta \in [\,X/\partial X, G/TOP\,]$

$$\alpha(\eta) = \theta(\eta \bullet x) - \theta(x)\ .$$

If X has the homotopy type of a manifold, x can be chosen so that $\theta(x) = 0$ and in general this approach divides the problem into a homotopy part and an algebraic part, A_{π,w_1}. Since A_{π,w_1} is a purely algebraic object, one can attack its analysis via algebra or via topology by using known structure set calculations. As an example, the Poincaré conjecture for $n \geq 5$ says $\mathcal{S}^{TOP}(S^n)$ has one point and one sees that the assembly map for the trivial group must be an isomorphism for this to work.

For analyzing the 4–dimensional case, we need to understand $\mathbf{L}\langle 1\rangle_4$ and $\mathbf{L}\langle 1\rangle_5$; the 3–dimensional case requires that we also understand $\mathbf{L}\langle 1\rangle_3$. The Atiyah–Hirzebruch spectral sequence for $\mathbf{L}\langle 1\rangle_*$ collapses for $* < 8$ since all the differentials are odd torsion: hence, for any space Y and $w_1 \in H^1(Y;\mathbf{Z}/2\mathbf{Z})$,

$$\begin{aligned}
&\mathbf{L}\langle 1\rangle_*^{w_1}(Y) = 0,\ * \leq 1 && \mathbf{L}\langle 1\rangle_3^{w_1}(Y) = H_1(Y;\mathbf{Z}/2\mathbf{Z})\\
&\mathbf{L}\langle 1\rangle_2^{w_1}(Y) = H_0(Y;\mathbf{Z}/2\mathbf{Z}) && \mathbf{L}\langle 1\rangle_4^{w_1}(Y) = H_0(Y;\mathbf{Z}^{w_1})\oplus H_2(Y;\mathbf{Z}/2\mathbf{Z})\\
& && \mathbf{L}\langle 1\rangle_5^{w_1}(Y) = H_1(Y;\mathbf{Z}^{w_1})\oplus H_3(Y;\mathbf{Z}/2\mathbf{Z})
\end{aligned}$$

Define $\mathcal{K}_n(\pi, w_1)$ and $Q_n(\pi, w_1)$ so as to make

(9)
$$0 \to \mathcal{K}_n(\pi, w_1) \to \mathbf{L}\langle 1\rangle_n^{w_1}(B\pi) \xrightarrow{A_{\pi,w_1}} \overset{s}{L}_n(\mathbf{Z}[\pi], w_1) \to Q_n(\pi, w_1) \to 0$$

exact.

The sequences (9) for various n clearly only depend on π and w_1. The groups needed for calculating the stable structure sets, $\bar{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h)$ should have the $\mathbf{L}\langle 1\rangle_n^{w_1}(B\pi)$ replace by $\mathbf{L}\langle 1\rangle_n^{w_1}(X)$ using the map u_*. In the 3–dimensional case, u_* is an isomorphism; for the 4–dimensional case u_* is still an epimorphism. For the dimensions considered here, the 5–dimensional case is only needed to compute the action of the L–group on the structure set. We want to identify the quotient group of $\overset{s}{L}_5$ which acts freely, but Q_5 is usually too small. The map $H_1(X;\mathbf{Z}^{w_1}) \to H_1(B\pi;\mathbf{Z}^{w_1})$ is an isomorphism, but the map $H_3(X;\mathbf{Z}/2\mathbf{Z}) \to H_3(B\pi;\mathbf{Z}/2\mathbf{Z})$ needs to be analyzed. The boundary of X may have several components, each with its own fundamental group: let $\cup B\pi_1(\partial X)$ be *notation* for the disjoint union of the classifying spaces for the fundamental groups of the various components of the boundary. There is a class

$$D_X \in H_4(B\pi_1(X), \cup B\pi_1(\partial X);\mathbf{Z}/2\mathbf{Z})$$

which is the image of the fundamental class of the Poincaré space. Cap product with D_X defines a homomorphism,

$$\cap D_X\colon H^1(B\pi_1(X), \cup B\pi_1(\partial X);\mathbf{Z}/2\mathbf{Z}) \to H_3(B\pi_1(X);\mathbf{Z}/2\mathbf{Z})$$

which is the image of u_*. Let

$$\bar{\mathbf{L}}\langle 1\rangle_5^{w_1}(B\pi) = H_1(B\pi;\mathbf{Z}^{w_1}) \oplus H^1(B\pi_1(X), \cup B\pi_1(\partial X);\mathbf{Z}/2\mathbf{Z})$$

and let $\bar{\mathbf{L}}\langle 1\rangle_5^{w_1}(B\pi) \to \mathbf{L}\langle 1\rangle_5^{w_1}(B\pi)$ be the map which is the identity on H_1 and $\cap D_X$ on H^1. Let

$$\bar{A}_{\pi,w_1}\colon \bar{\mathbf{L}}\langle 1\rangle_5^{w_1}(B\pi) \to \mathbf{L}\langle 1\rangle_5^{w_1}(B\pi) \xrightarrow{A_{\pi,w_1}} L_5^s(\mathbf{Z}\pi, w_1),$$

and define $\bar{\mathcal{K}}_5(\pi, w_1, D_X)$ and $\bar{Q}_5(\pi, w_1, D_X)$ as the kernel and cokernel of $\bar{A}_{\pi.\,w_1}$. Define

$$\bar{\gamma}(\pi_1(X), w_1(X), D_X) = H_1(B\pi_1(X); \mathbf{Z}/2\mathbf{Z})/p_1(\bar{\mathcal{K}}_5(\pi_1(X), w_1(X), D_X))$$

where

$$p_1\colon H_1(B\pi_1(X); \mathbf{Z}^{w_1(X)}) \oplus H^1(B\pi_1(X), \cup B\pi_1(\partial X); \mathbf{Z}/2\mathbf{Z}) \\ \to H_1(B\pi_1(X); \mathbf{Z}/2\mathbf{Z})$$

denotes the evident projection.

As we shall see, this $\bar{\gamma}$ describes the difference between the TOP and DIFF–structure sets. Define two pairs of groups depending only on π and w_1 so that

$$0 \to \hat{\mathcal{K}}_5(\pi, w_1) \to H_1(B\pi; \mathbf{Z}^{w_1}) \to L_5^s(\mathbf{Z}\pi, w_1) \to \hat{Q}_5(\pi, w_1) \to 0$$

is exact and define $\hat{\gamma}(\pi, w_1) = H_1(B\pi_1(X); \mathbf{Z}/2\mathbf{Z})/p_1(\hat{\mathcal{K}}_5(\pi_1(X), w_1(X)))$ and $\gamma(\pi, w_1) = H_1(B\pi_1(X); \mathbf{Z}/2\mathbf{Z})/p_1(\mathcal{K}_5(\pi_1(X), w_1(X)))$.

Proposition 10. *There are epimorphisms $\hat{\gamma} \to \bar{\gamma} \to \gamma$ and $\hat{Q}_5 \to \bar{Q}_5 \to Q_5$.*

1. *If $L_1^s(\mathbf{Z}\pi, w_1) = 0$, then $\hat{Q}_5 = \bar{Q}_5 = Q_5 = 0$ and* $\hat{\gamma} = \bar{\gamma} = \gamma = \begin{cases} 0 & \textit{if } w_1 \textit{ is trivial} \\ \mathbf{Z}/2\mathbf{Z} & \textit{otherwise} \end{cases}$
2. *If $H_3(B\pi; \mathbf{Z}/2\mathbf{Z}) = 0$, or if $D_X = 0$, or if*

   $$H^1(B\pi_1(X), \cup B\pi_1(\partial X); \mathbf{Z}/2\mathbf{Z}) = 0,$$

   or if $L_1^s(\mathbf{Z}\pi, w_1)$ has no 2–torsion, then $\hat{Q}_5 \to \bar{Q}_5$ and $\hat{\gamma} \to \bar{\gamma}$ are isomorphisms.

Two of the big conjectures in surgery theory have direct implications here. The Novikov conjecture says that the A_{π,w_1} are injective after tensoring with $\mathbf{Q}$. The Borel conjecture implies that, if $B\pi$ is a finite Poincaré complex, then A_{π,w_1} is split injective. Both of these conjectures are known to be true in many examples.

Here is a table of some sample calculations. In all cases of Table 11, Proposition 10 applies: moreover, the Whitehead group vanishes and $\mathcal{K}_2 = 0$ for all the listed groups: $Q_2 = 0$ for all the listed groups except $\mathbf{Z} \oplus \mathbf{Z}$. The displayed calculations are drawn from many sources.

