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Algebraic and geometric surgery. (English)

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The aim of this beautiful book is to present a first but extensive introduction to Algebraic and Geometric Surgery Theory for a reader who already has some background in Topology. The book is also an excellent framework for various courses in Surgery Theory allowing students to consider the main literature on the topic from a worthy perspective. Furthermore, it represents a very readable book on Surgery Theory, and serves also to facilitate the reading of the famous and celebrated book of *C. T. C. Wall* [Surgery on compact manifolds, Math. Surv. Monogr. 69 (1999; Zbl 0935.57003)]. Through the book one can find basic treatments of Morse Theory (Chp. 2), Homotopy and Homology (Chp. 3), Poincaré Duality (Chp. 4), Bundles (Chp. 5), Cobordism Theory (Chp. 6), Embeddings, Immersions and Singularities (Chp. 7), Whitehead Torsion (Chp. 8), Poincaré Complexes and Spherical Fibrations (Chp. 9), Surgery on Maps (Chp. 10). The Exact Surgery Sequence is presented in Chp. 1 and proved in Chp. 13 after a detailed discussion on surgery obstructions in even- and odd-dimensions given in Chps. 11 and 12.

Surgery Theory developed by Browder, Novikov, Sullivan, and Wall in the 1960s is one of the most important tools in the study of compact n-dimensional manifolds in some category $\mathbb{H}$ ($\mathbb{H} = \mathrm{TOP}, \mathrm{PL}$ or DIFF). The interest of the present book is essentially concentrated in differentiable manifolds of dimensions greater than or equal to 5. For Surgery Theory on 4-dimensional topological manifolds see for example the book of *M. H. Freedman* and *F. S. Quinn* [Topology of 4-manifolds, Princeton Math. Ser. 39 (1990; Zbl 0705.57001)].

A classification of manifolds up to some equivalence (diffeomorphism, homeomorphism, homotopy, h- and s-cobordism etc) requires the construction of a complete set of algebraic invariants such that:

the invariants of a manifold are computable;

two manifolds are equivalent if and only if they have the same invariants;

there is given a list of non-equivalent manifolds realizing every possible set of invariants.