π	$\{e\}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}/2 \oplus \mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$
w_1	0	*iso.*	0	0	*epi.*	0	0
$L_0(\mathbf{Z}\pi, w_1)$	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/2$	$\mathbf{Z} \oplus \mathbf{Z}/2$
$L_1(\mathbf{Z}\pi, w_1)$	0	0	0	$\mathbf{Z}$	0	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$
$L_2(\mathbf{Z}\pi, w_1)$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z} \oplus \mathbf{Z}/2$
$L_3(\mathbf{Z}\pi, w_1)$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$
$\mathbf{L}\langle 1\rangle_4^{w_1}(B\pi)$	$\mathbf{Z}$	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$	$\mathbf{Z} \oplus \mathbf{Z}/2$	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z} \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2$	$\mathbf{Z} \oplus \mathbf{Z}/2$
$H_1(B\pi; \mathbf{Z}^{w_1})$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}$	0	$\mathbf{Z} \oplus \mathbf{Z}/2$	$\mathbf{Z} \oplus \mathbf{Z}$
$\mathbf{L}\langle 1\rangle_3^{w_1}(B\pi)$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$
$\mathcal{K}_4$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	0	$\mathbf{Z}/2$	0
Q_4	0	0	$\mathbf{Z}$	0	0	$\mathbf{Z}$	0
$\mathcal{K}_3$	0	$\mathbf{Z}/2$	0	0	0	0	0
Q_3	0	0	0	0	0	0	0
$\hat{\gamma}$	0	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$
$\hat{Q}_5$	0	0	0	0	0	$\mathbf{Z}$	0

Table 11: Sample calculations

There are some results of a general nature which follow from naturality and the above calculations. If w_1 is trivial, $\mathcal{K}_4$ is a subgroup of $H_2(B\pi; \mathbf{Z}/2\mathbf{Z})$ and $\mathcal{K}_2 = \mathcal{K}_3 = 0$. If w_1 is non–trivial, $\mathcal{K}_4$ is at most $H_2(B\pi; \mathbf{Z}/2\mathbf{Z}) \oplus \mathbf{Z}/2\mathbf{Z}$ and $\mathcal{K}_2 = 0$. More calculations for finite groups can be deduced from [43].

§4. Computation of Stable Structure Sets.

The stable TOP–structure sets can now be "computed". First of all there is nothing to do if $\mathcal{L}^{TOP}(X; \mathrm{rel}\ h) = \emptyset$ so assume it is non–empty (as it always is in the 3–dimensional and the orientable 4–dimensional cases) and let

$$\hat{\theta}: \mathcal{L}^{TOP}(X; \mathrm{rel}\ h) \xrightarrow{\theta} \overset{s}{L}_n(\mathbf{Z}[\pi_1(X)], w_1(X)) \to Q_n(\pi_1(X), w_1(X))\ .$$

By the surgery theory in the last section, the image of $\hat{\theta}$ is a single point, denoted $\hat{\theta}(X, \mathrm{rel}\ h)$.

Theorem 12: TOP–structures for $n = 4$. *$\bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h) \neq \emptyset$ if and only if $\hat{\theta}(X, \mathrm{rel}\ h)$ is the 0 element in Q_4. If the stable structure set is non–empty, $\bar{Q}_5(\pi_1(X), w_1(X), D_X)$ acts freely on it. Choose a base point $*$ in* *it. Then* $N_{N(*)}: \bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h) \to [X/\partial X; G/TOP]$ *induces a bijection between the orbit space and the subgroup of* $H_2(X; \mathbf{Z}/2\mathbf{Z})$ *which maps onto* $\mathcal{K}_4(\pi_1(X), w_1(X)) \subset H_2(B\pi_1(X); \mathbf{Z}/2\mathbf{Z})$.

Remark: If $\pi_1(X)$ is trivial, $N_{N(*)}$ identifies $\bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h)$ with $H_2(X; \mathbf{Z}/2\mathbf{Z})$. In Corollary 20 below, it is shown that although the structure set can be large there is always just one or two distinct manifolds in it.

The 3–dimensional case is even easier.

Theorem 13: TOP–structures for $n = 3$**.** $\bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h) \neq \emptyset$ *if and only if* $\hat{\theta}(X, \mathrm{rel}\ h)$ *is the* 0 *element in* Q_3*. If the stable structure set is non–empty,* $Q_4(\pi_1(X), w_1(X))$ *acts freely on it. Choose a base point* $*$ *in it. Then* $N_{N(*)}$ *induces a bijection between the orbit space and* $\mathcal{K}_3(\pi_1(X), w_1(X))$.

To analyze the stable smooth structure set, we need good criteria to see if it is non–empty. Assuming $\bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h) \neq \emptyset$, the stable smoothing obstruction is a function $k: \bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h) \to H^4(X, \partial X; \mathbf{Z}/2\mathbf{Z})$ and $\bar{\mathcal{S}}^{DIFF}(X; \mathrm{rel}\ h) \neq \emptyset$ if and only if $k^{-1}(0) \neq \emptyset$ (see Theorem 5). In particular, it is non–empty in the 3–dimensional case. In the simply connected, 4–dimensional, case, Freedman, [30], argues that k is constant if and only if X is *Spin*, and he constructs examples where the constant is 0 and others where the constant is 1. In the non–simply connected case, $v_2(X) = 0$ still implies k constant, but life is more complicated when $v_2(X) \neq 0$. To describe the situation, let $\widetilde{X} \to X$ denote the universal cover. If $\widetilde{X}$ is not *Spin*, then k is not constant. If $\widetilde{X}$ is *Spin*, then there exists a unique class $v \in H^2(B\pi_1(X); \mathbf{Z}/2\mathbf{Z})$ such that $u^*(v) = v_2(X)$ under the map $u: X \to B\pi_1(X)$ which classifies the universal cover. Evaluation yields a map $\cap v: H_2(B\pi_1(X); \mathbf{Z}/2\mathbf{Z}) \to \mathbf{Z}/2\mathbf{Z}$.

Lemma 14. *k is constant if and only if* $\widetilde{X}$ *is Spin and* $\mathcal{K}_4(\pi_1(X), w_1(X)) \subset \ker(\cap v)$.

Remark: If π is finitely presented, any classes $w \in H^1(B\pi, \mathbf{Z}/2\mathbf{Z})$ and $v \in H^2(B\pi; \mathbf{Z}/2\mathbf{Z})$ can be w_1 and v_2 for a manifold with universal cover *Spin*. Hence, as soon as $\mathcal{K}_4(\pi, w_1) \neq H_2(B\pi; \mathbf{Z}/2\mathbf{Z})$, there are examples of manifolds with constant k for which $v_2 \neq 0$. From Table 11, $\mathbf{Z} \oplus \mathbf{Z}$ is such a group. For an explicit example, recall $CP^2 \# \bar{C}P^2 \to S^2$ is a 2–sphere bundle with $w_2 \neq 0$. Pull this bundle back over the degree one map $T^2 \to S^2$ and let M^4 denote the total space. Then $\widetilde{M}$ is *Spin*, but M is not: nevertheless, k is constant.

Theorem 15: DIFF–structures for $n = 4$**.** *If* $k^{-1}(0) \subset \bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h)$

is non–empty, $\bar{\mathcal{S}}^{DIFF}(X;\mathrm{rel}\ h) \neq \emptyset$. The group $\bar\gamma(\pi_1(X), w_1(X), D_X)$ acts freely on $\bar{\mathcal{S}}^{DIFF}(X;\mathrm{rel}\ h)$; the orbit space is the subset $k^{-1}(0)$.

Theorem 16: DIFF–structures for $n = 3$**.** *If $\bar{\mathcal{S}}^{TOP}(X;\mathrm{rel}\ h)$ is non–empty, then $\rho\colon \bar{\mathcal{S}}^{DIFF}(X;\mathrm{rel}\ h) \to \bar{\mathcal{S}}^{TOP}(X;\mathrm{rel}\ h)$ is onto. If $w_1(X)^2 = 0$, ρ is 2 to 1; if $w_1(X)^2 \neq 0$, ρ is a bijection.*

Remark: By Poincaré duality,

$$H_1(B\pi_1(X);\mathbf{Z}/2\mathbf{Z}) = H_1(X;\mathbf{Z}/2\mathbf{Z}) = H^3(X,\partial X;\mathbf{Z}/2\mathbf{Z}),$$

so the action of $\bar\gamma$ gives an action of $H^3(X,\partial X;\mathbf{Z}/2\mathbf{Z})$ on $\bar{\mathcal{S}}^{DIFF}(X;\mathrm{rel}\ h)$ which is the Kirby and Siebenmann action as extended by Lashof and Shaneson to dimension 4. In dimension 3, $\mathbf{Z}/2\mathbf{Z}$ acts on $\bar{\mathcal{S}}^{DIFF}(X;\mathrm{rel}\ h)$ by forming the connected sum with the Poincaré sphere. If $w_1(X)^2 \neq 0$, this action is trivial, otherwise it is free.

The proofs of these results are fairly straightforward. The TOP–results follow from the sequence (3) for TOP and the results from §5. The DIFF–results follow from comparing the sequences (3) for DIFF and TOP using the Kirby and Siebenmann action of $[\,X/\partial X, TOP/O\,]$ on both the normal maps and the structure sets Theorem 16 needs an additional remark. The outline above shows that a quotient of $H_0(B\pi_1;\mathbf{Z}/2\mathbf{Z})$ acts freely on the 3–dimensional structure set and this quotient can be compared with the quotient for fundamental group with $\mathbf{Z}/2\mathbf{Z}$ and w_1 non–trivial.

§5. A Construction of Novikov, Cochran and Habegger.

As we have seen above, the stable structure set in the simply connected case, while finite, can be arbitrarily large. However, Freedman, [30], says that there are either one or two manifolds in each homotopy type. The resolution of this conundrum is the following.

Let $HE^+(X;\mathrm{rel}\ \partial X)$ denote the group of degree one, simple homotopy automorphisms of X, $\ell\colon (X,\partial X) \to (X,\partial X)$, with $\ell|_{\partial X} = 1_{\partial X}$. Let ℓ act on $f\colon (M,L) \to (X,\partial X) \in \mathcal{S}^{CAT}(X;\mathrm{rel}\ h)$ via composition:

$$\ell \bullet f\colon (M,L) \xrightarrow{f} (X,\partial X) \xrightarrow{\ell} (X,\partial X)\ .$$

This group, $HE^+(X;\mathrm{rel}\ \partial X)$, acts on the stable structure sets, and even on each of the ${}^r\widetilde{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h)$, as follows. If $\ell \in HE^+(X;\mathrm{rel}\ \partial X)$, there is a well–defined element in $HE^+(X\#rS^2\times S^2;\mathrm{rel}\ \partial X)$, $\ell\#id\colon X\#r(S^2\times S^2) \to X\#r(S^2\times S^2)$ and we let ℓ act on $f\colon M \to X\#r(S^2\times S^2)$ in ${}^r\widetilde{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h)$ as the composite $\ell\bullet f\colon M \xrightarrow{f} X\#r(S^2\times S^2) \xrightarrow{\ell\#\mathrm{rid}} X\#r(S^2\times S^2)$. The maps ${}^{r}\widetilde{\mathcal{S}}^{CAT}(X; \mathrm{rel}\ h) \to {}^{r+1}\widetilde{\mathcal{S}}^{CAT}(X; \mathrm{rel}\ h)$ and the maps from the DIFF to the TOP structure sets are equivariant with respect to these actions, so there are also actions on the stable structure sets.

The set of CAT–manifolds homotopy equivalent to X, rel h, is just the orbit space of this action. The action preserves the stable triangulation obstruction, so there is a set map

$$k: \mathcal{S}^{TOP}(X; \mathrm{rel}\ h)/\mathit{HE}^{+}(X; \mathrm{rel}\ \partial X) \to \mathbf{Z}/2\mathbf{Z}$$

and Freedman's classification follows from Corollary 20 below that k is injective in the simply connected case plus the discussion of the image of k in Lemma 14 above. Check that the embedding of $\mathit{HE}^{+}(X; \mathrm{rel}\ \partial X)$ in $\mathit{HE}^{+}(X \# S^2 \times S^2; \mathrm{rel}\ \partial X)$ defined by $\ell \mapsto \ell \# 1_{S^2 \times S^2}$ defines an action of $\mathit{HE}^{+}(X; \mathrm{rel}\ \partial X)$ on $\bar{\mathcal{S}}^{CAT}(X; \mathrm{rel}\ h)$. Theorems 19 and 21 below give a partial calculation of $\bar{\mathcal{S}}^{CAT}(X; \mathrm{rel}\ h)/\mathit{HE}^{+}(X; \mathrm{rel}\ \partial X)$.

Let X be a CAT–manifold and use the identity as a base point in $\mathcal{S}^{CAT}(X; \mathrm{rel}\ h)$. Brumfiel, [9], shows that, in $[X/\partial X, G/CAT]$,

$$N_{N_{1_X}}(\ell \bullet f) = N_{N_{1_X}}(\ell) + (\ell^{-1})^{*}\big(N_{N_{1_X}}(f)\big)\ . \tag{17}$$

A similar formula holds for the action on the stable structure sets. Observe that any $\ell \in \mathit{HE}^{+}(X; \mathrm{rel}\ \partial X)$ preserves $w_1(X)$ and so induces an automorphism of the Wall group $L^{s}_{n}(\mathbf{Z}[\pi_1(X)], w_1(X))$. One can check that with these definitions the sequences (3) are $\mathit{HE}^{+}(X; \mathrm{rel}\ \partial X)$ equivariant.

There is a construction due to Novikov, [66], with the details finally worked out by Cochran and Habegger, [19]. Given any $\alpha \in \pi_2(X)$, let ℓ_α denote the following composite

$$X \to X \vee S^4 \xrightarrow{1_X \vee \eta^2} X \vee S^2 \xrightarrow{1_X \vee \alpha} X$$

where $\eta^2 \in \pi_4(S^2) = \mathbf{Z}/2\mathbf{Z}$ denotes the non–trivial element and the map $X \to X \vee S^4$ just pinches the boundary of a disk in the top cell to a point.

One point of Cochran and Habegger's paper is to compute the normal invariant of ℓ_α. This result requires no fundamental group hypotheses and yields:

Theorem 18.

$$N_{1_X}(\ell_\alpha) = (1 + \langle v_2(X), \alpha\rangle)\bar{\alpha}$$

where $\bar{\alpha} \in [X/\partial X, G/TOP]$ denotes the image of α in $H_2(X; \mathbf{Z}/2\mathbf{Z}) \subset [X/\partial X, G/TOP]$ and $\langle v_2(X), \alpha\rangle \in \mathbf{Z}/2\mathbf{Z}$ denotes the evaluation of the cohomology class on the homotopy class.

Remarks: Since ℓ_α can be checked to induce the identity on $[X/\partial X, G/CAT]$, this formula and (17) determine the action of ℓ_α on the TOP–normal maps. If X is oriented, the DIFF–normal maps are a subset of the TOP ones, so this formula determines the action on the DIFF–normal maps as well. In the non–orientable case, there is a $\mathbf{Z}/2\mathbf{Z}$ in the kernel of the map from the DIFF–normal maps to the TOP ones and the Novikov–Cochran–Habegger formula does not determine the normal invariant.

Let $HE_1^+(X; \mathrm{rel}\, \partial x)$ denote the subgroup of $HE^+(X; \mathrm{rel}\, \partial X)$ generated by the ℓ_α.

Theorem 19.

$$\bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h)/HE_1^+(X; \mathrm{rel}\ \partial X)$$
$$\xrightarrow{N} \begin{cases} \mathcal{K}_4(\pi_1(X), w_1(X)) & \text{if } v_2(\widetilde{X}) = 0 \\ \mathcal{K}_4(\pi_1(X), w_1(X)) \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } v_2(\widetilde{X}) \neq 0 \end{cases}$$

is onto. In the second case, the stable triangulation obstruction is onto the $\mathbf{Z}/2\mathbf{Z}$*: in the first case, k may or may not be constant as discussed in Lemma 14 above. Moreover $\bar{Q}_5(\pi_1(X), w_1(X), D_X)$ acts transitively on the orbits of this map.*

Remark: Theorem 19 shows that except for a $\mathbf{Z}/2\mathbf{Z}$ related to stable triangulation, there is an upper bound for $\bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\, h)/HE_1^+(X; \mathrm{rel}\, \partial X)$ which depends only on "fundamental group data".

Corollary 20. *Suppose that $\bar{Q}_5(\pi_1(X), w_1(X), D_X) = 0$ and $\mathcal{K}_4(\pi_1(X), w_1(X)) = 0$. Then the set*

$$\bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h)/HE^+(X; \mathrm{rel}\ \partial X) = \bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h)/HE_1^+(X; \mathrm{rel}\ \partial X)$$

has one element if $\widetilde{X}$ is Spin, and two elements with different triangulation obstructions if it is not. Any simple homotopy equivalence f is homotopic to the composition of a homeomorphism and an element in HE_1^+.

Notice that the action of $\bar{\gamma}$ on $\bar{\mathcal{S}}^{DIFF}$ preserves the HE_1^+ orbits, so

Theorem 21. *The group $\bar{\gamma}$ acts on $\bar{\mathcal{S}}^{DIFF}(X; \mathrm{rel}\, h)/HE_1^+(X; \mathrm{rel}\, \partial X)$ and the orbit space injects into $\bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\, h)/HE_1^+(X; \mathrm{rel}\, \partial X)$.*

The action by the full group, HE^+, is more subtle and often involves the homotopy of X, not just "fundamental group data". Let X be a TOP–manifold and define

$$HE_0^+(X; \mathrm{rel}\, \partial X) = \{\ell \in HE^+(X; \mathrm{rel}\, \partial X) \mid N_{N(1_X)}(\ell) = 0 \text{ and } \ell_* = 1_{\pi_1(X)}\}\,.$$

It follows from Brumfiel's formula (17) that HE_0^+ is a subgroup of HE^+. Let $HE_{\%}^+$ denote the subgroup of HE^+ generated by HE_1^+ and HE_0^+. Theorems 19 and 21 continue to hold with $HE_{\%}^+$ replacing HE_1^+. The actual homotopy type of X can be seen to effect $\bar{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h)/HE_{\%}^+(X;\mathrm{rel}\ \partial)$ via the following observation. The evident map $\bar{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h) \to \bar{\mathcal{S}}^{CAT}(X\#S^2 \times S^2;\mathrm{rel}\ h)$ induces a map

$$\iota_X\colon \bar{\mathcal{S}}^{CAT}(X;\mathrm{rel}\ h)/HE_{\%}^+(X;\mathrm{rel}\ \partial) \\ \to \bar{\mathcal{S}}^{CAT}(X\#S^2 \times S^2;\mathrm{rel}\ h)/HE_{\%}^+(X\#S^2 \times S^2;\mathrm{rel}\ \partial).$$

Let $WSE^{CAT}(X;\mathrm{rel}\ h)$ denote the limit of the maps

$$\iota_X,\ \iota_{X\#S^2\times S^2},\ \cdots,\ \iota_{X\#rS^2\times S^2},\ \cdots.$$

Theorem 22. *The evident quotient of the normal map,*

$$WSE^{TOP}(X;\mathrm{rel}\ h) \xrightarrow{N} \begin{cases} \mathcal{K}_4(\pi_1(X), w_1(X)) & \text{if } v_2(\widetilde{X}) = 0 \\ \mathcal{K}_4(\pi_1(X), w_1(X)) \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } v_2(\widetilde{X}) \neq 0 \end{cases}$$

is a bijection. If $k^{-1}(0) \neq \emptyset$, then $WSE^{DIFF}(X;\mathrm{rel}\ h) \to k^{-1}(0)$ is a bijection.