One way to prove that smooth manifolds are diffeomorphic is to first decide whether they are cobordant, and then to decide whether some cobordism can be modified by a sequence of surgeries on its interior to be an s-cobordism. It is well known that there is a complete diffeomorphism classification of n-manifolds for any $n \leq 2$, and this coincides with the homotopy classification. For any $n \geq 3$, there exist n-manifolds which are homotopy equivalent but not diffeomorphic. For $n = 3$ complete classifications are theoretically possible but have not been achieved in practice (also for the unsolved status of the three-dimensional Poincaré Conjecture). For $n \geq 4$, a complete classification is not possible since there does not exist a complete set of invariants for distinguishing the isomorphism class of a group from a finite presentation of it (on the other hand, such a group is a fundamental group of a closed connected 4-manifold). Hence, surgery methods for classifying manifolds are concerned with a different but interesting question. The problem is to say when a homotopy equivalence between closed manifolds in the category $\mathbb{H}$ is homotopic to a morphism in $\mathbb{H}$. Surgery theory provides a systematic procedure for deciding (at least in higher dimensions) whether a normal map between n-manifolds in the category $\mathbb{H}$ (that is, a map satisfying certain bundle-theoretic conditions) is bordant to a homotopy equivalence, and whether the bordism can be chosen to be a homotopy. The obstructions take values in the topological K-theory of linear bundles and in the algebraic L-theory of quadratic forms. The book under review develops sufficient machinery to prove the main result of the theory in the category DIFF which is the famous Surgery Exact Sequence. The formulations and proofs of the results are very precise and carefully written. Here we give an outline of the main result in Surgery Theory. Let X^n be a closed connected n-dimensional manifold in the category $\mathbb{H}$ ($\mathbb{H}$ = TOP, PL or DIFF) with fundamental group $\pi_1 = \pi_1(X)$, and orientation character $w : \pi_1 \to \{\pm 1\}$. In higher dimensions, the problem of determining the homotopy type and the cobordism class of X was successfully reduced to the determination of normal invariants and surgery obstruction groups. This reduction works also in dimension four, provided π_1 is good in the sense of [Freedman-Quinn, loc. cit.]. In recent years, many extensions of surgery obstruction groups have been successfully introduced to study various geometric problems in the theory of compact manifolds (see for example [Freedman-Quinn, loc. cit.; the author, Exact sequences in the algebraic theory of surgery, Math. Notes, Princeton 26 (1981; Zbl 0471.57012); Algebraic L-theory and topological manifolds, Camb. Tracts Math. 102 (1992; Zbl 0767.57002); C. T. C. Wall, loc. cit.; *C. T. C. Wall*, Lect. Notes Math. 343, 266-300 (1973; Zbl 0269.18010)], and the book under review together with its references). Let ξ_X^k be a linear bundle over X. Let us consider the set of triples (M, f, b), where M is a closed n-manifold in the category $\mathbb{H}$, $f : M \to X$ is a degree one map, and $b : \nu_M^k \to \xi_X^k$ is a linear bundle map covering f. Here ν_M^k denotes the stable normal bundle of an embedding of M into the $(n+k)$-sphere $\mathbb{S}^{n+k}$, for k sufficiently large with respect to n. Two triples (M_0, f_0, b_0) and (M_1, f_1, b_1) are said to be equivalent if there exist a cobordism $W \subset \mathbb{S}^{n+k} \times I$ ($I = [0,1]$) between M_0 and M_1, a map $F : W \to X$ extending f_0 and f_1, and a linear bundle map $B : \nu_W^k \to \xi_X^k$ extending b_0 and b_1. Let us denote by $\Omega_n^{\mathbb{H}}(X, \xi_X^k)$ the set of equivalence classes. Disjoint union gives the operation of addition which passes to equivalence classes. Let $\mathcal{N}_n^{\mathbb{H}}(X)$ be the union of the sets $\Omega_n^{\mathbb{H}}(X, \xi_X^k)$ over all k-plane bundles ξ_X^k. The elements of this set are called the normal invariants of X in the category $\mathbb{H}$. In the relative case, we include the condition that the normal map (M, f, b) in $\mathcal{N}_n^{\mathbb{H}}(X, \partial X)$, represented by the degree one map $f : (M, \partial M) \to (X, \partial X)$, induces a homotopy equivalence when restricted to ∂M. A theorem of Sullivan identifies normal invariants with homotopy classes of maps from X to $G/\mathbb{H}$, where $G/\mathbb{H}$ is the fiber of the classifying map $B\mathbb{H} \to BG$ (see for example [*Ib Madsen* and *R. J. Milgram*, The classifying spaces of surgery and cobordism of manifolds, Ann. Math. Stud. 92 (1979; Zbl 0446.57002)] and [C. T. C. Wall, 1999, loc. cit.]). Here, G denotes the stable group of homotopy self-equivalences of k-spheres $\mathbb{S}^k$. Thus we have isomorphisms $\mathcal{N}_n^{\mathbb{H}}(X) \cong [X, G/\mathbb{H}]$ and $\mathcal{N}_{n+1}^{\mathbb{H}}(X \times I, \partial(X \times I)) \cong [\Sigma X, G/\mathbb{H}]$, where ΣX denotes the suspension of X. Let $\mathcal{S}_n^h(X)$ be the set of h-cobordism classes in the category $\mathbb{H}$ of orientation preserving homotopy equivalences $h : M \to X$, where M is a closed connected n-manifold in $\mathbb{H}$. Two such maps are said to be equivalent if there is an h-

cobordism W between them, with a map from W to X extending the ones given on the boundary. The set $\mathcal{S}_n^h(X)$ is also called the structure set of X. Results on the structure set for a Poincaré n-complex X can be found in Chp. 13 of the book under review together with the surgery classification of smooth manifolds homotopy equivalent to spheres (according to Kervaire and Milnor), and the proof of the Generalized Poincaré Conjecture due to Smale. There is an obvious map

$$\eta_n : \mathcal{S}_n^h(X) \to \mathcal{N}_n^{\mathbb{H}}(X) \cong [X, G/\mathbb{H}]$$

which sends every pair (M, h) to the normal invariant (M, h, h^*), where h^* is the bundle map induced by the homotopy equivalence h on the corresponding bundles. To obtain from a normal invariant a map close to a homotopy equivalence, we would like to kill the homotopy classes below the middle dimension by surgery. In fact, if (M, f, b) is a normal invariant in $\mathcal{N}_n^{\mathbb{H}}(X)$, surgery along immersed spheres in M yields a degree one map $\bar{f} : \bar{M} \to X$ such that the induced homomorphisms $\bar{f}_* : \pi_1(\bar{M}) \to \pi_1(X)$ and $\bar{f}_{*i} : H_i(\bar{M}; \Lambda) \to H_i(X; \Lambda)$ are bijective, for every $i < [n/2]$. Moreover, the triples (M, f, b) and $(\bar{M}, \bar{f}, \bar{b})$ are equivalent in $\mathcal{N}_n^{\mathbb{H}}(X)$. Here $\Lambda = \mathbb{Z}[\pi_1]$ is the group ring of π_1 according to the identifications $\pi_1 = \pi_1(\bar{M}) = \pi_1(X)$. In even dimension $n = 2k$, let us consider the kernel $K_k = K_k(f; \Lambda)$ of the homomorphism