Remarks: The stable triangulation obstruction is onto the $\mathbf{Z}/2\mathbf{Z}$ if $v_2(\widetilde{X}) \neq 0$: otherwise, k may or may not be constant as discussed in Lemma 14 above. Note that $\mathcal{K}_4$ is always a $\mathbf{Z}/2\mathbf{Z}$ vector space of dimension at most $H_2(B\pi;\mathbf{Z}/2\mathbf{Z}) \oplus \mathbf{Z}/2\mathbf{Z}$, and hence finite. If $\hat{Q}_5(\pi_1(X), w_1(X))$ is finitely generated, then there exists an r such that

$$\bar{\mathcal{S}}^{CAT}(X\#rS^2\times S^2;\mathrm{rel}\ h)/HE_{\%}^+(X\#rS^2\times S^2;\mathrm{rel}\ \partial X) \to WSE^{CAT}(X;\mathrm{rel}\ h)$$

is a bijection.

§6. Examples.

Here are some calculations for some specific manifolds. The quoted values of $\bar{Q}_5$, $\mathcal{K}_4$ and $\bar{\gamma}$ can be obtained from Table 11, after noting Proposition 10 applies so $\bar{Q}_5 = \hat{Q}_5$ and $\bar{\gamma} = \hat{\gamma}$.

Example: RP^4**.** Here $\pi = \mathbf{Z}/2\mathbf{Z}$ and w_1 is an isomorphism. Then $\bar{Q}_5 = 0$, $\mathcal{K}_4 = \mathbf{Z}/2\mathbf{Z}$ and $\bar{\gamma} = \mathbf{Z}/2\mathbf{Z}$. For the normal maps, $[\,RP^4, G/TOP\,] = \mathbf{Z}/2\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}$; $[\,RP^4, G/O\,] = \mathbf{Z}/4\mathbf{Z}$ and it is a useful exercise to understand how the DIFF and TOP versions of sequence (3) work in this case without relying on the general theory.

Hence $\bar{\mathcal{S}}^{TOP}(RP^4) = \mathbf{Z}/2\mathbf{Z}$ and k is a bijection. The non–triangulable example was constructed by Ruberman, [72], using only Freedman's simply connected results. In the smooth case, $\bar{\mathcal{S}}^{DIFF}(RP^4) = \mathbf{Z}/2\mathbf{Z}$ as well, but the map from the smooth to the topological sets takes both elements of the smooth set to one element in the topological set. Cappell and Shaneson, [12], constructed an element in $\mathcal{S}^{DIFF}(RP^4)$ which hits the "other element" in $\bar{\mathcal{S}}^{DIFF}(RP^4)$.

Example: $S^3 \times S^1$**.** Here $\pi = \mathbf{Z}$ and w_1 trivial. Then $\bar{Q}_5 = 0$, $\mathcal{K}_4 = 0$ and $\bar{\gamma} = \mathbf{Z}/2\mathbf{Z}$.

It follows that $\bar{\mathcal{S}}^{TOP}(S^3 \times S^1)$ is one point and $\bar{\mathcal{S}}^{DIFF}(S^3 \times S^1)$ is two points. The "other element" in $\bar{\mathcal{S}}^{DIFF}(S^3 \times S^1)$ was constructed in ${}^1\widetilde{\mathcal{S}}^{CAT}(S^3 \times S^1)$ by Scharlemann, [74]. It is an open question as to whether this element is in the image from $\mathcal{S}^{DIFF}(S^3 \times S^1)$.

Example: $S^3 \widetilde{\times} S^1$**.** Here $\pi = \mathbf{Z}$ and w_1 non–trivial. Then $\mathcal{K}_4 = 0$, $\bar{Q}_5 = 0$ and $\bar{\gamma} = \mathbf{Z}/2\mathbf{Z}$.

Hence $\bar{\mathcal{S}}^{TOP}(S^3 \widetilde{\times} S^1)$ consists of one point, while $\bar{\mathcal{S}}^{DIFF}(S^3 \widetilde{\times} S^1)$ consists of two points, distinguished by the smooth normal invariant. In this case, Akbulut, [1], constructed the "other element" in ${}^1\widetilde{\mathcal{S}}^{DIFF}(S^3 \widetilde{\times} S^1)$.

Remark: If one could find a manifold to show ψ_{DIFF} were onto for $S^3 \widetilde{\times} S^1$, Lashof and Taylor, [57], observed that $\bar{\gamma}$ would act freely on $\mathcal{S}^{DIFF}(X; \text{rel } h)$ as soon as this structure set is non–empty. It does act freely on

$$\mathcal{S}^{DIFF}(X \# S^2 \times S^2; \text{rel } h).$$

Cappell and Shaneson's work [12] shows that $\bar{\gamma}$ acts freely on $\mathcal{S}^{DIFF}(X; \text{rel } h)$ if $\pi_1(X) = \mathbf{Z}/2\mathbf{Z}$ and w_1 is non–trivial.

Example: $RP^3 \times S^1$**.** Here $\pi = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}$ and w_1 is trivial. Then $\mathcal{K}_4 = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$, $\bar{Q}_5 = \mathbf{Z}$ and $\bar{\gamma} = \mathbf{Z}/2\mathbf{Z}$.

The manifold $RP^3 \times S^1$ is *Spin*, so $\bar{\mathcal{S}}^{DIFF}(RP^3 \times S^1) \to \bar{\mathcal{S}}^{TOP}(RP^3 \times S^1)$ is onto. There are two elements in $\bar{\mathcal{S}}^{DIFF}(RP^3 \times S^1)$ over each element of $\bar{\mathcal{S}}^{TOP}(RP^3 \times S^1)$. Each orbit of the Wall group has countable many elements falling into 4 orbits, distinguished by the normal invariant. For some r, $\bar{\mathcal{S}}^{CAT}(RP^3 \times S^1 \# rS^2 \times S^2)/HE^+(RP^3 \times S^1 \# rS^2 \times S^2)$ contains at most 4 elements.

§7. The Topological Case in General.

In a series of papers, Freedman, [30], [31], [32], showed that the high dimensional theory of surgery and the high dimensional s–cobordism theorem hold in the TOP–category in dimension 4 for certain fundamental groups. As of this writing, there are no known failures of either surgery theory or the s–cobordism theorem in the TOP–category in dimension 4.

We say *CAT–surgery works in dimension n for fundamental group π*, provided that, for any n–dimensional Poincaré space X with fundamental group π, the map

$$\psi_{CAT}\colon \mathcal{S}^{CAT}(X; \mathrm{rel}\ h) \to \bar{\mathcal{S}}^{CAT}(X; \mathrm{rel}\ h)$$

is a surjection; we say the *CAT–s–cobordism works in dimension n for fundamental group π*, provided that, for any n–dimensional Poincaré space X with fundamental group π, the map

$$\psi_{CAT}\colon \mathcal{S}^{CAT}(X; \mathrm{rel}\ h) \to \bar{\mathcal{S}}^{CAT}(X; \mathrm{rel}\ h)$$

is an injection.

The first of Freedman's theorems is

Theorem. *TOP–surgery and the TOP–s–cobordism theorem work in dimension 4 for trivial fundamental group.*

It took some work to get to this statement. Freedman began with the simply connected, smooth case, building on work of Casson, [14]. By showing that Casson handles were topologically standard, Freedman showed that surgery theory and the h–cobordism theorem held *topologically* for simply connected, *smooth* manifolds.

Quinn melded these results with his controlled results to prove

$$\pi_i(TOP(4)/O(4)) = 0, \quad i = 0, 1, 2;$$

$i = 0$ is the annulus conjecture in dimension 4. Lashof and Taylor, [57], showed $\pi_3(TOP(4)/O(4)) = \mathbf{Z}/2\mathbf{Z}$ and reproved Quinn's result for $i = 2$. Finally, Quinn, [68], showed $\pi_4(TOP(4)/O(4)) = 0$, thus computing the last of the "geometrically interesting" homotopy groups. Using these results, Quinn, [32], then went on to show that transversality works inside of topological 4–manifolds. Freedman had already completed a program of Scharlemann, [73], by showing that transversality worked in other dimensions when the expected dimension of the result was 4. After this, the standard geometric tools were available in dimension 4 and TOP–surgery and the TOP–s–cobordism theorem now worked for trivial fundamental group.

Freedman, [31], then introduced capped–grope theory which he used to extend the fundamental groups for which TOP–surgery theory and the

TOP–s–cobordism theorem work. There is a nice general result, explained in [34]. Following that exposition, we say that a group, π, is NDL, for *Null Disk Lemma*, provided that, for any height 2 capped grope, G, and any homomorphism, $\psi\colon \pi_1(G) \to \pi$, we can find an immersed core disk, so that all the double point loops map to 0 under ψ.

Theorem 23. *If π is an NDL group, then TOP–surgery and the TOP–s–cobordism theorem work in dimension 4 for π.*

Freedman and Teichner, [34], check that any extension of an NDL group by another NDL group is itself NDL, and they check that a direct limit of NDL groups is NDL. Transparently, subgroups of NDL groups are NDL, and, since $\pi_1(G)$ is a free group, quotients of NDL groups are NDL. Hence subquotients of NDL groups are NDL and a group is NDL if and only if all its finitely–generated subgroups are. Finally, the main result of [34], is

Theorem 24. *Groups of subexponential growth are NDL.*

It is possible that all groups are NDL. Since any finitely–generated group is a subquotient of the free group on 2 generators, all groups are NDL if and only if the free group on 2 generators is. An equivalent formulation, which might make the result seem less likely, is that all groups are NDL if and only if each height 2 capped grope has an immersed core disk with all double point loops null homotopic.

Among the groups satisfying NDL are the finite groups, $\mathbf{Z}$, $\mathbf{Q}$ and nilpotent groups. There do exist nilpotent groups of exponential growth [47, Problem 4.6].

Free groups on more than one generator are not known to be NDL and this causes a great many other geometrically interesting groups to be on the unknown list. Surface groups for genus 2 or more are examples of such groups. The free product of two groups, neither of which is trivial, is either $\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/2\mathbf{Z} = \mathbf{Z}\rtimes\mathbf{Z}/2\mathbf{Z}$ (and is NDL) or has a free subgroup of rank 2 (and is not known to be NDL). Hence the fundamental groups of most connected sums of 3–manifolds are not known to be NDL. Among the irreducible 3–manifolds, many are hyperbolic by Thurston, [82], and many of these have incompressible surfaces: the fundamental groups of such manifolds are not known to be NDL. Even if some group fails to be NDL, it is not clear that TOP–surgery must therefore fail for it. In Freedman and Quinn [32] there is a different condition whose truth would yield surgery and the s–cobordism theorem. It is possible that this condition could yield results even if the Null Disk Lemma were to fail. Quinn [69] has a nice discussion of the current state of affairs regarding the groups for which surgery and/or the s–cobordism theorem works.

In dimension 3, TOP–surgery sometimes holds for trivial reasons: for fundamental group trivial, $\mathbf{Z}$ (with either w_1) or groups satisfying the Borel conjecture (which is conjectured to hold for all irreducible 3–manifold groups), the stable TOP structure set is trivial and so TOP–surgery holds. For non–trivial, finite fundamental group, TOP–surgery fails for closed manifolds. As an example $\bar{\mathcal{S}}^{TOP}(RP^3) = \mathbf{Z}$ but Casson, [3], shows that $\psi_{TOP}: \mathcal{S}^{TOP}(RP^3) \to \bar{\mathcal{S}}^{TOP}(RP^3)$ cannot hit an element of odd order since the double cover of any such element would be a homotopy 3–sphere of Rochlin invariant 1. This line of argument works for any other finite fundamental group. The DIFF–case is even worse since $\bar{\mathcal{S}}^{DIFF}(S^3) = \mathbf{Z}/2\mathbf{Z}$ and the result of Casson's used above also shows that ψ_{DIFF} is not onto.

To say that the s–cobordism theorem holds in dimension 3 is a bit of a misnomer. If TOP–surgery works for $\pi_1(X)$, then two elements in $\mathcal{S}^{TOP}(X; \mathrm{rel}\ h)$ which hit the same element in $\bar{\mathcal{S}}^{TOP}(X; \mathrm{rel}\ h)$ differ by an s–cobordism. However, as we saw above, we do not know whether TOP–surgery holds for many 3–manifold groups and hence we do not usually know that there is an s–cobordism between the two elements. Still, we retain the terminology despite its drawbacks.

For π trivial, the s–cobordism theorem holds in dimension 3 if and only if the Poincaré conjecture holds. In general, the s–cobordism fails in the strict sense that there are 4–dimensional TOP–s–cobordisms which are not products. The first such examples are due to Cappell and Shaneson, [13], with a much larger collection of examples worked out by Kwasik and Schultz, [55]. Surprisingly, there are no counterexamples known to us of the smooth s–cobordism theorem failing in dimension 4, but this is probably due to our inability to construct smooth s–cobordisms.

It may be worth remarking that two 4–dimensional results from the past now can be pushed down one dimension. Barden's old observation that an h–cobordism from S^4 to itself is a smooth product can be made again to observe any h–cobordism from S^3 to itself is a topological product. Thomas's techniques, [80], can be applied to show that any 4–dimensional s–cobordism with NDL fundamental group is invertible.

There has been a great deal of work using Freedman's ideas to attack old problems in four manifolds. A complete survey of such results would require more than our allotted space. Here are some examples which have lead to further work. Hambleton, Kreck and Teichner classify non–orientable 4–manifolds with fundamental group $\mathbf{Z}/2\mathbf{Z}$, [42]. Hambleton and Kreck also classify orientable 4–manifolds with fundamental group $\mathbf{Z}/N\mathbf{Z}$,

[41], as the start of a general program to extend Freedman's simply connected classification to manifolds with finite fundamental group. Kreck's reformulation of surgery theory works very well here, [51].

Lee and Wilczynski, [59], have largely solved the problem of finding a minimal genus surface representing a 2–dimensional homology class in a simply connected 4-manifold. Askitas [4] and [5] considers some cases of trying to represent several homology classes at once.

The slicing of knots and links is an active area as well. The first results here were negative. Casson and Gordon's examples [15] [16] of algebraically slice knots that were not slice showed that there does not exist enough embedding theory in dimension four to do Γ–group surgery in the style of Cappell and Shaneson [11].

One of Freedman's striking results [31] is that knots of Alexander polynomial 1 are topologically slice. Casson and Freedman [17] found links which would be slice if and only if surgery theory worked in dimension 4 for all groups.

On the other hand, it was known in the 1970's to Casson (and others?) that in a smooth 4-manifold M with no 1-handles, the only obstruction to representing a characteristic class of square one by a PL embedded 2-sphere with one singularity with link a knot of Alexander polynomial one, was the Arf invariant of the knot (that is, $\sigma M \equiv 1 \bmod 16$). Once Donaldson showed that non-diagonal definite forms were not realized by smooth 4-manifolds, then in CP^2 blown up at 16 points, any characteristic class of square 1 cannot be represented by a smoothly embedded 2-sphere. Hence there must be an Alexander polynomial one knot which is not smoothly slice in a homology 4-ball. (See Problem 1.37, page 61 in [47])

§8. The Smooth Case in Dimension 4.

Shortly after Freedman's breakthrough in 1981, Donaldson made spectacular progress in the smooth case. We soon learned that neither DIFF–surgery nor the DIFF–s–cobordism theorem holds, even for simply connected smooth manifolds. In the next fifteen years, we learned a great deal more, but the overall situation has only become more complex from the point of view of surgery theory.

Existence: Donaldson's first big theorem, [22], severely limited the forms which could be the intersection form of a smooth, simply connected 4-manifold. Any form can be stably realized and as soon as the form is indefinite, they are completely classified. In the *Spin* case, the forms are $2mE_8 \oplus rH_2$ where E_8 is the famous definite even form of signature 8 and H_2 is the dimension 2 hyperbolic. Donaldson, [23], proved that if $m = 1$, then

$r \geq 3$, and there is a conjecture, the 11/8–th's conjecture ($b_2/|\sigma| \geq 11/8$), which says that $r \geq 3m$ in general. At this time Furuta, [36], has proved the 10/8–th's conjecture, which says that $r \geq 2m$. See [47], Problems 4.92 and 4.93. In particular, there exists a simply connected, TOP manifold, M_{2mE_8} with form $2mE_8$; from Theorem 15, $\bar{\mathcal{S}}^{DIFF}(M_{2mE_8}) = 16m(\mathbf{Z}/2\mathbf{Z})$, but ${}^r\widetilde{\mathcal{S}}^{DIFF}(M_{2mE_8}) = \emptyset$ for $r < 2m$. In the simply connected case, we also know that, for each integer $r \geq 0$, either ${}^r\widetilde{\mathcal{S}}^{DIFF}(M) = \emptyset$ or else $\psi^r_{DIFF}\colon {}^r\widetilde{\mathcal{S}}^{DIFF}(M) \to \bar{\mathcal{S}}^{DIFF}(M)$ is onto.

Scharlemann, [74], showed $\psi^1_{DIFF}\colon {}^1\widetilde{\mathcal{S}}^{DIFF}(S^3 \times S^1) \to \bar{\mathcal{S}}^{DIFF}(S^3 \times S^1) = \mathbf{Z}/2\mathbf{Z}$ is onto: $\mathcal{S}^{DIFF}(S^3 \times S^1)$ is certainly non–empty, but as of this writing, ψ_{DIFF} is not known to be onto. Wall, [85, §16], shows all homotopy equivalences are homotopic to diffeomorphisms, so $HE^+(S^3 \times S^1)$ acts trivially on the smooth structure set. Interestingly, a folk result of R. Lee, [10], says that $HE^+(S^3 \times S^1 \# S^2 \times S^2)$ acts transitively on ${}^1\widetilde{\mathcal{S}}^{DIFF}(S^3 \times S^1)$.

The above gives many examples of simply connected smooth manifolds which topologically decompose as connected sums, but have no corresponding smooth decomposition. Works of Freedman and Taylor, [33], and Stong, [76], show that one can still mimic this decomposition by decomposing along homology 3–spheres into simply connected pieces.

Uniqueness: Donaldson, [24], also proved that the h–cobordism theorem fails for smooth, 5–dimensional, simply connected h–cobordisms. Note however that a smooth h-cobordism between simply connected 4-manifolds is unique up to diffeomorphism, [49]. There is another classification theorem of simply connected h–cobordism due to Curtis, Freedman, Hsiang and Stong, [20], in terms of Akbulut's corks, [46], [2], [63].

We know of no case in which ψ_{DIFF} is not ∞–to–one and we know of no case where all the elements in $\mathcal{S}^{DIFF}(M)$ have been described. The smooth Poincaré conjecture, unresolved at the time of this writing, says $\mathcal{S}^{DIFF}(S^4)$ has one element. The uniqueness result for $\mathbf{R}^4$ is known to fail spectacularly, [38], [21]. In contrast to the existence question, where we know examples for which we need arbitrarily many $S^2 \times S^2$'s before a particular stable element exists, for all we know, ${}^r\widetilde{\mathcal{S}}^{DIFF}(M) \to \bar{\mathcal{S}}^{DIFF}(M)$ and ${}^r\widetilde{\mathcal{S}}^{DIFF}(M) \to {}^{r+1}\widetilde{\mathcal{S}}^{DIFF}(M)$ have the same image. Some works of Mandelbaum and Moishezon, [62], and Gompf, [39], give many examples in which this one–fold stabilization suffices.