$$\bar{f}_{*k} : H_k(\bar{M}; \Lambda) \to H_k(X; \Lambda).$$

By the Hurewicz theorem, every element of K_k can be represented, up to homotopy, by a k-sphere immersed in $\bar{M}$. Let λ denote the bilinear map on K_k which associates to every pair of transverse k-spheres, immersed in $\bar{M}$, their intersection number in Λ. Self-intersection numbers can also be defined, but they lie in the quotient $\Lambda/\{a - (-1)^k \bar{a}\}$, where $^- : \Lambda \to \Lambda$ is the natural anti-automorphism of the group ring Λ sending every element $\sum n_g g$ to $\sum n_g w(g) g^{-1}$, for every $n_g \in \mathbb{Z}$ and $g \in \pi_1$. The indeterminacy results from the fact that at every self-intersection there is not a natural way to choose a first and a second sheet; the quotient is obtained simply by dividing out differences between the two possible choices. This yields a map $\mu : K_k \to \Lambda/\{a - (-1)^k \bar{a}\}$. By definition, the Wall obstruction surgery group $L_{2k}^h(\pi_1, w)$ is the set of stable equivalence classes of all $(-1)^k$-Hermitian forms (K_k, λ, μ), with the sum operation given by the orthogonal direct sum of representatives. There exists a surgery obstruction map

$$\sigma_n : \mathcal{N}_n^{\mathbb{H}} \cong [X, G/\mathbb{H}] \to L_n^h(\pi_1, w)$$

which sends every normal invariant (M, f, b) to the stable equivalence class of the corresponding $(-1)^k$-Hermitian forms (K_k, λ, μ). The odd dimensional case is a bit more complicated, but it is possible to adjust any construction to define the corresponding surgery obstruction groups $L_n^h(\pi_1, w)$ (see for example [C. T. C. Wall, 1999, loc. cit.] and Chp. 12 of the book under review for more details). We also have the Wall surgery obstruction group $L_{n+1}^h(\pi_1, w)$ obtained from the bordered $(n+1)$-manifold $X \times I$ by using the identification $\pi_1 = \pi_1(X \times I)$. This group acts on the set $\mathcal{S}_n^h(X)$, and the orbits of this action are precisely the point-inverses of the map η_n. Therefore, there is an induced map

$$\omega_n : L_{n+1}^h(\pi_1, w) \to \mathcal{S}_n^h(X).$$

The following is the basic result in Surgery Theory of compact manifolds.
The main theorem. Let X^n be a closed connected n-manifold in the category $\mathbb{H}$ ($\mathbb{H} =$ TOP, PL, or DIFF) with fundamental group $\pi_1 = \pi_1(X)$, and orientation character $w : \pi_1 \to \{\pm 1\}$. Then the surgery sequence

$$\cdots \to \mathcal{S}^h_{n+1}(X \times I, \partial(X \times I)) \to [\Sigma X, G/\mathbb{H}] \xrightarrow{\sigma_{n+1}} L^h_{n+1}(\pi_1, w) \xrightarrow{\omega_n} \mathcal{S}^h_n(X) \xrightarrow{\eta_n} [X, G/\mathbb{H}] \xrightarrow{\sigma_n} L^h_n(\pi_1, w)$$

is exact, for every $n \geq 5$. If $n = 4$, then the sequence is exact provided π_1 is good in the sense of [Freedman-Quinn, loc. cit.].
The surgery sequence allows us to study the problem of determining a homotopy equivalence in a fixed cobordism class, and also provides the conditions under which two homotopy equivalences are cobordant. There is a version for simple structures obtained by replacing L^h and $\mathcal{S}^h$ with L^s and $\mathcal{S}^s$. The superscript s means that the elements of the group L^s are the obstructions to surgery, up to simple homotopy equivalence. The definition of $\mathcal{S}^s$ uses the concept of s-cobordism which replaces that of h-cobordism. Finally, recall that the obstruction groups are periodic of period four, that is, $L_m \cong L_{m+4}$.
In conclusion, I read this fine and carefully written book with great pleasure, and highly recommend it for everyone who wants to undertake a deeper study of Surgery Theory and its Applications.

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