It follows from Cochran and Habegger, [19], that the group of homotopy automorphisms of a closed, simply connected 4–manifold, M, is the semidirect product of the Novikov maps, $HE^+_1(M)$, and the automorphisms of $H_2(M;\mathbf{Z})$ which preserve the intersection form. Moreover, Cochran and Habegger show that all the non–trivial elements of $HE_1^+(M)$ are detected by normal invariants and so are not homotopic to homeomorphisms. Now it follows, as observed by Freedman, [30], that the automorphisms of $H_2(M;\mathbf{Z})$ which preserve the intersection form are realized by homeomorphisms, unique up to homotopy. Further work by Quinn, [68], shows that they are in fact unique up to isotopy.

When M is also smooth and of the form $P\#S^2 \times S^2$, Wall, [83], and Freedman and Quinn, [32], showed that any homeomorphism is isotopic to a diffeomorphism. But when M is not of the form $P\#S^2 \times S^2$, then there are often severe restrictions on realizing a homotopy equivalence by a diffeomorphism due to the existence of basic classes in $H^2(M,\mathbf{Z})$. These classes were defined for Donaldson theory by Kronheimer and Mrowka, [52], [53]. Conjecturally equivalent basic classes were also defined using Seiberg-Witten invariants, [86], and these classes were shown to be equivalent by Taubes, [78], to classes defined via Gromov's pseudoholomorphic curves. Although the set of basic classes can be as simple as the zero class in $H^2(M,\mathbf{Z})$ for the $K3$ surface, the classes can be as complicated as Alexander polynomials are, [29]. The isometry induced on $H^2(M;\mathbf{Z})$ by a diffeomorphism must take each basic class to $\pm$(a, possibly different, basic class).

There can be further restrictions, beyond those determined by the basic classes, to realizing homotopy equivalences by diffeomorphisms. For example, any $K3$ surface has additional restrictions, see [25, Corollary 9.14, p.345]. The homeomorphism of $K3$ which is the identity except on an $S^2 \times S^2$ summand and is *antipodal* $\times$ *antipodal* on the $S^2 \times S^2$ summand cannot be realized by a diffeomorphism. However, it follows from [35], that a subgroup of finite index in the group of isometries of the intersection form of $K3$ is realized by diffeomorphisms.

As of this writing, work in the smooth case is continuing at a feverish pace and is hardly ripe for a survey. For many smooth manifolds we now know the minimal genus smooth embeddings representing any homology class; see Kuga [54], Li and Li [60][61], Kronheimer and Mrowka [52], and Morgan, Szabó and Taubes [65]. Some work on simultaneous representation of several classes in the smooth case is in [4]. The xxx Mathematics Archive at Los–Alamos (see `http://front.math.ucdavis.edu/`) is a useful resource for those wishing to remain current.

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Problems in 4-dimensional topology

Frank Quinn

Introduction

The early 1980's saw enormous progress in understanding 4-manifolds: the topological Poincaré and annulus conjectures were proved, many cases of surgery and the s-cobordism theorem were settled, and Donaldson's work showed that smooth structures are stranger than anyone had imagined. Big gaps remained: topological surgery and s-cobordisms with arbitrary fundamental group, and general classification results for smooth structures. Since then the topological work has been refined and applied, but the big problems are still unsettled. Gauge theory has flowered, but has had more to say about geometric structures (esp. complex or symplectic) than basic smooth structures. So on the foundational questions not much has happened in the last fifteen years. We might hope that this has been a period of consolidation, providing foundations for the next generation of breakthroughs.

Kirby has recently completed a massive review of low-dimensional problems [Kirby]. Here the focus is on a shorter list of "tool" questions, whose solution could unify and clarify the situation. These are mostly well-known, and are repeated here mainly to give a context for comments and status reports. We warn that these formulations are implicitly biased toward positive solutions. In other dimensions when tool questions turn out to be false they still frequently lead to satisfactory solutions of the original problems in terms of obstructions (eg. surgery obstructions, Whitehead torsion, characteristic classes, etc). In contrast, failures in dimension four tend to be indirect inferences, and study of the failure leads nowhere. For instance the failure of the disk embedding conjecture in the smooth category was inferred from Donaldson's nonexistence theorems for smooth manifolds. Some direct information about disks is now available, eg. [Kr], but it does not particularly illuminate the situation.

Topics discussed are: in section 1, embeddings of 2-disks and 2-spheres needed for surgery and s-cobordisms of 4-manifolds. Section 2 describes uniqueness questions for these, arising from the study of isotopies. Section 3 concerns handlebody structures on 4-manifolds. Section 4 concerns invariants. Finally section 5 poses a triangulation problem for certain low-dimensional stratified spaces.

I would like to expand on the dedication of this paper to C. T. C. Wall. When I joined the mathematical community in the late 1960s the development of higher-dimensional topology was in full swing. Surgery was hot: "everybody" seemed to be studying Wall's monograph [W1], the solution of the Hauptvermutung was just around the corner, and the new methods were revolutionizing the study of transformation groups. However little or none of it applied to low dimensions. Few people seemed to be bothered by excluding dimensions below 5, 6 or 7, and in some quarters there was even disdain for them as an old-fashioned distraction from the "big picture." Wall, in contrast, systematically explored low-dimensional consequences of each new technique. His work, for instance the stable 5-dimensional s-cobordism theorem, or the results on diffeomorphisms of connected sums, exposed the key problems and showed progress was possible. This made a lasting impression on the students then studying high-dimensional topology, and prepared us for our later focus on low dimensions. Without Wall we might very well still be wondering about the 4-dimensional Poincaré conjecture.

An early version of this paper appeared in the proceedings of the International Conference on Surgery and Controlled Topology, held at Josai University in September 1996 [Q5].

1: 2-disks and spheres in 4-manifolds

The target results here are surgery and the s-cobordism theorem. In general these are reduced, via handlebody theory, to questions about disks and spheres in the middle dimension of the ambient manifold. Two n-dimensional submanifolds of a manifold of dimension $2n$ will usually intersect themselves and each other in isolated points. The "Whitney trick" uses an isotopy across an embedded 2-disk to simplify these intersections. Roughly speaking this reduces the study of n-dimensional embeddings to embeddings of 2-disks. But this is not a reduction when the dimension is 4: the 2-disks themselves are middle-dimensional, so trying to embed them encounters exactly the same problems they are supposed to solve. This is the phenomenon that separates dimension 4 from others. The central conjecture is that some embeddings exist in spite of this problem.

1.1 Disk conjecture. *Suppose A is an immersion of a 2-disk into a 4-manifold, boundary going to boundary, and there is a framed immersed 2-sphere B with trivial algebraic selfintersection and algebraic intersection 1 with A. Then there is a topologically embedded 2-disk with the same framed boundary as A.*

If this were true as stated then the whole apparatus of high-dimensional topology would apply in dimension 4. It is known when the fundamental group is "small", [FQ, FT1]. It is expected to be false for other fundamental groups, but no demonstration is in sight. It is false for smooth embeddings, since it would imply existence and uniqueness results that are known to be false [Kirby 4.1, 4.6].

There are very interesting generalizations of 1.1, which for example ask about the minimal genus of an embedded surface with a given boundary, or in a given homology class (cf. [Kirby, 4.36]), or drop the hypothesis about a dual sphere B. However the data in 1.1 is available in the Whitney disk applications, so its inclusion reflects the "tool" orientation of this paper.

The current best results on 1.1 are by Freedman and Teichner [FT1], who show it holds if the fundamental group of the 4-manifold has "subexponential growth." We briefly discuss the proof. For surfaces in 4-manifolds here is a correspondence between intersections and fundamental group of the image: adding an intersection point enlarges the fundamental group of the image by one free generator (if the image is connected). Freedman's work roughly gives a converse: in order to remove intersections in M, it is sufficient to kill the image of the fundamental group of the data, in the fundamental group of M. More precisely, if we add the hypothesis that $A \cap B$ is a single point, and π_1 of the image $A \cup B$ is trivial in $\pi_1 M$ then there is an embedded disk. However applications of this depend on the technology for reducing images in fundamental groups. Freedman's earlier work showed (essentially) how to change A and B so the fundamental group image becomes trivial under any $\phi\colon \pi_1 M \to G$, where G is poly-(finite or cyclic). [FT1] improves this to allow G of subexponential growth. Quite a lot of effort is required for this rather minute advance, giving the impression that we are near the limits of validity of the theorem. In a nutshell, the new ingredient is the use of (Milnor) link homotopy. Reduction of fundamental group images is achieved by trading an intersection with a nontrivial loop for a great many intersections with trivial, or at least smaller, loops. The delicate point is to avoid reintroducing big loops through unwanted intersections. The earlier argument uses explicit moves. The approach in [FT1] uses an abstract existence theorem. The key is to think of a collection of disks as a nullhomotopy of a link. Selfintersections are harmless, while intersections between different components are deadly. Thus the nullhomotopies needed are exactly the ones studied by Milnor, and existence of the desired disks can be established using link homotopy invariants.

While the conjecture is expected to be false for arbitrary fundamental groups, no proof is in sight. Constructing an invariant to detect failure is a delicate limit problem. The fundamental group of the image of the data can be compressed into arbitrarily far-out terms in the lower central series of the fundamental group of M. If it could be pushed into the intersection of all terms then the general conjecture would follow. (This is because it is sufficient to prove the conjecture for M with free fundamental group, eg. by restricting to a regular neighborhood of the data, and the intersection of the lower central series of a free group is trivial). So we need an invariant that prevents descent to the intersection but not to any finite stage. Determined efforts to modify traditional link invariants to do this have failed so far. The smooth invariants (Donaldson, Seiberg-Witten, "quantum") do not apply directly since this is a purely topological question. There are smooth reformulations [F], but so far these give little indication of making contact with the invariants.

There is a modification of the conjecture, in which we allow the ambient manifold to change by s-cobordism. This form implies that "surgery" works, but not the s-cobordism theorem. I personally believe this one is true.

1.2 Embedding up to s-cobordism. *Suppose the embedding data of 1.1 is given in a 4-manifold M. Then there is a topological s-cobordism with a product structure on the boundary, to a manifold N with a topologically embedded 2-disk with the same framed boundary as A.*

Partial results are in [FQ , FT2]. [FQ, §6] shows that if the fundamental group of the image of the data of 1.1 is trivial in the whole manifold, then there is an embedding up to s-cobordism. This differs from the partial result on 1.1 in that $A \cap B$ is not required to be one point, just algebraically 1. This modest relaxation has applications, but does not give surgery for a larger class of fundamental groups. Application to surgery still depends on the technology for reducing the fundamental group of the image, and the weaker hypotheses have not helped with this.

The improvement of [FT2] over the earlier result is roughly that infinitesimal holes are allowed in the data. A regular neighborhood of the data gives a 4-manifold with boundary, and carrying certain homology classes. In the regular neighborhood the homology class is represented by a sphere, since a sphere is given in the data. The improvement relaxes this: the homology class is required to be in a certain subgroup of H_2, but not necessarily in the image of π_2. Heuristically we can drill a hole in the sphere, as long as it is small enough not to move the homology class too far out of π_2 (technically, still in the ω term of Dwyer's filtration on H_2).

The improved version has applications, but again falls short of the full conjecture. Again it is a limit problem: they show that one can start with arbitrary data and drill very small holes to get the image π_1 trivial in M. The holes can be made "small" enough that the resulting homology classes are in an arbitrarily far-out term in the Dwyer filtration, but maybe not in the infinite intersection.

Here is a suggestion for a new approach. The old approach combines many parallel copies of the data in careful ways to get improvement. It is a bit like winding a spring up until something snaps. Unfortunately, years of winding has failed to produce any snaps when the fundamental group is large. Maybe instead of winding tighter we should be trying to spread things out. Consider a selfintersection point x of A (or technically, caps in a grope representing A) with a non-trivial loop passing through it. Change A to remove the intersection, at the cost of introducing a new sphere A_x. The union $A \cup A_x$ still contains the loop, but it passes through A_x. Persistent iteration is supposed to give a great many spheres with the property that a nontrivial loop in the union must pass through at least n of the spheres. Think of these as located on a wedge of circles, and so small that it takes n of them to go around a circle. Technically we would want to get a family of spheres "controlled" over the wedge of circles in the sense of [Q1] or [FQ 5.4]. A controlled embedding theorem should then provide the desired embeddings.

2: Uniqueness

The uniqueness question we want to address is: when are two homeomorphisms of a 4-manifold topologically isotopic? The answer is is known for compact 1-connected 4-manifolds [Q2], but not for nontrivial groups even in the good class for surgery. Neither is there a controlled version, not even in the 1-connected case. The controlled version may be more important than general fundamental groups, since it is the main missing ingredient in a general topological isotopy extension theorem for stratified sets [Q3].

The study of isotopies is approached in two steps. First determine if two homeomorphisms are concordant (pseudoisotopic), then see if the concordance is an isotopy. The first step still works for 4-manifolds, since it uses 5-dimensional surgery. The high-dimensional approach to the second step [HW] reduces it to a "tool" question. However the uniqueness tool question is *not* simply the uniqueness analog of the existence question. In applications Conjecture 1.1 would be used to find Whitney disks to manipulate 2-spheres. The tool question needed to analyse isotopies directly concerns these Whitney disks.

Conjecture 2.1. *Suppose A and B, are framed embedded families of 2-spheres, and V, W are two sets of Whitney disks for eliminating AB intersections. Each set of Whitney disks reduces the intersections to make the families transverse: the spheres in A and B are paired, and the only intersections are a single point between each pair. Then the sets V, W are* *equivalent up to isotopy and disjoint replacement.*

"Isotopic" means there is an ambient isotopy that preserves the spheres A, B setwise, and takes one set of disks to the other. Note that $A \cap B$ must be pointwise fixed under such an isotopy. "Disjoint replacement" means we declare two sets to be equivalent if the only intersections are the endpoints (in $A \cap B$). Actually there are further restrictions on framings and π_2 homotopy classes, related to Hatcher's secondary pseudoisotopy obstruction [HW]. In practice these do not bother us because the work is done in a relative setting that encodes a vanishing of the high-dimensional obstruction: we try to show that a 4-dimensional concordance is an isotopy if and only if the product with a disk is an isotopy. In [Q2] this program is reduced to conjecture 2.1, and the conjecture itself is proved for simply connected manifolds and A, B each a single sphere.

The (classical) reduction of the pseudoisotopy problem to the conjecture goes as follows: a pseudoisotopy is an isomorphism of $M \times I$ with itself. Think of this as two handlebody structures on $M \times I$, both without any handles (a handlebody is a collar $M \times I$ with handles attached to it). Join these by a 1-parameter family of handlebody structures. This family can be visualized as follows: begin with the collar on one end. Change the handlebody structure (not the manifold) by introducing lots of handles, and let them interact in elaborate ways. Finally they all cancel to leave us with the second collar structure. There is also an isotopy of the base collars. If we can deform this family to get one with no handles at all then what is left is an isotopy of the base collar structures, and thus of the original isomorphisms. In (base) dimension four the family of handlebodies can be deformed to a more restrained one: first a lot of disjoint cancelling pairs of 2- and 3-handles appear. Next, in the level between the 2- and 3-handles the attaching maps of the 3-handles are isotoped (all together) by doing finger moves to introduce new intersections with the dual spheres of the 2-handles. Then there is the inverse of such a move: all the 3-handle attaching spheres are moved by pushing across Whitney disks to get back a geometrically cancelling situation. Finally these are cancelled. The view at the center point is: we have a family A of dual spheres of the 2-handles, a family B of attaching spheres for the 3-handles, and two complete sets of Whitney disks (V and W) that eliminate extra intersections in two different ways. We would like to eliminate the handles completely, to show there is a 1-parameter family without handles. We could do this if the two sets of disk were disjoint except for mandatory intersections in $A \cap B$, or at the other extreme, if the two sets were equal.

Each disk in V or W has boundary given as two arcs, one on A and one on B, and the endpoints of these arcs are intersection points in $A \cap B$.

Focus on the arcs on A. An intersection point is the endpoint of a V arc if the intersection is eliminated by pushing across the V disks. Similarly it is a W endpoint if it is eliminated by the other move. This means each intersection point is an endpoint of at most two of these arcs. Therefore the two families of arcs fit together to form circles and arcs, and the endpoints of the arcs are $A \cap B$ intersections that are not eliminated in one or the other move. Since each family has exactly one such special intersection point on each sphere, there is exactly one union arc on each sphere.

The proof of [Q2] employs the arcs of boundary $V \cup W$ curves. Focus on a single pair of spheres. The 1-connectedness is used to merge the circles into the arc. Intersections among Whitney disks strung out along the arc are then "pushed off the end" of the arc. This makes the two sets of disks equivalent in the sense of 2.1, and allows the pair of handles to be eliminated from the 1-parameter family. This process can be iterated to eliminate finitely many pairs, and the compact 1-connected case follows.

This iterative procedure cannot be done with control since each cancellation will greatly rearrange the remaining spheres. It cannot be done with nontrivial fundamental group because the circles of $V \cup W$ curves cannot be absorbed into the arcs. To treat either nontrivial fundamental groups or control will require dealing directly with the circles of Whitney arcs. But the proof of [Q2] gets stuck because circles have no ends to push things off. Still, they can be manipulated quite a bit, and it may well be possible to extract an invariant from them. The best current guess is that such an invariant will show the conjecture is false.

3: 4-dimensional handlebodies

Handlebody structures on 4-manifolds correspond exactly to smooth structures. The targets in studying handlebody structures are therefore the detection and manipulation of smooth structures. However these are much more complicated than in other dimensions, and not well enough understood even to confidently identify tool questions that might unravel them.

4-dimensional handlebodies are described by their attaching maps, embeddings of circles and 2-spheres in 3-manifolds. The dimension is low enough to draw explicit pictures of many of these. Kirby developed notations and a "calculus" of such pictures for 1- and 2-handles. This approach has been used to analyse specific manifolds: see [HKK] for pictures of complex surfaces, and Gompf's identification of some homotopy spheres as standard [Gf]. It was also used in Freedman's original proof of the disk embedding theorem. However it has been limited even in the study of examples because:

(1) it only effectively tracks 1- and 2-handles, and Gompf's example shows one cannot afford to ignore 3-handles;
(2) it is a non-algorithmic "art form" that can hide mistakes from even skilled practitioners; and
(3) there is little clue how the pictures relate to effective (eg. Donaldson and Seiberg-Witten) invariants.

The most interesting possibility for manipulating handlebodies is suggested by the work of Poenaru on the 3-dimensional Poincaré conjecture. The following is suggested as a test problem to develop the technique:

3.2 Conjecture. *A 4-dimensional (smooth) s-cobordism without 1-handles is a product.*

Settling this would be an important advance, but a lot of work remains before it would have profound applications. To some extent it would show that the real problem is getting rid of 1-handles ([Kirby 4.18, 4.88, 4.89]; see below). It might have some application to this: if we can arrange that some subset of the 2-handles together with the 1-handles forms an s-cobordism, then the dual handlebody structure has no 1-handles and the conjecture would apply. Replacing these 1- and 2-handles with a product structure gives a new handlebody without 1-handles. The problem encountered here is control of the fundamental group of the boundary above the 2-handles. The classical manipulations produce a homology s-cobordism (with $\mathbb{Z}[\pi_1]$ coefficients), but to get a genuine s-cobordism we need the new boundary to have the same π_1. Thus to make progress we would have to understand the relationship between things like Seiberg-Witten invariants and restrictions on fundamental groups of boundaries of sub-handlebodies.

To analyse the conjecture consider the level between the 2- and 3-handles in the s-cobordism. The attaching maps for the 3-handles are 2-spheres, and the dual spheres of the 2-handles are circles. The usual manipulations arrange the algebraic intersection matrix between these to be the identity. In other dimensions the next step is to realize this geometrically: find an isotopy of the circles so each has exactly one point of intersection with the family of spheres. But the usual methods fail miserably in this dimension. V. Poenaru has attacked this problem in the special case of $\Delta \times I$, where Δ is a homotopy 3-ball, [Po, Gi]. The rough idea is an infinite process in which one repeatedly introduces new cancelling pairs of 2- and 3-handles, then damages these in order to fix the previous ones. The limit has an infinite collections of circles and spheres with good intersections. Unfortunately this limit is a real mess topologically, in terms of things converging to each other. The goal is to see that, by being incredibly clever and careful, one can arrange the spheres to converge to a singular lamination with control on the fundamental groups of the complementary components. As an outline this makes a lot of sense. Unfortunately Poenaru's manuscript is extremely long and complicated and, as a result of many years of work without feedback from the rest of the mathematical community, quite idiosyncratic. It would probably take years of effort to extract clues on how to deal with the difficult parts.

We end the section with an historical note continuing the theme of the dedication. Wall extracted a clean statement of what is essentially the induction hypothesis in the proof of the s-cobordism theorem, in his paper [Wa2]. Suppose M is a manifold and $W \subset \partial M$ is a codimension-0 submanifold. Wall showed that if (M, W) is homotopically r-connected with $r \leq \dim(M) - 4$ then it is *geometrically* r-connected in that it has a handlebody structure without handles of index $\leq r$. In typical fashion he investigated consequences of the techniques in low dimensions. [Wa3] gives a version for 3-manifolds. A third preprint in the series asserted that a 1-connected 4-manifold pair has a handlebody structure without 1-handles. Unfortunately this relied on an attractive and oft-rediscovered error, and had to be withdrawn. Nonetheless the paper made a big impression on many of us, and posed what has turned out to be one of the key problems in the area.

4: Invariants

Some compact topological 4-manifolds have infinitely many smooth structures, and many non-compact ones have uncountably many. At present this is inferred from Donaldson and Seiberg-Witten invariants, defined using global differential geometry. Since a handlebody structure determines a smooth structure these invariants are somehow encoded in the handle structure, and for a "topological" understanding we would need to decode some of this. We already know that the tools that work in higher dimensions — homology, characteristic classes, etc. — are too simple for dimension 4. The invariants we know do work lie at the other extreme: behavior with respect to geometric decompositions is still largely unclear but already too complicated for a useful topological theory. The best hope seems to be with a class of theories of intermediate complexity:

4.1: Problem. *Find a combinatorially-defined "topological quantum field theory" that detects exotic smooth structures.*

Three-dimensional combinatorial field theories were pioneered by Reshetikhin and Turaev [RT], followed by [KM, L] and many others. A number of axiom systems have been proposed; the mathematically precise versions (cf. [Q4], [Kl]) are fairly complicated but have enough interesting structure to suggest that they may be both useful and comprehensible. Originally it was hoped that this would provide a context for the Donaldson (and later, Seiberg-Witten) invariants, but unfortunately they are yet more complicated.

The phrase "combinatorially-defined" in 4.1 should be interpreted loosely. Field theories involve cutting manifolds into pieces. "Modular" field theories involve cutting boundaries of these pieces, so cutting to codimension 2 in the original manifold. Bimodular theories go to codimension 3, etc. Deep structure provides good tools but rules out many theories: few general theories are modular, and fewer yet are bimodular. Taken literally the "combinatorial" in 4.1 might suggest definition in terms of simplices, ie. cutting to codimension 4. There has been quite a bit of work on this ("solving the simplex equations"), but early indications [CKY] suggest that only the classical invariants survive such deep cuts. Bimodular theories (cutting to down to circles, in codimension 3) are next. Some abstract work has been done, but so far no serious examples have been developed.

Returning to our historical subtheme, "Novikov additivity" of the signature describes field-like behavior with respect to cutting to codimension 1. Wall's analysis of the signature when cutting to codimension 2 [W4] seems to be the first foreshadowing of modular field theories.

5: Stratified spaces

A class of stratified spaces with a relatively weak relationship between the strata has emerged as the proper setting for purely topological stratified questions, see eg. [Q3, We]. The analysis of these sets, to obtain results like isotopy extension theorems, uses a great deal of handlebody theory, and as a result often requires the assumption that all strata have dimension 5 or greater. This restriction is acceptable in some applications, for example in group actions, but not in others like smooth singularity theory, algebraic varieties, and limit problems in Riemannian geometry. The suggestion here is that many of the low-dimensional issues can be reduced to (usually easier) PL and differential topology. The conjecture, as formulated, is a tool question for applications of stratified sets. After the statement we discuss the dissection into topological tool questions.

5.1: Conjecture. *A three-dimensional homotopically stratified space with manifold strata is triangulable. A 4-dimensional space of this type is triangulable in the complement of a discrete set of points.*

As stated this implies the 3-dimensional Poincaré conjecture. To avoid this assume either that there are no fake 3-balls below a certain diameter, or change the statement to "obtained from a polyhedron by replacing sequences of balls converging to the 2-skeleton by fake 3-balls."

The "Hauptvermutung" for 3-dimensional polyhedra [Pa] asserts that homeomorphisms are isotopic to PL homeomorphisms. This reduces the 3-dimensional version to showing that stratified spaces are locally triangulable. The 2-skeleton and its complement are both triangulable, so the problem concerns how the 3-dimensional part approaches neighborhoods of points in the 2-skeleton.

We begin with a manifold point in the skeleton, so a neighborhood in the skeleton is isomorphic with $\mathbb{R}^n$ for $n = 0, 1$, or 2. Near this the 3-stratum looks locally homotopically like a fibration over $\mathbb{R}^n$ with fiber a Poincaré space of dimension $3 - n - 1$. We can reduce to the case where the fiber is connected by considering components of the 3-stratum one at a time. If $n = 2$ then the fiber is a point, and the union of the two strata is a homology 3-manifold with $\mathbb{R}^2$ as boundary. Thus the question: is this union a manifold, or equivalently, is the $\mathbb{R}^2$ collared in the union? This is a very classical question, and may already be known. If $n = 1$ then the fiber is S^1, and the union gives an arc homotopically tamely embedded in the interior of a homology 3-manifold. Is it locally flat? Finally if $n = 0$ then the fiber is a surface (2-dimensional Poincaré spaces are surfaces, [EL]). This is an end problem: if a 3-manifold has a tame end homotopic to $S \times \mathbb{R}$, S a surface, is the end collared? This seems to follow easily from standard embedded surface theory, but I do not know a reference. The next step is to consider a point in the closure of strata of three different dimensions. There are three cases: $(0, 1, 3)$, $(0, 2, 3)$ and $(1, 2, 3)$. Again each case can be described quite explicitly, and should either be known or accessible to standard 3-manifold techniques.

Now consider 4-dimensional spaces. 4-manifolds are triangulable in the complement of a discrete set, so again the question concerns neighborhoods of the 3-skeleton. In dimension 4 homeomorphism generally does not imply PL isomorphism, so this does not immediately reduce to a local question. However the objective is to construct bundle-like structures in a neighborhood of the skeleton, and homeomorphism of total spaces of bundles in most cases will imply isomorphism of bundles. So the question might be localized in this way, or just approached globally using relative versions of the local questions.

As above we start with manifold points in the skeleton. If the point has a 2- or 3-disk neighborhood then the question reduces to local flatness of boundaries or 2-manifolds in a homology 4-manifold, see [Q2, FQ 9.3A]. If the point has a 1-disk neighborhood then a neighborhood looks homotopically like the mapping cylinder of a surface bundle over $\mathbb{R}$. This leads to the question: is it homeomorphic to such a mapping cylinder? If the surface fundamental group has subexponential growth (ie. the surface is S^2, $\mathbb{RP}^2$, T^2, or the Klein bottle) then this probably can be settled by current techniques, but the general case may have to wait on solution of the conjectures of section 1. Finally neighborhoods of isolated points in the skeleton correspond exactly to tame ends of 4-manifolds. Some of these are known not to be triangulable, so these would have to be among the points that the statement allows to be deleted. From here the analysis progresses to points in the closure of strata of three or four different dimensions. Again there are a small number of cases, each of which has a detailed local homotopical description.

We close with another historical note. After topology and algebra Wall progressed to the study of singularities of smooth maps. This area depends heavily on understanding stratified sets, and Wall's interest in characterizing topological stability [dPW] was a major motivation for conjecture 5.1.

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